Interchangeability of Relevant Cycles in Graphs: Erratum

Petra M. Gleiss[a], Josef Leydold[b,*] and Peter F. Stadler[a,c]

[a]Institute for Theoretical Chemistry and Molecular Structural Biology,
University of Vienna, Währingerstrasse 17, A-1090 Vienna, Austria
Phone: **43 1 4277-52737 Fax: **43 1 4277-52793
E-Mail: {pmg,studla}@tbi.univie.ac.at
URL: http://www.tbi.univie.ac.at/~{pmg,studla}

[b]Dept. for Applied Statistics and Data Processing
University of Economics and Business Administration
Augasse 2-6, A-1090 Wien, Austria
Phone: **43 1 31336-4695 Fax: **43 1 31336-738
E-Mail: Josef.Leydold@statistik.wu-wien.ac.at
URL: http://statistik.wu-wien.ac.at/staff/leydold
*Address for correspondence

[c]The Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501, USA
Phone: (505) 984 8800 Fax: (505) 982 0565
E-Mail: stadler@santafe.edu
URL: http://www.tbi.univie.ac.at/~studla

Abstract

We provide a correction for the incomplete proof of Lemma 7 of *Interchangeability of Relevant Cycles in Graphs*, *Elec. J. Comb.* **7** (2000), #R16.

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We consider unweighted simple undirect connected graphs G. A cycle C in G is identified with its edge set and considered as an element the cycle vector space defined over GF(2). We write $X \oplus Y$ for the symmetric difference of the edge sets X and Y. The length $|C|$ of a cycle is the number of its edges. A cycle is *relevant* if it cannot be represented as a $\oplus$-sum of strictly shorter cycles. The set of relevant cycles is denoted by $\mathcal{R}$. For a given length l define $\mathcal{R}_< = \{C \in \mathcal{R} \big| |C| < l\}$ and $\mathcal{R}_= = \{C \in \mathcal{R} \big| |C| = l\}$.

For further definitions and references we refer to the main text *Elec. J. Comb.* **7** (2000), #R16.

For the purpose of this erratum it is convenient to reformulate Definition 6 in the form:

Definition 1. *Two relevant cycles $C', C'' \in \mathcal{R}$ are* interchangeable, *$C' \leftrightarrow C''$, if (i) $|C'| = |C''|$ and (ii) there exists a minimal linearly dependent set of relevant cycles that contains C' and C'' and with each of its elements not longer than C'.*

We claimed that $\leftrightarrow$ is an equivalence relation. The proof of this statement in the main text, however, is incomplete.

Let us fix a length l. Then two cycles C_{j_1} and C_{j_2} of length l are interchangeable if and only if the equation

$$x_1 C_1 \oplus \cdots \oplus x_M C_M \oplus \cdots \oplus x_{j_1} C_{j_1} \oplus \cdots \oplus x_{j_2} C_{j_2} \oplus \cdots \oplus x_N C_N = 0 \qquad (1)$$

has a solution with $x_{j_1} = x_{j_2} = 1$ and with the following properties:

(1) $\{C_1, \ldots C_M\}$ is the intersection of $\mathcal{R}_<$ with an arbitrary but fixed minimal cycle basis, and $\{C_{M+1}, \ldots, C_N\} = \mathcal{R}_=$. The fact that instead of $\mathcal{R}_<$ we can restrict ourselves to a subset of a minimal cycle basis follows from the *matroid property.*

(2) The solution is minimal in the following sense: if we take any strict subset of the coefficients with $x_k = 1$ then there is no solution with exactly these coefficients being nonzero. This is equivalent to the fact that we have a minimally linearly dependent set of cycles.

Let $A = (C_1, \ldots, C_M, C_{M+1}, \ldots, C_N)$ be the $(|E| \times N)$-matrix with the cycles C_k represented as column vectors. A can be transformed into the reduced row echelon form $\tilde{A}$ by Gauß-Jordan elimination. Then exactly the first $R = \operatorname{rank}(A)$ rows of $\tilde{A}$ are nonzero. Notice that the upper-left $M \times M$-matrix of $\tilde{A}$ is the identity matrix since $\{C_1, \ldots, C_M\}$ is a subset of a cycle basis by construction, see Fig. 1.

We introduce a coloring of the columns $M+1, \ldots, N$ of $\tilde{A}$:
(1) Two columns j' and j'' ($> M$) have the same color if there exists a row i such that $\tilde{A}_{ij'} = \tilde{A}_{ij''} = 1$.
(2) Use as many colors as possible.

Definition 2. *Two relevant cycles $C', C'' \in \mathcal{R}$ are* color-related, *if (i) $|C'| = |C''|$ and (ii) they have the same color (as described above).*

It is clear from the definition that color-related is an equivalence relation. The definition of color-relatedness, however, depends explicitly on a prescribed ordering of the cycles $C_{M+1}, \ldots, C_N$. We proceed by showing that color-relatedness is in fact independent of this ordering and that it is equivalent to interchangeability.

Lemma 3. *If two cycles C_{j_1} and C_{j_2} are interchangeable w.r.t. any ordering of the cycles then C_{j_1} and C_{j_2} are color-related.*

Proof. Fix an arbitrary ordering of the cycles and assume that two interchangeable cycles C_{j_1} and C_{j_2} are not color-related. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $\{C_i \colon i \in \mathcal{J}_1\}$ and $\{C_i \colon i \in \mathcal{J}_2\}$ are the respective color-equivalence classes of C_{j_1} and C_{j_2}. Then there

$$
\begin{array}{cccccc|cccccccc|cccccc}
C_1 & & & & & C_M & \multicolumn{8}{c|}{C_{j_1}} & \multicolumn{6}{c}{C_{j_2}} \\
1&0&0&0&0&0&0&0&0&1&1&0&0&0& \multicolumn{6}{c}{} \\
0&1&0&0&0&0&0&0&0&0&1&0&1&0& \multicolumn{6}{c}{0} \\
0&0&1&0&0&0&0&0&0&1&1&0&1&0& \multicolumn{6}{c}{} \\
0&0&0&1&0&0& \multicolumn{8}{c|}{} &0&0&0&1&0&1 \\
0&0&0&0&1&0& \multicolumn{8}{c|}{0} &0&0&0&1&1&1 \\
0&0&0&0&0&1& \multicolumn{8}{c|}{} &0&0&0&1&0&0 \\
\hline
\multicolumn{6}{c|}{} &1&0&0&0&1&0&1&0& \multicolumn{6}{c}{} \\
\multicolumn{6}{c|}{} &0&1&0&1&1&0&0&0& \multicolumn{6}{c}{} \\
\multicolumn{6}{c|}{0} &0&0&1&1&0&0&1&0& \multicolumn{6}{c}{0} \\
\multicolumn{6}{c|}{} &0&0&0&0&0&1&1&0& \multicolumn{6}{c}{} \\
\multicolumn{6}{c|}{} &0&0&0&0&0&0&0&1& \multicolumn{6}{c}{} \\
\hline
\multicolumn{6}{c|}{} & \multicolumn{8}{c|}{} &1&0&0&1&1&0 \\
\multicolumn{6}{c|}{0} & \multicolumn{8}{c|}{0} &0&1&0&0&1&1 \\
\multicolumn{6}{c|}{} & \multicolumn{8}{c|}{} &0&0&1&1&1&1
\end{array}
$$

Figure 1. Example of a reduced echelon form $\tilde{A}$ for the special case where the cycles of each color-equivalence class are consecutive in the chosen ordering. For the general case the situation is analogous with columns and rows permutated.

is no row r in $\tilde{A}$ with two coefficients $\tilde{A}_{rk_1} = \tilde{A}_{rk_2} = 1$ such that $k_1 \in \mathcal{J}_1$ and $k_2 \in \mathcal{J}_2$, see Fig. 1.

Now suppose C_{j_1} and C_{j_2} are interchangeable. Then there exists a minimal solution of equ. (1) with $x_{j_1} = x_{j_2} = 1$. Set $x_k = 0$ for all $k \in \mathcal{J}_2$ in this solution (this includes $x_{j_2} = 0$). If the resulting vector (x'_i) is a solution of equ. (1), the original solution was not minimal, contradicting the assumption that C_{j_1} and C_{j_2} were interchangeable.

Hence we assume that the resulting vector (x'_i) may not be a solution any more. This happens when there is a row r with an odd number of coefficients $\tilde{A}_{rn}$ for which $x'_n \tilde{A}_{rn} = 1$. In this case, however, we must have $r \leq M$ and $x'_r \tilde{A}_{rr} = 1$. Hence we can set $x'_r = 0$, since the upper-left $M \times M$-matrix is the identity matrix. Since this holds for every such row r we end up with a new solution (x''_i) of equ. (1) with $x''_{j_1} = 1$ and $x''_{j_2} = 0$. Again the original solution (x_i) was not minimal, a contradiction to our assumption. $\square$

Lemma 4. *If two cycles C_{j_1} and C_{j_2} are color-related w.r.t. a given ordering of the cycles, then C_{j_1} and C_{j_2} are interchangeable.*

Proof. Assume C_{j_1} and C_{j_2} are color-related and let $\mathcal{J}$ denote the set of indices of the cycles C_i in the color-equivalence class of C_{j_1}. Then there exists a sequence $\sigma = \{j_1 = k_0, k_1, \ldots, k_m = j_2\} \subseteq \mathcal{J}$, such that for each $i = 0, \ldots, m-1$ there exists a row r with $\tilde{A}_{r,k_i} = \tilde{A}_{r,k_{i+1}} = 1$ (otherwise the cycles C_{k_i} would not be color-related). Assume that our sequence is minimal (in the sense that no other sequence connecting j_1 and j_2 consists of fewer elements).

Set all $x_{k_i} = 1$ for $k_i \in \sigma$ and $x_p = 0$ for all other $p > M$. Then for each row $r > M$ there are only two (or zero) columns with $x_k \tilde{A}_{rk} = 1$ (i.e., $\neq 0$). If there were more such columns, say at k_1, k_3, k_9, then σ would not be minimal, since we could then remove $k_2, \ldots, k_8$ from σ. By the same argument there are at most two columns with $x_k \tilde{A}_{rk} = 1$ for $r \leq M$. For the rows $r \leq M$ with only one such column we set $x_r = 1$ and $x_r = 0$ otherwise. Thus (x_i) is a solution of equ. (1). Moreover (x_i) has the property that for each row r there are either 2 or 0 columns with $x_k \tilde{A}_{rk} = 1$ and for each column k there are either 2 or 0 rows with $x_k \tilde{A}_{rk} = 1$.

Now we show that this solution is minimal. If we change one of these x_k from 1 to 0 then we obtain a row r with an odd number of coefficients with $x_k \tilde{A}_{rk} = 1$, i.e., we do not have a solution any more. Thus, if we want to construct a new solution (x'_i) of equ. (1) by changing x_j from 1 to 0 we have to change the other x_i in row r with $x_i \tilde{A}_{ri} = 1$ from 1 to 0 as well. If we still find a row r' with an odd number of coefficients with $x_n \tilde{A}_{r'n} = 1$ we have to repeat this procedure. As a consequence, if $\tilde{A}_{r,k_i} = \tilde{A}_{r,k_{i+1}} = 1$ and $x_{k_i} = x_{k_{i+1}} = 1$ then any modified solution (x'_i) must satisfy $x'_{k_i} = x'_{k_{i+1}}$ and therefore all coordinates x_k for $k \in \sigma$ must be equal, i.e., either $(x'_i) = (x_i)$ or (x'_i) is the trivial solution. Hence the original solution was minimal. $\square$

It follows that color-relatedness is independent of the ordering the cycles and the particular reduced echelon form $\tilde{A}$ that we have obtained by Gauß-Jordan elimination. Furthermore, color-relatedness and interchangeability are equivalent. Hence we have

Corollary 5. *Interchangeability is an equivalence relation on $\mathcal{R}$.*

Remark. Lemmata 3 and 4 replace the longer proof of lemma 22 in the main text.

Remark. The proofs of lemmata 3 and 4 explicitly uses the properties of a vector space over GF(2).