On Some Partitions Related to $\mathbb{Q}(\sqrt{2})$

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Abstract

We offer some new identities for a bipartition function, which has a relation to a Hecke-type identity of Andrews. Further, we show this partition function is lacunary, and relate it to a real quadratic field.

1. Introduction and Statement of Results

In the last two decades, several authors [2, 6] have observed certain q-series and q-products have relations to the arithmetic of real quadratic fields. This observation was initiated in [2], where it was discovered that certain q-series are related to the real quadratic field $\mathbb{Q}(\sqrt{6})$.

The objective of this paper is to offer a partition theoretic interpretation of a generating function related to a Hecke-type identity given by Andrews [1]

$$\prod_{n=1}^{\infty}(1-q^n)(1-q^{2n}) = \sum_{r\geq 2|n|}(-1)^{r+n}q^{r(r+1)/2-n^2}, \tag{1}$$

which is related to the arithmetic of $\mathbb{Q}(\sqrt{2})$. For the left side of (1), we find that the product generates a bipartition $\pi=(\pi_1,\pi_2)$ counted with weight $(-1)^{n(\pi_1)+n(\pi_2)}$, where π_1 is a partition into distinct parts, and π_2 is a partition into distinct even parts. Here we let $n(\pi_1)$ denote the number of parts taken from π_1.

For relevant material, and an introduction to partition theory, we refer the reader to [4]. Also, we shall use standard notation throughout [7, 8]

$$(a;q)_n = (a)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}),$$

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

Definition 1.1. *Let $\phi_{m,k}(l,n)$ be the number of bipartitions $\sigma = (\mu, \lambda)$ of n where μ is a partition into distinct parts with minimal part k, and λ is a partition into distinct even parts where all parts are $> m$ plus twice the minimal part of μ, counted with weight $(-1)^{n(\mu)}$. Moreover, l keeps track of the number of parts from λ.*

We note here that m is taken to be a positive even integer. The generating function for $\phi_{m,k}(l,n)$ will be given in the next section in the proof of Theorem 1.3.

Definition 1.2. *We define*

$$\Phi_m(l,n) := \sum_{k \geq 0} \phi_{m,k}(l,n).$$

Theorem 1.3. *Let $\Phi_0(l,n)$ be the $m = 0$ case of Definition 1.2. Then $\Phi_0(l,n)$ equals the sum of $(-1)^{r+j}$ over all pairs (r,j) such that $n = 2r^2 + r - j^2$, $|j| \leq r$, $r = l$.*

Before proceeding to the next theorem, we mention in passing that

$$\Phi_m(n) := \sum_{k,l \geq 0} \phi_{m,k}(l,n),$$

and

$$\chi_m(n) := \sum_{k,l \geq 0} (-1)^l \phi_{m,k}(l,n).$$

Theorem 1.4. *We have that $\chi_0(n) - \chi_2(n-1)$ is equal to the number of inequivalent solutions of $x^2 - 2y^2 = k$ with norm $8k+1$ in which $x + y \equiv 1 \pmod 4$ over the number in which $x + y \equiv 3 \pmod 4$.*

We mention that the generating function for $\chi_0(n) - \chi_2(n-1)$ is equal to (1). A brief outline of an analytic proof of this is given at the end of the proof of this theorem. Also, the weight for this partition function should be easily recognized to be $(-1)^{n(\mu)+n(\lambda)}$.

Corollary 1.5. $\chi_0(n) = \chi_2(n-1)$ *for almost all natural n.*

Theorem 1.6. $\Phi_0(n) - \Phi_2(n-1)$ *is equal to the excess of the number of inequivalent solutions of $x^2 - 2y^2 = k$ with norm $8k+1$ in which $x + 2y \equiv 1 \pmod 8$ or $x + 2y \equiv 7 \pmod 8$ over the number in which $x + 2y \equiv 3 \pmod 8$ or $x + 2y \equiv 5 \pmod 8$.*

Corollary 1.7. $\Phi_0(n) = \Phi_2(n-1)$ *for almost all natural n.*

Theorem 1.8. $\Phi_0(n) + \Phi_2(n-1)$ *is equal to the excess of the number of inequivalent solutions of $x^2 - 2y^2 = k$ with norm $8k+1$ in which $x + 2y \equiv 1 \pmod 8$ or $x + 2y \equiv 3 \pmod 8$ over the number in which $x + 2y \equiv 5 \pmod 8$ or $x + 2y \equiv 7 \pmod 8$.*

Corollary 1.9. $\Phi_0(n) + \Phi_2(n-1) = 0$ *for almost all natural n.*

2. Proofs of Theorems

In this section we will use a lemma given by Lovejoy [8] to prove the q-series identities that generate the desired partition functions. However, first we need to obtain a new Bailey pair by appealing to a result found in [5].

Lemma 2.1. *If $n \geq 0$ and*

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_n(a,q)}{(aq)_{n+r}(q)_{n-r}}, \tag{2}$$

then $(\alpha'_n(a,q), \beta'_n(a,q))$ forms a Bailey pair with respect to a where

$$\alpha'_n(a,q) = \alpha_n(a^2,q^2),$$

and

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(-aq)_{2k} q^{n-k}}{(q^2;q^2)_{n-k}} \beta_k(a^2,q^2).$$

From here we can change the base of a known pair from q^2 to q to obtain the following new Bailey pair:

Lemma 2.2. *The pair of sequences (α_n, β_n) form a Bailey pair with respect to q where*

$$\alpha_n = q^{2n^2+n}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-j^2},$$

and

$$\beta_n = \sum_{k=0}^{n} \frac{q^{n-k}}{(q^2;q^2)_{n-k}}.$$

Proof of Lemma 2.2: Take the Bailey pair with respect to q^2 (with q replaced by q^2 in the definition) from [3], given by

$$\alpha_n = q^{2n^2+n}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-j^2},$$

and

$$\beta_n = \frac{1}{(-q^2)_{2n}},$$

and insert it in Lemma 2.1 (with $a = q$).

Our last lemma is the $b = q$ case of the lemma given in [9].

Lemma 2.3. *If the pair of sequences (α_n, β_n) form a Bailey pair with respect to q then*

$$(1-q)\sum_{n=0}^{\infty}\frac{\alpha_n z^n}{1-q^{2n+1}} = (z, q; q)_\infty \sum_{j,n=0}^{\infty}\frac{q^{n+2jn}z^j\beta_j}{(z)_n(q)_n}. \tag{3}$$

Proof of Theorem 1.3: Inserting the pair given in Lemma 2.2 into Lemma 2.3 gives

$$\sum_{n\geq 0} z^n q^{2n^2+n}\sum_{j=-n}^{n}(-1)^j q^{-j^2} = \sum_{n=0}^{\infty} q^n(zq^{2n+2}; q^2)_\infty (q^{n+1}; q)_\infty, \tag{4}$$

since

$$\sum_{n=0}^{\infty}\beta_n z^n = \frac{1}{(1-z)(zq; q^2)_\infty},$$

and

$$\sum_{n=0}^{\infty}\frac{q^n(zq^n; q)_\infty(q^{n+1}; q)_\infty}{(1-zq^{2n})(zq^{2n+1}; q^2)_\infty} = \sum_{n=0}^{\infty} q^n(zq^{2n+2}; q^2)_\infty(q^{n+1}; q)_\infty.$$

Now we consider the right hand side of the series above. First, recall [7, p.56] that $q^k(1+q^{k+1})(1+q^{k+2})\cdots$ generates a partition into distinct parts with minimal part k. Also, $(zq^{2k+2}; q^2)_\infty$ generates a partition into distinct even parts $\geq 2k+2$, and z keeps track of the number of parts. Thus, replacing z by $-z$ we find

$$q^k(-zq^{2k+2}; q^2)_\infty(q^{k+1}; q)_\infty \tag{5}$$

generates a bipartition (μ, λ) where μ is a partition into distinct parts with minimal part k, and λ is a partition into distinct even parts where all parts are $>$ twice the minimal part of μ, with weight $(-1)^{n(\mu)}$, and z still keeping track of the number of even parts from λ. The generating function for definition 1.1 should be clear after replacing z by zq^m in (5), where m is taken to be a positive even integer. Summing over all k in (5) (with z replaced by zq^m) gives the generating function for $\Phi_m(l, n)$, where l is the number of parts of λ.

Proof of Theorem 1.4: Recall the ring of integers $\mathbb{Z}[\sqrt{2}]$ has its norm function equal to $x^2 - 2y^2$. In [10] it was shown that

$$\sum_{\substack{n\geq 0\\ |j|\leq n}}^{\infty}(-1)^j\left(q^{(4n+1)^2-2(2j)^2} - q^{(4n+3)^2-2(2j)^2}\right), \tag{6}$$

generates the number of inequivalent solutions of $x^2 - 2y^2 = k$ with norm $8k+1$ in which $x + y \equiv 1 \pmod 4$ over the number in which $x + y \equiv 3 \pmod 4$. So the remainder of the proof requires us to show the generating function for $\chi_0(n) - \chi_2(n-1)$ is equal to (6). To see this, add (5) to itself when z is replaced by zq^2 and multiplied by $-q$, after summing over k, to get

$$\sum_{n=0}^{\infty} q^n (zq^{2n+2};q^2)_\infty (q^{n+1};q)_\infty - q \sum_{n=0}^{\infty} q^n (zq^{2n+4};q^2)_\infty (q^{n+1};q)_\infty$$
$$= q^{-1/8} \sum_{\substack{n \geq 0 \\ |j| \leq n}}^{\infty} z^n (-1)^j (q^{[(4n+1)^2 - 2(2j)^2]/8} - q^{[(4n+3)^2 - 2(2j)^2]/8}). \quad (7)$$

After setting $z = 1$, the first sum on the left hand side is easily seen to be the generating function for $\chi_0(n)$. The weight here being -1 raised to the number of parts of λ plus the number of parts of μ. Now the next sum is the generating function for $\chi_2(n)$ multiplied by q. This is clear to see since the number of parts of λ are all even and > 2 plus twice the minimal part of μ.

Before proceeding to the next proof, we mention that the corollaries easily follow from the lacunarity of the series involving indefinite quadratic forms. Further, it has been noted in [10] that (6) is equivalent to the right side of (1) when q is replaced by q^8 and multiplied by q. Thus, our claim following Theorem 1.4 is easily established analytically.

Proof of Theorem 1.6: The generating function for the number of inequivalent solutions of $x^2 - 2y^2 = k$ with norm $8k + 1$ in which $x + 2y \equiv 1 \pmod 8$ or $x + 2y \equiv 7 \pmod 8$ over the number in which $x + 2y \equiv 3 \pmod 8$ or $x + 2y \equiv 5 \pmod 8$ was given in [6]:

$$\sum_{\substack{n \geq 0 \\ |j| \leq n}}^{\infty} (-1)^{n+j} (q^{(4n+1)^2 - 2(2j)^2} - q^{(4n+3)^2 - 2(2j)^2}),$$

and follows from the special case $z = -1$ of (8). This time, the generating functions for the first two sums in (8) only have weight $(-1)^{n(\mu)}$.

Proof of Theorem 1.8: The proof is identical to the proof of Theorem 1.4, except now we add (5) to itself when z is replaced by zq^2 and multiplied by q to get

$$\sum_{n=0}^{\infty} q^n (zq^{2n+2};q^2)_\infty (q^{n+1};q)_\infty + q \sum_{n=0}^{\infty} q^n (zq^{2n+4};q^2)_\infty (q^{n+1};q)_\infty$$
$$= q^{-1/8} \sum_{\substack{n \geq 0 \\ |j| \leq n}}^{\infty} z^n (-1)^j (q^{[(4n+1)^2 - 2(2j)^2]/8} + q^{[(4n+3)^2 - 2(2j)^2]/8}). \quad (8)$$

Now the last sum is similar to the last sum in (8), but with a different weight function. In particular, taking $z = -1$ we see that the sum generates the number of inequivalent solutions of $x^2 - 2y^2 = k$ with norm $8k + 1$ in which $x + 2y \equiv 1 \pmod 8$ or $x + 2y \equiv 3$

(mod 8) over the number in which $x+2y \equiv 5 \pmod 8$ or $x+2y \equiv 7 \pmod 8$. To see this, we only need to inspect when $(-1)^{n+j}$ is $+1$ and when it is -1. We leave the details to the reader.

3. Conclusions

The partition functions contained in this paper are rather curious in that they are all intimately related to the arithmetic of $\mathbb{Z}[\sqrt{2}]$. Unfortunately we have little information on the combinatorial behavior of these functions. We also mention that we may easily manipulate (5) to obtain more of the type of results offered by Lovejoy [9]. For example, replacing q by q^2 and setting $z = \frac{-a}{q}$ in (5) gives the function

$$\sum_{n=0}^{\infty} q^{2n}(-aq^{4n+3};q^4)_\infty (q^{2n+2};q^2)_\infty,$$

which generates a partition into distinct parts, where odd parts are $\equiv 3 \pmod 4$, minimal part even, and a keeps track of the number of parts $\equiv 3 \pmod 4$.

References

[1] G. E. Andrews, *Hecke modular forms and the Kac-Peterson identities,* Amer. Math. Soc. 283:2 (1984), 451-458.

[2] G. E. Andrews, F. J. Dyson, and D. Hickerson, *Partitions and indefinite quadratic forms,* Invent. Math. 91 (1988), no. 3, 391-407.

[3] G. E. Andrews and D. Hickerson, *Ramanujans lost notebook. VII. the sixth order mock theta functions,* Adv. Math. 89 (1991), no. 1, 60-105.

[4] G. E. Andrews. *The Theory of Partitions.* Addison-Wesley, Reading, Mass., 1976; reprinted, Cambridge University Press, 1998.

[5] D. M. Bressoud, M. Ismail and D. Stanton, *Change of base in Bailey pairs*, Ramanujan J. 4. (2000), 435453.

[6] H. Cohen, D. Favero, K. Liesinger, and S. Zubairy, *Characters and q-series in* $\mathbb{Q}(\sqrt{2})$, J. Number Theory. 107 (2004), pages 392-405.

[7] N. J. Fine, *Basic hypergeometric series and applications*, Math. Sruveys and Monographs, Vol. 27, Amer. Math.Soc., Providence, 1988.

[8] G. Gasper and M. Rahman, *Basic hypergeometric series,* Encyclopedia of Mathematics and its Applications, 35. Cambridge University Press, Cambridge, 1990.

[9] J. Lovejoy, *More Lacunary Partition Functions,* Illinois J. Math. 47 (2003), 769-773.

[10] A. E. Patkowski, *A Family of Lacunary Partition Functions*, to appear.