Multipartite separability of Laplacian matrices of graphs

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Submitted: Sep 4, 2008; Accepted: Apr 29, 2009; Published: May 15, 2009
Mathematics Subject Classifications: 81P15, 81P68

Abstract

Recently, Braunstein et al. introduced normalized Laplacian matrices of graphs as density matrices in quantum mechanics and studied the relationships between quantum physical properties and graph theoretical properties of the underlying graphs. We provide further results on the multipartite separability of Laplacian matrices of graphs. In particular, we identify complete bipartite graphs whose normalized Laplacian matrix is multipartite entangled under any vertex labeling. Furthermore, we give conditions on the vertex degrees such that there is a vertex labeling under which the normalized Laplacian matrix is entangled. These results address an open question raised in Braunstein et al. Finally, we show that the Laplacian matrix of any product of graphs (strong, Cartesian, tensor, lexicographical, etc.) is multipartite separable, extending analogous results for bipartite and tripartite separability.

1 Introduction

The objects of study in this paper are density matrices of quantum mechanics. Density matrices are used to describe the state of a quantum system and are fundamental mathematical constructs in quantum mechanics. They play a key role in the design and analysis of quantum computing and information systems [1].

Definition 1 *A complex matrix A is a* density matrix *if it is Hermitian, positive semidefinite and has unit trace.*

Remark: In this paper we will often use the following simple fact: $\frac{1}{tr(A)}A$ is a density matrix if A is Hermitian, positive semidefinite and has a strictly positive trace. We will refer to $\frac{1}{tr(A)}A$ as a *normalization* of A.

Definition 2 *A complex matrix A is row diagonally dominant if $A_{ii} \geq \sum_{j \neq i} |A_{ij}|$ for all i.*

By Gershgorin's circle criterion, all the eigenvalues of a row diagonally dominant matrix has nonnegative real parts. Thus a nonzero Hermitian row diagonally dominant matrix is positive semidefinite and has a strictly positive trace, and such a matrix normalized is a density matrix.

A key property of a density matrix is its separability. The property of nonseparability plays an important role in generating the myriad of counterintuitive phenomena in quantum mechanics.

Definition 3 *A density matrix A is* separable *in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$ if it can be written as $A = \sum_i c_i A_i^1 \otimes \cdots \otimes A_i^m$ where $c_i \geq 0$, $\sum_i c_1 = 1$ and A_i^j are density matrices in $\mathbb{C}^{p_j \times p_j}$. A density matrix is* entangled *if it is not separable.*

2 Laplacian matrices as density matrices

The Laplacian matrix of a graph is defined as $L = D - A$, where D is the diagonal matrix of the vertex degrees and A is the adjacency matrix. The matrix L is symmetric and row diagonally dominant, and therefore for a nonempty[1] graph the matrix $\frac{1}{trL}L$ is a density matrix. In Ref. [2], such normalized Laplacian matrices are studied as density matrices and quantum mechanical properties such as entanglement of various types of graph Laplacian matrices are studied. This approach was further investigated in [3] where it was shown that the Peres-Horodecki necessary condition for separability is equivalent to a condition on the partial transpose graph, and that this condition is also sufficient for separability of block tridiagonal Laplacian matrices and Laplacian matrices in $\mathbb{C}^2 \times \mathbb{C}^q$. In [4] the tripartite separability of normalized Laplacian matrices is studied. In [5] several classes of graphs were identified whose separability are easily determined.

As noted in [2], the separability of a normalized Laplacian matrix of a graph depends on the labeling of the vertices. In the sequel, unless otherwise noted (for example, in Theorem 3), we will assume a specific Laplacian matrix (and thus a specific vertex labeling) when we discuss separability of Laplacian matrices of graphs. A vertex labeling can be defined as:

Definition 4 *For $n = p_1 p_2 \cdots p_m$, a* vertex labeling *is a bijection between $\{1, \ldots, n\}$ and $\{1, \ldots, p_1\} \times \{1, \ldots, p_2\} \times \cdots \times \{1, \ldots, p_m\}$.*

3 Conditions for multipartite entanglement

In this section, we consider unweighted graphs, i.e. the adjacency matrix is a 0-1 matrix.

[1] A graph is *empty* if it has no edges. In this case the Laplacian matrix is the zero matrix and has zero trace.

Definition 5 *Given a graph $\mathcal{G}$ with vertices $V \times W$, the partial transpose graph $\mathcal{G}^{pT}$ is a graph with vertices $V \times W$ and edges defined by:*

$\{(u,v),(w,y)\}$ *is an edge of* $\mathcal{G}$ *if and only if* $\{(u,y),(w,v)\}$ *is an edge of* $\mathcal{G}^{pT}$.

Note that the partial transpose graph depends on the specific labeling of the vertices. The partial transpose graph is useful in determining separability of the Laplacian matrix of a graph with the same vertex labeling. In [5, 3] the following necessary condition for separability is shown:

Theorem 1 *If the normalized Laplacian matrix of $\mathcal{G}$ is separable then each vertex of $\mathcal{G}$ has the same degree as the same vertex of $\mathcal{G}^{pT}$.*

Corollary 1 *If the normalized Laplacian matrix of $\mathcal{G}$ is separable then each vertex of $\overline{\mathcal{G}}$ has the same degree as the same vertex of $\overline{\mathcal{G}}^{pT}$.*

Proof: Follows from the fact that the degree condition in Theorem 1 is true for a graph $\mathcal{G}$ if and only if it is true for the complement graph $\mathcal{G}$. □

In [3] the following sufficient condition for separability is shown:

Theorem 2 *For a graph $\mathcal{G}$, if for all $1 \leq i,j \leq p_1$, $1 \leq k \leq p_2$, $i \neq j$, the number of edges from vertex (v_i, w_k) to vertices of the form $(v_j, \cdot)$ is the same as the number of edges from vertex (v_j, w_k) to vertices of the form $(v_i, \cdot)$, then the normalized Laplacian matrix of $\mathcal{G}$ is separable in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2}$.*

3.1 Complete bipartite graphs

Theorem 3 *Let $n = p_1 p_2 \cdots p_m$, where $p_i \geq 2$. If there exists i such that $1 \leq r < \frac{n}{p_i}$ and $r \not\equiv 0 \mod p_i$, then the normalized Laplacian matrices of the complete bipartite graph $\mathcal{K}_{r,n-r}$ and its complement graph $\overline{\mathcal{K}_{r,n-r}}$ are entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$ for all vertex labelings.*

Proof: If A is entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2 p_3}$, then it is entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \mathbb{C}^{p_3}$. So we only need to consider the case $n = p_1 p_2$. Without loss of generality we assume $1 \leq r < p_1$ and $r \not\equiv 0 \mod p_2$. Let the vertices of $\mathcal{K}_{r,n-r}$ be partitioned into two disjoint sets of vertices A and B, with edges from every member of A to every member of B and $|A| = r$. Since $r \not\equiv 0 \mod p_2$, there exists v_u such that (v_u, w_a) and (v_u, w_b) are vertices in A and B respectively. The degree of (v_u, w_b) is r in $\mathcal{G}$. Let us look at the degree of (v_u, w_b) in $\mathcal{G}^{pT}$. Consider the vertices (v_y, w_b) for $v_y \neq v_u$. If $(v_y, w_b) \in A$, then $\{(v_u, w_b), (v_y, w_b)\}$ is an edge of both $\mathcal{G}$ and $\mathcal{G}^{pT}$. If $(v_y, w_b) \in B$, then $\{(v_u, w_a), (v_y, w_b)\}$ is an edge of $\mathcal{G}$ and thus $\{(v_u, w_b), (v_y, w_a)\}$ is an edge of $\mathcal{G}^{pT}$. Thus we have identified $p_1 - 1$ edges connected to (v_u, w_b) in $\mathcal{G}^{pT}$. Finally $\{(v_u, w_a), (v_u, w_b)\}$ is an edge in both $\mathcal{G}$ and $\mathcal{G}^{pT}$. Thus the degree of (v_u, w_b) in $\mathcal{G}^{pT}$ is at least $p_1 > r$ and thus by Theorem 1 the normalized Laplacian matrix is entangled. The part about $\overline{\mathcal{K}_{r,n-r}}$ follows from Corollary 1. □

Note that $\overline{\mathcal{K}_{r,n-r}} = \mathcal{K}_r \cup \mathcal{K}_{n-r}$ is the union of two complete graphs. Since the normalized Laplacian matrix of a complete graph is separable [2], this means that the union of graphs does not necessarily preserve separability of Laplacian matrices.

Corollary 2 *Let $n = p_1 p_2 \cdots p_m$, where $p_i \geq 2$. If $1 \leq r < \min_i p_i$, then the normalized Laplacian matrices of the graph $\mathcal{K}_{r,n-r}$ and its complement graph $\overline{\mathcal{K}_{r,n-r}}$ are entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$ under all vertex labelings.*

Ref. [2] shows that the bipartite separability of $\mathcal{K}_{1,n-1}$ and $\mathcal{K}_n$ do not depend on the vertex labeling. The normalized Laplacian matrix of $\mathcal{K}_{1,n-1}$ is entangled for all vertex labelings and the normalized Laplacian matrix of $\mathcal{K}_n$ is separable for all vertex labelings. It was posed as an open question in Ref. [2] whether there are other classes of graphs with this property. Theorem 3 and Corollary 2 list additional classes of graphs whose normalized Laplacian matrices are entangled under any vertex labeling.

If $r \equiv 0 \mod p_2$, then it is easy to find a vertex labeling such that $\mathcal{K}_{r,n-r}$ is separable in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2}$. In fact, since in this case $r = kp_2$ for some integer k, we assign the set A to be vertices with coordinates (i, j), $1 \leq i \leq k$, $1 \leq j \leq p_2$ and this vertex labeling will result in a separable density matrix by Theorem 2. On the other hand, the following result shows that for nontrivial graphs $\mathcal{K}_{r,n-r}$ of more than 4 vertices, there exists a vertex labeling such that the normalized Laplacian matrix is entangled.

Theorem 4 *Let $n = p_1 p_2 \cdots p_m > 4$, where $p_i \geq 2$. For all nonempty complete bipartite graphs $\mathcal{K}_{r,n-r}$, there exists a vertex labeling such that the normalized Laplacian matrices of $\mathcal{K}_{r,n-r}$ and $\overline{\mathcal{K}_{r,n-r}}$ are entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$.*

Proof: As before, we only need to consider the case $n = p_1 p_2$. Without loss of generality, let us assume that $p_1 \leq p_2$ and $r \leq \frac{n}{2}$. Since $\mathcal{K}_{r,n-r}$ is not empty, $0 < r < n$. Let A and B be defined as in Theorem 3. If $r < p_1$, then the result follows from Corollary 2. Suppose $r = p_1$. Assign to the elements of A the labeling $(v_1, w_2), (v_2, w_1), \cdots, (v_{p_1}, w_1)$. Assign an element of B the labeling (v_1, w_1). This vertex has degree r in $\mathcal{G}$. An edge in $\mathcal{G}$ from this vertex to each of the vertices of A in will remain an edge in $\mathcal{G}^{pT}$. Since $p_2 \geq 3$, we can assign another vertex in B to (v_1, w_3). There is an edge $\{(v_2, w_1), (v_1, w_3)\}$ in $\mathcal{G}$, so there is an edge $\{(v_1, w_1), (v_2, w_3)\}$ in $\mathcal{G}^{pT}$ and (v_1, w_1) has degree at least $r + 1$ in $\mathcal{G}^{pT}$. By Theorem 1 the normalized Laplacian matrix is entangled. Suppose $r > p_1$. Let p_1 elements from A be assigned the labeling $(v_1, w_1), (v_2, w_1), \cdots, (v_{p_1}, w_1)$. Since $n - r \geq \frac{n}{2} \geq p_2 > p_2 - 1$, we can pick $p_2 - 1$ elements from B and assign them the labeling $(v_1, w_2), \cdots, (v_1, w_{p_2})$. Since each element of A is connected to each element of B, $\{(v_i, w_1), (v_1, w_j)\}$ is an edge in $\mathcal{G}$ for $1 \leq i \leq p_1$, $2 \leq j \leq p_2$. Thus $\{(v_1, w_1), (v_i, w_j)\}$ is an edge in $\mathcal{G}^{pT}$ which means that (v_1, w_1) is connected to $p_1(p_2 - 1) = n - p_1$ nodes, i.e. it has degree at least $n - p_1$ in $\mathcal{G}^{pT}$. The vertex (v_1, w_1) is in A so it has degree $n - r < n - p_1$ in $\mathcal{G}$. Again the normalized Laplacian matrix is entangled by Theorem 1. The part about $\overline{\mathcal{K}_{r,n-r}}$ follows from Corollary 1. $\square$

In Section 5 we will show that the normalized Laplacian matrix of the complete graph $\mathcal{K}_n$ is multipartite separable under any vertex labeling (Corollary 7). Are there graphs besides the complete graph whose Laplacian matrix is multipartite separable for all vertex labeling? Theorem 4 shows that they will not be complete bipartite graphs nor their complement graphs. The results in the following section identify other classes of graphs whose normalized Laplacian matrix is entangled for some vertex labeling.

3.2 Vertex degree conditions for multipartite entanglement

For a vertex v, let $deg(v)$ denote its vertex degree.

Theorem 5 *Let $n = p_1 p_2 \cdots p_m$, where $p_i \geq 2$. Let $\mathcal{G}$ be a nonempty graph such that $\min_w deg(w) < \frac{n}{p_i} - 1$ for some i. Then there is a vertex labeling such that the normalized Laplacian matrix of the graph $\mathcal{G}$ is entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$.*

Proof: As in Theorem 3 we only need to consider the case $n = p_1 p_2$. Suppose that w is the vertex with minimal degree d in $\mathcal{G}$ and without loss of generality, $d < p_2 - 1$. We will construct a vertex labeling such that the normalized Laplacian matrix is entangled. Let $N(w)$ be the set of neighbors of w. By definition, $|N(w)| = d$. We assign $w = (v_1, w_1)$, and $(v_1, w_i), 2 \leq i \leq d+1$ for the vertices in $N(w)$. Since $\{(v_1, w_1), (v_1, w_i)\}, 2 \leq i \leq d+1$ is an edge of both $\mathcal{G}$ and $\mathcal{G}^{pT}$, w has degree at least d in $\mathcal{G}^{pT}$. Finally for a vertex $u \notin \{w\} \cup N(w)$, $deg(u) \geq d$. Since the graph is not empty, we can find u such that $deg(u) > 0$. We set $u = (v_2, w_1)$. There are 2 cases to consider. In case 1, u is connected to a vertex $y \notin \{w\} \cup N(w)$. We set $y = (v_1, w_{p_2})$. This means that $\{(v_1, w_1), (v_2, w_{p_2})\}$ is an edge in $\mathcal{G}^{pT}$. In case 2, u is connected to a vertex in $N(w)$, say (v_1, w_c). Then $\{(v_1, w_1), (v_2, w_c)\}$ is an edge in $\mathcal{G}^{pT}$. In either case w has degree strictly larger than d in $\mathcal{G}^{pT}$ and the result follows from Theorem 1. □

Corollary 3 *Let $n = p_1 p_2 \cdots p_m$, where $p_i \geq 2$. Let $\mathcal{G}$ be a noncomplete graph such that $\max_w deg(w) > n - \frac{n}{p_i}$ for some i. Then there is a vertex labeling such that the normalized Laplacian matrix of the graph $\mathcal{G}$ is entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$.*

Proof: Follows from Theorem 5 and Corollary 1. □

Theorem 6 *Let $n = p_1 p_2 \cdots p_m > 4$, where $p_i \geq 2$. Let $\mathcal{G}$ be a nonempty graph such that $\min_w deg(w) < p_i + \frac{n}{p_i} - 2$ for some i. Then there is a vertex labeling such that the normalized Laplacian matrix of the graph $\mathcal{G}$ is entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$.*

Proof: We assume without loss of generality that $n = p_1 p_2$ and $p_2 > 2$. Suppose vertex w has minimal degree d. If $d < p_2 - 1$ then the result follows from Theorem 5. Therefore we assume that $d \geq p_2 - 1$. Assign w to (v_1, w_1). Let $N(w)$ be the neighbors of w which we partition into two sets A and B of size $p_2 - 2$ and $d - p_2 + 2$ respectively. Since $d \geq p_2 - 1$ and $p_2 > 2$ the sets A and B are both nonempty. We assign vertices in A to $(v_1, w_2), \cdots, (v_1, w_{p_2 - 1})$. We assign vertices in B to $(v_2, w_1), \cdots, (v_{d - p_2 + 3}, w_1)$. Note that $d - p_2 + 3 \leq p_1$ by hypothesis. It is clear that an edge in $\mathcal{G}$ from w to the elements in $N(w)$ will remain an edge in $\mathcal{G}^{pT}$. Thus w has degree at least d in $\mathcal{G}^{pT}$. Consider the vertex (v_2, w_1) in B. It has degree $\geq d \geq p_2 - 1 \geq 2$. This means that it is connected to a vertex $(v_u, w_u) \neq (v_1, w_1)$. There are 3 cases to consider. In case 1, (v_u, w_u) is in A, i.e. $v_u = v_1$. Thus there is an edge $\{(v_1, w_u), (v_2, w_1)\}$ in $\mathcal{G}$ and an edge $\{(v_1, w_1), (v_2, w_u)\}$ in $\mathcal{G}^{pT}$. In case 2, (v_u, w_u) is in B. In this case, switch the assignment with a vertex in A and this reduces it to case 1. In case 3, $(v_u, w_u) \notin A \cup B$. We reassign it to (v_1, w_{p_2}). Then $\{(v_2, w_1), (v_1, w_{p_2})\}$ is an edge in $\mathcal{G}$ and $\{(v_1, w_1), (v_2, w_{p_2})\}$ is an edge in $\mathcal{G}^{pT}$. In all cases, the degree of w is strictly larger than d in $\mathcal{G}^{pT}$ and the result follows from Theorem 1. □

Corollary 4 *Let $n = p_1 p_2 \cdots p_m > 4$, where $p_i \geq 2$. Let $\mathcal{G}$ be a noncomplete graph such that $\max_w deg(w) > n - p_i - \frac{n}{p_i} + 1$ for some i. Then there is a vertex labeling such that the normalized Laplacian matrix of the graph $\mathcal{G}$ is entangled in $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$.*

Proof: Follows from Theorem 6 and Corollary 1. □

3.3 Bipartite separability in $\mathbb{C}^2 \times \mathbb{C}^2$ and $\mathbb{C}^2 \times \mathbb{C}^3$

The criterion that each vertex of $\mathcal{G}$ has the same degree as the same vertex in $\mathcal{G}^{pT}$ was shown in [3] to be a necessary and sufficient condition for normalized Laplacian matrices to be separable in $\mathbb{C}^2 \times \mathbb{C}^q$. Checking this vertex degree criterion on all graphs of 4 vertices shows that the complete graph $\mathcal{K}_4$ and the two graphs in Fig. 1 are separable in $\mathbb{C}^2 \times \mathbb{C}^2$ for all vertex labelings. For the graphs in Fig. 2 the normalized Laplacian matrix is entangled for all vertex labelings. For all other graphs[2] with 4 vertices, there exists a vertex labeling such that the normalized Laplacian matrix is entangled and a vertex labeling such that it is separable in $\mathbb{C}^2 \times \mathbb{C}^2$.

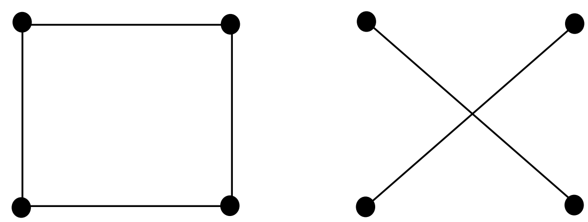

Figure 1: Noncomplete graphs on 4 vertices whose normalized Laplacian matrices are separable $\mathbb{C}^2 \times \mathbb{C}^2$ regardless of the vertex labeling.

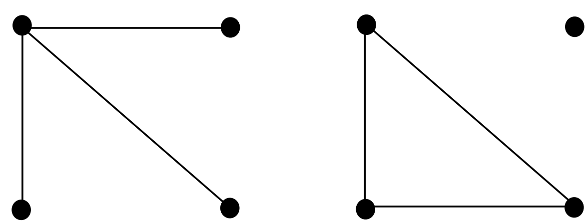

Figure 2: Graphs on 4 vertices whose normalized Laplacian matrices are entangled $\mathbb{C}^2 \times \mathbb{C}^2$ regardless of the vertex labeling.

For the case $\mathbb{C}^2 \times \mathbb{C}^3$, the complete graph $\mathcal{K}_6$ is the only graph with 6 vertices such that its normalized Laplacian matrix is separable in $\mathbb{C}^2 \times \mathbb{C}^3$ for all vertex labelings. The 6 graphs in Fig. 3 and their complement graphs have normalized Laplacian matrices that are entangled under all vertex labelings. For all other graphs with 6 vertices, there exists vertex labelings which results in a separable normalized Laplacian matrix and vertex labelings which results in an entangled normalized Laplacian matrix.

[2] We exclude the empty graph from consideration, since the Laplacian matrix is the zero matrix and has zero trace.

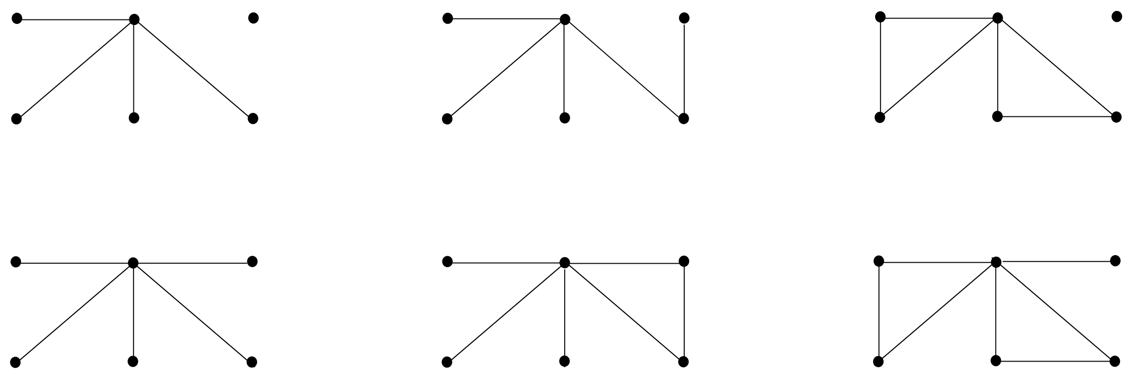

Figure 3: Graphs on 6 vertices whose normalized Laplacian matrices are entangled in $\mathbb{C}^2 \times \mathbb{C}^3$ regardless of the vertex labeling. Their complement graphs also have this property.

3.4 Multipartite separability in $\mathbb{C}^2 \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$, $n > 4$

Theorem 7 *Let $n = 2p_2p_3 \cdots p_m > 4$. If $\mathcal{G}$ is a graph that is not complete nor empty, then there is a vertex labeling such that the normalized Laplacian matrix is entangled in $\mathbb{C}^2 \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$.*

Proof: By Theorem 6 such a vertex labeling exists if $\min_w deg(w) \leq \frac{n}{2} - 1$. By Corollary 4 such a vertex labeling exists if $\max_w deg(w) \geq \frac{n}{2}$. It is clear that one of these two inequalities must be satisfied for any graph. $\square$

Is Theorem 7 true for the general case $\mathbb{C}^{p_1} \times \mathbb{C}^{p_2} \times \cdots \times \mathbb{C}^{p_m}$? Computer experiments indicate that for $4 < n \leq 9$, n composite, for all noncomplete graphs there exists a vertex labeling such that the normalized Laplacian matrix is entangled. It remains to be seen whether this is true for all noncomplete graphs with $n > 4$.

4 A joint matrix decomposition result

Let I denote the identity matrix and J denote the matrix of all 1's. The i-th unit vector is denoted as e_i.

Definition 6 *For a complex matrix A, let $|A|$ denote the real nonnegative matrix B such that $B_{ij} = |A_{ij}|$. Let $|A|_*$ be the matrix B such that $B_{ii} = A_{ii}$ and $B_{ij} = |A_{ij}|$ for $i \neq j$.*

Theorem 8 *Let D be a diagonal real matrix and A be a Hermitian matrix. If $D - A$ is row diagonally dominant, then there exists λ_i, μ_i real numbers and v_i n-vectors such that*

$$\begin{aligned} D &= \sum_i \mu_i v_i v_i^T \\ A &= \sum_i \lambda_i v_i v_i^T \qquad (1) \\ \mu_i &\geq \lambda_i \end{aligned}$$

Proof: To prove that the pair (D, A) satisfy the conditions in Eqs. (1), we decompose A and D into $A = A_1 + A_2$ and $D = D_1 + D_2$ such that the pairs (D_1, A_1) and (D_2, A_2) each satisfies the conditions in Eqs. (1). Let $r_1, r_2, \cdots, r_n$ be the row sums of $|A|_*$. Let $A_1 = A - A_2$ where

$$A_2 = \begin{pmatrix} 0 & & & & \\ & r_2 - r_1 & & & \\ & & r_3 - r_1 & & \\ & & & \ddots & \\ & & & & r_n - r_1 \end{pmatrix} \tag{2}$$

It is clear that all row sums of $|A_1|_*$ is equal to r_1. Let v_i be the normalized eigenvectors of A_1 with eigenvalues λ_i. Thus $A_1 = \sum_i \lambda_i v_i v_i^T$. Let $D_1 = r_1 I = r_1 \sum_i v_i v_i^T$. By Gershgorin's circle criterion, $\lambda_i \leq r_1$, so (D_1, A_1) satisfies the conditons in Eqs. (1). Let $D_2 = D - D_1$. The row diagonally dominant condition of $D - A$ ensures that D_2 is a diagonal matrix such that $(D_2)_{ii} \geq (A_2)_{ii}$. Since D_2 and A_2 can both be expressed as $D_2 = \sum_i (D_2)_{ii} e_i e_i^T$ and $A_2 = \sum_i (A_2)_{ii} e_i e_i^T$, the proof is complete. □

Corollary 5 *Let D_j be diagonal real matrices and A_j be Hermitian matrices. If $D_j - A_j$ is row diagonally dominant, then there exists λ_i, μ_i real numbers and v_i n-vectors such that*

$$\begin{aligned} \sum_j D_j &= \sum_i \mu_i v_i v_i^T \\ \sum_j A_j &= \sum_i \lambda_i v_i v_i^T \\ \mu_i &\geq \lambda_i \end{aligned} \tag{3}$$

Theorem 9 *Let D_i be diagonal real matrices and P_i be Hermitian matrices in $\mathbb{C}^{p_i \times p_i}$. Suppose $D_i - P_i$ is row diagonally dominant for $1 \leq i \leq m$. If $A = D_1 \otimes \cdots \otimes D_m - P_1 \otimes \cdots \otimes P_m$, then $\frac{1}{tr(A)} A$ is a separable density matrix in $\mathbb{C}^{p_i} \times \cdots \times \mathbb{C}^{p_m}$, provided $tr(A) \neq 0$.*

Proof: By Theorem 8 the matrices $D_1 \otimes \cdots \otimes D_m$ and $P_1 \otimes \cdots \otimes P_m$ can be decomposed as $\otimes_j \sum_i \mu_i^j v_i^j {v_i^j}^T$ and $\otimes_j \sum_i \lambda_i^j v_i^j {v_i^j}^T$ where $\mu_i^j \geq \lambda_i^j$. This means that A can be written as $A = \otimes_j \sum_i \left(\mu_i^j - \lambda_i^j\right) v_i^j {v_i^j}^T$, i.e. $\frac{1}{tr(A)} A$ is separable. □

Corollary 6 *Let D_i^j be diagonal real matrices and P_i^j be Hermitian matrices for $1 \leq i \leq m$, $1 \leq j \leq k$. Suppose $D_i^j - P_i^j$ is row diagonally dominant for $1 \leq i \leq m$, $1 \leq j \leq k$. If $A = \sum_{j=1}^k D_1^j \otimes \cdots \otimes D_m^j - P_1^j \otimes \cdots \otimes P_m^j$, then $\frac{1}{tr(A)} A$ is a separable density matrix in $\mathbb{C}^{p_i} \times \cdots \times \mathbb{C}^{p_m}$, provided $tr(A) \neq 0$.*

5 Multipartite separability of graph products

Definition 7 *For a graph with adjacency matrix A, a matrix of the form $L = D - A$ where D is a diagonal real matrix such that L is row diagonally dominant is called a* generalized Laplacian matrix *of the graph.*

Definition 7 is different from the definition in [6] in that here we assume A to be complex matrices and require row diagonal dominance. Clearly this definition does not define L uniquely since there are many choices for the matrix D. However this will not matter for the results in this section.

In the rest of this section we assume that the adjacency matrix A of a graph is a real nonnegative matrix. In this case, a matrix element $A_{ij} > 0$ can be considered as an edge from vertex i to vertex j with weight A_{ij}. Without loss of generality, we assume that $0 \leq A_{ij} \leq 1$.

Definition 8 *For a complex matrix A, $r(A)$ is the diagonal matrix such that $r(A)_{ii} = \sum_j A_{ij}$.*

It is clear that $r(|A|) - A$ is a Laplacian matrix of a graph with adjacency matrix A. The following Lemma is easy to show.

Lemma 1 $r(A \otimes B) = r(A) \otimes r(B)$, $r(|A \otimes B|) = r(|A|) \otimes r(|B|)$

A *graph product* of $\mathcal{G}$ and $\mathcal{H}$ (denoted as $\mathcal{G} \diamond \mathcal{H}$) is defined as a graph with vertices $V(\mathcal{G}) \times V(\mathcal{H})$ and edges defined by:

$\{(u, v), (w, y)\}$ is an edge if and only if Q is true. The relation Q is of the form $P_1 \vee P_2 \vee \cdots$, where each P_i is one of 8 conditions:

- $R_1 : (u \quad \text{adj} \quad w) \wedge (v \quad \text{adj} \quad y)$
- $R_2 : (u \quad \text{adj} \quad w) \wedge (v = y)$
- $R_3 : (u \quad \text{adj} \quad w) \wedge (v \quad \neg\text{adj} \quad y)$
- $R_4 : (u = w) \wedge (v \quad \text{adj} \quad y)$
- $R_5 : (u = w) \wedge (v \quad \neg\text{adj} \quad y)$
- $R_6 : (u \quad \neg\text{adj} \quad w) \wedge (v \quad \text{adj} \quad y)$
- $R_7 : (u \quad \neg\text{adj} \quad w) \wedge (v = y)$
- $R_8 : (u \quad \neg\text{adj} \quad w) \wedge (v \quad \neg\text{adj} \quad y)$

Thus there are $2^8 = 256$ different types of graph products. Table 1 lists some possibilities for Q and the common names associated to the corresponding graph product [7].

Let the adjacency matrices of $\mathcal{G}$ and $\mathcal{H}$ be G and H respectively. It is easy to show that the adjacency matrix of a graph product $\mathcal{G} \diamond \mathcal{H}$ is $\sum_i T_i$ where to each condition P_i in Q corresponds a matrix T_i according to Table 2.

Table 1: Commonly used graph products.

Condition Q	name(s)
R_1	tensor product, categorical product, direct product, cardinal product
$R_1 \vee R_2 \vee R_4$	strong product, normal product
$R_2 \vee R_4$	Cartesian product
$R_1 \vee R_2 \vee R_3 \vee R_4$	lexicographical product
$R_1 \vee R_2 \vee R_3 \vee R_4 \vee R_6$	disjunctive product, co-normal product

Table 2: Matrices T_i corresponding to each of the 8 conditions R_i.

Condition	T_i
$(u \quad \text{adj} \quad w) \wedge (v \quad \text{adj} \quad y)$	$G \otimes H$
$(u \quad \text{adj} \quad w) \wedge (v = y)$	$G \otimes I$
$(u \quad \text{adj} \quad w) \wedge (v \quad \neg\text{adj} \quad y)$	$G \otimes (J - I - H)$
$(u = w) \wedge (v \quad \text{adj} \quad y)$	$I \otimes H$
$(u = w) \wedge (v \quad \neg\text{adj} \quad y)$	$I \otimes (J - I - H)$
$(u \quad \neg\text{adj} \quad w) \wedge (v \quad \text{adj} \quad y)$	$(J - I - G) \otimes H$
$(u \quad \neg\text{adj} \quad w) \wedge (v = y)$	$(J - I - G) \otimes I$
$(u \quad \neg\text{adj} \quad w) \wedge (v \quad \neg\text{adj} \quad y)$	$(J - I - G) \otimes (J - I - H)$

Theorem 10 *The complement graph of a graph product is another graph product.*

Proof: We will prove this for a graph product of two graphs, as the general case is similar. The adjacency matrix of $\mathcal{G} \diamond \mathcal{H}$ is $\sum_i T_i$. Its complement graph has adjacency matrix $J - I - \sum_i T_i = J \otimes J - I \otimes I - \sum_i T_i$. Let $c(P) = J - I - P$. The matrix J can be decomposed as $J = c(P) + I + P$ and $J \otimes J - I \otimes I = (c(G) + I + G) \otimes (c(H) + I + H) - I \otimes I$ is exactly the sum of the 8 possible T_i's in Table 2. This means that the adjacency matrix of the complement graph is also of the form $\sum_i T_i$ and the proof is complete. □

The following theorem shows that the normalized Laplacian matrix of a graph product is multipartite separable.

Theorem 11 *For a set of graphs $\mathcal{P}_i$, if A is a Laplacian matrix of $\mathcal{P}_1 \diamond \mathcal{P}_2 \diamond \cdots \diamond \mathcal{P}_m$ under the canonical vertex labeling, where $\diamond$ is a graph product and $tr(A) \neq 0$, then $\frac{1}{tr(A)}A$ is a separable density matrix in $\mathbb{C}^{p_1} \times \cdots \times \mathbb{C}^{p_m}$, where p_i is the order of $\mathcal{P}_i$.*

Proof: Let B be the adjacency matrix of $\mathcal{P}_1 \diamond \mathcal{P}_2 \diamond \cdots \diamond \mathcal{P}_m$. By Lemma 1, the diagonal matrix of the row sums of $|B|$ can be written as $\sum_j \otimes_i D_i^j$ and A can be written as $\sum_j \otimes_i D_i^j - \sum_j \otimes_i T_i^j$. The theorem is then a direct consequence of Corollary 6. □

The special cases of tensor products ($Q = R_1$) and $m = 2$ (bipartite separability), $m = 3$ (tripartite separability) were proven in [2] and [4] respectively. The proof of Theorem 11 can be used to show that Theorem 11 is true for generalized Laplacian matrices of graph products even when the adjacency matrices are complex matrices.[3]

[3]In which case we define graph products via the matrices T_i's.

Note that the graphs in Fig. 1 are graph products.

The next result shows that the normalized Laplacian matrix of a complete graph is multipartite separable.

Corollary 7 *Let $A = nI - J$ be the Laplacian matrix of the complete graph $\mathcal{K}_n$, and $n = p_1 p_2 \cdots p_m$. Then $\frac{1}{tr(A)}A$ is a separable density matrix in $\mathbb{C}^{p_1} \times \cdots \times \mathbb{C}^{p_m}$.*

Proof: This follows from Theorem 11 and the fact that the complete graph $\mathcal{K}_{pq}$ is the strong product of $\mathcal{K}_p$ and $\mathcal{K}_q$. □

Corollary 7 for the cases of $m = 2$ and $m = 3$ were proven in [2] and [4] respectively.

6 Conclusions

We continue the study of normalized Laplacian matrices of graphs as density matrices and analyze their entanglement properties. In particular, we identify graphs whose normalized Laplacian matrices are entangled for every vertex labeling or whose Laplacian matrices are entangled for some vertex labeling. Furthermore, we show that normalized Laplacian matrices of graph products are multipartite separable.

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