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Existence and regularity of positive solutions for an elliptic system *

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Abstract

In this paper, we study the existence and regularity of positive solution for an elliptic system on a bounded and regular domain. The non linearities in this equation are functions of Caratheodory type satisfying some exponential growth conditions.

1 Introduction

In this work, we study the elliptic system

$$\begin{aligned} -\Delta_p u &= f(x,u,v) \quad \text{in } \Omega \\ -\Delta_p v &= g(x,u,v) \quad \text{in } \Omega \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, $1 < p < +\infty$, and f and g are Carathéodory functions satisfying some growth conditions specified later.

In the recent years; the existence and non existence for the scalar case have been studied by several author's by using various approaches [9, 5]. For the system case, we mention the recent work of Bechah [4]. He study the local and global behaviour of solutions of systems involving the p-Laplacian operator in unbounded domains with f , g functions satisfying some growth conditions of polynomial type. Also, we cite the work of Ahammou [2], where he studied the positive radial solutions of nonlinear elliptic systems (1.1) using the method of topological degree. There Ω is a ball in $\mathbb{R}^N$ and f, g are positive functions satisfying $f(x,0,0) = g(x,0,0) = 0$ under some growth conditions of polynomial type.

Here we study the existence and regularity of positive solutions of (1.1) in a regular bounded domain and f, g are functions of Carathéodory type satisfying some growth conditions of exponential type. We extend the results of De Thelin [8] for the problem

$$\Delta_p u + g(x,u) = 0$$

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in the case when the growth of $g(x,.)$ is allowed to be of exponential type.

The rest of this paper is organized as follows: In section 2 we introduce the assumptions and some results preliminaries. In section 3 we introduce the main results of this paper.

2 Assumptions and preliminaries

Let X be a closed subspace of $W_0^{1,p}(\Omega)$; f and g be two positives Carathéodory functions satisfying the growth conditions:

(H1) For all $K > 0$, there exists $m > 0$ such that for all $(\xi, \eta) \in \mathbb{R}\times\mathbb{R}$, satisfying $|\xi| + |\eta| \leq K$ and for almost every where $x \in \Omega$ we have

$$f(x,\xi,\eta) \leq m \quad \text{and} \quad g(x,\xi,\eta) \leq m.$$

(H2) There exist $\sigma_0 > 2p-1$, $\theta_0 > 2p-1$ and $R > 0$ such that for all $(\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^+$ satisfying $\xi + \eta \geq R$ we have

$$\xi f(x,\xi,\eta) \geq (\sigma_0 + 1)G(x,\xi,\eta) \text{ a.e } x \in \Omega \tag{2.1}$$

$$\eta g(x,\xi,\eta) \geq (\theta_0 + 1)G(x,\xi,\eta) \text{ a.e } x \in \Omega \tag{2.2}$$

where $\frac{\partial G(x,\xi,\eta)}{\partial \xi} = f(x,\xi,\eta)$, and $\frac{\partial G(x,\xi,\eta)}{\partial \eta} = g(x,\xi,\eta)$.

Definition We say that (u, v) is a weak solution of elliptic system (1.1) if for all $(\phi, \psi) \in (W_0^{1,p}(\Omega))^2$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi = \int_\Omega f(x,u,v)\phi$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\psi = \int_\Omega g(x,u,v)\psi$$

Theorem 2.1 (Mountain Pass [3]) *Let I be a C^1-differentiable functional on a Banach space E and satisfying the Palais-Smale condition (PS), suppose that there exists a neighbourhood U of 0 in E and a positive constant α satisfying the following conditions:*

(I1) $I(0) = 0$.

(I2) $I(u) \geq \alpha$ *on the boundary of* U.

(I3) There exists an $e \in E\backslash U$ *such that* $I(e) < \alpha$.

Then

$$c = \inf_{\gamma\in\Gamma} \sup_{y\in[0,1]} I(\gamma(y))$$

is a critical value of I with $\Gamma = \{g \in C([0,1]); g(0) = 0, g(1) = e\}$.

3 Main result

The case $p \neq N$.

Set

$$J(u,v) = \frac{1}{p}\int_\Omega (|\nabla u|^p + |\nabla v|^p)\,dx - \int_\Omega G(x,u,v)\,dx$$

J is well define in $(W_0^{1,p}(\Omega))^2$. In this subsection we have the following result

Theorem 3.1 *Let f and g are two Carathéodory functions satisfying (H1), (H2) and suppose that*

i) $X \subset L^\infty(\Omega)$.

ii) There exist some $r_0 > 0$, $\sigma > p-1$, $\theta > p-1$ and $c > 0$ such that, for almost every where $x \in \Omega$ and for all $|\xi| + |\eta| < r_0$ we have

$$G(x,\xi,\eta) \le c(\xi^{\sigma+1} + \eta^{\theta+1}).$$

Then, there is at least one positive solution $(u,v) \in (X \cap C^{1,\nu}(\bar\Omega))^2$ of (1.1).

Remark. The condition i) is true for $X = W_0^{1,p}(\Omega)$ where Ω is an open bounded domain in $\mathbb{R}^N$ and $p > N$.

The following proposition gives another interesting example of the space X with $p > 1$.

Proposition 3.2 ([8]) *Let $0 < \rho < R < +\infty$ and $\Omega = \{x \in \mathbb{R}^N : \rho < |x| < R\}$ an annulus in $\mathbb{R}^N$. Let X be the set of radially symmetric functions in $W_0^{1,p}(\Omega)$. Then, there exist a positive constant $c(N,\rho,p,R) > 0$ such that, for all $u \in X$ and for almost every where $x \in \Omega$ we have*

$$|u(x)| \le c(N,\rho,p,R)\|\nabla u\|_p.$$

To prove Theorem 3.1 we prove some preliminary lemmas.

Lemma 3.3 *Let $u \in X$. Suppose that f and g satisfy (H1) and (H2). Then, any sequence $\{(u_j,v_j)\}_{j\ge 0} \in X \times X$ satisfying the following two hypotheses:*

$$|J(u_j,v_j)| \le K \tag{3.1}$$

and for all $\epsilon > 0$ there exist $j_0 \in \mathbb{N}^*$ such that $\forall j \ge j_0$,

$$|\langle J'(u_j,v_j),(u_j,v_j)\rangle| \le \epsilon\|(u_j,v_j)\|, \tag{3.2}$$

is bounded in $X \times X$.

Proof. Set $\|(u,v)\| = (\|\nabla u\|_p^p + \|\nabla v\|_p^p)^{1/p}$. This is a norm in the product space $X \times X$, and $\|\nabla u\|_p = \|u\|_X$. Now we proceed by contradiction. Suppose that a subsequent denoted by $\{(u_j, v_j)\}_{j\geq 0}$ be such that

$$\lim_{j\to+\infty} \|(u_j, v_j)\| = +\infty,$$

In virtue (3.1), we get

$$\frac{-K}{\|(u_j,v_j)\|^p} \leq \frac{1}{p} - \frac{\int_\Omega G(x,u_j,v_j)dx}{\|(u_j,v_j)\|^p} \leq \frac{K}{\|(u_j,v_j)\|^p}.$$

By passing to limit we deduce that

$$\lim_{j\to+\infty} \frac{\int_\Omega G(x,u_j,v_j)dx}{\|(u_j,v_j)\|^p} = \frac{1}{p}. \tag{3.3}$$

On the other hand, (3.2) implies

$$\frac{-\varepsilon}{\|(u_j,v_j)\|^{p-1}} \leq 1 - \frac{\int_\Omega (u_j f(x,u_j,v_j) + v_j g(x,u_j,v_j))dx}{\|(u_j,v_j)\|^p} \leq \frac{\varepsilon}{\|(u_j,v_j)\|^{p-1}}.$$

By passing to limit, we obtain

$$\lim_{j\to+\infty} \frac{\int_\Omega (u_j f(x,u_j,v_j) + v_j g(x,u_j,v_j))dx}{\|(u_j,v_j)\|^p} = 1. \tag{3.4}$$

Combining (2.1), (2.2), (3.3) and (3.4) we deduce that

$$\frac{1}{p} \leq \frac{1}{\sigma_0 + 1} + \frac{1}{\theta_0 + 1} < \frac{1}{p}.$$

A contradiction, whence $\|(u_j, v_j)\|_X$ is bounded. □

Lemma 3.4 *Let f and g be two Carathéodory functions satisfying the hypothesis of Theorem 3.1 and let $\{(u_j, v_j)\}_{j\geq 0}$ be a sequence in $X \times X$ such that $(u_j, v_j) \rightharpoonup (u, v)$ weakly in $X \times X$. Then*

$$\lim_{j\to+\infty} \int_\Omega f(x,u_j,v_j)(u_j - u) = 0, quad \lim_{j\to+\infty} \int_\Omega g(x,u_j,v_j)(v_j - v) = 0.$$

Proof. By using Hölder's inequality we obtain

$$\left| \int_\Omega f(x,u_j,v_j)(u_j - u) \right| \leq \|f(x,u_j,v_j)\|_{p'} \|u_j - u\|_p.$$

In virtue i), (H1) and by using the imbedding Sobolev space we have $(u_j, v_j) \to (u, v)$ strongly in $L^p(\Omega) \times L^p(\Omega)$. Then, by Lebesgue's theorem, as $j \to +\infty$,

$$\lim_{j\to+\infty} \int_\Omega f(x,u_j,v_j)(u_j - u) = 0.$$

The proof of the second limit in this Theorem is the same. □

Lemma 3.5 *Under the hypothesis of Theorem 3.1, $J \in C^1(X \times X)$ and satisfies the Palais-Smale condition.*

Proof. In virtue of the preceding lemma we have $J \in C^1(X \times X)$. Let $\{(u_j, v_j)\}_{j\geq 0}$ be a sequence of element in $X \times X$ satisfying the conditions (3.1) and (3.2). Hence, by Lemma 3.3 the sequence (u_j, v_j) is bounded, then, there exist a subsequent still denoted $\{(u_j, v_j)\}_{j\geq 0}$ weakly convergent to $(u, v) \in X \times X$ and strongly in $L^p(\Omega) \times L^p(\Omega)$. On the other hand, since

$$\langle -\Delta_p u_j, u_j - u\rangle = \langle J'(u_j, v_j), (u_j, v_j) - (u, v)\rangle - \int_\Omega f(x, u_j, v_j)(u_j - u).$$

As $j \to +\infty$, in virtue of Lemma 3.4 and (3.2) we have

$$\lim_{j\to+\infty} \langle -\Delta_p u_j, u_j - u\rangle = 0.$$

Or the p-Laplacian operator satisfies the condition (S_+), thus

$$u_j \to u \text{ strongly in } X.$$

The same way, we prove that $v_j \to v$ strongly in X. □

Proof of Theorem 3.1 It suffices to prove that the functional J satisfies the conditions for the Pass-Mountain lemma [3]:
J satisfies condition of Palais-Smale and $J(0) = 0$ (see Lemma 3.5).
For $\|(u, v)\| = r$ sufficiently small, we have $J(u, v) \geq \alpha > 0$.

We prove this second conditions first. By i), for all $x \in \Omega$, there exist $c' > 0$ such that $|u(x)|+|v(x)| \leq c'\|(u, v)\|$; and for $\|(u, v)\| \leq \frac{r_0)}{c'}$. Using ii) we deduce that

$$\begin{aligned} G(x, u(x), v(x)) &\leq c(|u(x)|^{\sigma+1} + |v(x)|^{\theta+1}) \\ &\leq c[(c')^{\sigma+1}\|(u, v)\|^{\sigma+1} + (c')^{\theta+1}\|(u, v)\|^{\theta+1}] \\ &\leq c''(\|(u, v)\|^{\sigma+1} + \|(u, v)\|^{\theta+1}). \end{aligned}$$

Then

$$\begin{aligned} J(u, v) &\geq \frac{1}{p}\|(u, v)\|^p - c''(\|(u, v)\|^{\sigma+1} + \|(u, v)\|^{\theta+1}) \\ &\geq \frac{1}{p}\|(u, v)\|^p - 2c''\|(u, v)\|^l \end{aligned}$$

with $l = \min(\sigma + 1, \theta + 1)$. It suffices to take $r \leq \min(\frac{r_0)}{c'}, (\frac{1}{2pc''})^{\frac{1}{l-p}})$.
Finally, for $\|(u, v)\| \leq r$, we have

$$J(u, v) \geq \alpha = \frac{r^p}{p} > 0.$$

Now, we prove the first condition. Let $(u_0, v_0) \in X \times X$ such that for almost every where $x \in \Omega_0$ with $\text{meas}(\Omega_0) > 0$ we have $u_0(x) + v_0(x) > \alpha_0 > 0$with some $\alpha_0 > 0$. For t large enough we have $tu_0 > \xi_0$, $tv_0 > \eta_0$ with $\xi_0 + \eta_0 > R$. From (2.1) and (2.2) we get

$$\xi \to \frac{G(x,\xi,\eta)}{|\xi|^{\sigma_0+1}} \quad \text{and} \quad \eta \to \frac{G(x,\xi,\eta)}{|\eta|^{\theta_0+1}}$$

are increasing, then

$$\int_\Omega G(x,tu_0,tv_0) \geq \int_{\Omega_0} G(x,tu_0,tv_0) \geq \beta(t^{\sigma_0+1} + t^{\theta_0+1}),$$

with

$$\begin{aligned}\beta = \frac{1}{2}\inf\Big(&\frac{1}{\xi_0^{\sigma_0+1}}\int_{\Omega_0} G(x,\xi_0,\eta_0)|u_0(x)|^{\sigma_0+1},\\ &\frac{1}{\eta_0^{\theta_0+1}}\int_{\Omega_0} G(x,\xi_0,\eta_0)|v_0(x)|^{\theta_0+1}\Big).\end{aligned}$$

Consequently,

$$J(tu_0,tv_0) \leq \frac{t^p}{p}\|(u,v)\|^p - \beta(t^{\sigma_0+1} + t^{\theta_0+1}).$$

by passing to the limit, as $t \to +\infty$ we have $\lim_{t\to+\infty} J(tu_0,tv_0) = -\infty$. Then, there exist some $(e_1,e_2) \in X \times X$, with $e_1 \neq 0$ and $e_2 \neq 0$, such that $J(e_1,e_2) < 0$.

By the Pass-Mountain theorem, there exists $(u_0,v_0) \in X \times X$ $u_0 \neq 0$, $v_0 \neq 0$, such that $J'(u_0,v_0) = 0$, i.e for all $(\phi,\psi) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \nabla\phi - \int_\Omega f(x,u_0,v_0)\phi = 0,\\ \int_\Omega |\nabla v_0|^{p-2}\nabla v_0 \nabla\psi - \int_\Omega g(x,u_0,v_0)\psi = 0.\end{aligned}$$

In virtue of Tolksdorf regularity [10], $(u_0,v_0) \in C^{1,\nu}(\bar\Omega) \times C^{1,\nu}(\bar\Omega)$ and by Vazquez's maximum principle [11], $u_0 > 0$ and $v_0 > 0$. $\square$

Example Let $f(x,\xi,\eta) = \xi^\sigma \exp(\xi^q + \eta^r)$, $g(x,\xi,\eta) = \eta^\theta exp(\xi^q + \eta^r)$, $\sigma > 2p-1$, $\theta > 2p-1$, $r,q > 0$. The functions f and g satisfy the hypotheses (H1), (H2), (ii), and X the space defined in Proposition 3.2. Hence, for $\sigma,\theta > 1$;

$$\begin{gathered}-\Delta_p u = u^\sigma \exp(u^q + v^r) \quad \text{in } \Omega\\ -\Delta_p v = v^\theta \exp(u^q + v^r) \quad \text{in } \Omega\\ u = v = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

has a positive solution $(u,v) \in (X \times C^{1,\nu}(\overline{\Omega}))^2$.

The case $p = N$

Recall that a Young function is an even convex function from $\mathbb{R}$ into $\mathbb{R}^+$, such that

$$\lim_{\xi \to 0} \frac{M(\xi)}{\xi} = 0 \quad \text{and} \quad \lim_{\xi \to +\infty} \frac{M(\xi)}{\xi} = +\infty.$$

The conjugate function of M is defined as

$$M^*(\xi) = \sup_{s \in \mathbb{R}} [\xi s - M(s)].$$

The Orlicz space $L_M(\Omega)$ is the set of measurable function u defined on $\mathbb{R}$ such that, there is some $\lambda > 0$ with

$$\int_\Omega M(\frac{u}{\lambda}) < +\infty.$$

This is a Banach space for the norm

$$\|u\|_M = Inf \left\{ \lambda > 0 : \int_\Omega M(\frac{u}{\lambda}) < 1 \right\}.$$

Let $E_M(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $L_M(\Omega)$.
We say that M is super-homogenous of degree $(\sigma + 1)$ [8] if there exists $K > 0$ such that

$$M(h\xi) \le h^{\sigma+1} M(K\xi), \quad \forall \xi \in \mathbb{R},\ \forall h \in [0,1].$$

Let Ω be a bounded regular domain in $\mathbb{R}^N$. In this case $W_0^{1,p}(\Omega) \not\subset L^\infty(\Omega)$ but $W_0^{1,p}(\Omega) \subset E_{M_1}(\Omega)$ [1] where

$$M_1(\xi) = \exp(|\xi|^{p'}) - 1, \quad \text{or} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

So, we can get the following Theorem.

Theorem 3.6 *Let f and g be two positive functions which are Caratheodory and satisfy (H1) and (H2). Assume also that there exists a Young function of exponential type M such that:*

i) The imbedding $W_0^{1,p}(\Omega) \hookrightarrow E_M(\Omega)$ is compact.

ii) M is super-homogeneous of degree $\sigma_1 + 1 > p$.

iii) There are some $c_1 > 0$ and $K_1 > 0$ such that for a.e $x \in \Omega$ and for all $(\xi, \eta) \in \mathbb{R}^2$,

$$\xi f(x, \xi, \eta) \le c_1 M(\frac{\xi}{K_1}) \quad \textit{and} \quad \eta g(x, \xi, \eta) \le c_1 M(\frac{\eta}{K_1}).$$

iv) For all $K>0$, we have

$$\lim_{|\xi|+|\eta|\to+\infty}\frac{f(x,\xi,\eta)}{M'(\frac{\xi}{K})}=0 \quad \textit{and} \quad \lim_{|\xi|+|\eta|\to+\infty}\frac{g(x,\xi,\eta)}{M'(\frac{\eta}{K})}=0$$

almost every where in $x\in\Omega$.

Then there is at least one positive solution $(u,v)\in(W_0^{1,p}(\Omega)\cap C^{1,\nu}(\bar{\Omega}))^2$ of (1.1).

The proof of this Theorem needs the following lemma.

Lemma 3.7 *Under the hypotheses of Theorem 3.6, $J\in C^1((W_0^{1,p}(\Omega))^2$ and satisfies the Palais-Smale condition.*

Proof. Let $\{(u_j,v_j)\}_{j\geq 0}$ be a bounded sequence in $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$. By i) there exist some $K>0$ such that

$$\int_\Omega M(\frac{u_j}{K})\leq 1,\quad \int_\Omega M(\frac{v_j}{K})\leq 1.$$

Let $c>0$ be large enough such that $M^*(\frac{1}{c})\,\mathrm{meas}(\Omega)<1$. From iv) for all $(\xi,\eta)\in\mathbb{R}^2$ and for a.e $x\in\Omega$ we have

$$|f(x,\xi,\eta)|+|g(x,\xi,\eta)|\leq\frac{c}{2}+\frac{1}{4}\big(M'(\frac{\xi}{K})+M'(\frac{\eta}{K})\big),$$

or M^* is a Young function satisfies the "Δ_2-condition". Then

$$\begin{aligned}
M^*(\frac{f(x,u_j,v_j)}{c^2}) &\leq \frac{1}{2}M^*(\frac{1}{c})+\frac{1}{2}M^*\big(\frac{1}{2}M'(\frac{u_j}{c^2K})+\frac{1}{2}M'(\frac{v_j}{c^2K})\big)\\
&\leq \frac{1}{2}M^*(\frac{1}{c})+\frac{1}{4}\big(M(\frac{2u_j}{c^2K})+M(\frac{2v_j}{c^2K})\big)\\
&\leq \frac{1}{2}M^*(\frac{1}{c})+\frac{1}{4}\big(M(\frac{u_j}{K})+M(\frac{v_j}{K})\big).
\end{aligned}$$

Hence

$$\int_\Omega M^*(\frac{f(x,u_j,v_j)}{c^2})\leq 1. \tag{3.5}$$

In the same we obtain

$$\int_\Omega M^*(\frac{g(x,u_j,v_j)}{c^2})\leq 1. \tag{3.6}$$

Let $\{(u_j,v_j)\}_{j\geq 0}$ be a subsequent of the least sequence of element in $(W_0^{1,p}(\Omega))^2$ converges to $(u,v)\in(W_0^{1,p}(\Omega))^2$. For $\delta>0$ sufficiently small, for all $\epsilon>0$ and

$A \subset \Omega$ such that $\text{meas}(A) \leq \delta$ we have

$$\begin{aligned}
&\int_A M^*\big(\frac{f(x,u_j,v_j)}{c^2}\big)\\
\leq& \frac{1}{2}M^*\big(\frac{1}{c}\big)\,\text{meas}(A) + \frac{1}{4}\int_A \Big[M\big(\frac{2u_j}{c^2K}\big) + M\big(\frac{2v_j}{c^2K}\big)\Big]\\
\leq& \frac{1}{2}M^*\big(\frac{1}{c}\big)\,\text{meas}(A) + \frac{1}{8}\int_A \Big[M\big(\frac{u_j-u}{K}\big) + M\big(\frac{u}{K}\big) + M\big(\frac{v_j-v}{K}\big) + M\big(\frac{v}{K}\big)\Big]\\
\leq& \epsilon,
\end{aligned}$$

then $M^*\big(\frac{f(x,u_j,v_j)-f(x,u,v)}{c^2}\big)$ is equi-summable and

$$\lim_{j\to+\infty}\int_\Omega M^*\big(\frac{f(x,u_j,v_j)-f(x,u,v)}{c^2}\big) = 0$$

By ii) and since M^* satisfies "Δ_2-condition" we have

$$\lim_{j\to+\infty}\|f(.,u_j,v_j)-f(.,u,v)\|_{M^*} = 0.$$

In the same way we have

$$\lim_{j\to+\infty}\|g(.,u_j,v_j)-g(.,u,v)\|_{M^*} = 0$$

Whence $J \in C^1((W_0^{1,p}(\Omega))^2$. Let $\{(u_j,v_j)\}_{j\geq 0}$ be a sequence satisfying (3.1) and (3.2) then by lemma 3.3 the sequence $\{(u_j,v_j)\}_{j\geq 0}$ is bounded in $(W_0^{1,p}(\Omega))^2$,hence $\{(u_j,v_j)\}_{j\geq 0}$ converges weakly to $(u,v) \in (W_0^{1,p}(\Omega))^2$ and strongly in $(E_M(\Omega))^2$. In view of (3.5) (3.6) we deduce that $f(x,u_j,v_j)$, $g(x,u_j,v_j)$ converge with $\sigma(L_M \times L_M,\, E_M \times E_M)$. So the same proof of lemma 3.5 shows that the Palais-Smale condition is satisfied. $\square$

Proof of Theorem 3.6 Let us show that for $\|(u,v)\| = r$ sufficiently small, $J(u,v) \geq \alpha > 0$. By (H2) and iii), for a.e $x \in \Omega$, for all $\xi \in \mathbb{R}$,and for all $h \in [0,\ 1]$, we have

$$\begin{aligned}
G(x,\xi,\eta) &\leq \frac{1}{2}\Big[\frac{1}{\sigma_0+1}\xi f(x,\xi,\eta) + \frac{1}{\theta_0+1}\eta g(x,\xi,\eta)\Big]\\
&\leq \frac{c_1}{2}\Big[\frac{1}{\sigma_0+1}M\big(\frac{\xi}{K_1}\big) + \frac{1}{\theta_0+1}M\big(\frac{\eta}{K_1}\big)\Big]\\
&\leq \frac{c_1}{2}\Big[h^{\sigma_1+1}M\big(\frac{K\xi}{hK_1}\big) + h^{\theta_1+1}M\big(\frac{K\eta}{hK_1}\big)\Big]
\end{aligned}$$

on the other hand, in virtue of i) there exists $c > 0$ such that for all $(u,v) \in W_0^{1,p}(\Omega)^2$ we have

$$\|u\|_M + \|v\|_M \leq c\|(u,v)\|.$$

Whence for $\|(u,v)\| = r \leq \frac{K_1}{cK}$ and $h = \frac{cKr}{K_1}$ we get

$$\int_\Omega M(\frac{u}{cr}) \leq 1 \quad \text{and} \quad \int_\Omega M(\frac{v}{cr}) \leq 1.$$

Hence

$$\begin{aligned}\int_\Omega G(x,u,v)dx &\leq \frac{c_1}{2}\left[h^{\sigma_1+1} + h^{\theta_1+1}\right] \\ &\leq c'\left[\|(u,v)\|^{\sigma_1+1} + \|(u,v)\|^{\theta_1+1}\right]\end{aligned}$$

The same proof as in Theorem 3.1 gives $(u,v) \in (W_0^{1,p}(\Omega))^2$, $u \not\equiv 0$, $v \not\equiv 0$, solution of (1.1). The rest of the proof is a consequence of the following lemma. $\square$

Lemma 3.8 *Under the hypotheses of Theorem 3.6, if (u,v) is a solution of (1.1) then $(u,v) \in C^{1,\nu}(\bar{\Omega}) \times C^{1,\nu}(\bar{\Omega})$.*

Proof. This proof is inspired by the work of De Thélin [8] and Otani [6] (see also [7]).
In view of iii), there exists $s > 1$ such that $uf(x,u,v) \in L^s(\Omega)$ and $vg(x,u,v) \in L^s(\Omega)$. Consider the following sequences:

$$q_1 = 2ps^* = \frac{2ps}{s-1}, \quad q_{k+1} = 2(p+q_k)$$
$$m_k = s^* q_k.$$

Multiplying the first equation of (1.1) by $|u|^{q_k}u$ and the second equation by $|v|^{q_k}v$, we obtain:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(|u|^{q_k}u) = \int_\Omega uf(x,u,v)|u|^{q_k}$$
$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla(|v|^{q_k}v) = \int_\Omega vg(x,u,v)|u|^{q_k}$$

by Hölder's inequality we deduce that

$$\begin{aligned}(\frac{p}{p+q_k})^p \int_\Omega |\nabla u^{\frac{p+q_k}{p}}|^p = &\int_\Omega f(x,u,v)|u|^{q_k}u \\ &\leq \|uf(.,u,v)\|_s \|u^{q_k}\|_{s^*} \\ &\leq c\|u\|_{s^*}^{q_k}.\end{aligned} \tag{3.7}$$

Since the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{2ps^*}(\Omega)$ is compact, there exists $K > 0$ such that

$$\|u\|_{2s^*(p+q_k)}^{p+q_k} \leq K^p \int_\Omega |\nabla u^{\frac{p}{p+q_k}}|^p. \tag{3.8}$$

By combining (3.7) and (3.8) we have

$$\|u\|_{2s^*(p+q_k)}^{m_{k+1}/(2s^*)} \leq c\big(\frac{K(p+q_k)}{p}\big)^p \|u\|_{m_k}^{m_k/s^*}.$$

Since $p + q_k \leq 4^k ps^*$ we get

$$\|u\|_{m_{k+1}}^{m_k+1} \leq c^{2s^*}(4Ks^*)^{2ps^*} 4^{2(k-1)ps^*} \|u\|_{m_k}^{2m_k}.$$

Set $E_k = m_k \log \|u\|_{m_k}$, $a = 4^{2ps^*}$, $b = \log[c^{2s^*}(2Ks^*)^{2ps^*}]$ and $r_k = b + (k-1)\log a$. We obtain

$$E_{k+1} \leq r_k + 2E_k$$

then, by the result's of Otani [6] we deduce that

$$\|u\|_\infty \leq \limsup_{k\to+\infty} \exp\big(\frac{E_k}{m_k}\big) < +\infty.$$

Finally, by the regularity of Tolksdorf's results $u \in C^{1,\nu}(\bar{\Omega})$. In the same way we have $v \in C^{1,\nu}(\bar{\Omega})$. □

Example Let $f(x,\xi,\eta) = \xi^\sigma \exp(\xi^q - \eta^r)$, $g(x,\xi,\eta) = \eta^\theta \exp(-\xi^q + \eta^r)$, $\sigma > 2p-1$, $\theta > 2p-1$, $N \geq 2$, $0 < r, q < \frac{p}{p-1}$, and $M(\xi) = |\xi|^{\sigma+\theta+1-l}(e^{|\xi|^l} - 1)$ with $\max(p,r) < l < 2$ hence the functions f and g satisfy the hypothesis (H1), (H2), (i), (ii), (iii) and (iv). Then

$$\begin{aligned} -\Delta_p u &= u^\sigma \exp(u^q - v^r) \quad \text{in } \Omega \\ -\Delta_p v &= v^\theta \exp(-u^q + v^r) \quad \text{in } \Omega \\ u &= v = 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has a positive solution $(u,v) \in (W_0^{1,p}(\Omega) \times C^{1,\nu}(\bar{\Omega}))^2$.

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