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Existence of stable periodic solutions for quasilinear parabolic problems in the presence of well-ordered lower and upper-solutions *

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Abstract

We present existence and stability results for periodic solutions of quasilinear parabolic equation related to Leray-Lions's type operators. To prove existence and localization, we use the penalty method; while for stability we use an approximation scheme.

1 Introduction

In the last few years, many works have been devoted to the existence and stability of periodic solutions of problem

$$\begin{gathered}\frac{\partial u}{\partial t}+A(u)+F(u,\nabla u)=0 \quad \text{in } \Omega\times\mathbb{R}^+,\\ u=0 \quad \text{on } \partial\Omega\times\mathbb{R}^+,\\ u(0)=u(T) \quad \text{in } \Omega,\end{gathered} \tag{1.1}$$

where Ω is a bounded and open subset of $\mathbb{R}^N$, $N\geq 1$. For the usual Leray-Lions's operator A, Deuel and Hess [4] obtained existence of periodic solutions under the presence of well-ordered lower and upper-solutions. Unfortunately, uniqueness and therefore stability, can not be derived from the definition they used for solutions of (1.1).

For $A(u)=-\Delta g(u)$ and F depending only on (x,t), Harraux and Kenmochi [7] proved both existence and stability results by using subdifferential theory on Hilbert spaces.

Recently, Boldrini and Crema [2] considered the case where $A(u)$ is the p-laplacian operator, with $p\geq 2$, and F is independent of ∇u. They obtained an existence result via Shauder's fixed point theorem.

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More recently, De coster and Omari [3] considered problem (1.1) with a linear uniformly elliptic operator A

$$Au := -\sum_{i,j=1}^{N} \partial_{x_i}(a_{i,j}\partial_{x_j}u) + \sum_{i=1}^{N} a_i\partial_{x_i}u + a_0 u$$

and F independent of ∇u. These last authors obtained stability in a suitable sense of the maximal and minimal solutions in the presence of well-ordered lower and upper-solutions.

The aim of this paper is to show that the result of De Coster and Omari still holds for general problem (1.1). Our existence result is obtained by using a classical method of penalization as it was done by Grenon in [4]; while the stability one follows essentially the principal arguments of De Coster and Omari with some changes imposed by the nonlinear character of the equation in (1.1).

This paper is organized as follows: In section 2 we recall some known results related to the initial boundary value problem associated with (1.1), and give hypotheses and definitions of solutions. In section 3, we give existence and uniqueness results concerning periodic solutions of problem (1.1), while section 4 is devoted to the stability result of periodic solutions. Finally, in section 5 we give an application to a periodic-parabolic problem associated to the p-laplacian operator.

2 Hypotheses, definitions, and known results

Let Ω be an open bounded subset of $\mathbb{R}^N$ with boundary $\partial\Omega$ and $T > 0$ a fixed real. We shall denote

$$Q_T := \Omega\times]0,T[, \quad \Sigma_T := \partial\Omega\times]0,T[,$$

and for a real p with $1 < p < +\infty$, we denote by V the space $V := L^p(0,T;W_0^{1,p})$ and by $V' := L^{p'}(0,T;W^{-1,p'})$ its dual, with p' the real conjugate of $p : \frac{1}{p}+\frac{1}{p'} = 1$.

Let us consider the Leray-Lions's operator

$$A(v) := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,t,v,\nabla v), \text{ for each } v \in V. \tag{2.1}$$

We shall use the following assumptions:

(A1) A_i are caratheodory functions such that there exists $\beta_i > 0$, and $k_i \in L^{p'}(Q_T)$, so that for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$:

$$|A_i(x,t,s,\xi)| \le \beta_i(|s|^{p-1} + |\xi|^{p-1} + k_i(x,t)), \forall i = 1,\dots,N,$$

(A2) For all $s \in \mathbb{R}$ and all $\xi, \xi^* \in \mathbb{R}^N$, with $\xi \neq \xi^*$, we have

$$\sum_{i=1}^{N}[A_i(x,t,s,\xi) - A_i(x,t,s,\xi^*)](\xi - \xi^*) > 0 \quad \text{a.e. in } Q_T.$$

(A3) There exists $\alpha > 0$, so that for all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$, we have

$$\sum_{i=1}^{N} A_i(x,t,s,\xi)\xi_i \geq \alpha|\xi|^p \quad \text{a.e. in } Q_T.$$

(A4) The function f is of caratheodory type on $Q_T \times \mathbb{R} \times \mathbb{R}^N$, and there exist functions $b : \mathbb{R}^+ \to \mathbb{R}^+$ increasing, and $h \in L^1(Q_T)$, $h \geq 0$, such that

$$|f(x,t,s,\xi)| \leq b(|s|)(h(x,t) + |\xi|^p), \quad \text{for } (x,t,s,\xi) \in Q_T \times \mathbb{R} \times \mathbb{R}^N.$$

We denote by F the Nemyskii operator related to f and defined by

$$F(u, \nabla u)(x,t) := f(x,t,u,\nabla u).$$

To obtain (among other results) global existence for initial boundary-value problems associated with (1.1), we shall assume the following.

(A5) There exists $c_i > 0$ and $l_i \in L^{p'}(Q_T)$ with $l_i \geq 0$, such that for all $s, s^* \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$,

$$|A_i(x,t,s,\xi) - A_i(x,t,s^*,\xi)| \leq c_i|s - s^*|[l_i(x,t) + |\xi|^{p-1}] \quad \text{a.e. in } Q_T.$$

(A6) All data (coefficients and second member) are periodic in time with period T.

We are interested in the existence and stability of the solutions of problem

$$\begin{gathered} \frac{\partial u}{\partial t} + A(u) + F(u, \nabla u) = 0 \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \Sigma_T, \\ u(0) = u(T) \quad \text{in } \Omega. \end{gathered} \tag{2.2}$$

To this end, we consider the problem $(\mathcal{P}_{t_1,t_2;u_0})$:

$$\begin{gathered} \frac{\partial u}{\partial t} + A(u) + F(u, \nabla u) = 0 \quad \text{in } \Omega \times]t_1, t_2[, \\ u = 0 \quad \text{on } \partial\Omega \times]t_1, t_2[, \\ u(t_1) = u_0 \quad \text{in } \Omega, \end{gathered} \tag{2.3}$$

where $0 \leq t_1 < t_2 \leq +\infty$ and u_0 is a given function in $L^\infty(\Omega)$.

Definition We say that α is a lower-solution of (2.2) if

$$\alpha \in V \cap L^\infty(Q_T), \quad \frac{\partial \alpha}{\partial t} \in V' + L^1(Q_T)$$

and (in the distributional sense)

$$\begin{gathered} \frac{\partial \alpha}{\partial t} + A(\alpha) + F(\alpha, \nabla \alpha) \leq 0 \quad \text{in } Q_T, \\ \alpha(0) \leq \alpha(T) \quad \text{in } \Omega. \end{gathered}$$

An upper-solution is defined by reversing the sense of inequalities. And a solution is a function which is simultaneously lower and upper-solution.

Definition We say that α is a lower-solution of problem $(\mathcal{P}_{t_1,t_2;u_0})$ if: $\alpha \in L^p(t_1,t_2;W^{1,p}) \cap L^\infty(\Omega\times]t_1,t_2[)$, $\frac{\partial\alpha}{\partial t} \in L^{p'}(t_1,t_2;W^{-1,p'}) + L^1(\Omega\times]t_1,t_2[)$ and

$$\frac{\partial\alpha}{\partial t} + A(\alpha) + F(\alpha,\nabla\alpha) \leq 0 \quad \text{in } \Omega\times]t_1,t_2[,$$
$$\alpha \leq 0 \quad \text{on } \partial\Omega\times]t_1,t_2[,$$
$$\alpha(0) \leq u_0 \quad \text{in } \Omega.$$

Upper-solutions and solutions of $(\mathcal{P}_{t_1,t_2;u_0})$ are defined exactly as in the periodic case.

Remarks 1.) As the function f does not satisfy any Lipschitz condition, the use of systematic results concerning stability questions, as the Poincaré operator in connection with the theory of monotone operators and discrete dynamical systems (see [8] or [1]), seems not to be possible.
2.) As in [5], our definitions allow us to use solutions as lower or upper-solutions. This enables us to prove an uniqueness result among a class of periodic solutions. This can not be done when using the definitions of [4], where it is supposed that lower and upper-solutions are more regular than solutions.

Now, we recall some known results concerning solutions of $(\mathcal{P}_{t_1,t_2;u_0})$ with $T' > 0$ and $u_0 \in L^\infty(\Omega)$. We refer the reader to [5] for proofs.

Lemma 2.1 *Assume (A1)–(A5) and let (α_1,α_2) and (β_1,β_2) be respectively pairs of lower and upper-solutions of $(\mathcal{P}_{0,T';u_0})$ such that*

$$\sup(\alpha_1,\alpha_2) \leq \inf(\beta_1,\beta_2) \quad \textit{a.e. in } Q_{T'}.$$

Then, there exists a solution $u \in C([0,T'];L^q(\Omega))$ for any $q \geq 1$, of $(\mathcal{P}_{t_1,t_2;u_0})$ such that $\sup(\alpha_1,\alpha_2) \leq u \leq \inf(\beta_1,\beta_2)$ a.e. in $Q_{T'}$. Moreover, when $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, the Hypothesis (A5) can be removed.

Lemma 2.2 *Assume (A1)–(A5) and let α and β be respectively lower and upper-solutions of $(\mathcal{P}_{0,T';u_0})$, for any $T' > 0$. Then, there exists u (resp. v) $\in C([0,+\infty[;L^q(\Omega))$, $\forall q \geq 1$ such that for any $T' > 0$, the restriction of u (resp. v) on $[0,T']$ is the minimal (resp. maximal) solution of $(\mathcal{P}_{0,T';u_0})$ located between α and β. Moreover, if u_0 and v_0 are in $L^\infty(\Omega)$ and satisfy*

$$\alpha(0) \leq u_0 \leq v_0 \leq \beta(0) \quad \textit{a.e. in } \Omega$$

and $u_{\min}(u_0)$ (resp. $u_{\min}(v_0)$) is the minimal solution of $(\mathcal{P}_{0,T';u_0})$ with u_0 (resp. $(\mathcal{P}_{0,T';v_0})$ with v_0) laying between α and β, then

$$u_{\min}(u_0) \leq u_{\min}(v_0), \qquad \textit{a.e. in } Q_{T'}.$$

Furthermore, the same holds for maximal solutions.

Lemma 2.3 *Assume (A1)–(A5). Let $0 < T_1 < T_2$ and α and β be respectively lower and upper-solution of $(\mathcal{P}_{0,T';u_0})$ with $T' > 0$. Let u_1 (resp. u_2) be the minimal solution of $(\mathcal{P}_{0,T_1;u_0})$ (resp. $(\mathcal{P}_{0,T_2;u_0})$) located between α and β. Then u_1 is the restriction of u_2 on $[0,T_2]$ and the same holds for maximal solutions.*

3 Existence and uniqueness of periodic solutions

The first result of this section is the following.

Theorem 3.1 *Assume (A1)–(A4) and let α and β be respectively lower and upper-solutions of* (2.2) *with $\alpha \leq \beta$ a.e. in Q_T. Then, problem* (2.2) *has a weak solution u satisfying $u \in C([0,T]; L^q(\Omega))$, for any $q \geq 1$, and $\alpha \leq u \leq \beta$ a.e. in Q_T.*

Proof The proof is similar to that of the corresponding initial boundary-value problem treated in [5]. We shall give here only a sketch.
(i) We regularize (2.2) by taking

$$A_i^*(u, \nabla u) := A_i(Su, \nabla u), \ i \in \{1, \dots, N\}$$
$$F_\epsilon^*(u, \nabla u) := \frac{F(Su, \nabla Su)}{1 + \epsilon |F(Su, \nabla Su)|}$$

where $Su := u + (\alpha - u)^+ - (u - \beta)^-$, $\epsilon > 0$, and using the penalization operator θ_η related to the convex

$$\mathcal{K} := \{v \in V \text{ such that } -k \leq v \leq k \text{ a.e. in } Q_T\},$$

where k is such that $-k \leq \alpha - 1 \leq \beta + 1 \leq k$. For η and $\epsilon > 0$ fixed, consider the problem

$$\begin{gathered} u_{\eta,\epsilon} \in V, \quad \frac{\partial u_{\eta,\epsilon}}{\partial t} \in V', \\ \frac{\partial u_{\eta,\epsilon}}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} A_i^*(u_{\eta,\epsilon}, \nabla u_{\eta,\epsilon}) + F_\epsilon^*(u_{\eta,\epsilon}, \nabla u_{\eta,\epsilon}) + \theta_\eta(u_{\eta,\epsilon}) = 0 \quad \text{in } Q_T, \\ u_{\eta,\epsilon}(0) = u_{\eta,\epsilon}(T) \quad \text{in } \Omega. \end{gathered} \tag{3.1}$$

By [9, Theorem 1.1] (see also section 2.2 of chapter 3, p. 328), this problem has a solution $u_{\eta,\epsilon}$. moreover the estimates of [5, lemmas 3.6, 39] still apply and eventually after extracting a subsequence, we get

$$\lim_{\eta \to 0^+} u_{\eta,\epsilon} = u_\epsilon \text{ in } V,$$

with u_ϵ a solution of the variational inequality

$$\begin{gathered} \langle \frac{\partial u_\epsilon}{\partial t}, v - u_\epsilon \rangle + \int_{Q_T} A^*(u_\epsilon, \nabla u_\epsilon) \nabla (v - u_\epsilon) + \int_{Q_T} F_\epsilon^*(u_\epsilon, \nabla u_\epsilon)(v - u_\epsilon) \geq 0 \\ u_\epsilon \in \mathcal{K}, \quad \text{for } v \in \mathcal{K}. \end{gathered} \tag{3.2}$$

and of the system of equations

$$\begin{gathered} \frac{\partial u_\epsilon}{\partial t} - \operatorname{div}(A^*(u_\epsilon, \nabla u_\epsilon)) + F_\epsilon^*(u_\epsilon, \nabla u_\epsilon) + g_\epsilon = 0 \quad \text{in } Q_T \\ u_\epsilon = 0 \quad \text{on } \Sigma_T, \\ u_\epsilon(0) = u_\epsilon(T) \quad \text{in } \Omega, \end{gathered} \tag{3.3}$$

where
$$\lim_{\eta\to 0^+} \theta_\eta(u_{\eta,\epsilon}) = g_\epsilon \quad \text{in } L^{p'}(Q_T) \text{ weak }.$$

As in [5, p. 93], there exists $u \in V$ such that $\lim_{\epsilon\to 0^+} u_\epsilon = u$ in V and $\lim_{\epsilon\to 0^+} \frac{\partial u_\epsilon}{\partial t} = \frac{\partial u}{\partial t}$ in $V' + L^1(Q_T)$, with u satisfying

$$\frac{\partial u}{\partial t} = \operatorname{div}(A(Su, \nabla u)) + F(Su, \nabla Su).$$

To conclude that $u \in C([0,T]; L^q(\Omega))$ for any $q \geq 1$, it suffices to show that $u(0) \in L^\infty(\Omega)$ and then use [5, Lemma 3.2]. In fact, $u_\epsilon \in L^p(0,T; W_0^{1,p}(\Omega) \cap L^\infty(\Omega))$, and $\frac{\partial u_\epsilon}{\partial t} \in V'$ so that $u_\epsilon \in C([0,T]; L^2(\Omega))$ by Lions's lemma [9, p. 156]. But $u_\epsilon \in \mathcal{K}$, so the following claim gives $-k \leq u_\epsilon(0) \leq k$ a.e. in Ω.

Claim. Let $u, v \in C([0,T]; L^1(\Omega))$ with $u \geq v$ a.e. in Q_T. Then $u(t) \geq v(t)$ a.e. in Ω for all $t \in [0,T]$.
To prove this claim take $w := (v-u)^+$, so that $w = 0$ a.e. in Q_T. The continuity and the non negativity of $t \to \int_\Omega w(x,t)dx$ on $[0,T]$ gives the result.
(ii) A careful application of [5, Lemma 3.1] shows that

$$\langle\langle \frac{\partial \alpha}{\partial t} - \frac{\partial u_\epsilon}{\partial t}, (\alpha - u_\epsilon)^+ \rangle\rangle \geq 0.$$

Where $\langle\langle .,. \rangle\rangle$ is the duality between $V \cap L^\infty(Q_T)$ and $V' + L^1(Q_T)$. So we get: $\alpha \leq u$ a.e. in Q_T and by similar arguments, we also obtain $u \leq \beta$ a.e. in Q_T.

Now we state a uniqueness result concerning maximal and minimal solutions.

Theorem 3.2 *Assume (A1)–(A5) and let α and β be respectively lower and upper-solutions of* (2.2) *such that $\alpha \leq \beta$. Then, there exist a minimal solution v and a maximal solution w of* (2.2) *such that $\alpha \leq v \leq w \leq \beta$ a.e. in Q_T.*

The proof is based on the following lemma.

Lemma 3.3 *Assume (A1)–(A5) and let α_1, α_2 be two lower-solutions and β be an upper-solutions of* (2.2) *such that $sup(\alpha_1, \alpha_2) \leq \beta_1$ a.e. in Q_T. Then, there exists at least one weak solution of* (2.2) *such that* $\sup(\alpha_1, \alpha_2) \leq u \leq \beta$ *a.e. in Q_T*

The proof of this lemma is the same as that in [5, Theorem 3.2], except for what concerns the inequality of [5, Lemma 3.18], which must be replaced by

$$\langle\langle \frac{\partial \alpha_1}{\partial t}, [1 - \beta_\delta(\alpha_2 - \alpha_1)]\omega_\delta \rangle\rangle + \langle\langle \frac{\partial \alpha_2}{\partial t}, \beta_\delta(\alpha_2 - \alpha_1)\omega_\delta \rangle\rangle + \langle -\frac{\partial u_\epsilon}{\partial t}, \omega_\delta \rangle \geq \varphi(\delta)$$

where γ_δ, β_δ and ω_δ are defined as in [5, p. 31], φ is given by the uniform continuity of the function $s \to s^+$ on some compact set associated to $\mathcal{K}$ and is such that $\varphi(\delta) \to 0$ as $\delta \to 0^+$, and where $\langle .,. \rangle$ designates the duality between V an V'.

4 Stability result

The aim of this section is to prove the following theorem.

Theorem 4.1 *Assume (A1)–(A6) and let α and β be respectively lower and upper-solution of* (2.2) *with $\alpha \leq \beta$ a.e. in Q_T and $\alpha(0), \beta(0) \in L^\infty(\Omega)$. Denote by v (resp. ω) the minimal (resp. maximal) solution of* (2.2) *located between α and β. Then, for all $u_0 \in L^\infty(\Omega)$ satisfying $\alpha(0) \leq u_0 \leq v(0)$ (resp. $\omega(0) \leq u_0 \leq \beta(0)$), the set $\mathcal{U}(u_0, \alpha, v)$ (resp. $\mathcal{U}(u_0, \beta, \omega)$) of all solutions u of $(\mathcal{P}_{0,+\infty;u_0})$ satisfying $\alpha \leq u \leq v$ (resp. $\omega \leq u \leq \beta$) in $\Omega \times (0, +\infty)$, is nonempty and is such that for any $q \geq 1$, we have*

$$\begin{aligned} &\lim_{t\to+\infty} \|u(.,t) - v(.,t)\|_{L^q(\Omega)} = 0 \\ &(\textit{resp.}\ \lim_{t\to+\infty} \|u(.,t) - \omega(.,t)\|_{L^q(\Omega)} = 0), \end{aligned} \tag{4.1}$$

This theorem is a consequence of the following lemma.

Lemma 4.2 *Assume (A1)–(A6) and let Z be a solution of* (2.2) *such that $Z(0) \in L^\infty(\Omega)$. Then, we have:*
(a) If α is a lower-solution of (2.2) *with $\alpha(0) \in L^\infty(\Omega)$ such that $\alpha \leq Z$ a.e. in Q_T, with strict inequality in a subset of positive measure, and such that every solution v of* (2.2) *satisfying $\alpha \leq v \leq Z$ is equal to Z. Then the minimal solution $\tilde{\alpha}$ of $(\mathcal{P}_{0,+\infty;\alpha(0)})$ is such that $\alpha \leq \tilde{\alpha} \leq Z$, and*

$$\lim_{t\to+\infty} \|\tilde{\alpha}(.,t) - Z(.,t)\|_{L^q(\Omega)} = 0, \ \forall q \geq 1. \tag{4.2}$$

(b) If β is an upper-solution of (2.2) *with $\beta(0) \in L^\infty(\Omega)$ such that $Z \leq \beta$ a.e. in Q_T, with strict inequality in a subset of positive measure, and such that every solution v of* (2.2) *satisfying $Z \leq v \leq \beta$ is equal to Z. Then the maximal solution $\tilde{\beta}$ of $(\mathcal{P}_{0,+\infty;\beta(0)})$ is such that $Z \leq \tilde{\beta} \leq \beta$, and*

$$\lim_{t\to+\infty} \|\tilde{\beta}(.,t) - Z(.,t)\|_{L^q(\Omega)} = 0, \ \forall q \geq 1.$$

Proof. With the help of the lemmas in section 2, we apply the method of De coster and Omari [3]. First we show (a), and then (b) can be obtained by similar way. The proof is divided into three steps.
(i) We construct a sequence of lower-solutions of (2.2) converging to Z: Let α be a lower-solution of $(\mathcal{P}_{0,T;\alpha(0)})$, and Z verify $Z(0) \geq \alpha(0)$. Then Z is an upper-solution of $(\mathcal{P}_{0,T;\alpha(0)})$. By lemma 2.2, there exists a minimal solution $\tilde{\alpha}_0$ of $(\mathcal{P}_{0,T;\alpha(0)})$ such that $\alpha \leq \tilde{\alpha}_0 \leq Z$ a.e. in Q_T. So $\tilde{\alpha}_0(T) \geq \alpha(T) \geq \alpha(0) = \tilde{\alpha}_0(0)$. Now, we define by induction, the sequence $(\tilde{\alpha}_n)_n$ such that $\tilde{\alpha}_n$ is the minimal solution u of

$$\begin{gathered} \frac{\partial u}{\partial t} + A(u) + F(u, \nabla u) = 0 \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \Sigma_T, \\ u(0) = \tilde{\alpha}_{n-1}(T) \quad \text{in } \Omega, \end{gathered}$$

satisfying

$$\tilde{\alpha}_{n-1} \leq u \leq Z \quad \text{a.e. in } Q_T.$$

Hence $\tilde{\alpha}_n$ is a lower-solution of (2.2). Consequently,

$$\alpha \leq \tilde{\alpha}_{n-1} \leq \tilde{\alpha}_n \leq Z, \text{ for all } n. \tag{4.3}$$

and

$$\tilde{\alpha}_{n-1}(T) = \tilde{\alpha}_n(0), \quad \text{for all } n. \tag{4.4}$$

By Lebesgue dominated convergence theorem, there exists $u \in L^\infty(Q_T)$ such that $\alpha \leq u \leq Z$ a.e. in Q_T and $\lim_{n\to+\infty} \tilde{\alpha}_n = u$ in $L^q(Q_T)$, for any $q \geq 1$. Moreover, $\tilde{\alpha}_n$, $u \in C([0,T]; L^q(\Omega))$. By (4.3) and this claim, we get

$$\lim_{n\to+\infty} \tilde{\alpha}_n(t) = u(t) \quad \text{in } L^q(\Omega), \forall q \geq 1. \tag{4.5}$$

Let $f_n(t) := \int_\Omega (u - \tilde{\alpha}_n)^q(x,t)dx$, for any $n \geq 1$. We have, $(f_n)_n \subset C([0,T]; \mathbb{R})$ and converges simply to zero. By Dini's theorem one has

$$\lim_{n\to+\infty} \sup_{[0,T]} \|\tilde{\alpha}_n(t) - u(t)\|_q = 0.$$

(ii) Using [5, Theorem 3.6], we deduce that u satisfies the first two equations in (2.2). The third equation, the periodicity condition, is a consequence of (4.4). Then u is a solution of (2.2) with $\alpha \leq u \leq \beta$. Therefore, we have $u = Z$ a.e. in Q_T and

$$\lim_{n\to+\infty} \sup_{[0,T]} \|\tilde{\alpha}_n(t) - Z(t)\|_q = 0. \tag{4.6}$$

Let $\tilde{\alpha}(x,t) := \tilde{\alpha}_n(x, t - nT)$ for $(x,t) \in \Omega \times [nT, (n+1)T[$. Then $\tilde{\alpha}$ is a solution of $(\mathcal{P}_{0,\alpha(0)})$ satisfying (4.2). Indeed, we have

$$\|\tilde{\alpha}(.,t) - Z(.,t)\|_{L^q(\Omega)} \leq \sup_{\theta\in[0,T]} \|\tilde{\alpha}_{n_t}(.,\theta) - Z(.,\theta)\|_{L^q(\Omega)},$$

where $n_t = [t/T]$ is the integer part of t/T. Now, (4.2) is a consequence of (4.6).
(iii)The minimality of $\tilde{\alpha}$ as a solution of $(\mathcal{P}_{0,\alpha(0)})$ satisfying $\alpha \leq \tilde{\alpha} \leq Z$ is obtained exactly as in [3]. □

Remark In the sequel we shall identify a lower or an upper-solution ϕ defined on $\Omega \times [0,T)$ to its prolongment on $\Omega \times [0, +\infty)$ defined by $\tilde{\phi}(x,t) := \phi(x, t - nT)$ $\forall (x,t) \in \Omega \times [nT, (n+1)T[$.

Proof of Theorem 4.1 We prove the result concerning the minimal solution, the one corresponding to the maximal solution is obtained in a similar way. Let u_0 be such that $\alpha(0) \leq u_0 \leq v(0)$. We first show that: $\mathcal{U}(u_0, \alpha, v) \neq \emptyset$. v (resp. α) is an upper (resp. lower) solution of $(\mathcal{P}_{0,T';u_0})$, for any $T' > 0$. By lemma 2.3 the maximal and minimal solutions of $\mathcal{P}_{0,+\infty;u_0}$ are defined globally. Let

$u \in \mathcal{U}(u_0, \alpha, v)$ and $u_{\min}$ the minimal solution of $(\mathcal{P}_{0,T';u_0})$, $T' > 0$. We have $\alpha \leq u_{\min} \leq u \leq v$ on $\Omega \times (0, +\infty)$. And from lemma 2.2, we get

$$\alpha \leq \tilde{\alpha} \leq u_{\min} \leq u \leq v, \tag{4.7}$$

where $\tilde{\alpha}$ is the minimal solution of $(\mathcal{P}_{0,\alpha(0)})$ and u_0 satisfying $\alpha(0) \leq u_0$. Hence the proof is completed □

5 Applications

In this section we give some sufficient conditions on the data in order to obtain existence of lower and upper-solutions for a periodic-parabolic problem associated with the p-laplacian operator. Consider the problem

$$\begin{gathered} \frac{\partial u}{\partial t} - \Delta_p u + g(u) = h(x,t) \quad \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(0) = u(T) \quad \text{in } \Omega, \end{gathered} \tag{5.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, with p such that $1 < p < +\infty$ and T a fixed positive real number. $h \in L^\infty(\Omega \times \mathbb{R})$ is a caratheodory function which is T-periodic in time, and g is continuous function from $[0, +\infty[$ to $[0, +\infty[$ such that there is a non decreasing function b from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $g(s) \leq b(|s|)$ for any $s \in \mathbb{R}$. We denote by G the primitive of g vanishing at zero: $G(t) := \int_0^t g(s)ds$. By applying [6, Theorem 2.1] successively to the elliptic problems

$$\begin{gathered} -\Delta_p u + g(u) = -\|h\|_\infty \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{5.2}$$

and

$$\begin{gathered} -\Delta_p u + g(u) = \|h\|_\infty \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{5.3}$$

we obtain the following

Theorem 5.1 *Suppose that*

$$\liminf_{|s| \to +\infty} \frac{pG(s)}{|s|^p} < \mu' := \frac{1}{R(\Omega)^p} \frac{p-1}{p} \beta(\frac{1}{p}, 1 - \frac{1}{p}),$$

where $R(\Omega)$ and $\beta(r,s)$ are as in [6]. Then the conclusions of Theorem 4.1 are verified.

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