2001-Luminy conference on Quasilinear Elliptic and Parabolic Equations and Systems,
Electronic Journal of Differential Equations, Conference 08, 2002, pp 103–120.
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Geometry of the energy functional and the Fredholm alternative for the p-Laplacian in higher dimensions *

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Abstract

In this paper we study Dirichlet boundary-value problems, for the p-Laplacian, of the form

$$\begin{gathered} -\Delta_p u - \lambda_1 |u|^{p-2}u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geq 1, p > 1$, $f \in C(\bar{\Omega})$ and $\lambda_1 > 0$ is the first eigenvalue of Δ_p. We study the geometry of the energy functional

$$E_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega fu$$

and show the difference between the case $1 < p < 2$ and the case $p > 2$. We also give the characterization of the right hand sides f for which the Dirichlet problem above is solvable and has multiple solutions.

1 Introduction and statement of the results

Our aim is to study the solvability of the Dirichlet boundary-value problem

$$\begin{gathered} -\Delta_p u - \lambda_1 |u|^{p-2}u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Here $p > 1$ is a real number, Ω is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $f \in C(\bar{\Omega})$. We assume that if $N \geq 2$ then $\partial\Omega$ is a compact connected manifold of class C^2. By λ_1 we denote the first eigenvalue of the related homogeneous eigenvalue problem

$$\begin{gathered} -\Delta_p u - \lambda |u|^{p-2}u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

* *Mathematics Subject Classifications:* 35J60, 35P30, 35B35, 49N10.
Key words: p-Laplacian, variational methods, PS condition, Fredholm alternative, upper and lower solutions

Published October 21, 2002.

In this paper, the function u is said to be a (*weak*) *solution* of (1.1) if $u \in W_0^{1,2}(\Omega)$ and the integral identity

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v - \lambda_1 \int_\Omega |u|^{p-2}uv = \int_\Omega fv \tag{1.3}$$

holds for all $v \in W_0^{1,p}(\Omega)$.

As for the properties of λ_1 (see e.g. [2, 17]), let us mention that λ_1 is positive, simple and isolated and the corresponding eigenfunction φ_1 (associated with λ_1) satisfies $\varphi_1 > 0$ in Ω, $\frac{\partial \varphi_1}{\partial n} < 0$ on $\partial\Omega$, where n denotes the exterior unit normal to $\partial\Omega$. One also has $\varphi_1 \in C^{1,\nu}(\bar\Omega)$ with some $\nu \in (0,1)$ (see e.g. [9, Lemma 2.2, p. 115]). Moreover, λ_1 can be characterized as the best (the greatest) constant $C > 0$ in the Poincaré inequality

$$\int_\Omega |\nabla u|^p \geq C \int_\Omega |u|^p \tag{1.4}$$

for all $u \in W_0^{1,p}(\Omega)$, where identity

$$\int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p = 0$$

holds exactly for the multiples of the first eigenfunction φ_1.

Let us recall (see e.g. [9, pp. 114, 115]) that, for every $h \in L^\infty(\Omega)$, the problem

$$\begin{aligned} \Delta_p u &= h \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

has a unique solution $u \in W_0^{1,p}(\Omega) \cap C^{1,\nu}(\bar\Omega)$. Moreover, since $C^{1,\nu}(\bar\Omega)$ is compactly imbedded into $C^1(\bar\Omega)$, we can introduce the compact operator

$$\Delta_p^{-1} \colon L^\infty(\Omega) \to C^1(\bar\Omega)$$

such that $u = \Delta_p^{-1}h$ is the unique solution of (1.5). In particular, every solution of (1.1) belongs to $C_0^1(\bar\Omega)$.

In our further considerations we will use the standard spaces $W_0^{1,p}(\Omega)$, $L^p(\Omega), C(\bar\Omega)$ and $C^1(\bar\Omega)$ (or $C_0^1(\bar\Omega)$, respectively), with corresponding norms

$$\|u\| = \Big(\int_\Omega |\nabla u|^p\Big)^{1/p}, \quad \|u\|_{L^p} = \Big(\int_\Omega |u|^p\Big)^{1/p},$$
$$\|u\|_C = \max_{x\in\bar\Omega} |u(x)|, \quad \|u\|_{C^1} = \|u\|_C + \max_{x\in\bar\Omega} |\nabla u(x)|,$$

respectively, (here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}$ or $\mathbb{R}^N$). The subscript 0 indicates that the traces (or values) of functions are equal zero on $\partial\Omega$. Moreover, for the element h of any of the above mentioned space we use the following (L^2–orthogonal) decomposition

$$h(x) = \tilde h(x) + \bar h \varphi_1(x),$$

and also L^2–nonorthogonal decomposition

$$h(x) = \tilde{h}(x) + \hat{h},$$

where $\bar{h}, \hat{h} \in \mathbb{R}$ and

$$\int_\Omega \tilde{h}(x)\varphi_1(x)dx = 0.$$

The particular subspaces formed by $\tilde{h}(x)$ will be denoted by $\tilde{W}_0^{1,p}(\Omega), \tilde{C}(\bar{\Omega})$, and $\tilde{C}_0^1(\bar{\Omega})$, respectively.

By $B_X(v, \rho)$ we denote the open ball in the space X with the center v and radius ρ, where $X = C(\bar{\Omega})$ or $X = C_0^1(\bar{\Omega})$. We introduce the energy functional associated with (1.1):

$$E_f(u) \colon = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega fu, \ u \in W_0^{1,p}(\Omega).$$

This functional is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ and its *critical points* correspond one–to–one to *solutions* of (1.1).

Our main results concern the geometry of E_f and the structure of the set of its critical points on one hand and the solvability properties of (1.1) on the other hand. They are formulated in theorems below.

Theorem 1.1 *Let $1 < p < 2$ and $0 \neq \tilde{f} \in \tilde{C}(\bar{\Omega})$. Then there exists $\rho = \rho(\tilde{f}) > 0$ such that for any $f \in B_C(\tilde{f}, \rho)$ the functional E_f is unbounded from below and has at least one critical point. Moreover, for $f \in B_C(\tilde{f}, \rho) \setminus \tilde{C}(\bar{\Omega})$ the functional E_f has at least two distinct critical points.*

Theorem 1.2 *Let $p > 2$ and $0 \neq \tilde{f} \in \tilde{C}(\bar{\Omega})$. Then the functional $E_{\tilde{f}}$ is bounded from below and has at least one critical point (which is the global minimizer). Moreover, there exists $\rho = \rho(\tilde{f}) > 0$ such that for $f \in B_C(\tilde{f}, \rho) \setminus \tilde{C}(\bar{\Omega})$ the functional E_f has at least two distinct critical points.*

Theorem 1.3 *Let $p > 1, p \neq 2, \tilde{f} \in \tilde{C}(\bar{\Omega})$. Then the problem (1.1) has at least one solution if $f = \tilde{f}$. For $0 \neq \tilde{f} \in \tilde{C}(\bar{\Omega})$ there exists $\rho = \rho(\tilde{f}) > 0$ such that (1.1) has at least one solution for any $f \in B_C(\tilde{f}, \rho)$. Moreover, there exist real numbers $F_- < 0 < F_+$ (see Fig. 1) such that the problem (1.1) with $f = \tilde{f} + \hat{f}$ has*

(i) No solution for $\hat{f} \notin [F_-, F_+]$

(ii) At least two distinct solutions for $\hat{f} \in (F_-, 0) \cup (0, F_+)$

(iii) At least one solution for $\hat{f} \in \{F_-, 0, F_+\}$.

Remark 1.4 Note that standard bootstrap regularity argument implies that any solution from Theorems 1.1–1.3 belongs to $L^\infty(\Omega)$ (cf. Drábek, Kufner, Nicolosi [10]). It follows then from the regularity results of Tolksdorf [23] (see also Di Benedetto [6] and Liebermann [16]) that it belongs to $C^{1,\nu}(\bar{\Omega})$ with some $\nu \in (0, 1)$. In particular, our solution is an element of $C_0^1(\bar{\Omega})$.

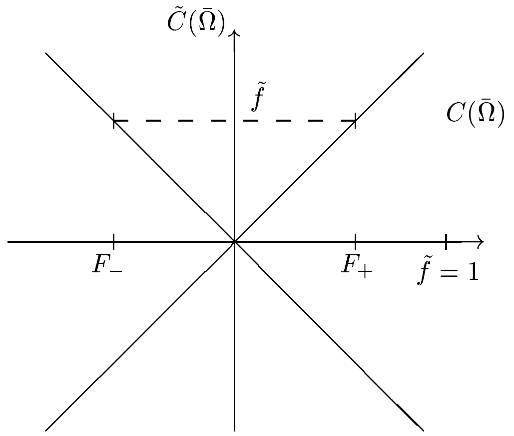

Figure 1: "Slice" of $C(\bar{\Omega})$ containing all constants and one fixed $\tilde{f} \in \tilde{C}(\bar{\Omega})$.

Remark 1.5 In particular, it follows from our results that the set of $f \in C(\bar{\Omega})$ for which (1.1) has at least one solution has a nonempty interior in $C(\bar{\Omega})$.

Remark 1.6 Note that Theorem 1.3 provides necessary and sufficient condition for solvability of the problem (1.1). This condition is in fact of Landesman–Lazer type (see [15], cf. also [11]). Indeed, given $\tilde{f} \in \tilde{C}(\bar{\Omega}), \tilde{f} \neq 0$, the problem (1.1) with the right hand side $f(x) = \tilde{f}(x) + \hat{f}$ has a solution if and only if

$$F_-(\tilde{f}) \leq \frac{1}{\|\varphi_1\|_{L^1}} \int_\Omega f(x)\varphi_1(x)dx \leq F_+(\tilde{f}).$$

However, it should be pointed out that this condition differs from the original condition of Landesman and Lazer due to the fact that F_- and F_+ depend on the component $\tilde{f}$ of the right hand side f and not on the perturbation term (which is actually not present in our problem (1.1)). By homogeneity we have that for any $t > 0$,

$$F_\pm(t\tilde{f}) = tF_\pm(\tilde{f}).$$

Our proofs rely on the combination of the variational approach and the method of lower and upper solutions. We also use essentially the results obtained by Drábek and Holubová [8], Takáč [21] and Fleckinger–Pellé and Takáč [14]. In fact, Theorem 1.1 was proved already in [8], however, here different approach is used. During the preparation of this manuscript the author received preprint of Takáč [22], where similar result to our Theorem 1.3 is proved. However, the approach used in [22] is very different from ours.

Our objective in this paper is to *avoid* complicated *technical assumptions*. For this reason we restrict to rather special domains Ω and right hand sides f. On the other hand, we belive that in our approach the main ideas appear more clearly and that possible generalization of Ω or f will not bring any new insight neither into the geometry of E_f nor to the solvability of (1.1).

It should be mentioned that our approach covers also the case $N = 1$, and completes thus previous results in this direction proved by Del Pino, Drábek and Manásevich [5], Drábek, Girg and Manásevich [7], Manásevich and Takáč [18], Binding, Drábek and Huang [3], Drábek and Takáč [12]. In fact, the first relevant result which led to better understanding of the problem appeared in [5].

Note also that our Theorems 1.1, 1.2 and 1.3 express not only the difference between the linear case $p = 2$ and the nonlinear case $p \neq 2$ but also the striking difference between the case $1 < p < 2$ and the case $p > 2$. The main goal of this paper is actually to emphasize this fact.

2 Auxiliary assertions, survey of known facts

It should be pointed out that E_f is continuously differentiable and weakly lower semicontinuous functional on $W_0^{1,p}(\Omega)$. The following notions are crutial in the study of the geometry of the functional E_f.

Definition 2.1 We say that the functional

$$E_f : W_0^{1,p}(\Omega) \to \mathbb{R}$$

has a *local saddle point geometry* if we can find $u, v \in W_0^{1,p}(\Omega)$ which are separated by $\tilde{W}_0^{1,p}(\Omega)$ in the sense that

$$E_f(u) < \inf_{w \in \tilde{W}_0^{1,p}(\Omega)} E_f(w), \quad E_f(v) < \inf_{w \in \tilde{W}_0^{1,p}(\Omega)} E_f(w)$$

and any continuous path from u to v in $W_0^{1,p}(\Omega)$ has a nonempty intersection with $\tilde{W}_0^{1,p}(\Omega)$.

We say that E_f has a *local minimizer geometry* if we can find open bounded set $D \subset W_0^{1,p}(\Omega)$ such that

$$\inf_{u \in D} E_f(u) < \inf_{u \in \partial D} E_f(u).$$

The following lemma is crutial for application of variational methods. Its proof can be found in [8, Lemma 2.2] (or in [7, Proposition 2.1] in one dimensional case).

Lemma 2.2 *Let $p > 1$, $f = \tilde{f} + \bar{f}\varphi_1$ with $\bar{f} \neq 0$. Then E_f satisfies Palais–Smale (PS) condition, i.e. if $E_f(u_n) \to c \in \mathbb{R}$, $E_f'(u_n) \to 0$ then $\{u_n\}$ contains strongly convergent subsequence in $W_0^{1,p}(\Omega)$.*

Note that the assertion of Lemma 2.2 is not true if $\bar{f} = 0$ (see [5]). The following assertion deals with the case $1 < p < 2$ and provides the information about the geometry of the energy functional E_f.

Lemma 2.3 (see [8, Lemma 2.1]) *Let $1 < p < 2$ and $\tilde{f} \in \tilde{C}(\bar{\Omega}), \tilde{f} \neq 0$. Then $E_{\tilde{f}}$ has a local saddle point geometry. Moreover, there are two sequences $\{u_n\}, \{v_n\} \subset W_0^{1,p}(\Omega)$ such that for any $n \in \mathbb{N}$, u_n and v_n are separated by $\tilde{W}_0^{1,p}(\Omega)$ and*

$$E(u_n) \to -\infty, \quad E(v_n) \to -\infty.$$

Later, in Section 4, we show that the situation is different if $p > 2$ and prove that E_f has a local minimizer geometry in this case.

The following notions are crutial in the application of the method of lower and upper solutions.

Definition 2.4 A function $u_s \in C^1(\bar{\Omega})$ is an *upper solution* of (1.1) if

$$\int_\Omega |\nabla u_s|^{p-2} \nabla u_s \cdot \nabla v - \lambda_1 \int_\Omega |u_s|^{p-2} u_s v \geq \int_\Omega f v \quad \forall v \in W_0^{1,p}(\Omega), v \geq 0,$$
$$u_s \geq 0 \quad \text{on } \partial\Omega.$$

In an analogous way we define *a lower solution* u_l of (1.1).

Definition 2.5 Let $u, v \in C^1(\bar{\Omega})$. We say that $u \prec v$ if $u(x) < v(x)$ on Ω, and for $x \in \partial\Omega$ either $u(x) < v(x)$, or $u(x) = v(x)$ and $(\partial u/\partial n)(x) > (\partial v/\partial n)(x)$.

Definition 2.6 A *lower solution* u_l of (1.1) is said to be *strict* if every solution u of (1.1) such that $u_l \leq u$ on Ω satisfies $u_l \prec u$. In an analogous way we define a *strict upper solution* of (1.1).

For $h \in C(\bar{\Omega})$ we define an operator $T_f\colon C_0^1(\bar{\Omega}) \to C_0^1(\bar{\Omega})$ as $T_f(v) = u$ where u satisfies

$$\Delta_p u = f(x) - \lambda_1 |v|^{p-2} v \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

The operator T_f is compact and its fixed points, i.e. $u = T_f(u)$ $u \in C_0^1(\Omega)$, correspond to solutions of the original problem (1.1). The following assertions are proved in [8], the idea comes from [4].

Lemma 2.7 (Well–Ordered Lower and Upper Solutions) *Let u_l and u_s be lower and upper solutions, respectively, of (1.1) such that $u_l \leq u_s$. Then the problem (1.1) has at least one solution u satisfying*

$$u_l \leq u \leq u_s \quad \text{in } \Omega.$$

If, moreover, u_l and u_s are strict and satisfy $u_l \prec u_s$, then there exists $R_0 > 0$ such that for all $R \geq R_0$

$$\deg[I - T_f; \mathcal{M}_1, 0] = 1,$$

where $\mathcal{M}_1 = \{u \in C_0^1(\bar{\Omega}); u_l \prec u \prec u_s\} \cap B_{C_0^1}(0, R)$.

Lemma 2.8 (Non–Ordered Lower and Upper Solutions) *Let u_l and u_s be lower and upper solutions, respectively, of (1.1) and $u_l(x_0) > u_s(x_0)$ for some $x_0 \in \Omega$. Then (1.1) has at least one solution in the closure (with respect to C^1-norm) of the set*

$$S = \{u \in C_0^1(\bar{\Omega}); x_1, x_2 \in \Omega : \ u(x_1) < u_l(x_1), u(x_2) > u_s(x_2)\}.$$

Set $\mathcal{M}_2 = S \cap B_{C_0^1}(0, R)$ and assume that there is no solution of (1.1) on $\partial\mathcal{M}_2$. Then there exists $R_0 > 0$ such that for all $R \geq R_0$

$$\deg[I - T_f; \mathcal{M}_2, 0] = -1.$$

As an immediate consequence of Lemmas 2.7 and 2.8 we have the following proposition.

Proposition 2.9 *Let (1.1) be solvable for $f_1 \in C(\bar{\Omega})$ and $f_2 \in C(\bar{\Omega})$ such that $f_1(x) \leq f_2(x)$, $x \in \bar{\Omega}$. Then it is also solvable for any $f \in C(\bar{\Omega})$ such that $f_1(x) \leq f(x) \leq f_2(x), x \in \bar{\Omega}$.*

Proof. Let u_i be a solution of (1.1) with $f_i, i = 1, 2$. Then $u_l = u_1$ and $u_s = u_2$ are lower and upper solutions, respectively, of (1.1) with f. Then either Lemma 2.7 or 2.8 applies to get a solution. □

The following assertion deals with the case $p > 2$ and helps to get the information about the geometry of the energy functional E_f.

Proposition 2.10 ([14, Theorem 1.1]) *There exists a positive constant $C = C(p, \Omega)$ such that for all $u \in W_0^{1,p}(\Omega), u(x) = \tilde{u}(x) + \bar{u}\varphi_1(x)$,*

$$\int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p \geq C\Big(|\bar{u}|^{p-2} \int_\Omega |\nabla\varphi_1|^{p-2}|\nabla\tilde{u}|^2 + \int_\Omega |\nabla\tilde{u}|^p\Big).$$

We will need also the following imbedding type inequality (see [21, Lemma 4.2], [14, Lemma 4.2]): Let $p > 2$, then there exists $\tilde{C} > 0$ such that for all $u \in W_0^{1,p}(\Omega)$,

$$\Big(\int_\Omega |u|^2\Big)^{1/2} \leq \tilde{C}\Big(\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla u|^2\Big)^{1/2}. \tag{2.1}$$

The last assertion of this section is related to the application of the degree argument in the proof of Theorem 1.3.

Proposition 2.11 (see [21, Theorems 2.3 and 2.8]) *Let $p > 1$ and K be a compact set in $C(\bar{\Omega})$ and $\int_\Omega f\varphi_1 \neq 0$ for any $f \in K$. Then there exists a constant $\tilde{C}_1 = \tilde{C}_1(K) > 0$ such that*

$$\|u\|_{C_0^1} \leq \tilde{C}_1$$

for any possible solution u of (1.1) with $f \in K$.

3 Proof of Theorem 1.1

For the case $1 < p < 2$, consider the energy functional

$$E_{\tilde{f}}(u) :\ = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega \tilde{f}u,\ u \in W_0^{1,p}(\Omega),$$

where $\tilde{f} \in \tilde{C}(\bar{\Omega}), \tilde{f} \not\equiv 0$. It was proved in Drábek and Holubová [8] that this functional has a *local saddle point geometry* and, in particular, it is *unbounded from below* (see also Lemma 2.3). It is also known (see DelPino, Drábek and Manásevich [5]) that $E_{\tilde{f}}$ does not satisfy (PS) condition in general. So we cannot deduce the existence of critical point of $E_{\tilde{f}}$ directly.

It follows from [8, proof of Lemma 2.1] that

$$\lim_{|\bar{u}|\to\infty} \inf_{\tilde{u}\in\tilde{W}_0^{1,p}(\Omega)} \{\frac{1}{p}\int_\Omega |\bar{u}\nabla\varphi_1 + \nabla\tilde{u}|^p - \frac{\lambda_1}{p}\int_\Omega |\bar{u}\varphi_1 + \tilde{u}|^p - \int_\Omega \tilde{f}\tilde{u}\} = -\infty. \quad (3.1)$$

Moreover, the infimum is achieved for any fixed $\bar{u} \in \mathbb{R}$ at some $\tilde{u}_{\bar{u}} \in \tilde{W}_0^{1,p}(\Omega)$. Indeed, for fixed $\bar{u} \in \mathbb{R}$ the functional

$$\tilde{u} \mapsto \frac{1}{p}\int_\Omega |\bar{u}\nabla\varphi_1 + \nabla\tilde{u}|^p - \frac{\lambda_1}{p}\int_\Omega |\bar{u}\varphi_1 + \tilde{u}|^p - \int_\Omega \tilde{f}\tilde{u}$$

is weakly lower semicontinuous and coercive on $\tilde{W}_0^{1,p}(\Omega)$. Weak lower semicontinuity follows from the same property of the norm on $\tilde{W}_0^{1,p}(\Omega)$ and compactness of the imbedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$. Coercivity is proved via contradiction. Assume that there is a sequence $\{\tilde{u}_n\} \subset \tilde{W}_0^{1,p}(\Omega)$ such that $\|\tilde{u}_n\| \to \infty$, and

$$\frac{1}{p}\int_\Omega |\bar{u}\nabla\varphi_1 + \nabla\tilde{u}_n|^p - \frac{\lambda_1}{p}\int_\Omega |\bar{u}\varphi_1 + \tilde{u}_n|^p - \int_\Omega \tilde{f}\tilde{u}_n \le C$$

for some constant $C > 0$ independent of n. Dividing the last inequality by $\|\tilde{u}_n\|^p$ and passing to the limit for $n \to \infty$, we obtain

$$\lim_{n\to\infty}\{\frac{1}{p}\int_\Omega |\frac{\bar{u}\nabla\varphi_1}{\|\tilde{u}_n\|} + \nabla\hat{\tilde{u}}_n|^p - \frac{\lambda_1}{p}\int_\Omega |\frac{\bar{u}\varphi_1}{\|\tilde{u}_n\|} + \hat{\tilde{u}}_n|^p - \int_\Omega \tilde{f}\frac{\tilde{u}_n}{\|\tilde{u}_n\|^p}\} \le 0,$$

where $\hat{\tilde{u}}_n = \frac{\tilde{u}_n}{\|\tilde{u}_n\|}$. The closedness of $\tilde{W}_0^{1,p}(\Omega)$ and the compactness of the imbedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ imply that there exists $\tilde{u}_0 \in \tilde{W}_0^{1,p}(\Omega), \|\tilde{u}_0\| = 1$, such that

$$\frac{1}{p}\int_\Omega |\nabla\tilde{u}_0|^p - \frac{\lambda_1}{p}\int_\Omega |\tilde{u}_0|^p = 0.$$

However, this contradicts the variational characterization and the simplicity of λ_1.

Lemma 3.1 *Let $\tilde{u}_{\bar{u}} \in \tilde{W}_0^{1,p}(\Omega)$ be as above. Then $\|\tilde{u}_{\bar{u}}\|_{L^p} = o(\bar{u})$ as $|\bar{u}| \to \infty$.*

Proof. (i) Assume that there exists $\{\bar{u}_n\} \subset \mathbb{R}$ such that $\bar{u}_n \to \infty$ and

$$\frac{\bar{u}_n}{\|\tilde{u}_{\bar{u}_n}\|} \to 0. \tag{3.2}$$

Set $\hat{\tilde{u}}_{\bar{u}_n} = \tilde{u}_{\bar{u}_n}/\|\tilde{u}_{\bar{u}_n}\|$. It follows from (3.1) that

$$\liminf_{\bar{u}_n \to \infty} \Big\{ \frac{1}{p} \int_\Omega |\frac{\bar{u}_n}{\|\tilde{u}_{\bar{u}_n}\|} \nabla\varphi_1 + \nabla\hat{\tilde{u}}_{\bar{u}_n}|^p - \frac{\lambda_1}{p} \int_\Omega |\frac{\bar{u}_n}{\|\tilde{u}_{\bar{u}_n}\|}\varphi_1 + \hat{\tilde{u}}_{\bar{u}_n}|^p - \frac{1}{\|\tilde{u}_{\bar{u}_n}\|^{p-1}} \int_\Omega \tilde{f}\hat{\tilde{u}}_{\bar{u}_n} \Big\} \le 0. \tag{3.3}$$

Passing to a subsequence if necessary we conclude $\hat{\tilde{u}}_{\bar{u}_n} \rightharpoonup u_0$ in $W_0^{1,p}(\Omega), \hat{\tilde{u}}_{\bar{u}_n} \to u_0$ in $L^p(\Omega)$ and

$$\int_\Omega u_0\varphi_1 = 0. \tag{3.4}$$

At the same time, for large $u \in \mathbb{N}$, we have

$$\frac{1}{p} \int_\Omega |\frac{\bar{u}_n}{\|\tilde{u}_{\bar{u}_n}\|} \nabla\varphi_1 + \nabla\hat{\tilde{u}}_{\bar{u}_n}|^p \ge \varepsilon$$

with some $\varepsilon > 0$. It follows then from (3.3) that

$$\frac{\lambda_1}{p} \int_\Omega |u_0|^p \ge \varepsilon$$

which means that $u_0 \ne 0$. At the same time we get from (3.3) that

$$\frac{1}{p} \int_\Omega |\nabla u_0|^p - \frac{\lambda_1}{p} \int_\Omega |u_0|^p \le 0$$

and so the variational characterization and simplicity of λ_1 imply that $u_0 = k\varphi_1, k \ne 0$. But this contradicts (3.4).

(ii) Assume that $\bar{u}_n \to \infty$ and there exist constant $C > 0$ independent of n such that

$$\frac{\|\tilde{u}_{\bar{u}_n}\|}{\bar{u}_n} \le C. \tag{3.5}$$

It follows from (3.1) that

$$\lim_{\bar{u}_n \to \infty} \inf\{\frac{1}{p} \int_\Omega |\nabla\varphi_1 + \nabla(\frac{\tilde{u}_{\bar{u}_n}}{\bar{u}_n})|^p - \frac{\lambda_1}{p} \int_\Omega |\varphi_1 + \frac{\tilde{u}_{\bar{u}_n}}{\bar{u}_n}|^p - \int_\Omega \tilde{f}\frac{\tilde{u}_{\bar{u}_n}}{\bar{u}_n^p}\} \le 0. \tag{3.6}$$

Passing to a subsequence if necessary, we conclude that there is $u_0 \in W_0^{1,p}(\Omega)$ such that $\frac{\tilde{u}_{\bar{u}_n}}{\bar{u}_n} \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$, $\frac{\tilde{u}_{\bar{u}_n}}{\bar{u}_n} \to u_0$ in $L^p(\Omega)$ and

$$\int_\Omega u_0\varphi_1 = 0.$$

Let $u_0 \neq 0$. Then we get from (3.6) that

$$\frac{1}{p}\int_\Omega |\nabla\varphi_1 + \nabla u_0|^p - \frac{\lambda_1}{p}\int_\Omega |\varphi_1 + u_0|^p \leq 0,$$

which contradicts the variational characterization and simplicity of λ_1. Hence $u_0 = 0$, i.e.

$$\frac{\tilde{u}_{\bar{u}_n}}{\bar{u}_n} \to 0 \quad \text{in } L^p(\Omega). \tag{3.7}$$

Assume now that the assertion of lemma is not true. Then there is a sequence $\{\bar{u}_n\} \subset \mathbb{R}$, $\bar{u}_n \to \infty$, such that for some $\tilde{C}_2 > 0$ we have

$$\frac{\|\tilde{u}_{\bar{u}_n}\|_{L^p}}{\bar{u}_n} \geq \tilde{C}_2.$$

For such a sequence we have that either (3.2) or (3.5) holds. The former case is impossible by (i) the latter case contradicts (3.7). □

As a consequence of Lemma 3.1 we have

$$\min_{\tilde{u}\in\tilde{W}_0^{1,p}(\Omega)} \{\frac{1}{p}\int_\Omega |\bar{u}\nabla\varphi_1 + \nabla\tilde{u}|^p - \frac{\lambda_1}{p}\int_\Omega |\bar{u}\varphi_1 + \tilde{u}|^p - \int_\Omega \tilde{f}\tilde{u}\} = o(\bar{u}),\ |\bar{u}| \to \infty. \tag{3.8}$$

Lemma 3.2 *For a given $T > 0$ there exists $R > 0$ such that for any $\bar{u} \in [0, T]$ and $\tilde{u} \in \tilde{W}_0^{1,p}(\Omega), \|\tilde{u}\| = R$, we have*

$$\frac{1}{p}\int_\Omega |\bar{u}\nabla\varphi_1 + \nabla\tilde{u}|^p - \frac{\lambda_1}{p}\int_\Omega |\bar{u}\varphi_1 + \tilde{u}|^p - \int_\Omega \tilde{f}\tilde{u} \geq 0. \tag{3.9}$$

Proof. Assume that there is $T > 0, \bar{u}_n \in [0, T], \|\tilde{u}_n\| \to \infty$ such that

$$\frac{1}{p}\int_\Omega |\bar{u}_n\nabla\varphi_1 + \nabla\tilde{u}_n|^p - \frac{\lambda_1}{p}\int_\Omega |\bar{u}_n\varphi_1 + \tilde{u}_n|^p - \int_\Omega \tilde{f}\tilde{u}_n < 0. \tag{3.10}$$

Set $\hat{\tilde{u}}_n = \tilde{u}_n/\|\tilde{u}_n\|$. Passing to subsequences if necessary we can assume that $\hat{\tilde{u}} \rightharpoonup u_0$ in $W_0^{1,p}(\Omega), \int_\Omega u_0\varphi_1 = 0$, $\bar{u}_n \to \bar{u}_0 \in [0, T]$. At the same time, dividing (3.10) by $\|\tilde{u}_n\|^p$, passing to the limit for $n \to \infty$ we derive that $u_0 \neq 0$ and

$$\frac{1}{p}\int_\Omega |\nabla u_0|^p - \frac{\lambda_1}{p}\int_\Omega |u_0|^p \leq 0$$

which contradicts the variational characterization and simplicity of λ_1. □

Let $\rho > 0$ be small enough (to be specified later) and consider $f \in B_C(\tilde{f}, \rho)\setminus \tilde{C}(\bar{\Omega})$. Then f splits as follows:

$$f(x) = \tilde{f}(x) + \bar{f}\varphi_1(x)$$

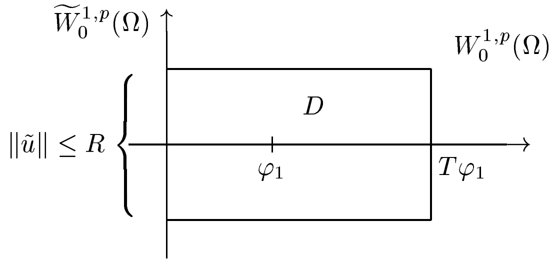

Figure 2: The set D constructed in the proof of Theorem 1.1

with $|\bar{f}|$ small, $\bar{f} \neq 0$. Then

$$\begin{aligned} E_f(u) =& \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega \tilde{f}\tilde{u} - \bar{u}\int_\Omega \bar{f}\varphi_1 \\ =& E_{\tilde{f}}(u) - \bar{u}\int_\Omega \bar{f}\varphi_1, \quad u \in W_0^{1,p}(\Omega), \end{aligned}$$

where $u = \bar{u}\varphi_1 + \tilde{u}$. Let $\bar{f} < 0$, so $\bar{f} \in (-\bar{\rho}, 0)$ with small $\bar{\rho} > 0$. We shall construct the set

$$D = \{u \in W_0^{1,p}(\Omega) \colon u = \bar{u}\varphi_1 + \tilde{u}, \bar{u} \in (0,T), \|\tilde{u}\| < R\}$$

with $T > 0$ and $R > 0$ to be specified later. We choose $T_1 > 0$ so that

$$E_{\tilde{f}}(\tilde{u}_{T_1}) \le 2E_{\tilde{f}}(\tilde{u}_0) \tag{3.11}$$

(this is possible due to (3.1), remind that $\tilde{u}_{T_1}$ and $\tilde{u}_0$ are the points where $\inf_{\tilde{u}\in\tilde{W}_0^{1,p}(\Omega)} E_{\tilde{f}}(\bar{u}\varphi_1 + \tilde{u})$ is achieved for $\bar{u} = T_1$ and $\bar{u} = 0$, respectively). Then take $\rho > 0$ (and hence $\bar{\rho} > 0$) so small that

$$E_f(T_1\varphi_1 + \tilde{u}_{T_1}) \le \frac{3}{2}E_{\tilde{f}}(\tilde{u}_0) \tag{3.12}$$

if $f \in B_C(\tilde{f}, \rho) \setminus \tilde{C}(\bar{\Omega})$. Now we choose $T > 0$ so that

$$E_f(T\varphi_1 + \tilde{u}_T) \ge 0 \tag{3.13}$$

(this is possible due to (3.8) and $\bar{f} < 0$). Finally, we choose $R = R(T) > 0$ according to Lemma 3.2 (see Fig. 2). Then it follows from Lemma 3.2, (3.12) and (3.13) that

$$\inf_{u\in D} E_f(u) < \inf_{u\in\partial D} E_f(u). \tag{3.14}$$

Since E_f is weakly lower semicontinuous functional on D there exists a global minimizer of E_f in D. Let $u_D \in D$ be the point of global minimum, i.e.

$$E_f(u_D) = \min_{u\in D} E_f(u).$$

Note that E_f is unbounded from below. This is easy to see, choosing e.g. $u_n = \bar{u}_n\varphi_1, \bar{u}_n \to -\infty$, we obtain $E_f(u_n) \to -\infty$. So, E_f has a Mountain Pass Theorem Geometry. Because E_f satisfies also (PS) condition according to Lemma 2.2, we can apply the results of Rabinowitz [20] to derive the existence of $u_0 \in W_0^{1,p}(\Omega), u_0 \neq u_D$, which is also a critical point of E_f. To summarize, we proved that for $f \in B_C(\tilde{f},\rho) \setminus \tilde{C}(\bar{\Omega})$ the functional E_f has at least two distinct critical points. The case $\bar{f} > 0$ is similar.

It remains to prove that $E_{\tilde{f}}$ has at least one critical point. This follows from the argument based on the method of upper and lower solutions. It follows from the previous considerations that there is $\bar{f} > 0$ small enough such that $E_{\tilde{f}\pm\bar{f}\varphi_1}$ has critical points $u_\pm \in W_0^{1,p}(\Omega)$, i.e.

$$\int_\Omega |\nabla u_\pm|^{p-2}\nabla u_\pm \cdot \nabla v - \lambda_1 \int_\Omega |u_\pm|^{p-2}u_\pm v = \int_\Omega \tilde{f}v \pm \int_\Omega \bar{f}\varphi_1 v$$

holds for any $v \in W_0^{1,p}(\Omega)$. It follows from Proposition 2.9 that there is a solution $u \in W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v - \lambda_1 \int_\Omega |u|^{p-2}uv = \int_\Omega \tilde{f}v$$

for any $v \in W_0^{1,p}(\Omega)$. This is equivalent to the fact that u is a critical point of $E_{\tilde{f}}$. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

We consider the case $p > 2$ and the energy functional $E_{\tilde{f}}$ with $\tilde{f} \in \tilde{C}(\bar{\Omega}), \tilde{f} \neq 0$. Let us choose a function $\varphi \in W_0^{1,p}(\Omega), \varphi \geq 0$ in Ω and such that

$$\{x \in \Omega \colon \varphi(x) > 0\} \subset \{x \in \Omega \colon \tilde{f}(x) > 0\}$$

(note that this is possible because the latter set is an open subset of Ω). Then there exists $t > 0$ (small enough) such that for $v = t\varphi$ we have

$$E_{\tilde{f}}(v) < 0. \tag{4.1}$$

Making use of Proposition 2.10 the Hölder and Young inequalities we have the following estimate

$$\begin{aligned} E_{\tilde{f}}(u) \geq & \frac{C}{p}\Big[|\bar{u}|^{p-2}\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla\tilde{u}|^2 + \int_\Omega |\nabla\tilde{u}|^p\Big] - \Big(\int_\Omega |\tilde{f}|^{p'}\Big)^{1/p'}\Big(\int_\Omega |\tilde{u}|^p\Big)^{1/p} \\ \geq & \frac{C}{p}\Big[|\bar{u}|^{p-2}\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla\tilde{u}|^2 + \int_\Omega |\nabla\tilde{u}|^p\Big] - \frac{C_1^p\varepsilon^p}{p}\|\tilde{u}\|^p - \frac{1}{\varepsilon^p p'}\|\tilde{f}\|_{L^{p'}}^{p'}, \end{aligned}$$

where $C_1 > 0$ is the constant of the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$. Choosing $C_1^p\varepsilon^p = \frac{C}{2}$ we arrive at

$$E_{\tilde{f}}(u) \geq \frac{C}{2p}\|\tilde{u}\|^p + \frac{C}{p}|\bar{u}|^{p-2}\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla\tilde{u}|^2 - \frac{(\frac{2}{C})^{\frac{1}{p-1}}C_1^{1-\frac{1}{p}}}{p'}\|\tilde{f}\|_{L^{p'}}^{p'}. \tag{4.2}$$

It follows from here that there exists $R = R(\tilde{f}) > 0$ such that for any $u = \bar{u}\varphi_1 + \tilde{u} \in W_0^{1,p}(\Omega)$ with $\|\tilde{u}\| = R$ we have

$$E_{\tilde{f}}(u) > 0. \tag{4.3}$$

Let us consider now $u = \bar{u}\varphi_1 + \tilde{u} \in W_0^{1,p}(\Omega)$ for which

$$\frac{C}{p}|\bar{u}|^{p-2}\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla\tilde{u}|^2 \le C_2\|\tilde{f}\|_{L^{p'}}^{p'} \tag{4.4}$$

where we denoted $C_2 = \frac{1}{p'}(\frac{2}{C})^{\frac{1}{p-1}}C_1^{1-\frac{1}{p}}$. It follows then from the Hölder inequality that $E_{\tilde{f}}(u) \ge -\|\tilde{f}\|_{L^2}\|\tilde{u}\|_{L^2}$. If we combine this with (2.1) and (4.4) we get

$$E_{\tilde{f}}(u) \ge -\frac{\tilde{C}p^{1/2}C_2^{1/2}\|\tilde{f}\|_{L^2}\|f\|_{L^{p'}}^{\frac{p'}{2}}}{C^{1/2}|\bar{u}|^{\frac{p-2}{2}}}. \tag{4.5}$$

Let us define the set

$$D = \{u \in W_0^{1,p}(\Omega) \colon u = \bar{u}\varphi_1 + \tilde{u}, \quad \bar{u} \in (-T, T), \|\tilde{u}\| < R\}$$

with R mentioned above and $T > 0$ to be fixed later (see Fig. 3). It follows from (4.1) that

$$i \colon = \inf_{u \in D} E_{\tilde{f}}(u) < 0$$

independently of $T \gg 1$. It follows from (4.5) that for $u = \pm T\varphi_1 + \tilde{u}$ satisfying (4.4) we have

$$E_{\tilde{f}}(u) > i \tag{4.6}$$

if T is large enough. On the other hand we have directly from (4.2) that

$$E_{\tilde{f}}(u) \ge 0 > i \tag{4.7}$$

for $u = \pm T\varphi_1 + \tilde{u}$ which do not satisfy (4.4). Now, if we combine (4.3), (4.6) and (4.7), we get

$$i < \inf_{u \in \partial D} E_{\tilde{f}}(u). \tag{4.8}$$

Thus $E_{\tilde{f}}$ has a local minimizer geometry. In particular, it follows also from above considerations that $E_{\tilde{f}}$ is bounded from below on $W_0^{1,p}(\Omega)$. Since $E_{\tilde{f}}$ is weakly lower semicontinuous functional on the bounded, convex and closed set $\bar{D}$, it has to achieve its minimum there. Due to (4.8) the minimizer is an interior point of D and due to the differentiability of $E_{\tilde{f}}$ it is a critical point at the same time.

Let $\rho > 0$ and consider $f \in B_C(\tilde{f}, \rho) \setminus \tilde{C}(\bar{\Omega})$. Then, as in Section 3, split f as follows:

$$f(x) = \tilde{f}(x) + \bar{f}\varphi_1(x)$$

with $\bar{f} \ne 0$. Then

$$E_f(u) = E_{\tilde{f}}(u) - \bar{u}\int_\Omega \bar{f}\varphi_1$$

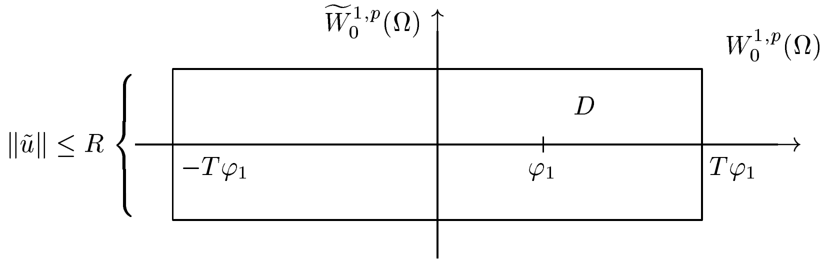

Figure 3: The set D constructed in the proof of Theorem 1.2

and thus E_f is unbounded from below (we can use the same reasoning as in the previous section). If ρ is small enough (and so is $|\bar{f}|$) then inequality (4.8) still holds. This means that E_f has a Mountain Pass Theorem Geometry and we proceed exactly as in the previous section to conclude the existence of at least two distinct critical points of $E_{\tilde{f}}$. This completes the proof of Theorem 1.2.

5 Proof of Theorem 1.3

Let $\tilde{f} \in \tilde{C}(\bar{\Omega})$. Then it follows from Theorems 1.1 and 1.3 that the problem (1.1) has at least one weak solution. It follows from these theorems that for $\tilde{f} \neq 0$ there exists $\rho = \rho(\tilde{f}) > 0$ such that (1.1) has at least one solution for any $f \in B_C(\tilde{f}, \rho)$. So we shall concentrate to the proof of the second part of Theorem 1.3. To this end we shall split $f \in C(\bar{\Omega})$ as follows

$$f(x) = \tilde{f}(x) + \hat{f}. \tag{5.1}$$

Define

$$F_- = F_-(\tilde{f}) := \inf \hat{f}, \quad F_+ = F_+(\tilde{f}) := \sup \hat{f},$$

where the infimum and the supremum are taken over all $\hat{f}$ for which (1.1) (with $f(x)$ given above) has a solution. It follows directly from the first part of Theorem 1.3 that $F_- < 0 < F_+$. To prove that $F_\pm$ are finite we argue by contradiction. Let us suppose that there exist sequences $\{\hat{f}_n\} \subset \mathbb{R}, \{u_n\} \subset C_0^1(\bar{\Omega})$, such that $\hat{f}_n \to \infty$ and u_n is a solution to (1.1) with the right hand side $f_n(x) = \tilde{f}(x) + \hat{f}_n$. Dividing the equation in (1.1) by $\hat{f}_n$, setting $v_n := \hat{f}_n^{-\frac{1}{p-1}} u_n$, and using the compactness of Δ_p^{-1}, we find that $v_n \to v_0$ in $C_0^1(\bar{\Omega})$ (at least for a subsequence). Moreover, v_0 satisfies

$$\begin{aligned} -\Delta_p v_0 - \lambda_1 |v_0|^{p-2} v_0 = 1 \quad &\text{in } \Omega, \\ v_0 = 0 \quad &\text{on } \partial\Omega. \end{aligned}$$

But this is a contradiction with the nonexistence result proved e.g. in [1, 13]. Hence (1.1) has no solution provided $\hat{f} \notin [F_-, F_+]$ which proves (i).

It follows directly from Proposition 2.9 that (1.1) is solvable for any $\hat{f} \in (F_-, F_+)$. Let now $\hat{f} = F_-$. Consider $\hat{f}_n > F_-, \hat{f}_n \to F_-$ and denote by $u_n \in C_0^1(\bar{\Omega})$ corresponding solutions of (1.1) with $f(x) = \tilde{f}(x) + \hat{f}_n$. According to Proposition 2.11 the sequence $\{u_n\}$ is bounded in $C_0^1(\bar{\Omega})$. Compactness of Δ_p^{-1} implies the existence of a subsequence (denoted again by $\{u_n\}$) for which $u_n \to u_-$ in $C_0^1(\bar{\Omega})$ for some $u_- \in C_0^1(\bar{\Omega})$. Moreover, similarly as above, u_- satisfies

$$\begin{gathered} -\Delta_p u_- - \lambda_1 |u_-|^{p-2} u_- = \tilde{f}(x) + F_- \quad \text{in } \Omega \\ u_- = 0 \quad \text{on } \partial\Omega \end{gathered}$$

Similarly, we prove that (1.1) is solvable for $f(x) = \tilde{f}(x) + F_+$. This proves (iii).

It remains to prove the multiplicity result stated in (ii). We proceed via contradiction. To this end we apply the degree theory combined with Lemmas 2.7, 2.8 and Propositions 2.9 and 2.11. Let us assume that $\hat{f} \in (0, F_+)$ (the proof in case $\hat{f} \in (F_-, 0)$ is similar). Then the problem (1.1) with $f(x) = \tilde{f}(x) + \hat{f}$ has a solution u and there exist $0 < \hat{f}_1 < \hat{f} < \hat{f}_2 < F_+$ such that (1.1) has also solutions u_i for $f_i(x) = \tilde{f}(x) + \hat{f}_i$, $i = 1, 2$. It is straightforward to verify that u_1 and u_2 are lower and upper solutions, respectively, of (1.1) with the right hand side f. We assume that u is unique solution of (1.1) obtained by Proposition 2.9, i.e. it is either $u_1 \le u \le u_2$ in Ω or $u \in \bar{S}$ (with S defined in Lemma 2.8). Assume that the former case occurs, u_1, u_2 are strict, and $u_1 \prec u_2$, i.e. $u \notin \partial\mathcal{M}_1$ with $R = R_0$ large enough (with $\mathcal{M}_1$ defined in Lemma 2.7). Then according to Lemma 2.7, we have that

$$\deg[I - T_f; \mathcal{M}_1, 0] = 1. \tag{5.2}$$

Let us choose $\hat{f}_3 > F_+$. It follows from above considerations that (1.1) with $f_3(x) = \tilde{f}(x) + \hat{f}_3$ has no solution. Hence

$$\deg[I - T_{f_3}; B_{C_0^1}(0, R), 0] = 0 \tag{5.3}$$

for arbitrary $R > 0$. Consider now the family of functions

$$f_t(x) = \tilde{f}(x) + t\hat{f} + (1-t)\hat{f}_3, \quad t \in [0, 1].$$

Then $K = \{f_t \in C(\bar{\Omega}) : t \in [0, 1]\}$ is a compact subset of $C(\bar{\Omega})$ and

$$H(t, \cdot) = I - T_{f_t}, \quad t \in [0, 1],$$

is a homotopy of compact perturbations of the identity. It follows from Proposition 2.11 that for $R = R_1 > R_0$ large enough we have that

$$\deg[I - T_{f_t}; B_{C_0^1}(0, R_1), 0]$$

is constant for $t \in [0, 1]$. Due to (5.3) we have also

$$\deg[I - T_f; B_{C_0^1}(0, R_1), 0] = 0. \tag{5.4}$$

Additivity property of the degree and (5.2), (5.4) imply that there is $\check{u}$ in $B_{C_0^1}(0,R_1)\setminus\mathcal{M}_1$ which is a solution of (1.1) and evidently $\check{u}\neq u$ which is a contradiction with uniqueness of u.

The proof follows the same lines if $u\in\bar{S}$ and $u\notin\partial\mathcal{M}_2$ (with $\mathcal{M}_2$ defined in Lemma 2.8). The only difference consists in substituting (5.2) by

$$\deg[I-T_f;\mathcal{M}_2,0]=-1.$$

Assume, now, that unique solution u is obtained by means of Lemma 2.7 but $u\in\partial\mathcal{M}_1$. Since R_0 can be chosen large enough this means that $u_1\not\prec u$ or $u\not\prec u_2$. Let us assume $u_1\not\prec u$ (the other case is similar). This means that either there exists $x_0\in\Omega$ such that $u_1(x_0)=u(x_0)$ or there exists $\check{x}_0\in\partial\Omega$ such that $\frac{\partial u_1}{\partial n}(\check{x}_0)=\frac{\partial u}{\partial n}(\check{x}_0)$. We choose $\delta>0$ small enough (to be specified later) and define $u_1^\delta(x)=u_1(x)-\delta, x\in\Omega$. Then $u_1^\delta\in C^1(\bar{\Omega})$ and $u_1^\delta\prec u$. We prove that for δ small this new function u_1^δ is lower solution of (1.1). Indeed, since $u_1\in C(\bar{\Omega})$, there exists a constant $C=C(\|u_1\|_C)>0$ such that for any $x\in\bar{\Omega}$,

$$\left||u_1(x)-\delta|^{p-2}(u_1(x)-\delta)-|u_1(x)|^{p-2}u_1(x)\right|\le|\delta|^{p-1},$$

for $1<p<2$, and

$$\left||u_1(x)-\delta|^{p-2}(u_1(x)-\delta)-|u_1(x)|^{p-2}u_1(x)\right|\le C|\delta|,$$

for $p>2$. In either case, there exists $\delta_0>0$ such that for all $0<\delta<\delta_0$ we have

$$\int_\Omega\left||u_1^\delta(x)|^{p-2}u_1^\delta(x)-|u_1(x)|^{p-2}u_1(x)\right|\psi(x)dx\le\frac{\hat{f}-\hat{f}_1}{2\lambda_1}\int_\Omega\psi(x)dx \qquad (5.5)$$

for all $\psi\ge0,\psi\in W_0^{1,p}(\Omega)$.
Since $\nabla u_1^\delta(x)=\nabla u_1(x), x\in\Omega$, it follows from (5.5) that

$$\int_\Omega|\nabla u_1^\delta|^{p-2}\nabla u_1^\delta\cdot\nabla\psi-\lambda_1\int_\Omega|u_1^\delta|^{p-2}u_1^\delta\psi\le\int_\Omega\tilde{f}\psi+\bar{f}\int_\Omega\psi,$$

for any $\psi\ge0,\psi\in W_0^{1,p}(\Omega)$, i.e. u_1^δ is a lower solution of (1.1).

Similarly we can define an upper solution $u_2^\delta=u_2+\delta$ such that $u\prec u_2^\delta$. We define then a new set $\mathcal{M}_1^\delta$ by means of u_1^δ,u_2^δ, with $u_1^\delta\prec u_2^\delta$, and since $u\notin\partial\mathcal{M}_1^\delta$, we proceed as above to get a contradiction with the uniqueness of u.

Assume, now, that unique solution u is obtained by means of Lemma 2.8 but $u\in\partial\mathcal{M}_2$. Since R_0 can be chosen large enough this means that we have two similar possibilities (which can occur simultaneously):
(i) either $u(x)\ge u_1(x), x\in\Omega$, and there exists $x_0^l\in\Omega$ such that $u(x_0^l)=u_1(x_0^l)$ or there exists $\check{x}_0^l\in\partial\Omega$ such that $\frac{\partial u_1}{\partial n}(\check{x}_0^l)=\frac{\partial u}{\partial n}(\check{x}_0^l)$,
(ii) either $u(x)\le u_2(x), x\in\Omega$, and there exists $x_0^s\in\Omega$ such that $u(x_0^s)=u_2(x_0^s)$ or there exists $\check{x}_0^s\in\partial\Omega$ such that $\frac{\partial u_2}{\partial n}(\check{x}_0^s)=\frac{\partial u}{\partial n}(\check{x}_0^s)$.

Let us assume that the first possibility (i) occurs. Then for δ small we define a function $u_1^\delta=u_1-\delta$. If the second possibility (ii) occurs then we define $u_2^\delta=u_2+\delta$. By the same reason as above, u_1^δ and u_2^δ are lower and upper solutions of (1.1), respectively, and they are still non-ordered if δ is small enough. Moreover, for $\mathcal{M}_2^\delta$ defined by means of u_1^δ, u_2^δ, we have that $u \notin \overline{\mathcal{M}}_2^\delta$. By Lemma 2.8 there must be $\breve{u} \in \overline{\mathcal{M}}_2^\delta$ which is a solution of (1.1) and $\breve{u} \neq u$. This contradicts again the uniqueness of u.

The proof of multiplicity result stated in Theorem 1.3 (ii) is thus proved and so the whole proof is finished. $\square$

Acknowledgements. This research was partially supported by grant number 201/00/0376 from the Grant Agency of the Czech Republic and Ministry of Education of Czech Republic, number MSM 235200001. The results in this article were presented in September 2001 at the Workshop "Quasilinear Elliptic and Parabolic Equations and Systems" held in C.I.R.M., Luminy, Marseille, France.

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