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Strongly nonlinear elliptic problem without growth condition *

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Abstract

We study a boundary-value problem for the p-Laplacian with a nonlinear term. We assume only coercivity conditions on the potential and do not assume growth condition on the nonlinearity. The coercivity is obtained by using similar non-resonance conditions as those in [1].

1 Introduction

Consider the boundary-value problem

$$\begin{gathered} -\Delta_p u = f(x,u) + h \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $-\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is the p-Laplacian operator defined by

$$\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad 1 < p < \infty.$$

The p-Laplacian is a degenerated quasilinear elliptic operator that reduces to the classical Laplacian when $p = 2$. The notation $\langle .,. \rangle$ stands hereafter for the duality pairing between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. While $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $h \in W^{-1,p'}(\Omega)$.

Consider the energy functional $\Phi\colon W_0^{1,p}(\Omega) \to \overline{\mathbb{R}}$ associated with the problem

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \int_\Omega F(x,u)\, dx - \langle h, u\rangle,$$

where $F(x,s) = \int_0^s f(x,t)\, dt$. We are interested in conditions to be imposed on the nonlinearity f in order that problem (1.1) admits at least one solution $u(x)$ for any given h. Such conditions are usually called *non-resonance conditions.*

When the nonlinearity satisfies a growth condition of the type

$$|f(x,s)| \le a|s|^{q-1} + b(x) \quad \text{for all } s \in \mathbb{R}, \text{ and a.e. in } \Omega, \tag{1.2}$$

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with $q < p^*$ where the Sobolev exponent $p^* = \frac{Np}{N-p}$ when $p < N$ and $p^* = +\infty$ when $p \geq N$ and $b(x) \in L^{(p^*)'}(\Omega)$, the functional Φ is well defined and is of class $\mathcal{C}^1$, l.s.c. and its critical points are weak solutions of (1.1) in the usual sense.

However, when this growth condition is not satisfied, Φ is not necessarily of class $\mathcal{C}^1$ on $W_0^{1,p}(\Omega)$ and may take infinite values. The first eigenvalue of the p-Laplacian characterized by the variational formulation

$$\lambda_1 = \lambda_1(-\Delta_p) = \min\left\{\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}\,dx;\ u \in W_0^{1,p}(\Omega) \setminus \{0\}\right\}$$

is known to be associated to a simple eigenfunction that does not change sign [4].

A procedure used to treat (1.1) when the nonlinearity lies asymptotically on the left of λ_1 consists in supposing a "coercivity" condition on F of the type

$$\limsup_{s\to\pm\infty} \frac{pF(x,s)}{|s|^p} < \lambda_1 \quad \text{for almost every } x \in \Omega \tag{1.3}$$

and minimizing Φ on $W_0^{1,p}(\Omega)$. The minimum being a weak solution of (1.1) in an appropriate sense [1, 2, 3]. Another way is to obtain *a priori* estimates on the solutions of some equations approximating (1.1) and to show that their weak limit is indeed a weak solution.

Note that with the help of the conditions (1.2) and (1.3), we know since the work of Hammerstein (1930) that (1.1) admits a weak solution that minimizes the functional Φ on $W_0^{1,p}(\Omega)$. The condition (1.3) does not imply a growth condition on f unless $f(x,u)$ is convex in u (see for example [5]).

In [1], Anane and Gossez supposed only a one-sided growth condition with respect to the Sobolev (conjugate) exponent that do not suffice to guarantee the differentiability of Φ, which may even take infinite values. Nevertheless, they showed that any minimum of Φ solves (1.1) in a suitable sense.

Here, we assume $1 < p < \infty$ and only that f maps $L^\infty(\Omega)$ into $L^1(\Omega)$; i.e.,

$$\sup_{|s|\leq R} |f(.,s)| \in L^1_{\rm loc}(\Omega), \quad \forall R > 0 \tag{1.4}$$

and a coercivity condition of the type (1.3). We prove that any minimum u of Φ, which is not of class $\mathcal{C}^1$ on $W_0^{1,p}(\Omega)$ and may take infinite values too, is a weak solution of (1.1) in the sense

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \int_\Omega f(x,u)v\,dx + \langle h, v\rangle,$$

for v in a dense subspace of $W_0^{1,p}(\Omega)$. This result is proved by Degiovanni-Zani [2] in the case $p = 2$.

In the autonomous case $f(x,s) = f(s)$, De Figueiredo and Gossez [6] have proved the existence of solutions for any $h \in L^\infty(\Omega)$ by a topological method. They supposed only a coercivity condition and established that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,dx = \int_\Omega f(x,u)v\,dx + \langle h, v\rangle$$

for all $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \cup \{u\}$ but the solution obtained may not minimize Φ. Indeed, an example is given in [6] in the case $p = 2$ and an other one is given in [3] where p may be different from 2.

Note that in our case, the condition (1.4) implies no growth condition on f as it may be seen in the following example.

Example Consider the function

$$f(x,s) = \begin{cases} d(x)\Big(\sin(\frac{\pi s}{2}) - \frac{\operatorname{sign}(s)}{2}\Big)\exp\Big(\frac{2\cos(\frac{\pi s}{2})}{\pi} + \frac{|s|-1}{2}\Big) & \text{if } |s| \geq 1 \\ d(x)\frac{s}{2}(10s^2 - 9) & \text{if } |s| \leq 1\,, \end{cases}$$

where $d(x) \in L^1_{\rm loc}(\Omega)$ and $d(x) \geq 0$ almost everywhere in Ω, so that

$$F(x,s) = \begin{cases} -d(x)\exp\Big(\frac{2\cos(\frac{\pi s}{2})}{\pi}\Big)\exp\Big(\frac{|s|-1}{2}\Big) & \text{if } |s| \geq 1 \\ -d(x)\frac{s^2}{4}(-5s^2 + 9) & \text{if } |s| \leq 1\,. \end{cases}$$

Then $F(x,s) \leq 0$ for all $s \in \mathbb{R}$ almost everywhere in Ω. So, Φ is coercive. Nevertheless, as we can check easily, f satisfies no growth condition.

2 Theoretical approach

We will show that when (1.4) is fulfilled, any minimum u of ϕ is a weak solution of (1.1) in an acceptable sense.

Definition The space $L_0^\infty(\Omega)$ is defined by

$$L_0^\infty(\Omega) = \big\{v \in L^\infty(\Omega);\ v(x) = 0 \text{ a.e. outside a compact subset of } \Omega\big\}.$$

For $u \in W_0^{1,p}(\Omega)$, we set

$$V_u = \big\{v \in W_0^{1,p}(\Omega) \cap L_0^\infty(\Omega);\ u \in L^\infty(\{x \in \Omega;\ v(x) \neq 0\})\big\}.$$

Proposition 2.1 (Brezis-Browder [7]) *If $u \in W_0^{1,p}(\Omega)$, there exists a sequence $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that:*

(i) $(u_n)_n \subset W_0^{1,p}(\Omega) \cap L_0^\infty(\Omega)$.

(ii) $|u_n(x)| \leq |u(x)|$ *and* $u_n(x).u(x) \geq 0$ *a.e. in* Ω.

(iii) $u_n \to u$ *in* $W_0^{1,p}(\Omega)$, *as* $n \to \infty$.

The linear space V_u enjoys some nice properties.

Proposition 2.2 *The space V_u is dense in $W_0^{1,p}(\Omega)$. And if we assume that* (1.4) *holds, then*

$$A_u = \big\{\varphi \in W_0^{1,p}(\Omega);\ f(x,u)\varphi \in L^1(\Omega)\big\}$$

is a dense subspace of $W_0^{1,p}(\Omega)$ as $V_u \subset A_u$. More precisely, Brezis-Browder's result holds true if we replace $W_0^{1,p}(\Omega) \cap L_0^\infty(\Omega)$ by V_u.

Proof It suffices to show that V_u is dense in $W_0^{1,p}(\Omega)$ and that $V_u \subset A_u$ when (1.4) holds.
The density of V_u in $W_0^{1,p}(\Omega)$: We have to show that for any $\varphi \in W_0^{1,p}(\Omega)$, there exists a sequence $(\varphi_n)_n \subset V_u$ satisfying (ii) and (iii). This is done in two steps. First, we show it is true for all $\varphi \in W_0^{1,p}(\Omega) \cap L_0^\infty(\Omega)$. Then, using Proposition 2.1, we show it is true in $W_0^{1,p}(\Omega)$.
First Step: Suppose $\varphi \in W_0^{1,p}(\Omega) \cap L_0^\infty(\Omega)$ and consider a sequence $(\Theta_n)_n \subset \mathcal{C}_0^\infty(\mathbb{R})$ such that:
(1) $\operatorname{supp}\Theta_n \subset [-n, n]$,
(2) $\Theta_n \equiv 1$ on $[-n+1, n-1]$,
(3) $0 \le \Theta_n \le 1$ on $\mathbb{R}$ and
(4) $|\Theta_n'(s)| \le 2$.
The sequence we are looking for is obtained by setting

$$\varphi_n(x) = (\Theta_n \circ u)(x)\varphi(x) \quad \text{for a.e. } x \text{ in } \Omega.$$

Indeed, let's check the following three statements
(a) $\varphi_n \in V_u$,
(b) $|\varphi_n(x)| \le |\varphi(x)|$ and $\varphi_n(x)\varphi(x) \ge 0$ a.e. in Ω and
(c) $\varphi_n \to \varphi$ in $W_0^{1,p}(\Omega)$.
For (a), since $\varphi \in L_0^\infty(\Omega)$, we have that $\varphi_n \in L_0^\infty(\Omega)$ and it's clear by (4) that $\varphi_n \in W_0^{1,p}(\Omega)$. Finally, by (1), $u(x) \in [-n, n]$ for a.e. x in $\{x \in \Omega;\ \varphi_n(x) \neq 0\}$. The assumption (b) is a consequence of (3). For (c), by (2), $\varphi_n(x) \to \varphi(x)$ a.e. in Ω and

$$\frac{\partial \varphi_n}{\partial x_i}(x) = \Theta_n'(u(x))\frac{\partial u}{x_i}\varphi(x) + \Theta_n(u(x))\frac{\partial \varphi}{\partial x_i} \ \to\ \frac{\partial \varphi}{\partial x_i} \text{ in } \Omega.$$

And by (4),

$$\Big|\frac{\partial \varphi_n}{\partial x_i}(x)\Big| \le 2\Big|\frac{\partial u}{\partial x_i}(x)\Big||\varphi(x)| + \Big|\frac{\partial \varphi}{\partial x_i}(x)\Big| \in L^p(\Omega).$$

Finally, by the dominated convergence theorem we get (c).
Second Step: Suppose that $\varphi \in W_0^{1,p}(\Omega)$. By Proposition 2.1, there is a sequence $(\psi_n)_n \subset W_0^{1,p}(\Omega)$ satisfying (i), (ii) and (iii).
For $k = 1, 2, \ldots$, there is $n_k \in \mathbb{N}$ such that $||\psi_{n_k} - \varphi||_{1,p} \le 1/k$. Since $\psi_{n_k} \in W_0^{1,p}(\Omega) \cap L_0^\infty(\Omega)$, by the first step, there is $\varphi_k \in V_u$ such that $|\varphi_k(x)| \le |\psi_{n_k}(x)|$ and $\varphi_k(x)\psi_{n_k}(x) \ge 0$ almost everywhere in Ω and $||\varphi_k - \psi_{n_k}||_{1,p} \le 1/k$, so that $(\varphi_k)_k$ is the sequence we are seeking. Indeed, $|\varphi_k(x)| \le |\psi_{n_k}(x)| \le |\varphi(x)|$, $\varphi_k(x)\varphi(x) \ge 0$ a.e. in Ω and $||\varphi_k - \varphi(x)||_{1,p} \le ||\varphi_k - \psi_{n_k}||_{1,p} + ||\psi_{n_k} - \varphi(x)||_{1,p} \le 2/k$.
The inclusion $V_u \subset A_u$: Indeed, for $\varphi \in V_u$, set $E = \big\{x \in \Omega;\ \varphi(x) \neq 0\big\}$ so that

$$\begin{aligned} |f(x,u)\varphi| &= |f(x,u)\chi_E \varphi(x)| \\ &\le \max\big\{|f(x,s)\varphi(x)|; |s| \le ||u||_{L^\infty(E)}\big\} \end{aligned}$$

where χ_E is the characteristic function of the set E. By (1.4), the last term lies to $L^1(\Omega)$, so that $\varphi \in A_u$. $\square$

Theorem 2.3 *Assume* (1.4). *If* $u \in W_0^{1,p}(\Omega)$ *is a minimum of* Φ *such that* $F(x,u) \in L^1(\Omega)$, *then*

(i) $\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi\,dx = \int_\Omega f(x,u)\phi\,dx + \langle h,\phi\rangle$ *for all* $\phi \in A_u$.

(ii) $f(x,u) \in W^{-1,p'}(\Omega)$ *in the sense that the mapping* $T : V_u \to \mathbb{R} : T(\phi) = \int_\Omega f(x,u)\phi\,dx$ *is linear, continuous and admits an unique extension* $\tilde{T}$ *to the whole space* $W_0^{1,p}(\Omega)$.

(iii) $\langle f(x,u),\phi\rangle = \int_\Omega f(x,u)\phi\,dx \quad \forall\phi \in A_u$.

(iv) $-\Delta_p u = f(x,u) + h$ *in* $W^{-1,p'}(\Omega)$.

Remark There are in In [1] some conditions that guarantee the existence of a minimum u of Φ in $W_0^{1,p}(\Omega)$ and consequently $F(x,u) \in L^1(\Omega)$.

Proof of Theorem 2.3 We will prove that the assertion (i) holds for all $\phi \in V_u$ as a first step, then prove (iii), (iv) and (i). Let $\phi \in V_u$ and $s \in \mathbb{R}$ such that $0 < s < 1$. There exists $\beta = \beta(x,s,\phi,u) \in [-1,1]$ such that

$$\begin{aligned}\Big|\frac{F(x,u+s\phi)-F(x,u)}{s}\Big| &= |f(x,u+\beta\phi)\phi| \\ &\le \max\big\{|f(x,t)\phi(x)|;\ |t| \le \|u\|_{L^\infty(E)} + \|\phi\|_{L^\infty(\Omega)}\big\},\end{aligned}$$

where $E = \{x \in \Omega;\ \phi(x) \neq 0 \ \text{ a.e. }\}$. Since $F(x,u) \in L^1(\Omega)$, by (1.4), we have $F(x,u+s\phi) \in L^1(\Omega)$ for all $0 < s < 1$. On the other hand

$$\lim_{s\to 0}\frac{F(x,u(x)+s\phi(x)) - F(x,u(x))}{s} = f(x,u(x))\phi \quad \text{a.e. in } \Omega.$$

It follows from Lebesgue's dominated convergence that

$$\lim_{s\to 0}\frac{F(x,u+s\phi)-F(x,u)}{s} = f(x,u)\phi \quad \text{strongly in } L^1(\Omega).$$

Since $u \in W_0^{1,p}(\Omega)$ is a minimum point of Φ, we get

$$\frac{\Phi(u+s\phi)-\Phi(u)}{s} \ge 0 \quad \text{for all } 0 < s < 1,$$

then, we get (i) for all $\phi \in V_u$.

The linear mapping defined by $T(\phi) = \int_\Omega f(x,u)\phi$ is continuous, because for all $\phi \in V_u$,

$$|T(\phi)| = \Big|\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi - \langle h,\phi\rangle\Big| \le \big(\|u\|_{1,p}^{p/p'} + \|h\|_{W^{-1,p'}(\Omega)}\big)\|\phi\|_{1,p}.$$

By Proposition 2.2, T admits an unique extension $\tilde{T}$ to the whole space $W_0^{1,p}(\Omega)$. Henceforth, we will make the identification $f(x,u) = \tilde{T}$. Since

$$\langle -\Delta_p u,\phi\rangle = \langle f(x,u),\phi\rangle - \langle h,\phi\rangle \ \ \forall\phi \in V_u,$$

we conclude **(iv)**. Let $\phi \in W_0^{1,p}(\Omega)$ such that $f(x,u)\phi \in L^1(\Omega)$, i.e. $\phi \in A_u$. By Proposition 2.2 there exists $(\phi_n) \subset V_u$. We can suppose that $\phi_n \to \phi$ almost everywhere, $|f(x,u)\phi_n| \leq |f(x,u)\phi|$ and $f(x,u)\phi_n \to f(x,u)\phi$ a.e.. By the dominated convergence theorem,

$$f(x,u)\phi_n \to f(x,u)\phi \quad \text{in } L^1(\Omega).$$

Since $\langle f(x,u), \phi_n\rangle = \int_\Omega f(x,u)\phi_n$ for all $n \in \mathbb{N}$ and $f(x,u) \in W^{-1,p'}(\Omega)$ we get (iii). Finally, (i) is an immediate consequence of (iii) and (iv). $\square$

3 Description of the space A_u

Now, we will see some condition that guarantee some properties of A_u.

Proposition 3.1 *Assume* (1.4)*. Let u be a minimum of Φ in $W_0^{1,p}(\Omega)$ with $F(x,u) \in L^1(\Omega)$. And let $\phi \in W_0^{1,p}(\Omega)$, $v \in L^1(\Omega)$ such that $f(x,u(x))\phi(x) \geq v(x)$ or $f(x,u(x))\phi(x) \leq v(x)$ a.e. in Ω, then $\phi \in A_u$.*

Proof Suppose $f(x,u(x))\phi(x) \geq v(x)$ a.e. in Ω (the same argument works if $f(x,u(x))\phi(x) \leq v(x)$ a.e. in Ω). By Proposition 2.2, there exists $(\phi_n) \subset V_u$ such that $\phi_n \to \phi$ in $W_0^{1,p}(\Omega)$, $|\phi_n| \leq |\phi|$ and $\phi_n(x)\phi(x) \geq 0$ a.e. in Ω. We have

$$\begin{aligned} f(x,u(x))\phi_n(x) &= f^+(x,u(x))\phi_n(x) - f^-(x,u(x))\phi_n(x) \\ &\geq -f^+(x,u(x))\phi^-(x) - f^-(x,u(x))\phi^+(x) \\ &\geq -v^-(x). \end{aligned}$$

By Fatou lemma, we have

$$\begin{aligned} -\infty < \int_\Omega f(x,u(x))\phi(x) &\leq \liminf_n \int_\Omega f(x,u(x))\phi_n(x) \\ &= \liminf_n \langle f(x,u), \phi_n\rangle < +\infty, \end{aligned}$$

which implies $f(x,u)\phi \in L^1(\Omega)$, i.e. $u \in A_u$. $\square$

Corollary 3.2 *If η, η_1 and η_2 in $L^1_{\mathrm{loc}}(\Omega)$, such that one of the following conditions is satisfied:*

(1) $f(x,u(x)) \geq \eta(x)$ *a.e. in* Ω

(2) $f(x,u(x)) \leq \eta(x)$ *a.e. in* Ω

(3) $f(x,u(x)) \leq \eta_1(x)$ *a.e. in* $\{x \in \Omega;\ u(x) < 0\}$ *and* $f(x,u(x)) \geq \eta_2(x)$ *a.e. in* $\{x \in \Omega;\ u(x) > 0\}$,

(4) $f(x,u(x)) \geq \eta_1(x)$ *a.e. in* $\{x \in \Omega;\ u(x) < 0\}$ *and* $f(x,u(x)) \leq \eta_2(x)$ *a.e. in* $\{x \in \Omega;\ u(x) > 0\}$.

Then $f(x,u) \in L^1_{\mathrm{loc}}(\Omega)$ and consequently $L^\infty_c(\Omega) \cap W_0^{1,p}(\Omega) \subset A_u$.

Proof Assume (3) (the same argument works for (4)). Let $\phi \in C_c^\infty(\Omega)$. We set $\Omega_1 = \{x \in \Omega;\ u(x) \leq -1 \text{ a.e.}\}$, $\Omega_2 = \{x \in \Omega;\ |u(x)| \leq 1 \text{ a.e.}\}$ and $\Omega_3 = \{x \in \Omega;\ u(x) \geq 1 \text{ a.e.}\}$. It suffices to prove that $f(x,u)|\phi|\chi_{\Omega_i} \in L^1(\Omega)$ for $i = 1, 2, 3$. By (1.4) we have $f(x,u)\phi\chi_{\Omega_2} \in L^1(\Omega)$. Let $\theta \in C^\infty(\mathbb{R})$:

$$\theta(s) = \begin{cases} 1 & \text{if } s \geq 1, \\ 0 \leq \theta(s) \leq 1 & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s \leq 0. \end{cases}$$

It is clear that $(\theta \circ u)|\phi| \in W_0^{1,p}(\Omega)$ and that

$$f(x,u(x))(\theta \circ u(x))|\phi(x)| \geq (\theta \circ u(x))|\phi(x)|\eta_2(x) \in L^1(\Omega).$$

By Proposition 3.1, we have $f(x,u)(\theta \circ u)|\phi| \in L^1(\Omega)$, then $f(x,u)\phi\chi_{\Omega_3} \in L^1(\Omega)$ (the same argument to prove $f(x,u)\phi\chi_{\Omega_1} \in L^1(\Omega)$). We conclude that $f(x,u)\phi \in L^1(\Omega)$ for all $\phi \in C_c^\infty(\Omega)$, which implies $f(x,u) \in L^1_{\text{loc}}(\Omega)$.

Now assume (1) (the same argument works for (2)). For all $\phi \in C_c^\infty(\Omega)$ we have $f(x,u)|\phi| \geq \eta(x)|\phi| \in L^1(\Omega)$, then $f(x,u)|\phi| \in L^1(\Omega)$; therefore, $f(x,u)\phi \in L^1(\Omega)$. Then we conclude that $f(x,u) \in L^1_{\text{loc}}(\Omega)$. $\square$

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