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# Nonlinear initial-value problems with positive global solutions *

John V. Baxley & Cynthia G. Enloe

**Abstract**

We give conditions on $m(t)$, $p(t)$, and $f(t,y,z)$ so that the nonlinear initial-value problem

$$\frac{1}{m(t)}(p(t)y')' + f(t,y,p(t)y') = 0, \quad \text{for } t > 0,$$
$$y(0) = 0, \quad \lim_{t\to 0^+} p(t)y'(t) = B,$$

has at least one positive solution for all $t > 0$, when $B$ is a sufficiently small positive constant. We allow a singularity at $t = 0$ so the solution $y'(t)$ may be unbounded near $t = 0$.

## 1 Introduction

We consider the initial-value problem

$$\frac{1}{m(t)}(p(t)y')' + f(t,y,p(t)y') = 0, \quad t > 0, \tag{1.1}$$
$$y(0) = 0, \quad \lim_{t\to 0^+} p(t)y'(t) = B, \quad B > 0. \tag{1.2}$$

We allow a singularity at $t = 0$, and so $y'(t)$ may not be bounded near $t = 0$. However, we require of a solution that it be continuous at $t = 0$, satisfy (1.1) a.e. on some interval $(0, \delta)$, and satisfy (1.2). The singularity may be caused by the behavior of $m$ or $p$ or $f$ near $t = 0$ or by some combination of them.

This problem was considered earlier by Zhao [5] and by Maagli and Masmoudi [4]. In particular, [5] considered the case that $m \equiv p \equiv 1$ while [4] required that $m \equiv p$. In each of these papers, only one of the initial conditions ($y(0) = 0$) was imposed and conditions were specified which guaranteed that this "incomplete" initial-value problem has infinitely many positive solutions existing on the entire interval $(0, \infty)$. Both papers viewed the problem as a boundary-value problem by imposing a condition at $\infty$, namely

$$\lim_{t\to\infty} \frac{y(t)}{r(t)} = c > 0, \tag{1.3}$$

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where $r(t) = \int_0^t (p(s))^{-1}\,ds$. In [5], $r(t)$ reduces to $r(t) = t$. In both [4] and [5], the Schauder fixed point theorem is the main tool and the hypotheses imposed allow the authors to prove existence of at least one solution of the boundary-value problem for $c$ sufficiently small.

Here, we shall treat the problem in the initial-value form (1.1), (1.2). We shall impose conditions rather close to those of [4] and [5], and prove that our initial-value problem has at least one positive solution for $B$ sufficiently small. Our methods use initial-value techniques, similar to those used already in [1, 2], and are completely different from the previous papers discussed above. To get started, we must have a local solution on some interval $(0, \delta)$ and for that purpose, we need a slight generalization of the classical theorem of Carathéodory [3], which we provide in Section 2.

In Section 3, we prove our main result, which we state below. Let $r(t) = \int_0^t (p(s))^{-1}\,ds$ and assume that

M1: $p(t)$ and $m(t)$ are positive and continuous on $(0, \infty)$;

M2: $\frac{1}{p(t)} \in L^1(0, 1)$;

M3: for some positive number, $D < \infty$,

$$f : (0, \infty) \times (0, Dr(\infty)) \times (0, D) \to \mathbb{R}$$

is a measurable function on $(0, \infty) \times (0, Dr(\infty)) \times (0, D)$ and $f(t, \cdot, \cdot)$ is continuous on $(0, Dr(\infty)) \times (0, D)$ for each fixed $t \in (0, \infty)$;

M4:

$$|f(t, y, z)| \le h_1(t, y, z)y + h_2(t, y, z)z$$

where $h_1(t, y, z) \to 0$ and $h_2(t, y, z) \to 0$ as $(y, z) \to (0, 0)$, $h_1$ and $h_2$ are nonnegative, and for $\alpha > 0$, let $h(t, y, z) = h_1(t, y, z)r(t) + h_2(t, y, z)$,

$$g_\alpha(s) = \sup\{h(s, y, z) : 0 < y < \alpha r(s), 0 < z < \alpha\}, s > 0,$$

and $m(s)g_\alpha(s) \in L^1(0, \infty)$ for sufficiently small $\alpha > 0$.

**Theorem 1.1** *Under assumptions M1–M4, there exists $\gamma > 0$ so that $B \in (0, \gamma)$ implies that the initial-value problem (1.1), (1.2) has at least one solution existing for $0 < t < \infty$ and satisfying*

$$\frac{B}{2} < p(t)y'(t) < \frac{3B}{2},$$
$$\frac{Br(t)}{2} < y(t) < \frac{3Br(t)}{2},$$

*for $0 < t < \infty$. Moreover, the two limits*

$$\lim_{t\to\infty} \frac{y(t)}{r(t)}, \quad \lim_{t\to\infty} p(t)y'(t)$$

*exist, and if $r(\infty) = \infty$, the two limits are equal.*

Other than the fact that [4] requires that $m \equiv p$, the only substantive difference in our hypotheses is that we do not require that $h_1$, $h_2$ be nondecreasing with respect to $y$ and $z$, as they do. Of course, we prove existence for an initial-value problem, not a boundary-value problem as they do.

The key to our proof is that our local existence theorem in Section 2 is formulated carefully to provide a lower bound on the length of the interval of existence. In applying it in Section 3, we show that this lower bound gives us a uniform lower bound on the length of the interval of existence, regardless of where in the interval $[0, \infty)$ we start the solution. Thus, we are able to step from 0 to $\infty$ inductively, without fear that the sum of the lengths of our intervals will converge, to complete the proof.

## 2 Local Solutions

In this section, we consider the initial-value problem

$$\frac{1}{m(t)}(p(t)y')' + f(t, y, p(t)y') = 0, \quad t > t_0, \tag{2.1}$$

$$y(t_0) = A, \quad \lim_{t \to t_0^+} p(t)y'(t) = B. \tag{2.2}$$

We use $x_1 = y$, $x_2 = p(t)y'$ to transform to the two-dimensional system

$$\begin{aligned} x_1' &= \frac{x_2}{p(t)} \\ x_2' &= -m(t)f(t, x_1, x_2) \end{aligned} \tag{2.3}$$

with initial conditions

$$\lim_{t \to t_0^+} x_1(t) = A, \quad \lim_{t \to t_0^+} x_2(t) = B. \tag{2.4}$$

Let $R(t) = \int_{t_0}^t (p(s))^{-1}\, ds$. We shall assume that

L1: There exists $b > t_0$ such that $p(t)$ and $m(t)$ are positive and continuous on $(t_0, b)$.

L2: $\frac{1}{p(t)} \in L^1(t_0, b)$.

L3: $f : S \to \mathbb{R}$, where $S = \{t_0 < t \le b, A + cR(t) < y < A + dR(t), c < z < d\}$, and $f$ is measurable in $t$ for each fixed $(y, z)$ and continuous in $(y, z)$ for each fixed $t$.

L4: There exists $h(t) \in L^1(t_0, b)$ such that $m(t)|f(t, y, z)| \le h(t)$, almost everywhere on the set $S$.

We shall prove the following generalization of Carathéodory's local existence theorem. The proof follows the same general lines as the well-known proof in [3].

**Theorem 2.1** *Suppose hypotheses L1-L4 are satisfied. Let* $0 < d^* < \min\{d - B, B - c\}$ *and suppose* $\beta \in (0, b)$ *and satisfies* $\int_{t_0}^{t_0+\beta} h(s)ds < d^*$*. Then, the initial-value problem (2.1), (2.2) has a solution existing on the interval* $[t_0, t_0+\beta]$ *and satisfies*

$$\begin{gathered} A + cR(t) < x_1(t) < A + dR(t) \\ c < x_2(t) < d \end{gathered}$$

*for* $t_0 < t \leq t_0 + \beta$.

**Proof:** Choose a fixed integer $n > 1$. Let $h_n = \beta/n$ and let $t_k = t_0 + kh_n$ for $k = 1, 2, \cdots, n$. Define

$$u_{2,n}(t) = B, \quad \text{for} \quad t_0 \leq t \leq t_1.$$

Note that $B - d^* < u_{2,n}(t) < B + d^*$ for $t_0 \leq t \leq t_1$. Also define

$$u_{1,n}(t) = A + \int_{t_0}^{t} \frac{u_{2,n}(s)}{p(s)}\, ds, \quad \text{for} \quad t_0 \leq t \leq t_1.$$

It follows that

$$(B - d^*)R(t) < u_{1,n}(t) - A < (B + d^*)R(t)$$

and so

$$A + (B - d^*)R(t) < u_{1,n}(t) < A + (B + d^*)R(t).$$

Thus, $(t, u_{1,n}(t), u_{2,n}(t)) \in S$ for $t_0 \leq t \leq t_1$.

We extend the pair $(u_{1,n}, u_{2,n})$ to the entire interval $[t_0, t_0+\beta]$ by recursively defining the pair on the subintervals $[t_{j-1}, t_j]$. Thus, for each $j = 2, 3, \cdots, n$, we define

$$\begin{aligned} u_{2,n}(t) &= B - \int_{t_0}^{t-h_n} m(s) f(s, u_{1,n}(s), u_{2,n}(s))\, ds, \quad \text{for} \quad t_{j-1} \leq t \leq t_j, \\ u_{1,n}(t) &= A + \int_{t_0}^{t} \frac{u_{2,n}(s)}{p(s)}\, ds, \quad \text{for} \quad t_{j-1} \leq t \leq t_j. \end{aligned}$$

(The measurability of the integrand in the integral for $u_{2,n}$ follows from L3 by approximating with simple functions.) Using L4, we have

$$\begin{aligned} |u_{2,n}(t) - B| &\leq \int_{t_0}^{t-h_n} m(s)|f(s, u_{1,n}(s), u_{2,n}(s))|ds \\ &\leq \int_{t_0}^{t_0+\beta} h(s)ds < d^*, \end{aligned}$$

and therefore, $B - d^* < u_{2,n}(t) < B + d^*$. Further,

$$(B - d^*)R(t) < u_{1,n}(t) - A < (B + d^*)R(t),$$

and so

$$A + (B - d^*)R(t) < u_{1,n}(t) < A + (B + d^*)R(t).$$

These inequalities show that $(t, u_{1,n}(t), u_{2,n}(t))$ remains in $S$ on each subinterval and the recursive definition is allowed. Moreover, the two sequences $\{u_{1,n}\}$, $\{u_{2,n}\}$ are uniformly bounded on $t_0 \le t \le t_0 + \beta$.

We shall show that these sequences are equicontinuous so that Ascoli's theorem may be applied. Suppose $t_0 \le t \le t^* \le t_0 + \beta$. Then

$$|u_{1,n}(t) - u_{1,n}(t^*)| = \Big| \int_t^{t^*} \frac{u_{2,n}(s)}{p(s)}\, ds \Big| \le \int_t^{t^*} \frac{Q}{p(s)}\, ds,$$

where $Q = \max\{|B - d^*|, |B + d^*|\}$. Moreover,

$$\begin{aligned} |u_{2,n}(t) - u_{2,n}(t^*)| &= \Big| \int_{t-h_n}^{t^*-h_n} m(s) f(s, u_{1,n}(s), u_{2,n}(s))\, ds \Big| \\ &\le \int_{t-h_n}^{t^*-h_n} h(s)\, ds. \end{aligned}$$

The desired equicontinuity follows from absolute continuity of the integral. Using Ascoli's theorem, we may assume without loss of generality that both sequences converge uniformly on $[t_0, t_0 + \beta]$ to limit functions $u_1(t)$, $u_2(t)$. We may use the Lebesgue dominated convergence theorem (for $u_{2,n}$ the dominating function is $h(s)$; for $u_{1,n}$, the dominating function is $(p(s))^{-1}$) to take limits under each integral sign as $n \to \infty$ to show that

$$\begin{aligned} u_1(t) &= \int_{t_0}^{t} \frac{u_2(s)}{p(s)}\, ds, \\ u_2(t) &= B - \int_{t_0}^{t} m(s) f(s, u_1(s), u_2(s))\, ds, \end{aligned}$$

for $t_0 \le t \le t_0 + \beta$, from which we obtain

$$\begin{aligned} u_2'(t) &= -m(t) f(t, u_1(t), u_2(t)), \\ u_1'(t) &= \frac{u_2(t)}{p(t)}, \end{aligned}$$

almost everywhere on $[t_0, t_0 + \beta]$.

The specific size of $\beta$ provided by the hypotheses of this last theorem is crucial for our main proof in the next section.

# 3 Proof of Main Theorem

First note that the hypotheses M1-M4 imply that the earlier hypotheses L1-L4 hold on any interval $(t_0, b)$ with $0 \le t_0 < b < \infty$, so we may apply Theorem 2.1 as needed.

From hypothesis M4, we have $m(s)g_{\alpha_0}(s) \in L^1(0,\infty)$ if $\alpha_0 > 0$ is sufficiently small. Further, M4 implies that $m(s)g_\alpha(s) \le m(s)g_{\alpha_0}(s)$ whenever $0 < \alpha < \alpha_0$ and also that $m(s)g_\alpha(s) \to 0$ as $\alpha \to 0$, for all $s > 0$. Thus, by the Lebesgue Dominated Convergence Theorem,

$$\int_0^\infty m(t)g_\alpha(t)dt \to 0 \quad \text{as } \alpha \to 0.$$

Hence, there exists $\delta \in (0,\alpha_0]$ such that $0 < \alpha < \delta$ implies

$$\int_0^\infty m(t)g_\alpha(t)dt < \frac{1}{4}.$$

We shall show that $\gamma = \frac{1}{2}\min\{D,\delta\}$, where $D$ is the number from our hypothesis M3, satisfies the requirements of our theorem. To apply Theorem 2.1, we pick $0 < C \le \gamma$, $d^* = C/2$, $c = 0$, $d = 2C$, $t_0 = 0$, $b = 1$, $B = C$, and $A = 0$. Note that $d^* = \frac{d}{4} < d - d/2 = d - C = d - B$ and $d^* < C = B - c$. So $d^* < \min\{d - B, B - c\}$.

By absolute continuity of the integral, there exists $\beta \in (0,b) = (0,1)$ so that for $k = 0,1,\cdots$,

$$\int_{k\beta}^{(k+1)\beta} \alpha m(s)g_\alpha(s)\,ds < d^*. \tag{3.1}$$

This last inequality, for $k = 0$ allows us to apply Theorem 2.1 to get a solution $y_1(t)$ on $[0,\beta]$ so that for $0 < t \le \beta$,

$$0 < y_1(t) < 2Cr(t), \quad 0 < p(t)y_1'(t) < 2C\,. \tag{3.2}$$

Integrating (1.1) from 0 to $t$ and using M4, we obtain

$$\begin{aligned} |p(t)y_1'(t) - C| &\le \int_0^t m(s)|f(s,y_1(s),p(s)y_1'(s))|\,ds \\ &= 2C\int_0^t m(s)h(s,y_1(s),p(s)y_1'(s))\,ds \\ &< 2C\int_0^t m(s)g_{2C}(s)\,ds < \frac{C}{2}, \end{aligned}$$

if $t \in [0,\beta]$. Hence,

$$\frac{C}{2} < p(t)y_1'(t) < \frac{3C}{2}.$$

Then

$$y_1(t) = \int_0^t \frac{p(s)y_1'(s)}{p(s)}ds$$

and so

$$\frac{C}{2}r(t) < y_1(t) < \frac{3C}{2}r(t).$$

We claim that for $k = 2, 3, \dots$, there exists a solution $y_k(t)$ of (1.1) on the interval $(k-1)\beta \le t \le k\beta$ so that

$$\begin{aligned} y_{k+1}(k\beta) &= y_k(k\beta), \\ y'_{k+1}(k\beta) &= y'_k(k\beta), \end{aligned}$$

for $k \ge 1$, and

$$\begin{aligned} \frac{C}{2}r(t) < y_k(t) < \frac{3C}{2}r(t), \\ \frac{C}{2} < p(t)y'_k(t) < \frac{3C}{2}, \end{aligned} \tag{3.3}$$

for $(k-1)\beta \le t \le k\beta$.

Noting that $y_1(t)$ has already been constructed, we continue by induction and assume that $y_1(t), y_2(t), \dots, y_n(t)$ have been constructed. Next, we construct $y_{n+1}(t)$. To use Theorem 2.1, we keep $C$, $d^*$, $c$, $d$, $b$, and $\beta$ as before and let $t_0 = n\beta$, $A = A_n = y_n(n\beta)$, and $B = B_n = p(n\beta)y'_n(n\beta)$. Inequality (3.1) for $k = n$ allows us to apply Theorem 2.1 to get a solution $y_{n+1}(t)$ of (1.1) on $[t_0, t_0 + \beta] = [n\beta, (n+1)\beta]$ so that $y_{n+1}(n\beta) = y_n(n\beta)$ and

$$\begin{aligned} A_n \le y_{n+1}(t) < A_n + 2C \int_{t_0}^{t} \frac{1}{p(s)} ds, \\ 0 < p(t)y'_{n+1}(t) < 2C. \end{aligned}$$

To complete the induction, we must verify that $y_{n+1}(t)$ satisfies (3.3). Define $y(t)$ for $0 \le t \le (n+1)\beta$ by $y(t) = y_k(t)$ for $(k-1)\beta \le t \le k\beta$.

Since

$$\frac{C}{2} < p(s)y'(s) < \frac{3C}{2}, \quad \text{for } 0 \le s \le n\beta$$

and

$$0 < p(s)y'(s) < 2C, \quad \text{for } n\beta \le s \le (n+1)\beta,$$

it follows that

$$0 < p(s)y'(s) < 2C$$

for the larger interval, $0 \le s \le (n+1)\beta$, and it follows by integrating that

$$0 < y(t) < 2Cr(t), \quad \text{for } 0 \le t \le (n+1)\beta.$$

The calculation appearing just after (3.2) may now be repeated to show that

$$\frac{C}{2} < p(t)y'(t) < \frac{3C}{2}, \quad \text{for } 0 \le t \le (n+1)\beta,$$

which implies, as before, that

$$\frac{C}{2}r(t) < y(t) < \frac{3C}{2}r(t), \quad \text{for } 0 \le t \le (n+1)\beta.$$

Finally, we define $y(t)$ for $0 \le t < \infty$ by $y(t) = y_k(t)$ for $(k-1)\beta \le t < k\beta$, and each $k = 1, 2, \dots$. Clearly, $y(t)$ is the desired solution.

To investigate the limit of $p(t)y'(t)$ at infinity, we examine

$$\begin{aligned} p(t)y'(t) &= p(t_0)y'(t_0) + \int_{t_0}^{t} (p(s)y'(s))'\,ds \\ &= p(t_0)y'(t_0) - \int_{t_0}^{\infty} (m(s)f(s,y(s),p(s)y'(s))\mathcal{X}_{[t_0,t]}(s)\,ds. \end{aligned}$$

Since $(m(s)f(s,y(s),p(s)y'(s))\mathcal{X}_{[t_0,t]}(s) < m(s)g_{2C}(s)$, and $m(s)g_{2C}(s)$ is in $L^1(0,\infty)$, we can use the Lebesgue Dominated Convergence Theorem to take the limit of both sides as $t$ approaches $\infty$ and conclude that $\lim_{t\to\infty} p(t)y'(t)$ exists. If $r(\infty) = \infty$, then

$$\lim_{t\to\infty} \frac{y(t)}{r(t)} = \lim_{t\to\infty} \frac{y'(t)}{r'(t)} = \lim_{t\to\infty} p(t)y'(t),$$

which we have already proven to exist.

If $r(\infty) < \infty$, then $y(t)$ is a monotone increasing function which is bounded above by $\frac{3C}{2}r(\infty)$. Hence, $\lim_{t\to\infty} y(t)/r(t)$ exists.

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John V. Baxley (email: baxley@mthcsc.wfu.edu)
Cynthia G. Enloe
Department of Mathematics, Wake Forest University
Winston-Salem, NC 27109, USA