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Description of regional blow-up in a porous-medium equation *

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Abstract

We describe the (finite-time) blow-up phenomenon for a non-negative solution of a porous medium equation of the form

$$u_t = \Delta u^m + u^m$$

in the entire space. Here $m > 1$ and the initial condition is assumed compactly supported. Blow-up takes place exactly inside a finite number of balls with same radii and exhibiting the same self-similar profile.

1 Introduction

This paper deals with the description of the blow-up phenomenon in the porous-medium equation in $\mathbb{R}^N$, $N \geq 1$,

$$\begin{aligned} u_t &= \Delta u^m + u^m \\ u(x,0) &= u_0(x) \end{aligned} \tag{1.1}$$

where $m > 1$ and $u_0(x)$ is a compactly supported, not identically zero nonnegative function whose regularity will be specified later. This gives rise to the interesting phenomenon of *regional blow-up*, meaning this blow-up taking place only in a compact set with nonempty interior.

The purpose of this note is to describe the blow-up in (1.1) in the following sense: we show that for any initial condition the solution u develops (exactly) a finite number of similar *spherical hot spots*: more precisely, there is a finite number of disjoint balls with common radii R^* outside which the solution remains uniformly bounded, while inside each of them it develops a common self-similar radially symmetric profile $(\bar{T} - t)^{-1/(m-1)} w_*(r)$, where r is the distance to the center of these balls and w_* is a strictly positive function.

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The presence of regional blow-up in this equation was first observed and studied in the case $N=1$ in [11]. The elliptic problem found when searching by separation of variables a solution of the form

$$u(x,t)=(\bar{T}-t)^{-1/(m-1)}\theta(x)$$

has been studied for radial symmetry in [2, 3, 4, 15].

The result contained in this paper is a sequel of the work [5] where the following partial result was established: Let $\bar{T}>0$ be the time at which blow-up occurs. Let t_n be any sequence $t_n \uparrow \bar{T}$. Then there is a subsequence of t_n which we still denote t_n, and a nontrivial compactly supported solution $w(x)$ of the elliptic equation

$$\Delta w^m + w^m - \frac{1}{m-1}w = 0, \tag{1.2}$$

such that $(\bar{T}-t_n)^{1/(m-1)}u(x,t_n) \to w(x)$ uniformly.

On the other hand, it was established in [3] that the support of any finite energy solution w of (1.2) consists of a finite number of disjoint balls of the same radii and that the solution is radially symmetric inside each of them. This radially symmetric solution $w_*(|x|)$ turns out to be unique, as established in [4]. Thus w can be written as

$$w(x)=\sum_{j=1}^{k} w_*(|x-x_j|), \tag{1.3}$$

where, if R^* is the radius of the support of w_*, $|x_i-x_j|\geq 2R^*$ for $i\neq j$.

Let $BU(u_0)$ be the set of *blow-up points* of u, namely the set of points x for which there are sequences $x_n \to x$ and $t_n \to \bar{T}$ such that $u(x_n,t_n)\to +\infty$. It was also shown in [5] that this set is compact and it is precisely constituted by the union of the supports of all possible limiting w's. The important point unsolved in [5] was whether there is an *actual unique blow-up profile*, rather than oscillation between different limiting configurations. The question turns out to be rather subtle, and we answer it affirmatively in the following result.

Theorem 1.1 *Let $u(x,t)$ be the solution of* (1.1)*, where $u_0(x)$ is compactly supported, continuous and such that $u_0^m \in H^1(\mathbb{R}^N)$. Let $\bar{T}>0$ be the blow-up time of this solution. Then there are points $x_1,\dots x_k \in \mathbb{R}^N$ such that*

$$\lim_{t\to\bar{T}}(\bar{T}-t)^{1/(m-1)}u(x,t)=\sum_{j=1}^{k} w_*(|x-x_j|)$$

uniformly. Here $w_*(|x|)$ is the unique compactly supported, radially symmetric solution of (1.2)*. If R^* is the radius of its support then we also have $|x_i-x_j|\geq 2R^*$ for $i\neq j$. Moreover $u(x,t)$ remains uniformly bounded up to its blow-up time on compact subsets of $\mathbb{R}^N \setminus \bigcup_{j=1}^{k}\bar{B}(x_j,R^*)$. In other words,*

$$BU(u_0)=\bigcup_{j=1}^{k}\bar{B}(x_j,R^*).$$

Next we describe the proof of the above results. Let us introduce the change of variables

$$v(x,t) = (\bar{T}-\tau)^{1/(m-1)}u(x,\tau)\big|_{\tau=\bar{T}(1-e^{-t})}. \tag{1.4}$$

It is readily checked that v is globally defined in time and satisfies the equation

$$v_t = \Delta v^m + v^m - \frac{1}{m-1}v \tag{1.5}$$

$$v(x,0) = \bar{T}^{1/(m-1)}u_0(x). \tag{1.6}$$

From [5, Proposition 4.1], v is a bounded function and given a sequence $t_n \to +\infty$ there is a subsequence, which we denote in the same way, and a nontrivial, compactly supported solution of (1.2) so that

$$v(x,t_n)^m \to w(x)^m \quad \text{as } n \to \infty,$$

both in uniform and H^1-senses. Thus our task in establishing Theorem 1.1 is precisely to prove that the limit $w(x)$ is actually the same along *every* sequence $t_n \to +\infty$. Here $w(x)$ has the form (1.3), and for the sake of simplicity we restrict ourselves in what follows of this paper only to the case $k=1$. The proof for $k>1$ is similar if $|x_i - x_j| > 2R^*$ for all $i \neq j$, only to the expense of some extra notation. Instead, if $|x_i - x_j| = 2R^*$ for some $i \neq j$, the situation is substantially more involved. The complete proof is contained in the forthcoming work [7]. It should be mentioned that conditions for one-ball blow-up (i.e. $k=1$) are provided in [5]. For instance, if the support of the initial condition is contained in a ball of radius less than R^* this is the case. The one-ball blow up can be shown to be stable, with basically the same proof below. This means that if an initial condition u_0 leads to blow-up with $k=1$ in Theorem 1.1, then this is the case for all sufficiently close initial data, where the limiting ball is also close to that associated to u_0. Instead, the two-ball blow-up is not stable as the following example shows. Let us fix points x_1 and x_2 with $|x_1 - x_2| > 2R^*$. Then the function

$$u(x,t) = (\bar{T}_1 - t)^{-1/(m-1)}w_*(|x-x_1|) + (\bar{T}_2 - t)^{-1/(m-1)}w_*(|x-x_2|)$$

solves equation (1.1) for $0 < t < \min\{\bar{T}_1, \bar{T}_2\}$. If $\bar{T}_1 = \bar{T}_2$, then two-ball blow-up takes place, which however disappears as soon as $\bar{T}_1$ and $\bar{T}_2$ differ, no matter how close they are. This example suggests that one-ball blow-up may actually hold for "generic" initial data.

The main feature of equation (1.5) is the presence of a Lyapunov functional for it, namely

$$J(z) = \frac{1}{2}\int (|\nabla z^m|^2 - z^{2m})dx + \frac{m}{m^2-1}\int z^{m+1}dx. \tag{1.7}$$

In fact we have that the mapping $t \mapsto J(v(\cdot,t))$ is decreasing on $t>0$ and

$$\lim_{t\to+\infty} J(v(\cdot,t)) = J(w).$$

Here and in what follows the integral symbol without limits specified means integration on the whole $\mathbb{R}^N$. The presence of this functional implies that limit points of the trajectory must be steady states. The problem of uniqueness of asymptotic limits in nonlinear heat equations under the presence of a Lyapunov functional has been analyzed in a number of works. A general result due to L. Simon [24] shows the uniqueness of the limit for uniformly parabolic equations in a uniform real analytic setting on a compact manifold. Analiticity cannot be lifted in general in this result, at least in the non-autonomous setting, as shown in [22]. Needless to say, the compactly supported setting we deal with makes our situation highly non uniformly analytic.

Related uniqueness results in parabolic problems, nondegenerate and degenerate, are contained in the works [1, 6, 8, 9, 10, 13, 17, 16, 20]. In [6], a re-normalization method based on L. Simon's ideas, used in classifying singularities in an elliptic problem in [18] was adapted to a semilinear heat equation. The general framework of this method is what we will use here. Alternative methods for degenerate equations of porous-medium type, in one and higher dimensions, have been devised in [9], [10]. Those techniques do not apply to the nonlinearity of equation (1.6), in particular those in [10], based on analyticity, because of the presence of compactly supported steady states. This is explicitly commented in [10] and posed as an open question.

2 Proof of the main results

For functions v_1 and v_2 defined on $\mathbb{R}^N$ we consider the "distance" between them defined as

$$d(u_1, u_2) = \Big(\int (u_1(x)^m - u_2(x)^m)(u_1(x) - u_2(x))dx \Big)^{1/2}.$$

We observe that there is a constant $D > 0$ for which

$$d(u_1, u_3) \leq D(\, d(u_1, u_2) \, + \, d(u_2, u_3) \,).$$

The theorem stated in the previous section (in the case $k = 1$) will be a direct consequence of the following result,

Proposition 2.1 *There exist positive numbers T and C such that if v is a solution of equation* (1.6) *defined on $0 \leq t < +\infty$, such that for some sequence $t_n \to +\infty$, setting*

$$v_n(x,t) \equiv v(x, t_n + t), \quad w_n(x) = w_*(|x - x_n|),$$

one has

$$\eta_n \equiv \sup_{t \in [0,T]} d(v_n(t), w_n) \to 0 \ \textit{as} \ n \to \infty. \tag{2.1}$$

Then there exists a point $\bar{x}_n$ with $|\bar{x}_n - x_n| \leq C\eta_n$ and

$$\sup_{t \in [0,T]} d(v_n(t), \bar{w}_n) \leq \frac{\eta_n}{2}$$

for all n sufficiently large, and where $\bar{w}_n(x) = w_*(|x - \bar{x}_n|)$.

As a consequence, we obtain the validity of the following fact: There exist positive numbers T, δ, C and $t^*$with the following property:

Let $v(x,t)$ be a solution of (1.6), defined in $0 < t < \infty$ as in the statement of the theorem. Consider a point x_1 and set $w(x) = w_*(|x - x_1|)$. Assume that $t_0 > t^*$ is such that

$$\eta = \sup_{t_0 \le t \le t_0 + T} d(v(t), w) < \delta.$$

Then there exists a points $\bar{x}_1$ with $|\bar{x}_1 - x_1| \le C\eta$ such that

$$\sup_{t_0 + T \le t \le t_0 + 2T} d(v(t), \bar{w}) \le \frac{\eta}{2},$$

where $\bar{w}(x) = w_*(|x - \bar{x}_1|)$.

Let us see how Theorem 1.1 (for $k = 1$) follows from this assertion. Let ε be given and let us write w as $w(x) = w_*(|x - x_1|)$. Let $\delta_0 < \delta$, with δ the number predicted by Proposition 2.1. Assume that for some $t_0 > t^*$ we have $\eta_1 \equiv \sup_{t_0 \le t \le t_0 + T} d(v(t), w) \le \delta_0$, where T and t^* are the numbers given by Proposition 2.1. We find then that there is a point x_2 with $|x_1 - x_2| \le C\eta_1$ such that $\eta_2 \equiv \sup_{t_0 + T \le t \le t_0 + 2T} d(v(t), w_2) \le \frac{\eta_1}{2}$. where $w_2(x) = w_*(|x - x_2|)$. Since $\eta_2 \le \eta_1/2 < \delta$, we can apply again Proposition 2.1 to find a point x_3 with now $|x_3 - x_2| \le C\frac{\eta_1}{2}$ such that $\eta_3 = \sup_{t \in [t_0 + 2T, t_0 + 3T]} d(v(t), w_3) \le \frac{\eta_1}{4}$ where $w_3(x) = w_*(|x - x_3|)$. Iterating this procedure we find a sequence x_j, $j = 1, 2, \ldots$ such that $|x_{j+1} - x_j| \le C\frac{\eta}{2^j}$ and

$$\sup_{t \in [t_0 + jT, t_0 + (j+1)T]} d(v(t), w_j) \le \frac{\eta_1}{2^j}$$

with $w_j(x) = w_*(|x - x_j|)$.

Now let t be any number greater than t_0. Then $t \in (t_0 + jT, t_0 + (j+1)T]$ for some j. Using that w_* is Hölder continuous we see that $d(w, w_j) \le C|x_j - x_1|^a$ for some $a, C > 0$. Moreover, $|x_j - x_1| \le C\sum_{l=1}^{\infty} \eta_1 2^{-l} = C\eta_1$, hence

$$d(v(t), w) \le A\{(\eta_1)^2 + C^a \eta_1^a\}.$$

We conclude that there is a $\delta > 0$ such that if $\eta_1 < \delta$, then $d(v(t), w) < \varepsilon$ for all $t > t_0$. Finally, from [5, Proposition 4.1], there is a sequence $s_k \to \infty$ such that $v(x, s_k + \tau) \to w(x)$ for some nontrivial solution of (1.2), uniformly in x and for τ in bounded intervals, in particular for $\tau \in [0, T]$. We recall that the space support of v is contained inside a ball independent of the time variable. It follows that, given $\varepsilon > 0$, there is indeed a number $t_0 > 0$ such that $\eta_1 < \delta$. Since ε is arbitrary, we have actually established that w is the unique limit point of the trajectory $v(\cdot, t)$, and the proof of the theorem is complete. $\square$

The remaining of this paper will be devoted to the proof of Proposition 2.1.

3 Preliminaries

Let v_n be a sequence as in the statement of Proposition 2.1. Then we have that $v_n(x,t)^m - w_n(x)^m \to 0$ in L^∞ and H^1-senses in $\mathbb{R}^N$, uniformly locally in time $t \in [0,\infty)$, as it follows from the results in [5]. For $T > 0$ fixed, which we will choose later, we use in what follows the following notation.

$$\phi_n(x,t) \equiv \frac{v_n(x,t) - w_n(x)}{\eta_n}. \tag{3.1}$$

Then ϕ_n satisfies the equation

$$\frac{\partial \phi_n}{\partial t} = m\Delta(\tilde{w}_n^{m-1}\phi_n) + m\tilde{w}_n^{m-1}\phi_n - \frac{1}{m-1}\phi_n \tag{3.2}$$

where

$$\tilde{w}_n(x,t)^{m-1} \equiv \int_0^1 (w_n(x) + t(v_n(x,t) - w_n(x)))^{m-1}dt. \tag{3.3}$$

Let us observe that, by definition of the number η_n we have that

$$(m\int w_n(x,t)^{m-1}\phi_n(x,t)^2dx)^{1/2} \leq 1$$

for all $t \in [0,T]$. In the limit w_n must converges (up to subsequences) to a steady state w of equation (1.2). Then $w(x) = w_*(|x - x_1|)$ for certain $x_1 \in \mathbb{R}^N$. In what follows we shall assume, with no loss of generality that $x_1 = 0$

These facts suggest that on interior sets of the support B of the limit $w(x)$ of the w_n we should see convergence in certain sense of ϕ_n to a solution of the degenerate parabolic equation

$$\phi_t = m\Delta(w^{m-1}\phi) + mw^{m-1}\phi - \frac{1}{m-1}\phi. \tag{3.4}$$

By a *weak solution* of equation (3.4), we understand a function ϕ which is smooth in the interior of $\mathcal{B} = B \times (0,\infty)$, and satisfies (3.4) there, such that moreover

$$\int_0^s \int_B (m|\nabla(w^{m-1}\phi)|^2 m(w^{m-1}\phi)^2 + \frac{1}{m-1}\phi^2)dxdt < +\infty \quad \text{for all } s > 0. \tag{3.5}$$

Lemma 3.1 *There is a subsequence of ϕ_n which we denote in the same way, for which $\phi_n(x,t) \to \phi(x,t)$ (up to a subsequence) in the uniform C^1-sense over compact subsets of $\mathcal{B}$. Moreover, ϕ is a weak solution of* (3.4).

Before proving Lemma 3.1 we introduce some notation. Let us write $w_n = w_*(|x - x_n|)$ and consider the functions

$$\psi_n = \frac{v_n^m - w_n^m}{\eta_n}. \tag{3.6}$$

$$G_n = \frac{1}{\eta_n^2}\{\frac{v_n^{m+1}}{m+1} - \frac{w_n^{m+1}}{m+1} - w_n^m(v_n - w_n)\}. \tag{3.7}$$

$$H_n = \frac{(m-1)(v_n^{m+1} - w_n^{m+1}) + (m+1)(v_n w_n^m - v_n^m w_n)}{\eta_n^2}. \tag{3.8}$$

It is easily checked the existence of constants C_1 and C_2, depending only on m such that the following inequalities hold:

$$C_1 G_n \le \frac{(v_n^m - w_n^m)(v_n - w_n)}{\eta_n^2} \le C_2 G_n. \tag{3.9}$$

We have the validity of the following relation:

$$\int \frac{\partial}{\partial t} G_n \le \frac{1}{m^2-1} \int H_n. \tag{3.10}$$

In fact, let J be the Lyapunov functional for (1.5) given by (1.7). Let us set $I = 2(J(v_n) - J(w_n))$. Then $I \ge 0$. After integrating by parts and using the equations satisfied by v_n and w_n we get

$$I = -\int v_{nt}(v_n^m - w_n^m) - \frac{1}{m-1}\int (v_n + w_n)(v_n^m - w_n^m) + \frac{2m}{m^2-1}\int (v_n^{m+1} - w_n^{m+1}).$$

Now, we have that $\eta_n^2 \frac{\partial G_n}{\partial t} = v_{nt}(v_n^m - w_n^m)$, so

$$I = -\eta_n^2 \int \frac{\partial G_n}{\partial t} + \frac{1}{m-1}\int (v_n w_n^m - v_n^m w_n) + \frac{1}{m+1}\int (v_n^{m+1} - w_n^{m+1}).$$

Using the definition of H_n, inequality (3.10) follows.

Let us observe the following consequence of (3.10): since $\int G_n(x,0) \le C$, and by definition $|H_n| \le C G_n$, then there exist constants $a, b > 0$ such that for all n, t,

$$\int G_n(x,t) \le b e^{at}. \tag{3.11}$$

Proof of Lemma 3.1. Let us now analyze the convergence of ϕ_n. We note that ψ_n defined by (3.6) satisfies the equation

$$(\phi_n)_t = \Delta \psi_n + \psi_n - \frac{1}{m-1}\phi_n. \tag{3.12}$$

Integrating (3.12) against ψ_n, recalling that $\frac{\partial}{\partial t} G_n = \psi_n \frac{\partial \phi_n}{\partial t}$, we obtain

$$\frac{\partial}{\partial t}\int G_n dx = \int \frac{\partial \phi_n}{\partial t}\psi_n dx = -\int |\nabla \psi_n|^2 dx + \int \psi_n^2 - \frac{1}{m-1}\int \phi_n \psi_n. \tag{3.13}$$

Note that given $t > 0$, there exists $C(t) > 0$ such that

$$\int_0^t \int (|\nabla \psi_n(\cdot, s)|^2 + |\psi_n(\cdot, s)|^2) ds \le C(t) \tag{3.14}$$

for all n. In fact, let us recall that

$$\int G_n(s,x) dx \le b e^{as}.$$

Since the function v is bounded we see that $|\psi_n| \leq C|\phi_n|$. Hence, using (3.9), we get that $\psi_n\ \phi_n < CG_n$ and $\psi_n^2 \leq CG_n$. Now integrating relation (3.13) in time, between $s = 0$ and $s = t$ and using relation (3.11) we get (3.14).

As a consequence of the last result, the sequence ψ_n can be assumed, after passing to a subsequence, weakly convergent in $L^2((0,S);H^1(\mathbb{R}^N))$ for each $S > 0$. Let $\psi(x,s)$ be this limit. Recall that we are assuming $w(x) = w_*(|x|)$, so that the support of w is the ball $B = B(0,R^*)$. Then if we define $\phi = w^{-(m-1)}\psi$ on $\mathcal{B} = B \times (0,\infty)$ then

$$\int_0^S \int_B |\nabla(w^{m-1}\phi)|^2 + |w^{m-1}\phi|^2 < +\infty$$

for each $S > 0$. We recall that ϕ_n satisfies the equation

$$(\phi_n)_t = \Delta(a_n\phi_n) + a_n\phi_n - \phi_n \tag{3.15}$$

where $\frac{a_n=(v_n^m-w^m)/(}{v_n-w)}$. Then $a_n \to mw^{m-1}$ uniformly. Hence over compacts of $\mathcal{B}$ the coefficient a_n is uniformly positive and bounded. The standard theory for quasilinear nondegenerate parabolic equations, see [19], gives that this convergence is also uniform in the C^1-sense over compacts of $\mathcal{B}$, so that ∇a_n is also bounded there. Again the theory for nondegenerate parabolic equations in [19] provides uniform estimates for $C^{1,\alpha}$ norms over compacts of $\mathcal{B}$, from where C^1 convergence follows. The fact that ϕ is a weak solution of (3.4) is now easily checked. □

Let us consider the eigenvalue problem

$$m\Delta(w^{m-1}\phi) + mw^{m-1}\phi - \frac{1}{m-1}\phi + \lambda\phi = 0. \tag{3.16}$$

with ϕ such that $w^{m-1}\phi \in H^1(B)$ and $w^{\frac{m-1}{2}}\phi \in L^2(B)$. Here $w(x) = w_*(|x|)$ and $B = B(0,R^*)$.

Concerning this problem, we have the validity of the following result.

Lemma 3.2 *There is only a finite number of negative eigenvalues (repeated according to multiplicity)*

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k < 0,$$

with associated eigenfunctions $\phi_1, \ldots, \phi_k$*, normalized so that* $\int_B \phi_i\phi_j w^{m-1} = \delta_{ij}$*. Besides, the only eigenfunctions of* (3.16) *for* $\lambda = 0$ *are linear combinations of the functions* $\frac{\partial w}{\partial_{x_i}}$*.*

Let $\mathcal{N}$ *be the finite-dimensional vector space spanned by all eigenfunctions in* H *associated to non-positive eigenvalues. Then there exists a number* $\lambda^* > 0$ *such that*

$$\lambda^* \int_B w^{m-1}|\tilde{\phi}|^2 \leq \int_B m(|\nabla(w^{m-1}\tilde{\phi})|^2 + m|w^{m-1}\tilde{\phi}|^2 - \frac{w^{m-1}}{m-1}\tilde{\phi}^2)dx$$

for all $\tilde{\phi} \in H$ *with* $\int w^{m-1}\tilde{\phi}\zeta = 0$*, for all* $\zeta \in \mathcal{N}$*.*

The proof of the above lemma, which can be found in [7] is based on standard compactness arguments in Sobolev spaces. The fact that w^{m-1} vanishes quadratically on the boundary of its support plays a role here.

We can state now the following lemma.

Lemma 3.3 *Let $\phi(x,t)$ be the function found in Lemma 3.1, then*

$$\phi(x,t) = \sum_{j=1}^{k} D_j e^{-\lambda_j t}\phi_i(x) + \sum_{i=1}^{N} C_i \frac{\partial w(x)}{\partial x_i} + \theta(x,t), \tag{3.17}$$

where $\theta(x,t)$ converges to zero as $t \to +\infty$, exponentially uniformly inside compact sets of the set $\mathcal{A}$

Proof: Let us consider the expansion in B,

$$\phi(x,0) = \sum_{i=1}^{k} D_i\phi_i + \sum_{i=1}^{N} C_i \frac{\partial w}{\partial x_i} + \theta(x)$$

where

$$\int_B \theta\phi_i w^{m-1} = \int_B \theta \frac{\partial w}{\partial x_j} w^{m-1} = 0 \tag{3.18}$$

for all i, j. Now let us consider the function

$$\tilde{\phi}(x,t) = \phi(x,t) - \sum_{i=1}^{k} D_i e^{-\lambda_i t}\phi_i - \sum_{i=1}^{N} C_i \frac{\partial w}{\partial x_i}.$$

Let us observe that

$$\sum C_i^2 + \sum D_i^2 \leq \int_B w^{m-1}\phi^2(x,0)dx \leq 1.$$

Clearly $\tilde{\phi}(x,t)$ satisfies equation (3.4). We claim that

$$\int_B \tilde{\phi}(\cdot,s)\phi_i w^{m-1} = \int_B \tilde{\phi}(\cdot,s)\frac{\partial w}{\partial x_j} w^{m-1} = 0$$

for all $s > 0$. In fact, if we set for instance $\varphi(s) = \int_B \tilde{\phi}(\cdot,s)\phi_i w^{m-1}$, then $\varphi'(s) = -\lambda_i\varphi(s)$. Since $\varphi(0) = 0$, the claim follows. Now, let us set $\eta(s) = \int_B w^{m-1}\tilde{\phi}(\cdot,s)^2$. Then

$$\eta'(s) = -2\int_B m(|\nabla(w^{m-1}\tilde{\phi})|^2 - m(w^{m-1}\tilde{\phi})^2 + \frac{w^{m-1}}{m-1}\tilde{\phi}^2)dx.$$

Since $\tilde{\phi}$ satisfies the above orthogonality relations, it follows from Lemma 3.2 that

$$\eta'(s) \leq -2\lambda^*\eta(s),$$

with $\lambda^* > 0$, and hence

$$\eta(s) \leq \eta(0)e^{-2\lambda^* s}.$$

Also, $\eta(0) \leq 1$. Finally, linear parabolic regularity implies that exponential decay at this rate for $\tilde{\phi}$ also holds uniformly on compact subsets of B. □

4 Analysis near the boundary

In last section we have found the validity of convergence of ϕ_n to ϕ *essentially* in the interior of the support of the limiting w. Here we will show estimates which provide control of ϕ_n near the boundary of the support of w. Our main purpose is to show that the contribution of the region near the boundary on the integral of G_n is basically negligible.

Lemma 4.1 *Let $\varepsilon > 0$ be given. Then there exist numbers $0 < r_0 < R^*$ and $s^* > 0$ such that for each given $\bar{s} \geq s^*$ and all n sufficiently large we have*

$$\sup_{s\in[s^*,\bar{s}]} \int_{\{|x|>r_0\}} G_n(s,x)dx < \varepsilon.$$

The proof is divided into two steps.
Step 1: We make the following claim:

> There exist numbers A and c, depending only on m, with the following property: Given $\epsilon > 0$ and $0 < r_0 < R^*$, with r_0 sufficiently close to R^* and any $t > 0$, we have that for all n sufficiently large,
>
> $$\int_{|x|\geq r_1} G_n(x,t)dx \leq$$
>
> $$A\Big[e^{-ct} + \epsilon + w_*^{\frac{3m-3}{2}}(r_0) \sup_{(s,r)\in[0,t]\times[r_0,r_1]} \int_{|x|=r} \phi^2(x,s)d\sigma\Big] \qquad (4.1)$$
>
> where $r_1 = (r_0 + R^*)/2$.

To prove this claim we set $D_n = \{x/|x - x_n| \geq r_0\}$, and define for $x \in D_n$

$$g(x) = w_*^m(ar_0) - w_*^m(a|x - x_n|), \quad a \equiv \frac{2R^*}{R^* + r_0} > 1.$$

Let us multiply equation (3.12) by $\psi_n(x,t)g(x)$ and integrate on D_n. Since g vanishes on the boundary of this region, recalling that $(G_n)_t = \psi_n(\phi_n)_t$, we find

$$\frac{\partial}{\partial t}\int_{D_n} G_n(x,t)g(x)dx$$
$$= B_n(t) + \int_{D_n} \frac{\psi_n^2}{2}\Delta g(x)dx + \int_{D_n} \{-|\nabla\psi_n|^2 + \psi_n^2 - \frac{1}{m-1}\phi_n\psi_n\}g(x)\,dx. \qquad (4.2)$$

where

$$B_n(t) = -\frac{a\partial w_*^m}{\partial r}(ar_0)\int_{|x-x_n|=r_0} \frac{\psi_n^2}{2}d\sigma.$$

Now, $\Delta w_*^m = \frac{w_*}{m-1} - w_*^m \geq 0$ for $|x| \geq r_0$ if r_0 is sufficiently close to R^*, hence we assume $\Delta g(x) \leq 0$ on D_n. Using these observations and (4.2) we get

$$\frac{\partial}{\partial t}\int_{D_n} G_n(x,t)g(x)dx \leq B_n(t) + \int_{D_n} \{\psi_n^2 - \frac{1}{m-1}\phi_n\psi_n\}g(x)\,dx.$$

On the other hand, recalling the definition of ψ_n, estimate (3.10) then reads

$$0 \leq -\int \frac{\partial G_n}{\partial t} + \frac{1}{m-1}\phi_n\psi_n - \frac{2}{m-1}G_n. \tag{4.3}$$

We obtain then that

$$\int_{D_n} \frac{\partial G_n}{\partial t}(g(x) + w_*^m(ar_0))$$
$$\leq B_n(t) + \frac{1}{m-1}\int_{D_n} \phi_n\psi_n(w_*^m(ar_0) - g(x)) + \int_{D_n} \{\psi_n^2 g(x) - \frac{2w_*^m(ar_0)}{m-1}G_n.\}. \tag{4.4}$$

Here we observe that $w_*^m(ar_0) - g(x) = 0$ on $F_n = \{a|x - x_n| \geq R^*\}$. Now, given $t^* > 0$ we have that $v_n \to w$ uniformly on $[0, t^*] \times \mathbb{R}^N$. Thus, if r_0 is sufficiently close to R^* we obtain that $\psi_n^2 \leq \frac{1}{(m-1)}G_n$ for $|x| \geq r_0$ and $0 < t < t^*$ for large n. Hence

$$\psi_n^2 g(x) - \frac{2w_*^m(ar_0)}{m-1}G_n \leq -\frac{(w_*^m(ar_0) + g(x))}{(2(m-1))}G_n.$$

on this region. Substituting this information into relation (4.4), we obtain the following differential inequality for all sufficiently large n.

$$Y_n'(t) \leq B_n(t) + W_n(t) - cY_n(s), \quad 0 < t < t^*,$$

where c is a positive constant depending only on m and

$$Y_n(t) = \int_{D_n} G_n(x,t)(w_*^m(ar_0) + g(x)),$$
$$W_n(t) = \frac{1}{m-1}\int_{D_n \setminus F_n} \phi_n\psi_n(w_*^m(ar_0) - g(x)).$$

It follows that

$$Y_n(s) \leq Y_n(0)e^{-ct} + e^{-ct}\int_0^t e^{cs}(B_n(s) + W_n(s))ds. \tag{4.5}$$

We will estimate the right hand side of (4.5). First, we see that $B_n(s) \to B(s)$ and $W_n(s) \to W(s)$ uniformly on compact sets where

$$B(s) = -\frac{a\partial w_*^m}{\partial r}(ar_0)\int_{|x|=r_0} m^2 w_*^{2(m-1)}(r_0)\frac{\phi^2}{2}d\sigma,$$

and

$$W(s) = \frac{m}{m-1}\int_{r_0 \leq |x| \leq \frac{r_0+R^*}{2}} \phi^2\ (w_*^{m-1}(|x|)\ (w_*^m(ar_0) - g(x)).$$

Now,

$$(w^m)'(r_0)w^{2m-2}(r_0) = \frac{2}{m-1}(w^{\frac{m-1}{2}})'(r_0)w^m(r_0)w^{\frac{3m-3}{2}}(r_0),$$

and $w(ar_0) < w(r_0)$, for r_0 close enough to R^*, so that

$$|B(s)| \le Cw^m(r_0)w^{\frac{3m-3}{2}}(r_0)\int_{|x|=r_0}\phi^2 d\sigma.$$

On the other hand,

$$|W(s)| \le C(w_*^{m-1}(r_0)w_*^m(ar_0)(\frac{R^*-r_0}{2}) \sup_{r\in[r_0,\frac{r_0+R^*}{2}]}\int_{|x-x_i|=r}\phi^2 d\sigma$$

Since $\frac{\partial}{\partial r}w_*^{\frac{m-1}{2}}(R^*) < -c < 0$, we have

$$\frac{R^*-r_0}{2} \le Cw_*^{\frac{m-1}{2}}(r_0)$$

for some $C > 0$ depending only on m, provided that r_0 is sufficiently close to R^*. From these facts and (4.5) we see that for given $\varepsilon > 0$ and $t > 0$,

$$Y_n(t) \le Y_n(0)e^{-ct} + \epsilon + Cw_*^m(r_0)w_*^{\frac{3m-3}{2}}(r_0)\sup_{[0,t]}\sup_{r\in[r_0,\frac{r_0+R^*}{2}]}\int_{|x|=r}\phi^2 d\sigma,$$

for all sufficiently large n, where we have used again that $w_*(ar_0) < w_*(r_0)$. Finally it is easily checked that $1 \le \frac{w_*(r_0)}{w_*(ar_0)} \le C$ for some constant C independent of r_0 close to R^*. Since $g(x) = w_*^m(ar_0)$ for $|x| \ge \frac{r_0+R^*}{2}$, the resulting inequality (4.1) then follows.

Step 2: As a second step in the proof we claim that in the expansion (3.17), we have $D_i = 0$ for $i = 1, \ldots, k$.

We have that $\int G_n(x,t)dx \le \frac{1}{m^2-1}\int_0^t\int H_n + C$, for certain number C independent of t. Let us fix a number r_0 close to R^* as in the first step and set $r_1 = \frac{r_0+R^*}{2}$. We also write

$$B_{r_1} = \{x/|x| \le r_1\}.$$

Then there is a constant $D > 0$ depending on r_0 such that

$$|H_n| \le \frac{D}{\eta_n^2}|v_n - w|^3 \quad \text{on } B_{r_1},$$

from where it follows that $\int_0^t\int_{B_{r_1}} H_n \to 0$ as $n \to \infty$, for each fixed t. Then recalling that $|H_n| \le CG_n$, we find that for some $C > 0$ independent of r_0 and all sufficiently large n,

$$\int_{B_{r_1}} G_n(x,t)dx \le C[\frac{1}{m^2-1}\int_0^t\int_{\mathbb{R}^N\setminus B_{r_1}} G_n + 1].$$

Then, passing to the limit, recalling that G_n converges uniformly in $B_{r_1}\times[0,t]$ to $mw^{m-1}\phi^2$, we get

$$\int_{B_{r_1}} mw^{m-1}\phi^2(x,t)dx \le \limsup_{n\to\infty} C(\int_0^t\int_{\mathbb{R}^N\setminus B_{r_1}} G_n + 1).$$

Now, from the expression (3.17) for ϕ, we obtain

$$\phi(x,t)=\sum_{j=1}^{k} D_j e^{-\lambda_j t}\phi_i(x)+O(1).$$

with $O(1)$ uniformly bounded in time and space inside B_{r_1}. It follows that

$$\int_{\mathcal{A}\nabla_\infty} w^{m-1}\phi^2(x,t)dx=\sum_{j=1}^{k} D_j^2 e^{-2\lambda_j t}+O(1).$$

On the other hand, we can find numbers A and c which depend only on m so that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_{r_1}} G_n(x,s)dx$$
$$\leq A\{e^{-cs}+w_*^{\frac{3m-3}{2}}(r_0)\sup\{\int_{|x|=r}\phi^2(x,s)d\sigma : s\in[0,t],\ r\in[r_0,r_1],\ 1\leq i\leq k\}. \tag{4.6}$$

It can also be proved that if ϕ_i is an eigenfunction corresponding to a negative eigenvalue of (3.16), then $|\phi_i(x)|\leq Cw(|x|)^{\frac{3-m}{2}}$, hence

$$w^{\frac{3m-3}{2}}(r_0)\sup\{\int_{|x|=r}\phi^2 d\sigma : s\in[0,t],\ r\in[r_0,r_1],\ 1\leq i\leq k\}$$
$$\leq Cw(r_0)^{\frac{m+3}{2}}\sum_{j=1}^{k} D_j^2 e^{-2\lambda_j t}+O(1), \tag{4.7}$$

where C is independent of r_0. Combining the above relations, we get then that for certain constant C

$$\sum_{j=1}^{k} D_j^2 e^{-2\lambda_j t}\leq Cw(r_0)^{\frac{m+3}{2}}\sum_{j=1}^{k} D_j^2 e^{-2\lambda_j t}+O(1),$$

where C is independent of r_0. Since $w(r_0)$ may be chosen arbitrarily small, we obtain a contradiction from this last relation for all t sufficiently large if any of the D_j's was not zero. This completes the second step.

The result of the lemma follows now easily from combining steps one and two. □

5 Proof of Proposition 2.1

We define $\bar{x}_{in}=\eta_n(C_1,\ldots,C_N)$. Then $|\bar{x}_n|\leq C\eta_n$ with $C=C(m)$. Let us write $\bar{w}_n(x)=w_*^m(x+\bar{x}_n)$. We want to estimate the quantity

$$I_n(s)=\int(v_n(x,s)^m-\bar{w}_n(x)^m)(v_n(x,s)-\bar{w}_n(x))dx$$

Let us consider $r \in [0, R^*]$, to be determined later, and set $B_r = \{|x| < r\}$. Then

$$\begin{aligned} I_n(s) &= \int_{B_\delta} (v_n^m - \bar{w}_n^m)(v_n - \bar{w}_n)dx + \int_{\mathbb{R}^N \setminus B_\delta} (v_n^m - \bar{w}_n^m)(v_n - \bar{w}_n)dx \\ &= I_n^1(s) + I_n^2(s). \end{aligned}$$

We have

$$I_n^1(s) \le C[\int_{B_r} (v_n^m - w_n^m)(v_n - w_n)dx + \int_{B_r} (\bar{w}_n^m - w_n^m)(\bar{w}_n - w_n)dx].$$

Now, from 3.9, we get

$$\int_{B_r} (v_n^m - w_n^m)(v_n - w_n)dx \le C\eta_n^2 \int_{B_r} G_n(x,s)dx,$$

again with $C = C(m)$. Corollary 4.1 then implies that if r is chosen close enough to $R*$, depending on m, and $s \ge s^*$, with s^* also depending only on m then for all n sufficiently large

$$C \int_{B_r} G_n(x,s)dx < \frac{1}{8}.$$

Also,

$$C(\bar{w}_n^m - w_n^m)(\bar{w}_n - w_n) \le K\eta_n^2$$

with K depending on m and k only. Therefore, taking r closer to R^*, if necessary, we get

$$C \int_{B_r} (\bar{w}_n^m - w_n^m)(\bar{w}_n - w_n)dx \le \frac{\eta_n^2}{8}$$

if n is large enough. Putting these two estimates together we see that if we choose $T \ge s^*$, then

$$I_n^1(s) \le \frac{\eta_n^2}{4}$$

for all $s \in [T, 2T]$ provided that n is sufficiently large. On the other hand, we recall that

$$v_n(x,s) = w_n(x) + \eta_n \phi_n(x,s),$$

so that we can write

$$v_n(x,s) = w_n(x) + \eta_n \sum_{i=1}^N C_i \frac{\partial w_*}{\partial x_i}(x) + \eta_n \theta(x,s) + (\phi_n(x,s) - \phi(x,s)),$$

where θ decays exponentially in compact sets of $\mathcal{A}$.

Now, since r has been already fixed it follows that $\bar{w}_n(x) = w_*(x - x_n + \bar{x}_n)$ and

$$\lim_{n\to\infty} \eta_n^{-2} \int_{\mathbb{R}^N \setminus B_r} (v_n^m - \bar{w}_n^m)(v_n - \bar{w}_n)dx = m \int_{\mathbb{R}^N \setminus B_r} w^{m-1}(x)\theta^2(x,s)\,dx$$

uniformly on s on compact subsets of $(0, \infty)$. Since $\theta(x, s)$ decays exponentially we have that there are positive numbers A and a, depending only on m, such that

$$I_n^2(s) \leq \eta_n^2 A e^{-at}.$$

Consequently we have

$$I_n(s) \leq \eta_n^2 (\frac{1}{4} + A e^{-aT})$$

for all $s \in [T, 2T]$. Making T larger if necessary (depending only on m) we obtain that the quantity between brackets is less than $1/2$. This concludes the proof of the Proposition. □

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References

[1] J. Busca, M.A. Jendoubi, P. Polacik, *Convergence to equilibrium for semilinear parabolic problems in* $\mathbb{R}^N$. Preprint 2001.

[2] C. Cortázar, M. Elgueta, P. Felmer, *Symmetry in an elliptic problem and the blow-up set of a quasilinear heat equation.* Comm. in P.D.E.'s, 21, (1996), 507-520.

[3] C. Cortázar, M. Elgueta, P. Felmer. *Uniqueness of positive solutions of* $\Delta u + f(u) = 0$ *in* $\mathbb{R}^N$*,* $N \geq 3$. Arch. Rat. Mech. Anal. **142** (1998), 127-141.

[4] C. Cortázar, M. Elgueta, P. Felmer. *On a semilinear elliptic problem in* $\mathbb{R}^N$ *with a non-lipschitzian nonlinearity.* Adv. in Diff. Eq. **1** No 2 (1996), 199-218.

[5] C. Cortázar, M. del Pino, M. Elgueta. *On the blow-up set for* $u_t = \Delta u^m + u^m$*,* $m > 1$. Indiana Univ. Math. J. **47** (1998), 541-561.

[6] C. Cortázar, M. del Pino, M. Elgueta. *The problem of uniqueness of the limit in a semilinear heat equation* Comm. in Part. Diff. Equations **24**, (1999), 2147-2172.

[7] C. Cortázar, M. del Pino, M. Elgueta. *Uniqueness and stability of regional blow-up in a porous-medium equation.* To appear in Ann. Inst H. Poincaré, Analyse Nonlineaire.

[8] E. Feireisl y H.Petzeltova *Convergence to a ground state as threshold phenomenos in nonlinear parabolic equations*, Differential and Integral Equations 10 (1997), 181-196

[9] E. Feireisl, F. Simondon, *Convergence for degenerate parabolic equations.* J. Differential Equations 152 (1999), no. 2, 439-466.

[10] E. Feireisl, F. Simondon, *Convergence for semilinear degenerate parabolic equations in several space dimensions*, Preprint.

[11] V. Galaktionov. *On a blow-up set for the quasilinear heat equation* $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$. J. Diff. Equations 101 (1993), 66-79.

[12] V. Galaktionov, *Blow-up for quasilinear heat equations with critical Fujita's exponent.* Proc. Roy. Soc. Edinburgh 124A (1994), 517-525.

[13] V. Galaktionov, L.A. Peletier, *Asymptotic Behaviour near Finite-Time Extinction for the Fast Difussion Equation*, Arch. Rat. Mech. Anal. 139 (1997) 83-98.

[14] V. Galaktionov, J.L. Vazquez, *Continuation of blowup solutions of nonlinear heat equations in several space dimensions,* Comm. Pure Appl. Math. 50 (1997), no. 1, 1-67.

[15] C. Gui, *Symmetry of the Blow-up set of a porous medium equation.* Comm. Pure Appl. Math. 48, (1995), 471-500.

[16] J. Hale, G. Raugel, *Convergence in gradient-like and applications*, Z. Angew. Math. Phys., 43, (1992), 63-124.

[17] A. Haraux, P. Polacik, *Convergence to a positive equilibrium for some nonlinear evolution equations in a ball*, Acta Math. Univ. Comeniane, 61, (1992) 129-141.

[18] N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen, *Refined asymptotics for constant scalar curvature metrics with isolated singularities.* Invent. Mat. 135, No. 2, (1999), 233-272 .

[19] O.A. Ladyzenskaja, V.A.Solonnikov, N.N. Ural'ceva, *Linear y quasi- linear equations of parabolic type* Translations of Mathematical Monographs, 23, 1968.

[20] H. Matano, *Nonincrease of the lap number of a solution for a one-dimensional semilinear parabolic equation*, J. Fac. Sci. Univ. Tokyo 1A 29, (1982), 401-411.

[21] W.-M. Ni and I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. 70 (1993), 247-281.

[22] P. Polacik, K.P. Rybakowski, *Nonconvergent bounded trajectories in semilinear heat equations*, J. Differential Equations 124, (1996), 472-494.

[23] A. Samarskii, V. Galaktionov, V. Kurdyumov and A. Mikhailov. *Blow-up in problems for quasilinear parabolic equations,* (in Russian), Nauka, Moscow 1987.

[24] L. Simon, *Asymptotics for a class of non-linear evolution equations with applications to geometric problems*, Annals of Math. 118 (1983), 525-571.

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