Comment on
"Permutations which are the Union of an Increasing and a Decreasing Subsequence," by M.D. Atkinson.

by Volker Strehl
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These comments contain a somewhat shorter proof of Atkinson's Theorem 2 and give some pointers to closely related literature.

Let s_n denote the set of *skew-merged* permutations of $[1..n]$. The characteristic property of these permutations is precisely described in the title of Atkinson's article. Each such permutation, if represented as a cloud of points on a $n \times n$-grid in the traditional manner, has a number k (where $0 \leq k \leq n$) of *white* elements (see Atkinson, Theorem 1). These are the elements that simultaneously belong to an increasing subsequence of maximum length and to a decreasing subsequence of maximum length. We will denote by $t_k(n)$ the set elements of s_n with precisely k white elements (which form an increasing or decreasing sequence of "contiguous" elements). For further reference we introduce the set

$$y_n := t_1(n) + t_2(n) + \ldots + t_n(n)$$

of skew-merged permutations with *at least one* white element. In the sequel I use the symbols $s_n, t_k(n), y_n$ also to denote the cardinalities of the corresponding sets; this follows Atkinson's notation. Equality signs should then be read as "there exists a bijection". Finally I introduce the set z_n of skew-merged permutation where exactly one of the white elements has been *marked*. Using the convention just mentioned we may simply write

$$z_n = 1 \cdot t_1(n) + 2 \cdot t_2(n) + \ldots + n \cdot t_n(n)$$

More specifically, for a, b such that $0 \leq a, b < n$ and $a + b = n + 1$ $z_{a,b}$ will denote the set of all elements of z_n such that the underlying permutation is the union of an increasing subsequence of length $a + 1$ and a decreasing subsequence of length $b+1$ (both are necessarily monotone subsequences of either kind of maximal length).

Lemma 9 of Atkinson's article, proved using Schensted's theorem [2], stated:

$$y_n = \sum_{a=0}^{n-1} \binom{n-1}{a}^2 = \binom{2(n-1)}{n-1} \tag{1}$$

A proof of this is already contained in the paper by Baer and Brock [3], see Theorem 9, p. 292., but these authors already point to the work by Schensted in [2], even though Schensted's article had not yet appeared when they submitted their paper, see the footnote on p.286 of [3].

Our new proof of Atkinson's Theorem 2 is via Brock's identity [1],[3]:

$$z_{a,b} - z_{a-1,b} - z_{a,b-1} = \binom{a+b}{a}^2 \tag{2}$$

Equation (2) first appeared as a problem, posed by P. Brock, in the SIAM review [1]. It is not difficult to see that

$$z_{a,b} = \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{i+j}{j} \binom{a-i+j}{a-i} \binom{b-j+i}{i} \binom{a-i+b-j}{b-j}$$

But if these sums are substituted for the z-terms in (2), then a binomial identity of considerable complexity shows up — this is the way in which the problem was presented. A (rather involved) proof of (2) was given in the Appendix 1 of [3]. A proof of (2) using generating functions was given by Slepian [4], and this approach was then taken up by Carlitz [5], which marks only the beginning of interesting "generatingfunctionological" investigations, variations, extensions by Carlitz and many others. A combinatorial proof of (2) was first presented by Church, jr. in [6], and independently by Strehl in [7]. The idea is indeed very simple:

> Consider the elements of $z_{m,n}$. If the marked white element is not the leftmost white element, then the marked element and its predecessor (also a white element) can be lumped together. Carrying over the mark to this new element, we obtain either an element of $z_{a-1,b}$ or $z_{a,b-1}$, depending on whether the sequence of white elements was increasing or decreasing. This mapping is reversible. The elements of $z_{m,n}$ left untouched are those elements, where the leftmost white element is marked. But marking the leftmost white element is as good as not marking at all – so this set is in bijection with skew-merged permutations of length n which are unions of an increasing subsequence of length $a+1$ and a decreasing subsequence of length $b+1$.

Note that by summing over all a, b in (2) we get

$$z_n - 2z_{n-1} = \sum_{a+b=n-1} \binom{a+b}{a}^2 = \binom{2(n-1)}{n-1} = y_n$$

which can also be established directly using the combinatorial argument above. Induction now leads to

$$z_n = \sum_{m=1}^{n} 2^{n-m} y_m \tag{3}$$

We can now give our new proof of Atkinson's Theorem 2:

$$s_n = \binom{2n}{n} - \sum_{m=0}^{n-1} 2^{n-m-1} \binom{2m}{m}$$

or equivalently, by (1) and (3),

$$y_{n+1} = s_n + \sum_{m=1}^{n-1} 2^{n-m} y_m = s_n + z_n \tag{4}$$

By Lemmas 10 and 12 of Atkinson's paper

$$t_0(n) = t_1(n+1) \quad \text{and} \quad t_k(n) = t_{k+1}(n+1) \quad (k \geq 2)$$

so that (4) is equivalent

$$t_2(n+1) = t_1(n) + z_n$$

But this is true due to the simple combinatorial observation that the two white elements, forming an increasing (decreasing, resp.) sequence, from an element of the l.h.s. can be lumped together into one *marked* element of the decreasing (increasing, resp.) white sequence of an element on the r.h.s., which gives an element of z_n. Some care has to be taken if the white sequence on the r.h.s. degenerates into one element, which introduces the additional term $t_1(n)$. (A similar reasoning could already be applied to (4)).

What is perhaps more important than presenting a simpler proof is to point out that the enumeration of skew-merged permutations (with at least one white element), as initiated by the work of Baer and Brock, has a long history and has, via Brock's identity, given rise to many interesting investigations, in particular on the associated generating functions. The seminar report [7] contains much of the history and some of my own work in this field, a comprehensive treatment is given in my "Habilitationsschrift" [8], a short account of all that has been presented at the 4th FPSAC conference in Montréal[9].

References

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[9] V. Strehl, Combinatorics of Special Functions: Facets of Brock's Identity. Invited talk presented at the 4th Colloquium *Formal Power Series and Algebraic Combinatorics*, Montréal 1992. (see the conference proceedings published by LaCIM, pp. 363-378)

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