

Congruences involving alternating multiple harmonic sums

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Abstract

We show that for any prime $p \neq 2$,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} \equiv - \sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod{p^3}$$

by expressing the left-hand side as a combination of alternating multiple harmonic sums.

1 Introduction

In [8] Van Hamme presented several results and conjectures concerning a curious analogy between the values of certain hypergeometric series and the congruences of some of their partial sums modulo power of prime. In this paper we would like to discuss a new example of this analogy. Let us consider

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} &= \binom{1}{2} + \frac{1}{2} \binom{1 \cdot 3}{2 \cdot 4} + \frac{1}{3} \binom{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1}{4} \binom{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots \\ &= \int_0^{-1} \frac{1}{x} \left(\frac{1}{\sqrt{1+x}} - 1 \right) dx = -2 \left[\log \left(\frac{1 + \sqrt{1+x}}{2} \right) \right]_0^{-1} = 2 \log 2. \end{aligned}$$

Let p be a prime number, what is the p -adic analogue of the above result?

The real case suggests to replace the logarithm with some p -adic function which behaves in a similar way. It turns out that the right choice is the *Fermat quotient*

$$q_p(x) = \frac{x^{p-1} - 1}{p}$$

(which is fine since $q_p(x \cdot y) \equiv q_p(x) + q_p(y) \pmod{p}$), and, as shown in [7], the following congruence holds for any prime $p \neq 2$,

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} \equiv 2q_p(2) \pmod{p}.$$

Here we improve this result to the following statement.

Theorem 1.1. *For any prime $p > 3$,*

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} &\equiv 2q_p(2) - pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7}{12}p^2B_{p-3} \\ &\equiv - \sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod{p^3} \end{aligned}$$

where B_n is the n -th Bernoulli number.

In the proof we will employ some new congruences for alternating multiple harmonic sums which are interesting in themselves, such as

$$\begin{aligned} H(-1, -2; p-1) &= \sum_{0 < i < j < p} \frac{(-1)^{i+j}}{ij^2} \equiv -\frac{3}{4}B_{p-3} \pmod{p}, \\ H(-1, -1, 1; p-1) &= \sum_{0 < i < j < k < p} \frac{(-1)^{i+j}}{ijk} \equiv q_p(2)^3 + \frac{7}{8}B_{p-3} \pmod{p}. \end{aligned}$$

2 Alternating multiple harmonic sums

Let $r > 0$ and let $(a_1, a_2, \dots, a_r) \in (\mathbb{Z}^*)^r$. For any $n \geq r$, we define the *alternating multiple harmonic sum* as

$$H(a_1, a_2, \dots, a_r; n) = \sum_{1 \leq k_1 < k_2 < \dots < k_r \leq n} \prod_{i=1}^r \frac{\text{sign}(a_i)^{k_i}}{k_i^{|a_i|}}.$$

The integers r and $\sum_{i=1}^r |a_i|$ are respectively the *depth* and the *weight* of the harmonic sum. From the definition one derives easily the *shuffle relations*:

$$\begin{aligned} H(a; n) \cdot H(b; n) &= H(a, b; n) + H(b, a; n) + H(a \oplus b; n) \\ H(a, b; n) \cdot H(c; n) &= H(c, a, b; n) + H(a, c, b; n) + H(a, b, c; n) \\ &\quad + H(a \oplus b, c; n) + H(a, b \oplus c; n) \end{aligned}$$

where $a \oplus b = \text{sign}(ab)(|a| + |b|)$. Moreover, if p is a prime, by replacing k_i with $p - k_i$ we get the *reversal relations*:

$$\begin{aligned} H(a, b; p-1) &\equiv H(b, a; p-1)(-1)^{a+b}\text{sign}(ab) \pmod{p}, \\ H(a, b, c; p-1) &\equiv H(c, b, a; p-1)(-1)^{a+b+c}\text{sign}(abc) \pmod{p}. \end{aligned}$$

The values of several *non-alternating* (i.e. when all the indices are positive) harmonic sums modulo a power of a prime are well known:

(i). ([4], [11]) for $a, r > 0$ and for any prime $p > ar + 2$

$$H(\{a\}^r; p-1) \equiv \begin{cases} (-1)^r \frac{a(ar+1)}{2(ar+2)} p^2 B_{p-ar-2} & (\text{mod } p^3) \text{ if } ar \text{ is odd,} \\ (-1)^{r-1} \frac{a}{ar+1} p B_{p-ar-1} & (\text{mod } p^2) \text{ if } ar \text{ is even;} \end{cases}$$

(ii). ([6]) for any prime $p > 3$

$$H\left(1; \frac{p-1}{2}\right) \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2 B_{p-3} \pmod{p^3}.$$

and for $a > 1$ and for any prime $p > a + 1$

$$H\left(a; \frac{p-1}{2}\right) \equiv \begin{cases} -\frac{2^a-2}{a} B_{p-a} & (\text{mod } p) \text{ if } a \text{ is odd,} \\ \frac{a(2^{a+1}-1)}{2(a+1)} p B_{p-a-1} & (\text{mod } p^2) \text{ if } a \text{ is even;} \end{cases}$$

(iii). ([4], [10]) for $a, b > 0$ and for any prime $p > a + b + 1$

$$H(a, b; p-1) \equiv \frac{(-1)^b}{a+b} \binom{a+b}{a} B_{p-a-b} \pmod{p}$$

(note that $B_{2n+1} = 0$ for $n > 0$).

The following result will allow us to compute the mod p values of multiple harmonic sums of depth ≤ 2 when the indices are all negative.

Theorem 2.1. *Let $a, b > 0$; then for any prime $p \neq 2$,*

$$\begin{aligned} H(-a; p-1) &= -H(a; p-1) + \frac{1}{2^{a-1}} H\left(a; \frac{p-1}{2}\right), \\ 2H(-a, -a; p-1) &= H(-a; p-1)^2 - H(2a; p-1), \end{aligned}$$

and

$$\begin{aligned} H(-a, -b; p-1) &\equiv -\left(1 - \frac{1}{2^{a+b-1}}\right) H(a, b; p-1) \\ &\quad - \frac{(-1)^b}{2^{a+b-1}} H\left(a; \frac{p-1}{2}\right) H\left(b; \frac{p-1}{2}\right) \pmod{p}. \end{aligned}$$

Proof. The first shuffling relation applied to $H(-a; p-1)^2$ yields the second equation. As regards the first equation we simply observe that $(-1)^i/i^a$ is positive if and only if i is even. We use a similar argument for the congruence: since $(-1)^{i+j}/(i^a j^b)$ is positive if and only if i and j are both even or if $(p-i)$ and $(p-j)$ are both even then we have mod p ,

$$H(-a, -b; p-1) \equiv -H(a, b; p-1) + \frac{2}{2^{a+b}} \left(H\left(a, b; \frac{p-1}{2}\right) + (-1)^{a+b} H\left(b, a; \frac{p-1}{2}\right) \right).$$

Moreover, by decomposing the sum $H(a, b; p-1)$ we obtain

$$\begin{aligned} H(a, b; p-1) &\equiv H\left(a, b; \frac{p-1}{2}\right) + H\left(a; \frac{p-1}{2}\right) (-1)^b H\left(b; \frac{p-1}{2}\right) \\ &\quad + (-1)^{a+b} H\left(b, a; \frac{p-1}{2}\right), \end{aligned}$$

that is,

$$\begin{aligned} H\left(a, b; \frac{p-1}{2}\right) + (-1)^{a+b} H\left(b, a; \frac{p-1}{2}\right) \\ \equiv H(a, b; p-1) - H\left(a; \frac{p-1}{2}\right) (-1)^b H\left(b; \frac{p-1}{2}\right), \end{aligned}$$

and the desired congruence follows immediately. $\square$

Corollary 2.2. *For any prime $p > 3$,*

$$\begin{aligned} H(-1; p-1) &\equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2 q_p(2)^3 - \frac{1}{4}p^2 B_{p-3} \pmod{p^3}, \\ H(-1, -1; p-1) &\equiv 2q_p(2)^2 - 2pq_p(2)^3 - \frac{1}{3}p B_{p-3} \pmod{p^2}. \end{aligned}$$

Moreover, for $a > 1$ and for any prime $p > a + 1$,

$$H(-a; p-1) \equiv -\frac{2^a - 2}{a2^{a-1}} B_{p-a} \pmod{p}.$$

Proof. The proof is straightforward: apply Theorem 2.1, (i), (ii), and (iii). $\square$

The following theorem is a variation of a result presented in [9].

Theorem 2.3. *Let $r > 0$; then for any prime $p > r + 1$,*

$$H(\{1\}^{r-1}, -1; p-1) \equiv (-1)^{r-1} \sum_{k=1}^{p-1} \frac{2^k}{k^r} \pmod{p}.$$

Proof. For $r \geq 1$, let

$$F_r(x) = \sum_{0 < k_1 < \dots < k_r < p} \frac{x^{k_r}}{k_1 \dots k_r} \in \mathbb{Z}_p[x] \quad \text{and} \quad f_r(x) = \sum_{0 < k < p} \frac{x^k}{k^r} \in \mathbb{Z}_p[x].$$

We show by induction that

$$F_r(x) \equiv (-1)^{r-1} f_r(1-x) \pmod{p}$$

then our congruence follows by taking $x = -1$.

For $r = 1$, since $\binom{p}{k} = (-1)^{k-1} \frac{p}{k} \pmod{p^2}$ for $0 < k < p$ then

$$f_1(x) \equiv \frac{1}{p} \sum_{k=1}^{p-1} (-1)^{k-1} \binom{p}{k} x^k = -\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} (-x)^k = \frac{1 - (1-x)^p - x^p}{p} \pmod{p}.$$

Hence $F_1(x) = f_1(x) \equiv f_1(1-x) \pmod{p}$.

Assume that $r > 1$, then the formal derivative yields

$$\begin{aligned} \frac{d}{dx} F_r(x) &= \sum_{0 < k_1 < \dots < k_r < p} \frac{k_r x^{k_r-1}}{k_1 \dots k_r} = \sum_{0 < k_1 < \dots < k_{r-1} < p} \frac{1}{k_1 \dots k_{r-1}} \sum_{k_r=k_{r-1}+1}^{p-1} x^{k_r-1} \\ &= \sum_{0 < k_1 < \dots < k_{r-1} < p} \frac{1}{k_1 \dots k_{r-1}} \cdot \frac{x^{p-1} - x^{k_{r-1}}}{x-1} \\ &= \frac{x^{p-1}}{x-1} H(\{1\}^{r-1}; p-1) - \frac{1}{x-1} F_{r-1}(x) \equiv \frac{F_{r-1}(x)}{1-x} \pmod{p} \end{aligned}$$

where we used (i). Moreover

$$\frac{d}{dx} f_r(1-x) = - \sum_{0 < k < p} \frac{(1-x)^{k-1}}{k^{r-1}} = - \frac{f_{r-1}(1-x)}{1-x}.$$

Hence, by the induction hypothesis

$$(1-x) \frac{d}{dx} (F_r(x) + (-1)^r f_r(1-x)) \equiv F_{r-1}(x) + (-1)^{r-1} f_{r-1}(1-x) \equiv 0 \pmod{p}.$$

Thus $F_r(x) + (-1)^r f_r(1-x) \equiv c_1 \pmod{p}$ for some constant c_1 because this polynomial has degree $< p$. Substituting in $x = 0$ we find that by (i),

$$F_r(x) + (-1)^r f_r(1-x) \equiv c_1 \equiv F_r(0) + (-1)^r f_r(1) = (-1)^r H(r; p-1) \equiv 0 \pmod{p}.$$

□

With the next two corollaries we have a complete the list of the mod p values of the alternating multiple harmonic sums of depth and weight ≤ 3 .

Corollary 2.4. *The following congruences mod p hold for any prime $p > 3$,*

$$\begin{aligned} H(1, -1; p-1) &\equiv -H(-1, 1; p-1) \equiv q_p(2)^2, \\ H(-1, 2; p-1) &\equiv H(1, -2; p-1) \equiv H(2, -1; p-1) \equiv H(-2, 1; p-1) \equiv \frac{1}{4}B_{p-3}, \\ H(-1, -2; p-1) &\equiv -H(-2, -1; p-1) \equiv -\frac{3}{4}B_{p-3}. \end{aligned}$$

Proof. By Theorem 2.3 and by [2] we have that

$$H(1, -1; p-1) \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv q_p(2)^2 \pmod{p}.$$

By the first shuffling relation applied to the product $H(-1; p-1)H(2; p-1)$, by (i), and by Corollary 2.2 we get

$$H(-1, 2; p-1) = \frac{1}{2}H(-1; p-1)H(2; p-1) - \frac{1}{2}H(-3; p-1) \equiv \frac{1}{4}B_{p-3} \pmod{p}.$$

By (ii) and by Theorem 2.1

$$H(-1, -2; p-1) \equiv -\frac{3}{4}H(1, 2; p-1) - \frac{1}{4}H\left(1; \frac{p-1}{2}\right)H\left(2; \frac{p-1}{2}\right) \equiv -\frac{3}{4}B_{p-3} \pmod{p}.$$

The remaining congruences follow by applying the reversal relation of depth 2. $\square$

Corollary 2.5. *The following congruences mod p hold for any prime $p > 3$,*

$$\begin{aligned} H(-1, 1, -1; p-1) &\equiv 0, \\ H(1, 1, -1; p-1) &\equiv H(-1, 1, 1; p-1) \equiv -\frac{1}{3}q_p(2)^3 - \frac{7}{24}B_{p-3}, \\ H(-1, -1, 1; p-1) &\equiv -H(1, -1, -1; p-1) \equiv q_p(2)^3 + \frac{7}{8}B_{p-3}, \\ H(1, -1, 1; p-1) &\equiv \frac{2}{3}q_p(2)^3 + \frac{1}{12}B_{p-3}, \\ H(-1, -1, -1; p-1) &\equiv -\frac{4}{3}q_p(2)^3 - \frac{1}{6}B_{p-3}. \end{aligned}$$

Proof. By the reversal relation of depth 3, $H(-1, 1, -1; p-1) \equiv -H(-1, 1, -1; p-1) \equiv 0$. By Theorem 2.3, by Theorem 1 in [3], and by Corollary 2.2 we have that

$$H(1, 1, -1; p-1) \equiv \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3}q_p(2)^3 + \frac{7}{12}H(-3, p-1) \equiv -\frac{1}{3}q_p(2)^3 - \frac{7}{24}B_{p-3} \pmod{p}.$$

By the second shuffling relation applied to the products

$$H(1, -1; p-1)H(-1; p-1), \quad H(1, -1; p-1)H(1; p-1), \quad \text{and} \quad H(-1, -1; p-1)H(-1; p-1)$$

we respectively find that

$$\begin{aligned} 2H(1, -1, -1; p-1) &\equiv H(1, -1; p-1)H(-1; p-1) - H(1, 2; p-1) - H(-2, -1; p-1), \\ H(1, -1, 1; p-1) &\equiv -2H(1, 1, -1, p-1) - 2H(2, -1; p-1), \\ 3H(-1, -1, -1; p-1) &\equiv H(-1, -1; p-1)H(-1; p-1) - 2H(2, -1; p-1). \end{aligned}$$

The remaining congruences follow by applying the reversal relation of depth 3. $\square$

3 Proof of Theorem 1.1

The following useful identity appears in [7]. Here we give an alternative proof by using Riordan's array method (see [5] for more examples of this technique), hoping that it will help the interested reader to find some generalization.

Theorem 3.1. *Let $n \geq d > 0$,*

$$d \sum_{k=1}^n \binom{2k}{k+d} \frac{x^{n-k}}{k} = \sum_{k=0}^{n-d} \binom{2n}{n+d+k} v_k - \binom{2n}{n+d}$$

where $v_0 = 2$, $v_1 = x - 2$ and $v_{k+1} = (x-2)v_k - v_{k-1}$ for $k \geq 1$.

Proof. We first note that

$$\begin{aligned} \binom{2k}{k+d} &= \binom{2k}{k-d} = (-1)^{k-d} \binom{-k-d-1}{k-d} \\ &= [z^{k-d}] \frac{1}{(1-z)^{k+d+1}} = [z^{-1}] \frac{z^{d-1}}{(1-z)^{d+1}} \cdot \left(\frac{1}{z(1-z)} \right)^k. \end{aligned}$$

Since the residue of a derivative is zero then

$$\begin{aligned} d \sum_{k=1}^n \binom{2k}{k+d} \frac{x^{n-k}}{k} &= [z^{-1}] x^n \frac{dz^{d-1}}{(1-z)^{d+1}} G \left(\frac{1}{xz(1-z)} \right) \\ &= -[z^{-1}] x^n \frac{z^d}{(1-z)^d} G' \left(\frac{1}{xz(1-z)} \right) \cdot \left(\frac{1}{xz(1-z)} \right)' \\ &= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \frac{1 - x^n z^n (1-z)^n}{1 - xz + xz^2} \cdot (1-2z) \\ &= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \frac{1-2z}{1 - xz + xz^2} \end{aligned}$$

where $G(z) = \sum_{k=1}^n \frac{z^k}{k}$ and $G'(z) = \sum_{k=1}^n z^{k-1} = \frac{1-z^n}{1-z}$. Moreover

$$\begin{aligned} \binom{2n}{n+d+k} &= \binom{2n}{n-d-k} = (-1)^{n-d-k} \binom{-n-d-k-1}{n-d-k} \\ &= [z^{n-d-k}] \frac{1}{(1-z)^{n+d+k+1}} = [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \cdot \left(\frac{z}{1-z} \right)^k. \end{aligned}$$

Letting $F(z) = \sum_{k=0}^{\infty} v_k z^k = \frac{2-(x-2)z}{1-(x-2)z+z^2}$ then

$$\begin{aligned} \sum_{k=0}^{n-d} \binom{2n}{n+d+k} v_k - \binom{2n}{n+d} &= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \cdot F\left(\frac{z}{1-z}\right) - [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \\ &= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \left(\frac{(2-xz)(1-z)}{1-xz+xz^2} - 1 \right) \\ &= [z^{-1}] \frac{z^{d-n-1}}{(1-z)^{n+d+1}} \frac{1-2z}{1-xz+xz^2}. \end{aligned}$$

□

Notice that it is easy to verify by comparing the recurrence relations or the generating functions that

$$v_k = 2T_k\left(\frac{x}{2} - 1\right)$$

where $T_k(x)$ denote the Chebyshev polynomials of the first kind (I thank the referee for this valuable remark). We wonder if Theorem 3.1 can be interpreted as a known identity for Chebyshev polynomials.

Corollary 3.2. *For any $n > 0$,*

$$4^n \sum_{k=1}^n \binom{-\frac{1}{2}}{k} \frac{(-1)^k}{k} = -4(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \sum_{j=0}^{d-1} \binom{2n}{j} - 2(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \binom{2n}{d}.$$

Proof. Since

$$0 = \sum_{d=-k}^k (-1)^d \binom{2k}{k+d} = \binom{2k}{k} + 2 \sum_{d=1}^k (-1)^d \binom{2k}{k+d}$$

then for any $n \geq k$,

$$(-1)^k \binom{-\frac{1}{2}}{k} = 4^{-k} \binom{2k}{k} = -2 \cdot 4^{-k} \sum_{d=1}^k (-1)^d \binom{2k}{k+d}.$$

For $x = 4$ we have $v_k = 2$ for all $k \geq 0$ and with Theorem 3.1 we get

$$\begin{aligned} 4^n \sum_{k=1}^n \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} &= -2 \sum_{k=1}^n \frac{4^{n-k}}{k} \sum_{d=1}^n (-1)^d \binom{2k}{k+d} = -2 \sum_{d=1}^n (-1)^d \sum_{k=1}^n \frac{4^{n-k}}{k} \binom{2k}{k+d} \\ &= -2 \sum_{d=1}^n \frac{(-1)^d}{d} \left(2 \sum_{k=0}^{n-d} \binom{2n}{n+d+k} - \binom{2n}{n+d} \right) \\ &= -4 \sum_{d=1}^n \frac{(-1)^d}{d} \sum_{k=1}^{n-d} \binom{2n}{n-d-k} - 2 \sum_{d=1}^n \frac{(-1)^d}{d} \binom{2n}{n-d} \\ &= -4(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \sum_{j=0}^{d-1} \binom{2n}{j} - 2(-1)^n \sum_{d=0}^{n-1} \frac{(-1)^d}{n-d} \binom{2n}{d}. \end{aligned}$$

□

We will make use of the following lemma.

Lemma 3.3. *For any prime $p \neq 2$ and for $0 < j < p$,*

$$\binom{2p}{j} \equiv -2p \frac{(-1)^j}{j} + 4p^2 \frac{(-1)^j}{j} H(1; j-1) \pmod{p^3}$$

and

$$\binom{2p}{p} \equiv 2 - \frac{4}{3} p^3 B_{p-3} \pmod{p^4}.$$

Proof. It suffices to expand the binomial coefficient as follows

$$\binom{2p}{j} = -2p \frac{(-1)^j}{j} \prod_{k=1}^{j-1} \left(1 - \frac{2p}{k}\right) = \frac{(-1)^j}{j} \sum_{k=1}^{j-1} (-2p)^k H(\{1\}^{k-1}; j-1),$$

and apply (i). □

Proof of Theorem 1.1. Letting $n = p$ in the identity given by Corollary 3.2 we obtain

$$4^p \sum_{k=1}^p \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} = 4 \sum_{0 \leq j < d < p} \frac{(-1)^d}{p-d} \binom{2p}{j} + 2 \sum_{0 \leq d < p} \frac{(-1)^d}{p-d} \binom{2p}{d},$$

that is,

$$4^{p-1} \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} = \frac{2 - \binom{2p}{p}}{4p} - \sum_{0 < d < p} \frac{(-1)^d}{d} + \sum_{0 < j < d < p} \frac{(-1)^d}{p-d} \binom{2p}{j} + \frac{1}{2} \sum_{0 < d < p} \frac{(-1)^d}{p-d} \binom{2p}{d}.$$

Now we consider each term of the right-hand side separately. By Lemma 3.3,

$$\frac{2 - \binom{2p}{p}}{4p} \equiv \frac{1}{3} p^2 B_{p-3} \pmod{p^3}.$$

By Corollary 2.2,

$$\sum_{0 < d < p} \frac{(-1)^d}{d} = H(-1; p-1) = -2q_p(2) + pq_p(2)^2 - \frac{2}{3} p^2 q_p(2)^3 - \frac{1}{4} p^2 B_{p-3} \pmod{p^3}.$$

Since for $0 < d < p$,

$$\frac{1}{p-d} = -\frac{1}{d(1 - \frac{p}{d})} \equiv -\frac{1}{d} - \frac{p}{d^2} \pmod{p^2}$$

then by Lemma 3.3, (i), and (iii) we have that

$$\begin{aligned} \sum_{0 < d < p} \frac{(-1)^d}{p-d} \binom{2p}{d} &\equiv \sum_{0 < d < p} \left(-\frac{(-1)^d}{d} - p \frac{(-1)^d}{d^2} \right) \left(-2p \frac{(-1)^d}{d} + 4p^2 \frac{(-1)^d}{d} H(1; d-1) \right) \\ &\equiv 2pH(2; p-1) + 2p^2 H(3; p-1) - 4p^2 H(1, 2; p-1) \\ &\equiv -\frac{8}{3} p^2 B_{p-3}. \pmod{p^3}. \end{aligned}$$

In a similar way, by Lemma 3.3 and Corollaries 2.4 and 2.5 we get

$$\begin{aligned}
\sum_{0 < j < d < p} \frac{(-1)^d}{p-d} \binom{2p}{j} \\
&\equiv \sum_{0 < j < d < p} \left(-\frac{(-1)^d}{d} - p \frac{(-1)^d}{d^2} \right) \left(-2p \frac{(-1)^j}{j} + 4p^2 \frac{(-1)^j}{j} H(1; j-1) \right) \\
&\equiv 2pH(-1, -1; p-1) + 2p^2H(-1, -2; p-1) - 4p^2H(1, -1, -1; p-1) \\
&\equiv 4pq_p(2)^2 + \frac{4}{3}p^2B_{p-3} \pmod{p^3}.
\end{aligned}$$

Thus,

$$4^{p-1} \sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} = 2q_p(2) + 3pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

Since $4^{p-1} = (q_p(2)p + 1)^2 = 1 + 2q_p(2)p + q_p(2)^2p^2$ then

$$4^{-(p-1)} = (1 + 2q_p(2)p + q_p(2)^2p^2)^{-1} \equiv 1 - 2q_p(2)p + 3q_p(2)^2p^2 \pmod{p^3}.$$

Finally,

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{-\frac{1}{2}}{k} \\
&\equiv (1 - 2q_p(2)p + 3q_p(2)^2p^2) \left(2q_p(2) + 3pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7}{12}p^2B_{p-3} \right) \\
&\equiv 2q_p(2) - pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7}{12}p^2B_{p-3} \pmod{p^3}.
\end{aligned}$$

Note that by (ii) the right-hand side is just $-H(1, (p-1)/2) = -\sum_{k=1}^{(p-1)/2} \frac{1}{k}$. □

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