New Lower Bounds for Some Multicolored Ramsey Numbers

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Abstract

In this article we use two different methods to find new lower bounds for some multicolored Ramsey numbers. In the first part we use the finite field method used by Greenwood and Gleason [GG] to show that $R(5,5,5) \geq 242$ and $R(6,6,6) \geq 692$. In the second part we extend Fan Chung's result in [C] to show that,

$$R(3,3,3,k_1,k_2,\ldots,k_r) \geq 3R(3,3,k_1,k_2,\ldots,k_r) + R(k_1,k_2,\ldots,k_r) - 3$$

holds for any natural number r and for any $k_i \geq 3$, $i = 1,2,\ldots r$. This general result, along with known results, imply the following nontrivial bounds: $R(3,3,3,4) \geq 91$, $R(3,3,3,5) \geq 137$, $R(3,3,3,6) \geq 165$, $R(3,3,3,7) \geq 220$, $R(3,3,3,9) \geq 336$, and $R(3,3,3,11) \geq 422$.

Introduction

This paper is presented in two part which can be read independently of each other. Part one uses finite fields, and Part two extends an argument by Fan Chung.

Part one of this article is accompanied by the Maple package `RES`, available for download at the author's website.

Recall that $N = R(k_1,k_2,\ldots,k_r)$ is the minimal integer with the following property:

> **Ramsey Property**: *If we r-color the edges of the complete graph on N vertices, then there exists j, $1 \leq j \leq r$, such that a monochromatic j-colored complete graph on k_j vertices is a subgraph of the r-colored K_N.*

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Part One: The Finite Field Method

In this first part we add two more lower bounds to Radziszowski's Dynamic Survey [R] on the subject. We show, by using the finite field technique in [GG], that $R(5,5,5) \geq 242$ and $R(6,6,6) \geq 692$. The previous best lower bound for $R(5,5,5)$ was 169 given by Song [S], who more generally shows that $R(\underbrace{5,5,\ldots,5}_{r\ times}) \geq 4(6.48)^{r-1}+1$ holds for all r. For $R(6,6,6)$ there was no established nontrivial lower bound.

Consider the number $R(5,5,5)$. To find a lower bound, L, we are searching for a three coloring of K_L which avoids a monochromatic K_5. We use an argument of Greenwood and Gleason, which is reproduced here for the sake of completeness.

Let L be prime and consider the field of L elements, numbered from 0 to $L-1$. Associate each field element with a vertex of K_L. We require that 3 divides $L-1$. Now consider the cubic residues of the multiplicative group $\mathbf{Z}_L^* = \mathbf{Z}_L \setminus \{0\}$, which form a coset of $\mathbf{Z}_L^*$. Note that since 3 divides $L-1$, there must be 2 other cosets.

Let i and j be two vertices of K_L. Color the edges of K_L as follows: If $j-i$ is a cubic residue color the edge connecting i and j red, if it is in the second coset, color the edge blue, and if it is the third coset, color the edge green. (Note that the order of differencing is immaterial since -1 is a cubic residue.)

Now suppose that a monochromatic K_5 exists in this coloring. Without loss of generality we may call the five vertices 0, a, b, c, and d, with $0<a<b<c<d$. Then the set of edges, $E=\{a,b,c,d,b-a,c-a,d-a,c-b,d-b,d-c\}$, must be a subset of one of the cosets. Since $a \neq 0$, multiplication by a^{-1} is allowed. Set $B=ba^{-1}$, $C=ca^{-1}$, and $D=da^{-1}$. Then the set $a^{-1}E=\{1,B,C,D,B-1,C-1,D-1,C-B,D-B,D-C\}$ must be a subset of the cubic residues. Hence if we find an L for which there does not exist B, C, and D such that $a^{-1}E$ is a subset of the cubic residues, then we can conclude that $R(5,5,5)>L$. Of course, this argument holds for $R(\underbrace{t,t,\ldots,t}_{k\ times})$ for any k, and any t.

Using `RES`

We only acheived results when we restricted our search to fields of prime order (although any finite field can be explored using `RES` (or at least easily modified to do so)). Since we are considering the number $R(5,5,5)$, reject any prime, q, for which 3 does not divide $q-1$. This can be accomplished automaticly by using the procedure `pryme`. By using the procedure `res` we produce all of the cubic residues of $\mathbf{Z}_p^*$, for a given prime, p. We then use the procedure `siv` to discard any residue, R, for which $R-1$ is not a residue. We now have a much more manageable list to search. We then call the procedure `diffcheck` to choose all possible 3-sets (for B, C, and D) and check whether or not the differences between any two elements are *all* cubic residues. If such a 3-set exists, `diffcheck` will output the first 3-set it finds. However, in the event that no such 3-set exists, `diffcheck` will output 1.

`RES` can also be used to search finite fields whose order is not prime. For example, to verify that the field on 2^4 elements, avoids a monochromatic triangle by using cubic residues (this fact was proven in [GG]), type `GalField3(2,4,3)`.

By using `RES` we were able to find the following lower bounds: $R(5,5,5) \geq 242$ and $R(6,6,6) \geq 692$. These are obtained by the following colorings: (Since -1 is a cubic residue it suffices to list only entries up to 120 for $R(5,5,5)$ and 345 for $R(6,6,6)$.)

$R(5,5,5) > 241$:

Color 1: 1, 5, 6, 8, 17, 21, 23, 25, 26, 27, 28, 30, 33, 36, 40, 41, 43, 44, 47, 48, 57, 61, 64, 73, 76, 79, 85, 87, 91, 93, 98, 101, 102, 103, 105, 106, 111, 115, 116, 117

Color 2: 2, 7, 9, 10, 11, 12, 16, 19, 29, 31, 34, 35, 37, 39, 42, 45, 46, 50, 52, 54, 55, 56, 59, 60, 66, 67, 71, 72, 80, 82, 83, 86, 88, 89, 94, 95, 96, 113, 114, 119

Color 3: 3, 4, 13, 14, 15, 18, 20, 22, 24, 32, 38, 49, 51, 53, 58, 62, 63, 65, 68, 69, 70, 74, 75, 77, 78, 81, 84, 90, 92, 97, 99, 100, 104, 107, 108, 109, 110, 112, 118, 120

$R(6,6,6) > 691$:

Color 1: 1, 2, 4, 5, 8, 10, 16, 19, 20, 21, 25, 27, 31, 32, 33, 38, 39, 40, 42, 50, 51, 54, 62, 64, 66, 67, 69, 71, 73, 76, 78, 80, 83, 84, 87, 89, 95, 100, 102, 105, 107, 108, 109, 123, 124, 125, 128, 132, 134, 135, 138, 139, 142, 146, 149, 151, 152, 155, 156, 160, 163, 165, 166, 168, 173, 174, 178, 179, 181, 190, 191, 195, 199, 200, 204, 210, 214, 216, 218, 246, 248, 250, 255, 256, 259, 263, 264, 268, 270, 271, 276, 278, 283, 284, 291, 292, 293, 298, 301, 302, 304, 309, 310, 311, 312, 320, 326, 329, 330, 332, 333, 335, 336, 343, 345

Color 2: 7, 9, 11, 13, 14, 17, 18, 22, 23, 26, 28, 29, 34, 35, 36, 41, 44, 45, 46, 52, 55, 56, 58, 65, 68, 70, 72, 82, 85, 88, 90, 92, 97, 103, 104, 110, 111, 112, 115, 116, 127, 129, 130, 131, 133, 136, 140, 141, 144, 145, 147, 159, 164, 167, 170, 171, 175, 176, 177, 180, 183, 184, 189, 194, 197, 205, 206, 208, 209, 217, 220, 222, 224, 225, 227, 229, 230, 231, 232, 233, 237, 241, 243, 247, 251, 254, 257, 258, 260, 262, 266, 272, 273, 275, 279, 280, 281, 282, 288, 290, 294, 297, 303, 313, 318, 323, 325, 328, 331, 334, 337, 339, 340, 341, 342

Color 3: 3, 6, 12, 15, 24, 30, 37, 43, 47, 48, 49, 53, 57, 59, 60, 61, 63, 74, 75, 77, 79, 81, 86, 91, 93, 94, 96, 98, 99, 101, 106, 113, 114, 117, 118, 119, 120, 121, 122, 126, 137, 143, 148, 150, 153, 154, 157, 158, 161, 162, 169, 172, 182, 185, 186, 187, 188, 192, 193, 196, 198, 201, 202, 203, 207, 211, 212, 213, 215, 219, 221, 223, 226, 228, 234, 235, 236, 238, 239, 240, 242, 244, 245, 249, 252, 253, 261, 265, 267, 269, 274, 277, 285, 286, 287, 289, 295, 296, 299, 300, 305, 306, 307, 308, 314, 315, 316, 317, 319, 321, 322, 324, 327, 338, 344

Part Two: On the Ramsey Numbers $R(3,3,3,k_1,k_2,\ldots,k_r)$

Let $N = R(k_1, k_2, \ldots, k_r)$. The Ramsey Property implies that there must exist a graph on $N-1$ vertices which avoids the Ramsey Property. Using such a graph, along with the construction in [C], we will prove that, for any natural number r and for any $k_i \geq 3$, $i = 1, 2, \ldots r$,

$$R(3,3,3,k_1,k_2,\ldots,k_r) \geq 3R(3,3,k_1,k_2,\ldots,k_r) + R(k_1,k_2,\ldots,k_r) - 3.$$

The Construction

Fix $r \geq 1$, and $k_i \geq 3$ for $i = 1, 2, \ldots r$. Let $M = R(3,3,k_1,k_2,\ldots k_r)$. Then there exists a graph, G, on $M-1$ vertices which avoids the Ramsey Property. Call the incidence matrix of this graph $T_{r+2} = T_{r+2}(x_0, x_1, x_2, \ldots, x_{r+2})$, where x_0 are the diagonal entries only, and the x_i, for $i = 1, 2, \ldots r+2$, are the $r+2$ colors. By definition of G, there are no x_1-colored nor x_2-colored triangles, and no x_{i+2}-colored K_{k_i}, for $i = 1, 2, \ldots r$.

Now consider the following slightly modified construction from [C]:

$$T_{r+3}(0,1,2,\ldots,r+3) = \begin{array}{cccc} A & & & \\ D & B & & \\ E & F & C & \\ 1,\ldots,1 & 2,\ldots,2 & 3,\ldots,3 & \\ \vdots & \vdots & \vdots & G \\ 1,\ldots,1 & 2,\ldots,2 & 3,\ldots,3 & \end{array}$$

the incidence matrix of a graph H on $3M + R(k_1,k_2,\ldots,k_r) - 4$ vertices, where

$$\begin{aligned} A &= T_{r+2}(0,2,3,4,5,\ldots,r+3) \\ B &= T_{r+2}(0,3,1,4,5,\ldots,r+3) \\ C &= T_{r+2}(0,1,2,4,5,\ldots,r+3) \\ D &= T_{r+2}(3,2,1,4,5,\ldots,r+3) \\ E &= T_{r+2}(2,1,3,4,5,\ldots,r+3) \\ F &= T_{r+2}(1,3,2,4,5,\ldots,r+3) \end{aligned}$$

and G is any matrix on $R(k_1,k_2,\ldots,k_r) - 1$ vertices in the colors 4 through $r+3$ which avoids the Ramsey Property.

Using Fan Chung's result [C] we see that the graph H avoids 1-colored, 2-colored, and 3-colored triangles. We now argue that no $(j+3)$-colored K_{k_j} exists in H for $j = 1, 2, \ldots, r$: Assume there exists a J-colored K_{k_J} in H, for some J between 4 and $r+3$. Then there must exist $\binom{k_J}{2}$ entries in the $3(M-1)\times 3(M-1)$ upper left submatrix of T_{r+3}, all of value J, which form the edges of a complete graph on k_J vertices. However, by the construction of T_{r+3} we see that *all* of these entries can be taken modulo $M-1$, since the entries of value J in each block are in exactly the same places as in the upper left block, A. We further note that if (s,t) and (u,v) are two of the entries in question, then $(s,t) \not\equiv (u,v)$ $(mod\ M-1)$ (componentwise). Without loss of generality we may assume $s < u$. If we had $s \equiv u$ $(mod\ M-1)$, then since (u,s) must also have the same value as (u,v), we would have $(u,s) \equiv (u,u)$ $(mod\ M-1)$. This implies that the entry J is on the diagonal of $T_{r+2}(0,2,3,4,\ldots,r+3)$, a contradiction. Hence, if we have a J-colored K_{k_J} in the upper left submatrix of T_{r+3} (of size $3(M-1) \times 3(M-1)$), then there must be a J-colored K_{k_J} in $T_{r+2}(0,2,3,4,\ldots,r+3)$, contradicting the definition of T_{r+2}.

Remark: Up to the renaming of colors and vertices, the above permutation configuration of colors which defines T_{r+3} is the *only* configuration which will avoid monochromatic triangles.

Harvesting Some Lower Bounds for Ramsey Numbers

It is amazing that the result of this section has not been observed for the 25 years since [C] was published. Using this observation we will give 6 new lower bounds. Currently in Radziszowski's Survey [R], we have that $R(3,3,3,4) \geq 87$, due to Exoo [E1]. By applying the result of this section to $R(3,3,3,4)$ and using the fact that $R(3,3,4) \geq 30$ [K], we get the new lower bound: $R(3,3,3,4) \geq 91$. Using the bound $R(3,3,5) \geq 45$ [E2,KLR], we get the bound $R(3,3,3,5) \geq 137$. Finally, using $R(3,3,6) \geq 54$, $R(3,3,7) \geq 72$, $R(3,3,9) \geq 110$, and $R(3,3,11) \geq 138$ all from [SLZL], we get the lower bounds $R(3,3,3,6) \geq 165$, $R(3,3,3,7) \geq 220$, $R(3,3,3,9) \geq 336$, and $R(3,3,3,11) \geq 422$.

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