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Existence of weak solutions for the thermistor problem with degeneracy *

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Abstract

We prove the existence of weak solutions for the thermistor problem with degeneracy by using a regularization and truncation process. The solution of the regularized-truncated problem is obtained by using Schauder's fixed point theorem. Then the solutions of the thermistor problem are obtained by applying the monotonicity-compacity method of Lions.

1 Introduction

This paper is devoted to the study of coupled parabolic-elliptic system of partial differential equations related to the often so called thermistor problem. More precisely, we are interested in the existence of solutions of problem

$$\begin{aligned} &\frac{\partial u}{\partial t} - \triangle\theta(u) = \sigma(u)|\nabla\varphi|^2 \quad \text{in } \Omega\times(0,T),\\ &\operatorname{div}(\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega\times(0,T),\\ &u = \overline{u} \quad \text{on } \Gamma_D^u\times(0,T),\\ &\frac{\partial\theta(u)}{\partial n} + \beta(x,t)(u-\overline{u}) = 0 \quad \text{on } \Gamma_N^u\times(0,T),\\ &\varphi = \overline{\varphi} \quad \text{on } \Gamma_D^\varphi\times(0,T),\\ &\frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \Gamma_N^\varphi\times(0,T),\\ &u(x,0) = \overline{u}(x,0) \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a regular open bounded subset of R^N, $N \geq 1$, with smooth boundary $\partial\Omega$ and T a positive real. Here Γ_D^u, and Γ_D^φ are two nonempty open subsets of $\partial\Omega$ with smooth boundaries, $\Gamma_N^u = \partial\Omega - \overline{\Gamma_D^u}, \Gamma_N^\varphi = \partial\Omega - \Gamma_D^\varphi$ and $\frac{\partial}{\partial n}$ is the outward normal derivative to $\partial\Omega$. While $\theta, \sigma, \beta, \overline{u}$, and $\overline{\varphi}$ are known functions of their arguments.

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These problems arise from many applications in the automotive industry and in the field of physics, especially in the study of electrical heating of a conductor. In this situation, u is the temperature of the conductor, φ the potential and σ denotes the electrical conductivity. Problems of this type, under various assumptions on θ and σ and coupled with different types of boundary conditions, have received a lot of attention in the last decade by numerous authors. We quote in particular [3],[4], [5], [6],[7] ... and the references therein concerning problem (1.1) or it's corresponding stationary problems.

Our main goal here is to prove existence of weak solutions to (1.1), under the following hypotheses:

(H1) θ is a continuous nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$, with $\theta(0) = 0$.

(H2) σ is a real positive continuous function.

(H3) β is a continuous function from $\Omega \times [0, \infty[$ to $[0, \infty[$.

(H4) $\overline{u} \in W^{1,\infty}(\Omega \times (0, T))$ and $\overline{\varphi} \in L^\infty(0, T, W^{1,\infty}(\Omega))$ with $0 \le \overline{u}(x, t) \le M$ a.e. in $\Omega \times (0, T)$, where M is positive constant.

In [6] Xu obtained existence of weak solutions of (1.1) under (H2)–(H4) and the hypothesis

(H1') θ is an increasing C^1-function from $\mathbb{R}$ to $\mathbb{R}$, with $\theta(0) = 0$.

The result of Xu states that for each $M > \|\overline{u}\|_{L^\infty(\Omega\times(0,T))}$, there exists a $\delta > 0$ such that if $\|\overline{\varphi}\|_{L^\infty(\Omega\times(0,T))} < \delta$ one can find a weak solution (u, φ) with $\|u\|_{L^\infty(\Omega\times(0,T))} \le M$. That is to keep temperature from exceeding a certain value, it is enough to make the electrical potential drop applied suitably small.

Here, we obtain existence and boundedness of the temperature regardless to the smallness of the potential, provided that $\overline{u}$ is bounded. Moreover our result generalizes the one of Xu to the case where θ is not differentiable. The solution (u, φ) is obtained as a limit of a sequence of weak solutions (u_k, φ_k) of some regularized-truncated problem (3.1) associated with (P).

This paper is organized as follows: In section 2, we state our existence result concerning solutions of (1.1). Section 3 is devoted to the existence of weak solutions for problem (3.1). Then section 4 deals with a-priori estimates for solutions of (3.1). Finally section 5 is devoted to the proof of the main result.

2 Statement of the main Result

Let $Q_T = \Omega \times (0, T)$. Define the space

$$V = \{v \in H^1(\Omega), v = \overline{u} \text{ on } \Gamma_u^D\}$$

with inner product $((u, v)) = \Sigma_{i=1}^N \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, ds = \int_\Omega \nabla u \nabla v\, ds$, and $\langle , \rangle$ the duality bracket between V' and V. Also define the space

$$W(0, T) = \{v \in L^2(0, T, V) : \frac{\partial v}{\partial t} \in L^2(0, T, V')\}.$$

Definition 2.1 By a weak solution of (1.1), we mean a pair of function (u,φ) such that $u \in L^2(0,T,V)$, $\frac{\partial u}{\partial t} \in L^2(0,T,V')$, $\varphi \in L^2(0,T,H^1(\Omega))$, and for a.e. t

$$\langle \frac{\partial u}{\partial t}, v\rangle + \int_\Omega \nabla\theta(u)\nabla v\,ds + \int_{\Gamma_N^u} \beta(u-\overline{u})v\,ds - \int_{\Gamma_D^u} \frac{\partial\theta(\overline{u})}{\partial n} v\,ds$$
$$= -\int_\Omega \sigma(u)\varphi\nabla\varphi\nabla v\,ds + \int_{\Gamma_D^\varphi} \sigma(u)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n} v\ ds, \quad \text{for } v \in V \cap C^1(\overline{\Omega}),$$
$$\int_\Omega \sigma(u)\varphi\nabla\psi\ ds = \int_{\Gamma_D^\varphi} \sigma(u)\frac{\partial\overline{\varphi}}{\partial n}\psi\ ds, \quad \text{for } \psi \in H^1(\Omega).$$

The main result of this section is the following.

Theorem 2.2 *Under hypotheses (H1)–(H4), Problem* (1.1) *has a weak solution* (u,φ) *such that*

$$u \in L^2(0,T,V) \cap L^2(0,T,H^1(\Omega)) \cap L^2(0,T,W^{s,2}(\Omega)), \forall s : 0 < s < 1,$$
$$\frac{\partial u}{\partial t} \in L^2(0,T,V'), \theta(u) \in L^2(0,T,H^1(\Omega)) \text{ and } \varphi \in L^2(0,T,H^1(\Omega)).$$

Moreover, $0 \le u(x,t) \le M$ *a.e. in* Q_T.

Remark. If (u,φ) is a solution of problem (1.1), then $u \in W(0,T)$ which is compactly embedded in $L^2(0,T,L^2(\Omega))$. Thus from Lion's lemma of compacity (see [2], p. 58) we deduce that the initial condition of (1.1) makes sense.

3 Existence of solutions for regularized truncated problem

From θ, we construct a sequence $\theta_k \in C^\infty$ such that $\frac{1}{k} \le \theta_k'$, $\theta_k(0) = 0$ and $\theta_k \to \theta$ in $C_{\rm loc}(\mathbb{R})$, and from σ we introduce the truncated function

$$\widetilde{\sigma}(s) = \begin{cases} \sigma(M) & \text{if } s > M, \\ \sigma(s) & \text{if } 0 \le s \le M, \\ \sigma(0) & \text{if } s < 0. \end{cases}$$

Now, we define the regularized-truncated problem (3.1) associated with (1.1):

$$\begin{gathered} \frac{\partial u_k}{\partial t} - \triangle\theta_k(u_k) = \widetilde{\sigma}(u_k)|\nabla\varphi_k|^2 \quad \text{in } Q_T, \\ \operatorname{div}(\widetilde{\sigma}(u_k)\nabla\varphi_k) = 0 \quad \text{in } Q_T, \\ u_k = \overline{u} \quad \text{on } \Gamma_D^u \times (0,T), \\ \frac{\partial\theta_k(u_k)}{\partial n} + \beta(x,t)(u_k - \overline{u}) = 0 \quad \text{on } \Gamma_N^u \times (0,T), \\ \varphi_k = \overline{\varphi} \quad \text{on } \Gamma_D^\varphi \times (0,T), \\ \frac{\partial\varphi_k}{\partial n} = 0 \quad \text{on } \Gamma_N^\varphi \times (0,T), \\ u_k(x,0) = \overline{u}(x,0) \quad \text{in } \Omega. \end{gathered} \tag{3.1}$$

Lemma 3.1 *Let $u \in L^2(0,T,H^1(\Omega))$ and $\varphi \in L^2(0,T,H^1(\Omega))$. Then*

$$\operatorname{div}(\widetilde{\sigma}(u)\varphi\nabla\varphi) = \widetilde{\sigma}(u)|\nabla\varphi|^2$$

in the distributional sense.

For the proof of this lemma see ([6], p. 205).

Remarks. 1) By Definition 2.1, (u_k, φ_k) is a solution of (3.1) if and only

$$\begin{aligned}&\langle\frac{\partial u_k}{\partial t}, v\rangle + ((\theta_k(u_k), v)) + \int_{\Gamma_N^u} \beta(u_k - \overline{u})v\,ds - \int_{\Gamma_D^u} \theta_k'(\overline{u})\frac{\partial \overline{u}}{\partial n} v\,ds \\ &= -\int_\Omega \widetilde{\sigma}(u_k)\varphi_k\nabla\varphi_k\nabla v\ ds + \int_{\Gamma_D^\varphi} \widetilde{\sigma}(u_k)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n} v\,ds,\end{aligned} \tag{3.2}$$

for all $v \in V \cap C^1(\overline{\Omega})$, and

$$\int_\Omega \widetilde{\sigma}(u_k)\nabla\varphi_k\nabla\psi\,ds = \int_{\Gamma_D^\varphi} \widetilde{\sigma}(u_k)\frac{\partial\overline{\varphi}}{\partial n}\psi\,ds, \quad \text{for all } \psi \in H^1(\Omega). \tag{3.3}$$

2) The boundary integral in the right term of (3.3) makes sense since the operator trace from $H^1(\Omega)$ to the boundary space $L^2(\partial\Omega)$ is linear and compact. In fact, one can show that for each $\varphi \in L^\infty(0,T,H^1(\Omega))$, the restriction of φ to $\partial\Omega \times (0,T)$ belongs to the space $L^2(0,T,L^2(\partial\Omega))$.

For the rest of this paper, we shall denote by c_i different constants depending only on Ω and the data but not on k.

Theorem 3.2 *Under Hypotheses (H1)–(H4), there exists at least a weak solution (u_k, φ_k) of (3.1), such that*

$$\begin{aligned}u_k \in W(0,T), \varphi_k \in L^\infty(0,T,H^1(\Omega)),\\ u_k(x,0) = \overline{u}(x,0) \quad a.e.\ in\ \Omega,\end{aligned}$$

and satisfying (3.2)–(3.3). Moreover, $0 \le u_k \le M$, a.e. in Q_T.

Proof of Theorem 3.2 This is based on Schauder's fixed point theorem. We shall construct an appropriate mapping whose fixed points are solutions of (3.1). To this end let $U_k(w) = u_{k,w}$ where $u_{k,w}$ is the unique solution of

$$\begin{aligned}&\langle\frac{\partial u_{k,w}}{\partial t}, v\rangle + \int_\Omega \theta_k'(w)\nabla u_{k,w}\nabla v\,ds + \int_{\Gamma_N^u} \beta(u_{k,w} - \overline{u})v\,ds - \int_{\Gamma_D^u} \theta_k'(\overline{u})\frac{\partial\overline{u}}{\partial n} v\,ds \\ &= -\int_\Omega \widetilde{\sigma}(w)\varphi_{k,w}\nabla\varphi_{k,w}\nabla v\ ds + \int_{\Gamma_D^\varphi} \widetilde{\sigma}(w)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n} v\ ds, \quad \text{for all} v \in V \cap C^1(\overline{\Omega}).\end{aligned} \tag{3.4}$$

(It is easy to show that such a solution exists and is unique.)

Let $S_k : W(0,T) \to W(0,T)$ be the operator defined by $S_k(w) = \varphi_{k,w}$ where $\varphi_{k,w}$ is the unique solution of

$$\int_\Omega \widetilde{\sigma}(w)\nabla\varphi_{k,w}\nabla\psi\ ds = \int_{\Gamma_D^\varphi} \widetilde{\sigma}(w)\frac{\partial\overline{\varphi}}{\partial n}\psi\,ds, \quad \text{for } \psi \in H^1(\Omega). \tag{3.5}$$

By [3, Lemma 2.2], we have $\varphi_{k,w} \in L^\infty(\Omega)$ and the H^1-norm of $\varphi_{k,w}$ is bounded by a constant which is independent of w. Then, all terms in equations (3.4) and (3.5) are well defined.

To continue the proof of theorem 3.2, we need the following a-priori estimates.

Lemma 3.3 *Let $(u_{k,w}, \varphi_{k,w})$ be the solution of (3.4)–(3.5). Then, we have the following a-priori estimates*

$$\|\varphi_{k,w}\|_{L^2(0,T,H^1(\Omega))} \le c_1, \tag{3.6}$$

$$\|u_{k,w}\|_{L^2(0,T,V)} \le c_2, \tag{3.7}$$

$$\|u_{k,w}\|_{L^2(0,T,L^2(\Gamma_N^u))} \le c_3, \tag{3.8}$$

$$\|\frac{\partial u_{k,w}}{\partial t}\|_{L^2(0,T,V)} \le c_4, \tag{3.9}$$

where the positive constants c_i $(i = 1 \dots 4)$ are not depending on w, nor on k.

Proof. (i) Taking $\psi = \varphi_{k,w} - \overline{\varphi}$ in (2.11) and using the properties of $\widetilde{\sigma}$, the conditions of $\overline{\varphi}$ and young's inequality, we deduce that

$$\int_\Omega \widetilde{\sigma}(w)|\nabla\varphi_{k,w}|^2 \le c_5 \int_\Omega |\nabla\overline{\varphi}|^2,$$

where c_5 is a positive constant. Which obviously implies (3.6).
(ii) Taking $v = u_{k,w}$ in (3.4), it follows that

$$\begin{aligned}
&\langle\frac{\partial u_{k,w}}{\partial t}, u_{k,w}\rangle + \int_\Omega \theta_k'(w)|\nabla u_{k,w}|^2 \, ds + \int_{\Gamma_N^u} \beta(u_{k,w} - \overline{u})u_{k,w} \, ds \\
&= -\int_\Omega \widetilde{\sigma}(w)\varphi_{k,w}\nabla\varphi_{k,w}\nabla u_{k,w} \, ds + \int_{\Gamma_D^\varphi} \widetilde{\sigma}(w)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n}u_{k,w} \, ds \\
&\quad + \int_{\Gamma_D^u} \theta_k'(\overline{u})\frac{\partial\overline{u}}{\partial n}u_{k,w} \, ds.
\end{aligned}$$

So, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}|u_{k,w}|^2_{L^2(\Omega)} + \int_\Omega \theta_k'(w)|\nabla u_{k,w}|^2 \, ds + \int_{\Gamma_N^u} \beta|u_{k,w}|^2 \, ds \\
&= \int_{\Gamma_N^u} \beta u_{k,w}\overline{u} \, ds - \int_\Omega \widetilde{\sigma}(w)\varphi_{k,w}\nabla\varphi_{k,w}\nabla u_{k,w} \, ds \\
&\quad + \int_{\Gamma_D^\varphi} \widetilde{\sigma}(w)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n}u_{k,w} \, ds + \int_{\Gamma_D^u} \theta_k'(\overline{u})\frac{\partial\overline{u}}{\partial n}\overline{u} \, ds.
\end{aligned}$$

Using a lemma3.3 in [3, p. 245] and the hypotheses on θ_k' and β, and applying Young's inequality, there exist positive constants c_i, $(i = 6 \dots 12)$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}|u_{k,w}|^2_{L^2(\Omega)} + c_6\int_\Omega |\nabla u_{k,w}|^2\,ds + c_7\int_{\Gamma_N^u}|u_{k,w}|^2\,ds\\
&\leq \int_{\Gamma_N^u}\beta u_{k,w}\overline{u}\,ds - \int_\Omega \widetilde{\sigma}(w)\varphi_{k,w}\nabla\varphi_{k,w}\nabla u_{k,w}\,ds\\
&\quad + \int_{\Gamma_D^\varphi}\widetilde{\sigma}(w)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n}u_{k,w}\,ds + \int_{\Gamma_D^u}\theta_k'(\overline{u})\frac{\partial\overline{u}}{\partial n}\overline{u}\,ds\\
&\leq c_8\int_\Omega|\nabla\varphi_{k,w}||\nabla u_{k,w}|\,ds + \int_{\Gamma_D^\varphi}\widetilde{\sigma}(w)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n}u_{k,w}\,ds\\
&\quad + \int_{\Gamma_D^u}\theta_k'(\overline{u})\frac{\partial\overline{u}}{\partial n}\overline{u}\,ds + \int_{\Gamma_N^u}\beta u_{k,w}\overline{u}\\
&\leq \frac{c_6}{2}\int_\Omega|\nabla u_{k,w}|^2\,ds + c_9\int_\Omega|\nabla\varphi_{k,w}|^2\,ds + c_{10}\int_{\Gamma_D^\varphi}|\frac{\partial\overline{\varphi}}{\partial n}|^2\,ds\\
&\quad + \frac{c_7}{4}\int_{\Gamma_D^\varphi}|u_{k,w}|^2\,ds + \frac{c_7}{4}\int_{\Gamma_N^u}|u_{k,w}|^2\,ds + c_{11}\int_{\Gamma_N^u}|\overline{u}|^2\,ds + c_{12}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}|u_{k,w}|^2_{L^2(\Omega)} + \frac{c_6}{2}\int_\Omega|\nabla u_{k,w}|^2\,ds + c_7\int_{\Gamma_N^u}|u_{k,w}|^2\,ds\\
&\leq c_{13} + \frac{c_7}{4}\int_{\Gamma_D^\varphi}|u_{k,w}|^2\,ds + \frac{c_7}{4}\int_{\Gamma_N^u}|u_{k,w}|^2\,ds\\
&\leq c_{13} + \frac{c_7}{4}\int_{\partial\Omega}|u_{k,w}|^2\,ds + \frac{c_7}{4}\int_{\Gamma_N^u}|u_{k,w}|^2\,ds\\
&\leq c_{13} + \frac{c_7}{4}\int_{\Gamma_D^u}|u_{k,w}|^2\,ds + \frac{c_7}{2}\int_{\Gamma_N^u}|u_{k,w}|^2\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}|u_{k,w}|^2_{L^2(\Omega)} + \frac{c_6}{2}\int_\Omega|\nabla u_{k,w}|^2\,ds + \frac{c_7}{2}\int_{\Gamma_N^u}|u_{k,w}|^2\,ds&\\
\leq c_{13} + \frac{c_7}{4}\int_{\Gamma_D^u}|\overline{u}|^2\,ds \leq c_{14}.& \quad (3.10)
\end{aligned}$$

Integrating (3.10) on $(0,T)$, we conclude to the estimates (3.7) and (3.8) .
(iii) According to (3.4), (3.6), (3.7), (3.8) and a lemma3.3 in [3, p. 245], we obtain

$$\|\frac{\partial u_{k,w}}{\partial t}\|_{L^2(0,T,V')} \leq c_4.$$

Hence Lemma 3.3 is proved. □

Now, we define the space

$$W_0 := \left\{ \begin{array}{l} v \in W(0,T), \|v\|_{L^2(0,T,V)} \le c_2, \quad \|\frac{\partial v}{\partial t}\|_{L^2(0,T,V')} \le c_4, \\ 0 \le v(x,t) \le M \text{ a.e. in } Q_T, \quad v(0) = \overline{u}(x,0) \text{ in } \Omega \end{array} \right\}.$$

Note that W_0 is a non empty convex set, and by Lemma 3.3, the operator $U_k : W_0 \to W_0$ is well defined. To use Schauder's fixed point theorem, it remains to show that U_k is continuous with respect to the weak topology of $W(0,T)$. Then, using the weak compactness of W_0 we shall conclude that U_k has a fixed point w in the set W_0. To prove the weak continuity, assume that $(w_j)_j$ is a sequence in W_0 satisfying $w_j \to w$ weakly in $W(0,T)$ and let $(u_{k,w_j}, \varphi_{k,w_j})$ the corresponding sequence of solutions of (3.4)–(3.5). By estimates (3.6)–(3.9), there exists at least a subsequence denoted again by w_j such that as $j \to \infty$, we have

$$\begin{gathered} w_j \to w \quad \text{weak in } L^2(0,T,V), \\ \frac{\partial w_j}{\partial t} \to \frac{\partial w}{\partial t} \quad \text{weak in } L^2(0,T,V'), \\ u_{k,w_j} \to u_k \quad \text{weak in } L^2(0,T,V), \\ u_{k,w_j} \to u_k \quad \text{weak in } L^2(0,T,L^2(\Gamma_N^u)), \\ \frac{\partial u_{k,w_j}}{\partial t} \to \frac{\partial u_k}{\partial t} \quad \text{weak in } L^2(0,T,V'), \\ \varphi_{k,w_j} \to \varphi_k \quad \text{weak in } L^2(0,T,H^1(\Omega)). \end{gathered}$$

Note that one may assume, without loss of generality, that $w_j \to w$ strongly in $L^2(Q_T)$ and a.e. in Q_T. Since $\widetilde{\sigma}$ is continuous and bounded, then, thanks to the dominated convergence theorem of Lebesgue, it follows that

$$\widetilde{\sigma}(w_j) \to \widetilde{\sigma}(w) \quad \text{strongly in } L^2(Q_T).$$

By the trace theorem, we derive that

$$\widetilde{\sigma}(w_j)\frac{\partial \varphi_{k,w_j}}{\partial n} \to \widetilde{\sigma}(w)\frac{\partial \varphi_k}{\partial n} \text{ weak in } L^2(\Gamma_D^{\varphi}).$$

On the other hand, by vertue of the estimates of Lemma 3.3, we deduce that there exist functions $\alpha_1, \alpha_2, \alpha_3$, such that

$$\widetilde{\sigma}(w_j)\nabla\varphi_{k,w_j} \to \alpha_1 \text{ weak in } L^2(Q_T), \tag{3.11}$$

$$\theta_k'(w_j)\nabla u_{k,w_j} \to \alpha_2 \text{ weak in } L^2(Q_T), \tag{3.12}$$

$$\widetilde{\sigma}(w_j)\varphi_{k,w_j}\nabla\varphi_{k,w_j} \to \alpha_3 \text{ weak in } L^2(Q_T). \tag{3.13}$$

Now, as $\nabla\varphi_{k,w_j} \to \nabla\varphi_k$ weak in $L^2(Q_T)$, we have

$$\widetilde{\sigma}(w_j)\nabla\varphi_{k,w_j} \to \widetilde{\sigma}(w)\nabla\varphi_k \text{ in } D'(Q_T).$$

Consequently,

$$\alpha_1 = \widetilde{\sigma}(w)\nabla\varphi_k.$$

In a similar way, we obtain

$$\alpha_2 = \theta'_k(w)\nabla u_k,$$
$$\alpha_3 = \widetilde{\sigma}(w)\varphi_k\nabla\varphi_k.$$

Then, passing to the limit as $j \to \infty$ in the relations (3.4) and (3.5) satisfied by $(u_{k,w_j}, \varphi_{k,w_j})$, we deduce immediately that $u_k = U_k(w)$ and $\varphi_k = S_k(w)$. By the unique solvability of (3.4), all the sequence u_{k,w_j} converges to $u_k = U_k(w)$ weakly in $W(0,T)$. This completes the proof. □

Remark By theorem 3.2, we have $0 \leq u_k \leq M$. Hence $\widetilde{\sigma}(u_k) = \sigma(u_k)$.

4 Estimates on solutions of the regularized truncated problem

In this section, we obtain appropriate estimates on the solutions (u_k, φ_k) of the regularized-truncated problem (3.1).

Lemma 4.1 *Let (u_k, φ_k) be a solution of (3.1). Under the hypotheses (H1)–(H4), there exist constants c_i $(i = 15\ldots19)$ such that, for any $k \geq 1$, the following estimates hold*

$$|u_k(t)|^2_{L^2(\Omega)} \leq |\overline{u}(x,0)|^2_{L^2(\Omega)} + c_{15}, \tag{4.1}$$

$$\|u_k\|^2_{L^2(0,T,V)} \leq |\overline{u}(x,0)|^2_{L^2(\Omega)} + c_{15}, \tag{4.2}$$

$$\|\frac{1}{k}u_k\|^2_{L^2(0,T,V)} \leq \frac{c_{16}}{2k}(|\overline{u}(x,0)|^2_{L^2(\Omega)} + c_{17}), \tag{4.3}$$

$$\|\theta_k(u_k)\|^2_{L^2(0,T,H^1(\Omega))} \leq c_{18}, \tag{4.4}$$

$$\|\frac{\partial u_k}{\partial t}\|_{L^2(0,T,V')} \leq c_{19}. \tag{4.5}$$

The different constants are positive and not depending on k.

Proof. Choosing $v = u_k$ as a function test in (3.2), using the hypotheses on θ'_k and β and applying a lemma3.3 in [3, p. 245], we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}|u_k(t)|^2_{L^2(\Omega)} + c_{20}\int_\Omega |\nabla u_k|^2\,ds + c_{21}\int_{\Gamma^u_N} |u_k|^2\,ds \\
&\leq c_{22}\int_\Omega |\nabla\varphi_k||\nabla u_k|\,ds + c_{23}\int_{\Gamma^u_N} |u_k||\overline{u}|\,ds \\
&\quad + c_{24}\int_{\Gamma^\varphi_D} |u_k||\frac{\partial\overline{\varphi}}{\partial n}|\,ds + c_{25}\int_{\Gamma^u_D} |\frac{\partial\overline{u}}{\partial n}||\overline{u}|\,ds.
\end{aligned}$$

The same arguments as in the proof of Lemma 3.3 lead to

$$\frac{\partial}{\partial t}|u_k(t)|^2_{L^2(\Omega)} + \int_\Omega |\nabla u_k|^2\,ds + \int_{\Gamma^u_N} |u_k|^2\,ds \leq c_{26}. \tag{4.6}$$

Integrating (4.6) on $(0,t)$ yiel ds

$$|u_k(t)|^2_{L^2(\Omega)} + \int_{Q_t} |\nabla u_k|^2\, ds + \int_0^t \int_{\Gamma_N^u} |u_k|^2\, ds \le |\overline{u}(x,0)|^2_{L^2(\Omega)} + c_{15}.$$

Hence (4.1) and (4.2) are satisfied. On the other hand, by (3.2) we have

$$\begin{aligned}
&\langle \frac{\partial u_k}{\partial t}, u_k\rangle + ((\theta_k(u_k), u_k)) + \int_{\Gamma_N^u} \beta |u_k|^2 \\
&= -\int_\Omega \widetilde{\sigma}(u_k)\varphi_k \nabla\varphi_k \nabla u_k\, ds + \int_{\Gamma_N^u} \beta u_k \overline{u}\, ds \\
&\quad + \int_{\Gamma_D^\varphi} \widetilde{\sigma}(u_k)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n} u_k\, ds + \int_{\Gamma_D^u} \theta_k'(\overline{u})\frac{\partial \overline{u}}{\partial n}\overline{u}\, ds.
\end{aligned}$$

Therefore, arguing exactly as above, we deduce that

$$|u_k(t)|^2_{L^2(\Omega)} + 2\int_0^T ((\theta_k(u_k), u_k))dt + \int_0^T \int_{\Gamma_N^u} |u_k|^2\, ds \le c_{16}(|\overline{u}(x,0)|^2_{L^2(\Omega)} + c_{17}).$$

Furthermore,

$$\int_0^T ((\theta_k(u_k), u_k))dt = \int_0^T \Big(\int_\Omega \theta_k'(u_k)|\nabla u_k|^2\, ds\Big)dt$$

and $0 < \frac{1}{k} \le \theta_k'(u_k)$. Then

$$\frac{1}{k}\|u_k\|^2_{L^2(0,T,H^1(\Omega))} \le \frac{c_{16}}{2}(|\overline{u}(x,0)|^2_{L^2(\Omega)} + c_{17}).$$

Which gives estimate (4.3).

To obtain an estimate on $(\theta_k(u_k))_{k\ge 1}$, we take $\theta_k(u_k)$ as a test function in (3.2). We get

$$\begin{aligned}
&\langle \frac{\partial u_k}{\partial t}, \theta_k(u_k)\rangle + \|\theta_k(u_k)\|^2_{H^1(\Omega)} + \int_{\Gamma_N^u} \beta(u_k - \overline{u})\theta_k(u_k)\, ds \\
&= -\int_\Omega \widetilde{\sigma}(u_k)\varphi_k \nabla\varphi_k \nabla\theta_k(u_k)\, ds + \int_{\Gamma_D^u} \theta_k'(\overline{u})\frac{\partial\overline{u}}{\partial n}\theta_k(u_k)\, ds \\
&\quad + \int_{\Gamma_D^\varphi} \widetilde{\sigma}(u_k)\overline{\varphi}\frac{\partial\overline{\varphi}}{\partial n}\theta_k(u_k)\, ds.
\end{aligned}$$

Straightforward calculations and Bamberger's lemma [8, p. 8] give

$$\frac{d}{dt}\Big\{\int_\Omega \Big(\int_0^{u_k(x,.)} \theta_k(r)dr\Big)\, ds\Big\} + \frac{1}{2}\|\theta_k(u_k)\|^2_{H^1(\Omega)} \le c_{27}. \tag{4.7}$$

Integrating (4.7) from 0 to T yiel ds

$$\int_\Omega \Big(\int_0^{u_k(x,T)} \theta_k(r)dr\Big)\, ds + \frac{1}{2}\|\theta_k(u_k)\|^2_{L^2(0,T,H^1(\Omega))} \le \frac{c_{18}}{2}.$$

Hence estimate (4.4) follows. According to (3.2), (4.1), (4.2), (4.4) and Lemma 3.3, [3, p. 245], and the definition of dual norm, we get the desired relation (4.5). □

5 Passage to the limit in (3.1) as $k \to \infty$

The aim now is to pass to the limit in the process (3.1). By using estimates of Lemma 4.1 and standard compacity arguments, we deduce that there exists a subsequence (u_k, φ_k), not relabelled, such that, as $k \to \infty$, we have

$$\begin{gathered} u_k \to u \quad \text{weak in } L^2(0,T,V), \\ u_k \to u \quad \text{weak star } L^\infty(0,T,L^2(\Omega)), \\ u_k \to u \quad \text{weak in } L^2(0,T,L^2(\Gamma_N^u)), \\ \frac{\partial u_k}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weak in } L^2(0,T,V'), \\ \varphi_k \to \varphi \quad \text{weak star in } L^\infty(0,T,H^1(\Omega)). \end{gathered}$$

On the other hand, since the space $\{v \in L^2(0,T,V), \frac{\partial v}{\partial t} \in L^2(0,T,V')\}$ is compactly embedded in $L^2(Q_T)$, [2, p. 58], we can extract a subsequence from (u_k), not relabelled, such that

$$u_k \to u \quad \text{strongly and a.e. in } L^2(Q_T).$$

Moreover, we can assume that

$$\theta_k(u_k) \to \theta(u) \qquad \text{a.e. in } Q_T \text{ and weak in } L^2(0,T,H^1(\Omega)).$$

Indeed, we have

$$\begin{aligned} &\int_0^t \Big(\int_\Omega |\theta_k(u_k) - \theta(u)|\, dsdr \\ &\le \int_0^t \Big(\int_\Omega |\theta_k(u_k) - \theta(u_k)|\, ds\Big)dr + \int_0^t \Big(\int_\Omega |\theta(u_k) - \theta(u)|\, ds\Big)dr \\ &\le csup_{|r|\le M}|\theta_k(r) - \theta(r)| + \int_0^t \Big(\int_\Omega |\theta(u_k) - \theta(u)|\, ds\Big)dr. \end{aligned}$$

Now, arguing as in the proof of (3.11)–(3.13), we obtain

$$\begin{gathered} \widetilde{\sigma}(u_k)\nabla\varphi_k \to \sigma(u)\nabla\varphi \quad \text{weak in } L^2(Q_T), \\ \widetilde{\sigma}(u_k)\varphi_k\nabla\varphi_k \to \sigma(u)\varphi\nabla\varphi \quad \text{weak in } L^2(Q_T). \end{gathered}$$

Moreover, using Aubin's lemma (see [2, p. 7]), we obtain that $u_k \to u$ in $C([0,T];V')$. Then $u_k(0) \to u(0)$ weak in V'. We consequently obtain $u(x,0) = \overline{u}(x,0)$. Finally, we verify easily that the limit (u,φ) obtained is a solution of problem (1.1). This conclude to the proof of the main result. □

Remark. The question of uniqueness has been established in some special cases; see [4] and [8].

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