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# Strongly nonlinear degenerated unilateral problems with $L^1$ data *

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**Abstract**

In this paper, we study the existence of solutions for strongly nonlinear degenerated unilateral problems associated to nonlinear operators of the form $Au + g(x, u, \nabla u)$. Here $A$ is a Leray-Lions operator acting from $W_0^{1,p}(\Omega, w)$ into its dual, while $g(x, s, \xi)$ is a nonlinear term which has a growth condition with respect to $\xi$ and no growth condition with respect to $s$, the second term belongs to $L^1(\Omega)$.

## 1 Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $p$ be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions on $\Omega$, i.e. each $w_i(x)$ is a measurable a.e. strictly positive on $\Omega$. Let $W_0^{1,p}(\Omega, w)$ be the weighted Sobolev space associated with the vector $w$. Let $A$ be a nonlinear operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$, i.e.

$$Au = -\operatorname{div}(a(x, u, \nabla u)).$$

In the non-degenerated case, Bensoussan, Boccardo and Murat have proved in [4], the existence of a solution for the quasilinear equation of the form,

$$Au + g(x, u, \nabla u) = f.$$

They assume that $g$ is a nonlinearity having natural growth with respect to $|\nabla u|$ (of order $p$), and which satisfies the sign-condition and $f \in W^{-1,p'}(\Omega)$. Recently, in weighted case, Akdim, Azroul and Benkirane have first in [2] extended the last result to weighted Sobolev spaces and in [3] the authors have studied the following degenerated unilateral problem:

$$\begin{gathered}\langle Au, v - u\rangle + \int_\Omega g(x, u, \nabla u)(v - u)\, dx \geq \langle f, v - u\rangle \quad \forall\, v \in K_\psi \cap L^\infty(\Omega) \\ u \in W_0^{1,p}(\Omega, w) \quad u \geq \psi \text{ a.e. in } \Omega \\ g(x, u, \nabla u) \in L^1(\Omega) \quad g(x, u, \nabla u)u \in L^1(\Omega),\end{gathered}$$

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where $K_\psi = \{v \in W_0^{1,p}(\Omega, w),\ v \geq \psi$ a.e. in $\Omega\}$, with $\psi$ a measurable function on $\Omega$ such that $\psi^+ \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ and $f \in W^{-1,p'}(\Omega, w^*)$. The purpose of this paper, is to study the previous problems for $f \in L^1(\Omega)$. More precisely, we prove the existence theorem for the following degenerated unilateral problem:

$$\begin{aligned} &\langle Au, T_k(v-u)\rangle + \int_\Omega g(x,u,\nabla u)T_k(v-u)\,dx \geq \langle f, T_k(v-u)\rangle \\ &\qquad\qquad \text{for all} \quad v \in K_\psi \quad \text{and all } k > 0 \\ &\qquad\qquad u \in W_0^{1,p}(\Omega, w) \quad u \geq \psi \text{ a.e. in } \Omega \\ &\qquad\qquad g(x,u,\nabla u) \in L^1(\Omega). \end{aligned} \tag{1.1}$$

For that, we assume in addition that the nonlinearity $g$ satisfies some coercivity conditions (see (2.10)).

Concerning the existence result for the degenerated elliptic equations where the second member lies in the dual $W^{-1,p'}(\Omega, w^*)$ (resp. for the quasilinear equation where the second member is in $L^1(\Omega)$), we refer the reader to [6-7-8](resp. [1-2]).

**Remarks**

1) Note that the use of the truncation operator in (1.1) is justified by the fact that the solution does not in general belong to $L^\infty(\Omega)$ for $f \in L^1(\Omega)$.

2) An other work in this direction can be found in [5] in non-weighted case.

The paper is organized as follows: Section 2 contains some preliminaries and is concerned with the basic assumptions and some technical lemmas. In section 3, we state and prove main results. The last section is devoted to an example which illustrates our abstract conditions.

# 2 Preliminaries

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$, let $1 < p < \infty$, and let $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e. every component $w_i(x)$ is a measurable function which is strictly positive *a.e.* in $\Omega$ satisfying the integrability conditions

$$w_i \in L^1_{\text{loc}}(\Omega), \quad w_i^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\Omega) \tag{2.1}$$

for $0 \leq i \leq N$.

We define the weighted space $L^p(\Omega, \gamma)$, where $\gamma$ is a weight function on $\Omega$ by

$$L^p(\Omega, \gamma) = \{u = u(x),\ u\gamma^{1/p} \in L^p(\Omega)\}$$

with the norm

$$\|u\|_{p,\gamma} = \Big(\int_\Omega |u(x)|^p \gamma(x)\,dx\Big)^{1/p}.$$

We denote by $W^{1,p}(\Omega,w)$ the space of all real-valued functions $u \in L^p(\Omega,w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega,w_i) \quad \text{for all } i=1,\dots,N.$$

This set of functions defines a Banach space under the norm

$$\|u\|_{1,p,w} = \Big( \int_\Omega |u(x)|^p w_0(x)\,dx + \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx \Big)^{1/p}. \tag{2.2}$$

Since we shall deal with the Dirichlet problem, we shall use the space

$$X = W_0^{1,p}(\Omega,w)$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.2). Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega,w)$ and $(X,\|.\|_{1,p,w})$ is a reflexive Banach space.

We recall that the dual space of weighted Sobolev space $W_0^{1,p}(\Omega,w)$ is equivalent to $W^{-1,p'}(\Omega,w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ \forall i=0,\dots,N\}$, where $p'$ is the conjugate of $p$ i.e. $p' = p/(p-1)$ (for more details we refer to [7]).

Now we state the following assumptions:

**Assumption (H1)** The expression

$$|||u||| = \Big( \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx \Big)^{1/p}$$

is a norm defined on $W_0^{1,p}(\Omega,w)$ and is equivalent to the norm (2.2). There exists a weight function $\sigma$ on $\Omega$ such that $\sigma \in L^1(\Omega)$ and $\sigma^{1-q'} \in L^1(\Omega)$ for some parameter $q$, so that $1 < q < p + p'$ such that the Hardy inequality,

$$\Big( \int_\Omega |u(x)|^q \sigma(x)\,dx \Big)^{\frac{1}{q}} \le c \Big( \sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx \Big)^{1/p}, \tag{2.3}$$

holds for every $u \in W_0^{1,p}(\Omega,w)$ with a constant $c > 0$ independent of $u$, and moreover, the imbedding

$$W_0^{1,p}(\Omega,w) \hookrightarrow\hookrightarrow L^q(\Omega,\sigma), \tag{2.4}$$

determined by the inequality (2.3) is compact. Note that $(W_0^{1,p}(\Omega,w),|||.|||)$ is a uniformly convex and thus reflexive Banach space. Let $A$ be a nonlinear operator from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$ defined by

$$Au = -\mathrm{div}(a(x,u,\nabla u)),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-function satisfying for a.e $x \in \Omega$, for all $s \in \mathbb{R}$ and $\xi, \eta$ in $\mathbb{R}^N$ with $\xi \neq \eta$.

**Assumption (H2)**

$$|a_i(x,s,\xi)| \leq \beta w_i^{1/p}(x)[k(x) + \sigma^{1/p'}|s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}], \quad i = 1,\dots,N, \tag{2.4}$$

$$[a(x,s,\xi) - a(x,s,\eta)](\xi - \eta) > 0, \tag{2.5}$$

$$a(x,s,\xi).\xi \geq \alpha \sum_{i=1}^{N} w_i|\xi_i|^p, \tag{2.6}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and $\alpha,\ \beta$ are strictly positive constants.

**Assumption (H3)** Let $g(x,s,\xi)$ be a Carathéodory function satisfying

$$g(x,s,\xi)s \geq 0 \tag{2.7}$$

$$|g(x,s,\xi)| \leq b(|s|)\Big(\sum_{i=1}^{N} w_i|\xi_i|^p + c(x)\Big), \tag{2.8}$$

$$|g(x,s,\xi)| \geq \rho_2 \sum_{i=1}^{N} w_i|\xi_i|^p \quad \text{for } |s| > \rho_1 \tag{2.9}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(x)$ is a positive function which lies in $L^1(\Omega)$, $c \geq 0$ and $\rho_1 > 0$, $\rho_2 > 0$. We consider,

$$f \in L^1(\Omega). \tag{2.10}$$

Now we recall some lemmas which will be used later.

**Lemma 2.1 (cf. [2])** *Let $g \in L^r(\Omega,\gamma)$ and let $g_n \in L^r(\Omega,\gamma)$, with $\|g_n\|_{r,\gamma} \leq c$ $(1 < r < \infty)$. If $g_n(x) \to g(x)$ a.e. in $\Omega$, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega,\gamma)$, where $\gamma$ is a weight function on $\Omega$.*

**Lemma 2.2 (cf. [1])** *Assume that (H1) holds. Let $u \in W_0^{1,p}(\Omega,w)$, and let $T_k(u)$, $k \in \mathbb{R}^+$, be the usual truncation then $T_k(u) \in W_0^{1,p}(\Omega,w)$. Moreover, we have*

$$T_k(u) \to u \text{ strongly in } W_0^{1,p}(\Omega,w).$$

**Lemma 2.3 (cf. [2])** *Assume that (H1) and (H2) are satisfied, and let $(u_n)$ be a sequence of $W_0^{1,p}(\Omega,w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$ and*

$$\int_\Omega [a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)\,dx \to 0.$$

*Then, $u_n \to u$ strongly in $W_0^{1,p}(\Omega,w)$.*

**Lemma 2.4 (cf. [1])** *Assume that (H1) holds, let $(u_n) \in W_0^{1,p}(\Omega,w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$,then $T_k u_n \rightharpoonup T_k u$ weakly in $W_0^{1,p}(\Omega,w)$.*

## 3 Main result

Let $\psi$ be a measurable function with values in $\mathbb{R}$ such that,

$$\psi^+ \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \tag{3.1}$$

and let

$$K_\psi = \{v \in W_0^{1,p}(\Omega, w)\ v \geq \psi \text{ a.e. in } \Omega\}.$$

Consider the nonlinear problem with Dirichlet boundary condition,

$$\begin{aligned} \langle Au, T_k(v-u)\rangle + \int_\Omega g(x,u,\nabla u)T_k(v-u)\,dx \geq \langle f, T_k(v-u)\rangle \\ \text{for all } v \in K_\psi \text{ and all } k > 0 \\ u \in W_0^{1,p}(\Omega, w) \quad u \geq \psi \text{ a.e. in } \Omega \\ g(x,u,\nabla u) \in L^1(\Omega), \end{aligned} \tag{3.2}$$

where $u$ is the solution of this problem.

**Theorem 3.1** *Under the assumptions (H1)-(H3), (2.10) and (3.1), there exists at least one solution of (3.2).*

**Remarks**

1) Theorem 3.1 generalizes to weighted case the analogous in [5].
2) In the particular case when $w_0(x) \equiv 1$, we can replace (H1) by the conditions: There exists $s \in ]\frac{N}{p}, \infty[\cap[\frac{1}{p-1}, \infty[$ such that $w_i^{-s} \in L^1(\Omega)$ for all $i = 1, \dots, N$, (which is an integrability condition, stronger than (2.1)), since

$$|||u|||_X = \Big(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\,dx\Big)^{1/p}$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and equivalent to (2.2) and also the following imbeddings hold:

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega)$$

for $1 \leq q < p_1^*$, if $ps < N(s+1)$, and $q \geq 1$ is arbitrary if $ps \geq N(s+1)$, where $p_1 = ps/(s+1)$ and $p_1^*$ is devoted the Sobolev conjugate of $p_1$ (for more details see [2,7]). Hence the hypotheses (H1) is satisfied for $\sigma \equiv 1$.

**Proof of Theorem 3.1** Consider the sequence of approximate problems:

$$\begin{aligned} \langle Au_n, T_k(v-u_n)\rangle + \int_\Omega g(x,u_n,\nabla u_n)T_k(v-u_n)\,dx \geq \langle f_n, T_k(v-u_n)\rangle \\ \text{for all } v \in K_\psi \cap L^\infty(\Omega) \quad \text{and all } k > 0 \\ u_n \in W_0^{1,p}(\Omega, w) \quad u_n \geq \psi \quad \text{a.e. in } \Omega \\ g(x,u_n,\nabla u_n) \in L^1(\Omega), \end{aligned} \tag{3.3}$$

where $f_n$ is a sequence of smooth functions which converges strongly to $f$ in $L^1(\Omega)$ with $\|f_n\|_{L^1(\Omega)} \le C$. For some constant $C$. By Theorem 6.1 and Lemma 6.2 of [1] or via Theorem 4.1 of [3] there exists at least one solution $u_n$ of (3.3). In order to pass to the limit in the approximate problem (3.3), we claim that:
**Assertion(1)**

$$(u_n)_n \text{ is bounded in } W_0^{1,p}(\Omega, w) \tag{3.4}$$

**Assertion(2)**

$$\nabla T_k(u_n) \to \nabla T_k(u) \quad \text{strongly in } \prod_{i=1}^{N} L^p(\Omega, w_i) \tag{3.5}$$

which implies $\nabla u_n \to \nabla u$ a.e in $\Omega$
**Assertion(3)**

$$g(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega) \tag{3.6}$$

We can pass to the limit in the approximate problems (3.3), indeed,

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(v - u_n)\, dx + \int_\Omega g(x, u_n, \nabla u_n) T_k(v - u_n)\, dx \\ \ge \int_\Omega f_n T_k(v - u_n)\, dx$$

For all $v \in K_\psi \cap L^\infty(\Omega)$ and $k > 0$. From (2.5) and (3.4) we deduce that $a(x, u_n, \nabla u_n)$ is bounded in $\prod_{i=1}^{N} L^{p'}(\Omega, w_i^*)$. Using (3.5) we obtain

$$\nabla u_n \to \nabla u \quad \text{a.e in } \Omega. \tag{3.7}$$

Hence, we get

$$a(x, u_n, \nabla u_n) \to a(x, u, \nabla u) \quad \text{a.e in } \Omega \tag{3.8}$$

which implies with Lemma 2.1 that

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \quad \text{weakly in } \prod_{i=1}^{N} L^{p'}(\Omega, w_i^*). \tag{3.9}$$

On the other hand, let $v \in L^\infty(\Omega)$ and set $h = k + ||v||_\infty$, then

$$\begin{aligned} |\frac{\partial T_k(v - u_n)}{\partial x_i}| w_i^{1/p} &= (\chi_{|v-u_n| \le k} |\frac{\partial (v - u_n)}{\partial x_i}|) w_i^{1/p} \\ &\le \chi_{|u_n| \le k + ||v||_\infty} (|\frac{\partial v}{\partial x_i}| + |\frac{\partial u_n}{\partial x_i}|) w_i^{1/p} \\ &\le |\frac{\partial v}{\partial x_i}| w_i^{1/p} + |\frac{\partial T_h(u_n)}{\partial x_i}| w_i^{1/p} \end{aligned}$$

for $i = 1, \ldots, N$. Which implies by using the Vitali's theorem with (3.5) and (3.7) that

$$\nabla T_k(v - u_n) \to \nabla T_k(v - u) \quad \text{strongly in } \prod_{i=1}^{N} L^p(\Omega, w_i) \tag{3.10}$$

for any $v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$. From (3.9) and (3.10) we can pass to the limit in the first term of (3.3). Since $g(x, u_n, \nabla u_n) \to g(x, u, \nabla u)$ strongly in $L^1(\Omega)$ and $f_n \to f$ strongly in $L^1(\Omega)$, then we can pass to the limit in

$$\langle A(u_n), T_k(v - u_n)\rangle + \int_\Omega g(x, u_n, \nabla u_n) T_k(v - u_n)\, dx \geq \int_\Omega f_n T_k(v - u_n)\, dx.$$

This allows to consider the problem

$$\begin{gathered} \langle Au, T_k(v - u)\rangle + \int_\Omega g(x, u, \nabla u) T_k(v - u)\, dx \geq \langle f, T_k(v - u)\rangle \\ \text{for all } v \in K_\psi \cap L^\infty(\Omega) \text{ and all } k > 0 \\ u \in W_0^{1,p}(\Omega, w) \quad u \geq \psi \quad \text{a.e. in } \Omega \\ g(x, u, \nabla u) \in L^1(\Omega), \end{gathered} \tag{3.11}$$

Set $\phi = T_m(v)$ as a test function, where $m \geq \|\psi^+\|_\infty$ and $v \in K_\psi$, then $\phi \in K_\psi \cap L^\infty(\Omega)$. Multiplying (3.11) by $\phi$, we obtain

$$\begin{aligned} \langle Au, T_k(T_m(v_m) - u)\rangle + \int_\Omega & g(x, u, \nabla u) T_k(T_m(v_m) - u)\, dx \\ & \geq \int_\Omega f T_k(T_m(v_m) - u)\, dx \end{aligned} \tag{3.12}$$

From Lemma 2.2 and using the Vitali's theorem, we have

$$\nabla T_k(T_m(v) - u) \to \nabla T_k(v - u) \quad \text{strongly in } \prod_{i=1}^N L^p(\Omega, w_i).$$

Finally, passing to the limit in (3.12) as $m$ tends to infinity, we obtain:

$$\langle Au, T_k(v - u)\rangle + \int_\Omega g(x, u, \nabla u) T_k(v - u)\, dx \geq \int_\Omega f T_k(v - u)\, dx$$

for any $v \in W_0^{1,p}(\Omega, w)$, $v \geq \psi$ a.e. in $\Omega$.

**Proof of assertion 1:** We consider the sequence of approximate problems (3.3). By Theorem 6.1 and Lemma 6.2 of [1], there exists at least one solution $u_n$ of (3.3). Let $v = \psi^+$ as test function in (3.3), then

$$\langle Au_n, T_k(\psi^+ - u_n)\rangle + \int_\Omega g(x, u_n, \nabla u_n) T_k(\psi^+ - u_n)\, dx \geq \int_\Omega f_n T_k(\psi^+ - u_n)\, dx.$$

Since $u_n - \psi^+$ and $u_n$ have the same sign, we obtain by using (2.8) and $\|f_n\|_{L^1(\Omega)} \leq C$,

$$\int_{|u_n - \psi^+| \leq k} a(x, u_n, \nabla u_n) \nabla(u_n - \psi^+)\, dx \leq \int_\Omega f_n T_k(u_n - \psi^+)\, dx \leq kC \tag{3.13}$$

which implies that

$$\int_{|u_n-\psi^+|\le k} a(x,u_n,\nabla u_n)\nabla u_n\,dx \le Ck + \int_{|u_n-\psi^+|\le k} a(x,u_n,\nabla u_n)|\nabla\psi^+|\,dx$$

using Young's inequality, we obtain

$$\begin{aligned}&\int_{|u_n-\psi^+|\le k} a(x,u_n,\nabla u_n)\nabla u_n\,dx\\ \le& Ck + \sum_{i=1}^{N}\int_{|u_n-\psi^+|\le k}\frac{\eta^{p'}}{p'}|a_i(x,u_n,\nabla u_n)|^{p'} w_i^{1-p'}\,dx\\ &+\sum_{i=1}^{N}\int_{|u_n-\psi^+|\le k}\frac{1}{p}\frac{1}{\eta^p} w_i|\frac{\partial\psi^+}{\partial x_i}|^p\,dx,\end{aligned}$$

where $\eta$ is a positive constant. From (2.5) we have

$$\begin{aligned}&\int_{|u_n-\psi^+|\le k} a(x,u_n,\nabla u_n)\nabla u_n\,dx\\ \le& C_1 + \frac{\eta^{p'}}{p'}\beta^{p'}N\int_\Omega k^{p'}(x)\,dx + \frac{\eta^{p'}}{p'}\beta^{p'}N\int_{|u_n-\psi^+|\le k}\sigma|u_n|^q\,dx\\ &+\frac{\eta^{p'}}{p'}\beta^{p'}N\int_{|u_n-\psi^+|\le k}\sum_{i=1}^{N} w_i|\frac{\partial u_n}{\partial x_i}|^p\,dx\\ \le& C_2 + \frac{\eta^{p'}}{p'}\beta^{p'}N\int_{|u_n|\le||\psi^+||_\infty+k}\sigma|u_n|^q\,dx + \frac{\eta^{p'}}{p'}\beta^{p'}N\int_{|u_n-\psi^+|\le k}\sum_{i=1}^{N} w_i|\frac{\partial u_n}{\partial x_i}|^p\,dx\\ \le& C_2 + \frac{\eta^{p'}}{p'}\beta^{p'}N(||\psi^+||_\infty+k)^q\int_\Omega\sigma\,dx + \frac{\eta^{p'}}{p'}\beta^{p'}N\int_{|u_n-\psi^+|\le k}\sum_{i=1}^{N} w_i|\frac{\partial u_n}{\partial x_i}|^p\,dx.\end{aligned}$$

Consequently, using (2.7) and since $\sigma\in L^1(\Omega)$, we have

$$\int_{|u_n-\psi^+|\le k}\alpha\sum_{i=1}^{N} w_i|\frac{\partial u_n}{\partial x_i}|^p\,dx \le C_3 + \frac{\eta^{p'}}{p'}\beta^{p'}N\int_{|u_n-\psi^+|\le k}\sum_{i=1}^{N} w_i|\frac{\partial u_n}{\partial x_i}|^p\,dx,$$

we choose $0<\eta<\frac{1}{\beta}(\frac{\alpha p'}{N})^{1/p'}$, this implies

$$\int_{|u_n-\psi^+|\le k}\sum_{i=1}^{N} w_i|\frac{\partial u_n}{\partial x_i}|^p\,dx \le C \tag{3.14}$$

On the other hand, from (3.13) and $a(x,s,\xi)\xi\ge 0$

$$\begin{aligned}&\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-\psi^+)\,dx\\ \le& k\int_\Omega |f_n|\,dx - \int_{|u_n-\psi^+|\le k} a(x,u_n,\nabla u_n)\nabla(u_n-\psi^+)\,dx\\ \le& C_1 + \int_{|u_n-\psi^+|\le k} a(x,u_n,\nabla u_n)|\nabla\psi^+|\,dx.\end{aligned}$$

As in the proof of (3.14) we can show that

$$\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-\psi^+)\,dx$$
$$\leq C_2+\frac{\eta^{p'}}{p'}\beta^{p'}N\int_{|u_n-\psi^+|\leq k}\sum_{i=1}^N w_i|\frac{\partial u_n}{\partial x_i}|^p\,dx\leq C_3,$$

where $C_1, C_2$ and $C_3$ are positive constants. When $|u_n-\psi^+|>k$, we have $T_k(u_n-\psi^+)=+k(\text{or }-k)$, and since $T_k(u_n-\psi^+), u_n-\psi^+, u_n$ and $g(x,u_n,\nabla u_n)$ have the same sign, we obtain

$$\int_{|u_n-\psi^+|\leq k} g(x,u_n,\nabla u_n)T_k(u_n-\psi^+)\,dx$$
$$+\int_{|u_n-\psi^+|>k} g(x,u_n,\nabla u_n)T_k(u_n-\psi^+)\,dx\leq C_3$$

which gives

$$k\int_{|u_n-\psi^+|>k}|g(x,u_n,\nabla u_n)|\,dx\leq C_3.$$

From (2.10), we have

$$|g(x,u_n,\nabla u_n)|\geq\rho_2\sum_{i=1}^N w_i|\frac{\partial u_n}{\partial x_i}|^p\quad\text{for}\quad|u_n|\geq\rho_1$$

Choosing $k>\rho_1+||\psi^+||_\infty$, then $|u_n-\psi^+|>k$ implies $|u_n|>\rho_1$. We deduce that

$$\int_{|u_n-\psi^+|>k}\rho_2\sum_{i=1}^N w_i|\frac{\partial u_n}{\partial x_i}|^p\,dx\leq C_4. \tag{3.15}$$

Finally, combining (3.14) and (3.15) we have $|||u_n|||\leq C$.

**Proof of assertion 2** Let $k\geq\|\psi^+\|$ and $\delta=(\frac{b(k)}{2\alpha})^2$. Set $\varphi(s)=se^{\delta s^2}$, $z_n=T_k(u_n)-T_k(u)$, $\eta=e^{-4\delta k^2}$, and $v_n=u_n-\eta\varphi(z_n)$. By the choice of $k$, the above test function is admissible for (3.3). Multiplying (3.3) by $v_n$, for $h>0$, we obtain

$$\langle A(u_n),T_h(\eta\varphi(z_n))\rangle+\int_\Omega g(x,u_n,\nabla u_n)T_h(\eta\varphi(z_n))\,dx\leq\int_\Omega f_nT_h(\eta\varphi(z_n))\,dx.$$

Choosing $h>2k$, we have $|\eta\varphi(z_n)|\leq|z_n|\leq 2k<h$ and

$$\langle A(u_n),\varphi(z_n)\rangle+\int_\Omega g(x,u_n,\nabla u_n)\varphi(z_n)\,dx\leq\int_\Omega f_n\varphi(z_n)\,dx.$$

As $n\to\infty$, we have $f_n\to f$ strongly in $L^1(\Omega)$, and $\varphi(z_n)\rightharpoonup 0$ weak$^*$ in $L^\infty(\Omega)$. Hence we have

$$\int_\Omega f_n\varphi(z_n)\,dx\to 0.$$

Since $g(x,u_n,\nabla u_n)\varphi(z_n)\geq 0$ on the subset $|u_n(x)|>k$, this implies

$$\langle A(u_n),\varphi(z_n)\rangle+\int_{|u_n|\leq k} g(x,u_n,\nabla u_n)\varphi(z_n)\,dx\leq\varepsilon(n),\tag{3.16}$$

where $\varepsilon(n)$ is a real number which converge to zero when $n$ tends to infinity. On the other hand

$$\begin{aligned}\langle A(u_n),\varphi(z_n)\rangle&=\int_{|u_n|\leq k} a(x,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(z_n)\,dx\\&\quad+\int_{|u_n|>k} a(x,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(z_n)\,dx\\&=\int_\Omega a(x,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(z_n)\,dx\\&\quad-\int_{|u_n|>k} a(x,u_n,\nabla u_n)\nabla T_k(u)\varphi'(z_n)\,dx\\&=\int_\Omega (a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)))(\nabla T_k(u_n)\\&\quad-\nabla T_k(u))\varphi'(z_n)\,dx\\&\quad+\int_\Omega a(x,u_n,\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(z_n)\,dx\\&\quad-\int_{|u_n|>k} a(x,u_n,\nabla u_n)\nabla T_k(u)\varphi'(z_n)\,dx.\end{aligned}$$

Since $\nabla T_k(u)\chi_{\{|u_n|>k\}}\to 0$ strongly in $\prod_{i=1}^N L^p(\Omega,w_i)$, and from (2.3), (2.5) and (3.4) we have $(a(x,u_n,\nabla u_n)\varphi'(z_n))_n$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega,w_i^*)$, then

$$-\int_{|u_n|>k} a(x,u_n,\nabla u_n)\nabla T_k(u)\varphi'(z_n)\,dx=\varepsilon(n)\to 0\quad\text{as } n\to\infty.$$

Moreover, since $u_n\rightharpoonup u$ weakly in $W_0^{1,p}(\Omega,w)$, by Lemma 2.3 we have

$$T_k(u_n)\rightharpoonup T_k(u)\quad\text{weakly in } W_0^{1,p}(\Omega,w).$$

Then

$$\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)\quad\text{weakly in }\prod_{i=1}^N L^p(\Omega,w_i).$$

Since the sequence $(a(x,u_n,\nabla T_k(u))\varphi'(z_n))_n$ converges strongly in the space $\prod_{i=1}^N L^{p'}(\Omega,w_i^*)$,

$$\int_\Omega a(x,u_n,\nabla T_k(u))(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(z_n)\,dx=\varepsilon(n)\to 0\quad\text{as } n\to\infty,$$

and

$$\begin{aligned}\langle A(u_n),\varphi(z_n)\rangle=\int_\Omega&(a(x,u_n,\nabla T_k(u_n))-a(x,u_n,\nabla T_k(u)))\\&\times(\nabla T_k(u_n)-\nabla T_k(u))\varphi'(z_n)\,dx+\varepsilon(n).\end{aligned}\tag{3.17}$$

On the other hand

$$
\begin{aligned}
&\Big|\int_{|u_n|\le k} g(x,u_n,\nabla u_n)\varphi(z_n)\,dx\Big|\\
&\le \int_{|u_n|\le k} b(k)\Big(\sum_{i=1}^N w_i\Big|\frac{\partial u_n}{\partial x_i}\Big|^p + c(x)\Big)|\varphi(z_n)|\,dx\\
&\le \int_{|u_n|\le k} b(k)c(x)|\varphi(z_n)|\,dx + \int_{|u_n|\le k}\sum_{i=1}^N w_i\Big|\frac{\partial u_n}{\partial x_i}\Big|^p b(k)|\varphi(z_n)|\,dx\\
&\le \int_{|u_n|\le k} b(k)c(x)|\varphi(z_n)|\,dx + \int_{|u_n|\le k}\sum_{i=1}^N a_i(x,u_n,\nabla u_n)\frac{\partial u_n}{\partial x_i}\frac{b(k)}{\alpha}|\varphi(z_n)|\,dx\\
&\le \int_{|u_n|\le k} b(k)c(x)|\varphi(z_n)|\,dx + \int_{|u_n|\le k}\sum_{i=1}^N [a_i(x,u_n,\nabla T_k(u_n))\\
&\quad - a_i(x,u_n,\nabla T_k(u))]\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|\frac{b(k)}{\alpha}\,dx\\
&\quad + \int_{|u_n|\le k}\sum_{i=1}^N a_i(x,u_n,\nabla T_k(u))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|\frac{b(k)}{\alpha}\,dx\\
&\quad + \int_{|u_n|\le k}\sum_{i=1}^N a_i(x,u_n,\nabla T_k(u_n))\frac{\partial T_k(u)}{\partial x_i}|\varphi(z_n)|\frac{b(k)}{\alpha}\,dx.
\end{aligned}
$$

Since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $\prod_{i=1}^N L^p(\Omega,w_i)$ and

$$a(x,u_n,\nabla T_k(u))|\varphi(z_n)| \to 0 \quad \text{strongly in } \prod_{i=1}^N L^{p'}(\Omega,w_i^*),$$

it follows that

$$\int_\Omega \sum_{i=1}^N a_i(x,u_n,\nabla T_k(u))\Big(\frac{\partial T_k(u_n)}{\partial x_i}-\frac{\partial T_k(u)}{\partial x_i}\Big)|\varphi(z_n)|\frac{b(k)}{\alpha}\,dx = \varepsilon(n).$$

From (2.5) and (3.4), $(a(x,u_n,\nabla T_k(u_n)))_n$ converges weakly in $\prod_{i=1}^N L^{p'}(\Omega,w_i^*)$. Since $\nabla T_k(u)|\varphi(z_n)|\frac{b(k)}{\alpha} \to 0$ strongly in $\prod_{i=1}^N L^p(\Omega,w_i)$,

$$\int_\Omega \sum_{i=1}^N a_i(x,u_n,\nabla T_k(u_n))\frac{\partial T_k(u)}{\partial x_i})|\varphi(z_n)|\frac{b(k)}{\alpha}\,dx = \varepsilon(n).$$

Moreover, since $\int_{|u_n|\le k} b(k)c(x)|\varphi(z_n)|\,dx = \varepsilon(n)$, we have

$$\begin{aligned}
&|\int_{|u_n|\le k} g(x,u_n,\nabla u_n)\varphi(z_n)\,dx| \\
&\le \int_\Omega \sum_{i=1}^N \big[a_i(x,u_n,\nabla T_k(u_n)) - a_i(x,u_n,\nabla T_k(u))\big] \\
&\quad \times (\frac{\partial T_k(u_n)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i})|\varphi(z_n)|\frac{b(k)}{\alpha}\,dx + \varepsilon(n)
\end{aligned}$$

which with (3.16) and (3.17) gives

$$\begin{aligned}
&\int_\Omega \big[a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))\big] \\
&\qquad \times (\nabla T_k(u_n) - \nabla T_k(u))(\varphi'(z_n) - \frac{b(k)}{\alpha}|\varphi(z_n)|)\,dx \le \varepsilon(n).
\end{aligned}$$

Choosing, $\delta \ge (b(k)/(2\alpha))^2$, we obtain for all $s \in \mathbb{R}$

$$\varphi'(s) - \frac{b(k)}{\alpha}|\varphi(s)| \ge \frac{1}{2};$$

thus,

$$\frac{1}{2}\int_\Omega [a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))](\nabla T_k(u_n) - \nabla T_k(u))\,dx \le \varepsilon(n).$$

Then

$$\int_\Omega \big[a(x,u_n,\nabla T_k(u_n)) - a(x,u_n,\nabla T_k(u))\big](\nabla T_k(u_n) - \nabla T_k(u))\,dx \to 0$$

as $n \to \infty$. Moreover, $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega,w)$ and in view of Lemma 2.3, we have $T_k(u_n) \to T_k(u)$ as $n \to \infty$ strongly in $W_0^{1,p}(\Omega,w)$; hence,

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ strongly in } \prod_{i=1}^N L^p(\Omega,w_i).$$

Consequently, there exists a subsequence still denoted by $(u_n)_n$ such that, $\nabla u_n \to \nabla u$ a.e in $\Omega$.

**Proof of assertion 3** From (3.5) we deduce that

$$g(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{a.e in } \Omega. \tag{3.18}$$

For any measurable subset $E$ of $\Omega$ and any $m > 0$ we have

$$\begin{aligned}
&\int_E |g(x,u_n,\nabla u_n)|\,dx \\
&\le \int_{E\cap\{|u_n|\le m\}} |g(x,u_n,\nabla u_n)|\,dx + \int_{E\cap\{|u_n|>m\}} |g(x,u_n,\nabla u_n)|\,dx.
\end{aligned} \tag{3.19}$$

By (2.9), (3.5) and by using Vitali's theorem, we have for $\varepsilon > 0$ there exists $\rho(\varepsilon, m) > 0$ such that for $\rho(\varepsilon, m) > |E|$ we have

$$\int_{E\cap\{|u_n|\leq m\}} |g(x, u_n, \nabla u_n)|\, dx \leq \frac{\varepsilon}{2} \quad \forall n. \tag{3.20}$$

Now let $v_n = u_n - S_m(u_n)$ where for $m > 1$,

$$S_m(s) = \begin{cases} 0 & |s| \leq m-1 \\ \frac{s}{|s|} & |s| \geq m \\ S_m'(s) = 1 & m-1 \leq |s| \leq m. \end{cases}$$

Note that: If $u_n \leq m-1$, we have $S_m(u_n) \leq 0$ and $v_n \geq u_n \geq \psi$; if $u_n \geq m-1$, we have $0 \leq S_m(u_n) \leq 1$ and

$$u_n - S_m(u_n) \geq u_n - 1 \geq m - 2 \geq \psi \quad \text{for } m \geq 2 + ||\psi^+||_\infty.$$

Then, $v_n$ is admissible for (3.3). So, multiplying (3.3) by $v_n$ we obtain

$$\langle A(u_n), T_k(S_m(u_n))\rangle + \int_\Omega g(x, u_n, \nabla u_n) T_k(S_m(u_n))\, dx \leq \int_\Omega f_n T_k(S_m(u_n))\, dx.$$

Which by choosing $k \geq 1$ implies

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n s_m'(u_n)\, dx + \int_\Omega g(x, u_n, \nabla u_n) S_m(u_n)\, dx \leq \int_\Omega f_n S_m(u_n)\, dx,$$

i.e,

$$\int_{m-1\leq|u_n|\leq m} a(x, u_n, \nabla u_n)\nabla u_n\, dx + \int_{|u_n|>m-1} g(x, u_n, \nabla u_n) S_m(u_n)\, dx \leq \int_{|u_n|>m-1} |f_n|\, dx.$$

Since $a(x, u_n, \nabla u_n)\nabla u_n \geq 0$,

$$\int_{|u_n|>m-1} g(x, u_n, \nabla u_n) S_m(u_n)\, dx \leq \int_{|u_n|>m-1} |f_n|\, dx.$$

Since $S_m(u_n)$ and $u_n$ have the same sign,

$$\int_{|u_n|>m-1} |g(x, u_n, \nabla u_n)||S_m(u_n)|\, dx \leq \int_{|u_n|>m-1} |f_n|\, dx$$

and

$$\int_{|u_n|>m} |g(x, u_n, \nabla u_n)|\, dx \leq \int_{|u_n|>m-1} |f_n|\, dx.$$

Since $f_n \to f$ strongly in $L^1(\Omega)$ and since $|\{|u_n| > m-1\}| \to 0$ uniformly in $n$ when $m \to \infty$ (due to the fact that $\sigma^{1-q'} \in L^1(\Omega)$), there exists $m(\varepsilon) > 1$ such that

$$\int_{|u_n|>m-1} |f_n|\,dx \le \frac{\varepsilon}{2} \quad \forall n.$$

Then

$$\int_{|u_n|>m} |g(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2} \quad \forall n. \tag{3.21}$$

From (3.19), (3.20) and (3.21), we have

$$\int_E |g(x,u_n,\nabla u_n)|\,dx \le \varepsilon \quad \forall n. \tag{3.22}$$

Then $(g(x,u_n,\nabla u_n))_n$ is equi-integrable. Thanks to (3.18), (3.22) and Vitali's theorem yields,

$$g(x,u_n,\nabla u_n) \to g(x,u,\nabla u) \quad \text{strongly in } L^1(\Omega).$$

# 4 Example

The following example is closely inspired from the one used in [1,2]. Let $\Omega$ be a bounded domain of $\mathbb{R}^N\,(N \ge 1)$ satisfying the cone condition and let $\psi$ be a real valued measurable function such that $\psi^+ \in W_0^{1,p}(\Omega,w) \cap L^\infty(\Omega)$. Let us consider the Carathéodory functions

$$a_i(x,s,\xi) = w_i|\xi_i|^{p-1}\mathrm{sgn}(\xi) \quad \text{for } i = 0,\dots,N$$

and

$$g(x,s,\xi) = \rho s|s|^r \sum_{i=1}^N w_i|\xi_i|^p \quad \text{with} \quad \rho > 0,$$

where $w_i(x)$ are a given weight functions on $\Omega$ satisfying:

$$w_i(x) \equiv \text{some weight function } w(x) \text{ in } \Omega \text{ for all } i = 0,\dots,N.$$

Then, we consider the Hardy inequality (2.3) in the form,

$$\Big(\int_\Omega |u(x)|^q\sigma(x)\,dx\Big)^{1/q} \le c\Big(\int_\Omega |\nabla u(x)|^p w\Big)^{1/p}.$$

It is easy to show that the functions $a_i(x,s,\xi)$ satisfy the growth condition (2.5) and the coercivity (2.7). Also the Carathéodory function $g(x,s,\xi)$ satisfies the conditions (2.8), (2.9) and (2.10), in fact, concerning (2.10) we have,

$$|g(x,s,\xi)| = \rho|s|^{r+1}\sum_{i=1}^N w_i|\xi_i|^p.$$

Then

$$|g(x,s,\xi)| \geq \rho|\rho|^{r+1}\sum_{i=1}^{N} w_i|\xi_i|^p \quad \text{for } |s| > \rho_1 \geq 1.$$

Choosing for example $\rho_1 = 1$ and $\rho_2 = \rho > 0$. On the other hand, the monotonicity condition is satisfied. In fact,

$$\begin{aligned}&\sum_{i=1}^{N}(a_i(x,s,\xi) - a_i(x,s,\hat{\xi}))(\xi_i - \hat{\xi}_i)\\ &\qquad = w(x)\sum_{i=1}^{N}(|\xi_i|^{p-1}\operatorname{sgn}\xi_i - |\hat{\xi}_i|^{p-1}\operatorname{sgn}\hat{\xi}_i)(\xi_i - \hat{\xi}_i) > 0\end{aligned}$$

for almost all $x \in \Omega$ and for all $\xi, \hat{\xi} \in \mathbb{R}^N$ with $\xi \neq \hat{\xi}$, since $w > 0$ a.e. in $\Omega$. In particular, let us use the special weight functions $w$ and $\sigma$ expressed in terms of the distance to the boundary $\partial\Omega$. Denote $d(x) = \operatorname{dist}(x, \partial\Omega)$ and set

$$w(x) = d^\lambda(x), \quad \sigma(x) = d^\mu(x).$$

In this case, the Hardy inequality reads

$$\Big(\int_\Omega |u(x)|^q\, d^\mu(x)\, dx\Big)^{1/q} \leq c\Big(\int_\Omega |\nabla u(x)|^p\, d^\lambda(x)\, dx\Big)^{1/p}.$$

The corresponding imbedding is compact if:
(i) For, $1 < p \leq q < \infty$,

$$\lambda < p-1, \quad \frac{N}{q} - \frac{N}{p} + 1 \geq 0, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{N}{q} - \frac{N}{p} + 1 > 0, \tag{4.1}$$

(ii) For $1 \leq q < p < \infty$,

$$\lambda < p-1, \quad \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0, \tag{4.2}$$

(iii) For $q > 1$,

$$\frac{1}{q'-1} > \mu > -1. \tag{4.3}$$

**Corollary 4.1** *If $f \in L^1(\Omega)$, the following problem:*

$$\begin{gathered}\int_{|v-u|\leq k}\sum_{i=1}^{N} d^\lambda(x)|\frac{\partial u}{\partial x_i}|^{p-1}\operatorname{sgn}(\frac{\partial u}{\partial x_i})\frac{\partial(v-u)}{\partial x_i}\,dx\\ +\int_\Omega \rho u|u|^r\sum_{i=1}^{N} d^\lambda(x)|\frac{\partial u}{\partial x_i}|^p T_k(v-u)\,dx \geq \int_\Omega fT_k(v-u)\,dx.\\ u \in W_0^{1,p}(\Omega, d^\lambda), u \geq \psi, \quad \rho u|u|^r\sum_{i=1}^{N} d^\lambda(x)|\frac{\partial u}{\partial x_i}|^p \in L^1(\Omega)\\ v \in W_0^{1,p}(\Omega, d^\lambda), \quad \textit{for all } v \geq \psi \textit{ and all } k > 0,\end{gathered}$$

*has at least one solution.*

**Remarks**

1) Note that conditions (4.1) and (4.2) are sufficient to show the compact imbedding (2.4) (cf. [6, example 1], [8, example 1.5] and [10, theorem 19.17, 19.22]).

2) Condition (4.3) is sufficient for (2.3) to hold (cf. [9 p.p 40-41]).

# References

[1] Y. Akdim, *Sur certains problèmes quasi-linéaires et unilatéraux dégénérés ou singuliers,* Thèse de Doctorat, Université de Fès, (2002).

[2] Y. Akdim, E. Azroul and A. Benkirane, *Existence of solutions for quasilinear degenerated elliptic equations,* Electronic J. Diff. Eqns., vol. **2001** N 71 (2001) 1-19.

[3] Y. Akdim, E. Azroul and A. Benkirane, *Existence of solutions for quasilinear degenerated elliptic unilateral problem,* to appear in J. Math. Blaise-Pascal.

[4] A. Bensoussan, L. Boccardo and F. Murat, *On a non linear partial differential equation having natural growth terms and unbounded solution,* Ann. Inst. Henri Poincaré **5** $N4$ (1988), 347-364.

[5] A. Benkirane and A. Elmahi, *Strongly nonlinear elliptic unilateral problems having natural growth terms and $L^1$ data,* Rendiconti di Matematica, Serie VII vol. **18**, (1998), 289-303.

[6] P. Drabek, A. Kufner and V. Mustonen, *Pseudo-monotonicity and degenerated or singular elliptic operators,* Bull. Austral. Math. Soc. Vol. **58** (1998), 213-221.

[7] P. Drabek, A. Kufner and F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities,* De Gruyter Series in Nonlinear Analysis and Applications, Berlin New York, (1997).

[8] P. Drabek, A. Kufner and F. Nicolosi, *Nonlinear elliptic equations, singular and degenerate cases,* University of West Bohemia, (1996).

[9] A. Kufner, *Weighted Sobolev Spaces,* John Wiley and Sons, (1985).

[10] B. Opic and A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics Series **219** (Longman Scientific and Technical, Harlow, 1990).

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