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Local and global nonexistence of solutions to semilinear evolution equations *

Mohammed Guedda & Mokhtar Kirane

To Professor Bernard Risbourg, in memorium

Abstract

For a fixed p and $\sigma > -1$, such that $p > \max\{1, \sigma+1\}$, one main concern of this paper is to find sufficient conditions for non solvability of

$$u_t = -(-\Delta)^{\frac{\beta}{2}} u - V(x)u + t^\sigma h(x)u^p + W(x,t),$$

posed in $S_T := \mathbb{R}^N \times (0,T)$, where $0 < T < +\infty$, $(-\Delta)^{\frac{\beta}{2}}$ with $0 < \beta \leq 2$ is the $\beta/2$ fractional power of the $-\Delta$, and $W(x,t) = t^\gamma w(x) \geq 0$. The potential V satisfies $\limsup_{|x|\to+\infty} |V(x)||x|^a < +\infty$, for some positive a. We shall see that the existence of solutions depends on the behavior at infinity of both initial data and the function h or of both w and h. The non-global existence is also discussed. We prove, among other things, that if $u_0(x)$ satisfies

$$\lim_{|x|\to+\infty} u_0^{p-1}(x)h(x)|x|^{(1+\sigma)\inf\{\beta,a\}} = +\infty,$$

any possible local solution blows up at a finite time for any locally integrable function W. The situation is then extended to nonlinear hyperbolic equations.

1 Introduction

In this paper we consider the problem

$$\begin{gathered} u_t = -(-\Delta)^{\frac{\beta}{2}} u - V(x)u + t^\sigma h(x)u^p + W(x,t), \quad (x,t) \in \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^N, \end{gathered} \tag{1.1}$$

for some $0 < T \leq +\infty$, where $(-\Delta)^{\frac{\beta}{2}}$ with $0 < \beta \leq 2$ is the $\beta/2$ fractional power of the $-\Delta$, which stands for diffusion in media with impurities, $p > 1$, $\sigma > -1$, the functions h and u_0 are nonnegative and satisfy some growth conditions at

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infinity which will be specified later. The function $W(x,t) \geq 0$, which can be viewed as a noise or as a control, is locally integrable. Even if we can handle general $W(x,t)$, we will confine ourselves to the simple case where $W(x,t) = t^\gamma w(x)$, $\gamma > -1$. We assume that the potential V satisfies

$$\limsup_{|x|\to+\infty} |V(x)||x|^a < +\infty, \tag{1.2}$$

for some $a > 0$.

In the case $\beta = 2, \sigma = \gamma = 0$ and $V = 0$, Pinsky [7] proved that all nontrivial nonnegative solutions blow up at a finite time if $N \leq 2$ or $N \geq 3$ and the function $w(x)|x|^{2-N}$ is not integrable. It was also shown that if $h(x) \geq c|x|^m$ and $w(x) \geq c|x|^{-q}$, $2 < q < N$, for large $|x|$ there is no global solutions if $1 < p \leq 1 + \frac{2+m}{N-2}$.

In a recent paper [6], we studied the criticality for some evolution inequalities. It was shown, among other results, that for $V \leq 0, w(x) \geq 0$ if $h(x)$ behaves like $|x|^\gamma$ at infinity and if $1 < p \leq 1 + \frac{\gamma+\beta(1+\sigma)}{N} =: p_c$ then there is no global nonnegative weak solutions except the trivial one. In the case where $p > p_c$ solutions may exist, at least locally. More recently the first named author proved in [3] a similar result for $\sigma = 0$, but $V_+(x) \leq \frac{b}{1+|x|^a}$, where $V_+ = \max\{V, 0\}$, $a > \frac{N(p-1)}{p} > 0$, $b > 0$ and p small.

In [3],[6] the problem of nonexistence of global weak solution, with unsigned initial data and $w = V = 0$, is also considered. The authors obtained the absence of global solution for initial data satisfying

$$0 < \int_{\mathbb{R}^N} u_0(x)dx \leq +\infty,$$

and under some conditions on p and on the behavior at infinity of h.

In the present paper we are interested in conditions for local and global solvability of (1.1) from a different angle. We investigate, for any fixed $\sigma > -1$ and $p > \max\{1, 1+\sigma\}$, in contrast to the Fujita-type result, the effect of the behavior of u_0, h and w at infinity on the non existence of local and global weak solutions to (1.1).

This work is motivated by the paper [1] in which Baras and Kersner showed that the problem

$$u_t = \Delta u + h(x)u^p, \quad u(x,0) = u_0(x) \geq 0, \tag{1.3}$$

has no local weak solution if the initial data satisfies

$$\lim_{|x|\to+\infty} u_0^{p-1} h(x) = +\infty,$$

and any possible local weak solution blows up at a finite time if

$$\lim_{|x|\to+\infty} u_0^{p-1} h(x)|x|^2 = +\infty.$$

Here, we attempt to extend this result to (1.1). The methods used are some modifications and adaptations of ideas from [1] and [3].

Set

$$S_T := \mathbb{R}^N \times (0, T).$$

Definition We say that $u \geq 0$ is a local weak solution to (1.1), defined in $S_T, 0 < T < +\infty$, if it is a locally integrable function such that $u^p h \in L^1_{loc}(S_T)$, and

$$\int_{\mathbb{R}^N} u(x,0)\zeta(x,0)dx + \int_{S_T} t^\gamma wh\zeta\,dx\,dt + \int_{S_T} t^\sigma hu^p\zeta\,dx\,dt$$
$$= \int_{S_T} u(-\Delta)^{\frac{\beta}{2}}\zeta\,dx\,dt - \int_{S_T} u\zeta_t\,dx\,dt - \int_{S_T} uV(x)\zeta\,dx\,dt, \quad (1.4)$$

is satisfied for any $\zeta \in C_0^\infty(\overline{S_T})$ which vanishes for large $|x|$ and at $t = T$.

Definition We say that $u \geq 0$ is a global weak solution to (1.1), if it is a local solution to (1.1) defined in S_T for any $T > 0$.

Throughout this paper we may assume that there exists $R_0 > 0$ such that $w(x)$ are nonnegative for all $|x| \geq R_0$ and condition (1.2) is satisfied.

Theorem 1.1 *Let $\sigma > -1$, $p > \max\{1, 1+\sigma\}$. Assume that one of the following two conditions*

$$\lim_{|x|\to+\infty} u_0^{p-1}h(x) = +\infty, \quad (1.5)$$

$$\lim_{|x|\to+\infty} w^{p-1}h(x) = +\infty, \quad (1.6)$$

is satisfied. Then there is no $T > 0$ such that problem (1.1) *has a solution defined in S_T.*

This result shows in particular that any local solution to (1.1) blows up at $t = 0$. The proof of Theorem 1.1 is based on an upper estimate of the blowing up time as it is shown in the following theorem.

Theorem 1.2 *Let $\sigma > -1$, $p > \max\{1, 1+\sigma\}$. There exist positive constants K_1, K_2 such that if problem* (1.1) *has local solution defined in $S_T, T < +\infty$, the following two estimates hold:*

$$\liminf_{|x|\to+\infty} u_0^{p-1}h(x) \leq K_1\frac{1}{T^{1+\sigma}}, \quad (1.7)$$

$$\liminf_{|x|\to+\infty} w^{p-1}h(x) \leq K_2\frac{1}{T^{(1+\gamma)(p-1)}}. \quad (1.8)$$

We can deduce from the above result that if problem (1.1) has a global solution then the initial data and the function w must satisfy

$$\liminf_{|x|\to+\infty} u_0^{p-1}h(x) = \liminf_{|x|\to+\infty} w^{p-1}h(x) = 0.$$

But those conditions are not sufficient for the global existence as it can be seen from the following statement.

Theorem 1.3 *Let $\sigma > -1$, $p > \max\{1, 1+\sigma\}$. Assume that*

$$\lim_{|x|\to+\infty} u_0^{p-1}h(x)|x|^{(1+\sigma)\inf\{\beta,a\}} = +\infty, \tag{1.9}$$

or

$$\lim_{|x|\to+\infty} w^{p-1}h(x)|x|^{\inf\{\beta,a\}[\gamma(p-1)+p+\sigma]} = +\infty. \tag{1.10}$$

Then problem (1.1) *has no global solution.*

Remark 1.4 It is interesting to note here, that in some sense, there is no effect of V on the global solvability of (1.1) if $a \geq 2$. In case $V \leq 0$ assumptions (1.9),(1.10) have to be read with β instead of $\inf\{\beta, a\}$.

The second part of our paper deals with non existence results for the hyperbolic problem

$$\begin{aligned} u_{tt} &= \Delta u + V(x)u + t^\sigma h(x)u^p + t^\gamma w(x),\\ u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x). \end{aligned} \tag{1.11}$$

We use the similar approach to establish the

Theorem 1.5 *Let $p > \max\{1, 1+\sigma\}$, $\sigma > -1$. Assume that*

$$\limsup_{|x|\to\infty} |x|^{2a}|V(x)| < +\infty$$

and one of the following two conditions

$$\lim_{|x|\to+\infty} u_1 h^{1/(p-1)}|x|^{\inf\{1,a\}\frac{1+\sigma+p}{p-1}} = +\infty, \tag{1.12}$$

$$\lim_{|x|\to+\infty} w h^{1/(p-1)}|x|^{\inf\{1,a\}\left(1+\gamma+\frac{p+1+\sigma}{p-1}\right)} = +\infty, \tag{1.13}$$

holds. Then (1.11) *has no global weak solution.*

Observe that the conditions required does not involve the initial position u_0. The method of the proofs is based on a judicious choice of the test function in the form

$$\zeta(x,t) = \eta(t/T)\Phi(x),$$

where $\eta \in C_0^\infty([0,+\infty))$ and $\Phi \in C_0^\infty(\mathbb{R}^N)$. For Problem (1.11) we demand to η to satisfy $\eta'(0) = 0$ therefore $\zeta_t(x,0) = 0$, and this condition eliminates in the definition of solution to (1.12) the term which contains u_0. The strong point in this result is to obtain necessary conditions for non local and non global existence of solutions for any local integrable initial data even if u_0 has a compact support. This remark was first noticed by Pohozaev and Veron [8].

2 Nonexistence of local solutions

In this section, we provide a necessary condition for the local solvability of (1.1). We first obtain estimates (1.7), (1.8) and then prove Theorem 1.2. Without lost of generality we may assume that for large $|x|$,

$$|V(x)| \le |x|^{-a}, \quad a > 0.$$

Proof of Theorem 1.2 Suppose that u is a local solution to (1.1) defined in S_T, $0 < T < +\infty$. Let ζ be a test function which is nonnegative. According to (1.4) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} u(x,0)\zeta(x,0)dx + \int_{S_T} t^\gamma w\zeta\,dx\,dt + \int_{S_T} t^\sigma h u^p \zeta\,dx\,dt \\
&\qquad \le \int_{S_T} u((-\Delta)^{\frac{\beta}{2}}\zeta)_+\,dx\,dt + \int_{S_T} u|\zeta_t|\,dx\,dt + \int_{S_T} u|V|\zeta\,dx\,dt, \quad (2.1)
\end{aligned}$$

where $[.]_+ = \max\{.,0\}$. By considering the Young inequality, with $p' = p/(p-1)$, we obtain

$$\begin{aligned}
&\int_{S_T} u|\zeta_t|\,dx\,dt \\
&\le \frac{1}{3}\int_{S_T} t^\sigma u^p h\zeta\,dx\,dt + (p-1)3^{1/(p-1)}p^{-p/(p-1)}\int_{S_T} |\zeta_t|^{p'}(t^\sigma h\zeta)^{1-p'}\;,dx\,dt,
\end{aligned}$$

$$\begin{aligned}
\int_{S_T} u((-\Delta)^{\frac{\beta}{2}}\zeta)_+\,dx\,dt \le \frac{1}{3}\int_{S_T} t^\sigma u^p h\zeta\,dx\,dt + (p-1)3^{1/(p-1)}p^{-p/(p-1)} \\
\times \int_{S_T} ((-\Delta)^{\frac{\beta}{2}}\zeta)_+^{p'}(t^\sigma h\zeta)^{1-p'}\,dx\,dt,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{S_T} u|V(x)|\zeta\,dx\,dt \\
&\le \frac{1}{3}\int_{S_T} t^\sigma u^p h\zeta\,dx\,dt + (p-1)3^{1/(p-1)}p^{-p/(p-1)}\int_{S_T} |V|^{p'}\zeta(t^\sigma h)^{1-p'}\,dx\,dt.
\end{aligned}$$

Using the above estimates in (2.1), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} u(x,0)\zeta(x,0)dx + \int_{S_T} t^\gamma w\zeta\,dx\,dt \\
&\le (p-1)3^{1/(p-1)}p^{-p/(p-1)}\Big[\int_{S_T} |\zeta_t|^{p'}(t^\sigma h\zeta)^{1-p'}\,dx\,dt \\
&\quad + \int_{S_T} ((-\Delta)^{\frac{\beta}{2}}\zeta)_+^{p'}(t^\sigma h\zeta)^{1-p'}\,dx\,dt + \int_{S_T} |V|^{p'}\zeta(t^\sigma h)^{1-p'}\,dx\,dt\Big].
\end{aligned}$$

At this stage, let

$$\zeta(x,t) = (\eta(t/T))^{p'}\,\Phi(x),$$

where $\Phi \in C_0^\infty(\mathbb{R}^N)$, $\Phi \geq 0$ and $\eta \in C_0^\infty(\mathbb{R}_+)$, $0 \leq \eta \leq 1$, satisfying

$$\eta(r) = \begin{cases} 1 & \text{if } r \leq \frac{1}{2}, \\ 0 & \text{if } r \geq 1. \end{cases}$$

With the above choice of ζ, we obtain

$$\begin{aligned}
&\frac{1}{1+\gamma}\Big(\frac{T}{2}\Big)^{1+\gamma}\int_{\mathbb{R}^N}\Phi w\,dx + \int_{\mathbb{R}^N}\Phi u_0\,dx \\
&\leq (p-1)3^{1/(p-1)}p^{-p/(p-1)}C_\star^{p'-1}\Big\{(p')^{p'}T^{(1+\sigma)(1-p')} \in t_{\mathbb{R}^N}\Phi h^{1-p'}\,dx\,dt \\
&\quad + T^{1+\sigma(1-p')}\int_{\mathbb{R}^N}((-\Delta)^{\frac{\beta}{2}}\Phi)_+^{p'}(\Phi h)^{1-p'}\,dx + T^{1+\sigma(1-p')}\int_{\mathbb{R}^N}|V|^{p'}\Phi h^{1-p'}\,dx\Big\},
\end{aligned} \tag{2.2}$$

where

$$C_\star^{p'-1} = \frac{\max\Big\{1, \|\eta'\|_\infty^{p'}\Big\}}{1+\sigma(1-p')}.$$

Next, we consider $\Phi(x) = \varphi(x/R)$, $R > 0$, where

$$\varphi \in C_0^\infty(\mathbb{R}^N), \quad 0 \leq \varphi \leq 1, \quad \operatorname{supp}\varphi \subset \{1 < |x| < 2\}, \quad ((-\Delta)^{\frac{\beta}{2}})\varphi)_+ \leq \varphi.$$

Accordingly, via (2.2) we find

$$\inf_{|x|>R}\Big(u_0(x)h^{p'-1}\Big)\int_{\mathbb{R}^N}\Phi h^{1-p'}\,dx \leq (p-1)3^{1/(p-1)}p^{-p/(p-1)}C_\star^{p'-1}I(R), \tag{2.3}$$

and

$$\begin{aligned}
&\inf_{|x|>R}\Big(w(x)h^{p'-1}\Big)\int_{\mathbb{R}^N}\Phi h^{1-p'}\,dx \\
&\qquad \leq (\gamma+1)2^{1+\gamma}T^{-1-\gamma}(p-1)3^{1/(p-1)}p^{-p/(p-1)}C_\star^{p'-1}I(R),
\end{aligned} \tag{2.4}$$

for $R > R_0$, where

$$I(R) := \Big[\Big(\frac{p}{p-1}\Big)^{p'}T^{(1-p')(1+\sigma)} + T^{1+\sigma(1-p')}\Big\{\frac{1}{R^{\beta p'}} + \frac{1}{R^{ap'}}\Big\}\Big]\int_{\mathbb{R}^N}\Phi h^{1-p'}\,dx.$$

Then estimates (1.8), (1.9), with $K_1 = \frac{3C_\star}{p-1}, K_2 = (\gamma+1)^{p-1}2^{1+\gamma}(p-1)K_1$, are easily obtained by dividing (2.3) and (2.4) by $\int_{\mathbb{R}^N}\Phi h^{1-p'}dx$ and letting $R \to +\infty$. This completes the proof. □

Note that assumption (1.2) are only used to eliminate the second term of $I(R)$ when R tends to infinity. It is obvious that the conclusions of Theorems 1.1 and 1.2 remain true if we assume

$$\limsup_{|x|\to+\infty}|x|^a V_+(x) < +\infty,$$

or

$$\lim_{R\to+\infty} \max_{\{R<|x|<2R\}} V(x)_+ = 0,$$

instead of (1.2).

Remark 2.1 Following the above proof, the condition $\sigma > -1$ is not used. It is easily verified that estimates (1.7), (1.8) are satisfied for any σ. This leads in particular to

$$\liminf_{|x|\to+\infty} u_0^{p-1}h(x) = 0,$$

if $\sigma < -1$, or if $\sigma = -1$ the limit is finite. Therefore there is no local solution if $\liminf_{|x|\to+\infty} u_0^{p-1}h(x) > 0$ and $\sigma < -1$. For the case $\sigma = -1$ there is no local solution if

$$\liminf_{|x|\to+\infty} u_0^{p-1}h(x) > K_1.$$

3 Necessary conditions for global solvability

In this section, we discuss conditions for the non existence of global solution to (1.1).

Proposition 3.1 *Let $p > \max\{1, 1+\sigma\}$. Assume that* (1.1) *has a global solution. Then the following two limits are finite:*

$$\liminf_{|x|\to+\infty} w^{p-1}h(x)|x|^{\inf\{\beta,a\}(1+\sigma)}, \tag{3.1}$$

$$\liminf_{|x|\to+\infty} u_0^{p-1}h(x)|x|^{\inf\{\beta,a\}(\gamma(p-1)+p+\sigma)} \tag{3.2}$$

Proof Assume that (1.1)–(1.2) has a global weak solution. According to the proof of Theorem 1.2 we have, for any $T > 0$,

$$\int_{\Omega_R} \Phi u_0 dx \le C_1\Big\{\big(\frac{p}{p-1}\big)^{p'}T^{(1+\sigma)(1-p')} + 2R^{-\inf\{\beta,a\}p'}T^{1+\sigma(1-p')}\Big\} \times \int_{\Omega_R} \Phi h^{1-p'}\,dx\,dt \tag{3.3}$$

and

$$\int_{\Omega_R} \Phi w\,dx \le C_2\Big\{\big(\frac{p}{p-1}\big)^{p'}T^{\sigma(1-p')-p'-\gamma} + 2R^{-\inf\{\beta,a\}p'}T^{\sigma(1-p')-\gamma}\Big\} \times \int_{\Omega_R} \Phi h^{1-p'}\,dx\,dt, \tag{3.4}$$

where $\Omega_R = \{R < |x| < 2R\}$, $C_1 := (p-1)3^{1/(p-1)}p^{-p/(p-1)}C_\star^{p'-1}$, $C_2 = (\gamma+1)2^{\gamma+1}C_1$ and $\Phi(x) = \varphi(x/R)$, with $\varphi \in C_0^\infty(\mathbb{R}^N)$ nonnegative satisfying $((-\Delta)^{\beta/2}\varphi)_+ \le \varphi$ and $\operatorname{supp}\varphi \subset \{1 < |x| < 2\}$.

A simple minimization of the right hand side of (3.3) with respect to $T > 0$ yields

$$\int_{\Omega_R} \Phi u_0 dx \leq A_\star^{p'-1} R^{-\inf\{\beta,a\}\frac{1+\sigma}{p-1}} \int_{\Omega_R} \Phi h^{1-p'} dx,$$

where $A_\star = A_\star(p,\sigma)$ is a positive constant. This leads to the estimate

$$\inf_{|x|>R} \left(u_0(x) h^{p'-1}(x)|x|^{\inf\{\beta,a\}\frac{1+\sigma}{p-1}}\right) \int_{\Omega_R} \Phi h^{1-p'} |x|^{-\inf\{\beta,a\}\frac{1+\sigma}{p-1}} dx$$
$$\leq A_\star^{p'-1} \int_{\Omega_R} \Phi |x|^{-\inf\{\beta,a\}\frac{1+\sigma}{p-1}} h^{1-p'} dx.$$

Thus

$$\liminf_{|x|\to+\infty} \left(u_0(x) h^{p'-1}(x)|x|^{\inf\{\beta,a\}\frac{1+\sigma}{p-1}}\right) \leq A_\star^{p'-1}.$$

To confirm (3.2) we use (3.4), with $T = R^{\inf\{\beta,a\}}$, to deduce

$$\int_{\Omega_R} \Phi w dx \leq B_\star^{p'-1} R^{-\inf\{\beta,a\}(\gamma+p'+\sigma(p'-1))} \int_{\Omega_R} \Phi h^{1-p'} dx.$$

The rest of the proof as above. □

Remark 3.2 As in section 2, condition (1.2) can be relaxed to

$$\limsup_{|x|\to\infty} V_+(x)|x|^a < +\infty,$$

where $V_+ = \max\{V, 0\}$. For equation (1.1) with $W = 0$, i.e, equation

$$u_t = -(-\Delta)^{\frac{\beta}{2}} u - V(x)u + t^\sigma u^p, \tag{3.5}$$

we have no global solution whenever

$$\lim_{|x|\to\infty} u_0^{p-1}(x)|x|^{\inf\{\beta,a\}(1+\sigma)} > A_\star. \tag{3.6}$$

Now i f we keep the function $K_2^{1/(p-1)} T_0^{-\gamma-1} t^\gamma \frac{|x|}{1+|x|}, T_0 > 0$, in (1.1), any local solution ceases to exist before T_0.

Remark 3.3 In [7, 10, 11] a crucial role is played by some estimate of the heat kernel associated to the linear operator involved in the considered equations. The methods used in this paper seem to be more efficient because there are not based on a knowledge of the kernel of the involved operators. The methods have a remarkable degree of simplicity and versality. For instance, equations with nonlinear diffusion can be handled by the methods presented in here as we can see below while the methods adopted in [7, 10, 11] are clearly inoperative. For example, we can consider the

$$u_t \geq \Delta(a(x,t)u^m) + t^\sigma h(x) u^p,$$

where $a \geq 0$ in $L^\infty(\mathbb{R}^N \times (0,+\infty))$ and $0 < m < p$. These methods can also be used to derive a non global existence of weak solutions to

$$u_t = -|x|^\alpha(-\Delta)^{\beta/2}u - V(x)u + t^\sigma h(x)u^p + t^\gamma w(x), u(x,0) = u_0(x), \quad (3.7)$$

where $0 < \alpha < N$. We note that this equation has a diffusion that vanishes at the origin $x = 0$ [5]. Concerning the nonexistence of global solutions to the last problem, the following result can be established without any major difficulty.

Theorem 3.4 *Let $\sigma > -1, p > \max\{1, 1+\sigma\}$. Assume that*

$$\lim_{|x|\to+\infty} u_0^{p-1}h(x)|x|^{-\alpha p+(1+\sigma)\inf\{\beta,a+\alpha\}} = +\infty,$$

or

$$\lim_{|x|\to+\infty} w^{p-1}h(x)|x|^{\inf\{\beta,a\}[\gamma(p-1)+\sigma]-\alpha p} = +\infty.$$

Then problem (3.7) *has no global weak solution.*

Observe that we can also consider the equation

$$u_t = \Delta u + t^\sigma h(x)(1+u)\log(1+u)^p, \quad (3.8)$$

with an initial data $u_0 \geq 0$. We refer the reader to [9] for the case $\sigma = 0$ and $h = 1$. Equation (3.8) can be written

$$v_t = \Delta v + t^\sigma h(x)v^p + |\nabla v|^2 \geq \Delta v + t^\sigma h(x)v^p,$$

via the transformation $v = \log(u+1)$. According to the previous results, if

$$\liminf_{|x|\to+\infty} (\log(1+u_0))h(x)^{1/(p-1)}|x|^{2(1+\sigma)/(p-1)} > C_0,$$

for some positive constant C_0, where $p > \max\{1, 1+\sigma\}$, Equation (3.8) with initial value u_0 does not possess global solution.

4 Nonexistence results for nonlinear hyperbolic equations

This short section deals with the equivalent of Theorems 1.1 and 1.3 for nonlinear hyperbolic equations of the form

$$u_{tt} = \Delta u - V(x)u + t^\sigma h(x)u^p + t^\gamma w(x), \quad (4.1)$$

for $x \in \mathbb{R}^N$ and $t \in (0,T)$ subject to the conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N. \quad (4.2)$$

The extension of the results of Section 2 to equations of type (4.1) presents no conceptual difficulty. We will follow the routine calculations, except that we choose

$$\zeta(x,t) = \big(\eta(\frac{t^2}{T^2})\big)^{2p'}\Phi(x) \tag{4.3}$$

as a test function. Without loss of generality, we assume that the potential V satisfies

$$|V(x)| \le |x|^{-2a}, \quad a > 0, \tag{4.4}$$

for large $|x|$.

Proof of Theorem 1.5 Since the proof is similar to the one in the preceding sections, we only give here a sketch of the proof. First we assume on the contrary that problem (4.1)–(4.2) has a global solution, say u. Let ζ be a test function defined by (4.3) where η and Φ are defined in Sections 2 and 3. Observe that $\zeta_t(x,0) = 0$ for all x in $\mathbb{R}^N$. Therefore, we have the estimate

$$\begin{aligned}\int_{\mathbb{R}^N} u_1(x)\zeta(x,0)dx + \int_{S_T} t^\gamma w\zeta\, dx\, dt + \int_{S_T} t^\sigma h u^p \zeta\, dx\, dt \\ \le \int_{S_T} u(-\Delta\zeta)_+\, dx\, dt + \int_{S_T} u|\zeta_{tt}|\, dx\, dt + \int_{S_T} u|V(x)|\zeta\, dx\, dt,\end{aligned}$$

which leads to

$$\begin{aligned}&\int_{\mathbb{R}^N} u_1(x)\Phi(x)dx \\ &\quad \le \big[C_1 T^{-p'+(1+\sigma)(1-p')} + C_2 R^{-\inf\{2,2a\}p'} T^{1+\sigma(1-p')}\big] \int_{\mathbb{R}^N} \frac{\Phi(x)}{h^{p'-1}} dx, \end{aligned} \tag{4.5}$$

for some positive constants C_1, C_2, and for any $T > 0$. Therefore, by a minimization argument, we deduce that

$$\int_{\mathbb{R}^N} u_1(x)\Phi(x)dx \le K_1 R^{-\inf\{1,a\}(p+1+\sigma)(p'-1)} \int_{\mathbb{R}^N} \frac{\Phi(x)}{h^{p'-1}} dx.$$

Hence, as in section 2,

$$\liminf_{|x|\to+\infty} u_1^{p-1} h(x)|x|^{\inf\{1,a\}(p+1+\sigma)} < +\infty,$$

which is impossible. The rest of the proof is similar to that in Section 3 and is hence left to the reader. □

Remark 4.1 An immediate necessary conditions for the local existence can be obtained from (4.5) which leads to

$$\liminf_{|x|\to+\infty} u_1 h^{1/(p-1)} \le \frac{K_1}{T^{1+\sigma+2p}}.$$

Concerning the function w we have also a necessary condition for the local existence,

$$\liminf_{|x|\to+\infty} wh^{1/(p-1)} \leq \frac{K_2}{T^{\gamma(p-1)+\sigma+2p}}.$$

We illustrate our results with the example

$$u_{tt} = \Delta u + t^\sigma |u|^p + (1+|x|^2)^{-q}, u_t(x,0) = A(1+|x|^2)^{-(k+1)/2}.$$

If $k < \frac{2+\sigma}{p-1}$ or $q < \frac{1+\sigma+p}{2(p-1)}$ the problem has no global weak solution even if u_0 has a compact support or if $u_0 \equiv 0$. Now if $k = \frac{2+\sigma}{p-1}$ and $q \geq \frac{1+\sigma+p}{2(p-1)}$ the problem has no global weak solution if A is large enough.

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Mohammed Guedda
Lamfa, CNRS UMR 6140, Université de Picardie Jules Verne,
Faculté de Mathématiques et d'Informatique, 33,
rue Saint-Leu 80039 Amiens, France
e-mail: Guedda@u-picardie.fr

Mokhtar Kirane
Laboratoire de Mathématiques, Pôle Sciences et Technologies,
Université de La Rochelle,
Avenue Michel Crépeau, 17042 La Rochelle Cedex, France
e-mail: mokhtar.kirane@univ-lr.fr