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Sets of admissible initial data for porous-medium equations with absorption *

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Abstract

In this article, we study a porous-medium equation with absorption in $\mathbb{R}^N \times (0,T)$ or in $\Omega \times (0,T)$:

$$u_t - \Delta u^m + u^p = 0\,.$$

We give a rather complete qualitative picture of the initial trace problem in all the range $m > 1$, $p \geqslant 0$. We consider nonnegative Borel measures as initial data (not necessarily locally bounded) and discuss whether or not the Cauchy problem admits a solution. In the case of non-admissible data we prove the existence of some projection operators which map any Borel measure to an admissible measure for this equation.

1 Introduction

In this paper, we consider the equation

$$u_t - \Delta u^m + u^p = 0, \tag{1.1}$$

where $m > 1$ and $p > 0$. In particular, we look for nonnegative weak solutions $u = u(x,t)$ defined in $Q_T = \mathbb{R}^N \times (0,T)$ for some $T \in (0,\infty]$. We aim at describing the sets of nonnegative initial data for which there exists a solution of the Cauchy problem. We call these sets *admissible sets*; they can be quite different depending on the value of the exponents. For convenience we think of m as fixed and p as a variable parameter, hence we denote the admissible set by $\mathcal{A}^+(p)$.

1.1 Measures in pure diffusion

Our equation can be seen as a perturbation of the heat equation $u_t = \Delta u$, and it will be convenient to review the situation for this equation in order to present the main ideas. It is well-known [1] that any nonnegative distributional solution u in $\mathbb{R}^N \times (0,T)$ of the heat equation has an initial trace μ which is a Radon

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measure (i.e., a locally finite measure). This means that for any $\varphi \in C_0(\mathbb{R}^N)$, (i.e. continuous and compactly supported in $\mathbb{R}^N$) we have

$$\int_{\mathbb{R}^N} u(x,t)\varphi(x)dx \to \int_{\mathbb{R}^N} \varphi(x)d\mu(x) \quad \text{as } t \to 0. \tag{1.2}$$

Moreover, the initial trace satisfies the following growth property:

$$\int_{\mathbb{R}^N} e^{-|x|^2/4T}\, d\mu(x) < +\infty. \tag{1.3}$$

On the other hand, given a nonnegative measure μ which satisfies the above condition, there exists a unique nonnegative solution, so that the nonnegative admissible data are measures which are characterized by (1.3).

We have a similar result for the the porous-medium equation $u_t = \Delta u^m$, $m > 1$ [2, 4], where the admissible non-negative data are non-negative Radon measures which satisfy

$$\sup_{R \geqslant 1} R^{-\frac{2}{m-1}} \fint_{B_R} d\mu < +\infty, \tag{1.4}$$

where $\fint_E f$ denotes the average of f on E: $\fint_E f = |E|^{-1} \int_E f(x)dx$. On the other hand, the case $m < 1$ usually called *fast diffusion*, leads to difficulties that is better to postpone.

1.2 Complete equation and Borel measures

If we add an absorption term u^p, we get equation (1.1) and then there is a big extension of the class of initial data. Namely, we may have Borel measures, *i.e.*, not necessarily locally finite measures as initial data of standard weak solutions, even locally bounded and continuous.

A significant example of such a situation is well-known and goes back to the works of [5] and [13]. It concerns the heat equation with absorption

$$u_t - \Delta u + u^p = 0, \quad p > 1. \tag{1.5}$$

For any $c > 0$, this equation admits a solution u_c with initial data $u_c(0) = c\delta_0$, so-called fundamental solution with mass $c > 0$. Now, letting $c \to \infty$, one obtains a new kind of singular solution called *Very Singular Solution* (VSS for short) which is continuous (and locally bounded in Q_T), although it takes on the initial data "$+\infty \cdot \delta_0$". Such a measure is of course not locally bounded and is a basic example of Borel measure, that we briefly recall below. Later on, Marcus and Véron [16] proved that any Borel measure is admissible as initial data for this equation, under some capacity condition if $p \geqslant 1 + 2/N$.

We denote by $\mathcal{B}^+(\mathbb{R}^N)$ the set of Borel measures. Let us recall [16, 8] that any $\nu \in \mathcal{B}^+(\mathbb{R}^N)$ may be written as $\nu = (\mathcal{S}, \mu)$, where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ and $\mu \geqslant 0$ is a Radon measure on $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$. The set $\mathcal{S}$ is defined as follows:

$$\mathcal{S} = \{x \in \mathbb{R}^N : \forall r > 0, \quad \nu(B_r(x)) = +\infty\}.$$

Thus, a Radon measure is a Borel measure with $\mathcal{S} = \emptyset$. Let us now define what kind of solutions we consider.

Definition. By a solution we mean a function $u \in \mathrm{C}^0(Q_T)$, $u \geqslant 0$ such that (1.1) holds in the sense of distributions. We also consider $u \equiv +\infty$ as a solution to simplify the statements of our results.

It is important to notice that since u is continuous in Q_T, then it is bounded on every compact subset of Q_T, and thus it is the uniquely determined on such compact sets [8]. This proves that u can be viewed as a limit solution, *i.e.* as the limit of a sequence of smooth solutions.

We prove that the solutions we consider will take on initial data in the set of nonnegative Borel measures, $\mathcal{B}^+(\mathbb{R}^N)$. Now it means that there exists a measure $\nu \in \mathcal{B}^+(\mathbb{R}^N)$ such that for any $\varphi \in \mathrm{C}_0(\mathbb{R}^N)$, (i.e. continuous and compactly supported in $\mathbb{R}^N$), $\varphi \geqslant 0$, we have

$$\int_{\mathbb{R}^N} u(x,t)\varphi(x)dx \to \int_{\mathbb{R}^N} \varphi(x)d\nu(x) \quad \text{as } t \to 0.$$

Since $\nu = (\mathcal{S}, \mu)$ may have nonempty singular set $\mathcal{S}$, the right-hand may be infinite (it is well-defined in $\mathbb{R}_+ \cup \{+\infty\}$ since φ is nonnegative). The measure ν will be called the *initial trace* of u at $t = 0$, and we note $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu$.

Definition. Let $\nu \in \mathcal{B}^+(\mathbb{R}^N)$. Then ν is said to be admissible and we note $\nu \in \mathcal{A}^+(p)$ if there exists a solution u of (1.1) such that $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu$. We will consider that $+\infty$ is allowed as initial data, associated to the special solution $u \equiv +\infty$.

For some values of p, the set of admissible initial data is known:

- The case $p = 1$ is easily reduced to the pure diffusion case $u_t = \Delta u^m$ by the transformation $v(x,\tau) = e^t u(x,t)$, $d\tau/dt = e^{-(m-1)t}$, $\tau(0) = 0$ so that $\mathcal{A}^+(1)$ is given by (1.4) for $m > 1$, and (1.3) for $m = 1$, with $+\infty$ allowed for $u \equiv +\infty$.

- By [8] (and [16] in the case $m = 1$), it is known that if $m < p < m + 2/N$, any Borel measure is admissible as initial data, so that

$$\mathcal{A}^+(p) = \mathcal{B}^+(\mathbb{R}^N).$$

- The case $p = m$. It was shown in [9] the following exact condition

$$\mathcal{A}^+(m) = \{+\infty\} \cup \left\{ \int_{\mathbb{R}^N} \frac{e^{-|x|}}{1 + |x|^{\frac{N-1}{2}}} d\mu(x) < +\infty \right\},$$

and $+\infty$ is associated with the flat solution $u(x,t) = c_*(m)t^{-1/(m-1)}$, $c_* = (m-1)^{-1/(m-1)}$, and also with $u \equiv +\infty$.

Contents:

2 Main Results

Our results can be grouped in two areas: admissibility and the study of the projection operator.

Admissibility. In the case of very weak absorption, $0 \leqslant p \leqslant 1$, we prove that the admissible initial data are exactly the same as in the case of the purely diffusive equation:

$$\mathcal{A}^+(p) = \Big\{\mu \in \mathcal{B}^+(\mathbb{R}^N) : \sup_{R \geqslant 1} R^{-\frac{2}{m-1}} \fint_{B_R} d\mu < +\infty\Big\} \cup \{+\infty\},$$

where the measure $\nu = +\infty$ is associated to the "special solution" $u \equiv +\infty$. Moreover, we prove uniqueness of continuous weak solutions.

The situation is different in the case of weak absorption $1 < p < m$, because for example the infinite initial data is associated to a nontrivial solution, the flat solution

$$u(x,t) = c_* t^{-1/(p-1)}, \quad c_* = (p-1)^{-1/(p-1)}.$$

We do not have the exact description of the admissible initial traces in this range, but we can prove that the mapping

$$p \mapsto \mathcal{A}^+(p) \tag{2.1}$$

is strictly increasing in $[1, m]$. At least, it is known that the following rate is critical for functions [14]:

$$u_0(x) \underset{+\infty}{\backsim} c_0 |x|^{\frac{2}{m-p}}, \quad c_0(m,p,N).$$

Finally, in the case of super-critical absorption $p \geqslant m + 2/N$, we prove that the mapping (2.1) is non-increasing with p. Here also, we do not have the exact characterization of $\mathcal{A}^+(p)$, which is a question of local regularity of the measure (in terms of capacity - see [7]), however, we give some qualitative properties of this set, and some examples of admissible data.

Projections. Another important question that we investigate is what happens to non-admissible initial data when we perform an approximation process and we pass to the limit. We prove the following results:
(i) In the range $0 \leqslant p \leqslant m$, there exists a projection operator

$$\mathbb{P}_p : \mathcal{B}^+(\mathbb{R}^N) \to \mathcal{A}^+(p),$$

which maps any Borel measure to an admissible initial data. For $\nu \in \mathcal{B}^+(\mathbb{R}^N)$, $\mathbb{P}_p(\nu)$ is defined as the initial trace of the limit solution $u[\nu]$ obtained by approximation of ν with compactly supported data. It satisfies

$$\mathbb{P}_p \circ \mathbb{P}_p = \mathbb{P}_p, \quad \mathbb{P}_p(\nu) \geqslant \nu,$$

that is, $\mathbb{P}_p$ is a nondecreasing projection. More precisely, we have

Case $0 \leqslant p \leqslant 1$: if $\nu \notin \mathcal{A}^+(p)$, then $\mathbb{P}_p(\nu) = +\infty$ and $u[\nu] \equiv +\infty$. Case $1 < p \leqslant m$: if $\nu \notin \mathcal{A}^+(p)$, then $\mathbb{P}_p(\nu) = +\infty$ and $u[\nu](x,t) = c_* t^{-1/(p-1)}$. We can call the first case projection by complete blow-up and the second projection by instantaneous blow-up.
(ii) In the case $m < p < m + 2/N$, $\mathbb{P}_p$ is the identity operator in $\mathcal{B}^+(\mathbb{R}^N)$ since any Borel measure is admissible and the problem is well-posed.
(iii) In the case of super-critical absorption, we cannot prove that a projection operator is well-defined. This comes from the fact that for $0 \leqslant p \leqslant m$, admissibility is just a question of growth at infinity, hence it can be handled by monotonicity techniques (although we do not know explicitly $\mathcal{A}^+(p)$ for $1 < p < m$), whereas in the case $p \geqslant m + 2/N$, the local regularity of the initial data is involved. However, we prove that given an approximation process of ν by smooth function, the limit solution will have a trace $\nu' \leqslant \nu$. Hence, this may be viewed as a projection down towards an admissible initial data. We can say at least that ν' does not contain the part of ν which is singular with respect to the capacity in the Besov space $B^{2/q}_{q/(q-m),q/(q-1)}$. We refer to [7] for proofs of this latter fact.

3 Preliminaries

In this section, we recall some results which will be useful in the sequel.

3.1 Some Known Results

Let us begin with the following basic existence and uniqueness result taken from [8]:

Theorem 3.1 *Let $m > 1$ and $\leqslant p < m + 2/N$ and $\mu \geqslant 0$ be a compactly supported Radon measure in Ω. Then there exists a unique weak solution u with initial trace μ such that $u \in \mathrm{C}^0((0,T) \times \overline{\Omega})$, with $u = 0$ on $\partial\Omega \times (0,T)$. The same results holds when $\Omega = \mathbb{R}^N$ (with no boundary conditions).*

The following local estimate is taken from [19].

Lemma 3.2 *Let u be a smooth solution of (1.1) such that*

$$\sup_{0<t<T} \int_{B_R} u(x,t)dx \leqslant C(R),$$

then there exists a constant $C(R,\alpha)$ such that

$$\int_0^T \int_{B_R} u(x,t)^{m+2/N-\alpha} dxdt \leqslant C(R,\alpha),$$

for every $0 < \alpha < \min\{1/N; m-p+2/N\}$.

We will now study the limits of fundamental solutions when $0 \leqslant p \leqslant m$. In fact, when $p \geqslant m + 2/N$, those solutions do not exist, and we know that in the range $m < p < m + 2/N$, the limit of such solutions is the Very Singular Solution v_∞, which takes on the initial trace $\nu = +\infty\delta_0$. We thus note v_c the fundamental solution of (1.1) with initial data $c\delta_0$, in $\mathbb{R}^N$. We refer to [14] for the following result:

Theorem 3.3 *Let $\Omega = \mathbb{R}^N$.*

- *If $0 \leqslant p \leqslant 1$, then $v_c \to +\infty$ uniformly in $\mathbb{R}^N \times (0,T)$.*
- *If $1 < p \leqslant m$, then $v_c \to c(p)t^{-1/(p-1)}$ uniformly in $\mathbb{R}^N \times (0,T)$.*

3.2 Initial Trace

We will now prove that we can define an initial trace for continuous weak solutions of (1.1). In fact this result was proved in [8, 9] for the cases $1 < m < p$ and $1 < m = p$ respectively, and we first give below a localization Lemma valid for the range $0 \leqslant p \leqslant m$.

Lemma 3.4 *Let $0 \leqslant p \leqslant m$ and $u \geqslant 0$ be a solution of (1.1). Let U be a subset of $\mathbb{R}^N$. Then the following alternative holds:*

(i) If $\int_U u(x,s)dx$ remains bounded when s decreases to zero, then $u^m \in L^1(U \times (0,T))$.

(ii) If $u^m \in L^1(U \times (0,T))$, then for every $\zeta \in \mathrm{C}_0^2(U)$, the following limit exists:

$$\lim_{t\to 0} \int_U u(x,t)\zeta(x)\, dx = \ell(\zeta).$$

Proof. Since u is the limit of smooth solutions u_n, (i) is a direct consequence of Lemma 3.2. Indeed, we can assume with no restriction that the u_n satisfy

$$\int_U u_n(x,s)ds \leqslant 2 \int_U u(x,s)ds,$$

which remains bounded when $s \to 0$. Hence

$$\int_0^T \int_U u_n^{m+2/N-\alpha} \leqslant C(R,\alpha),$$

and in the limit, the same holds for u, so that in particular

$$\int_0^T \int_U u^m < \infty.$$

The proof of (ii) is done as in [9, lemma 4.1]. Indeed, we can show an integral version of the equation with function test ζ :

$$\int_U u(t)\zeta - \int_U u(s)\zeta - \int_s^t \int_U u^m \Delta\zeta + \int_s^t \int_U u^p \zeta = 0.$$

Moreover, there exists two constants c_1 and c_2 such that

$$\begin{aligned} |\zeta| &\leqslant c_1, \\ |\Delta\zeta| &\leqslant c_2, \end{aligned}$$

and thus

$$\int_0^t \int_U u^m |\Delta\zeta|,\ \int_0^t \int_U u^m |\zeta|,\ \int_0^t \int_U u^p \zeta \ < \infty,$$

so that if we let s decrease to zero,

$$\lim_{s\to 0} \int_U u(s)\zeta = \ell(\zeta) < \infty.$$

□

Thanks to this Lemma, we obtain the main initial trace result.

Theorem 3.5 *Let $m > 1$, $p \geqslant 0$ and $u \geqslant 0$ be a solution of $u_t - \Delta u^m + u^p = 0$. Then $u(t)$ has an initial trace when t decreases to zero which is a Borel measure $\nu \geqslant 0$ in $\mathbb{R}^N$, in the following sense: for any continuous, compactly supported $\varphi \in \mathrm{C}_0(\mathbb{R}^N)$, $\varphi \geqslant 0$,*

$$\int_{\mathbb{R}^N} u(x,t)\varphi(x)dx \underset{t\to 0}{\longrightarrow} \int_{\mathbb{R}^N} \varphi(x)d\nu(x), \tag{3.1}$$

whether the last integral is finite or not.

Proof. As was said, we already know the result if $1 < p < m$, so let us assume that $0 \leqslant p \leqslant m$ and for u as above, let us define

$$\mathcal{S} = \{x \in \mathbb{R}^N : \forall U \text{ neighborhood of } x,\ \int_0^T \int_U u^m = \infty\}.$$

Then $\mathcal{S}$ is a closed set, and for every $x \in \mathcal{S}$, U neighborhood of x, it is clear that

$$\int_U u(x,t)dx \underset{t\to 0}{\longrightarrow} +\infty,$$

because if the above integral remains bounded for some U, then by Lemma 3.4,

$$\int_0^T \int_K u^m < \infty,$$

for every $K \subset U$, which leads to a contradiction. On the other hand, if $x \in \mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$, then there exists a neighborhood U of x such that

$$\int_0^T \int_U u^m < \infty,$$

and again by Lemma 3.4, there exists a non negative linear functional ℓ_U defined on $\mathrm{C}_0^2(U)$.

As was done in [8], there exists a linear non negative functional ℓ on $\mathrm{C}_0(\mathcal{R})$, which can then be represented by a Radon measure μ on $\mathcal{R}$. If we set for every Borel subset E of $\mathbb{R}^N$,

$$\nu(E) = \begin{cases} \mu(E) & \text{if } E \subset \mathcal{R}, \\ +\infty & \text{if } E \cap \mathcal{S} \neq \emptyset, \end{cases}$$

then ν is a Borel measure in $\mathbb{R}^N$ and $u(t) \to \nu$ when $t \to 0$ in the following sense: for every $\varphi \in \mathrm{C}_0(\mathbb{R}^N)$, such that $\mathrm{supp}(\varphi) \subset \mathcal{R}$,

$$\int_{\mathbb{R}^N} u(t)\varphi \underset{t\to 0}{\longrightarrow} \int_{\mathbb{R}^N} \varphi d\mu,$$

and for every U such that $U \cap \mathcal{S} \neq \emptyset$,

$$\int_U u(t) \underset{t\to 0}{\longrightarrow} +\infty.$$

Then (3.1) follows for nonnegative continuous φ's with compact support in $\mathbb{R}^N$. $\square$

By convention, we define the initial trace of the infinite solution by:

$$\mathrm{tr}_{\mathbb{R}^N}(u \equiv +\infty) = +\infty.$$

Theorem 3.6 *The following results hold:*

i) If $0 \leqslant p \leqslant 1$ the initial trace of any solution $u \not\equiv +\infty$ is a Radon measure in $\mathbb{R}^N$

ii) If $1 < p \leqslant m$ the initial trace of any solution $u \not\equiv +\infty$ is either a Radon measure in $\mathbb{R}^N$, or $+\infty$, and in this latter case, we have

$$u(x,t) \geqslant c(p)t^{-1/(1-p)}.$$

Proof. This result relies upon comparison with fundamental solution, by using Theorem 3.3. Since such an argument was already used in [9], we only outline the proof.

If the trace of u contains a nonempty singular set $\mathcal{S}$, let $y \in \mathcal{S}$ and fix $c > 0$. Then there exists sequences $t_n \to 0$ and $r_n \to 0$ such that

$$\forall n \in \mathbb{N}, \quad \int_{B_{r_n}(y)} u(x, t_n)dx = c,$$

otherwise the integrals of u would remain bounded near y, which could not be a singular point. Now by comparison, this implies that u is not less than the solution u_n in $(t_n, T) \times \mathbb{R}^N$ with initial data $u_n(t_n) = u(t_n)\chi_n$, where χ_n is the characteristic function of $B_{r_n}(y)$. Then by concentration, u_n converges to the fundamental solution $v_{c,y}$ with initial data $c\delta_y$, so that $u \geqslant v_{c,\delta_y}$. Since c is arbitrary, we let $c \to \infty$, thanks to Theorem 3.3, which yields that u is either $+\infty$ everywhere or the flat solution whether p is less or greater than 1. $\square$

4 Very Weak Absorption: $0 \leqslant p \leqslant 1$

We consider equation (1.1) in the range $0 \leqslant p \leqslant 1$. We will see that in terms of initial data, the admissiblity condition is the same as for the diffusive case $u_t = \Delta u^m$ (see [2, 4]). Recall that in this range, solutions are only local *a priori*, as it is the case for the diffusive equation. A solution in Q_T is thus understood to be defined up to $t = T$ (see remark after Theorem 4.4).

4.1 Harnack Inequality and Admissibility

We prove now that when $p < 1$, $\mathcal{A}^+(p) = \mathcal{A}^+(1)$, in other terms, in this case, the absorption has no effect on admissibility of initial traces, compared with the purely diffusive case $u_t = \Delta u^m$ (we showed above that the case $p = 1$ can be reduced to the diffusive equation with a suitable change of variables and functions).

The following lemma reduces our study to the case $p = 0$.

Lemma 4.1 *Let μ be a Radon measure. Then for every $p \in [0, 1]$, the following inclusion holds:*

$$\mathcal{A}^+(1) \subset \mathcal{A}^+(p) \subset \mathcal{A}^+(0).$$

Proof. The first inclusion is obvious since $\mathcal{A}^+(1)$ is related to equation E_1 : $u_t - \Delta u^m + u = 0$, and this equation has the same admissibility set than the purely diffusive equation $u_t = \Delta u^m$. Indeed, there exists a change of time variable which maps solutions of $u_t = \Delta u^m$ to E_1 and preserves the initial data. So if a trace is admissible for E_1, it is also for the diffusive equation, and then it is admissible if we add any absorption term. We are left to prove the second inclusion.

Let $p \in [0,1]$, and $\mu \in \mathcal{A}^+(p)$. Then we have $u^p \leqslant u+1$, and thus

$$u_t - \Delta u^m + u + 1 \geqslant 0.$$

Now we use the same change of time variable (and of function) that maps E_1 to $u_t = \Delta u^m$: we get a function w which has initial data μ and

$$w_t - \Delta w^m + 1 \geqslant 0,$$

that is w is a super-solution of E_0. Now it is easy to construct a solution with initial data μ : let μ_n be a sequence of bounded measures converging monotonically to μ and v_n the sequence of minimal solutions of E_0 with initial data μ_n. Then v_n increases to some function v and since w is an upper bound for v, we know that v has locally finite initial trace. Then by monotonicity, it is obvious that $\mathrm{tr}_{\mathbb{R}^N}(v) = \mu$ (we have already proved this thing in the previous theorems). Thus $\mu \in \mathcal{A}^+(0)$ and the result is proved. ☐

We now consider the case $p = 0$, and more generally, we deal with nonnegative solutions of the equation

$$u_t = \Delta(u^m) - a\chi(u > 0), \qquad a \geqslant 0, \tag{4.1}$$

defined in $Q_T = \mathbb{R}^N \times (0,T)$. We obtain a Harnack inequality for this kind of equations, which includes in particular the porous medium equation when $a = 0$. In this case, it was already obtained by Aronson and Caffarelli [2], but our method is new for $a = 0$ and quite simple.

Lemma 4.2 *Let $u \in \mathrm{C}^0(\mathbb{R}^N \times [0,T])$ be a nonnegative solution of (4.1) in p_1 with $0 \leqslant a \leqslant A$. Let*

$$M = \int_{B_1} u(x,0)\,dx. \tag{4.2}$$

There exist positive constants $M_0 = M_0(N,m,A)$ and $k = k(N,m,A)$ such that for $M \geqslant M_0$

$$u(0,1) \geqslant k\,M^{2\lambda}, \quad \lambda = (N(m-1)+2)^{-1}. \tag{4.3}$$

Proof. It is the combination of several steps. The letter C will denote different positive constants that depend only on N and m.

• By comparison we may assume that u_0 is supported in the unit ball B_1. Indeed, for general u_0, then u_0 is greater than $u_0\eta$, η being a suitable cut-off function compactly supported in B_1 and less than one. Thus if v is the solution with initial data $u_0\eta$ (existence and uniqueness are well-known in this case), we obtain

$$\int_{B_1} u(x,0)dx \geqslant \int_{B_1} u_0\eta = M,$$

and if the lemma holds true for v, then

$$u(0,1) \geqslant v(0,1) \geqslant kM^{2\lambda}.$$

We may then take the domain of definition as $p = \mathbb{R}^N \times (0, \infty)$.

- By comparison with the porous medium equation without absorption, we know the a priori estimate for the solution [4, p. 54]

$$0 \leqslant u(x,t) \leqslant C\, M^{2\lambda}\, t^{-N\lambda} \tag{4.4}$$

and the a priori estimate for the support at time t,

$$\operatorname{supp} u(\cdot,t) \subset B_R(t), \quad R(t) = C\, M^{(m-1)\lambda}\, t^{\lambda}. \tag{4.5}$$

Hence, if M is large this radius is much larger than 1 at $t = 1$.

- The reflection argument of Aleksandrov used in Lemma 2.2 of [2] means that for $|x| \geqslant 2$ we have

$$u(0,t) \geqslant u(x,t). \tag{4.6}$$

- Let us now estimate the mass at time t

$$\int u(x,t)\, dx = \int u_0(x)\, dx - a \int_0^t \int \chi(u > 0)\, dx dt. \tag{4.7}$$

The last term is bounded above by $C\, a\, t\, R(t)^N$, while the first member can be split into the integrals

$$\int_{|x| \geqslant 2} u(x,t)\, dx + \int_{|x| \leqslant 2} u(x,t)\, dx,$$

and the last term can be estimated by

$$C\, M^{2\lambda}\, t^{-N\lambda} 2^N.$$

We conclude that

$$Cu(0,t)\, (R(t)^N - 2^N) \geqslant \int_{|x| \geq 2} u(x,t)\, dx \geqslant M - C\, a\, t\, R(t)^N - C\, M^{2\lambda}\, t^{-N\lambda} 2^N,$$

hence for $t = 1$,

$$C\, u(0,1)\, (M^{N(m-1)\lambda} - 2^N) \geqslant M - C\, a\, M^{N(m-1)\lambda} - C\, M^{2\lambda},$$

so that there are three constants $c_1, c_2, c_3(m,N)$ such that

$$u(0,1) \geqslant c_1 M^{2\lambda} - ac_2 - c_3 M^{\gamma\lambda}, \quad \gamma = 2 - (m-1)N.$$

Since $\gamma < 2$, there exists some constants M_0 and k such that

$$c_1 M^{2\lambda} - ac_2 - c_3 M^{\gamma\lambda} \geqslant k M^{2\lambda}$$

holds for every $M \geqslant M_0$, and this proves the Lemma. □

Now we give the Harnack-type inequality.

Lemma 4.3 *The estimate of the form*

$$\int_{B_r(x_0)} u(x,t)\, dx \leqslant C \left(r^{1/\lambda(m-1)} T^{-1/(m-1)} + T^{N/2} u^{1/2\lambda}(x_0, T) \right) \tag{4.8}$$

holds for all nonnegative solutions if $r \geqslant T^{m/2}$ *and* $t \leqslant T$.

Proof. We can use the previous lemma on (t,T) since $u \in \mathrm{C}^0(Q_T)$, perform the transformation

$$u^*(x,t) = r^{-2/(m-1)} T^{1/(m-1)} u(rx, Tt) \tag{4.9}$$

as in [2, p. 361], and look at the equation satisfied by u^*. Now it is the same (4.1) but for the constant a which becomes

$$a' = a\, r^{-2/(m-1)}\, T^{m/(m-1)}. \tag{4.10}$$

Hence, in order to apply the previous lemma we need to impose the condition $r \geqslant T^{m/2}$. □

Remark. The previous lemma gives a direct proof of the existence of the initial trace for E_0, which is a Radon measure.

Theorem 4.4 *For every $p \in [0,1]$, we have that following characterization:*

$$\mu \in \mathcal{A}^+(p) \Leftrightarrow \sup_{R \geqslant 1} R^{-\frac{2}{m-1}-N} \int_{B_R} d\mu(x) < \infty.$$

In other words, the admissibility condition is the same as in the purely diffusive equation.

Proof. By Lemma 4.1, we have only to prove the converse inclusion

$$\mathcal{A}^+(0) \subset \mathcal{A}^+(1).$$

If μ is admissible for E_0, there exists a minimal solution u. Hence by (4.8), we get

$$r^{1/\lambda(m-1)} \int_{B_r(x_0)} d\mu \leqslant C \left(T^{-1/(m-1)} + r^{-1/\lambda(m-1)} T^{N/2} u^{1/2\lambda}(x_0, T) \right), \tag{4.11}$$

and since $(\lambda(m-1))^{-1} = N + 2/(m-1)$, this implies that

$$\sup_{R \geqslant 1} R^{-\frac{2}{m-1}} \fint_{B_R} d\mu < \infty,$$

thus μ is admissible for E_1, and the theorem is proved. □

Remark. The existence of a solution of E_p with initial data μ is only valid up a time $T_p(\mu)$ in this range. It is obvious that $T_p(\mu)$ is not less that the blow-up time $T(\mu)$ in the case of the purely diffusive equation [4]:

$$T_p(\mu) \geqslant T(\mu) \geqslant C(m,N)/\ell(\mu)^{m-1},$$

where

$$\ell(\mu) = \lim_{r \to \infty} \sup_{R \geqslant r} R^{-\frac{2}{m-1}} \fint_{B_R} d\mu.$$

In fact, in the case $p = 1$, the blow-up time can be computed thanks to the exponential change of time variable, and we find:

$$T_1(\mu) \geqslant \frac{1}{m-1} \exp\big((m-1)\frac{C(m,N)}{\ell(\mu)}\big),$$

which is greater than $T(\mu)$.

4.2 Optimal Uniqueness

We have seen that in the case $0 \leqslant p \leqslant 1$, the admissibility condition for E_p was the same as for the case $u_t = \Delta u^m$, and since uniqueness holds for this diffusive equation with no growth restriction [11], one can reasonably think that the same holds for E_p. We prove here that it is indeed the case.

The following Lemma shows some *a priori* estimates for solutions of E_p, similar to the one satisfied by the solutions of the diffusive equation. For every $\alpha > 0$, we note

$$\rho_\alpha(x) = [1 + |x|^2]^\alpha.$$

Lemma 4.5 *Let u be a solution of E_p with initial data $\mu \in \mathcal{A}^+(p)$, i.e.*

$$\sup_{R \geqslant 1} R^{-\frac{2}{m-1}} \fint_{B_R} d\mu < \infty.$$

Then the following bound holds:

$$u(x,t) \leqslant C(t)\rho_{\frac{1}{m-1}}(x) \quad \textit{in} \quad \{|x| \geqslant 1\} \times (0,T),$$

where $C(\cdot) \in L^\infty_{\mathrm{loc}}(0,T)$. Moreover,

$$\int_{\mathbb{R}^N} u(x,t)\rho_\alpha dx \underset{t\to 0}{\longrightarrow} \int_{\mathbb{R}^N} \rho_\alpha(x) d\mu(x), \tag{4.12}$$

for every $\alpha > 1 + \frac{N}{2} + \frac{1}{m-1}$.

Proof. Let $s > 0$. Since u is a solution of E_p on (s,T), necessarily,

$$\sup_{R \geqslant 1} R^{-\frac{2}{m-1}} \fint_{B_R} u(s) < \infty,$$

and thus $u(s)$ is also an admissible initial data for $u_t - \Delta u^m = 0$. We call v_s the solution associated with $v_s(0) = u(s)$ for the diffusive equation (v_s is unique by the results of [11]). Moreover, since u is the limit of a sequence of smooth solutions on (s,T), we can compare u with v_s:

$$u(x,t) \leqslant v_s(x,t) \quad \text{in} \quad \mathbb{R}^N \times (s,t).$$

By the estimates on v_s [4, Rem. 3], we know that

$$v_s(x,t) \leqslant \frac{c(s)}{t^\lambda}[1 + |x|^2]^{\frac{1}{m-1}} \quad \text{in} \quad \{|x| \geqslant 1\} \times (s,T),$$

where $\lambda = \frac{N}{N(m-1)+2}$ and

$$c(s) \leqslant c(N,m)\Big[\sup_{R\geqslant 1} R^{-\frac{2}{m-1}} \fint_{B_R} u(s)\Big]^{2\lambda/N}.$$

The function $c(s)$ remains bounded when s decreases to zero (this is a consequence of the Harnack inequality (4.8)), we find that for some function $C(\cdot) \in L^\infty_{\rm loc}(0,T)$,

$$u(x,t) \leqslant C(t)[1+|x|^2]^{\frac{1}{m-1}} \quad \text{in} \quad \{|x| \geqslant 1\} \times (0,T).$$

Moreover, these techniques show that

$$u(x,t) \leqslant v(x,t) \quad \text{in} \quad Q_T,$$

where v is the unique solution of the diffusive equation $u_t = \Delta u^m$ with initial data μ. But the convergence property (4.12) holds for v (see [4, p. 81]), so that it also holds for u. Indeed, (all integrals are taken over $\mathbb{R}^N$)

$$\int u(t)\rho_\alpha - \int \rho_a d\mu = \int \underbrace{(u-v)}_{\leqslant 0}(t)\rho_a + \int v(t)\rho_\alpha - \int \rho_\alpha d\mu,$$

so that

$$\sup_{t\to 0}\lim \int u(t)\rho_\alpha \leqslant \int \rho_a d\mu,$$

and since $u(t) \to \mu$ weakly in measure, clearly

$$\int u(t)\rho_\alpha \underset{t\to 0}{\longrightarrow} \int \rho_a d\mu.$$

$\square$

Theorem 4.6 *Let $0 \leqslant p \leqslant 1$ and $\nu \in \mathcal{A}^+(p)$, i.e., ν satisfies*

$$\sup_{R\geqslant 1} R^{-\frac{2}{m-1}} \fint_{B_R} d\nu < \infty.$$

Then there exists a unique solution u to(1.1) such that $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu$.

Proof. Thanks to the previous a priori estimate, we can use the same techniques as in [4, Prop. 2.1], which consists in solving the dual problem. Before this, we need to construct a minimal solution, which can be obtained as in [8, Sec. 4.2]: if u is any solution, let $u_{R,\tau}$ be the unique solution of the problem

$$\begin{aligned} \partial_t u_{R,\tau} - \Delta u^m_{R,\tau} + u^p_{R,\tau} = 0 \quad &\text{in } B_R \times (\tau,T), \\ u_{R,\tau}(x,t) = 0 \quad &\text{on } \partial B_R \times (\tau,T), \\ u_{R,\tau}(\tau) = u(\tau) \quad &\text{in } B_R. \end{aligned}$$

By comparison in this set, since both solutions are bounded,

$$u_{R,\tau}(x,t) \leqslant u(x,t) \quad \text{in} \quad B_R \times (\tau, T).$$

Then if we let τ decrease to zero, we see that $u_{R,\tau}$ converges locally uniformly to a solution u_R with initial data $\mu/_{B_R}$ and zero lateral data on $\partial B_R \times (0,T)$. Moreover, u_R is uniquely determined, as was proved in [8, Theorem 6.2], so that it can be constructed independently of any solution, and in the limit,

$$u_R(x,t) \leqslant u(x,t) \quad \text{in} \quad B_R \times (0,T).$$

Finally, when R increase to $+\infty$, u_R increases to some solution $\underline{u}$ with initial data μ, which is the (unique) minimal solution since $\underline{u}(x,t) \leqslant u(x,t)$ in Q_T.

Now if u is any solution with initial data μ, we will prove that $u \equiv \underline{u}$, hence uniqueness since $\underline{u}$ can be constructed independently of any solution. Let us first fix $s > 0$ and $t \in (s,T)$. Since $u \geqslant \underline{u}$, we have

$$(u - \underline{u})_t - \Delta(u^m - \underline{u}) = \underline{u}^p - u^p \leqslant 0,$$

and thanks to the *a priori* estimate given by Lemma 4.5,

$$u\rho_{\frac{1}{m-1}}, \underline{u}\rho_{\frac{1}{m-1}} \in L^\infty(\mathbb{R}^N \times (s,t)).$$

Then the techniques of [4, Prop. 2.1] apply *verbatim* and give

$$\int_{\mathbb{R}^N} (u - \underline{u})(t)\theta \leqslant \int_{\mathbb{R}^N} |u - u\underline{u}|(s)\rho_\beta,$$

where $\theta \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \theta \leqslant 1$ is arbitrary and $\beta > \frac{N-1}{2} + \frac{m}{m-1}$ can also be chosen freely. In [4], since both solutions have the same value at $s = 0$ in L^1_{loc}, the conclusion is that the solutions coincide everywhere. But here we have to let s decrease to zero, so we use that fact that $u[\nu]$ is minimal:

$$\int |u - \underline{u}|(s)\rho_\beta = \int u(s)\rho_\beta - \int \underline{u}(s)\rho_\beta,$$

which both converge to the same value when s goes to zero, thanks to (4.12) with a suitable β. Hence in the limit,

$$\int_{\mathbb{R}^N} (u - \underline{u})(t)\theta \leqslant 0,$$

which proves that $u \equiv \underline{u}$ since $\theta \geqslant 0$ and $t > 0$ are arbitrary. □

5 Weak Absorption: $1 \leqslant p \leqslant m$

5.1 The Projection Operator

We now construct a projection operator $\mathbb{P}_p : \mathcal{B}^+(\Omega) \to \mathcal{B}^+(\Omega)$. Actually, the construction remains valid in the range $0 \leqslant p \leqslant 1$ previously studied, so that we give it in its full generality below. It is based on the following Lemma:

Lemma 5.1 *Let $0 \leqslant p \leqslant m$, and $\nu \in \mathcal{B}^+(\mathbb{R}^N)$. Let $\mu_n \geqslant 0$ be a sequence of compactly supported measures converging monotonically to ν and u_n the associated sequence of solutions. Then u_n converges locally uniformly to the minimal solution $u[\nu]$ with initial data ν. Moreover, if $\mathrm{tr}_{\mathbb{R}^N}(u[\nu])$ is not locally finite, then*

$$\begin{aligned} 0 \leqslant p \leqslant 1 &\Rightarrow u[\nu] \equiv +\infty, \\ 1 < p \leqslant m &\Rightarrow u[\nu] = c(p)t^{-1/(p-1)}. \end{aligned} \tag{5.1}$$

Proof. Let $\nu \in \mathcal{B}^+(\mathbb{R}^N)$. We will proceed in several steps:
Step 1 Let us assume that $1 < p \leqslant m$. We construct a minimal solution $u[\nu]$ by the following procedure: let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of bounded measures converging monotonically to ν, and $\{u_n\}_{n\in\mathbb{N}}$ the associated sequence of solutions of (1.1). Then u_n also converges monotonically to some function $u \in \mathrm{C}^0(\mathbb{R}^N \times (0,T))$. Indeed, we have the universal bound $u_n \leqslant c(p)t^{-1/(p-1)}$. We will prove later the minimality of u, so that we are left to show that $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu$. In fact, we know that either $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu'$ is locally finite, or $\mathrm{tr}_{\mathbb{R}^N}(u) = +\infty$.

We first assume that $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu'$ is locally finite. For every $\varphi \in \mathrm{C}_0^2(\mathbb{R}^N)$, we have both

$$\int_{\mathbb{R}^N} u_n(t)\varphi - \int_0^t \int_{\mathbb{R}^N} u_n^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} u_n^p \varphi = \int_{\mathbb{R}^N} \varphi d\mu_n, \tag{5.2}$$

and

$$\int_{\mathbb{R}^N} u(t)\varphi - \int_0^t \int_{\mathbb{R}^N} u^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} u^p \varphi = \int_{\mathbb{R}^N} \varphi d\nu'.$$

But since $u_n \leqslant u$ and $u \in L^m(0,T; L^m_{\mathrm{loc}}(\mathbb{R}^N))$ (because ν' is locally finite - see Lemma 3.4), then passing to the limit in the first equation and using the second leads to

$$\int_{\mathbb{R}^N} \varphi d\nu' = \int_{\mathbb{R}^N} \varphi d\nu,$$

which proves that $\nu' = \nu$, hence u is a solution with initial data ν. Now if $\mathrm{tr}_{\mathbb{R}^N}(u) = +\infty$, then u_n converges to a solution u which has initial trace $+\infty$. Indeed, if this were not true, then u_n would converge to a function u which has a locally finite initial trace and it would thus enjoy the following property:

$$u \in L^m(0,T; L^m_{\mathrm{loc}}(\mathbb{R}^N)).$$

Then choosing $\varphi \in \mathrm{C}_0^2(\mathbb{R}^N)^+$, and passing to the limit in equation (5.2) we would get (recall that μ_n converges to $+\infty$)

$$\int_{\mathbb{R}^N} u(t)\varphi - \int_0^t \int_{\mathbb{R}^N} u^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} u^p \varphi = +\infty,$$

which contradicts the fact that the trace of u is locally finite.
Step 2. Let us assume now that $0 \leqslant p \leqslant 1$. In this case, we prove that $u \equiv +\infty$. In fact, the limit u of the u_n cannot be locally bounded in Q_T

otherwise we would obtain a solution with trace ν (same proof as Step 1). Actually the exact condition for ν to be admissible is known and we can write (4.11) for the u_n under the form:

$$r^{1/\lambda(m-1)}\int_{B_r(x_0)} d\mu_n \leqslant C\big(t_0^{-1/(m-1)} + t_0^{N/2} r^{-1/\lambda(m-1)} u_n^{1/2\lambda}(x_0,t_0)\big).$$

Thus if we assume that there exists a point (x_0,t_0) such that $u(x_0,t_0)<\infty$, we get a bound for $r^{1/\lambda(m-1)}\int_{B_r(x_0)} d\mu_n$ as $n\to+\infty$, which says that ν satisfies the same bound. Then it is admissible by Theorem 4.4, and there exists a local solution u with trace ν. Thus we reach a contradiction, so that $u\equiv+\infty$ in Q_T.
Step 3. The minimality of u follows from uniqueness for finite measures (by Theorem 3.1). Thus, $u\geqslant u_n$ follows from uniqueness of u_n and the fact that $\mu_n\leqslant\mu$ (this kind of argument was used extensively in [8, 9], so that we omit the details). Passing to the limit in n gives the minimality of u. □

Note that the minimality of $u[\nu]$ guaranties that whatever the approximations μ_n are, we obtain always the same limit, so that the following definition makes sense.

Theorem 5.2 *For every $0\leqslant p\leqslant m$, we define a mapping $\mathbb{P}_p:\mathcal{B}^+(\mathbb{R}^N)\to\mathcal{B}^+(\mathbb{R}^N)$ by*

$$\mathbb{P}_p(\nu)=\mathrm{tr}_{\mathbb{R}^N}(u[\nu]),$$

with the convention that $\mathrm{tr}_{\mathbb{R}^N}(+\infty)=+\infty$. Then $\mathbb{P}_p(\mathcal{B}^+(\mathbb{R}^N))=\mathcal{A}^+(\mathbb{R}^N)$, and $\mathbb{P}_p$ is a projection, i.e., $\mathbb{P}_p\circ\mathbb{P}_p=\mathbb{P}_p$. Moreover, we have the characterization:

$$\nu\in\mathcal{A}^+(p)\Rightarrow\mathbb{P}_p(\nu)=\nu,$$
$$\nu\notin\mathcal{A}^+(p)\Rightarrow\mathbb{P}_p(\nu)=+\infty,$$

and in this latter case, (5.1) holds.

Proof. As we noticed, $\mathbb{P}_p$ is well-defined since the minimal solution does not depend on the approximations used. By construction, it is clear that $\mathbb{P}_p(\nu)$ is an admissible measure for any $\nu\in\mathcal{B}^+(\mathbb{R}^N)$.
Claim: $\nu\in\mathcal{A}^+(p)\Rightarrow\mathbb{P}_p(\nu)=\nu$.
Let us first assume that $1<p\leqslant m$ and let μ_n be a sequence of nonnegative compactly supported measures converging to ν. Let also u_n be the associated sequences of solutions. Then u_n converges to some function u' with trace $\mathbb{P}_p(\nu)=\nu'\leqslant\nu$. Then arguing as in Lemma 5.1 (Step 1), yields

$$\nu'=\nu,\quad i.e.,\quad \mathbb{P}_p(\nu)=\nu.$$

Now if $0\leqslant p\leqslant 1$, we use exactly the same argument if ν is locally finite. If $\nu\equiv+\infty$, then for any compact set K,

$$\int_K d\mu_n \underset{n\to\infty}{\longrightarrow} +\infty,$$

so that the Harnack Inequality (4.11) implies that $u_n \to +\infty$ everywhere. Here also, the argument follows Lemma 5.1 (Step 2).

This result proves several things: first that $\mathbb{P}_p(\mathcal{B}^+(\mathbb{R}^N)) = \mathcal{A}^+(\mathbb{R}^N)$, since any admissible data ν may be written under the form $\nu = \mathbb{P}_p(\nu)$. Second, $\mathbb{P}_p$ is a projector:

$$\forall \nu \in \mathcal{B}^+(\mathbb{R}^N), \quad \mathbb{P}_p(\mathbb{P}_p(\nu)) = \mathbb{P}_p(\nu), \quad \text{since} \quad \mathbb{P}_p(\nu) \in \mathcal{A}^+(\mathbb{R}^N).$$

Third, we have

$$\mathbb{P}_p(\nu) = \nu \iff \nu \in \mathcal{A}^+(\mathbb{R}^N).$$

It remains to prove that if $\nu \notin \mathcal{A}^+(\mathbb{R}^N)$, then $\mathbb{P}_p(\nu) = +\infty$. In the case $0 \leqslant p \leqslant 1$, we have the exact characterization, so that clearly, if $\nu \notin \mathcal{A}^+(\mathbb{R}^N)$, when we approximate it, the left-hand side of (4.11) goes to $+\infty$, so that $u_n \to +\infty$ everywhere, x_0 and T being arbitrary. Thus indeed, $\mathbb{P}_p(\nu) = +\infty$ since $u[\nu] \equiv +\infty$. Now in the case $1 < p \leqslant m$, u_n will always converge monotonically to some solution $u[\nu]$. And we saw that $\mathrm{tr}_{\mathbb{R}^N}(u)$ cannot be locally finite, otherwise monotonicity would imply that $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu$ (see Lemma 5.1). Thus $\mathrm{tr}_{\mathbb{R}^N}(u) = +\infty$, that is, $\mathbb{P}_p(\nu) = +\infty$ and the Theorem is completed. □

5.2 Admissibility Sets

We have seen that when $p \in [0, 1]$, the absorption term has no effect on admissibility of measures. Moreover, when $p = m$, we know the exact characterization of admissible data. Unfortunately, such a characterization is not known so far for measures when $1 < p < m$, but we have at least the following result which proves that the set of admissible data is increasing with p in this range.

Theorem 5.3 *For every $p, p' \in [1, m]$ such that $p < p'$, we have*

$$\mathcal{A}^+(p) \subsetneq \mathcal{A}^+(p').$$

In other words, the function $p \mapsto \mathcal{A}^+(p)$ is increasing in $[1, m]$.

Proof. Recall that thanks to Lemma 5.1, we can work with minimal solutions, hence limit solutions. We can thus use smooth solutions, the passage to the limit being automatic.

Let us assume that $p' > p$ and that $\mu \in \mathcal{A}^+(p)$. Then we shall show that μ is also an admissible data for $E_{p'}$, that is, $\mu \in \mathcal{A}^+(p')$. In fact, we will first construct a super-solution for equation $E_{p'}$ on some small interval $(0, t_0)$, with initial data μ, and then we will easily show that there exists a solution with initial data μ on $(0, T)$.

Let u be the minimal solution associated with μ, and let

$$v(x, t) = (1 + ct) \cdot u(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, t_0),$$

for some parameters $c, t_0 > 0$ which will be specified later on. Then a straightforward calculation gives

$$v_t = (1 + ct)^{1-m} \Delta v^m + c(1 + ct)^{-1} v - (1 + ct)^{1-p} v^p,$$

and changing the time variable in $\tau \in (0, \tau_0(t_0))$ such that

$$\frac{d\tau}{dt} = (1+ct)^{1-m},$$

we get, setting $v(x,t) = w(x,\tau)$,

$$w_\tau = \Delta w^m - w^p(1+ct)^{m-p} + cw(1+ct)^{m-2}.$$

Note that if we work with classical solutions,

$$w(x,0) = v(x,0) = u(x,0),$$

and this is again the case if the initial data is a measure, the above equality being understood in the sense of initial traces. Moreover, for a good choice of t_0 and c, we show that w is a super-solution of equation $E_{p'}$ with initial data μ : let c be a free parameter for the moment and put $t_0 = 1/c$, then for every $t \in (0, t_0)$,

$$\min\{1; 2^{m-2}\} < (1+ct)^{m-p}, (1+ct)^{m-2} < 2^m. \tag{5.3}$$

Now since $p' > p$, there exists a constant $k(m,p,p')$,such that for $w \geqslant k$,

$$w^{p'} \geqslant 2^m w^p,$$

and for $w \leqslant k$ (fixed above), thanks to (5.3) there exists some $c(m,p,p')$ (maybe big but finite), such that

$$cw(1+ct)^{m-2} \geqslant w^p(1+ct)^{m-p}.$$

Thus, we have obtained that whatever the value of w,

$$w^{p'} + cw(1+ct)^{m-2} \geqslant (1+ct)^{m-p}w^p, \quad \text{on} \quad (0,t_0),$$

so that there exists some $\tau_0 > 0$ (only depending on m, p and p' through c and k) such that for $\tau \in (0, \tau_0)$

$$w_\tau \geqslant \Delta w^m - w^{p'},$$

that is, w is a super-solution of $E_{p'}$ with initial data μ.

Now we can construct a solution of $E_{p'}$ with initial data μ : let μ_n be a sequence of bounded measures converging to μ monotonically and u_n be the associated sequence of minimal solutions. Then u_n also increases to some distributional solution $u \in \mathrm{C}^0(\mathbb{R}^N \times (0,T))$, and u_n has an initial trace $\nu \in \mathcal{B}^+(\mathbb{R}^N)$. We also construct the associated sequence of functions w_n on $(0, \tau_0)$ by the same process as above, taking the initial data μ_n. By construction, we have on $(0, \tau_0)$,

$$u_n(t) \leqslant w_n(t) \leqslant w(t),$$

and since $u \in L^m(0,T; L^m_{\text{loc}}(\mathbb{R}^N))$, then so is w, so that the $\{u_n\}$ remain uniformly bounded by w which is in $L^m(0,T; L^m_{\text{loc}}(\mathbb{R}^N))$, and in $L^1_{\text{loc}}(\mathbb{R}^N)$ for every

$\tau \in (0, \tau_0)$. As we already seen, this argument allows us to pass to the limit in the following equation, where $\varphi \in C_0^2(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} u_n(t)\varphi - \int_0^t \int_{\mathbb{R}^N} u_n^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} u_n^p \varphi = \int_{\mathbb{R}^N} \varphi d\mu_n,$$

and thus we get

$$\int_{\mathbb{R}^N} u(t)\varphi - \int_0^t \int_{\mathbb{R}^N} u^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} u^p \varphi = \int_{\mathbb{R}^N} \varphi d\mu,$$

hence initial trace of u is μ. Thus μ is an admissible data for $E_{p'}$, and the theorem is proved.
Finally, for $p < p'$ in $(1, m)$, the inclusion $\mathcal{A}^+(p) \subset \mathcal{A}^+(p')$ is strict, this can be easily seen on function initial data [15], and the limit cases $p = 1$ and $p = m$ where discussed above. $\square$

6 Super-Critical Absorption: $p \geqslant m + 2/N$

We will now concentrate on the case when the absorption exponent is bigger than the critical value $p_c = m + 2/N$. The operator is defined differently from the cases $0 \leqslant p \leqslant m$ since we cannot use monotonicity properties here.

6.1 The Operator $\mathbb{P}_p$

In this section, we will see that given an approximation process of the initial data $\nu \in \mathcal{B}^+(\Omega)$ by smooth functions, we obtain in the limit a solution which may have a smaller initial trace $\nu' \leqslant \nu$. As for the case $1 < p \leqslant m$, we will define $\mathbb{P}_p$ as follows: $\mathbb{P}_p(\nu) = \nu'$, but are not able to prove that $\mathbb{P}_p$ is a projector in $\mathcal{B}^+(\mathbb{R}^N)$. To fix ideas, we will use a sequence $\{\rho_n\}$ of nonnegative functions which approximate δ_0, with support $\mathrm{supp}(\rho_n) = B_{1/n}(0)$, but any other approximation process would give the same property. We point out however, that the value of $\mathbb{P}_p(\nu)$ may depend on the approximation process.

The definition of $\mathbb{P}_p$ will relies on the two following Lemmas.

Lemma 6.1 *Let $p > m$ and $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of solution of (1.1) in $\mathbb{R}^N \times (0, T)$ with initial trace ν_n such that u_n is increasing and converges locally uniformly to some u. Then u is a solution of (1.1) with initial trace*

$$\mathrm{tr}_\Omega(u) = \lim \nu_n.$$

Proof. Let us first notice that $\nu_n = \mathrm{tr}_\Omega(u_n)$ is well-defined since u_n is a weak solution, as well as $\{\mathrm{tr}_\Omega(u) = (\mathcal{S}, \mu)$. Let us call $\nu = \lim \nu_n$, which is well-defined by monotonicity. Now for any $\varphi \in C_0(\mathcal{R})$, and n big enough, we may write

$$\int_{\mathbb{R}^N} u_n(t)\varphi(t) + \int_0^t \int_{\mathbb{R}^N} \{-u_n\varphi_t - u_n^m \Delta\varphi + u_n^p \varphi\} = \int_{\mathbb{R}^N} \varphi d\nu_n < \infty. \tag{6.1}$$

Indeed, by monotonicity of u_n, the sequence of Borel measures $\nu_n = (\mathcal{S}^n, \mu_n)$ is also monotone so that φ has compact support in $\mathbb{R}^N \setminus \mathcal{S}^n$, for n sufficiently. To get some bounds, let us take $\varphi(x,t) = \varphi(x) \in \mathrm{C}_0^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi \leqslant 1$, with support in some ball $B_r \subset \mathcal{R}$, and let us define

$$X_n(t) = \int_0^t \int_{\mathbb{R}^N} u_n \varphi, \quad Y_n(t) = \int_0^t \int_{\mathbb{R}^N} u_n^p \varphi.$$

Then for a good choice of φ (see [8]), we have

$$\int_0^t \int_{\mathbb{R}^N} u_n^m |\Delta\varphi| \leqslant c(\varphi) Y_n(t)^{m/p},$$

so that we arrive at the following differential inequality, where $C = \nu(B_r) < \infty$:

$$\frac{dX_n(t)}{dt} - c(\varphi) Y_n(t)^{m/p} + Y_n(t) \leqslant C.$$

Then clearly, we get a uniform bound for $dX_n/dt = \int_{\mathbb{R}^N} u_n(t)\varphi(t)$, so that now we can use Lemma 3.2: it yields that u_n is uniformly bounded in $L^q(0,T; L^q_{\mathrm{loc}}(\mathcal{R}))$, for any $q < m + 2/N$, so that

$$\int_0^t \int_{\mathbb{R}^N} u_n \varphi_t \underset{n\to\infty}{\longrightarrow} \int_0^t \int_{\mathbb{R}^N} u\varphi_t, \quad \int_0^t \int_{\mathbb{R}^N} u_n^m \Delta\varphi \underset{n\to\infty}{\longrightarrow} \int_0^t \int_{\mathbb{R}^N} u^m \Delta\varphi.$$

If $m < p < m + 2/N$, we can use the same convergence for the absorption term, but for $p \geqslant m + 2/N$, we use the monotonicity to pass to the limit:

$$\int_0^t \int_{\mathbb{R}^N} u_n^p \varphi \underset{n\to\infty}{\longrightarrow} \int_0^t \int_{\mathbb{R}^N} u^p \varphi.$$

Passing to the limit in (6.1) yields

$$\int_{\mathbb{R}^N} u(t)\varphi(t) + \int_0^t \int_{\mathbb{R}^N} \{-u\varphi_t - u^m \Delta\varphi + u^p \varphi\} = \int_{\mathbb{R}^N} \varphi d\nu < \infty.$$

¿From this we deduce that on $\mathbb{R}^N \setminus \mathcal{S}$, $\mu = \nu$. It remains to show that the blow-up set of ν is exactly $\mathcal{S}$, which is easy: for any y in the blow-up set of ν, and any $r > 0$,

$$\int_{B_r(y)} u(x,t)dx \geqslant \int_{B_r(y)} u_n(x,t)dx,$$

so that as $t \to 0$,

$$\lim_{t\to 0} \int_{B_r(y)} u(x,t)dx \geqslant \nu_n(B_r(y)).$$

Thus when $n \to \infty$, we find

$$\lim_{t\to 0} \int_{B_r(y)} u(x,t)dx = +\infty,$$

which means that $y \in \mathcal{S}$. On the other hand, if we assume that $y \in \mathcal{S}$, there exists some $r > 0$ such that $\nu(B_r(y)) < \infty$. This implies that for any $n \in \mathbb{N}$,

$$\nu_n(B_r(y)) \leqslant \nu(B_r(y)) < \infty.$$

Thus by Lemma 3.2, we have a uniform bound for u_n^m in $L^1((0,T)\times B_r(y))$, and thus also for u_n^p (this derives from (6.1)), so that in the limit, u^m and u^p will be locally integrable near y. This proves that $y \notin \mathcal{S}$, and we reach a contradiction. $\square$

Lemma 6.2 *Let $1 < m < p$. Then for any $\nu_n = \rho_n \star \nu$ as above, there exists a solution u_n of (1.1) which takes on the initial trace ν_n.*

Proof. Let us decompose ν as $(\mathcal{S}, \mu)$, where $\mathcal{S} \subset \mathbb{R}^N$ is closed and μ is a Radon measure on $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$. Then $\nu_n = \rho_n \star \nu$ has the decomposition $(\mathcal{S}^n, \mu_n)$, where μ_n is smooth in $\mathcal{R}^n = \mathbb{R}^N \setminus \mathcal{S}^n$, and

$$\mathcal{S}^n = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \mathcal{S}) \leqslant 1/n\}.$$

This small expansion of $\mathcal{S}$ allows us to construct the following initial data:

$$u_{n,c}(0,x) = \begin{cases} \min\{c; \mu_n(x)\} & \text{on } \mathcal{R}^n, \\ c & \text{on } \mathcal{S}^n, \\ 0 & \text{otherwise,} \end{cases}$$

which clearly converges monotonically to $(\mathcal{S}^n, \mu_n)$ as c increases to $+\infty$. Then the associated sequence of solutions $u_{n,c}$ will also converge monotonically and locally uniformly to some function u, and monotonicity insures that the initial trace of u is exactly $(\mathcal{S}^n, \mu_n)$. $\square$

Now we can define $\mathbb{P}_p$ as follows: for any $\nu \in \mathcal{B}^+(\mathbb{R}^N)$, and $\nu_n = \rho_n \star \nu$, the sequence u_n of Lemma 6.2 converges locally uniformly to some weak solution $u[\nu]$ in $\mathbb{R}^N \times (0,T)$. Thus setting

$$\mathbb{P}_p(\nu) = \operatorname{tr}_{\mathbb{R}^N}(u[\nu]),$$

we have the following theorem:

Theorem 6.3 *For any $p \geqslant m + 2/N$, the operator $\mathbb{P}_p$ defined above satisfies $\mathbb{P}_p(\mathcal{B}^+(\mathbb{R}^N)) = \mathcal{A}^+(\mathbb{R}^N)$, and for any $\nu \in \mathcal{B}^+(\mathbb{R}^N)$,*

$$\mathbb{P}_p(\nu) = \nu' \leqslant \nu.$$

Proof. We note $\nu = (\mathcal{S}, \mu)$, $\nu' = (\mathcal{S}', \mu')$ and test functions $\varphi \geqslant 0$, with compact support in $\mathcal{R}$. Let also $\nu_n = \rho_n \nu$. Since $\int \varphi d\nu_n$ remains bounded (for n big enough), we get a bound for u in $L^q_{\mathrm{loc}}((\mathbb{R}^N \setminus \mathcal{S}) \times (0,T))$, for any $1 < q < m + 2/N$. Thus, in (6.1), we have convergence of u in L^1_{loc} and L^m_{loc} of the same set. This proves first that $\mathcal{S}' \subset \mathcal{S}$, and moreover using Fatou's Lemma for the term

$$\int_0^t \int_{\mathbb{R}^N} u_n^p \varphi,$$

one gets in the limit

$$\int_{\mathbb{R}^N} u(t)\varphi(t) + \int_0^t \int_{\mathbb{R}^N} \{-u\varphi_t - u^m \Delta\varphi + u^p \varphi\} \leqslant \int_{\mathbb{R}^N} \varphi d\nu,$$

which proves that on $\mathbb{R}^N \setminus \mathcal{S}$, $\mathrm{tr}_\Omega(u) \leqslant \nu$. If $p < m + 2/N$, then the absorption term also converges, and thus we pass to the limit with equality. The same happens if we assume monotonicity, by monotone convergence of the absorption term. The case when φ is not assumed to be nonnegative easily follows. □

6.2 Admissibility Sets

As we said in the introduction, in the super-critical case, the operator $\mathbb{P}_p$ is not well-defined. However, we prove here that the sets of admissible initial data do not increase with p in this range, and we give some properties of $\mathcal{A}^+(p)$ further.

Let us recall [8, Sec. 4.2] that uniqueness holds for locally finite measures also in the super-critical case. The proof was made under the assumption $1 < m < p < m + 2/N$, but the same arguments remain true if p is super-critical with no adaptations needed. In fact, this was stated in the introduction of [8]. Uniqueness is not known so far for general Borel measures, but we refer to [7] for further results.

Here is now the monotonicity property of the sets $\mathcal{A}^+(p)$.

Theorem 6.4 *Let $m + 2/N \leqslant p < p'$. Then $\mathcal{A}^+(p') \subset \mathcal{A}^+(p)$.*

Proof. Let $\nu \in \mathcal{A}(p')$, where $p' > p$, and v be the solution associated with equation $E_{p'}$. We would like to show that there exists a solution u of equation E_p.

Let us first remark that using the same technique as in Theorem 5.3, we can easily show that there exist some constants $t_0, c > 0$ such that the function

$$w(x,\tau) = (1 - ct) \cdot v(x,t), \quad \frac{d\tau}{dt} = (1 - ct)^{1-m},$$

satisfies

$$w_\tau - \Delta w^m + w^p \leqslant 0 \quad \text{in} \quad \mathbb{R}^N \times (0, \tau_0),$$

where $\tau_0 > 0$ depends on t_0 by the change of variable. Hence w is a sub-solution of equation E_p in $\mathbb{R}^N \times (0, \tau_0)$, and remembering that the constants t_0 and c depend only on m, p, p', we take n big enough so that $1/n < \tau_0$.

Now let u_n be the solution of the following problem:

$$\begin{aligned} \partial_t u_n - \Delta u_n^m + u_n^p &= 0 \quad \text{in } Q_T, \\ u_n(1/n) &= w(1/n) \quad \text{in } \mathbb{R}^N. \end{aligned}$$

It is well-known that u_n exists and is unique since $w(1/n)$ is continuous in $\mathbb{R}^N$ (see [18] or [8]). Using the same arguments as in the previous Subsection, it is clear that the u_n will converge (up to extraction) to a solution u such that

$$\mathrm{tr}_{\mathbb{R}^N}(u) \leqslant \nu.$$

But both u_n and v (thus w) are limits of regular solutions in $\mathbb{R}^N \times (1/n, T)$, and thus they can be compared in $\mathbb{R}^N \times (1/n, \tau_0)$, which gives that

$$u_n(x,\tau) \geqslant w(x,\tau) \quad \text{in} \quad \mathbb{R}^N \times (1/n, \tau_0).$$

Thus in the limit,

$$u(x,\tau) \geqslant w(x,\tau) \quad \text{in} \quad \mathbb{R}^N \times (0, \tau_0),$$

and this proves that the initial trace of u is controlled (from below) by w. Hence we have necessarily

$$\mathrm{tr}_{\mathbb{R}^N}(u) = \nu,$$

so that $\nu \in \mathcal{A}^+(p)$, and the result is proved. □

We now give some qualitative properties for the sets $\mathcal{A}^+(p)$ in the super-critical range. The first result is a consequence Lemma 6.1, concerning monotone convergence.

Lemma 6.5 *Let $p \geqslant m + 2/N$, and ν_n be an increasing sequence of admissible Borel measures such that μ_n converges in the sense of Borel measures to a measure $\nu \in \mathcal{B}^+(\mathbb{R}^N)$. Then $\nu \in \mathcal{A}^+(p)$, i.e., ν is also admissible.*

Proof. Let $\nu = (\mathcal{S}, \mu)$ and u_n be the sequence associated with ν_n. Then u_n converges monotonically and locally uniformly to a solution u of equation E_p. By Lemma 6.1, we know that

$$\mathrm{tr}_{\mathbb{R}^N}(u) \leqslant \nu,$$

but here we can use monotone convergence instead of Fatou's lemma in the proof of Lemma 6.1, which gives that on $\mathbb{R}^N \setminus \mathcal{S}$, the trace of u is exactly μ. It remains to prove that u blows up on $\mathcal{S}$. This is easy since for every open set U such that $U \cap \mathcal{S} \neq \emptyset$,

$$\int_U u(x,t)dx \geqslant \int_U u_n(x,t)dx,$$

and thus for every $n \in \mathbb{N}$,

$$\inf_{t\to 0}\lim \int_U u(x,t)dx \geqslant \nu_n(U \cap \mathcal{S}),$$

which blows up when n goes to infinity, because ν_n converges to $+\infty$ on $\mathcal{S}$. Thus we have obtained that $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu$, hence $\nu \in \mathcal{A}^+(p)$. □

Let us give now some basic examples of admissible measures, and admissible singular sets (see definition just below).

Definition. A closed set $\mathcal{S} \subset \mathbb{R}^N$ is said to be admissible as a singular set if the measure $\nu = (\mathcal{S}, 0)$ is admissible as initial data.

Proposition 6.6 *Every closed ball of positive radius is admissible as a singular set.*

Proof. Let $\overline{B}_r$ be such a ball, and χ be the characteristic function of this ball. Let u_n be the unique solution of equation E_p with initial data

$$u_n(0) = n \cdot \chi.$$

Then u_n is not zero since χ is not zero almost everywhere, and it is increasing, and by a direct application of Lemma 6.5, we obtain the result: u_n converges to a solution u with initial trace $+\infty \cdot \chi$, which is thus an admissible data. Of course, $+\infty \cdot \chi$ is the Borel measure which is represented by

$$\nu = (\overline{B}_r, 0),$$

hence the closed balls are admissible sets. $\square$

Remarks. 1. If we want to do the same with a set of zero Lebesgue measure, this method cannot work since $u_n(0) = 0$ almost everywhere, and then $u_n \equiv 0$. In fact it is clear that in the super-critical case, the singular set has to be dense enough to insure existence of a non trivial solution, since for instance a point is not admissible as a singular set (non existence of the *V.S.S.*). Obviously, balls are dense enough.
2. The same result holds true if we consider the Borel measure $\nu = (\overline{B}_r, \mu)$, where μ is an admissible Radon measure. The modification is the following: let v be the solution of E_p with initial data μ. Then if we put

$$u_n(0) = n \cdot \chi + v(1/n),$$

it is obvious that u_n will converge locally uniformly to a solution u with initial trace $+\infty$ on $\overline{B}_r$, and to check that the initial trace is not lost on the complement of $\overline{B}_r$, we use the fact that $u \geqslant v$ (by construction), so that finally, $\mathrm{tr}_{\mathbb{R}^N}(u) = \nu$. In the same spirit, one can easily prove the following results:

Proposition 6.7 *Let $\mathcal{S}$ be a closed set in $\mathbb{R}^N$ such that $\mathcal{S}$ is equal to the closure of its interior. Then $\mathcal{S}$ is admissible as a singular set.*

Proof. Let u_n be defined as in the case of a ball as the solution of E_p with initial data

$$u_n(0) = n \cdot \chi,$$

where χ is the indicator function of $\mathrm{int}(\mathcal{S})$. In particular, for every $y \in \mathrm{int}(\mathcal{S})$, there exists a ball B_y of positive Radius centered at y such that $B_y \subset \mathrm{int}(\mathcal{S})$, and if χ_y is the indicator function of B_y,

$$u_n(0) \geqslant n \cdot \chi_y.$$

Thus when n increases, u_n also increases to a limit solution u with initial trace ν, and clearly for every $y \in \mathrm{int}(\mathcal{S})$,$\nu = +\infty$ on B_y. Then it is obvious that the singular set of ν contains the closure of $\mathrm{int}(\mathcal{S})$, *i.e.* $\mathcal{S}$ itself. But on the complement of $\mathcal{S}$, it is also obvious that $\nu = 0$, by using exactly the same method as in Proposition 6.6 (or using directly Lemma 6.1). $\square$

Theorem 6.8 *Assume that $\nu = (\mathcal{S}, \mu)$, where $\mu \in \mathcal{A}^+(p)$ and $\mathcal{S}$ is admissible as a singular set, i.e., there exists a solution $V_{\mathcal{S}}$ of E_p such that*

$$\mathrm{tr}_{\mathbb{R}^N}(V_{\mathcal{S}}) = (\mathcal{S}, 0).$$

Then $\nu \in \mathcal{A}^+(p)$.

Proof. Since μ is admissible, there exists a solution v with initial data μ, and if we take the solution u_n of E_p with initial data

$$u_n(0) = V_{\mathcal{S}}(1/n) + v(1/n),$$

then obviously, $u_n \geqslant \max\{V_{\mathcal{S}}; v\}$ on $\mathbb{R}^N \times (1/n, T)$, so that u_n converges to a limit solution u locally uniformly, and $\mathrm{tr}_{\mathbb{R}^N}(u)$ is exactly $(\mathcal{S}, \mu)$. Indeed, it is obvious that u will blow-up on $\mathcal{S}$ since it is greater than $V_{\mathcal{S}}$, and on the regular set, there is no loss of initial trace because $u \geqslant v$, which takes on the initial data μ. $\square$

Remark. The Radon measure μ is defined on the complement of $\mathcal{S}$, hence it may not be a Radon measure in $\mathbb{R}^N$. Indeed, one may have $\mu(\partial\mathcal{S}) = +\infty$. Hence in this result, we say that $\mu \in \mathcal{A}^+(p)$ if μ, extended by zero on $\mathcal{S}$ is a Borel measure in $\mathbb{R}^N$ which is admissible. Note also that by Lemma 6.5, we know that if $\mu/_K$ is admissible for every compact set $K \subset \mathbb{R}^N \setminus \mathcal{S}$, then μ is admissible.

Proposition 6.9 *Let $p \geqslant m + 2/N$ and $\mu \in \mathcal{A}^+(p)$, locally finite. Then for every $\mu \in \mathcal{B}^+(\mathbb{R}^N)$,*

$$0 \leqslant \mu' \leqslant \mu \Rightarrow \mu' \in \mathcal{A}^+(p).$$

Proof. Let μ and μ' as above ($\mu' \leqslant \mu$ implies that μ' is also locally finite). Since $\mu \in \mathcal{A}^+(p)$, here exists a solution u of E_p such that $\mathrm{tr}_{\mathbb{R}^N}(u) = \mu$, and by the uniqueness result in [8, Sec. 4.2], recalled at the beginning of this section, u is unique. Thus the solution u can be viewed as the limit of the u_n, where u_n is the unique solution of E_p with initial data

$$\mu_n = \rho_n \star \mu,$$

ρ_n being a convolution kernel in $\mathbb{R}^N$. And if we call v_n the solution with initial data

$$\mu'_n = \rho_n \star \mu',$$

then by construction, $v_n \leqslant u_n$ in Q_T, and v_n will converge to a solution v of E_p. The problem is to check that v takes on the initial data μ', and to achieve this, it is sufficient to prove that the sequence v_n converges in $L^p(0,T;L^p_{\mathrm{loc}}(\mathbb{R}^N))$. Indeed, in this case, we can pass to the limit in the following weak formulation:

$$\int_{\mathbb{R}^N} v_n(t)\varphi - \int_0^t \int_{\mathbb{R}^N} v_n^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} v_n^p \varphi = \int_{\mathbb{R}^N} \varphi d\mu'_n,$$

which gives

$$\int_{\mathbb{R}^N} v(t)\varphi - \int_0^t \int_{\mathbb{R}^N} v^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} v^p \varphi = \int_{\mathbb{R}^N} \varphi d\mu',$$

hence $\mathrm{tr}_{\mathbb{R}^N}(v) = \mu'$. But since $v_n \leqslant u_n$, it is sufficient to prove that the u_n converge in $L^p(0,T;L^p_{\mathrm{loc}}(\mathbb{R}^N))$. So let us use the weak formulation for the u_n, where $\varphi \in \mathrm{C}^2(\mathbb{R}^N)$, $\mathrm{supp}(\varphi(t)) \subset K$ fixed:

$$\int_{\mathbb{R}^N} u_n(t)\varphi - \int_0^t \int_{\mathbb{R}^N} u_n \varphi_t - \int_0^t \int_{\mathbb{R}^N} u_n^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} u_n^p \varphi = \int_{\mathbb{R}^N} \varphi d\mu_n,$$

and in the limit, u is a solution with initial trace μ :

$$\int_{\mathbb{R}^N} u(t)\varphi - \int_0^t \int_{\mathbb{R}^N} u \varphi_t - \int_0^t \int_{\mathbb{R}^N} u^m \Delta\varphi + \int_0^t \int_{\mathbb{R}^N} u^p \varphi = \int_{\mathbb{R}^N} \varphi d\mu.$$

Since u_n, u_n^m converge in $L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^N))$ (from equi-integrability given by Lemma 3.2), then

$$\int_0^t \int_{\mathbb{R}^N} u_n^p \varphi \underset{n\to\infty}{\longrightarrow} \int_0^t \int_{\mathbb{R}^N} u^p \varphi. \tag{6.2}$$

Then it is easy to see that u_n converges in $L^p(0,T;L^p_{\mathrm{loc}}(\mathbb{R}^N))$: if we take $\varphi \geqslant 0$,

$$\int_0^t \int_{\mathbb{R}^N} (u^p - u_n^p)_+ \varphi \underset{n\to\infty}{\longrightarrow} 0$$

by dominated convergence, and combining this with (6.2), we obtain the same for $(u^p - u_n^p)_-$, which in turn implies the strong convergence:

$$\int_0^t \int_{\mathbb{R}^N} |u^p - u_n^p| \varphi \underset{n\to\infty}{\longrightarrow} 0.$$

Thus v is indeed a solution of E_p with initial data μ' (and in fact it is unique), hence $\mu' \in \mathcal{A}^+(p)$. □

Remark. In this result, the restriction to Radon measures is essential. Indeed, we have for instance (χ being the characteristic function of the ball $B_1(0)$):

$$0 \in \mathcal{A}^+(p), \quad \delta_0 \notin \mathcal{A}^+(p), \quad +\infty \cdot \chi \in \mathcal{A}^+(p),$$

although these measures are ordered.

7 Adaptations to the Case Ω Bounded

We give below the adapations of our results to the case when equation (1.1) is considered in $Q_T = \Omega \times (0,T)$, with Ω open, bounded and regular. Actually, the main difference is the existence of the so-called *Friendly Giant*, which was studied first by Dahlberg and Kenig [12] for the purely diffusive equation. We will use the same definition of weak solution with zero boundary data: $u \in \mathrm{C}^0(0,T;\mathrm{C}^0(\overline{\Omega})) \cap L^1_{\mathrm{loc}}(0,T;L^1(\Omega,\delta))$, where $\delta(x) = \mathrm{dist}(x,\partial\Omega)$, and for any $\varphi \in \mathrm{C}^0((0,T)\times\overline{\Omega})$, with compact support in some time interval $I \subset (0,T)$,

$$\iint_{Q_T} \{-u\varphi_t - u^m\Delta\varphi + u^p\varphi\} = 0.$$

The limit of fundamental solutions in the range $0 \leqslant p \leqslant m$ is thus the Friendly Giant:

Theorem 7.1 *Let $\Omega \subset \mathbb{R}^N$ be open, bounded, regular and $0 \leqslant p \leqslant m$. Then the following limit holds:*

$$v_c \to \mathcal{V}_p \quad \textit{locally uniformly} \quad \textit{in} \quad \overline{\Omega}\times(0,T),$$

where $\mathcal{V}_p$ is the so-called "Friendly Giant", which has the following properties: $\mathcal{V}_p$ takes on the initial trace $\nu = +\infty$ in Ω, $\mathcal{V}_p \in \mathrm{C}^0((0,T)\times\overline{\Omega})$, $\mathcal{V}_p = 0$ on $\partial\Omega\times(0,T)$.

Proof. It is clear by uniqueness of the v_c's we have comparison with the Friendly Giant for $u_t = \Delta u^m$, that is:

$$v_c(x,t) \leqslant t^{-\frac{1}{m-1}} f(x),$$

where f satisfies $-\Delta f - 1/(m-1)f = 0$ with zero boundary conditions on $\partial\Omega$. This gives a universal bound up to the boundary, hence by equi-continuity results [6], v_c will converge in $\mathrm{C}^0((0,T)\times\overline{\Omega})$ to some $\mathcal{V}_p$. Now, we are left to show that the initial trace of $\mathcal{V}_p$ is $+\infty$. We argue as in [9], with a little modification. For $k>1$, let T_k be the transformation

$$T_k(u)(x,t) = k^{1/(m-1)}u(x,kt).$$

The set Ω is invariant under T_k, and T_k maps v_c into a sub-solution with initial data $ck^{1/(m-1)}\delta_y$. Indeed, since $p \leqslant m$ and $k>1$, we have

$$\frac{\partial}{\partial t}T_k(v_c) - \Delta T_k(v_c)^m + T_k(v_c)^p = k^{m/(m-1)}u^p - k^{p/(m-1)}u^p \leqslant 0.$$

By uniqueness results in Ω and zero lateral data, we know that $v_{ck^{1/(m-1)}} \geqslant T_k(v_c)$, and thus in the limit as $c\to\infty$, we obtain

$$\mathcal{V}_p(x,t) \geqslant k^{1(m-1)}\mathcal{V}_p(x,kt).$$

So taking $k = t^{-1}$, which is greater than 1 for small t, it turns out that

$$\mathcal{V}_p(x,t) \geqslant t^{-1/(m-1)}\mathcal{V}_p(x,1).$$

In fact, it is clear by positivity properties that $\mathcal{V}_p(x,1) > 0$ at least on some open set $B_\eta(0) \subset \Omega$, which proves that the singular set of the initial trace of $\mathcal{V}_p$ contains $B_\eta(0)$. Now for any $z \in B_\eta(0)$, we can easily prove by comparison that for any $c > 0$,

$$\mathcal{V}_p(x,t) \geqslant v_c(x-y,t),$$

so that $\mathcal{V}_p(x,t) \geqslant \mathcal{V}_p(x-y,t)$. Hence, the singular set of $\mathrm{tr}_\Omega(\mathcal{V}_p)$ contains also $B_{2\eta}(0) \cap \Omega$, and by induction, we deduce that finally $\mathrm{tr}_\Omega(\mathcal{V}_p) = \Omega$. $\square$

The trace that we defined in $\mathbb{R}^N$ is based on purely local arguments, so that it is also valid in Ω bounded. Our solutions are clearly comparable with the solutions of the diffusive equation $u_t = \Delta u^m$, so that by the results of [12], the following result holds:

Theorem 7.2 *If $0 \leqslant p \leqslant m$, the initial trace of any solution $u \not\equiv \mathcal{V}_p$ in Ω is a Radon measure μ in Ω which satisfies*

$$\int_\Omega \mathrm{dist}(x, \partial\Omega)\, d\mu(x) < \infty$$

For $0 \leqslant p \leqslant m$, the operator $\mathbb{P}_p$ is constructed as in $\Omega = \mathbb{R}^N$.

Theorem 7.3 *Let $0 \leqslant p \leqslant m$. Then the operator $\mathbb{P}_p$ is a projection in $\mathcal{B}^+(\Omega)$, i.e. $\mathbb{P}_p \circ \mathbb{P}_p = \mathbb{P}_p$. Moreover,*

- *$\mathbb{P}_p(\nu) = \nu$ if and only if ν is admissible, and then $u[\nu]$ is the unique solution with trace ν,*

- *$\mathbb{P}_p(\nu) = +\infty$ if ν is not admissible and then $u[\nu] = \mathcal{V}_p$ (the Friendly Giant).*

Therefore, for bounded Ω's, we do not find any case of projection by complete blow-up.

The situation for $p > m$ is similar to the case $\Omega = \mathbb{R}^N$. Actually, the Friendly Giant gives a bound for $t > 0$, up to the boundary which allows to pass to the limit up to the boundary. Thus the limits are still weak solutions and the problem of the initial trace is exactly the same as for $\Omega = \mathbb{R}^N$. Thus, all the qualitative aspects of the sets $\mathcal{A}^+(p)$ for $p \geqslant m + 2/N$ remain valid in Ω. It is important to notice that contrary to what happens in the fast diffusion case ($m < 1$) [10], the initial Borel measure can have singular points at the boundary.

8 Some open problems and extensions

We list some of the questions that have not been solved in the preceding discussion.

- To characterize the admissibility set for $1 < p < m$.
- The same in the super-critical case $p \geqslant m + 2/N$ and to decide whether there is a projection operator.
- To describe the domains for which there exists a friendly giant and the results of the preceding section apply. For instance a domain bounded in one direction behaves like a bounded domain.
- A more difficult question is to describe admissibility sets for solutions with changing sign.

Concerning extensions to other equations, maybe the closest example is the p-Laplacian equation with absorption

$$u_t = \nabla \cdot (|\nabla u|^{p-2} \nabla u) - u^q.$$

We refer to [3] for a partial study. The super-critical exponent is now $q_c = p-1+p/N$. Different types of nonlinear equations involving absorption, reaction or convection terms come next to mind.

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