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Singular p-harmonic functions and related quasilinear equations on manifolds *

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Abstract

We give here an overview of some recent developments in the study of the description of singular solutions of

$$-\nabla.(|\nabla u|^{p-2}\nabla u)+\varepsilon|u|^{q-1}u=0$$

in $\mathbb{R}^N\setminus\{0\}$, where $p>1$, $\varepsilon\in\{0,1,-1\}$ and $q\geq p-1$.

1 Introduction

Let Ω be a domain in $\mathbb{R}^N$ containing 0, $N\geq 2$, and let

$$A:\Omega\times\mathbb{R}\times\mathbb{R}^N\mapsto\mathbb{R}^N,\quad\text{and}\quad B:\Omega\times\mathbb{R}\times\mathbb{R}^N\mapsto\mathbb{R},$$

be two Caratheodory functions. Then a classical problem is the study of the behaviour near 0 of a solution u of

$$-\nabla.A(x,u,\nabla u)+B(x,u,\nabla u)=0 \tag{1.1}$$

in $\Omega^*=\Omega\setminus\{0\}$. Besides the well known linear case, the first striking results in the nonlinear case were obtained by Serrin in 1964 in a series of celebrated articles [11, 12]. Under the assumptions

$$\begin{aligned}&(i) && A(x,r,Q).Q\geq c_1|Q|^p\\ &(ii) && |A(x,r,Q)|\leq c_2|Q|^{p-1}+c_3\\ &(iii) && |B(x,r,Q)|\leq c_4|Q|^{p-1}+c_5|r|^{p-1}+c_6\end{aligned} \tag{1.2}$$

for any $(x,r,Q)\in\Omega\times\mathbb{R}\times\mathbb{R}^N\mapsto\mathbb{R}^N$, where the c_i are positive constants and $N\geq p>1$. Serrin's results assert that any nonnegative weak solution u of (1.1) in Ω^* belonging to $W^{1,p}_{\mathrm{loc}}(\Omega^*)$ is either extendable by continuity as a $C(\Omega)\cap W^{1,p}_{\mathrm{loc}}(\Omega)$-solution of the same equation in whole Ω, or satisfies

$$\theta\leq\frac{u(x)}{\mu_p(x)}\leq\theta^{-1}, \tag{1.3}$$

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near 0, for some positive θ, in which formula the functions μ_p are defined in $\mathbb{R}^N \setminus \{0\}$ by

$$\mu_p(x) = \begin{cases} |x|^{(p-N)/(p-1)} & \text{if } 1 < p < N, \\ \ln(1/|x|) & \text{if } p = N. \end{cases} \tag{1.4}$$

A series of extensions were obtained in the eighties in the case

$$A(x, r, Q) = |Q|^{p-2}Q,$$

where the diffusion operator $\nabla.A(x, u, \nabla u)$ is called the p-Laplace: by Kichenassamy and Véron [9] in the case $B(x, r, Q) \equiv 0$; Vazquez and Véron [17], Friedman and Véron [5] in the case $B(x, r, Q) = |r|^{q-1}r$ with $q > p-1$; Guedda and Véron [7], Bidaut-Véron [1], Serrin and Zou [13] in the case $B(x, r, Q) = -|r|^{q-1}r$, always in assuming $q > p - 1$. We shall present below an overview or the results of these different authors, writing the equation (1.1) in the form

$$-\nabla.(|\nabla u|^{p-2}\nabla u) + \varepsilon|u|^{q-1}u = 0, \tag{1.5}$$

with $\varepsilon = 1, -1$ or 0. We put emphasis on separable solutions that are solutions of the form

$$u(r, \sigma) = r^{-\beta}\omega(\sigma), \quad (r, \sigma) \in (0, \infty) \times S^{N-1}.$$

Thus $\beta = \beta_q = p/(q + 1 - p)$ and the relation

$$\begin{aligned} &-\nabla_\sigma. \left((\omega^2 + |\nabla_\sigma\omega|^2)^{p/2-1}\nabla_\sigma\omega\right) + \varepsilon|\omega|^{q-1}\omega \\ &= \beta_q((\beta_q + 1)(p - 1) + 1 - N)(\omega^2 + |\nabla_\sigma\omega|^2)^{p/2-1}\omega, \end{aligned}$$

holds on S^{N-1}. This equation is not the usual Euler equation of a functional, which makes it more difficult study. However, we give a few results of existence and uniqueness of solutions.

2 Singular p-harmonic functions

By looking for radial solutions of the p-Laplace equation

$$-\nabla.(|\nabla u|^{p-2}\nabla u) = 0, \tag{2.1}$$

in $\mathbb{R}^N \setminus \{(0)\}$, we find that the only solutions are the functions

$$u = C_1\mu_p + C_2$$

where the C_i are arbitrary constants.The first result obtained by Kichenassamy and Véron in [9] pointed out that any nonnegative singular p-hamonic functions is asymptotically radial near its singularities. They proved the following result.

Theorem 2.1 *Assume $1 < p \leq N$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega^*)$ is nonnegative and satisfies (2.1) in Ω^*. Then there exists $\gamma \in \mathbb{R}_+$ such that*

$$u - \gamma\mu_p \in L^\infty_{\mathrm{loc}}(\Omega). \tag{2.2}$$

Moreover

$$\lim_{x\to 0} |x|^{(N-1)/(p-1)}\nabla(u - \gamma\mu_p)(x) = 0, \tag{2.3}$$

and the following equation holds in the sense of distributions in Ω

$$-\nabla.(|\nabla u|^{p-2}\nabla u) = c_{N,p}\gamma^{p-1}\delta_0, \tag{2.4}$$

for some positive constant $c_{N,p}$.

The proof is based on the a priori estimate

$$u(x) \leq C\mu_p(x)$$

for $0 < |x| \leq R$, for some $C > 0$ and $R > 0$ (this follows from Serrin's result), the scaling transformation

$$T_r(u)(\xi) = u(r\xi)/\mu(r)$$

and a version of the strong maximum principle which was first noticed by Tolksdorff [14]. Actually, the positivity assumption can be relaxed and replaced by

$$u/\mu_p \in L^\infty(B_R), \tag{2.5}$$

since Serrin's result asserts that any nonnegative singular p-harmonic function does satisfy this estimate. As a consequence, existence and uniqueness of a solution to the singular Dirichlet problem

$$\begin{gathered} -\nabla.(|\nabla u|^{p-2}\nabla u) = c_{N,p}|\gamma|^{p-2}\gamma\delta_0, \quad \text{in } \mathcal{D}'(\Omega), \\ u = g, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.6}$$

can be proved.

Corollary 2.2 *Assume $1 < p \leq N$, Ω is bounded with a C^2 boundary, $g \in L^\infty(\partial\Omega) \cap W^{1-1/p,p}(\partial\Omega)$ and $\gamma \in \mathbb{R}$. Then there exists a unique $u \in C^1(\Omega^*)$ such that $|\nabla u|^{p-1} \in L^1(\Omega)$ satisfying (2.6) and (2.5). Moreover (2.2) and (2.3) hold.*

Another consequence is the following singular Liouville type result.

Corollary 2.3 *Assume $1 < p \leq N$, and $u \in C^1(\mathbb{R}^N \setminus \{0\})$ is p-harmonic in $\mathbb{R}^N \setminus \{0\}$ and satisfies $|u(x)| \leq a|\mu_p(x)| + b$, for some positive constants a and b. Then there exist two real numbers α and β such that*

$$u = \alpha\mu_p + \beta.$$

If we look for singular p-harmonic functions u in $\mathbb{R}^N \setminus \{0\}$ under the form

$$u(x) = |x|^{-\beta}\omega(x/|x|) = r^{-\beta}\omega(\sigma), \tag{2.7}$$

where $(r, \sigma) \in (0, \infty) \times S^{N-1}$ are the spherical coordinates, then

$$-\nabla_\sigma . \left((\beta^2\omega^2 + |\nabla_\sigma\omega|^2)^{(p-2)/2}\nabla_\sigma\omega\right) = \lambda(\beta^2\omega^2 + |\nabla_\sigma\omega|^2)^{(p-2)/2}\omega, \tag{2.8}$$

where $\nabla_\sigma .$ is the divergence operator acting on C^1 vector fields on the unit (N-1)-sphere S^{N-1} and ∇_σ is the tangential gradient, identified with the covariant derivative on S^{N-1} for the Riemannian structure induced by the imbedding of S^{N-1} into $\mathbb{R}^N$, and

$$\lambda = \beta((\beta + 1)(p - 1) + 1 - N).$$

When $N = 2$ and $\omega(x/|x|) = \omega(\varphi)$ is a 2π- periodic function, equation (2.7) becomes

$$\left((\beta^2\omega^2 + \omega_\varphi^2)^{(p-2)/2}\omega_\varphi\right)_\varphi + ((\beta+1)(p-1)-1)\beta(\beta^2\omega^2 + \omega_\varphi^2)^{(p-2)/2}\omega = 0. \tag{2.9}$$

Putting $Y = \omega_\varphi/\omega$, and $\beta_0 = (2 - p)/(p - 1)$ yields to

$$\Big(\frac{\beta}{Y^2 + \beta^2} - \frac{\beta + 1}{Y^2 + \beta(\beta - \beta_0)}\Big)Y_\varphi = 1.$$

This equation is completely integrable [9], and the following result is proved.

Theorem 2.4 *Assume $p > 1$, then for each positive integer k there exist a β_k and $\omega_k : \mathbb{R} \mapsto \mathbb{R}$ with least period $2\pi/k$, of class C^∞ such that*

$$u(x) = |x|^{-\beta_k}\omega_k(x/|x|), \tag{2.10}$$

is p-harmonic in $\mathbb{R}^2 \setminus \{0\}$; β_k is the positive root of

$$(\beta + 1)^2 = (1 + 1/k)^2\left(\beta^2 + \beta(p - 2)/(p - 1)\right). \tag{2.11}$$

The couple (β_k, ω_k) is unique, up to translation and homothety over ω_k.

In the case of regular p-harmonic functions in the plane, which means that the exponent $\beta = -\tilde{\beta}$ in (2.7) is negative, the stationary equation becomes

$$\left((\tilde{\beta}^2\tilde{\omega}^2 + \tilde{\omega}_\varphi^2)^{(p-2)/2}\tilde{\omega}_\varphi\right)_\varphi + ((\tilde{\beta}-1)(p-1)-1)\tilde{\beta}(\tilde{\beta}^2\tilde{\omega}^2 + \tilde{\omega}_\varphi^2)^{(p-2)/2}\tilde{\omega} = 0. \tag{2.12}$$

Kroll and Mazja [8] obtained the complete set of solutions of (2.12):

Theorem 2.5 *For each positive integer k there exists a couple $(\tilde{\beta}_k, \tilde{\omega}_k)$, unique up to translation and homothety over $\tilde{\omega}_k$ such that*

$$x \mapsto u(x) = |x|^{\tilde{\beta}_k}\tilde{\omega}_k(x/|x|), \tag{2.13}$$

is p-harmonic in $\mathbb{R}^2$. The exponent $\tilde{\beta}_k$ is the root larger than 1 of the algebraic equation

$$(\tilde{\beta} - 1)^2 = (1 - 1/k)^2\left(\tilde{\beta}^2 - \tilde{\beta}(p - 2)/(p - 1)\right). \tag{2.14}$$

The derivation of regular or singular p-harmonic functions follows in higher dimension under a splitted form. For example, if $N = 3$ with (x_1, x_2, x_3) the canonical coordinates in $\mathbb{R}^3$, we put

$$x_1 = r\cos\varphi\sin\theta, \quad x_2 = r\sin\varphi\sin\theta, \quad x_3 = r\cos\theta,$$

where $r > 0$, $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$. Equation (2.8) takes the form

$$\begin{aligned} &-\frac{\partial}{\partial\theta}\left(\sin\theta\left(\beta^2\omega^2 + \omega_\theta^2 + \sin^{-2}\theta\,\omega_\varphi^2\right)^{(p-2)/2}\omega_\theta\right) \\ &-\frac{\partial}{\partial\varphi}\left(\sin^{-1}\theta\left(\beta^2\omega^2 + \omega_\theta^2 + \sin^{-2}\theta\,\omega_\varphi^2\right)^{(p-2)/2}\omega_\varphi\right) \\ &\quad=\beta(\beta(p-1)+p-3)\sin\theta\left(\beta^2\omega^2 + \omega_\theta^2 + \sin^{-2}\theta\,\omega_\varphi^2\right)^{(p-2)/2}\omega. \end{aligned} \tag{2.15}$$

We set

$$\omega(\varphi,\theta) = \sin^{-\beta}\theta\; v(\varphi) = \sin^{\tilde\beta}\theta\; v(\varphi),$$

then v satisfies (2.12). Thanks to Theorem 2.5 the set of singular (resp. regular) p-harmonic functions under the form

$$u(r,\varphi,\theta) = r^{-\beta}\sin^{-\beta}\theta\; v(\varphi),$$

resp.

$$u(r,\varphi,\theta) = r^{\tilde\beta}\sin^{\tilde\beta}\theta\; v(\varphi),$$

is explicitly known. Another way for constructing non-isotropic singular p-harmonic functions is to use Tolksdorf's shooting method [14].

Theorem 2.6 *Let $S \subset S^{N-1}$ be a connected and open, with a C^2 relative boundary ∂S. Then there exist a unique couple (β,ω), with $\beta > 0$, $\omega \in C^1(S)$, $\omega > 0$ in S, vanishing on ∂S, with maximal value 1 such that the function u defined by (2.7) is p-harmonic in $\mathbb{R}^N \setminus \{0\}$.*

Proof Put $K_S(R,R') = \{(r,\sigma) : \sigma \in S,\ R < r < R'\}$ and $B_S(R,R') = \{(r,\sigma) : \sigma \in \partial S,\ R < r < R'\}$. Let g be defined by

$$g(x) = \begin{cases} 2 - |x| & \text{if } |x| \le 2, \\ 0 & \text{if } |x| \ge 2. \end{cases}$$

For $n \ge 2$ we denote by u_n the unique solution of

$$\begin{aligned} -\nabla.(|\nabla u_n|^{p-2}\nabla u_n) = 0 \quad &\text{in } K_S(1,n), \\ u_n = g \quad &\text{on } B_S(1,n). \end{aligned}$$

Since Hopf maximum principle holds [14], u_n is positive in $K_S(1,n)$. The sequence $\{u_n\}$ is increasing and locally bounded in the $C^{1,\alpha}_{\text{loc}}$ topology of $\overline{K_S(1,\infty)}$.

Thus it converges in $C^1_{\mathrm{loc}}(\overline{K_S(1,\infty)})$ to some u which is positive and satisfies

$$\begin{gathered} -\nabla.(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in } K_S(1,\infty), \\ u = g \quad \text{on } B_S(1,\infty), \\ \lim_{|x|\to\infty} u(x) = 0. \end{gathered} \tag{2.16}$$

The function

$$R \mapsto C(R) = \sup_{x\in K_S(1,\infty)} u(x)$$

is decreasing and the supremum is achieved for $|x| = R$. One of the key idea is called the equivalence principle [14, Lemma 2.1], Lemma 2.1, which asserts that

$$u(Rx) \leq (1 - \varepsilon(R-1))u(x), \tag{2.17}$$

for some $\epsilon > 0$ and any $R \in (1,2)$. Thus there exists $k > 0$ such that $C(R) \leq kC(2R)$ for any $R \geq 3$. Then

$$|\nabla u(x)| \leq C(|x|)|x|^{-1}, \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \leq C(|x|)|x|^{-1-\alpha}|x-x'|^{\alpha},$$

for some $C > 0$ and $1 \leq |x| \leq |x'|$. Putting

$$u_R(x) = u(Rx)/C(R),$$

it follows that for any compact subset K of $\overline{K_S(0,\infty)}\setminus\{0\}$ there exists $C(K) > 0$ such that

$$\|u_R\|_{C^{1,\alpha}(K)} \leq C(K).$$

Thus there exist a sequence $R_n \to \infty$ and a p-harmonic function u^* in $K_S(0,\infty)$ such that $u_{R_n} \to u^*$ in the C^1_{loc} topology of $\overline{K_S(0,\infty)} \setminus \{0\}$. Moreover $u^* > 0$, and $\nabla u^* \neq 0$ because of (2.17).

In order to prove that there exists $\beta > 0$ such that

$$u^*(r,\sigma) = r^{-\beta}u^*(1,\sigma), \tag{2.18}$$

we define

$$\Sigma_R = \sup\left\{C > 0 : Cu^*(x) \leq u^*(Rx),\ \forall x \in \overline{K_S(0,\infty)} \setminus \{0\}\right\}.$$

Note that Σ_R exists because of (2.17). If we assume now that the equality

$$\Sigma_R u^*(x) = u^*(Rx), \tag{2.19}$$

does not hold in $\overline{K_S(0,\infty)}$, then

$$\Sigma_R u^*(x) < u^*(Rx), \tag{2.20}$$

from the strong maximum principle and Hopf lemma. Thus the function

$$\theta(\rho) = \min_{|x|=\rho} u^*(Rx)/u^*(x),$$

is strictly monotone and either (i) $\lim_{\rho\to\infty}\theta(\rho)=\Sigma_R$, or
(ii) $\lim_{\rho\to 0}\theta(\rho)=\Sigma_R$.

The treatment of the two cases is similar, then we assume (i). For any ρ, there exists $\sigma_\rho\in S$ such that

$$\theta(\rho)=u^*(R\rho\sigma_\rho)/u^*(\rho\sigma_\rho).$$

We can extract a sequence $\{R_{n_k}\}$ such that $\lim\limits_{n_k\to\infty} R_{n_k}/R_{n_{k+1}}=0$. Thus we set $\rho_{n_k}=R_{n_k}/R_{n_{k+1}}$ and assume that $\sigma_{\rho_{n_k}}\to\sigma_0\in\bar{S}$, by compactness. Because

$$\lim_{n_k\to\infty}\theta(\rho_{n_k})=\lim_{n_k\to\infty}\frac{C(R_{n_{k+1}})u(R_{n_{k+1}}R\sigma_{n_k})}{C(R_{n_{k+1}}R)u(R_{n_{k+1}}\sigma_{n_k})},$$

it implies

$$\Sigma_R=u^*(R,\sigma_0)<u^*(1,\sigma_0), \tag{2.21}$$

which contradicts (2.20).

The last point is to prove that

$$\Sigma_R=R^{-\beta} \tag{2.22}$$

for some $\beta>0$. Clearly $R\mapsto\Sigma_R$ is C^1 (as u^*) and decreases. For $k\in\mathbb{N}_*$ there holds

$$\Sigma_{R^k}u^*(x)=u^*(R^kx)=(\Sigma_R)^ku^*(x).$$

Then $\Sigma_{R^k}=(\Sigma_R)^k$. Consequently, for any $m\in\mathbb{N}_*$, $\Sigma_{R^{k/m}}=(\Sigma_R)^{k/m}$, and finally

$$\Sigma_{R^\alpha}=(\Sigma_R)^\alpha,$$

for any positive α. A straightforward consequence is that (2.22) holds for some $\beta>0$. If we set

$$\omega(\sigma)=u^*(1,\sigma), \tag{2.23}$$

then ω satisfies (2.8) in S, where it is positive, and vanishes on ∂S.

Uniqueness of the couple (β,ω) with $\sup_S\omega=1$ follows from the equivalence principle.

Remark Although the extension is far from being obvious, the regularity requirement on the domain S can be relaxed. It is possible to replace it by the assumption that ∂S is piecewise smooth. In dimension 3, Hopf lemma at a corner is replaced by an expansion in terms of conical functions as in Theorem 2.6. In higher dimension the proof goes by induction. However, uniqueness of the couple (β,ω) is not clear. From this observation, we can construct p-harmonic functions in $\mathbb{R}^N\setminus\{0\}$ under the form (2.7) with a finite symmetry group G generated by reflections through hyperplanes. Taking S to be a fundamental simplicial domain of G, we construct (β,ω) in S and then extend ω to the whole sphere by reflections through the edges.

It is natural to imbed this problem in a more general setting, by replacing (S^{N-1},g_0) by a compact and complete d-dimensional Riemannian manifold

(M, g). Let $\nabla_g.$ and ∇_g be respectively the divergence operator acting on vector fields on M and the gradient operator. For $\beta \in \mathbb{R}$ consider the equation

$$-\nabla_g.\left((\beta^2\psi^2 + |\nabla_g\psi|^2)^{(p-2)/2}\nabla_g\psi\right) = \beta((\beta+1)(p-1)-d)(\beta^2\psi^2 + |\nabla_g\psi|^2)^{(p-2)/2}\psi. \quad (2.24)$$

Definition We denote by $\mathfrak{S}_p(M)$ the set of couples $(\beta, \psi) \in \mathbb{R} \times C^1(M)$ satisfying (2.24) and call it the *p-quasi-spectrum* of M.

Theorem 2.7 *If* $(\beta, \psi) \in \mathfrak{S}_p(M)$, *then either* $\beta((\beta+1)(p-1)-d) = 0$ *and* ψ *is any constant, or* $\beta((\beta+1)(p-1)-d) > 0$ *and*

$$\int_M (\beta^2\psi^2 + |\nabla_g\psi|^2)^{(p-2)/2}\psi dv_g = 0. \quad (2.25)$$

Proof From (2.24),

$$\beta((\beta+1)(p-1)-d)\int_M (\beta^2\psi^2 + |\nabla_g\psi|^2)^{(p-2)/2}\psi dv_g = 0. \quad (2.26)$$

Thus if the integral term is not zero $\beta((\beta+1)(p-1)-d) = 0$. Clearly if $\beta = 0$, ψ is a constant. If $\beta \neq 0$, $(\beta+1)(p-1) = d$ and from (2.24) there holds

$$-\nabla_g.\left((\beta^2\psi^2 + |\nabla_g\psi|^2)^{(p-2)/2}\nabla_g\psi\right) = 0,$$

which implies

$$\int_M \left(\beta^2\psi^2 + |\nabla_g\psi|^2\right)^{(p-2)/2}|\nabla_g\psi|^2 dv_g = 0.$$

Thus ψ is constant. Moreover if $\beta((\beta+1)(p-1)-d) = 0$ any constant satisfies (2.24). Assume now that $\beta((\beta+1)(p-1)-d) \neq 0$. Then (2.25) holds. Moreover

$$\int_M \left(\beta^2\psi^2 + |\nabla_g\psi|^2\right)^{(p-2)/2}|\nabla_g\psi|^2 dv_g = \beta((\beta+1)(p-1)-d)\int_M (\beta^2\psi^2 + |\nabla_g\psi|^2)^{(p-2)/2}\psi^2 dv_g, \quad (2.27)$$

and the inequality $\beta((\beta+1)(p-1)-d) > 0$ follows.

Remark It should be interesting to study the links between $\mathfrak{S}_p(M)$ and the geometry of M, in particular the infimum of the $\beta((\beta+1)(p-1-d)$. Since we conjectured that the set of such β is unbounded, as on the sphere, their asymptotic distribution could be of interest. In the particular case where $p = d+1$, the $(d+1)$-quasi-spectrum of M is the set of couples (β, ψ) such that ψ is a solution of

$$-\nabla_g.\left((\beta^2\psi^2 + |\nabla_g\psi|^2)^{(d-1)/2}\nabla_g\psi\right) = d\beta^2(\beta^2\psi^2 + |\nabla_g\psi|^2)^{(d-1)/2}\psi. \quad (2.28)$$

As in the case $p = 2$, it should be interesting to study the invariance properties of $\mathfrak{S}_{d+1}(M)$ with respect to the conformal transformations of M.

3 Equations with strong absorption

In this section we assume $N \geq p > 1$ and $q > p-1$. If we look for solutions u of (1.5) with $\varepsilon = 1$ under the form (2.7) then $\beta = p/(q+1-p) = \beta_q$ and ω solves

$$-\nabla_\sigma . \left((\beta_q^2 \omega^2 + |\nabla_\sigma \omega|^2)^{(p-2)/2} \nabla_\sigma \omega \right) + |\omega|^{q-1}\omega = \lambda_q (\beta_q^2 \omega^2 + |\nabla_\sigma \omega|^2)^{(p-2)/2} \omega, \tag{3.1}$$

in S^{N-1}, where

$$\lambda_q = \beta_q((\beta_q + 1)(p-1) + 1 - N) = \Big(\frac{p}{q+1-p}\Big)\Big(\frac{pq}{q+1-p} - N\Big). \tag{3.2}$$

Since

$$\int_{S^{N-1}} \left((\beta_q^2 \omega^2 + |\nabla_\sigma \omega|^2)^{(p-2)/2} \left(|\nabla_\sigma \omega|^2 - \lambda_q \omega^2 \right) + |\omega|^{q+1} \right) d\sigma = 0,$$

there is no solution if $\lambda_q \leq 0$ or equivalently if $q \geq N(p-1)/(N-p)$. This fact corresponds to a removability result which was proved by Vazquez and Véron [17].

Theorem 3.1 *Let Ω be an open subset of $\mathbb{R}^N$ containing 0, $\Omega^* = \Omega \setminus \{0\}$, $N > p > 1$, $q \geq N(p-1)/(N-p) = p^\#$ and g a continuous real valued function satisfying*

$$\liminf_{r \to \infty} r^{-p^\#} g(r) > 0, \quad \text{and} \quad \limsup_{r \to -\infty} |r|^{-p^\#} g(r) < 0. \tag{3.3}$$

If $u \in C(\Omega^*) \cap W^{1,p}_{\text{loc}}(\Omega^*)$ is a weak solution of

$$-\nabla . \left(|\nabla u|^{p-2} \nabla u \right) + g(u) = 0, \quad \text{in } \Omega^*, \tag{3.4}$$

it can be extended to Ω as a continuous solution of the same equation in whole Ω.

On the contrary, if $p - 1 < q < p^\#$, the function

$$x \mapsto u_s(x) = \gamma_{N,p,q} |x|^{-\beta_q}, \tag{3.5}$$

with

$$\gamma_{N,p,q} = \Big(\big(\frac{p}{q+1-p}\big)^{p-1} \big(\frac{pq}{q+1-p} - N\big) \Big)^{1/(q+1-p)}, \tag{3.6}$$

is a singular solution of

$$-\nabla . (|\nabla u|^{p-2} \nabla u) + |u|^{q-1} u = 0. \tag{3.7}$$

in $\mathbb{R}^N \setminus \{0\}$. Friedman and Véron provided in [5] a full classification of singular nonnegative solutions of this equation.

Theorem 3.2 *Let Ω be an open subset of $\mathbb{R}^N$ containing 0, $\Omega^* = \Omega \setminus \{0\}$, $N \geq p > 1$, and $p-1 < q < p^\#$, $p-1 < q$ if $p = N$. If $u \in C^1(\Omega^*)$ is a nonnegative solution of (3.7) in Ω^*, the following dichotomy occurs.*

(i) Either $\lim_{x\to 0} |x|^{\beta_q} u(x) = \gamma_{N,p,q}$.

(ii) Either there exists $\gamma > 0$ such that $\lim_{x\to 0} u(x)/\mu_p(x) = \gamma$, and u satisfies

$$-\nabla.(|\nabla u|^{p-2}\nabla u) + |u|^{q-1}u = c_{N,p}|\gamma|^{p-2}\gamma\delta_0, \quad \text{in } \mathcal{D}'(\Omega). \tag{3.8}$$

(iii) Or u can be extended to whole Ω as a C^1 solution of (3.7) in Ω.

Proof By scaling we can always assume that $B_1 \subset \Omega$. The starting point is an a priori estimate of Keller-Osserman type due to Vazquez [16]: if u is any solution of (3.7) in $B_1^* = \{x \in \mathbb{R}^N : 0 < |x| < 1\}$, there exists a positive constant $K = K_{N,p,q}$ such that

$$|u(x)| \leq K|x|^{-\beta_q}, \tag{3.9}$$

for any $0 < |x| \leq 1/2$. By writting (3.7) under the form

$$-\nabla.(|\nabla u|^{p-2}\nabla u) + d(x)u^{p-1} = 0,$$

with $d(x) = u^{q+1-p}$, and using the Trudinger's estimate [15] in Harnack inequality, it follows that there exists some $A = A(N,p,q) > 0$ such that

$$\max_{|x|=r} u(x) \leq A \min_{|x|=r} u(x),$$

for any $0 < r \leq 1/4$.

Step 1 Assume that $u(x)/\mu_p(x)$ is not bounded in a neighborhood of 0. The previous estimate implies that there exists a sequence $r_n \to 0$ such that

$$\lim_{r_n\to 0} \min_{|x|=r_n} u(x)/\mu_p(r_n) = \infty.$$

Consequently, for any $k > 0$ there exists some n_k such that for $n \geq n_k$ the function u is bounded from below in $\bar{B}_1 \setminus B_{r_n}$ by the solution v_n of the Dirichlet problem

$$\begin{aligned} -\nabla.(|\nabla v_n|^{p-2}\nabla v_n) + |v|_n{}^{q-1}v_n &= 0, \quad \text{in } B_1 \setminus \bar{B}_{r_n}, \\ v_n(x) &= 0 \quad \text{if } |x| = 1, \\ v_n(x) &= k\mu_p(r_n) \quad \text{if } |x| = r_n. \end{aligned} \tag{3.10}$$

Note that v_n is positive, radial and bounded from above by $k\mu_p(x)$. Since $q < p^\#$ the absorption term v_n^q satisfies

$$\int_{r_n}^1 v_n^q r^{N-1} dr \leq k^q \int_0^1 \mu_p^q(r)^q r^{N-1} dr,$$

independently of n. This is sufficient to derive that there exists

$$\lim_{r_n\to 0} v_n = v,$$

where $v = v_{(k)}$ is a radial solution of

$$\begin{aligned} -\nabla.(|\nabla v|^{p-2}\nabla v) + |v|^{q-1}v = 0, \quad \text{in } B_1 \setminus \{0\}, \\ v(x) = 0 \quad \text{if } |x| = 1, \\ v(x) \approx k\mu_p(x) \quad \text{if } |x| \to 0. \end{aligned} \tag{3.11}$$

Actually, v is nonnegative, radial, bounded from above by u and solves

$$-\nabla.(|\nabla v|^{p-2}\nabla v) + v^q = c_{N,p}k^{p-1}\delta_0, \quad \text{in } \mathcal{D}'(B_1). \tag{3.12}$$

When $k \to \infty$, $v_{(k)}$ increases and converges to some $v_{(\infty)}$ which is a positive and radial solution of (3.7) in B_1^* such that

$$\lim_{r\to 0} v_{(\infty)}(r)/\mu_p(r) = \infty. \tag{3.13}$$

Moreover

$$v_{(\infty)}(|x|) \leq u(x) \leq u_s(x) = \gamma_{N,p,q}|x|^{-\beta_q} \quad \text{in } B_1^*. \tag{3.14}$$

The analysis of the behavior of $v_{(\infty)}$ near $r = 0$ is done either by a technical O.D.E. analysis, or a scaling invariance method based on uniqueness of the radial solution of (3.11) (see [4] for a proof in the case $p = 2$). From this analysis follows

$$\lim_{r\to 0} r^{\beta_q} v_{(\infty)}(r) = \gamma_{N,p,q}. \tag{3.15}$$

Consequently

$$\lim_{x\to 0} |x|^{\beta_q} u(x) = \gamma_{N,p,q}. \tag{3.16}$$

Step 2 Assume that $u(x)/\mu_p(x)$ is bounded near 0 (in this case, we need not impose the positivity of u). In such a case the absorption term $|u|^{q-1}u$ is dominated by $C\mu_p^q$ for some $C > 0$. By using the same scaling methods, estimates on ∇u, and the strict comparison principle as in the proof of Theorem 2.1, it can be proved that there exists a real number γ such that

$$\lim_{x\to 0} u(x)/\mu_p(x) = \gamma, \tag{3.17}$$

and

$$\lim_{x\to 0} (|x|)^{(N-1)/(p-1)} \nabla \left(u(x) - \gamma\mu_p(x)\right) = 0. \tag{3.18}$$

Thus u satisfies (3.8). If $\gamma = 0$, then

$$|u(x)| \leq \max_{|y|=1} |u(y)|, \quad \forall x \in B_1,$$

by the maximum principle. Thus u is $C^{1,\alpha}$ by the regularity theory of quasilinear equations. □

The construction of nodal singular solutions of (3.7) under the form (2.7) is done by a shooting technique, as for the p-Laplace equation.

Theorem 3.3 *Let $0 < p-1 < q < p^{\#}$ and $S \subset S^{N-1}$ be a domain with a C^2 relative boundary ∂S. Let $\beta = \beta_S > 0$ be the exponent defined in Theorem 2.6. If $\beta_q > \beta_S$ there exists a positive solution ω of (3.1) in S which vanishes on ∂S.*

Proof: **Step 1** Construction of an approximate solution. For $\varepsilon > 0$ small enough denote by $u = u_\varepsilon$ the unique solution of

$$\begin{aligned} -\nabla.\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{q-1}u = 0, \quad \text{in } K_S(1,\infty),\\ u = \varepsilon g^{\beta_q}, \quad \text{on } \partial K_S(1,\infty),\\ \limsup_{|x|\to\infty} |x|^{\beta_q}u(x) < \infty. \end{aligned} \tag{3.19}$$

By the monotone operator theory, u is unique and satisfies $0 \leq u < u_s$.

Step 2 Construction of a minorant subsolution. Let $\omega = \omega_S$ be the corresponding second element of the couple $(\beta,\omega) = (\beta_S,\omega_S)$ obtained in Theorem 2.6. Put $\theta = \beta_q/\beta_S$. We claim that for $\delta > 0$ small enough, the function

$$(r,\sigma) \mapsto w_\delta(x) = w_\delta(r,\sigma) = r^{-\beta_q}\delta\omega_S^\theta(\sigma) \tag{3.20}$$

satisfies

$$\begin{aligned} -\nabla.\left(|\nabla w_\delta|^{p-2}\nabla w_\delta\right) + |w_\delta|^{q-1}w_\delta \leq 0, \quad \text{in } K_S(1,\infty),\\ w_\delta = 0, \quad \text{on } B_S(1,\infty). \end{aligned} \tag{3.21}$$

Set

$$\mathcal{L}w_\delta = -\nabla.\left(|\nabla w_\delta|^{p-2}\nabla w_\delta\right) + |w_\delta|^{q-1}w_\delta.$$

Then $\mathcal{L}(w_\delta) = r^{-q\beta_q}\mathcal{T}(\delta\omega_S^\theta)$, where

$$\mathcal{T}(\eta) = -\nabla_\sigma.\left((\beta_q^2\eta^2 + |\nabla_\sigma\eta|^2)^{(p-2)/2}\nabla_\sigma\eta\right) - \lambda_q(\beta_q^2\eta^2 + |\nabla_\sigma\eta|^2)^{(p-2)/2}\eta + |\eta|^{q-1}\eta.$$

Putting $\eta = \delta\omega_S^\theta$,

$$(\beta_q^2\eta^2 + |\nabla_\sigma\eta|^2)^{(p-2)/2} = \delta^{p-2}\theta^{p-2}\omega_S^{(\theta-1)(p-2)}(\beta_S^2\omega^2 + |\nabla_\sigma\omega|^2)^{(p-2)/2},$$

and

$$\begin{aligned} &\nabla_\sigma.\left((\beta_q^2\eta^2 + |\nabla_\sigma\eta|^2)^{(p-2)/2}\nabla_\sigma\eta\right)\\ &=\delta^{p-1}\theta^{p-1}\nabla_\sigma.\left(\omega_S^{(\theta-1)(p-1)}(\beta_S^2\omega_S^2 + |\nabla_\sigma\omega_S|^2)^{(p-2)/2}\nabla_\sigma\omega_S\right)\\ &=\delta^{p-1}\theta^{p-1}\omega_S^{(\theta-1)(p-1)}\nabla_\sigma.\left((\beta_S^2\omega_S^2 + |\nabla_\sigma\omega_S|^2)^{(p-2)/2}\nabla_\sigma\omega_S\right)\\ &\quad + (\theta-1)(p-1)\delta^{p-1}\theta^{p-1}\omega_S^{(\theta-1)(p-1)-1}(\beta_S^2\omega_S^2 + |\nabla_\sigma\omega_S|^2)^{(p-2)/2}|\nabla_\sigma\omega_S|^2 \end{aligned}$$

But

$$-\nabla_\sigma.\left((\beta_S^2\omega_S^2 + |\nabla_\sigma\omega_S|^2)^{(p-2)/2}\nabla_\sigma\omega_S\right) = \lambda_S(\beta_S^2\omega_S^2 + |\nabla_\sigma\omega_S|^2)^{(p-2)/2}\omega_S,$$

with $\lambda_S = (\beta_S + 1)(p-1) + 1 - N)$. Thus,

$$\begin{aligned}\delta^{1-p}\mathcal{T}(\eta) &= \delta^{q+1-p}\omega_S^{\theta q} + \omega_S^{(\theta-1)(p-1)-1}\theta^{p-2}(\beta_S^2\omega_S^2 + |\nabla_\sigma \omega_S|^2)^{(p-2)/2} \\ &\quad \times \left((\theta\lambda_S - \lambda_q)\omega_S^2 - \theta(\theta-1)(p-1)|\nabla_\sigma\omega_S|^2\right).\end{aligned}$$

Since $\theta\lambda_S - \lambda_q = \beta_q(\beta_S - \beta_q)(p-1) = -\beta_S^2\theta(\theta-1)(p-1)$,

$$\begin{aligned}&\delta^{1-p}\mathcal{T}(\eta) \\ &= \delta^{q+1-p}\omega_S^{\theta q} - (p-1)(\theta-1)\theta^{p-1}\omega_S^{(\theta-1)(p-1)-1}(\beta_S^2\omega_S^2 + |\nabla_\sigma\omega_S|^2)^{p/2} \\ &\leq \delta^{q+1-p}\omega_S^{\theta q} - (p-1)(\theta-1)\theta^{p-1}\omega_S^{\theta(p-1)}.\end{aligned}$$

by assumption $\theta > 1$, therefore there exists $\delta > 0$ such that $\mathcal{T}(\eta) \leq 0$. Moreover it can also be assumed that $\delta\omega_S^\theta \leq \varepsilon$. Then $w_\delta(x) \leq u(x)$ if $|x| = 1$ and $w_\delta \leq u$ in $K_S(1,\infty)$ by the maximum principle. Henceforth

$$\delta\omega_S^\theta(x/|x|) \leq |x|^{\beta_q}u(x) \leq \gamma_{N,p,q} \quad \text{in} \;\; K_S(1,\infty). \tag{3.22}$$

Step 3 For $R > 0$, define the function u_R by $u_R = R^{\beta_q}u(Rx)$. The function u_R satisfies (3.7) in $K_S(1/R,\infty)$. By the degenerate elliptic equation regularity theory, the set of functions $\{u_R\}$ remains bounded in the $C^{1,\alpha}_{\text{loc}}$-topology of $\overline{K_S(0,\infty)}\setminus\{0\}$. Let $0 < R < R'$, in order to compare u_R and $u_{R'}$ in $K_S(1/R,\infty)$ we recall that $g(x) = (2-|x|)_+$. The relation

$$R'^{\beta_q}(2 - R'|x|)_+^{\beta_q} \leq R^{\beta_q}(2 - R|x|)_+^{\beta_q} \quad \text{for} \;\; |x| \geq 1/R,$$

implies

$$\frac{d}{dR}\left(R^{\beta_q}(2 - R|x|)_+^{\beta_q}\right) \leq 0 \quad \text{for} \;\; |x| \geq 1/R,$$

If and only if

$$\beta_q R(2 - R|x|)_+^{\beta_q - 1}(2 - 2R|x|) \leq 0 \quad \text{for} \;\; |x| \geq 1/R,$$

which holds true. By the maximum pinciple

$$R' \geq R \Longrightarrow u_{R'} \leq u_R \in K_S(1/R,\infty). \tag{3.23}$$

Thus there exists a function u^* such that u_R decreases and converges to u^* as $R \to \infty$ in $C^1_{\text{loc}}(\overline{K_S(0,\infty)}\setminus\{0\})$. The function u^* is a solution of (3.7) in $K_S(0,\infty)$ which vanishes on $B_S(0,\infty)$. Because of (3.22), u^* satisfies

$$\delta\omega_S^\theta(x/|x|) \leq |x|^{\beta_q}u^*(x) \leq \gamma_{N,p,q} \quad \text{in} \;\; K_S(0,\infty). \tag{3.24}$$

Finally,

$$\lim_{R\to\infty} R^{\beta_q}u(Rr,\sigma) = u^*(r,\sigma) = r^{-\beta_q}\lim_{R\to\infty}(Rr)^{\beta_q}u(Rr,\sigma) = r^{-\beta_q}u^*(1,\sigma).$$

Putting $\omega = u^*(1,\sigma)$ completes the proof. $\square$

In the next theorem we prove that the condition $\beta_q > \beta_S$ is sharp.

Theorem 3.4 *Let $0 < p-1 < q < p^{\#}$ and $S \subset S^{N-1}$ be a domain with a C^2 relative boundary ∂S. If $\beta_q \leq \beta_S$ there exists no solution ω of (3.1) in S which vanishes on ∂S.*

Proof Assume ω is a solution of (3.1). If $\theta = \beta_q/\beta_S$, then $0 < \theta \leq 1$. If we denote again $\eta = \delta\omega_S^{\theta}$, for some $\delta > 0$, it follows from the proof of Theorem 3.3-Step 2 that, for any $\delta > 0$,

$$\begin{aligned}\delta^{1-p}\mathcal{T}(\eta) &= \delta^{q+1-p}\omega_S^{\theta q} \\ &\quad +(p-1)(1-\theta)\theta^{p-1}\omega_S^{(\theta-1)(p-1)-1}(\beta_S^2\omega_S^2 + |\nabla_\sigma\omega_S|^2)^{p/2} > 0.\end{aligned}$$

We take $\delta = \delta_0$ as the smallest parameter such that $\eta = \eta_\delta \geq \omega$. Notice that such a choice is always possible since $\omega \in C^1(\bar{S})$, the normal derivative of ω_S on the relative boundary ∂S is negative from the Hopf boundary lemma and therefore $\omega_S^{\theta}(\sigma) \geq c(dist(\sigma, \partial S)^{\theta}$ for some $c > 0$. We shall distinguish according there exists $\sigma_0 \in S$ such that

$$\eta(\sigma) \geq \omega(\sigma),\ \forall \sigma \in \bar{S}, \quad \text{and} \quad \eta(\sigma_0) = \omega(\sigma_0), \tag{3.25}$$

or not. If (3.25) holds true, which is always the case if $\beta_S > \beta_q$, the function $\psi = \eta - \omega$ is nonnegative in $\bar{S}$, not identically 0 and achieves its minimal value 0 in an interior point σ_0. Let $g = (g_{ij})$ be the metric tensor on S^{N-1}. We write in local coordinates σ_j around σ_0,

$$|\nabla\varphi|^2 = \sum_{j,k} g^{jk}\frac{\partial\varphi}{\partial\sigma_j}\frac{\partial\varphi}{\partial\sigma_k},$$

$$\nabla.X = \frac{1}{\sqrt{|g|}}\sum_{\ell}\frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}X^{\ell}\right) = \frac{1}{\sqrt{|g|}}\sum_{\ell,i}\frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}g^{\ell i}X_i\right),$$

if we lower the indices by setting $X^{\ell} = \sum_i g^{\ell i}X_i$. From the Mean Value Theorem, we obtain

$$\begin{aligned}(\beta_q^2\eta^2 + |\nabla_\sigma\eta|^2)^{(p-2)/2}\frac{\partial\eta}{\partial\sigma_i} - (\beta_q^2\omega^2 + |\nabla_\sigma\omega|^2)^{(p-2)/2}\frac{\partial\omega}{\partial\sigma_i} \\ = \sum_j \alpha_j^i\frac{\partial(\eta-\omega)}{\partial\sigma_j} + b^i(\eta-\omega),\end{aligned}$$

where

$$\begin{aligned}b^i &= (p-2)\left(\beta_q^2(\omega + t(\eta-\omega))^2 + |\nabla_\sigma(\omega + t(\eta-\omega))|^2\right)^{(p-4)/2} \\ &\quad \times(\omega + t(\eta-\omega))\frac{\partial(\omega + t(\eta-\omega))}{\partial\sigma_i},\end{aligned}$$

and

$$\begin{aligned}\alpha_j^i &= (p-2)\left(\beta_q^2(\omega + t(\eta-\omega))^2 + |\nabla_\sigma(\omega + t(\eta-\omega))|^2\right)^{(p-4)/2} \\ &\quad \times\frac{\partial(\omega + t(\eta-\omega))}{\partial\sigma_i}\sum_k g^{jk}\frac{\partial(\omega + t(\eta-\omega))}{\partial\sigma_k} \\ &\quad +\delta_i^j\left(\beta_q^2(\omega + t(\eta-\omega))^2 + |\nabla_\sigma(\omega + t(\eta-\omega))|^2\right)^{(p-2)/2}.\end{aligned}$$

Since the graph of η and ω are tangent at σ_0,

$$\eta(\sigma_0) = \omega(\sigma_0) = P_0 > 0 \quad \text{and } \nabla\eta(\sigma_0) = \nabla\omega(\sigma_0) = Q.$$

Thus

$$b^i(\sigma_0) = (p-2)\left(\beta_q^2 P_0^2 + |Q|^2\right)^{(p-4)/2} P_0 Q_i,$$

and

$$\alpha_j^i(\sigma_0) = \left(\beta_q^2 P_0^2 + |Q|^2\right)^{(p-4)/2}\Big(\delta_i^j(\beta_q^2 P_0^2 + |Q|^2) + (p-2)Q_i \sum_k g^{jk} Q_k\Big).$$

Now

$$\begin{aligned}
&\mathcal{T}(\eta) - \mathcal{T}(\omega)\\
=&\frac{-1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial\sigma_\ell}\Big[\sqrt{|g|}g^{\ell i}\Big((\beta_q^2\eta^2 + |\nabla_\sigma \eta|^2)^{\frac{p}{2}-1}\frac{\partial\eta}{\partial\sigma_i} - (\beta_q^2\omega^2 + |\nabla_\sigma\omega|^2)^{\frac{p}{2}-1}\frac{\partial\omega}{\partial\sigma_i}\Big)\Big]\\
&- \lambda_q\left((\beta_q^2\eta^2 + |\nabla_\sigma\eta|^2)^{\frac{p}{2}-1}\eta - (\beta_q^2\omega^2 + |\nabla_\sigma\omega|^2)^{\frac{p}{2}-1}\omega\right) + \eta^q - |\omega|^{q-1}\omega),\\
=&-\frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial\sigma_\ell}\Big[\sqrt{|g|}g^{\ell i}\Big(\sum_j \alpha_j^i \frac{\partial(\eta-\omega)}{\partial\sigma_j} + b^i(\eta-\omega)\Big)\Big]\\
&+\sum_i C_i \frac{\partial(\eta-\omega)}{\partial\sigma_i} + C(\eta-\omega)\\
=&-\frac{1}{\sqrt{|g|}} \sum_{\ell,j} \frac{\partial}{\partial\sigma_\ell}\left[a_j^\ell \frac{\partial(\eta-\omega)}{\partial\sigma_j}\right] + \sum_i C_i \frac{\partial(\eta-\omega)}{\partial\sigma_i} + C(\eta-\omega),
\end{aligned}$$

where the C_i and C are continuous functions and

$$a_j^\ell = \sqrt{|g|}\sum_i g^{\ell i}\alpha_j^i.$$

The matrix $\left(\alpha_j^i(\sigma_0)\right)$ is symmetric, definite and positive since it is the Hessian of the strictly convex function

$$X = (X_1, \ldots, X_{n-1}) \mapsto \frac{1}{p}\left(P_0^2 + |X|^2\right)^{p/2} = \frac{1}{p}\Big(P_0^2 + \sum_{j,k} g^{jk}X_j X_k\Big)^{p/2}.$$

Therefore, $\left(\alpha_j^i\right)$ has the same property in some neighborhood of σ_0, and the same holds true with $\left(a_j^\ell\right)$. Finally the function $\psi = \eta - \omega$ is nonnegative, vanishes at σ_0 and satisfies

$$-\frac{1}{\sqrt{|g|}}\sum_{\ell,j}\frac{\partial}{\partial\sigma_\ell}\left[a_j^\ell \frac{\partial\psi}{\partial\sigma_j}\right] + \sum_i C_i\frac{\partial\psi}{\partial\sigma_i} + C_+\psi \geq 0. \tag{3.26}$$

Then $\psi = 0$ in a neighborhood of S. Since S is connected, ψ is identically 0, which a contradiction.

If (3.25) does not hold, then $\theta = 1$ and that the graphs of η and ω are tangent at some point σ_0 of the relative boundary ∂S. Proceeding as above and using the fact that $\partial\eta/\partial\nu$ exists and never vanishes on the boundary, we see that $\psi = \eta - \omega$ satisfies (3.26) with a strongly elliptic operator in a neighborhood $\mathcal{N}$ of σ_0. Moreover $\psi > 0$ in $\mathcal{N}$, $\psi(\sigma_0) = 0$ and $\partial\psi/\partial\nu(\sigma_0) = 0$. This is a contradiction, which ends the proof.

Remark The existence result of Theorem 3.3 is valid if S is no longer a C^2 domain but a domain with a piecewise regular boundary since only the existence of (β_S, ω_S) is needed. We conjecture that the condition $\beta_q > \beta_S$ is still necessary. As is section 2, we can construct nodal solutions of (3.1) with a finite symmetry group G generated by reflections through hyperplanes. Taking S to be a fundamental simplicial domain of G, we construct (β, ω) in S and then extend ω to the whole sphere by reflections through the edges. It follows that there exists nodal singular solutions of (3.7) in $\mathbb{R}^N \setminus \{0\}$.

Remark Under the assumptions of Theorem 3.3, we conjecture that uniqueness of the positive solution ω of (3.1) which vanishes on ∂S holds. If $S = S^{N-1}$ and $p-1 < q < p^{\#}$, an application of the maximum principle (or a consequence of Theorem 3.2) implies that the only positive solution of (3.1) on S^{N-1} is the constant function $\gamma_{N,p,q}$.

4 Equations with a source term

If we look for solutions of

$$\nabla.\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{q-1}u = 0 \tag{4.1}$$

under the form (2.7), then $\beta = p/(q+1-p) = \beta_q$ and ω solves

$$\nabla_\sigma.\left((\beta_q^2\omega^2 + |\nabla_\sigma\omega|^2)^{(p-2)/2}\nabla_\sigma\omega\right) + |\omega|^{q-1}\omega + \lambda_q(\beta_q^2\omega^2 + |\nabla_\sigma\omega|^2)^{(p-2)/2}\omega = 0, \tag{4.2}$$

on S^{N-1} with λ_q defined by (3.2). By integrating (4.2) we get

$$\lambda_q \int_{S^{N-1}} (\beta_q^2\omega^2 + |\nabla_\sigma\omega|^2)^{(p-2)/2}\omega d\sigma + \int_{S^{N-1}} |\omega|^{q-1}\omega d\sigma = 0.$$

Therefore, there exists no positive solution if $\lambda_q \geq 0$, or equivalently $q \leq N(p-1)/(N-p)$ (it is always assumed that $q > p-1$). In the range $1 < p < N$ and $q > N(p-1)/(N-p)$ the constant function

$$\omega_0 = (\beta_q^{p-1}(N - q\beta_q))^{1/(q+1-p)}$$

is a solution of (4.2), and a natural question is to look for nonconstant solutions. As in Section 2, we imbed this problem in the more general setting of a compact

d-dimensional Riemannian manifold (M,g) without boundary. For β and $\lambda \in \mathbb{R}$ consider the equation

$$-\nabla_g.\left((\beta^2\omega^2+|\nabla_g\omega|^2)^{(p-2)/2}\nabla_g\omega\right)+\lambda(\beta^2\omega^2+|\nabla_g\omega|^2)^{(p-2)/2}\omega=|\omega|^{q-1}\omega. \tag{4.3}$$

We shall assume $\lambda > 0$ in order for the constant solution

$$\omega_* = (\beta^{p-2}\lambda)^{1/(q+1-p)}$$

to exist. We assume also that the starting equation is super-quasilinear in the sense that $\beta > 0$ and $q > q+1-p$. We can linearize (4.3) in a neighborhood of ω_*, and we obtain

$$\frac{d}{dt}\nabla_g.\left((\beta^2(\omega_*+t\varphi)^2+|\nabla_g(\omega_*+t\varphi)|^2)^{(p-2)/2}\nabla_g(\omega_*+t\varphi)\right)\Big|_{t=0} = \beta^{p-2}\omega_*^{p-2}\Delta_g\varphi.$$

$$\frac{d}{dt}\left((\beta^2(\omega_*+t\varphi)^2+|\nabla_g(\omega_*+t\varphi)|^2)^{(p-2)/2}(\omega_*+t\varphi)\right)\Big|_{t=0}=(p-1)\beta^{p-2}\omega_*^{p-2}\varphi.$$

$$\frac{d}{dt}(\omega_*+t\varphi)^q\Big|_{t=0}=q\omega_*^{q-1}\varphi.$$

Since $\omega_* = (\beta^{p-2}\lambda)^{1/(q+1-p)}$, the linearized equation is

$$-\Delta_g\varphi = (q+1-p)\lambda\varphi. \tag{4.4}$$

where $\Delta_g = \nabla_i\nabla^i$ is the laplacian on M.

Theorem 4.1 *Let μ_1 be the first nonzero eigenvalue of Δ_g, and assume it is simple. Then for any $\lambda > \mu_1/(q+1-p)$ equation (4.3) admits a nonconstant positive solution ω_λ.*

Proof The existence of a global and unbounded branch of bifurcation $\mathcal{B} = \{(\lambda,\omega_\lambda)\} \subset \mathbb{R}\times C^1(M)$ issued from $(\mu_1/(q+1-p),\omega_*)$ follows from the application in the space $C^1(M)$ of the classical bifurcation theorem from a simple eigenvalue. □

Remark The condition on the simplicity of μ_1 can be avoided in many cases where symmetries occur. When $(M,g) = (S^{N-1},g_0)$, we have the parametric representation

$$S^{N-1} = \{\sigma = (\cos\varphi, \sin\varphi\sigma')\ :\ \varphi\in[0,\pi],\ \sigma'\in S^{N-2}\},$$

and

$$\Delta_{S^{N-1}}\omega = \sin^{2-N}\varphi\frac{\partial}{\partial\varphi}\left(\sin^{N-2}\varphi\frac{\partial\omega}{\partial\varphi}\right)+\sin^{-2}\varphi\Delta_{S^{N-2}}\omega.$$

If we only consider function depending on φ (they are called zonal functions), $\mu_1 = N-1$ is a simple eigenvalue. Moreover any eigenspace of S^{N-1} contains a 1-dimensional sub-eigenspace of functions depending only on φ. Therefore all the corresponding eigenvalues are simple. Thus from each of the couples $(\mu_k/(q+1-p),\omega_*)$ is issued a C^1 curve of positive solutions (λ,ω_λ) with $\lambda > \mu_k/(q+1-p)$.

Open question An interesting problem is to find sufficient conditions besides $\lambda \leq \mu_1/(q+1-p)$ and probably $q \leq dp/(d-p)-1$, in order the constant ω_* be the only positive solution of (4.3). We believe additional conditions linked to the curvature should be found (see [6], [2], [10] in the case $p=2$).

We define the critical Sobolev exponent q_c by

$$q_c = \frac{Np}{N-p} - 1 = \frac{N(p-1)+p}{N-p}. \tag{4.5}$$

A particular case of equation (4.1) is when $q = q_c$. Then

$$q_c + 1 - p = \frac{p^2}{N-p}, \quad \beta_{q_c} = \frac{N-p}{p} \quad \text{and} \quad \lambda_{q_c} = -\beta_{q_c}^2.$$

The critical equation is therefore

$$\nabla_\sigma.\Big((\beta_{q_c}^2\omega^2 + |\nabla_\sigma\omega|^2)^{p/2-1}\nabla_\sigma\omega\Big) + |\omega|^{q_c-1}\omega - \beta_{q_c}^2(\beta_{q_c}^2\omega^2 + |\nabla_\sigma\omega|^2)^{p/2-1}\omega = 0, \tag{4.6}$$

on S^{N-1}. A natural question is to explore the connection between the positive solutions of (4.6) and the positive solutions of

$$-\nabla.\Big(|\nabla u|^{p-2}\nabla u\Big)v = v^{q_c} \quad \text{in } \mathbb{R}^N. \tag{4.7}$$

Notice that the radial solutions of this equation, depending of a parameter $a > 0$, are known:

$$v_a(x) = \Big(Na(\frac{N-p}{p-1})^{p-1}\Big)^{(N-p)/p^2}\Big(a + |x|^{p/(p-1)}\Big)^{(p-N)/p}. \tag{4.8}$$

The solutions of (4.6) are the critical points of the functional

$$J_{q_c}(\psi) = \int_{S^{N-1}}\Big(\frac{1}{p}(\beta_{q_c}^2\psi^2 + |\nabla_\sigma\psi|^2)^{p/2} - \frac{1}{q_c+1}|\psi|^{q_c+1}\Big)\,d\sigma, \tag{4.9}$$

where $\psi \in W^{1,p}(S^{N-1})$.

Remark Let $0 < p-1 < q < q_c$ and $S \subset S^{N-1}$, it would be interesting to construct positive solutions ω of (4.2) in S which vanish on ∂S. In the case $p=2$, the equation becomes

$$\begin{aligned} -\Delta_\sigma\omega &= \beta_q(\beta_q + 2 - N)\omega + \omega^q, \quad \text{in } S,\\ \omega &= 0, \quad \text{on } \partial S, \end{aligned} \tag{4.10}$$

where Δ_σ is the Laplace-Beltrami operator on the sphere and $\beta_q = 2/(q-1)$. The solutions are constructed by a standard minimization process with a constraint. If $1 < q < (N+1)/(N-3)$, a necessary and sufficient condition for the existence of such a solution is

$$\beta_q < \beta_S,$$

and in that case $\beta_S = \lambda_1(S)$ is the first eigenvalue of Δ_σ in $W_0^{1,2}(S)$. When $p \neq 2$, this method no longer works. However under the same condition

$$\beta_q < \beta_S \quad \text{and} \quad q < q_c,$$

(adapted to the case of a general p) we have been able to prove the existence of positive super and subsolutions to equation (4.2). Unfortunately we do not know if they are ordered. We conjecture that, in the subcritical case, the condition $\beta_q \geq \beta_S$ is a necessary and sufficient condition for the existence of positive solutions to (4.2).

We want to mention another quasilinear equation of Emden type which admits specific solutions:

$$-\nabla.\big(|\nabla u|^{p-2}\nabla u\big) = \lambda e^u, \tag{4.11}$$

with $\lambda > 0$. If we look for particular solutions of (4.11) under the form

$$u(r,\sigma) = \alpha \ln r + bw(\sigma) + k,$$

where α, b and k are constants, one finds $\alpha = -p$ and

$$b\nabla_\sigma.\Big(\big[p^2 + b^2|\nabla_\sigma w|^2\big]^{p/2-1}\nabla_\sigma w\Big) + \lambda e^k e^{bw} - p(N-p)\big[p^2 + b^2|\nabla_\sigma w|^2\big]^{p/2-1} = 0$$

on S^{N-1}. A necessary condition for the existence of a solution is

$$p - N < 0. \tag{4.12}$$

Assuming this condition, we take $b = p$ and get

$$\nabla_\sigma.\Big(\big[1 + |\nabla_\sigma w|^2\big]^{p/2-1}\nabla_\sigma w\Big) - (N-p)\big[1 + |\nabla_\sigma w|^2\big]^{p/2-1} + \lambda p^{1-p} e^k e^{pw} = 0.$$

Now choose $k = \ln(p^{p-1}\lambda^{-1})$. Assuming $1 < p < N$, then w satisfies

$$\nabla_\sigma.\Big(\big[1 + |\nabla_\sigma w|^2\big]^{p/2-1}\nabla_\sigma w\Big) - (N-p)\big[1 + |\nabla_\sigma w|^2\big]^{p/2-1} + e^{pw} = 0 \tag{4.13}$$

on S^{N-1}. In the particular case $p = 2$, $N = 3$, this is the equation of conformal change of structures on S^2, and the set of all solutions can be endowed with a structure of a 3-dim non-compact Lie group. *We believe that the case $p = N - 1 = n$ should play a similar algebraic role.* The corresponding equation is

$$\nabla_\sigma.\Big(\big[1 + |\nabla_\sigma w|^2\big]^{n/2-1}\nabla_\sigma w\Big) - \big[1 + |\nabla_\sigma w|^2\big]^{n/2-1} + e^{nw} = 0 \tag{4.14}$$

on S^{N-1}.

In the case $1 < p < N$ and $p - 1 < q < N(p-1)/(N-p) = p^\#$, the classification of isolated singularities of positive solutions of (4.1) has been initiated by Guedda and Véron [7], under the priori bound assumption (4.18), and then completed by Bidaut-Véron [1].

Theorem 4.2 *Let Ω be an open subset of $\mathbb{R}^N$ containing 0, $\Omega^* = \Omega \setminus \{0\}$, $1 < p < N$ and $p - 1 < q < p^\#$, and let $u \in C^1(\Omega^*)$ be a nonnegative solution of (4.1) in Ω^*. Then the following dichotomy occurs.*

(i) Either there exists $\alpha > 0$ such that $\lim_{x \to 0} u(x)/\mu_p(x) = \alpha$, and u satisfies

$$-\nabla.(|\nabla u|^{p-2}\nabla u) - u^q = c_{N,p}\alpha^{p-1}\delta_0, \quad \text{in } \mathcal{D}'(\Omega). \tag{4.15}$$

(ii) Or u can be extended as a C^1 solution of (4.1) in Ω.

The general proof of this result is based upon the extension obtained in [1] of the Brezis-Lions lemma [3] dealing with singular super-harmonic functions.

Lemma 4.3 *Let $1 < p < N$ and $u \in C(\Omega^*) \cap W^{1,p}_{\mathrm{loc}}(\Omega^*)$ with $\nabla.\left(|\nabla u|^{p-2}\nabla u\right) \in L^1_{\mathrm{loc}}(\Omega^*)$ is a nonnegative solution of*

$$\nabla.\left(|\nabla u|^{p-2}\nabla u\right) \leq 0, \tag{4.16}$$

a.e. in Ω and in the sense of distributions in Ω^*. Then $u^{p-1} \in M^{N/(N-p)}_{\mathrm{loc}}(\Omega)$, $|\nabla u|^{p-1} \in M^{N/(N-1)}_{\mathrm{loc}}(\Omega)$, and there exists a nonnegative constant β and some $g \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$-\nabla.\left(|\nabla u|^{p-2}\nabla u\right) = g + \beta\delta_0, \tag{4.17}$$

in the sense of distributions in Ω.

From this result and using some test functions introduced by Serrin in [11], Harnack inequality and a method due to Benilan, it is possible to derive the key estimate that is satisfied by any positive solution u of (4.1) in this range of values of q : there exists some $C > 0$ such that

$$u(x) \leq C\mu_p(x), \tag{4.18}$$

holds in a neighborhood of 0. With this estimate, a scaling methods similar to the one used in [5] ends the proof. Actually, in [7], a more general convergence result is proved: if $1 < p \leq N$, $p - 1 < q < p^\#$ (no condition if $p = N$) and $u \in C^1(\Omega^*)$ is a signed solution of (4.1) in Ω^* such that

$$|u(x)| \leq C\mu_p(x),$$

near 0, then either
(i') there exists $\alpha \neq 0$ such that $\lim_{x \to 0} u(x)/\mu_p(x) = \alpha$, and u satisfies

$$-\nabla.(|\nabla u|^{p-2}\nabla u) - |u|^{q-1}u = c_{N,p}|\alpha|^{p-2}\alpha\delta_0, \quad \text{in } \mathcal{D}'(\Omega). \tag{4.19}$$

(ii') Or u can be extended as a C^1 solution of (4.1) in Ω.

In the case $q \geq p^\#$, the classification of isolated singularities of *radial* solutions of (4.1) has been performed by Guedda and Véron [7]. Latter on Guedda and Véron's results have been extended by Bidaut-Véron [1], with no restriction on q, but always when dealing with radial solutions.

Theorem 4.4 *Let $p^{\#} < p < q_c$, and let $u \in C^1(B_1^*)$ be a radial solution of (4.1) in B_1^*. Then the following occurs.*
(i) Either u is a regular solution of (4.1) in B_1.

(ii) Either

$$u(x) \equiv (\beta_q^{p-1}(N - q\beta_q)^{1/(q+1-p)}|x|^{-\beta_q},$$

or

$$u(x) \equiv -(\beta_q^{p-1}(N - q\beta_q)^{1/(q+1-p)}|x|^{-\beta_q}.$$

(iii) Or $|x|^{\beta_q}u(x)$ is not constant and

$$\lim_{x\to 0}|x|^{\beta_q}u(x) = (\beta_q^{p-1}(N - q\beta_q)^{1/(q+1-p)}|x|^{-\beta_q},$$

or

$$\lim_{x\to 0}|x|^{\beta_q}u(x) = -(\beta_q^{p-1}(N - q\beta_q)^{1/(q+1-p)}|x|^{-\beta_q}.$$

The results related to the cases $p^{\#} = p$, $p = Np/(N-p) - 1$ and $p > Np/(N-p) - 1$ can be found in [1]. For a long time, the non-radial case appeared out of reach up to the recent work of Serrin and Zou [13]. In this striking paper they proved, among other results, that Gidas and Spruck classical a priori estimate in the case $p = 2$, $N/(N-2) \leq q < q_c$ [6] still holds in the range $p > 1$ and $p^{\#} \leq p < Np/(N-p) - 1$ (under a form appropriate to the p-Laplace operator).
Any positive solution u of (4.1) in Ω^* satisfies

$$u(x) \leq C|x|^{-\beta_q}, \tag{4.20}$$

near 0.
The proof is an extremely clever (but difficult) adaptation of the proof given by Gidas and Spruck. Among other results Serrin and Zou provide also a description of entire solutions of the same equation in $\mathbb{R}^N$.

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