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A note on $W^{1,p}$ estimates for quasilinear parabolic equations *

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Abstract

This work deals with the study of the $W^{1,p}$ regularity for the solutions to parabolic equations in divergence form. An argument by perturbation based in real analysis is used.

1 Introduction

In this paper we study interior $W^{1,p}$ estimates for solutions to quasilinear parabolic equations in divergence form, namely, solutions to the equation

$$u_t - \operatorname{div} a(x,t,\nabla u) = 0, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

where $a : \Omega \times (0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$. We assume that $a(x,t,\xi)$ is a Caratheodory function (in the sense that it is measurable in (x,t) and continuous with respect to ξ for each x) and that satisfies the following conditions:

(a1) $a(x,t,0) = 0$

(a2) $\langle a(x,t,\xi) - a(x,t,\eta), (\xi - \eta)\rangle \geq \gamma|\xi|^2$.

(a3) $|a(x,t,\eta)| \leq \Gamma|\eta|$,

where γ and Γ are positive constants.

We will work under the following hypotheses.
(H1) (*Reference operator.*) For fixed a_0 satisfying (a1)–(a3), we consider the corresponding parabolic equation

$$w_t - \operatorname{div}(a_0(\nabla w)) = 0 \tag{1.2}$$

If u is a weak solution to (1.2) (see Definition 3.2 below) then there exists $\gamma > 0$ such that

$$\sup_{R'} |\nabla_x u(x,t)|^2 \leq \gamma \frac{1}{|R|} \int_R |\nabla_x u(x,t)|^2 dx\, dt, \tag{1.3}$$

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for all *parabolic rectangles*

$$R = \{(x,t) : |x_i - x_i^0| < \rho,\ i = 1, \dots N,\ t_0 - \rho^2 < t < t_0\} \subset \Omega,$$

where

$$R' = \{(x,t) : |x_i - x_i^0| < \frac{\rho}{2},\ i = 1, \dots N,\ t_0 - (\frac{\rho}{2})^2 < t < t_0\}.$$

(H2) (*Approximation property.*) The vector field $a(x,t,\xi)$ is close to $a_0(\xi)$ in the following sense:

$$|a(x,t,\xi) - a_0(\xi)| \leq \epsilon\, |\xi|, \tag{1.4}$$

uniformly in x and t.

The simplest example for $a_0(\xi)$ is just $a_0(\xi) = \xi$, which corresponds to the heat equation. A classical example for a is $a(x,t,\xi) = A(x,t)\xi$ with $A(x,t)$ a bounded $N \times N$ matrix such that

$$\|A(x,t) - I_{N\times N}\|_\infty \leq \epsilon.$$

Even in the linear case, the results presented here seem to be new. We also point out that more general situations can be considered by the method that we develop. For simplicity, we restrict ourselves to the *quadratic growth* case. With these hypotheses we will be able to show some sort of parabolic Meyers type inequalities. More precisely we have the following main result.

Theorem 1.1 *Let a be a vector field satisfying (a1)–(a3). Assume that (H1) holds. Given $p > 2$ there exists $\epsilon_0 > 0$ such that if for some $0 < \epsilon < \epsilon_0$ (H2) holds, then any weak solution to*

$$u_t - \operatorname{div}(a(x,t,\nabla_x u)) = 0,$$

satisfies that $|\nabla_x u| \in L^q_{\text{loc}}$, $2 < q < p$.

The idea is to estimate the level sets of $\nabla_x u$ and obtain the required growth of their measure to have the integrability property. To do that, we will use a form of the Calderón-Zygmund covering result. For the elliptic case see [1].

2 Preliminary results

In this section we present some tools that will be used along the paper. First of all we will prove the corresponding *Calderón-Zygmund* covering result and explain the properties of the maximal operator which naturally arises in the parabolic setting under study. To be systematic we will give a general *Covering Lemma* that includes the particular case that we will need.

The rectangle collection

For every $k \in \mathbb{Z}$, let $\mathcal{B}_k$ denote a collection of rectangles satisfying the following properties:

$i)$ All the rectangles of $\mathcal{B}_k$ have the same side lengths.

$ii)$ Any two distinct rectangles of $\mathcal{B}_k$ have disjoints interiors.

$iii)$ $\bigcup\{R \in \mathcal{B}_k\} = \mathbb{R}^N$

$iv)$ If $R \in \mathcal{B}_k$ then $R = \bigcup\{R' \in \mathcal{B}_{k-1} \,|\, R' \subset R\}$

$v)$ If δ_k denotes the length of the diagonal of any $R \in \mathcal{B}_k$ then $\delta_k \geq 2\delta_{k+1}$

Observe that given $R \in \mathcal{B}_k$ there exists a unique rectangle in $\mathcal{B}_{k-1}$, its *predecessor* which we denote by R^* so that $R \subset R^*$.

With these properties in hand we have the following observations.

Lemma 2.1 *Fix $R_0 \in \mathcal{B}_{k_0}$ and $0 < \delta < 1$. Assume that $A \subset R_0$ and that $0 < |A| < \delta |R_0|$. Then, there exists a sequence $\{R_d\} \subset \bigcup\limits_{k=k_0}^{\infty} \mathcal{B}_k$ such that:*

1. $|A \setminus \bigcup\limits_d R_d| = 0,$
2. $|A \bigcap R_d| \geq \delta |R_d|,$
3. *If $R_d \subset R$ for some $R \in \bigcup\limits_{k=k_0}^{\infty} \mathcal{B}_k$, with $R \subset R_0$, then $|A \bigcap R| \leq \delta |R|$ (in particular $|A \bigcap R^*| \leq \delta |R^*|$).*

Proof. The proof goes via the usual stopping-time argument. Observe that R_0 does not satisfies 2) by hypothesis. We select all the rectangles in $\mathcal{B}_{k_0+1}$ for which 2) holds.

If $R' \subset R_0$, $R' \in \mathcal{B}_{k_0+1}$ has not been selected we consider the partition of R' by rectangles of $\mathcal{B}_{k_0+2}$ and we take those satisfying 2).

In this way and by induction, we obtain a maximal sequence $\{R_d\}$ satisfying 2) and 3). We call $\mathcal{B}_A = \bigcup R_d$.

To see why $\{R_d\}$ satisfies 1) too, we observe that if $x \in R_0 \setminus \bigcup\{\partial R : R \in \mathcal{B}_A\}$ then there exists a unique sequence $\{R_k(x)\}$ so that

$$x \in R_k(x) \in \mathcal{B}_k, \ \forall k \geq k_0.$$

Moreover, since diam $R_k(x) = \delta_k \downarrow 0$ as $k \uparrow \infty$, we obtain that from the Lebesgue differentiation theorem,

$$\lim_{k \to \infty} \frac{|A \bigcap R_k(x)|}{|R_k(x)|} = 1, \quad a.e.\, x \in A.$$

In particular this says that $A \subset \bigcup R_d$ a.e. and, therefore, 1) holds. $\square$

Observe that the sequence $\{R_d\}$ in Lemma 2.1, is given by maximal rectangles in R_0 for which property 2) holds. We will call this sequence the *Calderón-Zygmund* covering of A. Note that given a rectangle R_k in a Calderón-Zygmund covering of the set A, there exists a nested finite family of parabolic rectangles

$$\tilde{R}_k^1 \supset \tilde{R}_k^2 \supset \cdots \supset \tilde{R}_k^{r(k)} \supset R_k,$$

for which

$$|\tilde{R}_k^l \cap A| \leq \delta |\tilde{R}_k^l|.$$

We say that $\tilde{R}_k^{r(k)}$ is the *predecessor* of R_k and for simplicity of notation we will write $\tilde{R}_k^{r(k)} \equiv \bar{R}_k$.

We give some applications of Lemma 2.1 that are relevant for this work.

Lemma 2.2 *The maximal operator*

$$\mathcal{M}f(x) = \sup_{x \in R \in \bigcup \mathcal{B}_k} \frac{1}{|R|} \int_R |f|$$

is of weak type $(1,1)$.

The proof uses the Vitali covering lemma and the observation that if $R, R' \in \mathcal{B} = \bigcup \mathcal{B}_k$ and with their interiors with nonempty intersection, then either $R \subset R'$ or $R' \subset R$.

Lemma 2.3 *Fix* $R_0 \in \mathcal{B}$ *and* $0 < \delta < 1$. *Given* $A \subset B \subset R_0$ *with the properties,*

a) $|A| < \delta |R_0|$,

b) If R_d *is a rectangle in the Calderón-Zygmund decomposition of* A *(with respect to* R_0 *and* δ*), then* $R_d^* \subset B$,

we conclude that $|A| \leq \delta |B|$

Proof. Consider the sequence $\{R_d^*\}$ of predecessors of the C-Z covering of A with respect to R_0 and δ. Select a subsequence $\{R_{d_l}^*\}$ with disjoint interiors so that

$$\bigcup R_{d_l}^* = \bigcup R_d^*,$$

(a maximal subsequence). Then,

$$|A| = \sum_l |A \bigcap R_{d_l}^*| \leq \delta \sum_l |R_{d_l}^*| = \delta |\bigcup R_{d_l}^*| \leq \delta |B|.$$

3 Approximations Results

Denote the open unit cube in $\mathbb{R}^{N+1}$ by $Q_0 = \{(x,t) \in \mathbb{R}^{N+1} \,||x_i| < 1, \, i = 1,2,\dots,N, \, |t| < 1\}$ and call $Q_0^+ = Q_0 \cap \{\, t > 0 \,\}$ and $Q_0^- = Q_0 \cap \{\, t < 0 \,\}$. A parabolic rectangle R is then the image of Q_0 through any transformation in $\mathbb{R}^{N+1}$ of the form

$$\phi_\alpha(x,t) = (x_0 + \alpha x, t_0 + \alpha^2 t), \quad \alpha > 0, \quad (x_0,t_0) \in \mathbb{R}^{N+1}.$$

Let us consider the following dyadic subdivision: Given a parabolic rectangle R

i) Divide into 2 equal parts each spatial side.

ii) Divide into 2^2 equal parts the temporal side.

We call this procedure a *parabolic subdivision*. It is obvious that with this procedure we obtain 2^{N+2} new parabolic subrectangles. Associated to this subdivision we have the corresponding Calderón-Zygmund decomposition as in the previous section.

Before we continue, some notation is in order. Given a parabolic rectangle

$$R = \{(x,t) : |x_i - x_i^0| < \rho, \, i = 1, \dots N, \, t_0 - \rho^2 < t < t_0\}$$

we define the parabolic boundary of R as

$$\partial_p R = \{(x,t) : |x_i - x_i^0| \le \rho, \, t_0 - \rho^2 = t\} \cup \{(x,t) : |x_i - x_i^0| = \rho, \, t_0 - \rho^2 < t < t_0\}.$$

Example 3.1 In the case of the heat equation (the simplest model) we have the following result by Moser [4]. Let w be a measurable function in Q_0 such that

(i) $w(x,y) \ge 0$

(ii) $w_t, \nabla_x w, w \in L^2(Q_0)$

(iii) $w_t - \Delta w \le 0$ in $\mathcal{D}'(Q_0)$.

We call such w a *positive subsolution* of the heat equation.

According to the classical results by Moser in [4] we have that there exists a positive constant γ such that if w is a positive subsolution to the heat equation then

$$\sup_{R'} w^2(x,t) \le \gamma \frac{1}{|R|} \int_R w^2(x,t) dx\, dt, \tag{3.1}$$

where

$$R = \big\{(x,t) : |x_i - x_i^0| < \rho, \, i = 1, \dots N, \, t_0 - \rho^2 < t < t_0\big\}$$

and $R' = \phi_{1/2}(R)$, that is,

$$R' = \big\{(x,t) : |x_i - x_i^0| < \frac{\rho}{2}, \, i = 1, \dots N, \, t_0 - (\frac{\rho}{2})^2 < t < t_0\big\}.$$

We point out that if u is a solution to the heat equation then $|\nabla_x u|^2$ is also a positive subsolution to the heat equation. Hence, by (2),

$$\sup_{R'} |\nabla_x u(x,t)|^2 \leq \gamma \frac{1}{|R|} \int_R |\nabla_x u(x,t)|^2 dx\, dt, \tag{3.2}$$

where R and R' are parabolic rectangles related as above.

It is clear that hypothesis $(H1)$ extends the previous Example.

Definition 3.2 We say that $u \in L^2(Q_0)$, with $u_t, \nabla_x u \in L^2(Q_0)$, is a weak solution to the equation

$$u_t - \operatorname*{div}_x(a(x,t,\nabla_x u)) = 0$$

if we have

$$\int_{Q_0} u\psi_t dx dt + \int_{Q_0} \langle a(x,t,\nabla_x u)\nabla\psi\rangle dx\, dt = 0,$$

for all ψ in $\mathcal{W}_0^{1,2}$, the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the L^2-norm of the function and its gradient.

Lemma 3.3 *Assume that (H1),(H2) hold. Let u be a weak solution to the equation*

$$u_t - \operatorname{div}_x(A(x,t,\nabla_x u) = 0, \tag{3.3}$$

such that for R, a parabolic rectangle contained in Q_0, u satisfies

$$\frac{1}{|R|} \int_R |\nabla_x u|^2 dx\, dt \leq \mu.$$

If v is the solution to the problem

$$v_t - \operatorname{div}_x(a_0(\nabla v)) = 0, \quad (x,t) \in R$$
$$v\big|_{\partial_p R} = u,$$

then

1. $\frac{1}{|R|}\int_R |\nabla_x(u-v)|^2 dx\, dt \leq \mu\epsilon^2$

2. $\frac{1}{|R|}\int_R |\nabla_x v|^2 dx\, dt \leq \mu(1+\epsilon)^2$

Proof. Write $R = Q \times (T_1, T_2)$. Observe that

$$\begin{aligned}
&\gamma\frac{1}{|R|}\int_R |\nabla(u-v)|^2 dx\, dt \\
\leq\ & \frac{1}{|R|}\int_R \langle a_0(\nabla u) - a_0(\nabla v), \nabla(u-v)\rangle dx\, dt \\
\leq\ & \frac{1}{|R|}\int_Q |u(x,T_2) - v(x,T_2)|^2 dx \\
& + \frac{1}{|R|}\int_R \langle a_0(\nabla u) - a_0(\nabla v), \nabla(u-v)\rangle dx\, dt \\
=\ & \frac{1}{|R|}\int_R [(u-v)_t - (\mathrm{div}\,_x a_0(\nabla u) - \mathrm{div}\,_x a_0(\nabla v))]\,(u-v) dx\, dt \\
=\ & \frac{1}{|R|}\int_R (u_t - \mathrm{div}\,_x a_0(\nabla u))(u-v) dx\, dt \\
& - \frac{1}{|R|}\int_R (v_t - \mathrm{div}\,_x a_0(\nabla v))(u-v) dx\, dt \\
=\ & \frac{1}{|R|}\int_R [u_t - \mathrm{div}\,_x (a(x,t,\nabla_x u)](u-v) dx\, dt \\
& + \frac{1}{|R|}\int_R \langle \nabla_x(u-v), (a_0(\nabla u) - a(x,t,\nabla_x u))\rangle dx\, dt \\
\leq\ & \epsilon\frac{1}{|R|}\Big(\int_R |\nabla_x(u-v)|^2 dx\, dt\Big)^{1/2}\Big(\int_R |\nabla_x u|^2 dx\, dt\Big)^{1/2},
\end{aligned}$$

where we have used for the last inequality that u is a solution to equation (1.1), condition (H2) and Cauchy-Schwarz inequality. We thus obtain (1). Now (2) is an easy consequence of (1) since, in fact,

$$\begin{aligned}
&\Big(\frac{1}{|R|}\int_R |\nabla_x v|^2 dx\, dt\Big)^{1/2} \\
=\ & \Big(\frac{1}{|R|}\int_R |\nabla_x(u + (v-u))|^2 dx\, dt\Big)^{1/2} \\
\leq\ & \Big(\frac{1}{|R|}\int_R [|\nabla_x u|^2 dx\, dt\Big)^{1/2} + \Big(\frac{1}{|R|}\int_R |\nabla_x(v-u))|^2 dx\, dt\Big)^{1/2} \\
\leq\ & (1+\epsilon)\mu^{1/2}.
\end{aligned}$$

□

We will now introduce the (dyadic) parabolic maximal operator, defined for $f \in L^1_{\mathrm{loc}}$ by

$$Mf(x) = \sup_{x\in R}\frac{1}{|R|}\int_R |f(y)|dy,$$

where the supremum is taken over all parabolic rectangles R containing $x \in \mathbb{R}^{N+1}$. Taking into account that $\mathbb{R}^{N+1}$ is a homogeneous space with respect to the seminorm

$$d(x,t) = \max\{|x_i|, |t|^{1/2},\ i = 1, 2\dots N\},$$

more precisely that

$$|\phi_2(R)| \le 2^{N+2}|R|,$$

we have that M is of weak type $(1,1)$ as a consequence of the usual Besicovich covering lemma. The following result is the main ingredient in Theorem 1.1 below.

Lemma 3.4 *Assume that u is a weak solution to*

$$u_t - \operatorname{div}_x(a(x,t,\nabla_x u)) = 0, \ \textit{in } Q_0.$$

Then, there exists a constant $C > 1$ so that for $0 < \delta < 1$ fixed, one can find $\epsilon_0 = \epsilon_0(\delta) > 0$ such that if $(H2)$ holds with $\epsilon < \epsilon_0$, for all parabolic rectangle R_k in the Calderón-Zygmund δ-covering of

$$\{(x,t) \in \mathbb{R}^{N+1} : M(|\nabla_x u|^2)(x,t) > C\mu\},$$

then its predecessor satisfies

$$\bar{R}_k \subset \{(x,t) : M(|\nabla_x u|^2)(x,t) > \mu\}.$$

In particular, we have

$$|\{(x,t) \in \mathbb{R}^{N+1} : M(|\nabla_x u|^2)(x,t) > C\mu\}| \le |\{(x,t) : M(|\nabla_x u|^2)(x,t) > \mu\}|.$$

Proof. The proof is similar to Lemma 3 in [1] and we sketch it here for the sake of completeness. Since we look for a local result, we can assume that a parabolic dilation of R_k is contained in Q_0, to be more precise, let us assume, say, that $Q = \phi_4(R_k) \subset Q_0$. We argue by contradiction. If R_k satisfies the hypothesis, namely,

$$|R_k \cap \{x : M(|\nabla_x u|^2) > C\mu\}| > \delta |R_k|$$

and $\bar{R}_k$ does not satisfy the conclusion, there exists $(x_0,t_0) \in \bar{R}_k$ for which,

$$\frac{1}{|R|}\int_R |\nabla_x u|^2 dx\,dt \le \mu, \ \text{for all parabolic rectangles } R \text{ with } (x_0,t_0) \in R.$$

We solve the problem

$$\begin{gathered} v_t - \operatorname{div}_x(a_0(\nabla v)) = 0, \quad (x,t) \in Q_0 \\ v\big|_{\partial_p Q_0} = u. \end{gathered}$$

Then, according to Lemma 3.3 we get:

1. $\frac{1}{|Q|}\int_Q |\nabla_x v|^2 dx\,dt \le (1+\epsilon)^2\mu$
2. $\frac{1}{|Q|}\int_Q |\nabla_x (u-v)|^2 dx\,dt \le \epsilon^2\mu$

Then the restricted maximal operator,

$$M^*(|\nabla_x u|^2)(x,t) = \sup_{x\in R,\, R\subset\phi_2(R_k)} \frac{1}{|R|}\int_R |\nabla_x u|^2 dyds,$$

satisfies

$$M(|\nabla_x u|^2)(x,t) \le \max\{M^*(|\nabla_x u|^2)(x,t), 4^{N+2}\mu\}.$$

Consider $C = \max\{4^{N+2}, 4(1+\epsilon)^2\}$. Then

$$\begin{aligned}
&|\{(x,t)\in R_k : M(|\nabla_x u|^2) > C\mu\}| \\
&\quad\le |\{(x,t)\in R_k : M^*(|\nabla_x u|^2) > C\frac{\mu}{2}\}| \\
&\qquad + |\{(x,t)\in R_k : M^*(|\nabla_x (u-v)|^2) > C\frac{\mu}{2}\}| \\
&\quad\le |\{(x,t)\in R_k : M^*(|\nabla_x (u-v)|^2) > C\frac{\mu}{2}\}|.
\end{aligned}$$

By the (1,1) weak type estimate for the maximal operator we conclude

$$\begin{aligned}
|\{x\in R_k : M(|\nabla_x u|^2) > C\mu\}| &\le |\{x\in R_k : M^*(|\nabla_x (u-v)|^2) > C\frac{\mu}{2}\}| \\
&\le A\frac{2}{C\mu}\int_{R_k} |\nabla_x (u-v)|^2 dx\, dt \\
&\le A\frac{2}{C\mu}\epsilon^2 |R_k|.
\end{aligned}$$

Taking $\epsilon > 0$ so that $A\frac{2}{C\mu}\epsilon^2 < \delta$ we reach a contradiction. $\square$

4 Proof of the main result

As a consequence of the approximation Lemma 3.3 and the behavior of the level sets of the maximal operator described in Lemma 3.4 we can formulate the following regularity result.

Theorem 4.1 *Assume that (H1) holds. Given $p > 2$ there exists $\epsilon_0 > 0$ such that if for some $0 < \epsilon < \epsilon_0$, (H2) holds, then any weak solution to*

$$u_t - \operatorname{div}(A(x,t)\nabla_x u) = 0,$$

satisfies that $|\nabla_x u| \in L^q_{\mathrm{loc}}$, $2 < q < p$.

Proof. For $s > 0$, call $\omega(s) = |\{(x,t) : M(|\nabla_x u|^2(x,t) > s\}|$, the distribution function of the maximal operator. Take $\delta \in (0,1)$ in such a way that $C^{q/2}\delta < 1$, where C is as in Lemma 3.4. Now, there exists ϵ_0, such that if $0 < \epsilon < \epsilon_0$ and (H2) holds, then Lemmas 3.4 and 2.3 imply

$$\omega(C\mu_0) \le \delta\omega(\mu_0).$$

Hence by recurrence

$$\omega(C^k\mu_0) \leq \delta^k \omega(\mu_0). \tag{4.1}$$

Now $|\nabla_x u|^q \in L^1$ if, in particular, $M(|\nabla_x u|^2) \in L^{q/2}$, and this is equivalent to the convergence of the series

$$\sum_{k=1}^{\infty} C^{k(q/2)} \omega(C^k\mu_0).$$

But, $C^{q/2}\delta < 1$ and from estimate (7) we obtain

$$\sum_{k=1}^{\infty} C^{k(q/2)} \omega(C^k\mu_0) \leq \sum_{k=1}^{\infty} (C^{q/2}\delta)^k \omega(\mu_0) < \infty$$

□

Corollary 4.2 *Assume $A(x,t)$ a $N \times N$ matrix which is continuous in Ω and such that*

$$\langle A(x,t)\xi, \xi\rangle \geq \gamma|\xi|^2.$$

Then if u is a weak solution to

$$u_t - \operatorname{div}_x(A(x,t)\nabla_x u) = 0,$$

we have $u \in W^{1,p}_{\mathrm{loc}}$ for all $1 < p < \infty$.

Proof. As reference equation we take the heat equation. The hypothesis $(H2)$ is obtained easily by an orthogonal change of variables in $\mathbb{R}^N$ and our assumptions on the continuity of A. □

By the same method we are able to get estimates for equations that are close to the p-Laplacian. For estimates to $u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ see [2].

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