A Generalization of Gosper's Algorithm to Bibasic Hypergeometric Summation

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Abstract

An algebraically motivated generalization of Gosper's algorithm to indefinite bibasic hypergeometric summation is presented. In particular, it is shown how Paule's concept of greatest factorial factorization of polynomials can be extended to the bibasic case. It turns out that most of the bibasic hypergeometric summation identities from literature can be proved and even found this way. A Mathematica implementation of the algorithm is available from the author.

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1 Introduction

Recently, Paule and Strehl [10] observed that the algorithm presented by Gosper [7] for indefinite hypergeometric summation extends quite naturally to the q-hypergeometric case by introducing a q-analogue of the canonical Gosper-Petkovšek (GP) representation for rational functions. Based on the new algebraic concept of greatest factorial factorization (GFF), Paule [8] developed an alternative but equivalent approach to hypergeometric telescoping. It was also shown by Paule (cf. Paule and Riese [9]) that the problem of q-hypergeometric telescoping can be treated along the same lines as the $q = 1$ case by making use of a q-version of GFF. Built on these concepts, a Mathematica implementation of q-analogues of Gosper's as well as of Zeilberger's [14] fast algorithm for definite q-hypergeometric summation has been carried out by the author (cf. Paule and Riese [9], and Riese [12]). The original approach to definite q-hypergeometric summation is due to Wilf and Zeilberger [13].

The object of this paper is to describe how the algorithm qTelescope presented in [9], a q-analogue of Gosper's algorithm, generalizes to the bibasic hypergeometric case. In Section 2, the underlying theoretical background based on a bibasic extension of GFF is discussed, which leads to the bibasic counterpart of the algorithm qTelescope. In Section 3, the degree setting for solving the bibasic key equation is established. Applications are given in Section 4 to illustrate the usage of the newly developed Mathematica implementation which is available by email request to `Axel.Riese@risc.uni-linz.ac.at`.

2 Theoretical Background

In this section, q-greatest factorial factorization (qGFF) of polynomials, which has been introduced by Paule (cf. Paule and Riese [9]) providing an algebraic explanation of q-hypergeometric telescoping, is extended to the bibasic hypergeometric case. It turns out that to this end the q-case argumentation can be carried over almost word by word.

2.1 Bibasic Greatest Factorial Factorization

Let $\mathbb{Z}$ denote the set of all integers, and $\mathbb{N}$ the set of all non-negative integers. Let p, q, x, and y be fixed indeterminates. Assume $K = L(\kappa_1, \ldots, \kappa_n)$ to be the field of rational functions in a fixed number of indeterminates $\kappa_1, \ldots, \kappa_n$, $n \in \mathbb{N}$, where $p \neq \kappa_i \neq y$ and $q \neq \kappa_i \neq x$, $1 \leq i \leq n$, over some computable field L of characteristic 0 and not containing p, q, x, and y. (For the sake of simplicity with regard to the implementation we will restrict ourselves to the case where L is the rational number field $\mathbb{Q}$.) The transcendental extension of K by the indeterminates p and q is denoted by F, i.e., $F = K(p, q)$.

For $P \in F[x, y]$, let the *bibasic shift operator* ϵ be given by $(\epsilon P)(x, y) = P(qx, py)$. The extension of this shift operator to the rational function field $F(x, y)$, the quotient field of the polynomial ring $F[x, y]$, will be also denoted by ϵ.

Definition 1. A polynomial $P \in F[x]$ (resp. $P \in F[y]$) is called *q-monic* (resp. *p-monic*) if $P(0) = 1$. A polynomial $P \in F[x, y]$ is called *bibasic monic* if $P(x, 0) \neq 0 \neq P(0, y)$ and either $P(0, 0) = 1$, or $P(0, 0) = 0$ and the coefficients of P are relatively prime polynomials in F.†

Example. (i) The following polynomials are bibasic monic:

$$P_1(x, y) = 1, \qquad P_2(x, y) = 1 - apqx^2y^3, \qquad P_3(x, y) = (1 - q)^2x^2 + py.$$

(ii) The following polynomials are not bibasic monic:

$$P_4(x, y) = q, \qquad P_5(x, y) = xy - apqx^2y^3, \qquad P_6(x, y) = (1 - q)^{-1}px^2 + py. \qquad \square$$

The properties of being q-monic, p-monic, and bibasic monic are clearly invariant with respect to the bibasic shift operator ϵ, i.e., if P is q-monic, p-monic, or bibasic monic, then the same holds true for ϵP. Furthermore, the product of two bibasic monic polynomials is again bibasic monic. Also note that a bibasic monic polynomial P satisfies $\gcd(x, P) = 1 = \gcd(y, P)$.

Evidently, any non-zero polynomial $P \in F[x, y]$ has a unique factorization, the *bibasic monic decomposition*, in the form

$$P = z \cdot x^\alpha \cdot y^\beta \cdot P^*,$$

where $z \in F$, $\alpha, \beta \in \mathbb{N}$, and $P^* \in F[x, y]$ is bibasic monic.

The bibasic monic decomposition of a polynomial $P \neq 0$ can be computed easily as follows. Define $\alpha := \max\{i \in \mathbb{N} : x^i | P\}$, $\beta := \max\{j \in \mathbb{N} : y^j | P\}$, and put $\bar{P} := x^{-\alpha} \cdot y^{-\beta} \cdot P$. If $\bar{P}(0, 0) \neq 0$ define $z := \bar{P}(0, 0)$, otherwise let l denote the least common multiple of all coefficient-denominators of $\bar{P}$, let g denote the greatest common divisor of all coefficients of $l \cdot \bar{P}$, and define $z := g/l$. Then, for $P^* := z^{-1} \cdot \bar{P}$, the bibasic monic decomposition of P is given by $P = z \cdot x^\alpha \cdot y^\beta \cdot P^*$.

†In other words, P is assumed to be primitive over $L[\kappa_1, \ldots, \kappa_n, p, q]$ in this case, which will guarantee the uniqueness of the so-called bibasic monic decomposition of a polynomial as shown below.

Example. The bibasic monic decompositions of the polynomials P_4, P_5, and P_6 from the example above are given by

$$P_4 = q \cdot x^0 \cdot y^0 \cdot 1, \quad P_5 = 1 \cdot x \cdot y \cdot (1 - apqxy^2), \quad P_6 = \frac{p}{1-q} \cdot x^0 \cdot y^0 \cdot (x^2 + (1-q)y). \quad \square$$

Moreover, we assume the result of any gcd computation over $F[x, y]$ as being normalized in the following sense. If $P_1 = z_1 \cdot x^{\alpha_1} \cdot y^{\beta_1} \cdot P_1^*$ and $P_2 = z_2 \cdot x^{\alpha_2} \cdot y^{\beta_2} \cdot P_2^*$ are the bibasic monic decompositions of $P_1, P_2 \in F[x, y]$, we define

$$\gcd_{p,q}(P_1, P_2) := \gcd(x^{\alpha_1}, x^{\alpha_2}) \cdot \gcd(y^{\beta_1}, y^{\beta_2}) \cdot \gcd_{p,q}(P_1^*, P_2^*),$$

where the $\gcd_{p,q}$ of two bibasic monic polynomials is understood to be bibasic monic.

The polynomial degree in x and y of any $P \in F[x, y]$ is denoted by $\deg_x(P)$ and $\deg_y(P)$, respectively.

Definition 2. For any bibasic monic polynomial $P \in F[x, y]$ and $k \in \mathbb{N}$, the *k-th falling bibasic factorial* $[P]_{p,q}^{\underline{k}}$ of P is defined as

$$[P]_{p,q}^{\underline{k}} := \prod_{i=0}^{k-1} \epsilon^{-i} P.$$

Note that by the null convention $\prod_{i \in \emptyset} P_i := 1$ we have $[P]_{p,q}^{\underline{0}} = 1$. In general, polynomials arising in bibasic hypergeometric summation have several different representations in terms of falling bibasic factorials. From all possibilities, we shall consider only the one taking care of maximal chains, which informally can be obtained as follows. One selects irreducible factors of P in such a way that their product, say

$$P_{k,1}(x, y) \cdot P_{k,1}(q^{-1}x, p^{-1}y) \cdots P_{k,1}(q^{-k+1}x, p^{-k+1}y),$$

forms a falling bibasic factorial $[P_{k,1}]_{p,q}^{\underline{k}}$ of maximal length k. For the remaining irreducible factors of P this procedure is applied again in order to find all k-th falling factorial divisors $[P_{k,1}]_{p,q}^{\underline{k}}, \ldots, [P_{k,l}]_{p,q}^{\underline{k}}$ of that type. Then $[P_k]_{p,q}^{\underline{k}} := [P_{k,1} \cdots P_{k,l}]_{p,q}^{\underline{k}}$ forms the bibasic factorial factor of P of maximal length k. Iterating this procedure one gets a factorization of P in terms of "greatest" factorial factors.

Definition 3. We say that $\langle P_1, \ldots, P_k \rangle$, $P_i \in F[x, y]$, is a *bibasic GFF-form* of a bibasic monic polynomial $P \in F[x, y]$, written as $\mathrm{GFF}_{p,q}(P) = \langle P_1, \ldots, P_k \rangle$, if the following conditions hold:

($\mathrm{GFF}_{p,q}$ 1) $P = [P_1]_{p,q}^{\underline{1}} \cdots [P_k]_{p,q}^{\underline{k}}$,

($\mathrm{GFF}_{p,q}$ 2) each P_i is bibasic monic, and $k > 0$ implies $P_k \neq 1$,

($\mathrm{GFF}_{p,q}$ 3) for $i \leq j$ we have $\gcd_{p,q}([P_i]_{p,q}^{\underline{i}}, \epsilon P_j) = 1 = \gcd_{p,q}([P_i]_{p,q}^{\underline{i}}, \epsilon^{-j} P_j)$.

Note that $\mathrm{GFF}_{p,q}(1) = \langle\rangle$. Condition ($\mathrm{GFF}_{p,q}$ 3) intuitively can be understood as prohibiting "overlaps" of bibasic factorials that violate length maximality. The following theorem states that, as in the q-hypergeometric case, the bibasic GFF-form is unique and thus provides a canonical form.

Theorem 1. *If $\langle P_1, \ldots, P_k \rangle$ and $\langle P_1', \ldots, P_l' \rangle$ are bibasic GFF-forms of a bibasic monic polynomial $P \in F[x, y]$, then $k = l$ and $P_i = P_i'$ for all $1 \leq i \leq k$.*

Proof. The corresponding result for the ordinary hypergeometric case ($p = q = 1$) has been proved by Paule [8, Thm. 2.1]. The arguments used there extend immediately to the bibasic hypergeometric case proceeding by induction on $d := \deg_x(P) + \deg_y(P)$. □

From algorithmic point of view it is important to note that the bibasic GFF-form can be computed in an iterative manner essentially involving only gcd computations.

In q-hypergeometric summation, the normalized gcd of a polynomial P and its q-shift ϵP plays a fundamental role, as the gcd of P and its shift EP does in ordinary hypergeometric summation, where $(EP)(x) = P(x+1)$. The same is true for bibasic hypergeometric summation with respect to the bibasic shift operator ϵ. The mathematical and algorithmic essence lies in the following lemma.

Lemma 1 (Fundamental $\mathbf{GFF}_{p,q}$ Lemma). *Let $P \in F[x,y]$ be a bibasic monic polynomial with* $\mathrm{GFF}_{p,q}(P) = \langle P_1, \ldots, P_k \rangle$. *Then*

$$\mathrm{gcd}_{p,q}(P, \epsilon P) = [P_1]^{\underline{0}}_{p,q} \cdots [P_k]^{\underline{k-1}}_{p,q}.$$

Proof. Due to the choice of the bibasic shift operator ϵ, the proof of the so-called Fundamental qGFF Lemma (cf. Paule and Riese [9, Lemma 1]) can be carried over to the bibasic hypergeometric case completely unchanged. □

Thus, if $\mathrm{GFF}_{p,q}(P) = \langle P_1, \ldots, P_k \rangle$, then $\mathrm{GFF}_{p,q}(\mathrm{gcd}_{p,q}(P, \epsilon P)) = \langle P_2, \ldots, P_k \rangle$. Consequently, dividing P with $\mathrm{GFF}_{p,q}(P) = \langle P_1, \ldots, P_k \rangle$ by $\epsilon^{-1} \mathrm{gcd}_{p,q}(P, \epsilon P)$ or $\mathrm{gcd}_{p,q}(P, \epsilon P)$ results in separating the product of the first, respectively last, falling bibasic factorial entries, or in other words

$$\frac{P}{\epsilon^{-1} \mathrm{gcd}_{p,q}(P, \epsilon P)} = P_1 \cdot P_2 \cdots P_k \quad \text{and} \quad \frac{P}{\mathrm{gcd}_{p,q}(P, \epsilon P)} = P_1 \cdot (\epsilon^{-1} P_2) \cdots (\epsilon^{-k+1} P_k).$$

2.2 Bibasic Hypergeometric Telescoping

A sequence $(f_k)_{k \in \mathbb{Z}}$ is said to be *bibasic hypergeometric* (see, e.g. Petkovšek, Wilf, and Zeilberger [11]) in p and q over F, if there exists a rational function $\rho \in F(x,y)$ such that $f_{k+1}/f_k = \rho(q^k, p^k)$ for all k where the quotient is well-defined.

Assume we are given a bibasic hypergeometric sequence $(f_k)_{k \in \mathbb{Z}}$. Then the problem of bibasic hypergeometric telescoping is to decide whether there exists a bibasic hypergeometric sequence $(g_k)_{k \in \mathbb{Z}}$ such that

$$g_{k+1} - g_k = f_k, \tag{1}$$

and if so, to determine $(g_k)_{k \in \mathbb{Z}}$ with the motive that for $a, b \in \mathbb{Z}$, $a \leq b$,

$$\sum_{k=a}^{b} f_k = g_{b+1} - g_a,$$

which solves the indefinite summation problem.

For the rational function ρ, related to f_{k+1}/f_k as above, there exists a representation $\rho(x,y) = z \cdot x^\alpha \cdot y^\beta \cdot A^*(x,y)/B^*(x,y)$ with bibasic monic $A^*, B^* \in F[x,y]$, $z \in F$, and $\alpha, \beta \in \mathbb{Z}$, which we call a *rational representation* of the bibasic hypergeometric sequence $(f_k)_{k \in \mathbb{Z}}$. If additionally A^* and B^* are relatively prime, then $\rho(x,y)$ is called the *reduced rational representation* of $(f_k)_{k \in \mathbb{Z}}$. For $\alpha \in \mathbb{Z}$, let $\alpha_+ := \max(\alpha, 0)$ and $\alpha_- := \max(-\alpha, 0)$.

It will be shown below that bibasic hypergeometric telescoping can be decided constructively as follows.

Algorithm Telescope$_{p,q}$. INPUT: *a bibasic hypergeometric sequence* $(f_k)_{k\in\mathbb{Z}}$ *specified by its reduced rational representation* $\rho = z \cdot x^\alpha \cdot y^\beta \cdot A^*/B^*$;
OUTPUT: *a bibasic hypergeometric solution* $(g_k)_{k\in\mathbb{Z}}$ *of* (1); *in case such a solution does not exist, the algorithm stops.*

(i) *Compute the bibasic GP form of* $(f_k)_{k\in\mathbb{Z}}$, *i.e.,*

(a) *determine unique bibasic monic polynomials* P^*, Q^*, $R^* \in F[x,y]$ *such that*

$$\frac{A^*}{B^*} = \frac{\epsilon P^*}{P^*} \cdot \frac{Q^*}{\epsilon R^*}, \tag{2}$$

where $\gcd_{p,q}(P^*, Q^*) = 1 = \gcd_{p,q}(P^*, R^*)$ *and* $\gcd_{p,q}(Q^*, \epsilon^j R^*) = 1$ *for all* $j \geq 1$, *and*

(b) *let* a_x, b_x, a_y, *and* b_y *denote the coefficients of the lowest occurring powers of* x *and* y *in* $A^*(x,0)$, $B^*(x,0)$, $A^*(0,y)$, *and* $B^*(0,y)$, *respectively. Define*

$$(\gamma,\delta) := \begin{cases} (\varphi,\psi) & \text{if } \alpha = 0 = \beta \text{ and } q^\varphi \cdot p^\mu \cdot b_y/a_y = z = p^\psi \cdot q^\nu \cdot b_x/a_x \\ & \text{for } \varphi,\psi \in \mathbb{N} \text{ and } \mu,\nu \in \mathbb{Z}, \\ (\varphi,0) & \text{if } \alpha = 0 \neq \beta \text{ and } z = q^\varphi \cdot p^\mu \cdot b_y/a_y \text{ for } \varphi \in \mathbb{N},\ \mu \in \mathbb{Z}, \\ (0,\psi) & \text{if } \alpha \neq 0 = \beta \text{ and } z = p^\psi \cdot q^\nu \cdot b_x/a_x \text{ for } \psi \in \mathbb{N},\ \nu \in \mathbb{Z}, \\ (0,0) & \text{otherwise,} \end{cases}$$

and put

$$\begin{aligned} P &:= x^\gamma \cdot y^\delta \cdot P^*, \\ Q &:= z \cdot q^{-\gamma} \cdot p^{-\delta} \cdot x^{\alpha+} \cdot y^{\beta+} \cdot Q^*, \\ \epsilon R &:= x^{\alpha-} \cdot y^{\beta-} \cdot \epsilon R^*, \end{aligned} \tag{3}$$

with the motive that then

$$\rho = \frac{\epsilon P}{P} \cdot \frac{Q}{\epsilon R}.$$

(ii) *Try to solve the bibasic key equation*

$$P = Q \cdot \epsilon Y - R \cdot Y \tag{4}$$

for a polynomial $Y \in F[x,y]$.

(iii) *If such a polynomial solution* Y *exists, then*

$$g_k = \frac{R(q^k,p^k) \cdot Y(q^k,p^k)}{P(q^k,p^k)} \cdot f_k \tag{5}$$

is a bibasic hypergeometric solution of (1), *otherwise no bibasic hypergeometric solution* $(g_k)_{k\in\mathbb{Z}}$ *exists.* □

The steps of Algorithm $\text{Telescope}_{p,q}$ are derived as follows. First, assume that a bibasic hypergeometric solution $(g_k)_{k\in\mathbb{Z}}$ with rational representation $g_{k+1}/g_k = \sigma(q^k, p^k)$ of (1) exists. Then evidently we have

$$g_k = \tau(q^k, p^k) \cdot f_k, \tag{6}$$

where $\tau(x, y) = 1/(\sigma(x, y) - 1) \in F(x, y)$.

By relation (6), equation (1) is equivalent to

$$z \cdot x^{\alpha+} \cdot y^{\beta+} \cdot A^* \cdot \epsilon\tau - x^{\alpha-} \cdot y^{\beta-} \cdot B^* \cdot \tau = x^{\alpha-} \cdot y^{\beta-} \cdot B^*, \tag{7}$$

where the reduced rational representation of $(f_k)_{k\in\mathbb{Z}}$ is given by $\rho = z \cdot x^\alpha \cdot y^\beta \cdot A^*/B^*$.

Vice versa, any rational solution $\tau \in F(x, y)$ of (7) gives rise to a bibasic hypergeometric solution $g_k := \tau(q^k, p^k) \cdot f_k$ of (1). This means, bibasic hypergeometric telescoping is equivalent to finding a *rational* solution τ of (7).

Any $\tau \in F(x, y)$ can be represented as the quotient of relatively prime polynomials in the form $\tau = \mathcal{U}/\mathcal{V}$ where $\mathcal{U}, \mathcal{V} \in F[x, y]$ with $\mathcal{V} = x^\varphi \cdot y^\psi \cdot \mathcal{V}^*$ the bibasic monic decomposition of $\mathcal{V}$. In case such a solution τ of (7) exists, assume we know $\mathcal{V}$ or a multiple $V \in F[x, y]$ of $\mathcal{V}$. Then by clearing denominators in

$$z \cdot x^{\alpha+} \cdot y^{\beta+} \cdot A^* \cdot \frac{\epsilon U}{\epsilon V} - x^{\alpha-} \cdot y^{\beta-} \cdot B^* \cdot \frac{U}{V} = x^{\alpha-} \cdot y^{\beta-} \cdot B^*,$$

the problem reduces further to finding a polynomial solution $U \in F[x, y]$ of the resulting difference equation with polynomial coefficients,

$$z \cdot x^{\alpha+} \cdot y^{\beta+} \cdot A^* \cdot V \cdot \epsilon U - x^{\alpha-} \cdot y^{\beta-} \cdot B^* \cdot (\epsilon V) \cdot U = x^{\alpha-} \cdot y^{\beta-} \cdot B^* \cdot V \cdot \epsilon V. \tag{8}$$

Note that at least one polynomial solution, namely $U = \mathcal{U} \cdot V/\mathcal{V}$, exists. Furthermore, equations of that type simplify by canceling $\gcd_{p,q}$'s. For instance, in order to get more information about the denominator $\mathcal{V}$, let $\mathcal{V}_i := \epsilon^i \mathcal{V}/\gcd_{p,q}(\mathcal{V}, \epsilon\mathcal{V})$, $i \in \{0, 1\}$. Then (7) is equivalent to

$$z \cdot x^{\alpha+} \cdot y^{\beta+} \cdot A^* \cdot \mathcal{V}_0 \cdot \epsilon\mathcal{U} - x^{\alpha-} \cdot y^{\beta-} \cdot B^* \cdot \mathcal{V}_1 \cdot \mathcal{U} = x^{\alpha-} \cdot y^{\beta-} \cdot B^* \cdot \mathcal{V}_0 \cdot \mathcal{V}_1 \cdot \gcd_{p,q}(\mathcal{V}, \epsilon\mathcal{V}). \tag{9}$$

Now, if $\langle \mathcal{P}_1, \ldots, \mathcal{P}_m \rangle$, $m \in \mathbb{N}$, is the bibasic GFF-form of $\mathcal{V}^*$, it follows from $\gcd_{p,q}(\mathcal{U}, \mathcal{V}) = 1 = \gcd_{p,q}(\mathcal{V}_0, \mathcal{V}_1)$ and the Fundamental $\text{GFF}_{p,q}$ Lemma that

$$\mathcal{V}_0 = (\epsilon^0 \mathcal{P}_1) \cdots (\epsilon^{-m+1} \mathcal{P}_m) \mid B^* \quad \text{and} \quad \mathcal{V}_1 = q^\varphi \cdot p^\psi \cdot (\epsilon \mathcal{P}_1) \cdots (\epsilon \mathcal{P}_m) \mid A^*.$$

This observation gives rise to a simple and straightforward algorithm for computing a multiple $V^* := [P_1]^{\underline{1}}_{p,q} \cdots [P_n]^{\underline{n}}_{p,q}$ of $\mathcal{V}^*$. For instance, if $P_1 := \gcd_{p,q}(\epsilon^{-1} A^*, B^*)$ then obviously $\mathcal{P}_1 | P_1$. Actually, one can iteratively extract bibasic monic $\mathcal{P}_i$-multiples P_i such that $\epsilon P_i | A^*$ and $\epsilon^{-i+1} P_i | B^*$ by the following algorithm.

Algorithm V*MULT. Input: *relatively prime and bibasic monic polynomials* $A^*, B^* \in F[x, y]$ *that constitute the bibasic monic quotient of* $\rho = z \cdot x^\alpha \cdot y^\beta \cdot A^*/B^* \in F(x, y)$*;*
Output: *bibasic monic polynomials* $P_1, \ldots, P_n$ *such that* $V^* := [P_1]^{\underline{1}}_{p,q} \cdots [P_n]^{\underline{n}}_{p,q}$ *is a multiple of* $\mathcal{V}^*$*, the bibasic monic part of the denominator* $\mathcal{V} = x^\varphi \cdot y^\psi \cdot \mathcal{V}^*$ *of* $\tau \in F(x, y)$*.*

(i) Compute $n = \min\{j \in \mathbb{N} \mid \gcd_{p,q}(\epsilon^{-1} A^*, \epsilon^{k-1} B^*) = 1$ *for all integers* $k > j\}$.

(ii) Set $A_0 = A^*$, $B_0 = B^*$, *and compute for* i *from* 1 *to* n*:*

$$\begin{aligned} P_i &= \gcd_{p,q}(\epsilon^{-1} A_{i-1}, \epsilon^{i-1} B_{i-1}), \\ A_i &= A_{i-1}/\epsilon P_i, \\ B_i &= B_{i-1}/\epsilon^{-i+1} P_i. \end{aligned}$$

□

A proof for the fact that the P_i are indeed multiples of the $\mathcal{P}_i$ has been worked out for the ordinary hypergeometric case by Paule [8, Lemma 5.1]. It can be carried over to the bibasic hypergeometric world almost word by word. Hence we leave the steps of the verification to the reader.

Note that in general step (i) of Algorithm V*MULT would be a rather time-consuming task involving resultant computations which could be solved by generalizing the univariate case (cf. Abramov, Paule, and Petkovšek [1]) in a straightforward way, for instance, as follows. Define $R_1(v,w) := \mathrm{Res}_x(A^*(x,y), B^*(vx,wy))$ and $R_2(v,w) := \mathrm{Res}_y(A^*(x,y), B^*(vx,wy))$, viewed as polynomials of v and w over $F[y]$, respectively $F[x]$. Then n is the maximal positive integer such that $R_1(q^n,p^n)\cdot R_2(q^n,p^n) = 0$ if such an integer exists, and $n = 0$ otherwise. However, in our implementation we make use of the fact that A^* and B^* already come in nicely factored form so that the computation of n boils down to a comparison of those factors.

Moreover, Algorithm V*MULT also delivers the constituents of the bibasic monic part of the GP representation (2) as stated in the following lemma.

Lemma 2. *Let n, A_n, B_n, and the tuple $\langle P_1,\dots,P_n\rangle$ be computed as in Algorithm* V*MULT. *Then for $P^* = V^*$, $Q^* = A_n$, and $R^* = \epsilon^{-1}B_n$ we have*

$$\frac{A^*}{B^*} = \frac{\epsilon P^*}{P^*}\cdot\frac{Q^*}{\epsilon R^*},$$

where $\gcd_{p,q}(P^*,Q^*) = 1 = \gcd_{p,q}(P^*,R^*)$ and $\gcd_{p,q}(Q^*,\epsilon^j R^*) = 1$ for all $j \geq 1$.

For more details on GP representations in the q-hypergeometric case, see Abramov, Paule, and Petkovšek [1], or Paule and Strehl [10]. The results obtained there also apply in the bibasic hypergeometric case.

With the multiple V^* of $\mathcal{V}^*$ in hands, all what is left for solving (7), and thus the bibasic hypergeometric telescoping problem (1), is to determine appropriate multiplicities γ and δ such that

$$V = x^\gamma\cdot y^\delta\cdot V^* \text{ is a multiple of } \mathcal{V} = x^\varphi\cdot y^\psi\cdot\mathcal{V}^*.$$

For that we consider equation (9) again in the equivalent version

$$z\cdot x^{\alpha_+}\cdot y^{\beta_+}\cdot A^*\cdot\mathcal{V}^*\cdot\epsilon\mathcal{U} - x^{\alpha_-}\cdot y^{\beta_-}\cdot B^*\cdot q^\varphi\cdot p^\psi\cdot(\epsilon\mathcal{V}^*)\cdot\mathcal{U} = x^{\alpha_-}\cdot y^{\beta_-}\cdot B^*\cdot\mathcal{V}^*\cdot\epsilon\mathcal{V}, \quad (10)$$

and distinguish the following cases corresponding to step (ib) of Algorithm $\mathrm{Telescope}_{p,q}$.

(i) Assume that either $\alpha_- \neq 0$ or $\alpha_+ \neq 0$. In the first case we have $\alpha_+ = 0$ and $x^{\alpha_-}\mid\mathcal{U}$, hence φ must be 0 because of $\gcd_{p,q}(\mathcal{U},\mathcal{V}) = 1$. This means, we can choose $\gamma := 0$. In the second case we have $\alpha_- = 0$ and $x^{\min(\alpha_+,\varphi)}\mid\mathcal{U}$, because of $\epsilon\mathcal{V} = x^\varphi\cdot y^\psi\cdot q^\varphi\cdot p^\psi\cdot\epsilon\mathcal{V}^*$. Again φ must be 0, and again we can choose $\gamma := 0$. Analogously, if $\beta \neq 0$ we can choose $\delta := 0$.

(ii) Assume that $\alpha = 0$ and $\beta \neq 0$, hence $\psi = 0$ by (i). For $\varphi > 0$, evaluating equation (10) at $x = 0$ results in

$$z\cdot y^{\beta_+}\cdot A^*(0,y)\cdot\mathcal{V}^*(0,y)\cdot\mathcal{U}(0,py) - y^{\beta_-}\cdot B^*(0,y)\cdot q^\varphi\cdot\mathcal{V}^*(0,py)\cdot\mathcal{U}(0,y) = 0. \quad (11)$$

In order to evaluate (11) at $y = 0$, note that $P \in F[x,y]$ being bibasic monic does not necessarily imply that $P(0,y) \in F[y]$ is p-monic. To overcome this problem, let us consider the p-monic decompositions of $\mathcal{U}(0,y)$ and $\mathcal{V}^*(0,y)$, say $\mathcal{U}(0,y) = u\cdot y^{\beta_u}\cdot\bar{U}(y)$

and $\mathcal{V}^*(0,y) = v \cdot y^{\beta_v} \cdot \bar{V}(y)$, respectively. Now, dividing equation (11) by $\mathcal{U}(0,y) \cdot \mathcal{V}^*(0,y) \neq 0$ leads to

$$z \cdot y^{\beta_+} \cdot A^*(0,y) \cdot p^{\beta_u} \cdot \frac{\bar{U}(py)}{\bar{U}(y)} - y^{\beta_-} \cdot B^*(0,y) \cdot q^{\varphi} \cdot p^{\beta_v} \cdot \frac{\bar{V}(py)}{\bar{V}(y)} = 0. \tag{12}$$

Additionally, let the p-monic decompositions of $A^*(0,y)$ and $B^*(0,y)$ be given by $A^*(0,y) = a_y \cdot y^{\beta_a} \cdot \bar{A}(y)$ and $B^*(0,y) = b_y \cdot y^{\beta_b} \cdot \bar{B}(y)$, respectively. Then the powers $y^{\beta_a+\beta_+}$ and $y^{\beta_b+\beta_-}$ must be equal, and after cancellation equation (12) at $y=0$ turns into

$$z \cdot a_y \cdot p^{\beta_u} - b_y \cdot q^{\varphi} \cdot p^{\beta_v} = 0.$$

This means, we obtain as a condition for $\varphi > 0$ that $z = q^{\varphi} \cdot p^{\mu} \cdot b_y/a_y$ with $\mu \in \mathbb{Z}$. Hence, in this case we choose $\gamma := \varphi$, i.e., we set γ to this q-power if z has this particular form, and $\gamma := 0$ otherwise. Analogously, if $\alpha \neq 0$ and $\beta = 0$ we define $\delta := \psi > 0$, if $z = p^{\psi} \cdot q^{\nu} \cdot b_x/a_x$ with $\nu \in \mathbb{Z}$, and $\delta := 0$ otherwise.

(iii) Finally, for the case $\alpha = 0 = \beta$ similar reasoning as in case (ii) leads to the conditions

$$q^{\varphi} \cdot p^{\mu} \cdot b_y/a_y = z = p^{\psi} \cdot q^{\nu} \cdot b_x/a_x, \tag{13}$$

for $\varphi > 0$ or $\psi > 0$, and $\mu, \nu \in \mathbb{Z}$. Thus, if both conditions (13) are satisfied we choose $\gamma := \varphi$ and $\delta := \psi$, and otherwise $\gamma = \delta := 0$.

The remaining steps of Algorithm Telescope$_{p,q}$ now are explained as follows. Once again, employing the GP representation for the bibasic monic quotient of ρ,

$$\frac{A^*}{B^*} = \frac{\epsilon P^*}{P^*} \cdot \frac{Q^*}{\epsilon R^*},$$

it is easily seen that equation (8) can be written as

$$z \cdot q^{-\gamma} \cdot p^{-\delta} \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot \frac{Q^*}{\epsilon R^*} \cdot \epsilon U - x^{\alpha_-} \cdot y^{\beta_-} \cdot U = x^{\gamma+\alpha_-} \cdot y^{\delta+\beta_-} \cdot P^*. \tag{14}$$

Because of relative primeness of certain polynomials, we observe that $x^{\alpha_-} \mid U$, $y^{\beta_-} \mid U$, and $\epsilon R^* \mid \epsilon U$. Hence by defining Y by the relation

$$U = x^{\alpha_-} \cdot y^{\beta_-} \cdot q^{-\alpha_-} \cdot p^{-\beta_-} \cdot R^* \cdot Y,$$

the task to solve equation (8) for U reduces to solve

$$z \cdot q^{-\gamma} \cdot p^{-\delta} \cdot x^{\alpha_+} \cdot y^{\beta_+} \cdot Q^* \cdot \epsilon Y - x^{\alpha_-} \cdot y^{\beta_-} \cdot q^{-\alpha_-} \cdot p^{-\beta_-} \cdot R^* \cdot Y = x^{\gamma} \cdot y^{\delta} \cdot P^* \tag{15}$$

for $Y \in F[x,y]$. By definition (3) of P, Q, and R, equation (15) immediately turns into the bibasic key equation (4),

$$Q \cdot \epsilon Y - R \cdot Y = P.$$

Finally, from $U/V = R \cdot Y/P$, again by definition (3), it follows directly that

$$g_k = \frac{R(q^k, p^k) \cdot Y(q^k, p^k)}{P(q^k, p^k)} \cdot f_k$$

as in (5) actually is a solution of the bibasic hypergeometric telescoping problem (1). This completes the proof of the correctness of Algorithm Telescope$_{p,q}$.

3 Degree Setting for Solving the Bibasic Key Equation

To solve the bibasic key equation

$$P = Q \cdot \epsilon Y - R \cdot Y \tag{16}$$

we first have to determine degree bounds d_1 and d_2, say, for the solution polynomial $Y \in F[x,y]$ with respect to x and y, respectively, as shown in Theorem 2 below. Then we put

$$Y(x,y) := \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} y_{i,j} \cdot x^i \cdot y^j$$

with undetermined $y_{i,j}$ and solve (16) for the $y_{i,j}$ by equating to zero all coefficients of $x^i y^j$ in the equation

$$P - Q \cdot \epsilon Y + R \cdot Y = 0,$$

which corresponds to solving a system of linear equations.

Theorem 2. *Let $l_Q^x(y)$, $l_Q^y(x)$, $l_R^x(y)$, and $l_R^y(x)$ denote the leading coefficient polynomials of Q and R with respect to x and y, respectively. Let $QR^+ := Q + R$ and $QR^- := Q - R$. Then bounds for $\deg_x(Y)$ and $\deg_y(Y)$ are given by:*

(A_x) If $\deg_x(QR^+) \neq \deg_x(QR^-)$, then

$$\deg_x(Y) \leq \max\{\deg_x(P) - \max\{\deg_x(QR^+), \deg_x(QR^-)\}, 0\}.$$

(A_y) If $\deg_y(QR^+) \neq \deg_y(QR^-)$, then

$$\deg_y(Y) \leq \max\{\deg_y(P) - \max\{\deg_y(QR^+), \deg_y(QR^-)\}, 0\}.$$

(B_x) If $\deg_x(QR^+) = \deg_x(QR^-)$, then

($B1_x$) if $\deg_x(Q) \neq \deg_x(R)$, then

$$\deg_x(Y) = \deg_x(P) - \deg_x(QR^+),$$

($B2_x$) if $\deg_x(Q) = \deg_x(R)$, then

($B2a_x$) if $l_R^x(y)/l_Q^x(y)$ is of the form $p^\mu \cdot q^\nu \cdot r(y)$ with $\mu, \nu \in \mathbb{N}$, and $r(y)$ a rational function with $r(0) = 1$, then

$$\deg_x(Y) \leq \max\{\deg_x(P) - \deg_x(QR^+), \nu\},$$

($B2b_x$) otherwise

$$\deg_x(Y) = \deg_x(P) - \deg_x(QR^+).$$

(B_y) If $\deg_y(QR^+) = \deg_y(QR^-)$, then

($B1_y$) if $\deg_y(Q) \neq \deg_y(R)$, then

$$\deg_y(Y) = \deg_y(P) - \deg_y(QR^+),$$

($B2_y$) if $\deg_y(Q) = \deg_y(R)$, then

($B2a_y$) if $l^y_R(x)/l^y_Q(x)$ is of the form $p^\mu \cdot q^\nu \cdot r(x)$ with $\mu, \nu \in \mathbb{N}$, and $r(x)$ a rational function with $r(0) = 1$, then

$$\deg_y(Y) \leq \max\{\deg_y(P) - \deg_y(QR^+), \mu\},$$

($B2b_y$) otherwise

$$\deg_y(Y) = \deg_y(P) - \deg_y(QR^+).$$

Proof. We rewrite the key equation to obtain

$$2\,P = QR^+ \cdot (\epsilon Y - Y) + QR^- \cdot (\epsilon Y + Y). \tag{17}$$

Cases (A_x) and (A_y) follow immediately. Note that it might happen that

$$\deg_x(QR^+) > \deg_x(P) \text{ and } \deg_x(QR^-) = \deg_x(P),$$

and simultaneously

$$\deg_y(QR^+) > \deg_y(P) \text{ and } \deg_y(QR^-) = \deg_y(P).$$

In this case, setting $\deg_x(Y) = \deg_y(Y) = 0$ could yield a solution, since $\epsilon Y - Y = 0$ then.

For Case ($\mathrm{B1}_x$) let $a := \deg_x(Q)$, $c := \deg_x(Y)$, and let $l^x_Y(y)$ denote the leading coefficient polynomial of Y with respect to x. Assume that $\deg_x(Q) > \deg_x(R)$. Then (17) gives

$$\begin{aligned} 2\,P(x,y) &= (l^x_Q(y)\,x^a + \ldots) \cdot [(l^x_Y(py)\,q^c - l^x_Y(y))\,x^c + \ldots] \\ &\quad + (l^x_Q(y)\,x^a + \ldots) \cdot [(l^x_Y(py)\,q^c + l^x_Y(y))\,x^c + \ldots] \\ &= 2\,l^x_Q(y)\,l^x_Y(py)\,q^c\,x^{a+c} + \ldots. \end{aligned} \tag{18}$$

Clearly, the coefficient of x^{a+c} in (18) will never vanish. Therefore we have

$$\deg_x(Y) = \deg_x(P) - \deg_x(Q).$$

Including the case $\deg_x(Q) < \deg_x(R)$, we obtain

$$\deg_x(Y) = \deg_x(P) - \max\{\deg_x(Q), \deg_x(R)\} = \deg_x(P) - \deg_x(QR^+).$$

Analogous reasoning leads to Case ($\mathrm{B1}_y$).

For Case ($\mathrm{B2}_x$) we similarly observe that

$$\begin{aligned} 2\,P(x,y) &= [(l^x_Q(y) + l^x_R(y))\,x^a + \ldots] \cdot [(l^x_Y(py)\,q^c - l^x_Y(y))\,x^c + \ldots] \\ &\quad + [(l^x_Q(y) - l^x_R(y))\,x^a + \ldots] \cdot [(l^x_Y(py)\,q^c + l^x_Y(y))\,x^c + \ldots] \\ &= 2\,[l^x_Q(y)\,l^x_Y(py)\,q^c - l^x_R(y)\,l^x_Y(y)]\,x^{a+c} + \ldots. \end{aligned} \tag{19}$$

Now we no longer have the guarantee that the coefficient of x^{a+c} in (19) does not vanish, but it is easily seen that this happens only for

$$q^c = \frac{l^x_R(y)}{l^x_Q(y)} \cdot \frac{l^x_Y(y)}{l^x_Y(py)}. \tag{20}$$

Note that $l^x_Y(y)$ is actually *not* known. However, for any non-zero polynomial $h(y) = h_0 + h_1\,y + \cdots + h_d\,y^d$, the quotient $h(y)/h(py)$ is of the form $p^{-m} \cdot s(y)$, where $s(y)$ is a rational function with $s(0) = 1$ and m is the zero-root multiplicity of $h(y)$. Hence, the rightmost fraction in (20) may eliminate only positive integer powers of p and a rational function of y but never introduce a power of q. This proves Case ($\mathrm{B2a}_x$), and after interchanging x and p with y and q, respectively, also Case ($\mathrm{B2a}_y$).

On the other hand, if the coefficient of x^{a+c} in (19) does not vanish, we obtain Case ($\mathrm{B2b}_x$) and analogously Case ($\mathrm{B2b}_y$). $\square$

4 Applications

In this section we shall illustrate the method of bibasic hypergeometric telescoping using the author's Mathematica implementation `qTelescope`, which is a bibasic extension of a q-analogue of Gosper's algorithm originally described in Paule and Riese [9].

Let the *q-shifted factorial* of $a \in F$ be defined as usual (see, e.g. Gasper and Rahman [5]) by

$$(a;q)_k := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{k-1}), & \text{if } k>0,\\ 1, & \text{if } k=0,\\ \left[(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{k})\right]^{-1}, & \text{if } k<0,\end{cases}$$

and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$

where products of q-shifted factorials will be abbreviated by

$$(a_1,a_2,\ldots,a_n;q)_k := (a_1;q)_k\,(a_2;q)_k\cdots(a_n;q)_k.$$

In the present implementation we allow as summand any bibasic hypergeometric sequence $(f_k)_{k\in\mathbb{Z}}$ of the form

$$\begin{aligned} f_k = {} & \frac{\prod_r (C_r\, q^{(c_r i_r)k+d_r}; q^{i_r})_{a_r k+b_r}}{\prod_s (D_s\, q^{(v_s j_s)k+w_s}; q^{j_s})_{t_s k+u_s}} \cdot \frac{\prod_r (C'_r\, p^{(c'_r i'_r)k+d'_r}; p^{i'_r})_{a'_r k+b'_r}}{\prod_s (D'_s\, p^{(v'_s j'_s)k+w'_s}; p^{j'_s})_{t'_s k+u'_s}} \\ & \times R(q^k,p^k)\cdot q^{\alpha\binom{k}{2}}\cdot p^{\beta\binom{k}{2}}\cdot z^k, \end{aligned}$$

with

C_r, D_s	power products in $K(p)$,
C'_r, D'_s	power products in $K(q)$,
a_r, t_s, a'_r, t'_s	specific integers (i.e., integers free of any parameters),
b_r, u_s, b'_r, u'_s	integer parameters free of k, or $\pm\infty$ if a_r (resp. t_s, a'_r, t'_s) $=0$,
c_r, v_s, c'_r, v'_s	specific integers,
d_r, w_s, d'_r, w'_s	integer parameters free of k,
i_r, j_s, i'_r, j'_s	specific non-zero integers,
R	a rational function in $F(q^k,p^k)$ such that the denominator factors completely into a product of terms of the form $(1-D\,q^{vk+w})$ and $(1-D'p^{v'k+w'})$,
α, β	specific integers, and
z	a rational function in F.

For the actual computation of the GP representation let $\rho(x,y)$ denote the possibly non-reduced rational representation of the summand f_k. It is obvious from the input specification that ρ can always be converted into the form

$$\begin{aligned} \rho(x,y) &= \frac{(\epsilon\bar{P})(x,y)}{\bar{P}(x,y)} \cdot \frac{\prod_i (1-\Gamma_i\, x^{\gamma_i})}{\prod_j (1-\Delta_j\, x^{\delta_j})} \cdot \frac{\prod_i (1-\Gamma'_i\, y^{\gamma'_i})}{\prod_j (1-\Delta'_j\, y^{\delta'_j})} \cdot x^{\bar{\alpha}} \cdot y^{\bar{\beta}} \cdot \bar{z} \\ &= \frac{(\epsilon\bar{P})(x,y)}{\bar{P}(x,y)} \cdot \frac{\bar{A}(x,y)}{\bar{B}(x,y)} \cdot x^{\bar{\alpha}} \cdot y^{\bar{\beta}} \cdot \bar{z}, \end{aligned}$$

where $\bar{P} \in F[x,y]$ is bibasic monic and satisfies $\gcd_{p,q}(\bar{P},\bar{A}) = 1 = \gcd_{p,q}(\epsilon\bar{P},\bar{B})$; the $\Gamma_i, \Delta_j, \Gamma'_i, \Delta'_j$ are power products in F, the $\gamma_i, \delta_j, \gamma'_i, \delta'_j$ are positive integers, $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}$, and $\bar{z} \in F$.

Concerning Algorithm V*MULT, it is clear from above that any $\bar{P} \neq 1$ will actually contribute to $[P_1]^{\frac{1}{P},q}$ and thus can be treated separately. Due to our input restrictions — this is the reason for admitting only power products instead of arbitrary rational functions — it is possible to find n in step (i) of Algorithm V*MULT simply by comparing all factors in $\bar{A}$ and $\bar{B}$ as already mentioned.

Furthermore, since $\bar{A}$ and $\bar{B}$ are both products of a q-monic and a p-monic polynomial, they will never contribute to b_x/a_x and b_y/a_y. Thus, b_x/a_x and b_y/a_y are in any case integer powers of q and p, respectively, coming from $\epsilon\bar{P}/\bar{P}$. Therefore, they do not take influence on the computation of γ and δ at all.

4.1 Bibasic Summation Formulas

In 1989, Gasper [3] derived the indefinite bibasic summation formula

$$\begin{aligned}\sum_{k=0}^{n} f_k &= \sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \, \frac{(a,b;p)_k\,(c,a/bc;q)_k}{(q,aq/b;q)_k\,(ap/c,bcp;p)_k}\, q^k \\ &= \frac{(ap,bp;p)_n\,(cq,aq/bc;q)_n}{(q,aq/b;q)_n\,(ap/c,bcp;p)_n} = g_n \end{aligned} \tag{21}$$

by showing that g_k is a bibasic hypergeometric solution of the equation $f_k = g_k - g_{k-1}$, however, without revealing how to come up with g_k. With our implementation the job of finding g_k is left to the computer.

```
In[1]:= <<qTelescope.m

Out[1]= Axel Riese's qTelescope implementation version 2.0 loaded

In[2]:= qTelescope[(1-a p^k q^k) (1-b p^k/q^k) qfac[a,p,k] qfac[b,p,k] qfac[c,q,k] *
              qfac[a/b/c,q,k] q^k / ((1-a) (1-b) qfac[q,q,k] qfac[a q/b,q,k] *
              qfac[a p/c,p,k] qfac[b c p,p,k]), {k, 0, n}]

                                          a q
        qfac[a p, p, n] qfac[b p, p, n] qfac[---, q, n] qfac[c q, q, n]
                                          b c
Out[2]= ----------------------------------------------------------------
             a p                                            a q
        qfac[---, p, n] qfac[b c p, p, n] qfac[q, q, n] qfac[---, q, n]
             c                                              b
```

Applying the same argumentation, Gasper and Rahman [4] generalized (21) to

$$\begin{aligned}&\sum_{k=-m}^{n} (1-adp^kq^k)(1-bp^k/dq^k)\,\frac{(a,b;p)_k\,(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k\,(adp/c,bcp/d;p)_k}\, q^k \\ &\quad = \frac{(1-a)(1-b)(1-c)(1-ad^2/bc)}{d(1-c/d)(1-ad/bc)} \\ &\qquad \times \left\{ \frac{(ap,bp;p)_n\,(cq,ad^2q/bc;q)_n}{(dq,adq/b;q)_n\,(adp/c,bcp/d;p)_n} - \frac{(c/ad,d/bc;p)_{m+1}\,(1/d,b/ad;q)_{m+1}}{(1/c,bc/ad^2;q)_{m+1}\,(1/a,1/b;p)_{m+1}} \right\}. \end{aligned} \tag{22}$$

Obviously, (21) is the case $d = 1$, $m = 0$ of (22). Since the output of `qTelescope` for identity (22) is quite lengthy, here we shall consider only the case $m = -1$ after dividing the summand by the constant fraction on the right hand side. Of course, the algorithm works for symbolic m as well.

```
In[3]:= qTelescope[(1-a d p^k q^k) (1-b/d p^k/q^k) qfac[a,p,k] qfac[b,p,k] *
                  qfac[c,q,k] qfac[a d^2/b/c,q,k] q^k d (1-c/d) (1-a d/b/c) /
                 (qfac[d q,q,k] qfac[a d q/b,q,k] qfac[a d p/c,p,k] *
                  qfac[b c p/d,p,k] (1-a) (1-b) (1-c) (1-a d^2/b/c)), {k, 1, n}]
```

```
                                                                        2
                                                                     a d  q
Out[3]= -1 + (qfac[a p, p, n] qfac[b p, p, n] qfac[c q, q, n] qfac[------, q, n]) /
                                                                      b c

         b c p            a d p                               a d q
   (qfac[-----, p, n] qfac[-----, p, n] qfac[d q, q, n] qfac[-----, q, n])
           d                c                                   b
```

4.2 Bibasic Matrix Inverses

Al-Salam and Verma [2] showed that the triangular matrices $H = (h_{n,k})$ and $G = (g_{k,n})$, where

$$h_{n,k} = \frac{(-1)^{n+k}\,(hqp^n;q)_{n-1}\,(1-hq^kp^k)}{(p;p)_{n-k}\,(hqp^n;q)_k}$$

and

$$g_{k,n} = \frac{(hp^nq^n;q)_{k-n}}{(p;p)_{k-n}}\,p^{\binom{k-n}{2}}$$

are inverse to each other. This result is equivalent to the fact that

$$\sum_{k=m}^{n} h_{n,k}\cdot g_{k,m} = \delta_{n,m}, \tag{23}$$

where $\delta_{n,m}$ denotes the Kronecker symbol. Running the algorithm we obtain:

```
In[4]:= qTelescope[(-1)^(n+k) qfac[h q p^n,q,n-1] (1-h q^k p^k) *
                  qfac[h p^m q^m,q,k-m] p^Binomial[k-m,2] /
                 (qfac[p,p,n-k] qfac[h q p^n,q,k] qfac[p,p,k-m]), {k, m, n}]

Out[4]= {0, {-m + n != 0}}
```

This means, we algorithmically proved identity (23) for $m \neq n$, but evaluation failed for $m = n$. However, it is easily seen that $h_{n,n}\cdot g_{n,n} = 1$, which completes the proof.

These matrices were used in a slightly modified form also by Gessel and Stanton [6] in the derivation of a family of q-Lagrange inversion formulas.

Al-Salam and Verma [2] employed the fact that the n-th q-difference of a polynomial of degree less than n is equal to zero, to show that

$$\left(1-\frac{a}{q}\right)\sum_{k=0}^{n}\frac{(-1)^k\,(ap^k;q)_{n-1}}{(p;p)_k\,(p;p)_{n-k}}\,p^{\binom{k}{2}} = \delta_{n,0}. \tag{24}$$

Unfortunately, for $d_k := (ap^k;q)_{n-1}$, we find that

$$\frac{d_{k+1}}{d_k} = \frac{(1-ap^{k+1})(1-ap^{k+1}q)\cdots(1-ap^{k+1}q^{n-2})}{(1-ap^k)(1-ap^kq)\cdots(1-ap^kq^{n-2})}$$

is a rational function of q^k and p^k only for fixed n. Therefore d_k is *not* a valid input for the algorithm. To overcome the problem, we replace k, n, and a in (24) by $k-m$, $n-m$, and $a^{-1}p^mq^{1-n}$, respectively, such that (24) turns into the orthogonality relation

$$c_{n,m} \sum_{k=m}^{n} a_{n,k} \cdot b_{k,m} = \delta_{n,m} \tag{25}$$

with

$$\begin{aligned} c_{n,m} &= (1-a^{-1}p^mq^{-n})\, a^{1+m-n}\, q^{\binom{m+1}{2}-\binom{n}{2}}, \\ a_{n,k} &= \frac{(ap^{-k};q)_n}{(p;p)_{n-k}}\, (-1)^{1+k+n}\, p^{\binom{n-k}{2}}, \\ b_{k,m} &= \frac{p^{-k(m+1)}}{(p;p)_{k-m}\,(ap^{-k};q)_{m+1}}. \end{aligned}$$

Note that $a_{n,k}$ and $b_{k,m}$ still do not fit into the input specification of the algorithm. For $A = (a_{n,k})$, $B = (b_{k,m})$, and $C = (c_{n,m})$, relation (25) could be rewritten as $A \cdot B = \operatorname{diag}(C)^{-1}$, showing that the matrix $\operatorname{diag}(C) \cdot A = (c_{n,n} \cdot a_{n,k})$ is inverse to the matrix B. Since inverse matrices commute, (25) is equivalent to

$$\sum_{k=m}^{n} c_{k,k} \cdot b_{n,k} \cdot a_{k,m} = \delta_{n,m},$$

or, in other words

$$\sum_{k=m}^{n} \frac{(-1)^{k+m}\,(1-ap^{-k}q^k)\,(ap^{-m};q)_k}{(p;p)_{n-k}\,(p;p)_{k-m}\,(ap^{-n};q)_{k+1}}\, p^{\binom{k-m}{2}-n(k+1)+k(m+1)} = \delta_{n,m}. \tag{26}$$

```
In[5]:= qTelescope[(-1)^(k+m) (1-a q^k/p^k) qfac[a/p^m,q,k] *
                p^(Binomial[k-m,2]-n(k+1)+k(m+1)) /
                (qfac[p,p,n-k] qfac[p,p,k-m] qfac[a/p^n,q,k+1]), {k, m, n}]

Out[5]= {0, {-m + n != 0}}
```

For $m=0$, (26) reduces to

$$\sum_{k=0}^{n} \frac{(-1)^k\,(1-ap^{-k}q^k)\,(a;q)_k}{(p;p)_{n-k}\,(p;p)_k\,(ap^{-n};q)_{k+1}}\, p^{\binom{n-k}{2}} = \delta_{n,0}.$$

```
In[6]:= qTelescope[(-1)^k (1-a q^k/p^k) qfac[a,q,k] p^Binomial[n-k,2] /
                (qfac[p,p,n-k] qfac[p,p,k] qfac[a/p^n,q,k+1]), {k, 0, n}]

Out[6]= {0, {n != 0}}
```

Proceeding in the same way, we can prove the bibasic identity (cf. Gasper [3])

$$\left(1-\frac{a}{q}\right)\left(1-\frac{b}{q}\right) \sum_{k=0}^{n} \frac{(-1)^k\,(ap^k, bp^{-k};q)_{n-1}\,(1-ap^{2k}/b)}{(p;p)_k\,(p;p)_{n-k}\,(ap^k/b;p)_{n+1}}\, p^{k(n-1)+\binom{n-k}{2}} = \delta_{n,0},$$

by transforming it into the equivalent version

$$\left(1-\frac{b}{a}\right)\sum_{k=0}^{n}(1-ap^{-k}q^k)(1-bp^kq^k)\,\frac{(-1)^k\,(a,b;q)_k\,(bp^{k+1}/a;p)_{n-1}}{(p;p)_k\,(p;p)_{n-k}\,(ap^{-n},bp^n;q)_{k+1}}\,p^{\binom{n-k}{2}}=\delta_{n,0}.$$

```
In[7]:= qTelescope[(1-b/a) (1-a q^k/p^k) (1-b p^k q^k) (-1)^k qfac[a,q,k] *
                   qfac[b,q,k] qfac[b/a p^(k+1),p,n-1] p^Binomial[n-k,2] /
                  (qfac[p,p,k] qfac[p,p,n-k] qfac[a/p^n,q,k+1] qfac[b p^n,q,k+1]),
                   {k, 0, n}]

Out[7]= {0, {n != 0}}
```

4.3 Open Problems

With the input specification described above we actually have not taken into account that a bibasic hypergeometric summand f_k could involve q-shifted factorials with mixed bases such as $(a;p^iq^j)_k$ for $i,j\in\mathbb{Z}$ as well. However, since to our knowledge applications of this type have not arisen in practice up to now, this feature has not been implemented yet.

For the sake of simplicity we restricted ourselves to discuss in detail the bibasic case. Nevertheless, the presented approach should easily extend to the *multi-basic* case, i.e., to sequences being hypergeometric in independent bases $q_1,\ldots,q_m$.

So far we found only one single bibasic example in the literature which we could not handle with our machinery, namely Gasper's [3] transformation formulas

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1-ap^kq^k}{1-a}\,\frac{(a;p)_k\,(c/b;q)_k}{(q;q)_k\,(abp;p)_k}\,b^k\\
&\quad=\frac{1-c}{1-b}\sum_{k=0}^{\infty}\frac{(ap;p)_k\,(c/b;q)_k}{(q;q)_k\,(abp;p)_k}\,(bq)^k\\
&\quad=\frac{1-c}{1-abp}\sum_{k=0}^{\infty}\frac{(ap;p)_k\,(cq/b;q)_k}{(q;q)_k\,(abp^2;p)_k}\,b^k\\
&\quad=\frac{(1-c)\,(ap;p)_\infty}{(1-b)\,(abp;p)_\infty}\sum_{k=0}^{\infty}\frac{(b;p)_k\,(cqp^k;q)_\infty}{(p;p)_k\,(bqp^k;p)_\infty}\,(ap)^k,
\end{aligned}$$

when $\max(|p|,|q|,|ap|,|b|)<1$.

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