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An adaptive numerical method for the wave equation with a nonlinear boundary condition *

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Abstract

We develop an efficient numerical method for studying the existence and non-existence of global solutions to the initial-boundary value problem

$$\begin{gathered} u_{tt} = u_{xx} \quad 0 < x < \infty,\ t > 0, \\ -u_x(0,t) = h(u(0,t)) \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad 0 < x < \infty. \end{gathered}$$

The results by this numerical method corroborate the theory presented in [1]. Furthermore, they suggest that blow-up can occur for more general nonlinearities $h(u)$ with weaker conditions on the initial data f and g.

1 Introduction

In this paper, we consider the initial-boundary value problem

$$\begin{gathered} u_{tt} = u_{xx} \quad 0 < x < \infty,\ t > 0, \\ -u_x(0,t) = h(u(0,t)) \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad 0 < x < \infty. \end{gathered} \tag{1.1}$$

Here we assume that $h(u)$ is continuous with $\lim_{u\to\infty} h(u) = \infty$, g is continuous, and f is continuously differentiable. To motivate our work for problem (1.1), we point out that this problem has been recently studied by the authors in [1]. For completeness, the main results obtained in that paper are presented as follows:

Theorem 1.1 *There exists at least one mild solution of (1.1) on* $[0,\infty)\times[0,T_0)$ *for some* $T_0 > 0$. *Moreover, if* $h(u)$ *is Lipschitz continuous, then the solution is unique.*

Theorem 1.2 *Suppose that* $|h(u)| \le \rho(|u|)$ *with* $\rho(r) > 0$ *continuous, nondecreasing on* $[0,\infty)$*, and such that*

$$\int^{\infty} \frac{dr}{\rho(r)} = \infty,$$

then all mild solutions of (1.1) are global.

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Theorem 1.3 *Suppose that* $f(t) + \int_0^t g(s)ds \geq 0$ ($\not\equiv 0$) *on* $[0,\infty)$ *and that* $h(u) \geq \sigma(|u|)$ *with* $\sigma(r) > 0$ *continuous, nondecreasing on* $[0,\infty)$*, and such that*

$$\int^{\infty} \frac{dr}{\sigma(r)} < \infty,$$

then every mild solution of (1.1) blows up in finite time.

Theorem 1.4 *Suppose that* $\int_0^\infty f(t)dt + \int_0^\infty \int_0^t g(s)dsdt > 0$ *and* $h(u) \geq c|u|^p$ ($p > 1$, $c > 0$)*, then the mild solution of (1.1) blows up in finite time.*

In [1], we point out that the blow-up occurs on the boundary $x = 0$ only. Moreover, using asymptotic techniques for integral equations [4] we establish the following blow-up rates: Letting T_b be the blow-up time,

- If $h(u) \sim u^p$, then $u(0,t) \sim \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} (T_b - t)^{-\frac{1}{p-1}}$ as $t \to T_b$;
- If $h(u) \sim e^u$, then $u(0,t) \sim \log\left(\frac{1}{T_b - t}\right)$ as $t \to T_b$.

The goal of this paper is to develop a numerical method for solving (1.1). In Section 2 we discuss the numerical approximation while in Section 3, we present numerical examples. In Section 4, we conclude with some remarks.

2 Time-Adaptive Method

We begin this section by integrating (1.1) along characteristics to obtain the following integral representation of solutions: For $t \leq x$,

$$u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds, \tag{2.1}$$

and for $t > x$,

$$\begin{aligned} u(x,t) =& \frac{1}{2}[f(t+x) + f(t-x)] + \frac{1}{2}\Big[\int_0^{t+x} g(s)ds + \int_0^{t-x} g(s)ds\Big] \\ &+ \int_0^{t-x} h(u(0,\tau))d\tau. \end{aligned} \tag{2.2}$$

A solution to the integral equations (2.1)-(2.2) defines a mild solution to the problem (1.1). Furthermore, if the initial data f and g are smooth and satisfy some compatibility conditions, then one can argue that a solution of (2.1)-(2.2) is also a strong solution of (1.1). Our numerical method will focus on the approximation of (2.1)-(2.2) rather than (1.1). This provides an efficient scheme which does not require a rather strong regularity assumption on the initial data.

Substituting $x = 0$ in (2.2), we get the Volterra integral equation

$$u(0,t) = f(t) + \int_0^t g(s)ds + \int_0^t h(u(0,\tau))d\tau. \tag{2.3}$$

Since blow-up occurs only on the boundary $x = 0$, a special attention will be devoted to the development of an approximation of $u(0,t)$ particularly near the blow-up time T_b. Once this is achieved, the approximations of the blow-up time T_b and $u(0,t)$ are used to compute $u(x,t)$ from the equations (2.1)-(2.2). To this end, differentiating (2.3) we get the following differential equation for $u(0,t)$:

$$\frac{du(0,t)}{dt} = \frac{df(t)}{dt} + g(t) + h(u(0,t)).$$

Let $\Delta t > 0$ be sufficiently small. Using Taylor approximation (formally) we observe that

$$u(0,t+\Delta t) - u(0,t) = \Delta t \frac{du(0,t)}{dt} + \frac{d^2u(0,\xi)}{dt^2}\Delta t^2, \quad \xi \in (t, t+\Delta t).$$

A key idea in our scheme is to adapt the time step in order to keep the quantity $|u(0,t+\Delta t) - u(0,t)| \sim |\Delta t \frac{du(0,t)}{dt}|$ bounded by a fixed constant. Since $h(u) \to \infty$ as $u \to \infty$ and blow-up occurs at T_b we see that $\frac{du(0,t)}{dt} \to \infty$, as $t \to T_b$. In particular, as $t \to T_b$ the size of the time step must approach zero if the magnitude of $\Delta t \frac{du(0,t)}{dt}$ is to remain bounded by a fixed constant. This forces the numerical approximation not to go beyond the blow-up time. Making use of this fact we now present a time-adaptive algorithm for computing $u(0,t)$ and the blow-up time T_b.

Let $\Delta t_{\min}$ and $\Delta t_{\max}$ be fixed numbers with $0 < \Delta t_{\min} < \Delta t_{\max} < \infty$. Let u_0^i be the approximation of $u(0,t_i)$ with $t_0 = 0$ and $\Delta t_i = t_i - t_{i-1} \in [\Delta t_{\min}, \Delta t_{\max}]$. Denote by

$$(u_t)_0^i = \frac{u_0^i - u_0^{i-1}}{\Delta t_i}$$

the difference approximations of $u_t(0,t_i)$. Guess an initial time step Δt_1 and fix a scaling factor $\alpha > 1$. Choose constants d_l and d_u such that $d_l < d_u$. The following is a pseudo code for the time-adaptive algorithm that we have developed:

for $i = 1, 2, \dots$
if $\Delta t_i |(u_t)_0^i| \le d_u$
then
  if $i \ge 2$
  then
    if $\Delta t_i < \Delta t_{\max}$
    then
      if $\Delta t_i |(u_t)_0^i|$ and $\Delta t_{i-1} |(u_t)_0^{i-1}| \le d_l$
      then
        $\Delta t_{i+1} = \min(\alpha\, \Delta t_i, \Delta t_{\max})$
      else
        $\Delta t_{i+1} = \Delta t_i$
      end
    else

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      Δt_{i+1} = Δt_i
    end
  else
    Δt_{i+1} = Δt_i
  end
  i = i + 1
else
  Δt_i = Δt_i / α
end
done
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Our adaptive method changes the current time step if one of the following two cases arises. The first case is that if $\Delta t_i \left|(u_t)_0^i\right| > d_u$ then the approximated quantity $|u_0^{i+1} - u_0^i| > d_u$. In this case the time step is decreased by a factor of $1/\alpha$ and the solution is recomputed at the new time step $(1/\alpha)\Delta t_i$. The second case is that if the current time step $\Delta t_i < \Delta t_{\max}$, $|u_0^{i+1} - u_0^i| \leq d_l$ and $|u_0^i - u_0^{i-1}| \leq d_l$, then this indicates that the time steps used for the last two iterations are very conservative. Hence, the scheme increases this time step to $\min(\alpha \Delta t_i, \Delta t_{\max})$ in order to save computation time. It is easy to see that near the blow-up time, the time step Δt_i will decrease until it reaches $\Delta t_{\min}$. When this happens the computation stops, and the current time is an approximation of the blow-up time T_b. We remark that the accuracy of the approximations of T_b depends on the choice of $\Delta t_{\min}$.

To compute u_0^i we combine the Runge-Kutta numerical method (see for example, [5]) with the above time-adaptive algorithm: Let $u_0^0 = f(0)$ and

$$\begin{aligned}
k_1 &= \Delta t_{i+1} y\big(t_i, u_0^i\big) \\
k_2 &= \Delta t_{i+1} y\big(t_i + \frac{\Delta t_{i+1}}{2}, u_0^i + \frac{1}{2}k_1\big) \\
k_3 &= \Delta t_{i+1} y\big(t_i + \frac{\Delta t_{i+1}}{2}, u_0^i + \frac{1}{2}k_2\big) \\
k_4 &= \Delta t_{i+1} y\big(t_{i+1}, u_0^i + k_3\big),
\end{aligned}$$

where $i = 0, 1, 2, \dots,$ and Δt_{i+1} is determined by the time-adaptive method developed above. Compute u_0^{i+1} as follows:

$$u_0^{i+1} = u_0^i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right).$$

Now, to approximate the solution of (2.1)-(2.2) we choose $x_{\max} > 0$ and divide the interval $[0, x_{\max}]$ into uniform mesh x_j with $\Delta x = x_j - x_{j-1}$, $j = 0, 1, \dots, m$. Denote by $S^n(a, b, I)$ the Simpson's numerical method for integrating a function $I(t)$ on the interval (a, b) with n subdivisions, and let $P^h(t)$ be the cubic interpolant of the function $h(u(0,t))$ at the mesh points t_i. Then we let u_j^i be the approximation of $u(x_j, t_i)$ and compute u_j^i as follows: For $t_i \leq x_j$,

$$u_j^i = \frac{1}{2}[f(x_j + t_i) + f(x_j - t_i)] + \frac{1}{2}S^n(x_j - t_i, x_j + t_i, g),$$

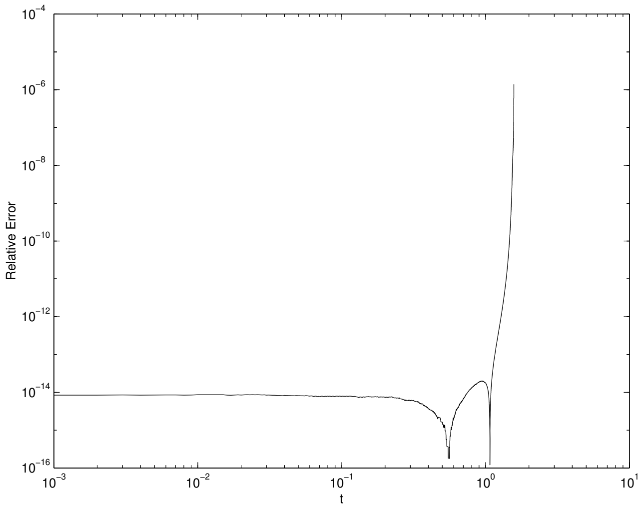

Figure 1: The relative error between the computed function $u(0,t)$ and the exact solution $\tan t$.

and for $t_i > x_j$,

$$\begin{aligned} u_j^i =& \frac{1}{2}[f(t_i + x_j) + f(t_i - x_j)] \\ &+ \frac{1}{2}[S^n(0, t_i + x_j, g) + S^n(0, t_i - x_j, g)] + S^n(0, t_i - x_j, P^h). \end{aligned}$$

In the next section we present numerical results which indicate the accuracy of such an adaptive numerical scheme in computing both $u(x,t)$ and the blow-up time T_b.

3 Numerical Results

The numerical method developed in the previous section is now used to corroborate and complement theoretical results in our earlier paper [1]. For the rest of this section let $\Delta t_{\max} = 10^{-3}$, $\Delta t_{\min} = 10^{-7}$, $\alpha = 2$, $d_u = 1$, $d_l = 0.1$, $n = 10$, $x_{\max} = 5$, and $m = 200$. In the first example we present the accuracy of our method. To this end, we choose $f = 0$, $g = 1$ and $h(u) = u^2$. It is not difficult to show that $u(0,t) = \tan t$, and hence blow-up occurs at $t = \pi/2$. In Figure 1 we show the relative error $\dfrac{|u_0^i - \tan t_i|}{\Delta t_i}$. The computed blow-up time $T_b = 1.5704$.

In the second example we let $f(x) = -(x-2)^2 + 4$, $g(x) = 0$ and $h(u) = u^3$. Notice that this choice of initial data does not satisfy the assumptions of Theorems 1.3-1.4 in Section 1. However, the numerical results presented in

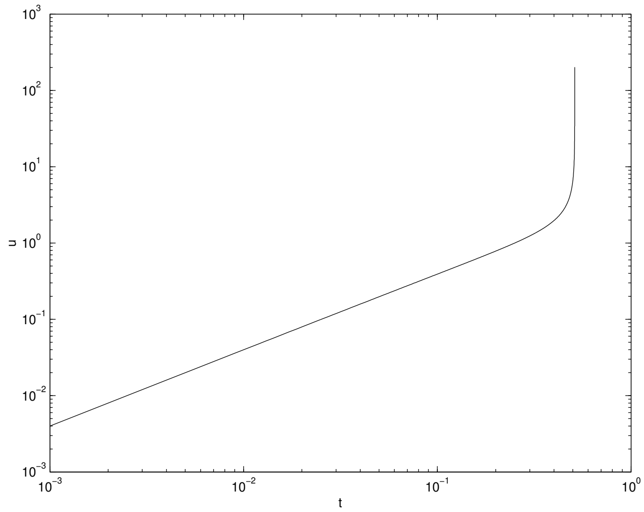

Figure 2: The computed function $u(0,t)$ for the data $f(x) = -(x-2)^2 + 4$, $g(x) = 0$ and $h(u) = u^3$.

Figures 2-3 indicate that blow-up occurs for this choice of functions with an approximated blow-up time $T_b = 0.5118$.

In our third numerical experiment we examine whether blow-up occurs for nonlinearities such as $h(u) = (1+u)[\log(1+u)]^p$ with initial data that do not satisfy the assumptions of Theorem 1.4. In Figure 4 we present the numerical results of $u(0,t)$ for the case $p = 6$, $f(x) = 3e^{-x}\cos(20x) - 0.1$ and $g(x) = 0$, and in Figure 5 we display the 3-D graph of the function $u(x,t)$. We remark that the blow-up time is $T_b = 0.22296$.

Using our numerical scheme, we have successfully verified the blow-up rates given in Section 1 for the functions e^u and u^p $(p > 1)$. We now use this method to examine the blow-up rate for the function $h(u) = (1+u)[\log(1+u)]^p$. Before presenting the numerical results we formally derive such a rate. Near the blow-up time the values $\frac{df(t)}{dt}$ and $g(t)$ are negligible when compared to $u(0,t)$, and hence

$$\frac{du(0,t)}{dt} \sim (1 + u(0,t))\,[\log(1 + u(0,t))]^p\,.$$

Integrating the above we find

$$\int_{u(0,t)}^{\infty} \frac{du}{(1+u)\,[\log(1+u)]^p} \sim \int_t^{T_b} dt.$$

Solving for u we get

$$u(0,t) \sim e^{\left(\frac{1}{(p-1)(T_b - t)}\right)^{\frac{1}{p-1}}} - 1. \tag{3.1}$$

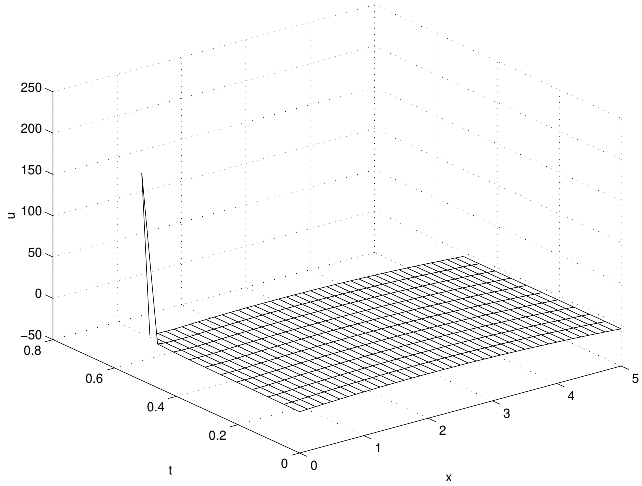

Figure 3: The solution $u(x,t)$ for the data $f(x) = -(x-2)^2 + 4$, $g(x) = 0$ and $h(u) = u^3$.

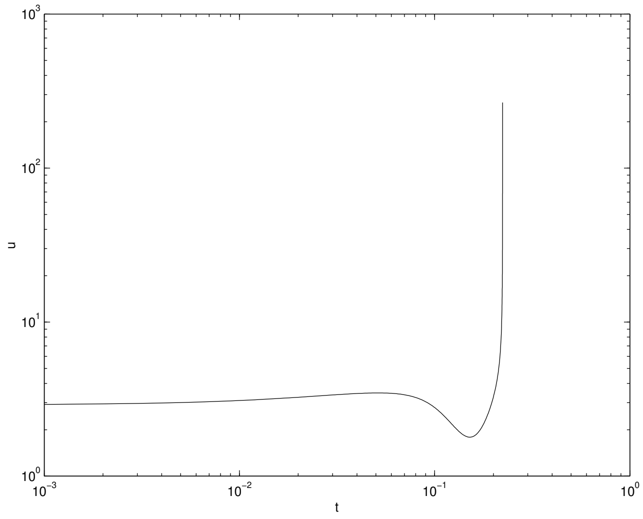

Figure 4: The computed function $u(0,t)$ for the data $f(x) = 3e^{-x}\cos(20x) - 0.1$ and $g(x) = 0$ and $h(u) = (1+u)[\log(1+u)]^6$.

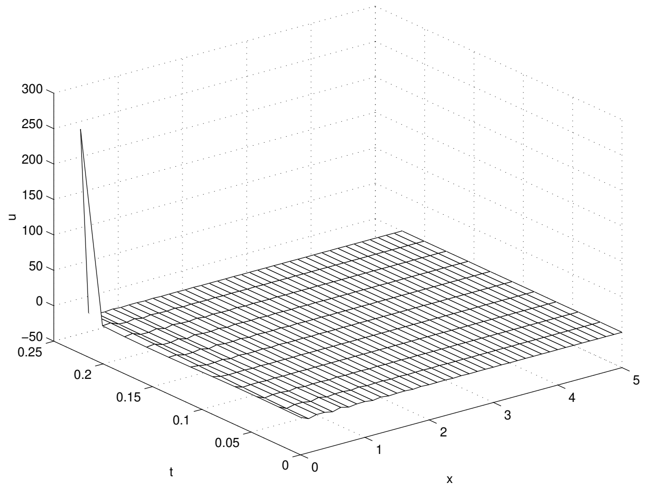

Figure 5: The computed solution $u(x,t)$ for the data $f(x) = 3e^{-x}\cos(20x) - 0.1$ and $g(x) = 0$ and $h(u) = (1+u)[\log(1+u)]^6$.

In Table 1 we give numerical results that verify such a blow-up rate. For this computational purpose we use the following equivalent form of (3.1)

$$\frac{1}{p-1} = (T_b - t)[\log(1+u(0,t))]^{p-1}.$$

Table 1: The blow-up rate for the function $h(u) = (1+u)(\log(1+u))^p$.

p	4	6	8	10
Conjectured: $\frac{1}{p-1}$	0.3333	0.2	0.1429	0.1111
Approximation	0.3205	0.1973	0.1411	0.1106

4 Concluding Remarks

The objective of this paper is to develop a numerical approximation for studying the existence and non-existence of global solutions to the wave equation with a nonlinear boundary condition. Our numerical results indicate that such a scheme is very accurate and efficient for computing the blow-up time, the blow-up rate, and the solution. These results also open up several theoretical questions: 1) How much can the conditions on the initial data f and g be relaxed for blow-up to occur? 2) Can one improve Theorem 1.4 for weaker nonlinearties such as $h(u) = (1+u)[\log(1+u)]^p$ $(p > 1)$? 3) Can one prove the blow-up rate given by (3.1) for such nonlinearities? Our future research efforts will focus on such questions as well as the application of time-adaptive methods to a system of wave equations coupled in the boundary conditions discussed in [2].

Finally, it is worth mentioning that one can also devise a numerical method by directly approximating the Volterra integral equation (2.3) using a combination of the time-adaptive method presented here and numerical quadrature methods for Volterra integral equations [3].

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