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A semilinear control problem involving homogenization *

Carlos Conca, Axel Osses, & Jeannine Saint Jean Paulin

Abstract

We consider a control problem involving a semilinear elliptic equation with a uniformly Lipschitz non-linearity and rapidly oscillating coefficients in a bounded domain of $\mathbb{R}^N$. The control is distributed on a compact subset interior to the domain. Given an $N-1$ dimensional hypersurface at the interior of the domain not intersecting the control zone, the trace of the solution on the curve has to be controlled. We prove that there exists a limit control as the homogenization parameter converges to zero, which results as the limit of fixed points for controllability problems. We link this limit control with the corresponding homogenized problem.

1 Introduction

Let Ω be a connected and open subset of $\mathbb{R}^N$ with smooth boundary Γ. Let $\omega \subset\subset \Omega$ be a non-empty open subset with indicatrix set 1_ω and let S be a $N-1$ dimensional manifold strictly included in Ω and not intersecting ω. Consider the following control problem. Given $\varepsilon > 0$, $\alpha > 0$ and $y_1 \in L^2(S)^N$ find a control function v^ε with support in ω such that

$$\begin{gathered} -\operatorname{div}(A^\varepsilon \nabla y^\varepsilon) + f(y^\varepsilon) = 1_\omega v^\varepsilon \quad \text{in } \Omega \\ y^\varepsilon = 0 \quad \text{on } \Gamma \end{gathered} \tag{1.1}$$

and

$$\|{y^\varepsilon}_{|S} - y_1\|_{0,S} \le \alpha, \tag{1.2}$$

where ${y^\varepsilon}_{|S}$ is the trace of y^ε on S and $\| \ \|_{0,S}$ denotes the standard L^2-norm on S. The nonlinear function f is such that

$$f \in C^0, \ f(0) = 0, \tag{1.3}$$

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and uniformly Lipschitz, that is

$$\exists \gamma > 0 \text{ such that } \forall s \in \mathbb{R} \setminus \{0\},\ 0 \leq \frac{f(s)}{s} \leq \gamma. \tag{1.4}$$

The coefficients of the symmetric matrix A^ε are real and piecewise C^1 in $\overline{\Omega}$. We assume the condition

$$\exists \alpha_m, \alpha_M > 0 \text{ such that } \forall \xi \in \mathbb{R}^N,\ |\xi| = 1,\ \alpha_m \leq \sum_{i,j=1}^{N} A^\varepsilon_{ij}(x)\xi_i\xi_j \leq \alpha_M, \tag{1.5}$$

for a.e. $x \in \Omega$. The following result can be established as in [7, 8] using a fixed point technique introduced in [2].

Theorem 1.1 *Assume that each point x_0 on S can be connected by an arc included in Ω to some point in ω without intersecting $S \setminus \{x_0\}$. Then, under the hypotheses* (1.3), (1.4) *and* (1.5), *there exists a control $v^\varepsilon \in L^2(\omega)^N$ satisfying* (1.1) *and* (1.2).

Moreover a control v^ε_* of minimal norm and solution of (1.1)-(1.2) can be constructed as follows. Using a density argument, we can assume $f \in C^1$. Define the real function

$$g(s) = \begin{cases} f(s)/s & \text{if } s \neq 0 \\ f'(0) & \text{if } s = 0. \end{cases}$$

For each $z \in L^2(\Omega)^N$ consider the following auxiliary control problem. Given $\varepsilon > 0$, $\alpha > 0$ and $y_1 \in L^2(S)^N$ find a control function v^ε supported in ω such that

$$\begin{gathered} -\operatorname{div}(A^\varepsilon \nabla y^\varepsilon) + g(z)y^\varepsilon = 1_\omega v^\varepsilon \quad \text{in } \Omega \\ y^\varepsilon = 0 \quad \text{on } \Gamma \end{gathered} \tag{1.6}$$

and

$$\left\| y^\varepsilon(z)_{|S} - y_1 \right\|_{0,S} \leq \alpha. \tag{1.7}$$

For the existence of these controls see [7]. Among the controls satisfying (1.6) and (1.7) we choose as an optimal the minimizer of the functional (see [4, 5])

$$I^\varepsilon_z(v) = \begin{cases} \frac{1}{2}\left\|v\right\|^2_{0,\omega} & \text{if (1.7) is satisfied} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.8}$$

We denote by $v^\varepsilon_*(z)$ the point of minimum value, which depends on z and ε of course. Associated to this control we have the solution of (1.6) that we denote by $y^\varepsilon_*(z)$. Now we define the mapping

$$\mathcal{F}^\varepsilon : z \in L^2(\Omega)^N \to y^\varepsilon_*(z) \in L^2(\Omega)^N. \tag{1.9}$$

We will show that it has a fixed point $\overline{z}^\varepsilon$, that is to say

$$\mathcal{F}^\varepsilon(\overline{z}^\varepsilon) = \overline{z}^\varepsilon. \tag{1.10}$$

An admissible control for the semilinear control problem (1.1) and (1.2) is simply

$$v_*^\varepsilon = v_*^\varepsilon(\overline{z}^\varepsilon). \tag{1.11}$$

Our main goal is to study the behavior of v_*^ε as $\varepsilon \to 0$.

Notation. We will denote by y^ε (or $y^\varepsilon(v^\varepsilon)$) the solution of the original problem (1.1) and by $y^\varepsilon(z)$ (or $y^\varepsilon(z, v^\varepsilon)$) the solution of the auxiliary problem (1.6).

2 Dual context

For each $z \in L^2(\Omega)^N$, $\varepsilon > 0$, $\alpha > 0$ and $y_1 \in L^2(S)^N$ the optimal control $v_*^\varepsilon(z)$ minimizing (1.8) and satisfying simultaneously (1.6) and (1.7) can be expressed in a dual context. Indeed, we have the relationship [7]

$$v_*^\varepsilon(z) = \varphi_*^\varepsilon(z)_{|\omega}, \tag{2.1}$$

where $\varphi_*^\varepsilon(z)$ is the solution of the following dual problem associated to (1.6) (δ_S is a Dirac mass concentrated on S)

$$\begin{aligned} -\operatorname{div}({}^tA^\varepsilon \nabla \varphi^\varepsilon) + g(z)\varphi^\varepsilon &= \delta_S \varphi_1 \quad \text{in } \Omega \\ \varphi^\varepsilon &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2.2}$$

for

$$\varphi_1 = \varphi_{1*}^\varepsilon(z), \tag{2.3}$$

where $\varphi_{1*}^\varepsilon(z)$ is the point of minimum in $L^2(S)^N$ of the following dual functional of (1.8)

$$J_z^\varepsilon(\varphi_1) = \frac{1}{2}\int_\omega |\varphi^\varepsilon|^2 \, dx + \alpha \|\varphi_1\|_{0,S} - \int_S y_1 \varphi_1 \, ds \tag{2.4}$$

in the sense of Fenchel-Rockafellar [1, 5]. Note that in order to evaluate this dual functional we have to solve the dual problem (2.2) for each $\varphi_1 \in L^2(S)^N$.

Notation. We will denote by $\varphi^\varepsilon(z)$ (or $\varphi^\varepsilon(z, \varphi_1)$) the solution of the auxiliary dual problem (2.2).

3 Main result

Our main result can be summarized as follows (the definition of H-convergence can be found in [6]).

Theorem 3.1 *Assume that A^ε H-converges to A^0 and that the hypotheses of Theorem 1.1 are satisfied, then up to a subsequence*

$$v_*^\varepsilon \rightharpoonup v_*^0 \text{ in } L^2(\omega)^N - \text{weakly} \quad \text{and} \quad y(v_*^\varepsilon) \rightharpoonup y_*^0 \text{ in } H_0^1(\Omega) - \text{weakly} \quad \text{as } \varepsilon \to 0,$$

where v_*^0 *has minimal norm among all controls* v *satisfying*

$$\left\| y_*^0(v)_{|S} - y^1 \right\|_{0,S} \le \alpha.$$

Moreover y_*^0 *is solution of the system*

$$\begin{aligned} -\operatorname{div}(A^0 \nabla y_*^0) + f(y_*^0) &= 1_\omega \varphi_*^0 \quad \text{in } \Omega \\ y_*^0 &= 0 \quad \text{on } \partial\Omega \\ -\operatorname{div}({}^tA^0 \nabla \varphi^0) + g(y_*^0)\varphi^0 &= \delta_S \varphi_1 \quad \text{in } \Omega \\ \varphi^0 &= 0 \quad \text{on } \partial\Omega \\ \varphi_{1*} = \operatorname{argmin} \left(\frac{1}{2} \int_\omega \left|\varphi^0\right|^2 dx + \alpha \left\|\varphi_1\right\|_{0,S} \right.&- \left. \int_S y_1 \varphi_1 \, ds \right), \end{aligned} \tag{3.1}$$

where φ_*^0 *is the solution of* (3.1c,d) *associated to* φ_{1*}. *In terms of this dual variable,*

$$v_*^0 = \varphi^0_{*|\omega}. \tag{3.2}$$

The proof of this theorem is developed in the rest of the paper and uses the following Lemma. The proof of this Lemma is similar to the one in [2] (see also [7]) taking care of the ε dependence in bounds and the regularity of A^ε.

Lemma 3.1 *Assume that the coefficients of A^ε are piecewise C^1 in $\overline{\Omega}$. Then, under the hypotheses of Theorem 1.1, we have*

$$\liminf_{\|\varphi_1\|_{0,S} \to \infty} \frac{J_z^\varepsilon(\varphi_1)}{\|\varphi_1\|_{0,S}} \ge \alpha > 0. \tag{3.3}$$

Proof. We have

$$\frac{J_z^\varepsilon(\varphi_1)}{\|\varphi_1\|_{0,S}} = \frac{1}{2} \int_\omega \frac{1}{\|\varphi_1\|_{0,S}} \left|\varphi^\varepsilon\right|^2 dx + \alpha - \int_S y_1 \frac{\varphi_1}{\|\varphi_1\|_{0,S}} \, ds.$$

Let

$$\widehat{\varphi}^\varepsilon = \frac{\varphi^\varepsilon}{\|\varphi_1\|_{0,S}} \quad \text{and} \quad \widehat{\varphi}_1 = \frac{\varphi_1^\varepsilon}{\|\varphi_1\|_{0,S}}.$$

Then

$$\frac{J_z^\varepsilon(\varphi_1)}{\|\varphi_1\|_{0,S}} = \frac{\|\varphi_1\|_{0,S}}{2} \int_\omega \left|\widehat{\varphi}^\varepsilon\right|^2 dx + \alpha - \int_S y_1 \, \widehat{\varphi}_1 \, ds. \tag{3.4}$$

We write that for a sequence $\varphi_{1,n}$ such that $\|\varphi_{1,n}\|_{0,S} \to \infty$ as $n \to \infty$. Since $\|\widehat{\varphi}_{1,n}\|_{0,S} = 1$ it is easy to see using (1.4) and (1.5) that the associated solutions of (2.2) satisfy

$$\|\widehat{\varphi}_n^\varepsilon\|_{1,\Omega} \leq C$$

where the constant C does not depend on n nor ε and only depends on α_m, γ and the norm of the trace operator from $H^1(\Omega)$ into $L^2(S)$. For a fixed ε up to a sequence (in n), we have

$$\begin{aligned} \widehat{\varphi}_{1,n} &\rightharpoonup \widetilde{\varphi}_1 \quad \text{in } L^2(S) - \text{weakly} \\ \widehat{\varphi}_n^\varepsilon &\rightharpoonup \widetilde{\varphi}^\varepsilon \quad \text{in } H^1(\Omega) - \text{weakly.} \end{aligned}$$

Then

$$\liminf_{\|\varphi_1\|_{0,S}\to\infty} \frac{J_z^\varepsilon(\varphi_1)}{\|\varphi_1\|_{0,S}} = \liminf_{n\to\infty} \frac{J_z^\varepsilon(\varphi_{1,n})}{\|\varphi_{1,n}\|_{0,S}}.$$

We consider two cases. Firstly, if

$$\lim_n \int_\omega |\widehat{\varphi}_n^\varepsilon|^2 \, dx = \int_\omega |\widetilde{\varphi}^\varepsilon|^2 \, dx > 0\,,$$

then

$$\|\varphi_{1,n}\|_{0,S} \int_\omega |\widehat{\varphi}_n^\varepsilon|^2 \, dx \to +\infty$$

and since $\int_S y_1 \widehat{\varphi}_{1,n} \to \int_S y_1 \widetilde{\varphi}_1$, from (3.4) we obtain (3.3). Secondly, if

$$\lim_n \int_\omega |\widehat{\varphi}_n^\varepsilon|^2 \, dx = \int_\omega |\widetilde{\varphi}^\varepsilon|^2 \, dx = 0$$

then $\widetilde{\varphi}^\varepsilon = 0$ in ω. Next, our aim is to prove that $\widetilde{\varphi}^\varepsilon = 0$ in the whole of Ω. The fact that we have supposed the coefficients of A^ε piecewise C^1, implies that $\widetilde{\varphi}^\varepsilon = 0$ till S. Indeed, the classical Holmgren's unique continuation property [3] shows that $\widetilde{\varphi}^\varepsilon$ is zero in the regions intersecting ω where A^ε is regular and the transmission conditions allow to extend $\widetilde{\varphi}^\varepsilon$ by zero to the contiguous regions till S. This gives the desired result if S is an open curve. Conversely, if S is closed, the geometrical hypothesis on S and ω introduced in Theorem 1.1 implies that $\widetilde{\varphi}^\varepsilon$ is zero in the whole Ω. This implies that $\widetilde{\varphi}_1 = 0$ on S, therefore

$$\liminf_n \frac{J_z^\varepsilon(\varphi_{1,n})}{\|\varphi_{1,n}\|_{0,S}} \geq \alpha + \liminf_n \left(\|\varphi_{1,n}\|_{0,S} \int_\omega |\widehat{\varphi}_n^\varepsilon|^2 \, dx \right) - 0 \geq \alpha,$$

which completes the proof of the lemma.

4 Step 1. Fixed point

We will establish that the operator $\mathcal{F}^\varepsilon$ defined in (1.9) has a fixed point using Schauder's theorem. We follow the ideas in [2] and [7], taking care of the ε dependence.

Let us prove that $\mathcal{F}^\varepsilon$ is continuous and maps $L^2(\Omega)^N$ into a relatively compact subset of $L^2(\Omega)^N$. Take

$$z_n \to z_0 \quad \text{in } L^2(\Omega)^N$$

and in order to simplify notations let us set

$$\varphi_n^\varepsilon = \varphi^\varepsilon(z_n)$$

the solution of (2.2) associated to z_n and to a fixed $\varphi_1 \in L^2(S)^N$. Now, taking φ_n^ε as a function test in (2.2) the following estimate is easily obtained

$$\|\varphi_n^\varepsilon\|_{1,\Omega} \leq C\, \|\varphi_1\|_{0,S}\,, \tag{4.1}$$

where the constant C depends only on the A^ε-ellipticity constant α_m, and on trace and Poincaré constants, but is independent on ε (we also use hypothesis (1.4) about f). Thanks to (4.1) we have up to a subsequence

$$\varphi_n^\varepsilon \rightharpoonup \varphi_0^\varepsilon \quad \text{in } H_0^1(\Omega) - \text{weakly.}$$

In order to pass to the limit in a variational formulation of (2.2), note that

$$\begin{aligned} &\int_\Omega g(z_n)\varphi_n^\varepsilon \varphi\, dx - \int_\Omega g(z_0)\varphi_0^\varepsilon \varphi\, dx \\ &= \int_\Omega g(z_n)(\varphi_n^\varepsilon - \varphi_0^\varepsilon)\varphi\, dx + \int_\Omega (g(z_n) - g(z_0))\varphi_0^\varepsilon \varphi\, dx, \end{aligned}$$

but $g(z_n)$ is bounded in $L^\infty(\Omega)$ and since z_n converges to z_0 a.e. then

$$g(z_n) \rightharpoonup g(z_0) \quad \text{in } L^\infty(\Omega) - \text{weakly*}. \tag{4.2}$$

Therefore

$$\int_\Omega g(z_n)\varphi_n^\varepsilon \varphi\, dx \to \int_\Omega g(z_0)\varphi_0^\varepsilon \varphi\, dx \quad \forall \varphi \in H_0^1(\Omega).$$

Remark 4.1 Convergence (4.2) implies weak but not strong convergence in $H^{-1}(\Omega)$.

Nevertheless, a technical argument allows to obtain the strong convergence in $H^{-1}(\Omega)$. Indeed, for all $\varphi \in H_0^1(\Omega)$, we have

$$\begin{aligned} &\left| \int_\Omega g(z_n)\varphi_n^\varepsilon \varphi\, dx - g(z_0)\varphi_0^\varepsilon \varphi\, dx \right| \leq \\ &\leq \left| \int_\Omega g(z_n)(\varphi_n^\varepsilon - \varphi_0^\varepsilon)\varphi\, dx \right| + \left| \int_\Omega (g(z_n) - g(z_0))\varphi_0^\varepsilon \varphi\, dx \right|. \end{aligned}$$

On the one hand

$$\left| \int_\Omega g(z_n)(\varphi_n^\varepsilon - \varphi_0^\varepsilon)\varphi\, dx \right| \leq \|g(z_n)\|_{L^{p_1}}\, \|\varphi_n^\varepsilon - \varphi_0^\varepsilon\|_{L^{p_2}}\, \|\varphi\|_{L^{p_3}}\,.$$

Choosing $p_1 = N$, $p_2 = p_3 = \frac{2N}{N-1}$ if $N \geq 2$ otherwise $p_1 = p_3 = 4$ and $p_2 = 2$, thanks to this choice of p_2, the injection from $H^1(\Omega)$ to $L^{p_2}(\Omega)$ is compact and then

$$\|\varphi_n^\varepsilon - \varphi_0^\varepsilon\|_{L^{p_2}} \to 0 \quad \text{as} \quad n \to \infty.$$

Note that g is bounded and

$$\|g(z_n)\|_{L^{p_1}} \leq \gamma \,\mathrm{meas}(\Omega)^{1/p_1}.$$

Finally, the injection from $H_0^1(\Omega)$ to $L^{p_3}(\Omega)$ is continuous so

$$\|\varphi\|_{L^{p_3}} \leq \|i\|_{\mathcal{L}(H_0^1(\Omega);L^{p_3}(\Omega))} \|\varphi\|_{1,\Omega}\,. \tag{4.3}$$

On the other hand

$$\Big\| \int_\Omega (g(z_n) - g(z_0))\varphi_0^\varepsilon \varphi \, dx \Big\| \leq \|g(z_n) - g(z_0)\|_{L^{q_1}} \|\varphi_0^\varepsilon\|_{L^{q_2}} \|\varphi\|_{L^{q_3}}$$

with $q_1 = \frac{N}{2}$, $q_2 = q_3 = \frac{2N}{N-2}$ for $N \geq 3$, otherwise $q_1 = 2$, $q_2 = q_3 = 4$. Thanks to this choice the injection from $H^1(\Omega)$ into $L^{q_3}(\Omega)$ is continuous and a bound can be obtained as in (4.3). In virtue of dominated convergence theorem and bounds on $g(z_n)$ we have

$$\|g(z_n) - g(z_0)\|_{L^{q_1}} \to 0 \quad \text{as} \quad n \to \infty.$$

From the above convergences we see that

$$g(z_n)\varphi_n^\varepsilon \to g(z_0)\varphi_0^\varepsilon \quad \text{in } H_0^1(\Omega) - \text{strongly} \quad \text{as} \quad n \to \infty.$$

Let us continue with our problem. Multiplying (2.2) by $\phi \in H_0^1(\Omega)$ and integrating by parts we obtain

$$\int_\Omega A^\varepsilon \nabla \varphi_n^\varepsilon \cdot \nabla \phi \, dx + \int_\Omega g(z_n)\varphi_n^\varepsilon \phi \, dx = \int_S \varphi_1 \phi \, d\sigma,$$

for a fixed ε and we take $n \to \infty$ to obtain

$$\int_\Omega A^\varepsilon \nabla \varphi_0^\varepsilon \cdot \nabla \phi \, dx + \int_\Omega g(z_0)\varphi_0^\varepsilon \phi \, dx = \int_S \varphi_1 \phi \, d\sigma,$$

and this shows that $\varphi_0^\varepsilon = \varphi^\varepsilon(z_0)$. Let us now show that

$$\varphi_n^\varepsilon \to \varphi_0^\varepsilon \quad \text{in } H_0^1(\Omega) - \text{strongly}. \tag{4.4}$$

Take φ_n^ε as a test function in the problem

$$- \mathrm{div}({}^t A^\varepsilon \nabla \varphi_n^\varepsilon) + g(z_n)\varphi_n^\varepsilon = \delta_S \varphi_1 \quad \text{in } \Omega$$
$$\varphi_n^\varepsilon = 0 \quad \text{on } \Gamma\,.$$

Passing to the limit, we obtain

$$\lim_{n\to\infty} \int_\Omega {}^t A^\varepsilon \nabla \varphi_n^\varepsilon \cdot \nabla \varphi_n^\varepsilon \, dx = \int_S \varphi_1 \varphi_0^\varepsilon \, d\sigma - \int_\Omega g(z_0)\varphi_0^\varepsilon \varphi_0^\varepsilon \, dx\,.$$

Now, taking φ_0^ε as a test function in

$$-\operatorname{div}({}^tA^\varepsilon \nabla \varphi_0^\varepsilon) + g(z_0)\varphi_0^\varepsilon = \delta_S \varphi_1 \quad \text{in } \Omega$$
$$\varphi_0^\varepsilon = 0 \quad \text{on } \Gamma\,,$$

we obtain

$$\int_\Omega A^\varepsilon \nabla \varphi_0^\varepsilon \cdot \nabla \varphi_0^\varepsilon \, dx = \int_S \varphi_1 \varphi_0^\varepsilon \, d\sigma - \int_\Omega g(z_0)\varphi_0^\varepsilon \varphi_0^\varepsilon \, dx\,.$$

By comparison

$$\lim_{n\to\infty} \int_\Omega {}^tA^\varepsilon \nabla \varphi_n^\varepsilon \cdot \nabla \varphi_n^\varepsilon \, dx = \int_\Omega {}^tA^\varepsilon \nabla \varphi_0^\varepsilon \cdot \nabla \varphi_0^\varepsilon \, dx.$$

We conclude (4.4) since $\left(\int_\Omega {}^tA^\varepsilon \nabla v \cdot \nabla v \, dx\right)^{1/2}$ is equivalent to the standard norm in $H_0^1(\Omega)$.

By a method analogous to the one that yields (4.4) from (4.1), we show that

$$\|\varphi_{1*}^\varepsilon\|_{0,S} \le C \tag{4.5}$$

with C independent of n and of ε, that is

$$\varphi_{1*}^\varepsilon(z_n) \rightharpoonup \xi^\varepsilon \quad \text{in } L^2(S) - \text{weakly}$$

and

$$\varphi^\varepsilon(z_n, \varphi_{1*}^\varepsilon(z_n)) \to \varphi^\varepsilon(z_0, \xi^\varepsilon) \quad \text{in } H_0^1(\Omega) - \text{strongly.} \tag{4.6}$$

Let us show by contradiction that (4.5) holds. Otherwise, there exists a sequence $\{\varphi_{1*}^\varepsilon(z_n)\}_{n\ge 0}$ such that

$$\|\varphi_{1*}^\varepsilon(z_n)\|_{0,S} \to +\infty \quad \text{as} \quad n \to \infty, \tag{4.7}$$

but for each z_n, the function $\varphi_{1*}^\varepsilon(z_n)$ minimizes $J_{z_n}^\varepsilon$ and consequently

$$J_{z_n}^\varepsilon(\varphi_{1*}^\varepsilon(z_n)) \le J_{z_n}^\varepsilon(\varphi_1) \quad \forall \varphi_1 \in L^2(S). \tag{4.8}$$

At the same time, we see that

$$J_{z_n}^\varepsilon(\varphi_1) = \frac{1}{2}\int_\omega |\varphi^\varepsilon(z_n)|^2 \, dx + \alpha \|\varphi_1\|_{0,S} - \int_S y_1 \varphi_1 \, d\sigma$$

converges as $n \to \infty$ to

$$J_{z_0}^\varepsilon(\varphi_1) = \frac{1}{2}\int_\omega |\varphi^\varepsilon(z_0)|^2 \, dx + \alpha \|\varphi_1\|_{0,S} - \int_S y_1 \varphi_1 \, d\sigma.$$

Therefore from (4.8), for each fixed φ_1

$$J_{z_n}^\varepsilon(\varphi_{1*}^\varepsilon(z_n)) \le C$$

with C independent of n (and of ε). This last upper bound contradicts (4.7) since

$$\liminf_{\|\varphi_{1*}^{\varepsilon}(z_n)\|_{0,S}\to\infty} \frac{J_{z_n}^{\varepsilon}(\varphi_{1*}^{\varepsilon}(z_n))}{\|\varphi_{1*}^{\varepsilon}(z_n)\|_{0,S}} \geq \alpha > 0. \tag{4.9}$$

Proof of (4.9) is similar to the proof of Lemma 3.1 since

$$\|\varphi_*^{\varepsilon}(z_n, \varphi_{1*}^{\varepsilon}(z_n))\|_{1,\Omega} \leq C \, \|\varphi_{1*}^{\varepsilon}(z_n)\|_{0,S}$$

with a constant C independent of n (and of ε). Since (4.7) does not hold, we have up to a subsequence

$$\varphi_{1*}^{\varepsilon}(z_n) \rightharpoonup \xi^{\varepsilon} \quad \text{in } L^2(S) - \text{weakly} \quad \text{as} \quad n \to \infty. \tag{4.10}$$

It remains to identify the limit. Let us show that ξ^{ε} minimizes $J_{z_0}^{\varepsilon}$, that is to say

$$J_{z_0}^{\varepsilon}(\xi^{\varepsilon}) \leq J_{z_0}^{\varepsilon}(\varphi_1) \quad \forall \varphi_1 \in L^2(S)^N. \tag{4.11}$$

First, note that $\varphi_{1*}^{\varepsilon}(z_n)$ is optimal for $J_{z_n}^{\varepsilon}$, that is

$$J_{z_n}^{\varepsilon}(\varphi_{1*}^{\varepsilon}(z_n)) \leq J_{z_n}^{\varepsilon}(\varphi_1) \quad \forall \varphi_1 \in L^2(S)^N$$

hence

$$\liminf_n J_{z_n}^{\varepsilon}(\varphi_{1*}^{\varepsilon}(z_n)) \leq \liminf_n J_{z_n}^{\varepsilon}(\varphi_1) = J_{z_0}^{\varepsilon}(\varphi_1) \quad \forall \varphi_1 \in L^2(S)^N.$$

In order to get (4.11) it remains to proof that

$$J_{z_0}^{\varepsilon}(\xi^{\varepsilon}) \leq \liminf_n J_{z_n}^{\varepsilon}(\varphi_{1*}^{\varepsilon}(z_n)). \tag{4.12}$$

Let us recall that

$$J_{z_n}^{\varepsilon}(\varphi_{1*}^{\varepsilon}(z_n)) = \frac{1}{2}\int_{\omega} |\varphi^{\varepsilon}(z_n, \varphi_{1*}^{\varepsilon}(z_n))|^2 \, dx + \alpha \, \|\varphi_{1*}^{\varepsilon}(z_n)\|_{0,S} - \int_S y_1 \varphi_{1*}^{\varepsilon}(z_n) \, d\sigma,$$

so from (4.10) we have

$$\liminf_n \alpha \, \|\varphi_{1*}^{\varepsilon}(z_n)\|_{0,S} - \int_S y_1 \varphi_{1*}^{\varepsilon}(z_n) \, d\sigma \geq \liminf_n \alpha \, \|\xi^{\varepsilon}\|_{0,S} - \int_S y_1 \xi^{\varepsilon} \, d\sigma$$

and from (4.6)

$$\liminf_n \int_{\omega} |\varphi^{\varepsilon}(z_n, \varphi_{1*}^{\varepsilon}(z_n))|^2 \, dx \geq \int_{\omega} |\varphi^{\varepsilon}(z_0, \xi^{\varepsilon})|^2 \, dx.$$

In this way, we obtain (4.12) and consequently (4.11), in other words

$$\xi^{\varepsilon} = \varphi_{1*}^{\varepsilon}(z_0).$$

With this relation, convergence in (4.6) becomes

$$\varphi^\varepsilon(z_n, \varphi^\varepsilon_{1*}(z_n)) \to \varphi^\varepsilon(z_0, \varphi^\varepsilon_{1*}(z_0)) \quad \text{in } H^1_0(\Omega) - \text{strongly.} \tag{4.13}$$

The rest of the proof is straightforward since

$$v^\varepsilon_*(z_n) = \varphi^\varepsilon(z_n, \varphi^\varepsilon_{1*}(z_n))_{|\omega}$$
$$v^\varepsilon_*(z_0) = \varphi^\varepsilon(z_0, \varphi^\varepsilon_{1*}(z_0))_{|\omega}$$

and it is clear from (4.13) that $v^\varepsilon_*(z_n) \to v^\varepsilon_*(z_0)$ in $H^1(\omega)$ - strongly. An analogous proof as for the adjoint problem shows that

$$y^\varepsilon(z_n, v^\varepsilon_*(z_n)) \to y^\varepsilon(z_0, v^\varepsilon_*(z_0)) \quad \text{in } H^1(\Omega) - \text{strongly,}$$

proving the continuity of the map $\mathcal{F}^\varepsilon$ for a fixed $\varepsilon > 0$.

Next we show that $\mathcal{F}^\varepsilon$ is compact (uniformly in ε). Let $z \in L^2(\Omega)^N$ since

$$\|g(z)\|_{\infty,\Omega} \leq \gamma$$

then

$$\|\varphi^\varepsilon(z, \varphi_1)\|_{1,\Omega} \leq C\, \|\varphi_1\|_{0,S}$$

with C independent of z (and of ε). This implies that $|J^\varepsilon_z(\varphi_1)| \leq C(\varphi_1)$, therefore

$$|J^\varepsilon_z(\varphi^\varepsilon_{1*}(z))| \leq C(\varphi_1).$$

Using again the coercitivity of J^ε_z we see that $\|\varphi^\varepsilon_{1*}\|_{0,S}$ is bounded independently of z (and ε). Then $\|\varphi^\varepsilon(z, \varphi^\varepsilon_{1*})\|_{1,\Omega}$ is bounded independently of z (and ε) and consequently the same is true for $v^\varepsilon_*(z) = \varphi^\varepsilon(z, \varphi^\varepsilon_{1*})_{|\omega}$ and $y^\varepsilon(z, v^\varepsilon_*(z))$. $\diamondsuit$

5 Step 2. H-convergence

We first consider the H-convergence in the original problem (1.1) with fixed control $v \in L^2(\omega)^N$, that is the H-convergence in the problem

$$\begin{aligned} -\operatorname{div}(A^\varepsilon \nabla y^\varepsilon) + f(y^\varepsilon) = 1_\omega v \quad \text{in } \Omega \\ y^\varepsilon = 0 \quad \text{on } \Gamma, \end{aligned} \tag{5.1}$$

under the hypotheses (1.3) and (1.4) on f. To have *a priori* estimates, we multiply (5.1) by y^ε and we integrate in Ω to obtain

$$\int_\Omega A^\varepsilon \nabla y^\varepsilon \cdot \nabla y^\varepsilon\, dx + \int_\Omega f(y^\varepsilon)\, y^\varepsilon\, dx = \int_\omega v\, y^\varepsilon\, dx,$$

but from (1.3)

$$f(y^\varepsilon) y^\varepsilon = \frac{f(y^\varepsilon)}{y^\varepsilon}\, |y^\varepsilon|^2 \geq 0$$

and it is true also in the case $y^\varepsilon(x) = 0$. Hence $\|y^\varepsilon\|_{1,\Omega} \le C\,\|v\|_{0,\omega}$ with C independent of ε. Up to a subsequence

$$y^\varepsilon \rightharpoonup y^0 \quad \text{in } H_0^1(\Omega) - \text{weakly.}$$

Now let us see which is the limit of $f(y^\varepsilon)$. Take $\varphi \in H_0^1(\Omega)$, then

$$\begin{aligned} \left|\int_\Omega (f(y^\varepsilon)\varphi - f(y^0)\varphi)\,dx\right| &= \left|\int_\Omega (g(y^\varepsilon)y^\varepsilon\varphi - g(y^0)y^0\varphi)\,dx\right| \le \\ \le \left|\int_\Omega (g(y^\varepsilon)(y^\varepsilon - y^0)\varphi)\,dx\right| &+ \left|\int_\Omega ((g(y^\varepsilon) - g(y^0))y^0\varphi)\,dx\right|. \end{aligned} \tag{5.2}$$

Starting from this and reasoning as in Remark 4.1 we can show that

$$f(y^\varepsilon) = g(y^\varepsilon)y^\varepsilon \to g(y^0)y^0 = f(y^0) \quad \text{in } H^{-1}(\Omega) - \text{strongly.}$$

Thanks to the H-convergence definition, we immediately deduce that

$$\begin{gathered} -\operatorname{div}(A^0\nabla y^0) + f(y^0) = 1_\omega v \quad \text{in } \Omega \\ y^0 = 0 \quad \text{on } \Gamma, \end{gathered} \tag{5.3}$$

where A^0 is the H-limit of A^ε (and $A^\varepsilon\nabla y^\varepsilon \rightharpoonup A^0\nabla y^0$ in $L^2(\Omega)^N$-weakly).

Consider now the H-convergence with the optimal control $v_*^\varepsilon = v_*^\varepsilon(\overline{z}^\varepsilon)$ satisfying (1.6)-(1.11) where $\overline{z}^\varepsilon$ is the fixed point of $\mathcal{F}^\varepsilon$. We have already seen at the end of the previous section that $\|v_*^\varepsilon(z)\|_{0,\omega}$ is bounded independently of $z \in L^2(\Omega)$ and ε. In particular

$$\|v_*^\varepsilon\|_{0,\omega} \le C$$

with C independent of ε. Hence there exists $v^0 \in L^2(\omega)^N$ such that

$$\begin{gathered} v_*^\varepsilon \rightharpoonup v^0 \quad \text{in } L^2(\omega) - \text{weakly} \\ 1_\omega v_*^\varepsilon \to 1_\omega v^0 \quad \text{in } H^{-1}(\Omega) - \text{strongly.} \end{gathered} \tag{5.4}$$

The same proof as in the case of a fixed v shows that the solution y_*^ε of

$$\begin{gathered} -\operatorname{div}(A^\varepsilon\nabla y_*^\varepsilon) + f(y_*^\varepsilon) = 1_\omega v_*^\varepsilon \quad \text{in } \Omega \\ y_*^\varepsilon = 0 \quad \text{on } \Gamma \end{gathered}$$

converges weakly to y_0, i.e.,

$$y_*^\varepsilon \rightharpoonup y^0 \quad \text{in } H_0^1(\Omega) - \text{weakly} \tag{5.5}$$

where y^0 is a solution of

$$\begin{gathered} -\operatorname{div}(A^0\nabla y^0) + f(y^0) = 1_\omega v^0 \quad \text{in } \Omega \\ y^0 = 0 \quad \text{on } \Gamma, \end{gathered} \tag{5.6}$$

and A^0 is the H-limit of A^ε.

Notation. In the following sections v^0 stands for the L^2-weak limit of the control in (5.4) and y^0 (or $y^0(v^0)$) stands for the weak H^1-limit of the solution in (5.5), which is solution of the limit problem (5.6).

6 Step 3. Limit of optimal controls

The objective is now to identify v^0. Is it an optimal solution? First at all, note that

$$\left\| y_*^\varepsilon(v_*^\varepsilon)_{|S} - y_1 \right\|_{0,S} \leq \alpha\,.$$

Since weak convergence in (5.5) implies

$$y^\varepsilon_{*|S} \rightharpoonup y^0_{|S} \quad \text{in } H^{1/2}(S)^N - \text{weakly } (L^2(S)^N\text{-strongly}),$$

we conclude that v^0 satisfies the approximate controllability inequality

$$\left\| y^0(v^0)_{|S} - y_1 \right\|_{0,S} \leq \alpha.$$

Also from (5.5), since $\overline{z}^\varepsilon$ is a fixed point (see (1.9), (1.10)), we have

$$\overline{z}^\varepsilon = y_*^\varepsilon \rightharpoonup y^0 \quad \text{in } H_0^1(\Omega) - \text{weakly.}$$

Let v_*^0 be the minimizer in $L^2(\omega)^N$ of the functional

$$I(v) = \begin{cases} \frac{1}{2}\left\|v\right\|_{0,\omega}^2 & \text{if } \left\| y^0(v)_{|S} - y^1 \right\|_{0,S} \leq \alpha \\ +\infty & \text{otherwise,} \end{cases} \tag{6.1}$$

where for each $v \in L^2(S)^N$, we denote $y^0(v)$ the solution of

$$\begin{gathered} -\operatorname{div}(A^0\nabla y^0) + f(y^0) = 1_\omega v \quad \text{in } \Omega \\ y^0 = 0 \quad \text{on } \Gamma. \end{gathered} \tag{6.2}$$

We will establish that

$$v^0 = v_*^0. \tag{6.3}$$

In virtue of Fenchel-Rockafellar duality, the minimum v_*^0 can be characterized as follows. Let us consider the dual problem associated to (6.2), that is, for each $\varphi_1 \in L^2(S)$, find $\varphi^0 \in L^2(\Omega)^N$ such that

$$\begin{gathered} -\operatorname{div}({}^tA^0\nabla\varphi^0) + g(y^0)\varphi^0 = \delta_S\varphi_1 \quad \text{in } \Omega \\ \varphi^0 = 0 \quad \text{on } \Gamma \end{gathered} \tag{6.4}$$

and let us define the respective dual functional of (6.1) as

$$J^0(\varphi_1) = \frac{1}{2}\int_\omega \left|\varphi^0\right|^2\, dx + \alpha\left\|\varphi_1\right\|_{0,S} - \int_S y_1\varphi_1\, ds. \tag{6.5}$$

If φ_{1*}^0 is the point of minimum of J^0 in $L^2(S)^N$, and if φ_*^0 is the solution of (6.4) associated to it, then the duality theory gives the relationship

$$v_*^0 = \varphi_{*|_\omega}^0. \tag{6.6}$$

We will pass to the limit in (2.1), (2.2), (2.3), (2.4) with $z = \overline{z}^\varepsilon$ as $\varepsilon \to 0$. An argument similar to the one used for obtaining (5.3) shows that if we pass to the limit in (2.2) with $z = \overline{z}^\varepsilon$ as $\varepsilon \to 0$ then

$$\varphi^\varepsilon(\overline{z}^\varepsilon) \rightharpoonup \varphi^0 \quad \text{in } H_0^1(\Omega) - \text{weakly}$$

where φ^0 is the solution of

$$\begin{aligned} -\operatorname{div}({}^tA^0\nabla\varphi^0) + g(y^0)\varphi^0 = \delta_S\varphi_1 \quad \text{in } \Omega \\ \varphi^0 = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6.7}$$

Taking the limit in (2.4) with φ_1 fixed,

$$J_{\overline{z}^\varepsilon}^\varepsilon(\varphi_1) \to J^0(\varphi_1).$$

Let us consider now the sequence $\varphi_{1*}^\varepsilon(\overline{z}^\varepsilon)$. From the uniform coecivity of $J_{\overline{z}^\varepsilon}^\varepsilon$ with respect to ε (an analogous to Lemma 3.1 with $z = \overline{z}^\varepsilon$), we deduce that $\varphi_{1*}^\varepsilon(\overline{z}^\varepsilon)$ is bounded in $L^2(S)^N$ then, up to a subsequence

$$\varphi_{1*}^\varepsilon(\overline{z}^\varepsilon) \rightharpoonup \varphi_1^0 \quad \text{in } L^2(S)^N - \text{weakly}.$$

Then (see the proof of (4.12)) for each $\varphi_1 \in L^2(S)$

$$J^0(\varphi_1^0) \le \liminf_\varepsilon J_{\overline{z}^\varepsilon}^\varepsilon(\varphi_{1*}^\varepsilon(\overline{z}^\varepsilon)) \le \liminf_\varepsilon J_{\overline{z}^\varepsilon}^\varepsilon(\varphi_1) = J^0(\varphi_1).$$

Therefore $\varphi_1^0 = \varphi_{1*}^0$ and consequently $\varphi^0 = \varphi_*^0$. Finally, passing to the limit as $\varepsilon \to 0$ in (2.1), we obtain

$$v_*^\varepsilon = \varphi_{*|_\omega}^\varepsilon \to \varphi_{*|_\omega}^0 = v_*^0 \quad \text{in } L^2(\omega) - \text{strongly}.$$

This, together with (5.4), implies (6.3).

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Carlos Conca & Axel Osses
Departamento de Ingeniería Matemática,
CMM, UMR 2071 CNRS-Uchile,
Universidad de Chile, Casilla 170/3 - Correo 3,
Santiago, Chile
e-mail: cconca@dim.uchile.cl, axosses@dim.uchile.cl

Jeannine Saint Jean Paulin
Département de Mathématiques, Université de Metz,
Ile du Saulcy, 57045 Metz Cedex 01, France
e-mail: sjpaulin@poncelet.sciences.univ-metz.fr