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Nonlinear equations with natural growth terms and measure data *

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Abstract

We consider a class of nonlinear elliptic equations containing a p-Laplacian type operator, lower order terms having natural growth with respect to the gradient, and bounded measures as data. The model example is the equation

$$-\Delta_p(u) + g(u)|\nabla u|^p = \mu$$

in a bounded set $\Omega \subset \mathbb{R}^N$, coupled with a Dirichlet boundary condition. We provide a review of the results recently obtained in the absorption case (when $g(s)s \geq 0$) and prove a new existence result without any sign condition on g, assuming only that $g \in L^1(\mathbf{R})$. This latter assumption is proved to be optimal for existence of solutions for any measure μ.

1 Introduction

In this work we focus our attention on nonlinear Dirichlet problems whose model is

$$\begin{gathered} -\Delta_p(u) + g(u)|\nabla u|^p = \mu \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $p > 1$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, and μ is a bounded Radon measure on Ω which is a bounded subset of $\mathbb{R}^N$.

Recently, many researchers have investigated the possibility to find solutions of (1.1) under the assumption that $g(s)s \geq 0$, in which case the term $g(u)|\nabla u|^p$ is said to be an absorption term. In this case a detailed picture of what happens is now available, according to the growth at infinity of $g(s)$ and to whether the measure μ charges or not sets of zero p-capacity (the capacity defined in $W_0^{1,p}(\Omega)$). In the next section, we try to give a quick review of these results and explain the main features of the problem in the absorption case, both for elliptic and for parabolic equations.

No results for general measures μ are known to our knowledge if the sign condition is not assumed to hold, possibly including the reaction case in which

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$g(s)s \le 0$. It is the purpose of the third section of this paper to give new results in this situation. Eventually, these new results seem to fit perfectly those proved in the absorption case, and we will prove (stated in more generality in Section 3) the following theorem, which extends that proved in [44] (under the same assumptions) for data $\mu \in L^1(\Omega)$.

Theorem 1.1 *Let μ be a nonnegative bounded Radon measure on Ω. Assume that $g \in L^1(\mathbb{R})$. Then there exists a distributional solution u of* (1.1).

Next we will give an example which somehow expresses that the assumption $g \in L^1(\mathbb{R})$ in Theorem 1.1 is optimal; if μ is the Dirac mass, we prove that no solution can be obtained by approximation. In particular, in the reaction case ($g(s)s \le 0$), if μ is approximated by a sequence of smooth functions, the sequence of approximating solutions converges to a solution of (1.1) if $g \in L^1(\mathbb{R})$, while it blows up everywhere in Ω if $g \notin L^1(\mathbb{R})$. We recall that in [34] the absorption case $g(s)s \ge 0$ had already been studied; in that situation if the Dirac mass is approximated by smooth functions, the approximated solutions still converge to a solution of the problem if $g \in L^1(\mathbb{R})$, while they converge to zero if $g \notin L^1(\mathbb{R})$. Thus, even if for different reasons, in both cases the assumption $g \in L^1(\mathbb{R})$ turns out to be optimal.

2 The absorption case: a quick review

A wide literature has dealt with elliptic and parabolic equations with measure data in the last decades. In particular, the techniques of a priori estimates and compactness of approximating solutions, firstly introduced in [14], have been proved to work well enough for pseudomonotone operators of Leray-Lions type ([32]), providing several existence results in case of L^1 data. The presence of absorbing lower order terms (i.e. satisfying a sign condition) often brings in this kind of problems new features; for instance, as in [18], [15], lower order terms may have a regularizing effect on solutions of problems with L^1 data. The two main examples are the following problems:

$$\begin{aligned} -\Delta_p u + |u|^{r-1}u = \mu \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega\,, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} -\Delta_p u + u|\nabla u|^p = \mu \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega\,. \end{aligned} \tag{2.2}$$

If $\mu \in L^1(\Omega)$, problem (2.1) has a solution in $W_0^{1,q}(\Omega)$ for any $q < \frac{pr}{r+1}$, while problem (2.2) has a finite energy solution u, which belongs to $W_0^{1,p}(\Omega)$. In general, if the lower order term is absorbing, one can prove the existence of a solution with $L^1(\Omega)$ data; for instance, the problem:

$$\begin{aligned} -\Delta_p(u) + H(x,u,\nabla u) = f \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \partial\Omega\,, \end{aligned} \tag{2.3}$$

with $\xi \mapsto H(x,s,\xi)$ growing at most like $|\xi|^p$ (the so-called natural growth), always admits a solution if $f \in L^1(\Omega)$ (see [42]). In fact, dealing with the limit growth for $H(x,s,\xi)$ is not that easy and requires the strong compactness of truncations in the energy space; on the other hand, these truncation methods can be adapted to several different contexts if still dealing with $L^1(\Omega)$ data, as obstacle problems or more general operators (see [9], [10], [28]).

When trying to extend the previous results to measure data, it turns out that precisely the regularizing effect mentioned above may be responsible for nonexistence of solutions. Actually, this fact was first observed in the pioneering works of H. Brezis ([20, 21]) and in a whole series of papers (see [4, 25, 30, 47, 45, 46] and the references therein) concerning problem (2.1) in the linear case $p = 2$. More recently, the nonlinear case $p \neq 2$ has been dealt with in [37, 38, 11]. Summing up these results, it is proved that problem (2.1) has a solution for every given bounded measure μ only if $r < \frac{N(p-1)}{N-p}$, while if $r \geq \frac{N(p-1)}{N-p}$ then no solution exists if μ charges sets of zero q-capacity with $\frac{q(p-1)}{q-p} < r$ (a necessary and sufficient condition in the linear case $p = 2$ is given in [30]). As an example, if μ is the Dirac mass, then a solution of (2.1) exists if and only if $r < \frac{N(p-1)}{N-p}$.

However, the statement of nonexistence of solutions needs to be suitably precised; how shall we express such a failure of existence? Three different ways have been suggested so far in previous works: firstly, nonexistence of solutions for a general problem as (2.3) may be deduced from removable singularity type results. This is a classical approach, and mostly used for linear operators; a set K is removable if any solution of

$$\begin{gathered} -\Delta_p(u) + H(x,u,\nabla u) = f \quad \text{in} \quad \Omega \setminus K, \\ u \in W^{1,p}(\Omega \setminus K)\,, \end{gathered}$$

can be proved to be a solution in the whole of Ω. If K is removable, then we cannot have a solution of the equation with data concentrated on K.

Alternatively, one studies the limit of approximating equations:

$$\begin{gathered} -\Delta_p(u_n) + H(x,u_n,\nabla u_n) = f_n \quad \text{in}\ \Omega\,, \\ u_n = 0 \quad \text{on}\ \partial\Omega\,, \end{gathered} \tag{2.4}$$

if f_n converges to a measure μ in the so-called narrow topology, which means

$$\int_\Omega f_n \varphi dx \overset{n\to+\infty}{\to} \int_\Omega \varphi d\mu\,, \tag{2.5}$$

for any function φ bounded and continuous on Ω. This is the most natural way to approximate a bounded Radon measure, so that, if a solution exists, we expect that we can prove the convergence of u_n towards a solution u of

$$\begin{gathered} -\Delta_p(u) + H(x,u,\nabla u) = \mu \quad \text{in}\ \Omega \\ u = 0 \quad \text{on}\ \partial\Omega\,, \end{gathered} \tag{2.6}$$

like it happens if f_n strongly converges in $L^1(\Omega)$. Thus studying the limit of u_n is a constructive way to see whether and why existence may fail; thanks to (2.5)

f_n is bounded in $L^1(\Omega)$ so that "a priori" estimates are available, and usually compactness of u_n can also be proved. The main task is to understand which is the limit of u_n and what equation it satisfies.

Finally, a third approach is in some sense the combination of the previous two. One studies (2.4) assuming only that f_n converges in $L^1_{\mathrm{loc}}(\Omega \setminus K)$ towards a function f, where K is a compact subset of Ω. Here no assumption is made on the behaviour of f_n on K, so that no estimates on K are obtained for u_n. If one proves that u_n still converges to a solution u of (2.3), this means that perturbations of f whatever singular, but localized on K, are not seen by the equation. As in the viewpoint of removable singularity, no solution can be expected for data concentrated on K. These three possible approaches were all investigated as far as problem (2.1) is concerned in some of the papers mentioned above.

In a series of recent works, these questions have been studied for problems with gradient dependent lower order terms. A particular case is given when the lower order term has natural growth. When trying to find solutions for the model equation

$$\begin{aligned} -\Delta_p u + g(u)|\nabla u|^p &= \mu \\ u &= 0, \end{aligned} \tag{2.7}$$

the growth at infinity of $g(s)$ and the regular or singular nature of μ play a crucial role. Removable singularity results were proved by H. Brezis and L. Nirenberg in [24] for $p = 2$, showing that if $sg(s) \geq \frac{\gamma}{s^2}$ with $\gamma > 1$, then any compact set of zero capacity (the standard Newtonnian capacity) is removable. In [17], [40], [34], the behaviour of sequences of approximating solutions was studied if μ is approximated in the narrow topology, say by a standard convolution. It is proved that if $g \in L^1(\mathbb{R})$, then there exists a solution u of (2.7) for any measure μ, while if $g \notin L^1(\mathbb{R})$ approximating solutions converge to a solution u of the same problem but with datum μ_0, the absolutely continuous part of μ with respect to p-capacity. Here p-capacity denotes the capacity defined in $W^{1,p}_0(\Omega)$ and we recall (see [29]) that any measure μ admits a unique decomposition as $\mu = \mu_0 + \lambda$, where λ is concentrated on a set of zero p-capacity and $\mu_0(E) = 0$ for any set E of zero p-capacity. In other words, in the approximation method, one looses the singular part of the measure which is concentrated on sets of zero p-capacity; if μ does not charge sets of zero p-capacity then existence is proved for any function $g(s)$.

Removability properties in the stability approach are investigated in [36], where the approximating equations of (2.7) are considered with data f_n only converging to a function f in $L^1_{\mathrm{loc}}(\Omega \setminus K)$, where K has p-capacity zero. It is proved that, setting $G(s) = \int_0^s g(t)dt$, if, roughly speaking, $\exp(-G(s)/(p-1)) \in L^1(\mathbb{R})$ then u_n still converges to a solution with datum f; thus, whatever singular perturbations, provided they are localized on sets of zero p-capacity, are not seen by the equation. This result somehow includes the removable singularity point of view, and extends the result in [24] since the assumption that $\exp(-G(s)/(p-1)) \in L^1(\mathbb{R})$, in the case $p = 2$, is weaker than assuming that $sg(s) \geq \frac{\gamma}{s^2}$ with $\gamma > 1$.

This kind of phenomena due to absorption terms has been investigated for parabolic equations as well. As it happens for the stationary case, the semilinear evolution problem

$$\begin{aligned} u_t - \Delta u + |u|^{r-1}u = \mu \quad \text{in } Q := \Omega \times (0,T) \\ u = 0 \quad \text{on } \Sigma := \partial\Omega \times (0,T) \\ u(0) = u_0 \quad \text{in } \Omega, \end{aligned} \tag{2.8}$$

does not always have a solution for any measure μ on Q and any measure initial datum u_0. In [22], the authors study the problem with $\mu = 0$ concentrating the attention on the initial measure u_0. They point out that, ir r is large enough, nonexistence phenomena may occurr, and can appear as initial layer phenomena. In fact, a singular measure as initial condition may be lost while approximating the problem with smooth approximating problems. Subsequently, in [5], necessary and sufficient conditions are given on the measures μ and u_0 in order to have a solution of (2.8); as expected, these conditions involve some notions of space-time dependent capacity. Further results on nonlinear analogue of (2.8) are proved in [3], [33], [12] (see also the references in these papers).

In view of the results mentioned above for elliptic equations, recent study has been devoted to evolution problems as the following:

$$\begin{aligned} u_t - \Delta_p u + g(u)|\nabla u|^p = 0 \quad \text{in } Q := \Omega \times (0,T) \\ u = 0 \quad \text{on } \Sigma := \partial\Omega \times (0,T) \\ u(0) = u_0 \quad \text{in } \Omega, \end{aligned} \tag{2.9}$$

in case u_0 is a bounded measure. The existence of a solution in case $u_0 \in L^1(\Omega)$ is proved in [41]. The possibility to extend this result to a general measure initial datum is studied in [13]. Again, under the assumption that $g \in L^1(\mathbb{R})$, it is proved the existence of a solution for any measure u_0. On the other hand, if $g \notin L^1(\mathbb{R})$, then initial layer phenomena occur; in particular, if u_{0n} is a convolution approximation of the measure u_0, the sequence of approximating solutions u_n of the same problem, with initial datum u_{0n}, converges to a solution u of the problem having, as initial value, the absolutely continuous part of u_0 with respect to Lebesgue measure. Sharp removable singularity type results, which in a stronger way express the nonexistence of solution, still depend on the growth at infinity of $g(s)$ and are obtained in [43].

Eventually, one obtains for the evolution problem (2.9) the same type of results obtained for the elliptic problem (2.7) replacing the role of μ with u_0 and the p-capacity (capacity in $W_0^{1,p}(\Omega)$) with the Lebesgue measure in Ω. Are then these results consistent? The answer has to be found in the study of the notion of capacity for parabolic equations. A functional type presentation and construction of the parabolic p-capacity (capacity defined in the space $W = \{u \in L^p(0,T;W_0^{1,p}(\Omega))\,, u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))\}$) is given in [39] for $p = 2$ and in [27] for $p \neq 2$. In this last paper, it is proved that given $B \subset \Omega$, the set $\{t = 0\} \times B$ has zero parabolic capacity in $(0,T) \times \Omega$ if and only if B has zero Lebesgue measure. Thus, if one looks at singularities at initial time as singularities on Q concentrated at $t = 0$, the results obtained on (2.9) reflect perfectly those on (2.7). Moreover, it becomes clear that in order to deal with problem (2.9) with interior space-time dependent measures as data, one has to follow the outlines of the stationary case and use a decomposition theorem for measures with respect to parabolic p-capacity. This latter result, which extends the stationary one given in [16], is proved in [27] and states that any measure μ on $(0, T) \times \Omega$ which does not charge sets of zero parabolic p-capacity admits the decomposition (as a distribution)

$$\mu = f + g_1 - (g_2)_t$$

with $f \in L^1(Q)$, $g_1 \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ and $g_2 \in L^p(0, T; W_0^{1,p}(\Omega))$.

Finally, let us mention that, in the linear case ($p = 2$), other existence and nonexistence results with gradient dependent lower order terms (absorbing or repulsive) and measure data are obtained in [1, 2, 6, 7] (see also the references cited therein). We point out that the techniques used in these papers are mainly based on a linear operator and on the concept of distributional solution (with two integration by parts), or on semigroup theory and the concept of integral solution. These approaches allow to have sharper nonexistence results especially for the case of subcritical growth, on the other hand their study is mostly restricted to the case $g(u) \equiv 1$.

2.1 Natural growth reaction terms and measure data

As explained in the previous section, if the term $H(x, u, \nabla u)$ is an absorption term and has natural growth, the borderline case which allows to have solutions of (2.6) for all measures μ is the case in which

$$|H(x, u, \nabla u)| \leq g(u)|\nabla u|^p, \quad \text{with } g \in L^1(\mathbb{R}).$$

Our aim is now to show that, somehow surprisingly, the same assumption is necessary and sufficient to have solutions for any measure even in the reaction case, that is without assuming any sign condition on $H(x, s, \xi)$. In particular, if we aim to have solutions of (2.7) for *any* given measure data, there is no difference between the reaction and the absorption case.

Heuristically, this feature can be easily explained. In fact, the model equation

$$-\Delta u = g(u)|\nabla u|^2 + \mu, \tag{2.10}$$

can be transformed, through a change of unknown, into the equation

$$-\Delta v = \exp(G(u))\mu, \tag{2.11}$$

with $v = \int_0^u \exp(G(s))ds$ and $G(s) = \int_0^s g(r)dr$.

In [34] we proved that equation (2.11) has a solution if $\exp(G(u))$ has a finite limit at infinity, which is the case whenever $g \in L^1(\mathbb{R})$, so that in this case (2.10) is also expected to have a solution. On the other hand, if $g \notin L^1(\mathbb{R})$, then the right hand side of (2.11) can be hardly handled since $\exp(G(u))$ is not bounded.

We are going to provide an example where μ is the Dirac mass and no solution of (2.11) can be found by approximation, precisely proving that approximated solutions of (2.10) in this case blow up completely (i.e. at every point of Ω).

We will prove our result in a more general situation. Assume that $a(x,s,\xi)$ and $H(x,s,\xi)$ are Carathéodory functions satisfying, for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, ξ, $\eta \in \mathbb{R}^N$ ($\xi \neq \eta$):

$$a(x,s,\xi)\cdot\xi \geq \alpha|\xi|^p\,, \quad \alpha > 0\,,\ p > 1\,, \tag{2.12}$$

$$|a(x,s,\xi)| \leq \beta(k(x) + |s|^{p-1} + |\xi|^{p-1}) \quad k(x) \in L^{p'}(\Omega),\ \beta > 0\,, \tag{2.13}$$

$$(a(x,s,\xi) - a(x,s,\eta))\cdot(\xi - \eta) > 0\,, \tag{2.14}$$

and

$$\begin{aligned} &|H(x,s,\xi)| \leq \gamma(x) + g(s)|\xi|^p\,,\ \gamma(x) \in L^1(\Omega) \\ &\text{and} \quad g : \mathbf{R} \to \mathbf{R}^+ \quad \text{continuous}\,, \quad g \geq 0\,, \quad g \in L^1(\mathbb{R})\,. \end{aligned} \tag{2.15}$$

In the following we denote by $\mathrm{cap}_p(B)$ the p-capacity of a borelian set $B \subset \Omega$, where the p-capacity is the standard notion of capacity defined in the Sobolev space $W_0^{1,p}(\Omega)$. Let us recall (see [29]) that any bounded Radon measure μ has a unique decomposition as

$$\mu = \mu_0 + \lambda\,, \tag{2.16}$$

where μ_0, λ are bounded measures such that μ_0 does not charge sets of zero p-capacity (i.e. $\mu_0(B) = 0$ for every B with $\mathrm{cap}_p(B) = 0$) and λ is concentrated on a set $E \subset \Omega$ such that $\mathrm{cap}_p(E) = 0$. Moreover, if μ is nonnegative, then both μ_0 and λ are nonnegative. For a presentation of the basic notions concerning measures and capacity the reader may refer to [31], [26]. We also have, from [16], that μ_0 furtherly admits a decomposition (in distributional sense) as

$$\mu_0 = f - \mathrm{div}(F)\,, \quad f \in L^1(\Omega),\ F \in L^{p'}(\Omega)^N. \tag{2.17}$$

Hereafter, let μ be a bounded nonnegative Radon measure on Ω. Referring to the previous decomposition of μ and μ_0 in (2.16), (2.17), there exists a sequence μ_n of bounded functions such that

$$\begin{gathered} \mu_n = \mu_{0n} + \lambda_n\,, \quad \mu_{0n} \geq 0,\ \lambda_n \geq 0, \\ \mu_{0n} = f_n - \mathrm{div}(F_n)\,, \quad f_n \in L^\infty(\Omega)\,,\ F_n \in L^\infty(\Omega)^N\,, \\ f_n \to f \quad \text{strongly in } L^1(\Omega), \\ F_n \to F \quad \text{strongly in } L^{p'}(\Omega)^N, \\ \int_\Omega \varphi\lambda_n dx \to \int_\Omega \varphi d\lambda \quad \forall\varphi \in C_b(\Omega)\,, \end{gathered} \tag{2.18}$$

where $C_b(\Omega)$ denotes the space of bounded continuous functions in Ω. Such a sequence μ_n can be constructed using convolution and a suitable compactly supported approximation of μ.

For fixed $n \in \mathbb{N}$, since $\mu_n \in L^\infty(\Omega)$, under the previous assumptions it is proved in [19] that there exists a weak solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ of the problem:

$$\begin{aligned} -\operatorname{div}(a(x,u_n,\nabla u_n)) &= H(x,u_n,\nabla u_n)+\mu_n \quad \text{in } \Omega\,,\\ u_n &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{2.19}$$

Our main result is the following.

Theorem 2.1 *Let $a(x,s,\xi)$ and $H(x,s,\xi)$ satisfy assumptions* (2.12)–(2.15). *Let μ be a nonnegative bounded Radon measure on Ω. Then there exists a solution u of the problem*

$$\begin{aligned} -\operatorname{div}(a(x,u,\nabla u)) &= H(x,u,\nabla u)+\mu \quad \textit{in } \Omega\,,\\ u &= 0 \quad \textit{on } \partial\Omega\,. \end{aligned} \tag{2.20}$$

Proof. We essentially follow the method used in [44], which consists in multiplying the equation (2.19) by $\exp(G(u_n))$ or by $\exp(-G(u_n))$, where $G(s)=\int_0^s g(t)/\alpha dt$ (the function g appears in (2.15)). In other words this replaces the idea of the change of unknown which transforms the model problem (2.10) into (2.11). After this multiplication, we will apply the techniques fully developed in [40], [34] to obtain the strong convergence of truncations.

In the following, we omit for shortness the dependence on x in the integrals, and we denote by c any positive constant independent on n. Let $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$; choosing $\exp(G(u_n))\varphi$ as test function in (2.19) we have

$$\begin{aligned} &\int_\Omega \exp(G(u_n))a(u_n,\nabla u_n)\nabla\varphi+\int_\Omega \frac{g(u_n)}{\alpha}\exp(G(u_n))a(u_n,\nabla u_n)\nabla u_n\varphi\\ &\quad=\int_\Omega H(u_n,\nabla u_n)\exp(G(u_n))\varphi+\int_\Omega \varphi\exp(G(u_n))\mu_n\,. \end{aligned}$$

For any $\varphi\geq 0$, thanks to (2.12) and (2.15) we obtain

$$\begin{aligned} \int_\Omega \exp(G(u_n))a(u_n,\nabla u_n)\nabla\varphi &\leq \int_\Omega \gamma(x)\varphi\exp(G(u_n))+\int_\Omega\varphi\exp(G(u_n))\mu_n\\ &\forall\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\,,\varphi\geq 0\,. \end{aligned} \tag{2.21}$$

Similarly, taking $\exp(-G(u_n))\varphi$ as test function in (2.19) we obtain

$$\begin{aligned} &\int_\Omega \exp(-G(u_n))a(u_n,\nabla u_n)\nabla\varphi+\int_\Omega\gamma(x)\varphi\exp(-G(u_n))\\ &\qquad\geq\int_\Omega\varphi\exp(-G(u_n))\mu_n \quad \forall\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\,,\varphi\geq 0\,. \end{aligned} \tag{2.22}$$

Let $\varphi=T_k(u_n)^+$ in (2.21) and $\varphi=T_k(u_n)^-$ in (2.22). Also let $G(\pm\infty)=\frac{1}{\alpha}\int_0^{\pm\infty}g(s)ds$ which are well defined since $g\in L^1(\mathbb{R})$. Since $\exp(G(-\infty))\leq\exp(G(s))\leq\exp(G(+\infty))$ and $\exp(|G(\pm\infty)|)\leq\exp(\|g\|_{L^1(\mathbb{R})}/\alpha)$, using (2.12), we obtain

$$\|T_k(u_n)\|^p_{W_0^{1,p}(\Omega)}\leq\frac{1}{\alpha}\exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)k(\|\gamma\|_{L^1(\Omega)}+\|\mu_n\|_{L^1(\Omega)})\leq ck\,. \tag{2.23}$$

Standard estimates (see [8]) imply that u_n is bounded in the Marcinkiewicz space $M^{\frac{N(p-1)}{N-p}}(\Omega)$ and $|\nabla u_n|$ is bounded in the Marcinkiewicz space $M^{\frac{N(p-1)}{N-1}}(\Omega)$. In particular we have from (2.13) that $a(x,u_n,\nabla u_n)$ is bounded in $L^q(\Omega)^N$ for any $q<\frac{N}{N-1}$. Furthermore, there exist a function u and a subsequence such that

$$u_n \to u \quad \text{a.e. in } \Omega,$$
$$T_k(u_n) \to T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega) \text{ and a.e. in } \Omega \text{ for any } k>0.$$

Let us take $\varphi = T_1(u_n - T_j(u_n))^-$ in (2.22); we obtain

$$\int_{\{-(j+1)\le u_n\le -j\}} a(u_n,\nabla u_n)\nabla u_n + \int_\Omega \exp(-G(u_n))T_1(u_n-T_j(u_n))^-\mu_n \le \gamma \int_\Omega \exp(-G(u_n))T_1(u_n-T_j(u_n))^- . \tag{2.24}$$

The term with μ_n can be neglected since it is nonnegative. In the right hand side we can pass to the limit in n and in j by Lebesgue's theorem, using that G is bounded; indeed, since

$$\exp(-G(u))T_1(u-T_j(u))^- \le \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)\chi_{\{u<-j\}}$$

we have

$$\int_\Omega \exp(-G(u_n))T_1(u_n-T_j(u_n))^- \overset{n\to\infty}{\to} \int_\Omega \exp(-G(u))T_1(u-T_j(u))^- \overset{j\to\infty}{\to} 0\,,$$

so that we deduce from (2.24)

$$\lim_{j\to\infty}\limsup_{n\to\infty} \int_{\{-(j+1)\le u_n\le -j\}} a(u_n,\nabla u_n)\nabla u_n = 0\,. \tag{2.25}$$

We are going now to prove that the truncations strongly converge in $W_0^{1,p}(\Omega)$. Following the idea introduced in [26], this is done by using a suitable sequence of cut-off functions. Indeed, let $\delta>0$; since λ is a regular measure concentrated on E and since E has zero p-capacity, there exist a compact set $K_\delta\subset E$ and a sequence $\{\psi_\delta\}$ of functions in $C_c^\infty(\Omega)$ with the properties that

$$\begin{gathered} \lambda(E\setminus K_\delta)<\delta\,, \quad 0\le\psi_\delta\le 1\,,\\ \psi_\delta\equiv 1 \text{ on an open neighbourhood } A_\delta \text{ of } K_\delta\\ \psi_\delta \overset{\delta\to 0}{\to} 0 \quad \text{strongly in } W_0^{1,p}(\Omega). \end{gathered} \tag{2.26}$$

Take now $\varphi = (k-T_k(u_n))(1-|T_1(u_n-T_j(u_n))|)\psi_\delta$ in (2.22), with $j>k$. Observe that $\varphi=(k-u_n)\psi_\delta$ if $|u_n|<k$ and $\varphi=0$ if $u_n>k$. Thus we get, using also that $\exp(-G(u_n)) \leq \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)$ and $\psi_\delta \leq 1$,

$$\begin{aligned}
&\int_\Omega \exp(-G(u_n))a(u_n, \nabla u_n)\nabla\psi_\delta(k - T_k(u_n))(1 - |T_1(u_n - T_j(u_n)|) \\
&+ 2k \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big) \int_{\{-(j+1)\leq u_n \leq -j\}} a(u_n, \nabla u_n)\nabla u_n + 2k \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big) \int_\Omega \gamma\psi_\delta \\
&\geq \exp\big(-\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big) \int_\Omega a(T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)\psi_\delta \qquad (2.27) \\
&\quad + \int_\Omega (k - T_k(u_n)) \exp(-G(u_n))(1 - |T_1(u_n - T_h(u_n)|)\psi_\delta \mu_n .
\end{aligned}$$

Since

$$|a(u_n, \nabla u_n)(1 - |T_1(u_n - T_j(u_n)|)| \leq |a(T_{j+1}(u_n), \nabla T_{j+1}(u_n))| ,$$

and since last term is bounded in $L^{p'}(\Omega)$ and G is bounded, we have that there exists $\Lambda_j \in L^{p'}(\Omega)^N$ such that

$$\exp(-G(u_n))a(u_n, \nabla u_n)(k - T_k(u_n))(1 - |T_1(u_n - T_j(u_n)|) \to \Lambda_j$$

weakly in $L^{p'}(\Omega)^N$. Thus we get

$$\lim_{n\to\infty} \int_\Omega \exp(-G(u_n))a(u_n, \nabla u_n)\nabla\psi_\delta(k - T_k(u_n))(1 - |T_1(u_n - T_j(u_n)|) = \int_\Omega \Lambda_j \nabla\psi_\delta ,$$

and then, as δ tends to zero, thanks to (2.26) we have

$$\lim_{\delta\to 0} \lim_{n\to\infty} \int_\Omega \exp(-G(u_n))a(u_n, \nabla u_n)\nabla\psi_\delta(k - T_k(u_n))(1 - |T_1(u_n - T_j(u_n)|) = 0 .$$

The third integral in (2.27) easily goes to zero since ψ_δ converges to zero and $\gamma \in L^1(\Omega)$. Furthermore, the term with μ_n can again be neglected since it is nonnegative. Therefore, passing to the limit first in n, then in δ we obtain from (2.27)

$$\begin{aligned}
&\lim_{\delta\to 0} \limsup_{n\to\infty} \int_\Omega a(T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)\psi_\delta \\
&\quad \leq \limsup_{n\to\infty} 2k \exp\big(\frac{2\|g\|_{L^1(\mathbb{R})}}{\alpha}\big) \int_{\{-(j+1)\leq u_n \leq -j\}} a(u_n, \nabla u_n)\nabla u_n .
\end{aligned}$$

Then, as j goes to infinity, using (2.25) and since $a(x, s, \xi) \cdot \xi \geq 0$, we get

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \int_\Omega a(T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)\psi_\delta = 0 . \qquad (2.28)$$

Let now $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$, we take $\varphi = w_n^+(1-\psi_\delta)$ in (2.21) and $\varphi = w_n^-(1-\psi_\delta)$ in (2.22) to obtain

$$\begin{aligned}&\int_{\{w_n\geq 0\}} \exp(G(u_n))a(u_n,\nabla u_n)\nabla w_n(1-\psi_\delta)\\ &\leq \int_\Omega \gamma w_n^+ \exp(G(u_n))(1-\psi_\delta) + \int_\Omega w_n^+ \exp(G(u_n))(1-\psi_\delta)\mu_n\\ &\quad + \int_\Omega \exp(G(u_n))a(u_n,\nabla u_n)\nabla\psi_\delta w_n^+\end{aligned}$$

and

$$\begin{aligned}&\int_{\{w_n\leq 0\}} \exp(-G(u_n))a(u_n,\nabla u_n)\nabla w_n(1-\psi_\delta)\\ &\leq \int_\Omega \gamma w_n^- \exp(-G(u_n))(1-\psi_\delta) - \int_\Omega w_n^- \exp(-G(u_n))(1-\psi_\delta)\mu_n\\ &\quad - \int_\Omega \exp(-G(u_n))a(u_n,\nabla u_n)\nabla\psi_\delta w_n^-\,.\end{aligned}$$

Setting $M = h + 4k$ and using $a(x,s,\xi)\cdot\xi \geq 0$, we have

$$\begin{aligned}a(u_n,\nabla u_n)\nabla w_n \geq\, &a(T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))\\ &- |a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}\,.\end{aligned}$$

Then

$$\begin{aligned}&\int_{\{w_n\geq 0\}} \exp(G(u_n))a(T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))(1-\psi_\delta)\\ &\leq \int_\Omega \gamma w_n^+ \exp(G(u_n))(1-\psi_\delta) + \int_\Omega w_n^+ \exp(G(u_n))(1-\psi_\delta)\mu_n\\ &\quad + \int_\Omega \exp(G(u_n))a(u_n,\nabla u_n)\nabla\psi_\delta w_n^+\\ &\quad + \int_\Omega \exp(G(u_n))|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}(1-\psi_\delta)\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}&\int_{\{w_n\leq 0\}} \exp(-G(u_n))a(T_k(u_n),\nabla T_k(u_n))\nabla(T_k(u_n)-T_k(u))(1-\psi_\delta)\\ &\leq \int_\Omega \gamma w_n^- \exp(-G(u_n))(1-\psi_\delta) - \int_\Omega w_n^- \exp(-G(u_n))(1-\psi_\delta)\mu_n\\ &\quad - \int_\Omega \exp(-G(u_n))a(u_n,\nabla u_n)\nabla\psi_\delta w_n^-\\ &\quad + \int_\Omega \exp(-G(u_n))|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}(1-\psi_\delta)\,.\end{aligned} \tag{2.30}$$

Since $a(u_n, \nabla u_n)$ is bounded in $L^q(\Omega)^N$ for any $q < \frac{N}{N-1}$, there exists $\nu \in L^q(\Omega)^N$ such that $a(u_n, \nabla u_n)$ weakly converges to ν in $L^q(\Omega)^N$. Since $\psi_\delta \in W_0^{1,\infty}(\Omega)$, and G is bounded, we get

$$\begin{aligned}&\int_\Omega \exp(G(u_n))a(u_n, \nabla u_n)\nabla\psi_\delta w_n^+ \\ &\overset{n\to\infty}{\to} \int_\Omega \exp(G(u))\nu\nabla\psi_\delta T_{2k}(u - T_h(u)) \overset{h\to\infty}{\to} 0\,.\end{aligned} \tag{2.31}$$

Using that $|\nabla T_k(u)|\chi_{\{|u_n|>k\}}$ strongly converges to zero in $L^p(\Omega)$ and that $\nabla T_M(u_n)$ is bounded in $L^{p'}(\Omega)^N$ we also have that

$$\int_\Omega \exp(G(u_n))|a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\chi_{\{|u_n|>k\}}(1 - \psi_\delta) \overset{n\to\infty}{\to} 0\,. \tag{2.32}$$

Similarly, using the weak convergence of $T_k(u_n)$ to $T_k(u)$ in $W_0^{1,p}(\Omega)$, we have

$$\int_\Omega \exp(-G(u_n))a(T_k(u_n), \nabla T_k(u))\nabla(T_k(u_n) - T_k(u))(1 - \psi_\delta) \overset{n\to\infty}{\to} 0\,, \tag{2.33}$$

and, since $\gamma \in L^1(\Omega)$,

$$\begin{aligned}&\int_\Omega \gamma w_n^+ \exp(G(u_n))(1 - \psi_\delta) \\ &\overset{n\to\infty}{\to} \int_\Omega \gamma \exp(G(u))(1 - \psi_\delta)T_{2k}(u - T_h(u))^+ \overset{h\to\infty}{\to} 0\,.\end{aligned} \tag{2.34}$$

Moreover, we have, using the decomposition of μ_n in (2.18),

$$\begin{aligned}&\int_\Omega w_n^+ \exp(G(u_n))(1 - \psi_\delta)\mu_n \\ &= \int_\Omega w_n^+ \exp(G(u_n))(1 - \psi_\delta)d\mu_{0n} + \int_\Omega w_n^+ \exp(G(u_n))(1 - \psi_\delta)\lambda_n \\ &\le \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)\int_\Omega w_n^+(1 - \psi_\delta)d\mu_{0n} + 2k\exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)\int_\Omega (1 - \psi_\delta)\lambda_n\,.\end{aligned}$$

Since w_n^+ converges to $T_{2k}(u - T_h(u))^+$ weakly-$*$ in $L^\infty(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$, using the convergence of μ_{0n} (which is strong in $L^1(\Omega) + W^{-1,p'}(\Omega)$) and λ_n we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \int_\Omega w_n^+ \exp(G(u_n))(1 - \psi_\delta)\mu_n \\ &\le \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)\int_\Omega T_{2k}(u - T_h(u))^+(1 - \psi_\delta)d\mu_0 \\ &\quad + 2k\exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)\int_\Omega (1 - \psi_\delta)\,d\lambda\,.\end{aligned} \tag{2.35}$$

Since $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$ and (2.23) holds true, we have (see e.g. Remark 2.11 in [26]) that u has a cap-quasi continuous representative which is cap-quasi everywhere finite, that is there exists a function $\tilde{u}$ such that $\tilde{u} = u$ almost everywhere and $\mathrm{cap}\{|\tilde{u}| = +\infty\} = 0$. In particular, since μ_0 does not charge sets of zero capacity, we have that $\tilde{u}$ is finite μ_0-quasi everywhere, hence $T_{2k}(\tilde{u} - T_h(\tilde{u}))$ converges to zero μ_0-quasi everywhere. Letting h go to infinity we deduce that

$$\lim_{h\to\infty} \int_\Omega T_{2k}(u - T_h(u))^+(1-\psi_\delta)\, d\mu_0 = 0\,,$$

so that (2.35) implies

$$\lim_{h\to\infty} \limsup_{n\to\infty} \int_\Omega w_n^+ \exp(G(u_n))(1-\psi_\delta)\mu_n \le 2k \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big) \int_\Omega (1-\psi_\delta)\, d\lambda \tag{2.36}$$

Then, as n and then h go to infinity, using (2.31), (2.32), (2.33), (2.34), (2.36), we obtain from (2.29),

$$\begin{aligned}
&\limsup_{h\to\infty} \limsup_{n\to\infty} \int_{\{w_n \ge 0\}} \exp(G(u_n))\big[a(T_k(u_n), \nabla T_k(u_n))\\
&- a(T_k(u_n), \nabla T_k(u))\big]\nabla(T_k(u_n) - T_k(u))(1-\psi_\delta)\\
&\quad \le 2k \exp(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}) \int_\Omega (1-\psi_\delta) d\lambda \le 2k \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)\lambda(\Omega \setminus K_\delta)\,.
\end{aligned}$$

By means of (2.14) and recalling (2.26) we deduce

$$\begin{aligned}
\limsup_{\delta\to 0} \limsup_{h\to\infty} \limsup_{n\to\infty} &\int_{\{w_n \ge 0\}} \big[a(T_k(u_n), \nabla T_k(u_n))\\
&- a(T_k(u_n), \nabla T_k(u))\big]\nabla(T_k(u_n) - T_k(u))(1-\psi_\delta) \le 0.
\end{aligned}$$

In the same way we work on (2.30), obtaining

$$\begin{aligned}
\limsup_{\delta\to 0} \limsup_{h\to\infty} \limsup_{n\to\infty} &\int_{\{w_n \le 0\}} \big[a(T_k(u_n), \nabla T_k(u_n))\\
&- a(T_k(u_n), \nabla T_k(u))\big]\nabla(T_k(u_n) - T_k(u))(1-\psi_\delta) \le 0.
\end{aligned}$$

Adding the two inequalities we conclude

$$\begin{aligned}
\limsup_{\delta\to 0} \limsup_{n\to\infty} &\int_\Omega \big[a(T_k(u_n), \nabla T_k(u_n))\\
&- a(T_k(u_n), \nabla T_k(u))\big]\nabla(T_k(u_n) - T_k(u))(1-\psi_\delta) = 0\,.
\end{aligned} \tag{2.37}$$

Now, we have

$$
\begin{aligned}
&\int_\Omega [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))]\nabla(T_k(u_n)-T_k(u))\\
&= \int_\Omega [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))]\nabla(T_k(u_n)-T_k(u))(1-\psi_\delta)\\
&\quad + \int_\Omega a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\psi_\delta - \int_\Omega a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\psi_\delta\\
&\quad - \int_\Omega a(T_k(u_n),\nabla T_k(u))\nabla(T_k(u_n)-T_k(u))\psi_\delta\,.
\end{aligned}
$$

Using the weak convergence of $T_k(u_n)$ to $T_k(u)$ last term converges to zero as n goes to infinity. Similarly, we have that $a(T_k(u_n),\nabla T_k(u_n))$ is bounded in $L^{p'}(\Omega)^N$ uniformly on n while $\nabla T_k(u)\psi_\delta$ converges to zero in $L^p(\Omega)^N$ as δ tends to zero. Using also (2.37) and (2.28) we finally get, letting first n go to infinity and then δ to zero,

$$
\lim_{n\to\infty}\int_\Omega [a(T_k(u_n),\nabla T_k(u_n)) - a(T_k(u_n),\nabla T_k(u))]\nabla(T_k(u_n)-T_k(u)) = 0.
$$

Under assumptions (2.12)–(2.14), it is well known that this implies

$$
T_k(u_n)\to T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega) \text{ for any } k>0. \tag{2.38}
$$

Moreover, using that meas$\{|u_n|>k\}$ goes to zero as k goes to infinity uniformly on n, as a consequence of (2.38) we also have that, up to subsequences, ∇u_n almost everywhere converges to ∇u in Ω. In turns, this implies that

$$
a(x,u_n,\nabla u_n)\to a(x,u,\nabla u) \quad \text{strongly in } L^q(\Omega)^N \text{ for any } q<\tfrac{N}{N-1}. \tag{2.39}
$$

Let $\varphi = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ in (2.21); since $|\varphi|\le\int_h^\infty g(s)ds$ we have

$$
\begin{aligned}
&\int_\Omega a(u_n,\nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}}\\
&\le \exp\big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\big)\Big(\int_h^\infty g(s)ds\Big)(\|\gamma\|_{L^1(\Omega)}+\|\mu_n\|_{L^1(\Omega)})\,.
\end{aligned}
$$

Using (2.12) and the fact that μ_n is bounded in $L^1(\Omega)$ gives

$$
\alpha\int_{\{u_n>h\}} g(u_n)|\nabla u_n|^p \le c\Big(\int_h^\infty g(s)ds\Big)\,,
$$

and then since $g\in L^1(\mathbb{R})$ we obtain

$$
\lim_{h\to\infty}\ \sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} g(u_n)|\nabla u_n|^p = 0\,.
$$

Similarly, taking $\varphi = \int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ in (2.22) we obtain the corresponding result on the set $\{u_n < -h\}$, hence

$$\lim_{h\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|u_n|>h\}} g(u_n)|\nabla u_n|^p = 0\,. \tag{2.40}$$

A standard argument allows to conclude from (2.38) and (2.40) that $g(u_n)|\nabla u_n|^p$ strongly converges in $L^1(\Omega)$ to $g(u)|\nabla u|^p$. Then from (2.15), the almost everywhere convergence of u_n and ∇u_n and Lebesgue's theorem we conclude that

$$H(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \tag{2.41}$$

Thanks to (2.39) and (2.41) we can pass to the limit in (2.19) and we obtain that u is a distributional solution of (2.20). □

Remark 2.2 The assumption that μ is nonnegative is not essential in Theorem 2.1. In order to deal with changing sign measures it is enough to follow the same lines of the previous proof with suitable modifications while proving the strong convergence of truncations similar to those developed in [26].

Example 2.3 Let $\mu = \delta_0$ be the Dirac mass at the origin and let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^N$, with $N \geq 3$. Let $\mu_n = n^N \chi_{B(0,\frac{1}{n})}$; clearly μ_n converges, in the narrow topology, to $\lambda\delta_0$ for some constant $\lambda > 0$. Note that in particular μ_n satisfies (2.18) (with $f_n = F_n = 0$). Let u_n be any sequence of solutions of

$$\begin{gathered} -\Delta u_n = g(u_n)|\nabla u_n|^2 + \mu_n \quad \text{in } \Omega\,, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.42}$$

We claim that if the following assumption holds:

$$\begin{gathered} \exists h \in C(\mathbb{R}, \mathbb{R}^+)\colon\ g(s) \geq h(s) \text{ for every } s \in \mathbb{R}^+, \\ h \text{ is nonincreasing, } \lim_{s\to+\infty} h(s) = 0 \text{ and } h \notin L^1(\mathbb{R}^+), \end{gathered} \tag{2.43}$$

then the sequence u_n blows up completely, namely $u_n(x) \to +\infty$ for every $x \in \Omega$.

As far as assumption (2.43) is concerned, observe that if g is nonincreasing, converges to zero at infinity and $g \notin L^1(\mathbb{R})$, we can clearly take $h = g$ in (2.43); this includes the main examples of g around the borderline case $g \in L^1(\mathbb{R})$, as $g(s) = 1/(|s|+1)$ or $g(s) = 1/((1+|s|)\log(1+|s|))$. Anyway, assumption (2.43) is stated in this generality to include most examples of g; in particular, note that the it requires g to be **larger** than a nonincreasing function which is not integrable, so that g itself may also be unbounded.

In order to prove our claim, we adapt an idea used in a context of sublinear equations by L. Orsina ([35]). Let us set $H(s) = \int_0^s h(\xi)d\xi$, $\psi(s) = \int_0^s \exp(H(\xi))d\xi$ and define $v_n := \psi(u_n)$ (the function h is defined in (2.43)). Observe that ψ is an increasing unbounded function, so that v_n goes to infinity if and only if u_n goes to infinity. Since $g(u_n) \geq h(u_n)$, v_n satisfies

$$\begin{gathered} -\Delta v_n \geq \exp(H(u_n))\mu_n \quad \text{in } \Omega\,, \\ v_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.44}$$

In particular, by definition of μ_n, we have that v_n is a supersolution of the problem

$$\begin{aligned} -\Delta z &= \exp(H(\psi^{-1}(z)))n^N \quad \text{in } B(0,\frac{1}{n}),\\ z &= 0 \quad \text{on } \partial B(0,\frac{1}{n})\,. \end{aligned} \tag{2.45}$$

Let $\varphi_{1,n}$ be the first eigenfunction of the Laplacian on $B(0,\frac{1}{n})$, normalized so that $\|\varphi_{1,n}\|_{L^\infty(\Omega)} = 1$, and let $\lambda_{1,n}$ be the first eigenvalue. Let us set

$$B(s) := \frac{\exp(H(\psi^{-1}(s)))}{s}.$$

Since h is nonincreasing we have

$$\begin{aligned} \frac{d}{dr}\Big(\frac{\exp(H(r))}{\psi(r)}\Big) =& \frac{\exp(H(r))}{\psi(r)^2}\Big(h(r)\int_0^r \exp(H(\xi))d\xi - \exp(H(r))\Big)\\ \leq& \frac{\exp(H(r))}{\psi(r)^2}\Big(\int_0^r \exp(H(\xi))h(\xi)d\xi - \exp(H(r))\Big) < 0\,, \end{aligned}$$

so that $B(\psi(s))$ is decreasing. Since ψ is increasing, we deduce that B is a decreasing function. Let us set $T_n = B^{-1}(\frac{\lambda_{1,n}}{n^N})$. Since B is decreasing, we deduce that

$$\frac{\lambda_{1,n}}{n^N} = B(T_n) = B(T_n\|\varphi_{1,n}\|_{L^\infty(\Omega)}) \leq B(T_n\varphi_{1,n}(x)) \quad \forall x \in B(0,\frac{1}{n})\,,$$

which implies, by definition of B,

$$\lambda_{1,n}T_n\varphi_{1,n}(x) \leq \exp(H(\psi^{-1}(T_n\varphi_{1,n}(x))))n^N \quad \forall x \in B(0,\frac{1}{n})\,.$$

Since $\lambda_{1,n}T_n\varphi_{1,n} = -\Delta(T_n\varphi_{1,n})$ we conclude that $T_n\varphi_{1,n}$ is a subsolution of (2.45). Since $\exp\big(H(\psi^{-1}(z))\big)/z = B(z)$ is decreasing, a well-known comparison principle holds for positive sub-super solutions of (2.45) (see for example [23]), so that we get $v_n \geq T_n\varphi_{1,n}$ in $B(0,\frac{1}{n})$. By scaling arguments we know that

$$\varphi_{1,n}(x) = \varphi_{1,1}(nx)\,, \quad \lambda_{1,n} = \lambda_{1,1}n^2\,,$$

hence we obtain

$$\forall x \in B(0,\frac{1}{2n}): \quad v_n(x) \geq B^{-1}(\frac{\lambda_{1,1}}{n^{N-2}}) \min_{B(0,\frac{1}{2})} \varphi_{1,1}\,.$$

Since $\varphi_{1,1}$ is radial, we have $\min_{B(0,\frac{1}{2})}\varphi_{1,1} = \varphi_{1,1}(\frac{1}{2})$, so that

$$\min_{B(0,\frac{1}{2n})} v_n \geq B^{-1}(\frac{\lambda_{1,1}}{n^{N-2}})\varphi_{1,1}(\frac{1}{2})\,. \tag{2.46}$$

Now observe that, using De L'Hospital's theorem and the fact that $h(s)$ goes to zero at infinity, we have $\lim_{s\to+\infty} B(s) = 0$. Since $\lambda_{1,1}/n^{N-2}$ converges to zero as n tends to infinity, we end up with

$$\lim_{n\to+\infty} B^{-1}\left(\frac{\lambda_{1,1}}{n^{N-2}}\right) = +\infty\,,$$

and then from (2.46)

$$\lim_{n\to+\infty} \min_{B(0,\frac{1}{2n})} v_n = +\infty\,. \tag{2.47}$$

Let now $G(x,y)$ be the kernel of the Laplacian with zero boundary condition; we have from (2.44)

$$\begin{aligned} v_n(x) &\geq \int_\Omega G(x,y)\exp(H(u_n))(y)\mu_n(y)dy \\ &\geq \min_{B(0,\frac{1}{2n})} \big(\exp(H(\psi^{-1}(v_n)))\big)\int_{B(0,\frac{1}{2n})} G(x,y)n^N\,dy\,. \end{aligned} \tag{2.48}$$

Since there exists a constant $c > 0$ such that

$$\int_{B(0,\frac{1}{2n})} G(x,y)n^N dy \to c\int_\Omega G(x,y)d\delta_0(y) > 0\,,$$

and since both ψ^{-1} and H go to infinity at infinity (because $h \notin L^1(\mathbb{R}^+)$), we deduce using (2.47) that the right hand side of (2.48) goes to infinity as n goes to infinity. We then conclude

$$\lim_{n\to+\infty} v_n(x) = +\infty \quad \forall x \in \Omega\,.$$

Since ψ is unbounded and $u_n = \psi^{-1}(v_n)$, we have proved that the solutions u_n of (2.42) blow up completely in Ω. This is in sharp contrast with what proved in Theorem 2.1 when $g \in L^1(\mathbb{R})$, so that this assumption is optimal in the existence result above.

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