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Some Liouville theorems for the p-Laplacian *

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Abstract

In this paper we propose a new proof for non-linear Liouville type results concerning the p-Laplacian. Our method differs from the one used by Mitidieri and Pohozaev because it uses a comparison principle that can be applied to nondivergence form operators.

1 Introduction

In 1981 Gidas and Spruck proved in their famous work [14] that for $1 < p < \frac{N+2}{N-2}$ there are no solutions to

$$\Delta u + u^p = 0,\ u > 0 \quad \text{in } \mathbb{R}^N.$$

The proof is very difficult but a simpler proof was given by Chen and Li using the moving plane method [7].

Similarly, non-existence results hold for the inequality

$$\Delta u + u^p \leq 0,\ u > 0 \quad \text{in } \Sigma$$

where Σ is a cone in $\mathbb{R}^N$ (see Berestycki, Capuzzo Dolcetta, Nirenberg [3]). The values of p for which there is no positive solution depend on the cone Σ. For example for $\Sigma = \mathbb{R}^N$, $p \in (0, \frac{N}{N-2})$.

The generalization of this result to the p-Laplacian ($\Delta_p = \mathrm{div}(|\nabla .|^{p-2}\nabla)$) is very recent. Mitidieri and Pohozaev proved among other things the following result.

Theorem 1.1 *1) Suppose that $N > p > 1$, and $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ is a nonnegative weak solution of*

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) \geq h(x)u^q \quad \textit{in } \mathbb{R}^N \tag{1.1}$$

with h satisfying

$$h(x) = a|x|^\gamma \quad \textit{for } |x| \textit{ large, } a > 0 \textit{ and } \gamma > -p. \tag{1.2}$$

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Suppose that

$$p-1 < q \leq \frac{(N+\gamma)(p-1)}{N-p}\,.$$

Then $u \equiv 0$.
2) Let $N \leq p$. *If* $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ *is a weak solution of*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq 0 \ \ in \ \ \mathbb{R}^N$$

and u *is bounded below then* u *is constant.*

In this paper we present a new simple proof of Theorem 1.1. The proof of Mitidieri and Pohozaev relies on variational methods and the use of global test function. On the other hand here we use the notion of viscosity solutions and therefore use local test functions.

This kind of technique should allow us to extend Theorem 1.1 to a large class of non divergence operators. An example of such operators is given by:

$$Lu = |\nabla u|^{\alpha}\big(\mathrm{Tr}(A(x)D^2u) + kD^2u : \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\big)$$

where $\alpha \in \mathbb{R}$, and $A(x)$ is a symmetric matrix with

$$\lambda|\xi|^2 \leq A\xi\cdot\xi \leq \Lambda|\xi|^2$$

and $k \in \mathbb{R}$ satisfies $\lambda + k > 0$.

More generally this kind of proof can be used for fully nonlinear equations: Suppose that we consider $F(x, \nabla u, D^2u)$ where for example $F(x, \xi, M)$ satisfies for some $\lambda > 0$

$$|\xi|^{\alpha}\lambda\mathrm{Tr}N \leq F(x,\xi,M+N) - F(x,\xi,M) \leq |\xi|^{\alpha}\Lambda\mathrm{Tr}N,$$
$$F(x,\xi,0) = 0$$

for any symmetric and positive matrix N.

Cutrì and Leoni [8] have used similar arguments to study Liouville theorems for fully non-linear operators $F(x, D^2u)$ which satisfy the above inequality for $\alpha = 0$.

We would like to remark that the first result of Theorem 1.1 is optimal in the sense that for any $q > (N+\gamma)(p-1)/(N-p)$ we construct a nonnegative solution of (1.1). A similar example was given in [5] when $p = 2$.

Let us also remark that the condition on γ in (1.2) is optimal. Indeed, for $\gamma < -p$, Drábek in [10] has proved the existence of non trivial weak solutions in $\mathbb{R}^N$ (see e.g. Theorem 4.1 of [11]).

When treating the equation instead of the inequality, the values of q for which non existence results hold true are not the same. Precisely for the following equation

$$-\Delta_p u = r^{\gamma}u^q, \ \ u \geq 0 \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

Serrin and Zou have proved in [20] that for $p-1 < q < \frac{(N+\gamma)(p-1)+p+\gamma}{N-p}$ and $\gamma \geq 0$ any non negative solution of (1.3) is identically zero.

Let us recall that Gidas and Spruck have used Liouville theorem (for $p=2$) to obtain a priori estimates for solutions of the following problem:

$$\begin{aligned} Lu + f(x,u) = 0 \quad \text{in } \Omega \\ u = \phi \quad \text{on } \partial\Omega \end{aligned} \tag{1.4}$$

where L is a second order uniformly elliptic operator and f satisfies some growth conditions. This is done through a blow up argument (see also [3]).

Analogously, Theorem 1.1 constitutes the first step to obtain a priori estimates for reaction diffusion equations involving p-Laplacian type operators in bounded domains. In the case of systems this was done by C. Azizieh and Ph. Clement in [1], it would be interesting to do it for general non divergence form operators.

2 The inequation

When $N > p$ our main non-existence result in this section is the following

Theorem 2.1 *Suppose that $N > p > 1$. Let $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ be a nonnegative weak solution of*

$$-\Delta_p u \geq h(x) u^q \quad \text{in } \mathbb{R}^N, \tag{2.1}$$

with h satisfying (1.2). If $0 < q \leq \frac{(N+\gamma)(p-1)}{N-p}$, then $u \equiv 0$.

The proof is inspired by the one given in [8], where the authors treat fully nonlinear strictly elliptic equations. Let us start by one remark and two propositions.

Remark 2.2 The following comparison result holds true: Let u and ϕ satisfy $u, \phi \in W^{1,p}(\Omega)$

$$\begin{aligned} -\Delta_p u \geq -\Delta_p \phi \quad \text{in } \Omega \\ u \geq \phi \quad \text{on } \partial\Omega\,. \end{aligned}$$

Then $u \geq \phi$ in Ω. This is a standard result and it is easy to see for example by multiplying $-\Delta_p u + \Delta_p \phi$ by $(\phi - u)^+$.

Proposition 2.3 *Let Ω be an open set in $\mathbb{R}^N$, and let $f \in \mathcal{C}(\overline{\Omega})$. Suppose that $u \in W^{1,p}_{\text{loc}}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a weak solution of $-\Delta_p u \geq f$ in Ω. If $x_0 \in \Omega$ and $\varphi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are such that*

$$\nabla\varphi(x_0) \neq 0, \; u(x_0) - \varphi(x_0) = \inf_{y \in \Omega} u(y) - \varphi(y)\,,$$

then $-\Delta_p \varphi(x_0) \geq f(x_0)$.

This proof is inspired by Juutinen [18].

Proof. Without loss of generality we can suppose that $u(x_0) = \varphi(x_0)$. Let us note first that it is sufficient to prove that the property holds for every φ such that $\varphi(y) < u(y)$ for all $y \neq x_0$ in a sufficiently small neighborhood of x_0. Indeed, suppose that the property holds for such functions then taking $\varphi_\epsilon(y) = \varphi(y) - \epsilon|y - x_0|^4$ and letting ϵ go to zero, one obtains the result for every φ.

Suppose by contradiction that there exists some $x_0 \in \Omega$ and some $\mathcal{C}^2$ function φ such that $\nabla\varphi(x_0) \neq 0$, $\varphi(x_0) = u(x_0)$ and $\varphi(y) < u(y)$ on some ball $B(x_0, r) \setminus \{x_0\}$ and $-\Delta_p\varphi(x_0) < f(x_0)$. By continuity, one can choose r sufficiently small such that $\nabla\varphi(y) \neq 0$, as well as

$$-\Delta_p\varphi(y) < f(y),$$

for all $y \in B(x_0, r)$. Let $m = \inf_{|x-x_0|=r}\{(u(x) - \varphi(x)) > 0\}$, and define

$$\bar{\varphi} = \varphi + \frac{m}{2}.$$

One has $-\Delta_p\bar{\varphi} < f$ in $B(x_0, r)$ and $\bar{\varphi} \leq u$ on $\partial B(x_0, r)$. Using the comparison principle one gets that $\bar{\varphi} \leq u$ in the ball which contradicts $\bar{\varphi}(x_0) = \varphi(x_0) + \frac{m}{2} > u(x_0)$. This ends the proof of Proposition 2.3. □

Finally let us recall that if v is radial i.e. $v(x) = V(|x|) \equiv V(r)$ for some function V in $\mathcal{C}^2$, then if x is such that $V'(|x|) \neq 0$,

$$\Delta_p v(x) = |V'(r)|^{p-2}\big((p-1)V''(r) + \frac{N-1}{r}V'(r)\big).$$

Hence for any constants C_1 and C_2 if $N \neq p$ and for $\lambda = \frac{p-N}{p-1}$ the function $\phi(x) = C_2|x|^\lambda + C_1$ satisfies $\Delta_p\phi = 0$ for $x \neq 0$.

Before giving the proof of Theorem 2.1 let us define $m(r) = \inf_{x\in B_r} u(x)$ and prove the following Hadamard type inequality

Proposition 2.4 *Let $N \neq p$. Suppose that $-\Delta_p u \geq 0$ and $u \geq 0$. Let $\lambda = \frac{p-N}{p-1}$. For any $0 < r_1 < r < r_2$:*

$$m(r) \geq \frac{m(r_1)(r^\lambda - r_2^\lambda) + m(r_2)(r_1^\lambda - r^\lambda)}{r_1^\lambda - r_2^\lambda}. \tag{2.2}$$

Let $N = p$. Then

$$m(r) \geq \frac{m(r_1)\log(\frac{r}{r_2}) + m(r_2)\log(\frac{r_1}{r})}{\log(\frac{r_1}{r_2})}. \tag{2.3}$$

Proof: Let $N \neq p$. Let $0 < r_1 < r_2$. Let us consider $\phi(r) = C_2 r^\lambda + C_1$ with C_2 and C_1 such that $\phi(r_1) = m(r_1)$ and $\phi(r_2) = m(r_2)$. It is easy to see that

$$\phi(r) = \frac{m(r_2)(r^\lambda - r_1^\lambda) + m(r_1)(r_2^\lambda - r^\lambda)}{r_2^\lambda - r_1^\lambda}.$$

Obviously $\phi > 0$ and for $i = 1$ and $i = 2$, $u(x) \geq m(r_i) = \phi(r_i)$ for $x \in \partial B_{r_i}$, hence u and ϕ satisfy the conditions of Remark 2.2. and $u(x) \geq \phi(|x|)$ in $B_{r_2} \backslash B_{r_1}$. Taking the infimum we obtain that $\inf_{|x|=r} u(x) \geq \phi(r)$ for $r \in [r_1, r_2]$. By the minimum principle $m(r) = \inf_{|x|=r} u(x)$. This completes the proof of the first part of proposition 2.4.

For $N = p$ consider

$$\psi(r) = \frac{m(r_1)\log(\frac{r}{r_2}) + m(r_2)\log(\frac{r_1}{r})}{\log(\frac{r_1}{r_2})}.$$

Remark that $\Delta_N \psi = 0$ and $\psi(r_1) = m(r_1)$ and $\psi(r_2) = m(r_2)$. Now proceed as above.

Remark 2.5 Clearly if $\lambda < 0$ i.e. $p < N$, then $g(r) := m(r)r^{-\lambda}$ is an increasing function. Just observe that $r_1^\lambda - r^\lambda \geq 0$ and let r_2 tend to $+\infty$ in (2.2) and one obtains for $r \geq r_1$:

$$m(r) \geq \frac{m(r_1)r^\lambda}{r_1^\lambda}.$$

Proof of Theorem 2.1. We suppose by contradiction that $u \not\equiv 0$ in $\mathbb{R}^n$, but since $u \geq 0$ by the strict maximum principle of Vasquez [22] we get that $u > 0$.

Let $0 < r_1 < R$, define $g(r) = m(r_1)\left\{1 - \frac{[(r-r_1)^+]^{k+1}}{(R-r_1)^{k+1}}\right\}$ with k such that

$$k \geq 3 \quad \text{and} \quad \frac{1}{k} < p - 1.$$

Let $\zeta(x) = g(|x|)$. Clearly for $|x| < r_1$, $u(x) > m(r_1) = \zeta(x)$ while for $|x| \geq R$, $\zeta(x) \leq 0 < u(x)$. On the other hand there exists $\tilde{x}$ such that $|\tilde{x}| = r_1$ and $u(\tilde{x}) = \zeta(\tilde{x})$. Hence the minimum of $u(x) - \zeta(x)$ occurs for some $\bar{x}$ such that $|\bar{x}| = \bar{r}$ with $r_1 \leq \bar{r} < R$.

Let $|x| = r$, it is an easy computation to see that for $r \geq r_1$

$$\begin{aligned} &-\Delta_p \zeta(x) \qquad\qquad (2.4)\\ &= \Big(\frac{(k+1)m(r_1)}{(R-r_1)^{k+1}}\Big)^{(p-1)} \big[2(p-1) + (N-1)\frac{(r-r_1)^+}{r}\big]((r-r_1)^+)^{kp-(k+1)}. \end{aligned}$$

Clearly with our choice of k, $kp - (k+1) > 0$ and hence, for $|x| = r_1$, $-\Delta_p \zeta(x) = 0$ while, of course, $\nabla \zeta(x) = 0$.

Now we have two cases: First case $\bar{r} = r_1$. This implies

$$u(\bar{x}) - m(r_1) = u(\bar{x}) - \zeta(\bar{x}) \leq u(x) - \zeta(x)$$

for all x. In particular choosing $x = \tilde{x}$, one gets

$$u(\bar{x}) - m(r_1) \leq u(\tilde{x}) - \zeta(\tilde{x}) = 0.$$

Finally $u(\bar{x}) = m(r_1)$ and $\bar{x}$ is a minimum for u on $B(0, r_1)$. Since $-\Delta_p u \geq 0$, Hopf's principle as stated in Vasquez [22] implies that $\nabla u(\bar{x}) \neq 0$. On the other hand $\nabla u(\bar{x}) = \nabla \zeta(\bar{x}) = 0$, a contradiction.

Second case: $r_1 < \bar{r} < R$. Now $\nabla\zeta(\bar{x}) \neq 0$, and using Proposition 2.3 one has

$$h(\bar{x})u^q(\bar{x}) \leq -\Delta_p\zeta(\bar{x}).$$

We choose r_1 and R sufficiently large in order that $h(x) = a|x|^\gamma$ for $|x| \geq \min(r_1, R/2)$. Combining this with (2.4), we obtain

$$a\bar{r}^\gamma m(\bar{r})^q \leq a\bar{r}^\gamma u^q(\bar{x}) \leq (k+1)^{(p-1)}(N+2p-3)m(r_1)^{(p-1)}(R-r_1)^{-p}.$$

Since m is decreasing we have obtained for some constant $C > 0$

$$m(R) \leq Cm(r_1)^{\frac{(p-1)}{q}}\bar{r}^{\frac{-\gamma}{q}}(R-r_1)^{\frac{-p}{q}}.$$

Now we choose $r_1 = \frac{R}{2}$, we use Remark 2.5 and the previous inequality becomes

$$m(R) \leq Cm(R)^{\frac{(p-1)}{q}}R^{\frac{-p-\gamma}{q}}. \tag{2.5}$$

First we will suppose that $q \leq p-1$; hence, using the monotonicity of $m(R)$, the above inequality becomes

$$R^{\frac{p+\gamma}{q}} \leq Cm(R)^{\frac{(p-1)}{q}-1} \leq Cu(0)^{\frac{(p-1)}{q}-1}.$$

But this is absurd since the left hand side tends to infinity when R does. This conclude the proof of this case.

Now suppose that $q > p-1$, then (2.5) becomes

$$m(R)R^{-\lambda} \leq CR^{-\lambda-\frac{p+\gamma}{q-p+1}}. \tag{2.6}$$

Clearly $-\lambda - \frac{p+\gamma}{q-p+1} = \frac{N-p}{p-1} - \frac{p+\gamma}{q-p+1} \leq 0$ when $q \leq \frac{(N+\gamma)(p-1)}{N-p}$.

If $q < (N+\gamma)(p-1)/(N-p)$ we have reached a contradiction since the right hand side of (2.6) tends to zero for $R \to +\infty$ while the left hand side is an increasing positive function as seen in Remark 2.5.

We now treat the case $q = (N+\gamma)(p-1)/(N-p)$. Let us remark that for this choice of q we have that for some $C_1 > 0$, $c > 0$ and $r > r_1 > 0$, with r_1 large enough:

$$-\Delta_p u \geq ar^\gamma u^q \geq C_1 r^{-N} \quad \text{since} \quad m(r) \geq cr^{\frac{p-N}{p-1}}. \tag{2.7}$$

We choose $\psi(x) = g(|x|)$ with

$$g(r) = \gamma_1 r^{\frac{p-N}{p-1}}\log^\beta r + \gamma_2$$

where γ_1 and γ_2 are two positive constants such that for some $r_1 > 1$ and some $r_2 > r_1$:

$$m(r_2) = g(r_2), \quad m(r_1) \geq g(r_1),$$

while β is a positive constant to be chosen later. It is easy to see that

$$\begin{aligned}\Delta_p\psi &= |\gamma_1|^{p-1}r^{-N}\Big|\frac{p-N}{p-1}\log^\beta r + \beta\log^{\beta-1} r\Big|^{p-2} \\ &\quad \times \Big[(p-1)\beta(\beta-1)\log^{\beta-2} r - \beta(3N-2p-2)\log^{\beta-1} r\Big]\end{aligned}$$

Suppose now that $p > 2$, and choose $0 < \beta < \frac{1}{p-1} < 1$, then there exists $C > 0$ such that

$$\Delta_p \psi \geq -|\gamma_1|^{p-1} C r^{-N} (\log r)^{\beta(p-1)-1} \geq -|\gamma_1|^{p-1} C r^{-N} (\log r_1)^{\beta(p-1)-1}.$$

On the other hand for $p \leq 2$ we can choose $\beta = 1$ and a calculation similar to the one above implies that

$$\Delta_p \psi \geq -c|\gamma_1|^{p-1} r^{-N} (\log r_1)^{p-2}.$$

In both cases we can choose γ_1 small enough to get

$$\Delta_p \psi \geq -C_1 r^{-N} \geq \Delta_p u.$$

Since $u \geq \psi$ on the boundary of $B_{r_2} \setminus B_{r_1}$, one obtains by the comparison principle (Remark 2.2) that $u \geq \psi$ everywhere in $B_{r_2} \setminus B_{r_1}$. When r_2 goes to infinity it is easy to see that $\gamma_2 \to 0$, and we obtain

$$u(x) \geq c|x|^{\frac{p-N}{p-1}} \log^\beta |x|,$$

for $|x| \geq r_1$. This implies that

$$m(r) \geq c r^{\frac{p-N}{p-1}} \log r$$

for $r > r_1$. We have reached a contradiction since

$$m(r) \leq C r^{\frac{p-N}{p-1}}.$$

Hence $u \equiv 0$. This concludes the proof of Theorem 2.1. □

We treat now the case $N \leq p$ where the result is much stronger.

Theorem 2.6 *Let $N \leq p$. If $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ is bounded below and is a weak solution of*

$$-\Delta_p u \geq 0 \quad \text{in} \quad \mathbb{R}^N,$$

then u is constant.

Remark 2.7 For $N \leq p$, for any $q > 0$ and for any nonnegative h, if $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap \mathcal{C}(\mathbb{R}^N)$ is a weak solution of

$$-\Delta_p u \geq h(x) u^q \quad \text{in} \quad \mathbb{R}^N$$

then $u \equiv 0$.

Proof of Theorem 2.6. Without loss of generality we can suppose that $u \geq 0$. First we will consider $N < p$. Let $m(r) = \inf_{x \in B_r(0)} u(x)$. From Proposition 2.4 we know that for $0 < r_1 < r < r_2$

$$m(r) \geq \frac{m(r_1)(r_2^\lambda - r^\lambda) + m(r_2)(r^\lambda - r_1^\lambda)}{r_2^\lambda - r_1^\lambda}, \tag{2.8}$$

where $\lambda = \frac{p-N}{p-1} > 0$. When we let $r_2 \to +\infty$ inequality (2.8) becomes

$$m(r) \geq m(r_1). \tag{2.9}$$

But of course $m(r)$ is decreasing hence (2.9) implies that $m(r)$ is constant i.e. $m(r) = m(0) = u(0)$ for any $r > 0$. Clearly this can be repeated with balls centered in any point of $\mathbb{R}^N$. Hence u is constant.

For the case $N = p$ just use inequality (2.3) in Proposition 2.4 and proceed as above. This concludes the proof of Theorem 2.6.

Counterexample

We are going to show that for $N > p$, $\gamma \geq 0$ and $q > (N+\gamma)(p-1)/(N-p)$ there exists a non-negative function u such that

$$-\Delta_p u \geq r^\gamma u^q \quad \text{in } \mathbb{R}^N$$

hence proving that $(N+\gamma)(p-1)/(N-p)$ is an optimal upper bound for q in Theorem 2.1.

Indeed consider $g(r) = C(1+r)^{-\alpha}$ with α and C two positive constants to be determined. Clearly $\Gamma(x) = g(|x|)$ satisfies

$$\begin{aligned} -\Delta_p \Gamma &= C^{p-1}\alpha^{p-1}(1+r)^{-(\alpha+1)(p-2)}[-(\alpha+1)(p-1)(1+r)^{-(\alpha+2)} + \\ &\quad + \frac{(N-1)}{r}(1+r)^{-(\alpha+1)}] \\ &\geq C^{p-1}\alpha^{p-1}(1+r)^{-\alpha(p-1)-p}[N-1-(\alpha+1)(p-1)] \end{aligned}$$

with $r = |x|$.

Now let $\epsilon > 0$ such that $q = (N+\gamma-\epsilon)(p-1)/(N-p-\epsilon)$ and let $\alpha = (N-p-\epsilon)/(p-1)$. Clearly we have $\alpha(p-1)+p+\gamma = N+\gamma-\epsilon = \alpha q$. Furthermore $N-1-(\alpha+1)(p-1) = N-p-\alpha(p-1) = \epsilon > 0$. Hence choosing C such that $C^{p-1}\alpha^{p-1}(\epsilon) = C^q$ we obtain that $\Gamma(x)$ satisfies

$$-\Delta_p \Gamma \geq C^q(1+r)^\gamma(1+r)^{-\alpha(p-1)-p-\gamma} \geq r^\gamma \Gamma^q \quad \text{in } \mathbb{R}^N.$$

3 The equation

In this section we are interested in studying non-existence results concerning the equation. Clearly in view of Theorem 2.6, we are only interested in the case $N > p$.

Theorem 3.1 *Suppose that $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ is nonnegative and satisfies*

$$-\Delta_p u = r^\gamma u^q, \tag{3.1}$$

for some $\gamma \geq 0$. If

$$p - 1 < q < \frac{(N+\gamma)(p-1)+p+\gamma}{N-p}$$

and u is radial then $u \equiv 0$.

Remark 3.2 One can get the same result for $-\Delta_p u = Cr^\gamma u^q$ by considering u multiplied by some convenient constant.

The proof given here is similar to the one given by Caffarelli, Gidas and Spruck in [6].

Proof. It is sufficient to consider the case $q \geq (N+\gamma)(p-1)/(N-p)$, since the other cases are proved in Theorem 2.1.

If u is a radial solution and satisfies (3.1) in a weak sense, then it is not difficult to see that it satisfies in the weak sense

$$-(r^{N-1}|u'|^{p-2}u')' = r^{N-1+\gamma}u^q$$

Integrating between 0 and r, one has

$$r^{N-1}|u'|^{p-2}u' = -\int_0^r s^{N-1+\gamma}u^q(s)ds.$$

Since $u' < 0$, u is decreasing and then,

$$r^{N-1}|u'|^{p-2}u' \leq -u(r)^q \frac{r^{N+\gamma}}{N+\gamma}.$$

Hence

$$u'u^{\frac{-q}{p-1}} \leq -cr^{\frac{1+\gamma}{p-1}}$$

and integrating one gets

$$u(r) \leq Cr^{\frac{\gamma+p}{p-1-q}}.$$

Coming back to the equation one obtains

$$r^{N-1}|u'|^{p-1} = \int_0^r s^{N-1+\gamma}u^q(s)ds \leq C\int_0^r s^{N-1+\gamma}s^{\frac{(\gamma+p)q}{(p-1-q)}}ds.$$

Clearly $N+\gamma+\frac{(\gamma+p)q}{p-1-q} \geq 0$ when $q \geq \frac{(N+\gamma)(p-1)}{N-p}$ and therefore

$$|u'(r)|^{p-1} \leq Cr^{\gamma+\frac{(\gamma+p)q}{p-1-q}+1}$$

and then

$$|u'| \leq Cr^{\frac{(\gamma+q+1)}{p-1-q}}.$$

In order to conclude, we need to use Pohozaiev identity:

$$(N-p)\int_B |\nabla u|^p + p\int_{\partial B} \sigma.n(\nabla u.x) = p\int_B \Delta_p u(\nabla u.x) + \int_{\partial B} |\nabla u|^p(x.\vec{n})$$

here $\sigma = |\nabla u|^{p-2}\nabla u$ and $B = B(0,R)$. From the equation we know that

$$\int_B |\nabla u|^p - \int_{\partial B}(\sigma.\vec{n})u = \int_B r^\gamma u^{q+1}$$

and then

$$(N-p)\Big(\int_B r^\gamma u^{q+1} + \int_{\partial B} \sigma.\vec{n}u\Big) + p\int_{\partial B}(\sigma.\vec{n})(\nabla u.x)$$
$$= -p\int_B r^\gamma u^q(\nabla u.x) + \int_{\partial B} |\nabla u|^p x.\vec{n}. \quad (3.2)$$

Using the fact that u is radial, for $\omega_n = |B_1|$ one gets

$$\begin{aligned}\frac{1}{\omega_n}\int_{B_R} r^\gamma u^q \nabla u.xdx &= \int_0^R r^{\gamma+N}u^q(r)u'(r)dr \\ &= \int_0^R r^{\gamma+N}(\frac{u^{q+1}(r)}{q+1})'dr \\ &= -\frac{\gamma+N}{q+1}\int_0^R r^{\gamma+N-1}u^{q+1} + \frac{1}{q+1}R^{\gamma+N}u^{q+1}(R).\end{aligned}$$

We have finally obtained

$$\begin{aligned}&\big(N-p-\frac{(\gamma+N)p}{q+1}\big)\int_0^R r^{\gamma+N-1}u^{q+1}dr \\ &= (N-p)|u'|(R)^{p-1}u(R)R^{N-1} + (1-p)|u'(R)|^pR^N - \frac{p}{q+1}R^{\gamma+N}u(R)^{q+1}.\end{aligned}$$

Let us note that since $q < \frac{(N+\gamma)(p)+p-N}{N-p}$, one has

$$\frac{(\gamma+N)p}{q+1} + p - N > 0.$$

Moreover the estimates on u and u' imply that the terms $|u'|^{p-1}u(R)R^{N-1}$, $|u'|^p(R)R^N$ and $R^{\gamma+N}u^{q+1}(R)$ behave respectively as $R^{N-1+\frac{\gamma+p}{p-1-q}+\frac{(\gamma+q+1)(p-1)}{p-1-q}}$, $R^{\gamma+N+\frac{\gamma+p}{p-1-q}(q+1)}$ and $R^{N-p\left(\frac{\gamma+q+1}{q-p+1}\right)}$. All the exponents are negative, and then $\int_0^R r^{\gamma+N-1}u^{q+1}dr \to 0$ when $R \to +\infty$, hence $u \equiv 0$. This concludes the proof. □

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