Towards the Albertson conjecture

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Submitted: Sep 2, 2009; Accepted: May 7, 2010; Published: May 14, 2010
Mathematics Subject Classification: 05C10, 05C15

Abstract

Albertson conjectured that if a graph G has chromatic number r, then the crossing number of G is at least as large as the crossing number of K_r, the complete graph on r vertices. Albertson, Cranston, and Fox verified the conjecture for $r \leqslant 12$. In this paper we prove it for $r \leqslant 16$.

Dedicated to the memory of Michael O. Albertson.

1 Introduction

Graphs in this paper are without loops and multiple edges. Every planar graph is four-colorable by the Four Color Theorem [2, 24]. The efforts to solve the Four Color Problem had a great effect on the development of graph theory, and FCT is one of the most important theorems of the field.

The *crossing number* of a graph G, denoted $\textsc{cr}(G)$, is the minimum number of edge crossings in a drawing of G in the plane. It is a natural relaxation of planarity, see [25] for a survey. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors in a proper coloring of G. The Four Color Theorem states: if $\textsc{cr}(G) = 0$, then $\chi(G) \leqslant 4$. Oporowski and Zhao [18] proved that every graph with crossing number at most two is 5-colorable. Albertson et al. [5] showed that if $\textsc{cr}(G) \leqslant 6$, then $\chi(G) \leqslant 6$. It

*Research is supported by OTKA Grants PD 75837 and K 76099, and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

†Research is supported by OTKA T 038397 and T 046246.

was observed by Schaefer that if $\text{CR}(G) = k$, then $\chi(G) = O(\sqrt[4]{k})$, and this is the correct order of magnitude [4].

Graphs with chromatic number r do not necessarily contain K_r as a subgraph, they can have clique number 2, see [27]. The Hajós conjecture proposed that graphs with chromatic number r contain a *subdivision* of K_r. This conjecture, whose origin is unclear but attributed to Hajós, turned out to be false for $r \geqslant 7$. Also, it was shown by Erdős and Fajtlowicz [9] that almost all graphs are counterexamples. Albertson posed the following

Conjecture 1 *If $\chi(G) = r$, then $\text{CR}(G) \geqslant \text{CR}(K_r)$.*

This statement is weaker than Hajós' conjecture: if G contains a subdivision of K_r, then $\text{CR}(G) \geqslant \text{CR}(K_r)$.

For $r = 5$, Albertson's conjecture is equivalent to the Four Color Theorem. Oporowski and Zhao [18] verified it for $r = 6$. Albertson, Cranston, and Fox [4] proved it for $r \leqslant 12$. In this note, we take one more little step.

Theorem 2 *For $r \leqslant 16$, if $\chi(G) = r$, then $\text{CR}(G) \geqslant \text{CR}(K_r)$.*

In their proof, Albertson, Cranston and Fox combined lower bounds for the number of edges of r-critical graphs, and lower bounds on the crossing number of graphs with given number of vertices and edges. Our proof is very similar, but we use better lower bounds in both cases.

Albertson et al. proved that any minimal counterexample to Conjecture 1 should have less than $4r$ vertices. We slightly improve this result as follows.

Lemma 3 *If G is an n-vertex, r-critical graph with $n \geqslant 3.57r$, then $\text{CR}(G) \geqslant \text{CR}(K_r)$.*

In Section 2, we review lower bounds for the number of edges of r-critical graphs. In Section 3, we discuss lower bounds on the crossing number. In Section 4, we combine these two bounds to obtain the proof of Theorem 2. In Section 5, we prove Lemma 3.

Let n always denote the number of vertices of G. In notation and terminology, we follow Bondy and Murty [6]. In particular, the *join* of two disjoint graphs G and H, denoted $G \vee H$, arises by adding all edges between vertices of G and H. A vertex v is of *full degree*, if it has degree $n - 1$. If a graph G contains a subdivision of H, then G *contains a topological* H. A vertex v is *adjacent to a vertex set* X means that each vertex of X is adjacent to v.

2 Color-critical graphs

A graph G is r-*critical*, if $\chi(G) = r$, but all proper subgraphs of G have chromatic number less than r. In what follows, let G denote an r-critical graph with n vertices and m edges. Since G is r-critical, every vertex has degree at least $r - 1$, therefore, $2m \geqslant (r - 1)n$. The value $2m - (r - 1)n$ is the *excess* of G. For $r \geqslant 3$, Dirac [7] proved the following: if G is not complete, then $2m \geqslant (r - 1)n + (r - 3)$. For $r \geqslant 4$, Dirac [8] gave a characterization of r-critical graphs with excess $r-3$. For a positive integer r, $r \geqslant 3$, let Δ_r be the following family of graphs. For any graph in the family, let the vertex set consist of three non-empty, pairwise disjoint sets A, B_1, B_2 and two additional vertices a and b. Here, $|B_1| + |B_2| = |A| + 1 = r - 1$. The sets A and $B_1 \cup B_2$ both span cliques, a is connected to $A \cup B_1$ and b is connected to $A \cup B_2$. See Figure 1. Graphs in Δ_r are called Hajós graphs of order $2r-1$. Observe, that these graphs have chromatic number r, and they contain a topological K_r. Hence they satisfy Hajós' conjecture.

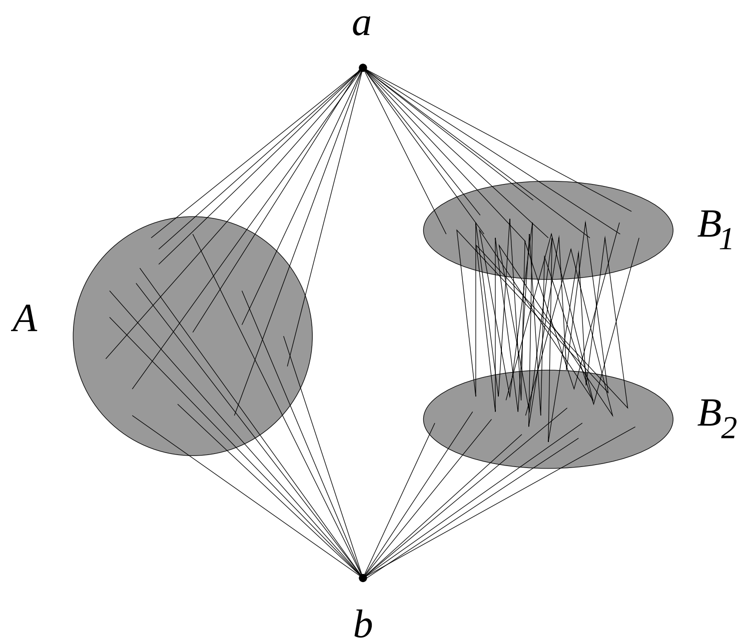

Figure 1: The family Δ_r

Gallai [10] proved that any r-critical graph with at most $2r-2$ vertices is the join of two smaller graphs. Therefore, the complement of any such graph is disconnected. Based on this observation, Gallai proved that non-complete r-critical graphs on at most $2r-2$ vertices have much larger excess than in Dirac's result.

Lemma 4 [10] *Let r, p be integers, $r \geqslant 4$ and $2 \leqslant p \leqslant r-1$. If G is an r-critical graph with n vertices and m edges, where $n = r+p$, then $2m \geqslant (r-1)n + p(r-p) - 2$. Equality holds if and only if G is the join of K_{r-p-1} and $G \in \Delta_{p+1}$.*

Since every G in Δ_{p+1} contains a topological K_{p+1}, the join of K_{r-p-1} and G contains a topological K_r. This yields a slight improvement for our purposes.

Corollary 5 *Let r, p be integers, $r \geqslant 4$ and $2 \leqslant p \leqslant r-1$. If G is an r-critical graph with n vertices and m edges, where $n = r+p$, and G does not contain a topological K_r, then $2m \geqslant (r-1)n + p(r-p) - 1$.*

We call the bound given by Corollary 5 *the Gallai bound*.

For $r \geqslant 3$, let $\mathcal{E}_r$ denote the family of the following graphs G. The vertex set of any G consists of four non-empty pairwise disjoint sets A_1, A_2, B_1, B_2, and one additional vertex c. Here $|B_1|+|B_2| = |A_1|+|A_2| = r-1$ and $|A_2|+|B_2| \leqslant r-1$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. The sets A and B each induce a clique in G. The vertex c is connected to $A_1 \cup B_1$. A vertex a in A is adjacent to a vertex b in B if and only if $a \in A_2$ and $b \in B_2$.

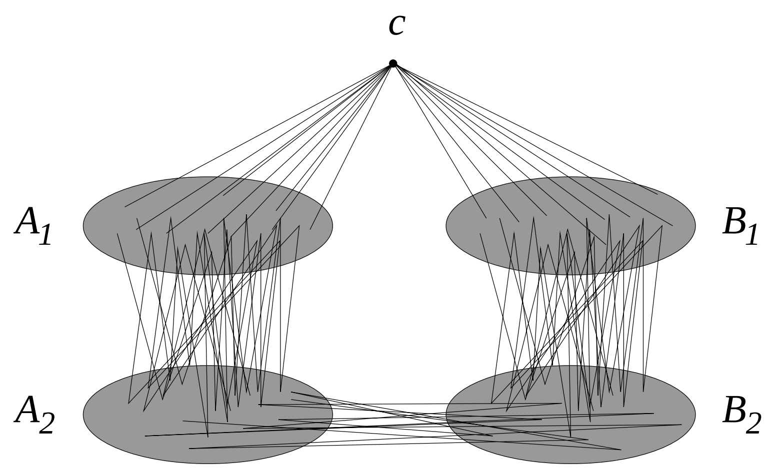

Figure 2: The family $\mathcal{E}_r$

Observe, that $\mathcal{E}_r \supset \Delta_r$, and every graph G in $\mathcal{E}_r$ is r-critical with $2r-1$ vertices. Kostochka and Stiebitz [15] improved Dirac's bound as follows.

Lemma 6 [15] *Let r be a positive integer, $r \geqslant 4$, and let G be an r-critical graph. If G is neither K_r nor a member of $\mathcal{E}_r$, then $2m \geqslant (r-1)n + (2r-6)$.*

Corollary 7 *Let r be a positive integer, $r \geqslant 4$, and let G be an r-critical graph. If G does not contain a topological K_r, then $2m \geqslant (r-1)n + (2r-6)$.*

Proof: We show that any member of $\mathcal{E}_r$ contains a topological K_r. The sets A and B both span a complete graph on $r-1$ vertices. We only have to show that vertex c is connected to A_2 or B_2 by vertex-disjoint paths. To see this, we observe that $|A_2|$ or $|B_2|$ is the smallest of $\{|A_1|, |A_2|, |B_1|, |B_2|\}$. Indeed, if $|B_1|$ was the smallest, then $|A_2| > |B_1|$ implies $|A_2|+|B_2| > |B_1|+|B_2| = r-1$ contradicting our assumption. We may assume that $|A_2|$ is the smallest. Now c is adjacent to A_1, and there is a matching of size $|A_2|$ between B_1 and B_2 and between B_2 and A_2. Therefore, we can find a set S of disjoint paths from c to A_2. In this way, $A \cup c \cup S$ is a topological r-clique. $\square$

The bound in Corollary 7 is the Kostochka, Stiebitz bound, or KS-bound for short.

In what follows, we obtain a complete characterization of r-critical graphs on $r+3$ or $r+4$ vertices.

Lemma 8 *For $r \geqslant 8$, there are precisely two r-critical graphs on $r+3$ vertices. They can be constructed from two 4-critical graphs on seven vertices by adding vertices of full degree.*

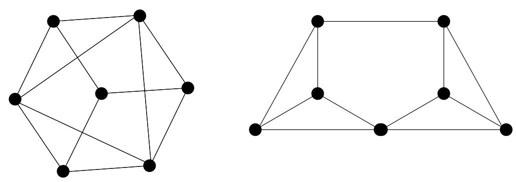

Figure 3: The two 4-critical graphs on seven vertices

Proof: The proof is by induction on r. For the base case $r = 8$, there are precisely two 8-critical graphs on 11 vertices, see Royle's complete search [22].

Let G be an r-critical graph with $r \geqslant 9$ and $n = r + 3 \geqslant 12$. The minimum degree is at least $r - 1$, and $r - 1 = n - 4$. If G has a vertex v of full degree, then we use induction. So we may assume that every vertex in $\overline{G}$, the complement of G, has degree 1, 2, or 3. By Gallai's theorem, $\overline{G}$ is disconnected. Observe the following: if there are at least four independent edges in $\overline{G}$, then $\chi(G) \leqslant n - 4 = r - 1$, a contradiction. That is, there are at most three independent edges in $\overline{G}$. Therefore, $\overline{G}$ has two or three components. If there is a triangle in the complement, then we can save two colors. If there were two triangles, then $\chi(G) \leqslant n - 4 = r - 1$, a contradiction.

Assume that there are three components in $\overline{G}$. Since each degree is at least one, there are at least three independent edges. Therefore, there is no triangle in $\overline{G}$ and no path with three edges. That is, the complement consists of three stars. Since the degree is at most three and there are at least 12 vertices, there is only one possibility: $\overline{G} = K_{1,3} \cup K_{1,3} \cup K_{1,3}$, see Figure 4.

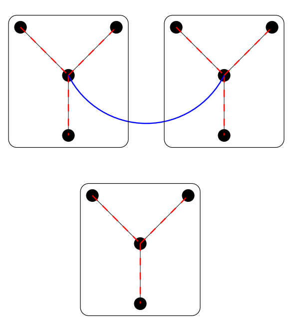

Figure 4: The complement and a removable edge

We have to check whether this concrete graph is indeed critical. Observe, that if we remove the edge connecting two centers of these stars, the chromatic number remains r. Therefore, our graph is not r-critical, a contradiction.

In the remaining case, $\overline{G}$ has two components H_1 and H_2. Since there are at most three independent edges, there is one in H_1 and two in H_2. It implies that H_1 has at most four vertices. Therefore, H_2 has at least eight vertices. Consider a spanning tree T of H_2 and remove two adjacent vertices of T, one of them being a leaf. It is easy to see that the remainder of T contains a path with three edges. Therefore, in total we found three independent edges of H_2, a contradiction. □

In Lemma 10, we characterize r-critical graphs on $r+4$ vertices. For that proof, we need the following result of Gallai.

Theorem 9 [10] *Let $r \geqslant 3$ and $n < \frac{5}{3}r$. If G is an r-critical, n-vertex graph, then it contains at least $\left\lceil \frac{3}{2}\left(\frac{5}{3}r - n\right)\right\rceil$ vertices of full degree.*

The existence of a vertex of full degree gives rise to an inductive proof of the following

Lemma 10 *For $r \geqslant 6$, there are precisely twenty-two r-critical graphs on $r+4$ vertices. Each of them can be constructed by adding vertices of full degree to a graph in the following list:*
- *the 3-critical graph on seven vertices,*
- *the four 4-critical graphs on eight vertices,*
- *the sixteen 5-critical graphs on nine vertices, or*
- *the 6-critical graph on ten vertices.*

Proof: For the base of induction, we use Royle's table again, see [22]. The full computer search shows that there are precisely twenty-two 6-critical graphs on ten vertices. One of them has three vertices of full degree, four of them has two, sixteen graphs have one vertex of full degree, and one graph has no such vertex. For the induction step, we use Theorem 9, and see that there are at least $r-6$ vertices of full degree. Since $r \geqslant 7$, there is always a vertex of full degree. We remove it, and use the induction hypothesis to finish the proof. □

There is an explicit list of twenty-one 5-critical graphs on nine vertices [22]. We had to check that each of those graphs contains a topological K_5. Mader [16] proved that any n-vertex graph with at least $3n-5$ edges contains the subdivision of K_5. We made a verification partly manually, partly using Mader's extremal result. Therefore, if we add $r-5$ vertices of full degree to any of these graphs, then the resulting graph contains a topological K_r. Also, the above mentioned 6-critical graph on ten vertices contains a topological K_6. These two observations imply the following

Corollary 11 *Any r-critical graph on at most $r+4$ vertices satisfy the Hajós conjecture.*

We believe that 4 can be replaced by any other constant in the above result.

Conjecture 12 *For every positive integer c, there exists a bound $r(c)$ such that for any r, where $r \geqslant r(c)$, any r-critical graph on $r+c$ vertices satisfies the Hajós conjecture.*

3 The crossing number

It follows from Euler's formula that a planar graph can have at most $3n-6$ edges. Suppose that G has more than $3n-6$ edges. By deleting crossing edges one by one, it follows by induction that for $n \geqslant 3$,

$$\text{cr}(G) \geqslant m - 3(n-2) \tag{1}$$

Pach et al. [19, 21] generalized this idea and proved the following lower bounds. Both of them holds for any graph G with n vertices and m edges, $n \geqslant 3$.

$$\text{cr}(G) \geqslant 7m/3 - 25(n-2)/3 \tag{2}$$

$$\text{cr}(G) \geqslant 4m - 103(n-2)/6 \tag{3}$$

$$\text{cr}(G) \geqslant 5m - 25(n-2) \tag{4}$$

Inequality (1) is the best for $m \leqslant 4(n-2)$,
(2) is the best for $4(n-2) \leqslant m \leqslant 5.3(n-2)$,
(3) is the best for $5.3(n-2) \leqslant m \leqslant 47(n-2)/6$,
(4) is the best for $47(n-2)/6 \leqslant m$.

It was also shown in [19] that (1) can not be improved in the range $m \leqslant 4(n-2)$, and (2) can not be improved in the range $4(n-2) \leqslant m \leqslant 5(n-2)$, apart from an additive constant. Inequalities (3) and (4) are conjectured to be far from optimal. Using the methods in [19], one can obtain an infinite family of such linear inequalities of the form $am - b(n-2)$. For instance, $\text{cr}(G) \geqslant 3m - 35(n-2)/3$.

The most important inequality for crossing numbers is undoubtedly the *Crossing Lemma*, first proved by Ajtai, Chvátal, Newborn, Szemerédi [1], and independently by Leighton [13]. If G has n vertices and m edges, $m \geqslant 4n$, then

$$\text{cr}(G) \geqslant \frac{1}{64}\frac{m^3}{n^2}. \tag{5}$$

The original constant was much larger. The constant $\frac{1}{64}$ comes from the well-known probabilistic proof of Chazelle, Sharir, and Welzl [3]. The basic idea is to take a random induced subgraph and apply inequality (1) for that.

The order of magnitude of this bound can not be improved, see [19]. The best known constant is obtained in [19]. If G has n vertices and m edges, $m \geqslant \frac{103}{16}n$, then

$$\text{cr}(G) \geqslant \frac{1}{31.1}\frac{m^3}{n^2}. \tag{6}$$

The proof is very similar to the proof of (5), the main difference is that instead of (1), inequality (3) is applied for the random subgraph. The proof of the following technical lemma is based on the same idea.

Lemma 13 *Suppose that $n \geqslant 10$, and $0 < p \leqslant 1$. Let*

$$\textsc{cr}(n, m, p) = \frac{4m}{p^2} - \frac{103n}{6p^3} + \frac{103}{3p^4} - \frac{5n^2(1-p)^{n-2}}{p^4}.$$

For any graph G with n vertices and m edges, the following holds:

$$\textsc{cr}(G) \geqslant \textsc{cr}(n, m, p).$$

Proof: Observe that inequality (3) does not hold for graphs with at most two vertices. For any graph G, let

$$\textsc{cr}'(G) = \begin{cases} \textsc{cr}(G) & \text{if } n \geqslant 3 \\ 4 & \text{if } n = 2 \\ 18 & \text{if } n = 1 \\ 35 & \text{if } n = 0 \end{cases}$$

It is easy to see that for *any* graph G

$$\textsc{cr}'(G) \geqslant 4m - \frac{103}{6}(n-2). \tag{7}$$

Let G be a graph with n vertices and m edges. Consider a drawing of G with $\textsc{cr}(G)$ crossings. Choose each vertex of G independently with probability p, and let G' be a subgraph of G induced by the selected vertices. Consider the drawing of G' *inherited* from the drawing of G. That is, each edge of G' is drawn exactly as it is drawn in G. Let n' and m' be the number of vertices and edges of G', and let x be the number of crossings in the present drawing of G'. Notice that $E(n') = pn$, $E(m') = p^2m$, $E(x) = p^4\textsc{cr}(G)$. Using inequality (7), and the linearity of expectations, the following holds:

$$\begin{aligned} E(x) &\geqslant E(\textsc{cr}(G')) \geqslant E(\textsc{cr}'(G')) - 4P(n' = 2) - 18P(n' = 1) - 35P(n' = 0) \\ &\geqslant 4p^2m - \frac{103}{6}pn + \frac{103}{3} - 4\binom{n}{2}p^2(1-p)^{n-2} - 18np(1-p)^{n-1} - 35(1-p)^n \\ &\geqslant 4p^2m - \frac{103}{6}pn + \frac{103}{3} - 5n^2(1-p)^{n-2}. \end{aligned}$$

Dividing by p^4, we obtain the statement of the lemma. □

Note that in our applications, p will be at least $1/2$, n will be at least 13. Therefore, the last term in the inequality, $\frac{5n^2(1-p)^{n-2}}{p^4}$, is negligible.

We also need some bounds on the crossing number of the complete graph, $\textsc{cr}(K_r)$. It is known that

$$\textsc{cr}(K_r) \leqslant Z(r) = \frac{1}{4}\left\lfloor\frac{r}{2}\right\rfloor\left\lfloor\frac{r-1}{2}\right\rfloor\left\lfloor\frac{r-2}{2}\right\rfloor\left\lfloor\frac{r-3}{2}\right\rfloor, \tag{8}$$

see [23]. Guy [11] conjectured $\textsc{cr}(K_r) = Z(r)$. It has been verified for $r \leqslant 12$, but still open for $r > 12$. The best known lower bound is due to de Klerk et al. [14]: $\textsc{cr}(K_r) \geqslant 0.86Z(r)$.

4 Proof of Theorem 2

Suppose that G is an r-critical graph. If G contains a topological K_r, then $\text{CR}(G) \geqslant \text{CR}(K_r)$. Suppose in the sequel that G does not contain a topological K_r. Therefore, we can apply the Kostochka, Stiebitz- and the Gallai bound on the number of edges. Next we use Lemma 13 to get the desired lower bound on the crossing number. Albertson et al. used the same approach in [4]. They used a weaker version of the bounds, and instead of Lemma 13, they applied the weaker inequality (3). In the tables below, we include the results of our calculations. For comparison, we also include the result Albertson et al. might have had using (3). In the appendix, we present our simple Maple program performing all calculations.

1. Let $r = 13$. By (8), we have $\text{CR}(K_{13}) \leqslant 225$. By Corollary 11, we only need to consider $n \geqslant r + 5 = 18$. If $n \geqslant 22$, then the KS-bound combined with (3) gives the desired result: $2m \geqslant 12n + 20 \Rightarrow \text{CR}(G) \geqslant 4(6n + 10) - 103/6(n - 2) \geqslant 224.67$.

For $18 \leqslant n \leqslant 21$ the result follows from the table below.

n	m	bound (3)	p	$\lceil \text{CR}(n, m, p) \rceil$
18	128	238	0.719	288
19	135	249	0.732	296
20	141	255	0.751	298
21	146	258	0.774	294

2. Let $r = 14$. By (8), we have $\text{CR}(K_{14}) \leqslant 315$. By Corollary 11, we only need to consider $n \geqslant r + 5 = 19$. If $n \geqslant 27$, then the KS-bound combined with (3) gives the desired result: $2m \geqslant 13n + 22 \Rightarrow \text{CR}(G) \geqslant 4(6.5n + 11) - 103/6(n - 2) \geqslant 316$.

For $19 \leqslant n \leqslant 26$ the result follows from the table below.

n	m	bound (3)	p	$\lceil \text{CR}(n, m, p) \rceil$
19	146	293	0.659	388
20	154	307	0.670	402
21	161	318	0.684	407
22	167	325	0.702	406
23	172	328	0.723	398
24	176	327	0.747	384
25	179	322	0.775	366
26	181	312	0.807	344

3. Let $r = 15$. By (8), we have $\text{CR}(K_{15}) \leqslant 441$. By Corollary 11, we only need to consider $n \geqslant r + 5 = 20$. Suppose now that G is 15-critical and $n \geqslant 28$. By the KS-bound we have $m \geqslant 7n + 12$. Apply Lemma 13 with $p = 0.764$ and a straightforward calculation gives $\text{CR}(G) \geqslant \text{CR}(n, m, 0.764) \geqslant 441$.

For $20 \leqslant n \leqslant 27$ the result follows from the table below.

n	m	bound (3)	p	$\lceil \text{CR}(n,m,p) \rceil$
20	165	351	0.610	510
21	174	370	0.617	531
22	182	385	0.623	542
23	189	396	0.642	545
24	195	403	0.659	539
25	200	406	0.678	526
26	204	404	0.700	508
27	207	399	0.725	484

4. Let $r = 16$. By (8), we have $\text{CR}(K_{16}) \leqslant 588$. By Corollary 11, we only need to consider $n \geqslant r+5 = 21$. Suppose now that G is 16-critical and $n \geqslant 32$. By the KS-bound we have $m \geqslant 7.5n + 13$. Apply Lemma 13 with $p = 0.72$ and again a straightforward calculation gives $\text{CR}(G) \geqslant \text{CR}(n, m, 0.72) \geqslant 588$.

For $21 \leqslant n \leqslant 31$ the result follows from the table below.

n	m	bound (4)	p	$\lceil \text{CR}(n,m,p) \rceil$
21	185	450	0.567	657
22	195	475	0.573	687
23	204	495	0.581	706
24	212	510	0.592	714
25	219	520	0.605	712
26	225	525	0.621	701
27	230	525	0.639	683
28	234	520	0.659	658
29	237	510	0.681	628
30	239	495	0.706	593
31	246	505	0.713	601

This concludes the proof of Theorem 2. □

Remark For $r \geqslant 17$, we could not completely verify Albertson's conjecture. By (8), $\text{CR}(K_{17}) \leqslant 784$. By Corollary 11, we only need to consider $n \geqslant r + 5 = 22$.

Lemma 14 *Let G be a 17-critical graph on n vertices. If $n \geqslant 35$, then* $\text{CR}(G) \geqslant 784 \geqslant \text{CR}(K_{17})$.

Proof: Let $p = 0.681$. Then $\text{CR}(G) \geqslant \text{CR}(n, m, 0.681) \geqslant 14.64n + 280.38$. Therefore, if $n \geqslant \frac{784-280.38}{14.64} \geqslant 34.4$, then we are done. □

The next table contains our calculations. There are three cases, $n = 32, 33, 34$, for which our approach is not sufficient.

n	m	bound (4)	p	$\lceil \text{CR}(n,m,p) \rceil$
22	206	530	0.530	832
23	217	560	0.534	874
24	227	585	0.541	902
25	236	605	0.550	917
26	244	620	0.560	920
27	251	630	0.573	913
28	257	635	0.588	897
29	262	635	0.604	872
30	266	630	0.622	840
31	269	620	0.643	802
32	271	605	0.665	759
33	278	615	0.672	765
34	286	630	0.677	779

Lemma 15 *Let G be a 17-critical graph on 32 vertices. Then* $\text{CR}(G) \geqslant \text{CR}(K_{17})$.

PROOF: Gallai [10] proved that any r-critical graph on at most $2r-2$ vertices is a join of two smaller critical graphs. In our case, $r = 17$, and $n = 2r - 2 = 32$. Assume that $G = G_1 \vee G_2$, where G_1 is r_1-critical on n_1 vertices, G_2 is r_2-critical on n_2 vertices, where $17 = r_1 + r_2$ and $32 = n_1 + n_2$. The sum of the degrees of G can be expressed as the sum of the degrees of the vertices in G_i, for $i = 1, 2$, plus twice the number of edges between G_1 and G_2:

$$2m \geqslant (r_1 - 1)n_1 + (r_2 - 1)n_2 + 2(r - 6) + 2n_1n_2.$$

Here, we used the KS-bound for the smaller parts, G_1, G_2. The right-hand side is minimal, if $r_1n_1 + r_2n_2 + 2n_1n_2$ is minimal. With equivalent modifications, we get the following: $n_1(r_1 + n_2) + n_2(r_2 + n_1) = n_1(r_1 + n - n_1) + (n - n_1)(r - r_1 + n_1) = (n_1 - r_1)(n - 2n_1) + nr + n_1(n - r)$. This expression is minimal, if n_1 is minimal and $n_1 = r_1$. This yields the following: $2m \geqslant n(r - 1) + 2n - r - n + 2(r - 6)$. In our case, it yields $m \geqslant 275$. Next we apply Lemma 13 with $p = 0.665$, and we get $\text{CR}(32, 275, 0.665) \geqslant 795 > \text{CR}(K_{17})$. □

5 Proof of Lemma 3

Suppose that $r \geqslant 17$, and G is an r-critical graph with n vertices and m edges. If $n \geqslant 4r$, then the statement of the lemma holds by [4]. Suppose that $n = \alpha r$ and $3.57 \leqslant \alpha \leqslant 4$. In order to estimate the crossing number of G, instead of the probabilistic argument in the proof of Lemma 13, we apply inequality (3) for each induced subgraph of G with exactly 52 vertices. Let $k = \binom{n}{52}$, and let $G_1, G_2, \ldots, G_k$ be the induced subgraphs of G with 52 vertices. Suppose that G_i has m_i edges. By (3), the following holds for any i:

$$\text{CR}(G_i) \geqslant 4m_i - \frac{103}{6} \cdot 50,$$

consequently,

$$
\begin{aligned}
\operatorname{CR}(G) &\geqslant \frac{1}{\binom{n-4}{48}} \sum_{i=1}^{k} \left(4m_i - \frac{103}{6}\cdot 50\right) = \frac{4m}{\binom{n-4}{48}}\binom{n-2}{50} - \frac{50}{\binom{n-4}{48}}\frac{103}{6}\binom{n}{52} \\
&= \frac{4(n-2)(n-3)m}{50\cdot 49} - \frac{103}{6}\frac{n(n-1)(n-2)(n-3)}{52\cdot 51\cdot 49} \\
&\geqslant \frac{2(n-2)(n-3)n(r-1)}{50\cdot 49} - \frac{103}{6}\frac{n(n-1)(n-2)(n-3)}{52\cdot 51\cdot 49} \\
&= \frac{n(n-2)(n-3)}{49}\left(\frac{r-1}{25} - \frac{103(n-1)}{6\cdot 52\cdot 51}\right)
\end{aligned}
$$

since we counted each possible crossing at most $\binom{n-4}{48}$ times, and each edge of G exactly $\binom{n-2}{50}$ times. Finally, some calculation shows that this lower bound is greater than

$$
\frac{r(r-2)(r-3)}{49}(r-1)\left(\frac{1}{25} - \frac{103\alpha}{6\cdot 52\cdot 51}\right) \geqslant \frac{1}{64}r(r-1)(r-2)(r-3) > \operatorname{CR}(K_r)
$$

for $3.57 \leqslant \alpha \leqslant 4$, which proves the lemma. □

Remarks

1. As we have already mentioned, see (6), the best known constant in the Crossing Lemma, 1/31.1, is obtained in [19]. Montaron [17] managed to improve it slightly for *dense* graphs, that is, in the case when $m = O(n^2)$. His calculations are similar to the proof of Lemma 3 and 13.

2. Our attack of the Albertson conjecture is based on the following philosophy. We calculate a lower bound for the number of edges of an r-critical n-vertex graph G. Next we substitute this into the lower bound given by Lemma 13. Finally, we compare the result and $Z(r)$. For large r, this method is not sufficient, but it gives the right order of magnitude, and the constants are roughly within a factor of 4.

Let G be an r-critical graph with n vertices, where $r \leqslant n \leqslant 3.57r$. Then $2m \geqslant (r-1)n$. We can apply (6):

$$
\operatorname{CR}(G) \geqslant \frac{1}{31.1}\frac{((r-1)n/2)^3}{n^2} = \frac{(r-1)^3 n}{31.1\cdot 8} \geqslant \frac{1}{250}r(r-1)^3 \geqslant \frac{Z(r)}{4}.
$$

3. Let $G = G(n,p)$ be a random graph with n vertices and edge probability $p = p(n)$. It is known [12], that there exists a constant $C_0 > 0$ such that if $np > C_0$, then asymptotically almost surely

$$
\chi(G) < \frac{np}{\log np}.
$$

Therefore, asymptotically almost surely

$$
\operatorname{CR}(K_{\chi(G)}) \leqslant Z(\chi(G)) < \frac{n^4 p^4}{64\log^4 np}.
$$

On the other hand, by [20], if $np > 20$, then almost surely

$$\text{CR}(G) \geqslant \frac{n^4p^2}{20000}.$$

Consequently, almost surely we have $\text{CR}(G) > \text{CR}(K_{\chi(G)})$. Roughly speaking, unlike in the case of the Hajós conjecture, a random graph almost surely satisfies the statement of the Albertson conjecture.

4. If we do not believe in Albertson's conjecture, we have to look for a counterexample in the range $n \leqslant 3.57r$. Any candidate must also be a counterexample for the Hajós Conjecture. It is tempting to look at Catlin's graphs.

Let C_5^k denote the graph arising from C_5 by repeating each vertex k times. That is, each vertex of C_5 is blown up to a complete graph on k vertices, and any edge of C_5 is blown up to a complete bipartite graph $K_{k,k}$.

Lemma 16 *Catlin's graphs satisfy the Albertson conjecture.*

PROOF: It is known that $\chi(C_5^k) = \lceil \frac{5}{2}k \rceil$. To draw C_5^k, we must draw two copies of K_{2k}, a K_k and three copies of $K_{k,k}$. Therefore,

$$cr(C_5^k) \geqslant 2Z(2k) + Z(k) + 3cr(K_{k,k}) \sim 2\frac{1}{4}k^4 + \frac{1}{4}\left(\frac{k}{2}\right)^4 + 3\left(\frac{k}{2}\right)^4 > 0.70k^4.$$

On the other hand

$$cr(K_{\chi(C_5^k)}) \sim cr(K_{\frac{5}{2}k}) \leqslant \frac{1}{4}\left(\frac{5}{4}k\right)^4 < 0.62k^4 \qquad (9)$$

which proves the claim. □

Acknowledgement. We are very grateful to the anonymous referee for his comments.

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Appendix

```
start:=proc(r,n)
local p,m,eredm,f,g,h,cr;
if (n<=2*r-2) then
p:=n-r;
m:=ceil(((r-1)*n+p*(r-p)-1)/2);
else
m:=ceil(((r-1)*n+2*(r-3))/2);
fi;
g:= ceil(5*m-25*(n-2));
print(m,g);
f:= 4*m*x^2-(103/6)*n*x^3+(103/3)*x^4;
eredm:=[solve((diff(f,x)/x)=0, x)];
print(evalf(eredm));
cr := min(eredm[1], eredm[2]);
print(evalf(1/cr));
h:= f-(5*n^2*(1-1/x)^(n-2))/(1/x)^4;
evalf((subs(x=cr, h)));
end:
```