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Instantaneous blow-up of solutions to a class of hyperbolic inequalities *

Stanislav I. Pohozaev & Alberto Tesei

Abstract

We prove instantaneous blow-up of nonnegative solutions for a class of semilinear hyperbolic inequalities with first order terms and singular coefficients. The present approach relies on a suitable choice of the functions used to test the differential inequalities.

1 Introduction

In this paper we investigate *instantaneous blow-up* of solutions to semilinear hyperbolic inequalities of the type

$$\begin{gathered} u_{tt} - \Delta u \geq \lambda |x|^{-\mu}(x, \nabla u) + |x|^{-\alpha} u^q \quad \text{in } Q := \Omega \times (0,T] \\ u \geq 0 \quad \text{in } Q. \end{gathered} \tag{1.1}$$

Here $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$ is a bounded smooth domain which contains the origin, $q > 1$ and λ, μ, α are real parameters. By $(\cdot, \cdot)$ we denote the scalar product in $\mathbb{R}^n$.

Instantaneous blow-up can be regarded as *nonexistence of local solutions* - namely, nonexistence of solutions in any neighbourhood of the origin. Similar nonexistence phenomena were investigated in [1] for the semilinear elliptic problem:

$$\begin{gathered} -\Delta u \geq |x|^{-2} u^2 \quad \text{in } \Omega \\ u \geq 0 \quad \text{in } \Omega \end{gathered} \tag{1.2}$$

moreover, instantaneous blow-up results were proved for the companion parabolic inequality (in this connection, see also [2]).

A major step of the proof in [1] was demonstrating a removable singularity result of solutions at the origin, then using comparison results for the extended solutions. In the present approach we take advantage of the local behaviour of solutions near the origin by a direct bootstrap argument, which relies on

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a proper choice of the test functions; no comparison results are needed. This allows us to deal not only with elliptic or parabolic inequalities (see [6]), but also with the hyperbolic case (1.1). Let us mention that the same approach was used elsewhere (e.g., see [3], [5]; a comprehensive account can be found in [4]).

A specific aim of the present paper is to investigate how singular first order terms affect existence of local solutions to problem (1.1). As long as $\mu \leq 2$, nonexistence results are proven only if $\alpha \geq 2$ (see assumptions (a)-(b) of Theorem 2.2 below); in this case nonexistence of local solutions depends on the singularity of the source term as in [1]. On the other hand, if $\mu > \frac{2q}{q-1}$ nonexistence can be proved also for $\alpha < 0$ (see assumption (c) of Theorem 2.2); in this case the coefficient of the source term is regular in Ω, thus the nonexistence result depends on the singularity of the first order term. The same situation occurs for elliptic or parabolic inequalities analogous to (1.1) (see [6]).

2 Mathematical background and results

Solutions to problem (1.1) are meant in the following sense.

Definition 2.1 By a solution to problem (1.1) in $Q := \Omega \times (0,T]$ we mean any function $u \in C([0,T]; H^1_{\mathrm{loc}}(\Omega \setminus \{0\})) \cap C^1([0,T]; L^q_{\mathrm{loc}}(\Omega \setminus \{0\}))$ such that:

(i) $u \geq 0$ almost everywhere in Q;

(ii) for any test function $\zeta \in C^{\infty,2}_{x,t}(\bar{Q})$, $\zeta \geq 0$, $\zeta(\cdot,t) \in C^\infty_0(\Omega \setminus \{0\})$ $(t \in [0,T])$, $\zeta(\cdot,T) = \zeta_t(\cdot,T) = 0$ there holds:

$$\begin{aligned}&\iint_Q (\nabla u, \nabla\zeta) + \lambda \iint_Q u \operatorname{div}\left(|x|^{-\mu} x\zeta\right)\\ &\quad\geq \iint_Q |x|^{-\alpha} u^q \zeta + \int_\Omega u_t(x,0)\zeta(x,0) - \int_\Omega u(x,0)\zeta_t(x,0) - \iint_Q u\zeta_{tt}. \quad (2.1)\end{aligned}$$

The following nonexistence result will be proven.

Theorem 2.2 *Let either of the following assumptions be satisfied:*

(a) $\mu < 2$, $\alpha \geq 2$

(b) $\mu = 2$ *and either* $\alpha > 2, \lambda \geq 2-n$, *or* $\alpha = 2, \lambda > 2-n$

(c) $\mu > 2$, $\alpha > \mu + (2-\mu)q$, *and* $\lambda > 0$.

Moreover, let

$$\liminf_{x\to 0} |x|^{-\gamma} u(x,0) > 0 \quad (2.2)$$

for some $\gamma < 0$ *and*

$$u_t(x,0) \geq 0 \quad (2.3)$$

in some neighbourhood of the origin. Then the only solution to problem (1.1) in any cylinder $\Omega_1 \times (0,\tau]$, $\Omega_1 \subseteq \Omega$ *containing the origin and* $\tau \in (0,T)$, *is trivial.*

Remark 2.3 With the exception of the case $\alpha = 2$, Theorem 2.2 still holds if we assume $\gamma < \frac{\alpha}{q-1}$ in (2.2).

The proof of Theorem 2.2 relies on a proper choice of the test function in inequality (2.1). For this purpose we introduce some preliminary material.

Let $\Omega_1 \subseteq \Omega$ be any neighbourhood containing the origin, $0 < \epsilon < \eta$, $\eta > 2\epsilon$ so small that $A_{\epsilon,\eta} := \{x \in \mathbb{R}^n : \epsilon < |x| < \eta\} \subseteq \Omega_1 \setminus \{0\}$. For $r \in [\epsilon, \eta]$ define

$$\phi_0(r) := r^\sigma - \eta^\sigma,$$

where $\sigma < 0$ will be fixed later. Define also

$$\phi_1(r) := \bar{\phi}(\frac{r}{\epsilon}) \quad (r \in [\epsilon, \eta]),$$

where $\bar{\phi} \in C^\infty([0, \frac{\eta}{\epsilon}])$ is nondecreasing, such that

$$\bar{\phi}(s) := \begin{cases} 0 & \text{if } s \in (0,1) \\ 1 & \text{if } s \in (2, \eta/\epsilon). \end{cases}$$

Finally, set

$$\bar{\zeta}(r) := r^\rho \phi_0(r)\phi_1(r) \quad (r \in [\epsilon, \eta]),$$

where ρ is a real parameter to be chosen later.

Remark 2.4 The following properties of the function $\bar{\zeta}$ are easily checked.
(i) There holds:

$$\bar{\zeta}(\epsilon) = \bar{\zeta}(\eta) = 0; \quad \frac{d\bar{\zeta}}{dr}(\epsilon) \geq 0, \quad \frac{d\bar{\zeta}}{dr}(\eta) \leq 0.$$

(ii) There exists a sequence $\{\zeta_k\} \subseteq C_0^\infty(A_{\epsilon,\eta})$, $\zeta_k \geq 0$ for any k, such that $\zeta_k \to \tilde{\zeta}$ in $W_0^{1,p}(A_{\epsilon,\eta})$ $(p \in (1,\infty))$, where

$$\tilde{\zeta}(x) := \bar{\zeta}(|x|) \quad (x \in \bar{A}_{\epsilon,\eta}). \tag{2.4}$$

3 Proofs

Let us prove the following result.

Proposition 3.1 *Let u be a solution to problem (1.1) in some cylinder $\Omega_1 \times (0,\tau] \subseteq Q$, Ω_1 containing the origin and $\tau \in (0,T)$. Then for any $0 < \epsilon < \eta$, η sufficiently small and any $\tau \in (0,T)$ there holds:*

$$\begin{aligned} &\int_0^\tau (\tau - t)^\beta\, dt \int_{A_{\epsilon,\eta}} |x|^{-\alpha} u^q(x,t)\tilde{\zeta}(x) \\ &\leq -\int_0^\tau (\tau - t)^\beta\, dt \int_{A_{\epsilon,\eta}} |x|^{-(n-1)} \frac{d\psi}{dr}(|x|) u(x,t) \\ &\quad + \beta(\beta-1)\int_0^\tau (\tau - t)^{\beta-2}\, dt \int_{A_{\epsilon,\eta}} u(x,t)\tilde{\zeta}(x) \\ &\quad - \tau^\beta \int_{A_{\epsilon,\eta}} u_t(x,0)\tilde{\zeta}(x) - \beta\tau^{\beta-1}\int_{A_{\epsilon,\eta}} u(x,0)\tilde{\zeta}(x), \end{aligned} \tag{3.1}$$

where $\beta > \frac{q+1}{q-1}$, $\tilde{\zeta}$ *is the function (2.4) and*

$$\psi(r) := r^{n-1}\frac{d\bar{\zeta}}{dr}(r) - \lambda r^{n-\mu}\bar{\zeta}(r) \quad (r \in [\epsilon,\eta]). \tag{3.2}$$

Proof. Let $\tau \in (0,T)$; set

$$\hat{\phi}(t) := \begin{cases} (\tau - t)^{\beta} & \text{if } t \in (0,\tau) \\ 0 & \text{if } t \in (\tau, T)\,. \end{cases}$$

Let $\{\zeta_k\} \subseteq C_0^{\infty}(A_{\epsilon,\eta})$ be the approximating sequence in Remark 2.4-(ii); set $\zeta(x,t) = \zeta_k(x)\hat{\phi}(t)$ in inequality (2.1). Letting $k \to \infty$ we obtain easily:

$$\begin{aligned}
&\int_0^{\tau} (\tau - t)^{\beta}\, dt \int_{A_{\epsilon,\eta}} |x|^{-\alpha} u^q(x,t)\tilde{\zeta}(x) \\
&\quad \leq \int_0^{\tau} (\tau - t)^{\beta}\, dt \Big(\int_{A_{\epsilon,\eta}} (\nabla u(x,t), \nabla\tilde{\zeta}) + \lambda \int_{A_{\epsilon,\eta}} u(x,t)\, \mathrm{div}\left(|x|^{-\mu} x\zeta\right)\Big) \\
&\qquad + \beta(\beta-1)\int_0^{\tau} (\tau - t)^{\beta-2}\, dt \int_{A_{\epsilon,\eta}} u(x,t)\tilde{\zeta}(x) \\
&\qquad - \tau^{\beta}\int_{A_{\epsilon,\eta}} u_t(x,0)\tilde{\zeta}(x) - \beta\tau^{\beta-1}\int_{A_{\epsilon,\eta}} u(x,0)\tilde{\zeta}(x)\,.
\end{aligned}$$

On the other hand, for any $t \in (0,\tau)$ there holds:

$$\begin{aligned}
&\int_{A_{\epsilon,\eta}} (\nabla u(x,t), \nabla\tilde{\zeta}) + \lambda \int_{A_{\epsilon,\eta}} u(x,t)\, \mathrm{div}\left(|x|^{-\mu} x\zeta\right) \\
&\qquad\qquad \leq - \int_{A_{\epsilon,\eta}} u(x,t) \Big\{ \Delta\tilde{\zeta} - \lambda\ \mathrm{div}\left(|x|^{-\mu} x\zeta\right)\Big\}\,.
\end{aligned}$$

An elementary calculation shows that

$$\Delta\tilde{\zeta} - \lambda\ \mathrm{div}\left(|x|^{-\mu} x\zeta\right) = |x|^{-(n-1)}\frac{d\psi}{dr}(|x|);$$

Then from the above inequalities the conclusion follows. $\square$

It is easily checked that the function ψ defined in (3.2) reads:

$$\psi = \phi_1\psi_1 + r^{n-1+\rho}\phi_0\frac{d\phi_1}{dr}, \tag{3.3}$$

where

$$\psi_1 = \psi_1(r) := r^{n-1}\frac{d}{dr}[r^{\rho}\phi_0(r)] - \lambda r^{n-\mu+\rho}\phi_0(r) \quad (r \in [\epsilon,\eta]).$$

The following technical lemma plays an important role in the sequel.

Lemma 3.2 *Let any of the following assumptions be satisfied:*

- *(i)* $\mu < 2$, $\rho \leq 2 - n$, $\sigma < 0$
- *(ii)* $\mu < 2$, $\rho \in (2-n, 0)$, $\rho \leq 4 - n - \mu$, $\sigma = -\rho + 2 - n$
- *(iii)* $\mu = 2$, $\rho \leq 2 - n \leq \lambda$, $\sigma < 0$
- *(iv)* $\mu = 2$, $\rho = \lambda > 2 - n$, $\sigma = -\rho + 2 - n$
- *(v)* $\mu > 2$, $\rho \leq \mu - n$, $\lambda > 0$, $\sigma < 0$.

Then there exists $\eta_0 > 0$ (depending on $n, \lambda, \mu, \rho, \sigma$) such that for any $\eta < \eta_0$ there holds

$$\frac{d\psi_1}{dr} \geq 0 \quad in\ (\epsilon, \eta)\,. \tag{3.4}$$

Proof . We deal only with cases (i)-(ii) for shortness. Observe that

$$\frac{d\psi_1}{dr} = r^{n-3+\rho+\sigma}\psi_0(\frac{r}{\eta}),$$

where

$$\begin{aligned}\psi_0(s) :=& (\rho+\sigma)(n-2+\rho+\sigma) - \rho(n-2+\rho)s^{-\sigma} \\ &- \lambda\Big[(n-\mu+\rho+\sigma) - (n-\mu+\rho)s^{-\sigma}\Big]\eta^{2-\mu}s^{2-\mu} \quad (s \in [0,1]),\end{aligned}$$

as an elementary calculation shows.
(i) Since $\rho \leq 2 - n \leq 0$ and $\sigma < 0$, there holds

$$\begin{aligned}&(\rho+\sigma)(n-2+\rho+\sigma) - \rho(n-2+\rho)s^{-\sigma} \\ &\geq (\rho+\sigma)(n-2+\rho+\sigma) - \rho(n-2+\rho) > 0\,;\end{aligned}$$

in fact, it is easily seen that the function $f(s) := s(n-2+s)$ is strictly increasing in the interval $[\rho+\sigma, \rho]$. Since by assumption $2 - \mu > 0$, choosing η sufficiently small, we prove the claim.
(ii) In this case $n - 2 + \rho + \sigma = 0$ and $-\rho(n-2+\rho) > 0$; since $2 - \mu + \sigma = 4 - n - \mu - \rho \geq 0$, the claim follows as in (i).

In the remaining cases we can argue similarly; hence the conclusion follows. □

Then we have the following result.

Proposition 3.3 *Let u be a solution to problem (1.1) in some cylinder $\Omega_1 \times (0,\tau] \subseteq Q$, Ω_1 containing the origin and $\tau \in (0,T)$. Let the assumptions of Lemma 3.2 be satisfied; choose $\eta < \eta_0$ accordingly. Moreover, let*

$$\rho + \sigma > -\frac{\alpha}{q-1} - n\,. \tag{3.5}$$

Then for any $\epsilon > 0$ sufficiently small and any $\tau \in (0,T)$ there holds:

$$\begin{aligned}
&\int_0^\tau (\tau-t)^\beta\,dt\Big(\int_{A_{\epsilon,\eta}} u(x,t)\tilde{\zeta}(x)\Big)^q \\
&\le M\tau^{\beta-1}C_1(\epsilon,\eta)\Big(C_1(\epsilon,\eta)^{\frac{1}{q-1}}\tau^{-\frac{2}{q-1}} + C_2(\epsilon,\eta)\tau^2 \\
&\quad -\tau\int_{A_{\epsilon,\eta}} u_t(x,0)\tilde{\zeta}(x) - \int_{A_{\epsilon,\eta}} u(x,0)\tilde{\zeta}(x)\Big)\,,
\end{aligned} \tag{3.6}$$

for some constant $M = M(\beta, q) > 0$. Here

$$C_1(\epsilon,\eta) := \Big(\int_\epsilon^\eta r^{\frac{\alpha}{q-1}+n-1}\bar{\zeta}(r)dr\Big)^{q-1}, \tag{3.7}$$

$$C_2(\epsilon,\eta) := \int_{A_{\epsilon,\eta}} |x|^{-(n-1)q'}\Big[|x|^{-\alpha}\tilde{\zeta}(x)\Big]^{-(q'-1)}\chi(|x|)^{q'}\,dx\,. \tag{3.8}$$

Proof. (i) Due to Lemma 3.2 and the choice $\eta < \eta_0$, for any $t \in (0,\tau)$ we have

$$-\int_{A_{\epsilon,\eta}} |x|^{-(n-1)}\frac{d\psi}{dr}(|x|)u(x,t) \le -\int_{A_{\epsilon,\eta}} |x|^{-(n-1)}\chi(|x|)u(x,t), \tag{3.9}$$

where

$$\chi(r) := \frac{d\phi_1}{dr}\psi_1 + \frac{d}{dr}[r^{n-1+\rho}\phi_0\frac{d\phi_1}{dr}] \quad (r \in [\epsilon,\eta]). \tag{3.10}$$

Using Hölder inequality, the right-hand side of inequality (3.9) can be estimated as follows

$$\begin{aligned}
&\Big|\int_{A_{\epsilon,\eta}} |x|^{-(n-1)}\chi(|x|)u(x,t)\Big| \\
&\le \Big(\int_{A_{\epsilon,\eta}} |x|^{-\alpha}u^q(x,t)\tilde{\zeta}(x)\Big)^{1/q}\Big(\int_{A_{\epsilon,\eta}} |x|^{-(n-1)q'}\big[|x|^{-\alpha}\tilde{\zeta}(x)\big]^{-(q'-1)}\chi(|x|)^{q'}\Big)^{1/q'} \\
&= \Big(\int_{A_{\epsilon,\eta}} |x|^{-\alpha}u^q\tilde{\zeta}(x)\Big)^{1/q}\Big(C_2(\epsilon,\eta)\Big)^{1/q'}
\end{aligned} \tag{3.11}$$

(see definition (3.8); here $q' := \frac{q}{q-1}$). Using inequality (3.11) and Young inequality we obtain

$$\Big|\int_{A_{\epsilon,\eta}} |x|^{-(n-1)}\chi(|x|)u(x,t)\Big| \le \frac{1}{q}\Big|\int_{A_{\epsilon,\eta}} |x|^{-\alpha}u^q(x,t)\tilde{\zeta}(x)\Big| + \frac{1}{q'}C_2(\epsilon,\eta) \tag{3.12}$$

for any $t \in (0,\tau)$. Then from inequalities (3.12), (3.9) we obtain for any $t \in (0,\tau)$:

$$\begin{aligned}
&-\int_0^\tau (\tau-t)^\beta\,dt\int_{A_{\epsilon,\eta}} |x|^{-(n-1)}\frac{d\psi}{dr}(|x|)u(x,t) \\
&\qquad \le \frac{1}{q}\int_0^\tau (\tau-t)^\beta\,dt\int_{A_{\epsilon,\eta}} |x|^{-\alpha}u^q(x,t)\tilde{\zeta}(x) + \frac{1}{q'}\frac{\tau^{\beta+1}}{\beta+1}C_2(\epsilon,\eta).
\end{aligned}$$

Substituting the above inequality in (3.1) easily gives

$$\begin{aligned}&\int_0^\tau (\tau-t)^\beta\,dt\int_{A_{\epsilon,\eta}}|x|^{-\alpha}u^q(x,t)\tilde\zeta(x)\,dt\\&\leq\frac{\tau^{\beta+1}}{\beta+1}C_2(\epsilon,\eta)+\beta(\beta-1)q'\int_0^\tau(\tau-t)^{\beta-2}\,dt\int_{A_{\epsilon,\eta}}u(x,t)\tilde\zeta(x)\\&\quad-q'\tau^\beta v'(0)-q'\beta\tau^{\beta-1}v(0)\,.\end{aligned}\tag{3.13}$$

(ii) Observe that for any $t\in(0,\tau)$,

$$\begin{aligned}\int_{A_{\epsilon,\eta}}u(x,t)\tilde\zeta(x)\leq&\Big(\int_{A_{\epsilon,\eta}}|x|^{-\alpha}u^q(x,t)\tilde\zeta(x)\Big)^{1/q}\Big(\int_{A_{\epsilon,\eta}}|x|^{\frac{\alpha}{q-1}}\tilde\zeta(x)\Big)^{1/q'}\\=&\Big(\int_{A_{\epsilon,\eta}}|x|^{-\alpha}u^q(x,t)\tilde\zeta(x)\Big)^{1/q}\Big(\int_\epsilon^\eta r^{\frac{\alpha}{q-1}+n-1}\bar\zeta(r)dr\Big)^{1/q'}\,.\end{aligned}$$

Set

$$v(t):=\int_{A_{\epsilon,\eta}}u(x,t)\tilde\zeta(x),\quad t\in(0,\tau)\,.$$

Then by definition (3.7) the above inequality reads

$$v^q(t)\leq C_1(\epsilon,\eta)\int_{A_{\epsilon,\eta}}|x|^{-\alpha}u^q(x,t)\tilde\zeta(x)\tag{3.14}$$

for any $t\in(0,\tau)$. Then from inequalities (3.14), (3.13) we get

$$\begin{aligned}\int_0^\tau(\tau-t)^\beta v^q(t)\,dt\leq&C_1(\epsilon,\eta)\Big[\frac{\tau^{\beta+1}}{\beta+1}C_2(\epsilon,\eta)+\beta(\beta-1)q'\int_0^\tau(\tau-t)^{\beta-2}\,dt\\&\times\int_{A_{\epsilon,\eta}}u(x,t)\tilde\zeta(x)-q'\tau^\beta v'(0)-q'\beta\tau^{\beta-1}v(0)\Big]\,.\end{aligned}\tag{3.15}$$

(iii) Due to Young inequality,

$$\begin{aligned}&\beta(\beta-1)q'C_1(\epsilon,\eta)\int_0^\tau(\tau-t)^{\beta-2}v(t)\,dt\\&\leq\frac1q\int_0^\tau(\tau-t)^\beta v^q(t)\,dt+\frac{[\beta(\beta-1)]^{q'}(q')^{q'-1}}{\beta-2q'+1}C_1^{q'}(\epsilon,\eta)\tau^{\beta-2q'+1}\end{aligned}$$

(here we used the assumption $\beta>\frac{q+1}{q-1}$). From the previous inequality and (3.13) we obtain

$$\begin{aligned}&(1-\frac1q)\int_0^\tau(\tau-t)^\beta v^q(t)\,dt\\&\leq\frac{\tau^{\beta+1}}{\beta+1}C_1(\epsilon,\eta)C_2(\epsilon,\eta)+\frac{[\beta(\beta-1)]^{q'}(q')^{q'-1}}{\beta-2q'+1}C_1^{q'}(\epsilon,\eta)\tau^{\beta-2q'+1}\\&\quad-q'C_1(\epsilon,\eta)\tau^\beta v'(0)-q'C_1(\epsilon,\eta)\beta\tau^{\beta-1}v(0)\,.\end{aligned}$$

Then the conclusion follows. □

Let us now proceed to prove Theorem 1.1. Observe that, due to assumption (2.3), from inequality (3.6), we obtain

$$\int_0^\tau (\tau - t)^\beta \, dt \Big(\int_{A_{\epsilon,\eta}} u(x,t)\tilde{\zeta}(x) \Big)^q \tag{3.16}$$
$$\le M\tau^{\beta-1} C_1(\epsilon,\eta) \Big\{ C_1(\epsilon,\eta)^{\frac{1}{q-1}} \tau^{-\frac{2}{q-1}} + C_2(\epsilon,\eta)\tau^2 - \int_{A_{\epsilon,\eta}} u(x,0)\tilde{\zeta}(x) \Big\} .$$

We will estimate the various terms in inequality (3.16) as $\epsilon \to 0^+$. This motivates the following considerations:
(α) Observe that for any $t \in [0,\tau]$,

$$\lim_{\epsilon \to 0^+} \int_{A_{\epsilon,\eta}} u(x,t)\tilde{\zeta}(x) = \int_{B_\eta} |x|^{\rho+\sigma} \Big[1 - \big(\frac{\eta}{|x|} \big)^\sigma \Big] u(x,t)$$

by monotonicity, due to the choice of the function $\tilde{\zeta}$. Moreover, due to assumption (2.2), there exist $k > 0$ and $\eta_1 > 0$ such that for any $|x| < \eta < \eta_1$ there holds: $u(x,0) \ge k|x|^\gamma$. Hence

$$\int_{B_\eta} |x|^{\rho+\sigma} \Big[1 - \Big(\frac{\eta}{|x|} \Big)^\sigma \Big] u(x,0) dx \ge k \int_0^\eta r^{\gamma+\rho+\sigma+n-1} \Big[1 - \Big(\frac{\eta}{r} \Big)^\sigma \Big] dr \, .$$

If $\gamma \le -2$, the integral in the right-hand side of the above inequality diverges. On the other hand, if $\gamma > -2$ we obtain:

$$\int_{B_\eta} |x|^{\rho+\sigma} \Big[1 - \Big(\frac{\eta}{|x|} \Big)^\sigma \Big] u(x,0) dx \ge K\eta^{\gamma+\rho+\sigma+n}$$

for some $K > 0$.
(β) Concerning the coefficient $C_1(\epsilon,\eta)$ we have (see definition (3.7))

$$C_1(\epsilon,\eta) \le \Big(\int_\epsilon^\eta r^{\frac{\alpha}{q-1}+\rho+\sigma+n-1} dr \Big)^{q-1} ;$$

thus, by monotonicity

$$\lim_{\epsilon \to 0^+} C_1(\epsilon,\eta) \le L\eta^{\alpha+(\rho+\sigma+n)(q-1)}$$

for some $L > 0$, provided that condition (3.5) is satisfied.
(γ) As for the coefficient $C_2(\epsilon,\eta)$, we claim that

$$C_2(\epsilon,\eta) = C_2(\epsilon,2\epsilon) \le \bar{C}\epsilon^\theta \tag{3.17}$$

for some constant $\bar{C} > 0$, where

$$\theta := n - \alpha + \rho + \sigma + (\alpha - 2)\frac{q}{q-1}. \tag{3.18}$$

In fact, let us estimate the integral in the right-hand side of inequality (3.8). To this purpose, set $s := \frac{r}{\epsilon} \in [1,2]$. It is easily seen that

$$\bar{\zeta}(\epsilon s) \geq c_1 \epsilon^{\rho+\sigma} \bar{\phi}(s),$$
$$\chi(\epsilon s) \leq c_2 \epsilon^{n-3+\rho+\sigma} [\bar{\phi}'(s) + \bar{\phi}''(s)],$$

for some $c_1, c_2 > 0$ and any $s \in [1,2]$; here we used the equalities

$$\phi_1(\epsilon s) = \bar{\phi}(s), \quad \frac{d\phi_1}{dr}(\epsilon s) = \frac{\bar{\phi}'(s)}{\epsilon}, \quad \frac{d^2\phi_1}{dr^2}(\epsilon s) = \frac{\bar{\phi}''(s)}{\epsilon^2}.$$

Moreover, choosing $\bar{\phi}(s) = O((s-1)^\gamma)$ with $\gamma > \max\{2, \frac{q+1}{q-1}\}$ as $s \to 1^+$, we obtain

$$\int_1^2 \Big[\frac{\bar{\phi}'(s)^q}{\bar{\phi}(s)}\Big]^{\frac{1}{q-1}} < \infty, \quad \int_1^2 \Big[\frac{\bar{\phi}''(s)^q}{\bar{\phi}(s)}\Big]^{\frac{1}{q-1}} < \infty.$$

It follows that

$$\int_\epsilon^{2\epsilon} \Big[r^{n-\alpha-1}\bar{\zeta}(r)\Big]^{-\frac{1}{q-1}} \chi(r)^{\frac{q}{q-1}}\, dr \leq \tilde{C}\epsilon^\theta \tag{3.19}$$

for some $\tilde{C} > 0$, where

$$\begin{aligned}\theta :=& (n-3+\rho+\sigma)\frac{q}{q-1} - (n-\alpha-1+\rho+\sigma)\frac{1}{q-1} + 1\\ =& n-\alpha+\rho+\sigma+(\alpha-2)\frac{q}{q-1}.\end{aligned}$$

This proves the claim. □

Proof of Theorem 1.1 Suppose that assumption (a) is satisfied. (the proof cases (b)-(c) being the same we omit them).
(i) Let $\mu < 2$, $\alpha > 2$. In this case we can choose the parameters ρ, σ so that both assumption (i) of Lemma 3.2 and condition (3.5) are satisfied, and moreover the exponent θ defined in (3.18) is positive .

Due to the above remarks (α)–(γ), taking the limit of inequality (3.16) as $\epsilon \to 0^+$ gives

$$\begin{aligned}\int_0^\tau (\tau-t)^\beta\, dt \Big\{ \int_{B_\eta} |x|^{\rho+\sigma} \Big[1 - \Big(\frac{\eta}{|x|}\Big)^\sigma\Big] u(x,t)dx\Big\}^q\\ \leq M\tau^\beta \eta^{\frac{\alpha}{q-1}+\rho+\sigma+n}\Big\{\tau^{-\frac{2}{q-1}} - K\eta^{\gamma-\frac{\alpha}{q-1}}\Big\}\end{aligned}$$

for any $\tau \in (0,T)$, if $\gamma > -2$. In this case the right-hand side of the above inequality is negative for any $\tau > \tau_* = \tau_*(\eta) := K^{-(q-1)}\eta^{\alpha-\gamma(q-1)}$; since $\tau_*(\eta) \to 0^+$ as $\eta \to 0^+$, the conclusion follows in this case.

On the other hand, if $\gamma \leq -2$ the right-hand side of inequality (3.16) tends to $-\infty$ as $\epsilon \to 0^+$, thus a contradiction follows in this case, too. This proves the result in the case $\alpha > 2$.

(ii) Let us assume $\mu < 2, \alpha = 2$. In this case we make the choice $\rho + \sigma + n = 2$, so that both assumption (ii) of Lemma 3.2 and condition (3.5) are satisfied; the above choice gives $\theta = 0$. Taking the limit of inequality (3.16) as $\epsilon \to 0^+$, we obtain

$$\int_0^\tau (\tau - t)^\beta \, dt \Big\{ \int_{B_\eta} |x|^{\rho+\sigma} \Big[1 - \Big(\frac{\eta}{|x|} \Big)^\sigma \Big] u(x,t) dx \Big\}^q \le M\tau^\beta \Big\{ g(\eta,\tau) - K\eta^{\gamma+2} \Big\} ,$$

where

$$g(\eta,\tau) := \eta^{\frac{2q}{q-1}} \tau^{-\frac{2}{q-1}} + \bar{C}\tau^2 .$$

It is easily seen that the function $g(\eta,\cdot)$ has a unique minimum $\tau_* = \tau_*(\eta) := [(q-1)\bar{C}]^{-\frac{q-1}{2q}}\eta$ in $[0,T]$; moreover, $g(\eta,\tau_*) = q\bar{C}\tau_*^2$. Then by the above inequality there holds:

$$\int_0^\tau (\tau - t)^\beta \, dt \Big\{ \int_{B_\eta} |x|^{\rho+\sigma} \Big[1 - \Big(\frac{\eta}{|x|} \Big)^\sigma \Big] u(x,t) \Big\}^q \le M'\tau^{\beta+2} \Big\{ \bar{C} - K\eta^\gamma \Big\}$$

for some $M' > 0$. Since $\gamma < 0$ and $\tau_*(\eta) \to 0^+$ as $\eta \to 0^+$, the conclusion follows in this case, too. This completes the proof. $\square$

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Stanislav I. Pohozaev
Steklov Mathematical Institute, Russian Academy of Sciences,
Gubkina 8, 117966 Moscow, Russia
e-mail: pohozaev@mi.ras.ru

Alberto Tesei
Dipartimento di Matematica "G. Castelnuovo",
Università di Roma "La Sapienza", P.le A. Moro 5,
I-00185 Roma, Italia
e-mail: tesei@mat.uniroma1.it