Subsequence Sums of Zero-sum-free Sequences

Pingzhi Yuan

School of Mathematics, South China Normal University,
Guangzhou 510631, P. R. CHINA

e-mail mcsypz@mail.sysu.edu.cn

Submitted: Apr 25, 2009; Accepted: Jul 30, 2007; Published: Aug 7, 2009
Mathematics Subject Classification: 11B75, 11B50

Abstract

Let G be a finite abelian group, and let S be a sequence of elements in G. Let $f(S)$ denote the number of elements in G which can be expressed as the sum over a nonempty subsequence of S. In this paper, we slightly improve some results of [10] on $f(S)$ and we show that for every zero-sum-free sequences S over G of length $|S| = \exp(G) + 2$ satisfying $f(S) \geqslant 4\exp(G) - 1$.

Key words: Zero-sum problems, Davenport's constant, zero-sum-free sequence.

1 Introduction

Let G be a finite abelian group (written additively)throughout the present paper. $\mathcal{F}(G)$ denotes the free abelian monoid with basis G, the elements of which are called *sequences* (in G). A sequence of not necessarily distinct elements from G will be written in the form $S = g_1 \cdot \cdots \cdot g_n = \prod_{i=1}^{n} g_i = \prod_{g\in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G)$, where $\mathsf{v}_g(S) \geqslant 0$ is called the *multiplicity* of g in S. Denote by $|S| = n$ the number of elements in S (or the *length* of S) and let $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$ be the *support* of S.

We say that S contains some $g \in G$ if $\mathsf{v}_g(S) \geqslant 1$ and a sequence $T \in \mathcal{F}(G)$ is a *subsequence* of S if $\mathsf{v}_g(T) \leqslant \mathsf{v}_g(S)$ for every $g \in G$, denoted by $T|S$. If $T|S$, then let ST^{-1} denote the sequence obtained by deleting the terms of T from S. Furthermore, by $\sigma(S)$ we denote the sum of S, (i.e. $\sigma(S) = \sum_{i=1}^{k} g_i = \sum_{g\in G} \mathsf{v}_g(S)g \in G$). By $\sum(S)$ we denote the set consisting of all elements which can be expressed as a sum over a nonempty subsequence of S, i.e.

$$\sum(S) = \{\sigma(T) : T \text{ is a nonempty subsequence of } S\}.$$

Supported by the Guangdong Provincial Natural Science Foundation (No. 8151027501000114) and NSF of China (No. 10571180).

We write $f(S) = |\sum(S)|$, $\langle S\rangle$ for the subgroup of G generated by all the elements of S.

Let S be a sequence in G. We call S a $zero-sum\ sequence$ if $\sigma(S) = 0$, a $zero-sum-free\ sequence$ if $\sigma(W) \neq 0$ for any subsequence W of S, and $squarefree$ if $\mathsf{v}_g(S) \leqslant 1$ for every $g \in G$.

Let $D(G)$ be the Davenport's constant of G, i.e., the smallest integer d such that every sequence S of elements in G with $|S| \geqslant d$ satisfies $0 \in \sum(S)$. For every positive integer r in the interval $\{1, \ldots, D(G)-1\}$, let

$$f_G(r) = \min_{S,\,|S|=r} f(S), \tag{1}$$

where S runs over all zero-sumfree sequences of r elements in G.

In 1972, Eggleton and Erdős (see [4]) first tackled the problem of determining the minimal cardinality of $\sum(S)$ for squarefree zero-sum-free sequences (that is for zero-sum-free subsets of G). In 2006, Gao and Leader [5] proved the following result.

Theorem A [5] *Let G be a finite abelian group of exponent m. Then*

(i) If $1 \leqslant r \leqslant m-1$ then $f_G(r) = r$.

(ii) If $\gcd(6, m) = 1$ and G is not cyclic then $f_G(m) = 2m-1$.

In 2007, Sun[11] showed that $f_G(m) = 2m-1$ still holds without the restriction that $\gcd(6, m) = 1$.

Using some techniques from the author [12], the author [13] proved the following two theorems.

Theorem B([9],[13]) *Let S be a zero-sumfree sequence in G such that $\langle S\rangle$ is not a cyclic group, then $f(S) \geqslant 2|S|-1$.*

Theorem C ([13]) *Let S be a zero-sumfree sequence in G such that $\langle S\rangle$ is not a cyclic group and $f(S) = 2|S|-1$. Then S is one of the following forms*

(i) $S = a^x(a+g)^y$, $x \geqslant y \geqslant 1$, where g is an element of order 2.

(ii) $S = a^x(a+g)^y g$, $x \geqslant y \geqslant 1$, where g is an element of order 2.

(iii) $S = a^x b$, $x \geqslant 1$.

However, Theorem B is an old theorem of Olson and White (see [10] Theorem 1.5) which has been overlooked by the author.

Recently, by an elegant argument, Pixton [10] proved the following result.

Theorem D ([10]) *Let G be a finite abelian group and S a zero-sum-free sequence of length n generating a subgroup of rank greater than 2, then $f(S) \geqslant 4|S|-5$.*

One purpose of the paper is to slightly improve the above result of Pixton. We have

Theorem 1.1 *Let $n \geqslant 2$ be a positive integer. Let G be a finite abelian group and $S = (g_i)_{i=1}^n$ a zero-sum-free sequence of length n generating a subgroup H of rank 2 and $H \ncong C_2 \oplus C_{2m}$, where m is a positive integer. Suppose that*

$$\sum(S) \neq A_a \cup (b + B_a),$$

where $a, b \in G$, A_a, B_a are some subsets of the cyclic group $\langle a\rangle$ generated by a and $b \notin \langle a\rangle$, then $f(S) \geqslant 3n-4$.

Theorem 1.2 *Let $n \geqslant 5$ be a positive integer. Let G be a finite abelian group and $S = (g_i)_{i=1}^n$ a zero-sum-free sequence of length n generating a subgroup H of rank 2 and $H \ncong C_2 \oplus C_{2m}, \ncong C_3 \oplus C_{3m}, \ncong C_4 \oplus C_{4m}$, where m is a positive integer. Suppose that*

$$\sum(S) \neq A_a \cup (b + B_a),\ A_a \cup (b + B_a) \cup (2b + C_a),\ A_a \cup (b + B_a) \cup (-b + C_a),$$

where $a, b \in G$, A_a, B_a, C_a are some subsets of the cyclic group $\langle a \rangle$ generated by a and $b \notin \langle a \rangle$, then $f(S) \geqslant 4n - 9$.

Theorem 1.3 *Let G be an abelian group and $S = (g_i)_{i=1}^n$ is a zero-sum-free sequence of length $n \geqslant 5$ that generating a subgroup of rank greater than 2 and $\langle S \rangle \ncong C_2 \oplus C_2 \oplus C_{2m}$, then $f(S)| \geqslant 4|S| - 3$ except when $S = a^x(a+g)^y c, a^x(a+g)^y gc, a^x bc$, where a, b, c, g are elements of G with $ord(g) = 2$, in these cases, $f(S) = 4|S| - 5$ when the rank of the subgroup generated by S is 3.*

Another main result of the paper runs as follows.

Theorem 1.4 *Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ be a finite abelian group with $1 < n_1 | \ldots | n_r$. If $r \geqslant 2$ and $n_{r-1} \geqslant 4$, then every zero-sum-free sequence S over G of length $|S| = n_r + 2$ satisfies $f(S) \geqslant 4n_r - 1$.*

This partly confirms a former conjecture of Bollobás and Leader [2] and a conjecture of Gao, Li, Peng and Sun [6], which is outlined in Section 5.

The paper is organized as follows. In Section 2 we present some results on Davenport's constant. In section 3 we prove more preliminary results which will be used in the proof of the main Theorems. The proofs of Theorems 1.1 to 1.3 are given in Section 4. In section 5 we will prove Theorem 1.4 and give some applications of Theorems 1.1 and 1.2.

2 Some bounds on Davenport's constant

Lemma 2.1 (see [8]) *Let G be a non-cyclic finite abelian group. Then $D(G) \leqslant \frac{|G|}{2} + 1$.*

Lemma 2.2 ([10] Lemma 4.1) *Let $k \in \mathbb{N}$. If $H \leqslant G$ are some finite abelian groups and $G_1 = G/H \simeq (\mathbb{Z}/2\mathbb{Z})^{k+1}$. Then $D(G) \leqslant 2D(H) + 2^{k+1} - 2$.*

Lemma 2.3 ([10] Lemma 2.3) *Let $H \leqslant G$ be some finite abelian groups and $G_1 = G/H$ is non-cyclic, then $D(G) \leqslant (D(G_1) - 1)D(H) + 1$.*

Lemma 2.4 *(i)Let G be a finite abelian group of rank 2 and $G \ncong C_2 \oplus C_{2m}$. Then (i) $D(G) \leqslant \frac{|G|}{3} + 2$.*

(ii) $D((\mathbb{Z}/p\mathbb{Z})^r) = r(p-1) + 1$ for prime p and $r \geqslant 1$.

(iii) $D(G) \leqslant |G|$.

Proof. (iii) is obvious. (i) and (ii) follow from Theorems 5.5.9 and 5.8.3 in [7]. □

Lemma 2.5 *If G is an abelian group of rank greater than 2 and $G \ncong C_2 \oplus C_2 \oplus C_{2m}$, then $D(G) \leqslant \frac{|G|+2}{4}$.*

Proof. Since G has rank greater than 2, then G has p-rank at least 3 for some prime p, and thus there exists a subgroup $H \leqslant G$ with $G/H \simeq (\mathbb{Z}/p\mathbb{Z})^3$. We can then apply Lemmas 2.3 and 2.4 (ii),(iii) to conclude that

$$D(G) \leqslant \frac{3(p-1)}{p^3}|G| + 1 \leqslant \frac{2}{9}|G| + 1 \leqslant \frac{|G|+2}{4}$$

when $p \geqslant 3$. If $p = 2$ we can apply Lemmas 2.1 and 2.2 to see that

$$D(G) \leqslant 2D(H) + 6 \leqslant 2 \cdot \left(\frac{|H|}{2} + 1\right) + 6 = \frac{|G|}{8} + 8 \leqslant \frac{|G|+2}{4}$$

when $|G| \geqslant 60$. Further, the only case with $|G| \leqslant 60$ and $G \ncong C_2 \oplus C_2 \oplus C_{2m}$ is that $G \cong C_2 \oplus C_4 \oplus C_4$, in this case $D(G) = 8 \leqslant \frac{32+2}{4}$. We are done. □

Lemma 2.6 ([10] Theorem 5.3) *If G is an abelian group of rank greater than 2, and let $X \subseteq G \backslash \{0\}$ be a generating set for G consisting only of elements of order greater than 2. Suppose $A \subset G$ satisfies $|(A+x)\backslash A| \leqslant 3$ for all $x \in X$. Then* $\min\{|A|, |G \backslash A|\} \leqslant 5$.

Lemma 2.7 ([10] Lemma 4.3) *Let G be a finite abelian group and let $X \subseteq G\backslash\{0\}$ be a generating set for G. Suppose A is a nonempty proper subset of G. Then*

$$\sum_{x \in X} |(A+x)\backslash A| \geqslant |X|.$$

Lemma 2.8 ([10] Lemma 4.4) *Let G be a finite abelian group and let $X \subseteq G\backslash\{0\}$ be a generating set for G. Suppose $f : G \to \mathbb{Z}$ is a function on G. Then*

$$\sum_{x \in X g \in G} \max\{f(g+x) - f(g), 0\} \geqslant (\max(f) - \min(f))|X|.$$

Using the technique in the proof of [10] Theorem 5.3, we have

Lemma 2.9 *Let $m > 0$ be a positive integer and G a finite abelian group, and let $X \subseteq G\backslash\{0\}$ be a generating set for G. Suppose $A \subseteq G$ satisfies $|(A+x)\backslash A| \leqslant m$ for all $x \in X$ and there exists a proper subset $Y \subset X$ such that $H = \langle Y \rangle$ and $G_1 = G/H$ both contain at least $(m+1)$ elements. Then* $\min\{|A|, |G\backslash A|\} \leqslant m^2$.

Proof. First, without loss of generality we may replace X by a minimal subset X_1 of X such that $\langle X_1 \cap Y \rangle = \langle Y \rangle$ and $\langle X_1 \rangle = G$.

Define a function $f : G_1 \to \mathbb{Z}$ by $f(g) = |A \cap (g+H)|$. Then we have that

$$\begin{aligned}
|(A-x)\backslash A| &= \sum_{g\in G_1} |((A-x)\backslash A) \cap (g+H)| \\
&= \sum_{g\in G_1} |(A-x) \cap (g+H)| - |(A-x) \cap A \cap (g+H)| \\
&= \sum_{g\in G_1} |(A) \cap (g+x+H)| - |(A-x) \cap A \cap (g+H)| \\
&\geqslant \sum_{g\in G_1} \max\{f(g+x) - f(g), 0\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
m|X\backslash Y| &\geqslant \sum_{x\in X\backslash Y} |(A-x)\backslash A| \\
&\geqslant \sum_{x\in X\backslash Y} \sum_{g\in G_1} \max\{f(g+x) - f(g), 0\} \\
&\geqslant (\max(f) - \min(f))|X\backslash Y|
\end{aligned}$$

by Lemma 2.8, since $X\backslash Y$ projects to $|X\backslash Y|$ distinct nonzero elements in G_1 because X is a minimal generating set with the property described in the first paragraph. Thus $(\max(f) - \min(f)) \leqslant m$. Then by replacing A by $G\backslash A$ if necessary, we can assume that $f(g) \neq |H|$ for any $g \in G_1$. The reason is that

$$(G\backslash A + x)\backslash(G\backslash A) = A\backslash(A+x),$$

so

$$|(G\backslash A + x)\backslash(G\backslash A)| = |A\backslash(A+x)| = |(A-x)\backslash A|.$$

Since for every $x \in Y$ we have

$$\begin{aligned}
|(A+x)\backslash A| &= \sum_{g\in G_1} |((A+x)\backslash A) \cap (g+H)| \\
&= \sum_{g\in G_1} |((A+x) \cap (g+H) - (A+x) \cap A \cap (g+H)| \\
&= \sum_{g\in G_1} |((A+x) \cap (g+H+x) - ((A+x) \cap (g+x+H)) \cap (A \cap (g+H))| \\
&= \sum_{g\in G_1} |((A \cap (g+H)) + x - (A \cap (g+H) + x) \cap (A \cap (a+H))| \\
&= \sum_{g\in G_1} |((A \cap (g+H) + x)\backslash(A \cap (g+H))|,
\end{aligned}$$

thus we can apply Lemma 2.7 to obtain that

$$\begin{aligned} m|Y| &\geqslant \sum_{x\in Y} |(A+x)\backslash A| \\ &= \sum_{g\in G_1}\sum_{x\in Y} |((A\cap(g+H)+x)\backslash(A\cap(g+H))| \\ &\geqslant |supp(f)|\,|Y|, \end{aligned}$$

where $supp(f) = \{g \in G_1 | f(g) \neq 0\}$ is the support of f. Since $|G_1| \geqslant m+1$, this implies that $f(g) = 0$ for some g, and thus $f(g) \leqslant m$ for all $g \in G_1$. Then $|A| = \sum_{g\in G_1} f(g) \leqslant \max(f)|supp(f)| \leqslant m^2$, as desired. □

3 Proof of Theorems 1.1 to 1.3

Proof of Theorem 1.1:

Proof. We first prove the theorem if S contains an element of order 2. Suppose that $S = (g_i)_{i=1}^n$ generates G, G has rank $2, 0 \notin \sum(S)$, and g_n has order 2. Let $\overline{G}$ be the quotient of G by the subgroup generated by g_n, then $\overline{G}$ has rank 2 since $G \ncong C_2 \oplus C_{2m}$. Let $\overline{S} = (\overline{g_i})_{i=1}^{n-1}$ be the projection of the first $n-1$ terms of S to $\overline{G}$. Then $0 \in \sum(\overline{S})$ would imply that either 0 or g_n lies in $\sum((g_i)_{i=1}^{n-1})$ and hence $0 \in \sum(S)$, so $\langle (g_i)_{i=1}^{n-1} \rangle$ is not a cyclic group and $\sum(S) = \sum((g_i)_{i=1}^{n-1}) \cup \{g_n\} \cup (\sum((g_i)_{i=1}^{n-1}) + g_n)$ is a disjoint union. Therefore, by Theorem B

$$f(S) \geqslant 2f((g_i)_{i=1}^{n-1}) + 1 \geqslant 2(2n-3) + 1 \geqslant 4n-5 \geqslant 3n-4,$$

as desired.

Now suppose for contradiction that the theorem fails for some abelian group G of minimum size. Choose $S = (g_i)_{i=1}^n$ to be a counterexample sequence of minimum length n, so $f(S) \leqslant 3n-5$. Also, S must generate G by the minimality of $|G|$, so G is noncyclic, $G \ncong C_2 \oplus C_{2m}$. Moreover, by the minimality of n we have that either the theorem holds for all Sg_i^{-1} $(1 \leqslant i \leqslant n)$; or $\langle Sg_i^{-1} \rangle \cong C_2 \oplus C_{2m}$, or $\sum(Sg_i^{-1}) = A_a \cup (b + B_a)$, where $a, b \in G$, A_a, B_a are some subsets of the cyclic group $\langle a \rangle$ generated by a and $b \notin \langle a \rangle$ for some $1 \leqslant i \leqslant n$. We divide the remaining proof into three cases.

Case 1: $\langle Sg_i^{-1} \rangle \cong C_2 \oplus C_{2m}$ for some $1 \leqslant i \leqslant n$. Then $S = (Sg_i^{-1})g_i$ and $g_i \notin \langle Sg_i^{-1} \rangle$ since $G \ncong C_2 \oplus C_{2m}$. It follows that $\sum(S) = \sum(Sg_i^{-1}) \cup \{g_i\} \cup (\sum(Sg_i^{-1}) + g_i)$ is a disjoint union, by Theorem B we have $f(S) \geqslant 2f(Sg_i^{-1}) + 1 \geqslant 2(2n-3) + 1 \geqslant 3n-4$, as desired.

Case 2: $\sum(Sg_i^{-1}) = A_a \cup (b + B_a)$ for some $1 \leqslant i \leqslant n$. Then $g_i \notin \langle a \rangle$ since $\sum(S) \neq A_a \cup (b + B_a)$. By the definitions of $\sum(Sg_i^{-1})$, we have $Sg_i^{-1} = S(g_ig_j)^{-1}g_j$, $g_j = b + la \notin \langle a \rangle$, $S(g_ig_j)^{-1} \subseteq \langle a \rangle$ and $j \neq i$. It follows that $\sum(Sg_i^{-1}) = A_a \cup \{g_j\} \cup (g_j + A_a) := A, A_a \subseteq \langle a \rangle$ is a disjoint union and

$$\sum(S) = A \cup \{g_i\} \cup B,\ B = (g_i + A_a) \cup \{g_i + g_j\} \cup (g_i + g_j + A_a).$$

If $g_i = g_j$ or $A \cap B \neq \emptyset$, then $x_i \in (b + \langle a \rangle) \cup (-b + \langle a \rangle)$, and thus

$$\sum(S) = A_a \cup (b + B_a) \cup (2b + C_a), \quad \text{or} \quad A_a \cup (b + B_a) \cup (-b + C_a),$$

where A_a, B_a, C_a are some subsets of $\langle a \rangle$.

If $g_i \in b + \langle a \rangle$, then $g_i = b + ka$ for some $k \in \mathbb{Z}$ and

$$\sum(S) \supset A_a \cup (b + B_a) \cup (2b + ka + B_a),$$

and the right hand side is a disjoint union, and thus

$$f(S) \geqslant |A_a| + |B_a| + |B_a| \geqslant n - 2 + 2(n-1) = 3n - 4.$$

If $g_i \in -b + \langle a \rangle$, then $g_i = -b + ka$ for some $k \in \mathbb{Z}$ and

$$\sum(S) \supseteq A_a \cup (b + B_a) \cup (-b + ka + (A_a \cup \{0\})$$

and $A_a \cup (b + B_a) \cup (-b + ka + (A_a \cup \{0\})$ is a disjoint union, and thus

$$f(S) \geqslant |A_a| + |B_a| + |A_a| + 1 \geqslant n - 2 + 2(n-1) = 3n - 4.$$

If $g_i \neq g_j$ and $A \cap B = \emptyset$, then $\sum(S) = A \cup \{g_i\} \cup B$, $B = (g_i + A_a) \cup \{g_i + g_j\} \cup (g_i + g_j + A_a)$ is a disjoint union, hence

$$f(S) = 4|A_a| + 3 \geqslant 4(n-2) + 3 \geqslant 3n - 4.$$

Case 3: If the theorem holds for all $Sg_i^{-1}, 1 \leqslant i \leqslant n$. Let $A = \sum(S) \subseteq G$. Then for any i we have $\sum(Sg_i^{-1}) \subseteq (A - g_i) \cap A$, so

$$|(A - g_i) \backslash A| \leqslant f(S) - f(Sg_i^{-1}) \leqslant 3n - 5 - (3(n-1) - 4) = 2.$$

It is easy to see that S satisfies the conditions of Lemma 2.9 since $\langle S \rangle \not\cong C_2 \oplus C_{2m}$. Applying Lemma 2.9 to $A \subseteq G$ with generating set S, we obtain that either A or $G \backslash A$ has cardinality at most 4. Since $|A| > 4$, so we have that $|G \backslash A| \leqslant 4$.

We now consider the two cases. If $|G \backslash A| = 1$, then $n \leqslant D(G) - 1 \leqslant \frac{|G|}{3} + 1$ by Lemma 2.4(i), and hence

$$|G| = |A| + 1 \leqslant 3n - 5 + 1 \leqslant |G| - 1,$$

which is a contradiction.

Otherwise, there is some nonzero element $y \in G \backslash A$, and S is still zero-sum free after appending $-y$, so $n \leqslant D(G) - 2 \leqslant \frac{|G|}{3}$ by Lemma 2.4(i) again, and thus

$$|G| \leqslant |A| + 4 \leqslant 3n - 5 + 4 \leqslant |G| - 1,$$

is again a contradiction. Theorem 1.1 is proved.

□

Proof of Theorem 1.2:

Proof. For $|S| = 5$, by Theorems 1.1, we have $f(S) \geqslant 3|S| - 4 = 4|S| - 9$, so the theorem holds for $n = 5$. If $S = (g_i)_{i=1}^n$ contains an element of order 2, say, $o(g_n) = 2$. By the similar argument as in Theorem 1.1 and by Theorem B, we have

$$f(S) \geqslant 2f((g_i)_{i=1}^{n-1}) + 1 \geqslant 2(2n-3) + 1 \geqslant 4n - 5,$$

as desired.

Now suppose for contradiction that the theorem fails for some abelian group G of minimum size. Choose $S = (g_i)_{i=1}^n$ to be a counterexample sequence of minimum length n, so $f(S) \leqslant 4n - 10$. Also, S must generate G by the minimality of $|G|$, so G is noncyclic, $G \ncong C_2 \oplus C_{2m}, \ncong C_3 \oplus C_{3m}, \ncong C_4 \oplus C_{4m}$. Moreover, by the minimality of n we have that either the theorem holds for all Sg_i^{-1} $(1 \leqslant i \leqslant n)$, or $\langle Sg_i^{-1}\rangle \cong C_2 \oplus C_{2m}$, or $\langle Sg_i^{-1}\rangle \cong C_3 \oplus C_{3m}$, or $\langle Sg_i^{-1}\rangle \cong C_4 \oplus C_{4m}$, or $\sum(Sg_i^{-1}) = A_a \cup (b + B_a)$, or $A_a \cup (b + B_a) \cup (2b + C_a)$, or $A_a \cup (b + B_a) \cup (-b + C_a)$, where $a, b \in G$, A_a, B_a, C_a are some subsets of the cyclic group $\langle a\rangle$ generated by a and $b \notin \langle a\rangle$ for some $1 \leqslant i \leqslant n$. We divide the remaining proof into five cases.

Case 1: $\langle Sg_i^{-1}\rangle \cong C_2 \oplus C_{2m}$, or $\langle Sg_i^{-1}\rangle \cong C_3 \oplus C_{3m}$ or $\langle Sg_i^{-1}\rangle \cong C_4 \oplus C_{4m}$ for some $1 \leqslant i \leqslant n$. Then $S = (Sg_i^{-1})g_i$ and $g_i \notin \langle Sg_i^{-1}\rangle$ since $G \ncong C_2 \oplus C_{2m}$, $G \ncong C_3 \oplus C_{3m}$ and $G \ncong C_4 \oplus C_{4m}$. It follows that $\sum(S) = \sum(Sg_i^{-1}) \cup \{g_i\} \cup (\sum(Sg_i^{-1}) + g_i)$ is a disjoint union, by Theorem B we have $f(S) \geqslant 2f(Sg_i^{-1}) + 1 \geqslant 2(2n-3) + 1 \geqslant 4n - 5$, as desired.

Case 2: $\sum(Sg_i^{-1}) = A_a \cup (b + B_a)$ for some $1 \leqslant i \leqslant n$. Then $g_i \notin \langle a\rangle$ since $\sum(S) \neq A_a \cup (b + B_a)$. By the definitions of $\sum(Sg_i^{-1})$, we have $Sg_i^{-1} = (S(g_ig_i)^{-1})g_j$, $g_j = b + la \notin \langle a\rangle$, $S(g_ig_j)^{-1} \subseteq \langle a\rangle$ and $j \neq i$. It follows that $\sum(Sg_i^{-1}) = A_a \cup \{g_j\} \cup (g_j + A_a) :=$ $A, A_a \subseteq \langle a\rangle$ is a disjoint union and

$$\sum(S) = A \cup \{g_i\} \cup B,\ B = (g_i + A_a) \cup \{g_i + g_j\} \cup (g_i + g_j + A_a).$$

If $g_i = g_j$ or $A \cap B \neq \emptyset$, then $g_i \in (b + \langle a\rangle) \cup (-b + \langle a\rangle)$, and thus

$$\sum(S) = A_a \cup (b + B_a) \cup (2b + C_a), \quad \text{or} \quad A_a \cup (b + B_a) \cup (-b + C_a),$$

where A_a, B_a, C_a are some subsets of $\langle a\rangle$, a contradiction. It follows that $\sum(S) = A \cup \{g_i\} \cup B$, $B = (g_i + A_a) \cup \{g_i + g_j\} \cup (g_i + g_j + A_a)$ is a disjoint union, and thus $f(S) = 4|A_a| + 3 \geqslant 4|S(g_ig_j)^{-1}| + 3 = 4(n-2) + 3 = 4n - 5$, as desired.

Case 3: $\sum(Sg_i^{-1}) = A_a \cup (b + B_a) \cup (2b + C_a) := A$ for some $1 \leqslant i \leqslant n$. Then $g_i \notin \langle a\rangle$ since $\sum(S) \neq A_a \cup (b + B_a) \cup (2b + C_a)$. By the definitions of $\sum(Sg_i^{-1})$, we have $Sg_i^{-1} = (S(g_ig_jg_k)^{-1})g_jg_k$, $g_j = b + la \notin \langle a\rangle$, $g_k = b + l_1a \notin \langle a\rangle$, $(S(g_ig_jg_k)^{-1}) \subseteq \langle a\rangle$ and $j \neq k \neq i$. It follows that $\sum(Sg_i^{-1}) = A_a \cup (b + B_a) \cup (2b + C_a) := A, A_a \subseteq \langle a\rangle$ is a disjoint union and $|A_a| \geqslant |S(g_ig_jg_k)^{-1}| = n - 3$, $|B_a| \geqslant |A_a| + 1 \geqslant n - 2$, $|C_a| \geqslant |A_a| + 1 \geqslant n - 2$. And

$$\sum(S) = A \cup \{g_i\} \cup B,\ B = (g_i + A).$$

If $g_i = g_j$ or $g_i = g_k$ or $A \cap B \neq \emptyset$, then $g_i \in (b+\langle a\rangle)\cup(-b+\langle a\rangle)\cup(2b+\langle a\rangle)\cup(-2b+\langle a\rangle)$ and b is an element of order at least 4 by the assumptions. If $g_i \in b+\langle a\rangle$, then $g_i = b+ka$ for some $k \in \mathbb{Z}$ and

$$\sum(S) = A_a \cup (b + B'_a) \cup (2b + C'_a) \cup (3b + ka + C_a), B_a \subseteq B'_a,\ C_a \subseteq C'_a$$

is a disjoint union, and thus

$$f(S) = |A_a| + |B'_a| + |C_a| + |C_a| \geqslant n - 3 + 3(n - 2) = 4n - 9.$$

If $g_i \in 2b + \langle a\rangle$, then $g_i = 2b + ka$ for some $k \in \mathbb{Z}$ and

$$\sum(S) \supseteq A_a \cup (b + B'_a) \cup (2b + C'_a) \cup (3b + ka + B_a), B_a \subseteq B'_a,\ C_a \subseteq C'_a$$

and $A_a \cup (b + B'_a) \cup (2b + C'_a) \cup (3b + ka + B_a)$ is a disjoint union, and thus

$$f(S) \geqslant |A_a| + |B'_a| + |C_a| + |B_a| \geqslant n - 3 + 3(n - 2) = 4n - 9.$$

If $g_i \in -b + \langle a\rangle$, then $g_i = -b + ka$ for some $k \in \mathbb{Z}$ and

$$\sum(S) = A'_a \cup (b + B'_a) \cup (2b + C_a) \cup (-b + ka + (A_a \cup \{0\})), A_a \subseteq A'_a,\ B_a \subseteq B'_a$$

is a disjoint union, and thus

$$f(S) \geqslant |A_a| + |B_a| + |C_a| + |A_a| + 1 \geqslant n - 3 + 3(n - 2) = 4n - 9.$$

If $g_i \in -2b + \langle a\rangle$, then $g_i = -2b + ka$ for some $k \in \mathbb{Z}$ and

$$\sum(S) \supseteq A'_a \cup (b + B'_a) \cup (2b + C_a) \cup (-b + ka + B_a), A_a \subseteq A'_a,\ B_a \subseteq B'_a$$

is a disjoint union, and thus

$$f(S) \geqslant |A_a| + |B_a| + |C_a| + |B_a| \geqslant n - 3 + 3(n - 2) = 4n - 9.$$

If $g_i \neq g_j$ and $g_i \neq g_k$ and $A \cap B = \emptyset$, then $\sum(S) = A \cup \{g_i\} \cup B,\ B = (g_i + A)$ is a disjoint union, hence

$$f(S) \geqslant 2(n - 3) + 4(n - 2) + 1 \geqslant 4n - 9.$$

Case 4: $\sum(Sg_i^{-1}) = A_a \cup (b+B_a) \cup (-b+C_a) := A$ for some $1 \leqslant i \leqslant n$. Then $g_i \notin \langle a\rangle$ since $\sum(S) \neq A_a \cup (b+B_a) \cup (-b+C_a)$. By the definitions of $\sum(Sg_i^{-1})$, we may assume that $Sg_i^{-1} = (S(g_ig_jg_k)^{-1})g_jg_k$, $g_j = b + la \notin \langle a\rangle$, $g_k = -b + l_1a \notin \langle a\rangle$, $(S(g_ig_jg_k)^{-1}) \subseteq \langle a\rangle$ and $j \neq k \neq i$. It follows that $\sum(Sg_i^{-1}) = (\sum(S(g_ig_jg_k)^{-1}(l + l_1)a)) \cup (b + (\sum(S(g_ig_jg_k)^{-1}) \cup \{0\})) \cup (-b + (\sum(S(g_ig_jg_k)^{-1}) \cup \{0\})) := A, (\sum(S(g_ig_jg_k)^{-1}) \subseteq \langle a\rangle$ is a disjoint union and $|S(g_ig_jg_k)^{-1}| = n - 3$. And

$$\sum(S) = A \cup \{g_i\} \cup B,\ B = (g_i + A).$$

The remaining proof of this case is similar to the proof of the case 3, we omit the detail.

Case 5: If the theorem holds for all $Sg_i^{-1}, 1 \leqslant i \leqslant n$. Let $A = \sum(S) \subseteq G$. Then for any i we have $\sum(Sg_i^{-1}) \subseteq (A - g_i) \cap A$, so

$$|(A - g_i)\backslash A| \leqslant |\sum(S)| - |\sum(Sg_i^{-1})| \leqslant 4n - 10 - (4(n-1) - 9) = 3.$$

It is easy to see that S satisfies all the conditions of Lemma 2.9 by the assumptions. Applying Lemma 2.9 to $A \subseteq G$ with generating set S, we obtain that either A or $G\backslash A$ has cardinality at most 9.

We now consider the two cases. If $|G\backslash A| = 1$, then $n \leqslant D(G) - 1 \leqslant \frac{|G|}{5} + 3$, and hence

$$|G| = |A| + 1 \leqslant 4n - 10 + 1 \leqslant \frac{4}{5}|G| + 3 \leqslant |G| - 1$$

since $|G| \geqslant 25$, which is a contradiction.

Otherwise, there is some nonzero element $y \in G\backslash A$, and S is still zero-sum free after appending $-y$, so $n \leqslant D(G) - 2 \leqslant \frac{|G|}{5} + 2$, and thus

$$|G| \leqslant |A| + 9 \leqslant 4n - 10 + 9 \leqslant \frac{4}{5}|G| + 7 \leqslant |G| - 1$$

when $|G| \geqslant 50$, which is again a contradiction.

The only left case is that $G \cong C_5 \oplus C_5$. If $n = 8 = D(G) - 1$ then $f(S) = 24 \geqslant 4 \times 8 - 9$. The case that $n = 7$ follows from [6] Lemma 4.5. The case that $n = 6$ follows from the proof of the above case 5 since $f(S) = |A| \geqslant |G| - 9 \geqslant 4 \times 6 - 9$. The case that $n = 5$ follows from Theorem 1.1 since $f(S) \geqslant 3 \times 5 - 4 = 11 = 4 \times 5 - 9$.

□

Proof of Theorem 1.3:

Proof. If there exists some integer $i, 1 \leqslant i \leqslant n$ such that the rank of $\langle Sg_i^{-1} \rangle$ is two and $f(Sg_i^{-1}) = 2|Sg_i^{-1}| - 1$, then by Theorem C we have $Sg_i^{-1} = a^x(a+g)^y, a^x(a+g)^y g, a^x b$, where a, b, g are elements of G with $ord(g) = 2$. It follows from our assumption that $g_i \notin \langle Sg_i^{-1} \rangle$, and thus

$$f(S) = 2f(Sx_i^{-1}) + 1 = 2(2n-3) + 1 = 4n - 5.$$

If $rank\langle Sg_i^{-1} \rangle = 2$ and $f(Sg_i^{-1}) \geqslant 2|Sg_i^{-1}|$, then

$$f(S) = 2f(Sg_i^{-1}) + 1 = 2(2n-2) + 1 = 4n - 3.$$

If $\langle Sg_i^{-1} \rangle \cong C_2 \oplus C_2 \oplus C_{2m}$ for some $i, 1 \leqslant i \leqslant n$, then $g_i \notin \langle Sg_i^{-1} \rangle$ since $\langle S \rangle \ncong C_2 \oplus C_2 \oplus C_{2m}$, and so

$$f(S) = 2f(Sg_i^{-1}) + 1 \geqslant 2(4(n-1) - 5) + 1 = 8n - 17 > 4n - 3$$

since $n \geqslant 4$, as desired.

Now we suppose that for all $i, 1 \leqslant i \leqslant n$, $\langle Sg_i^{-1}\rangle$ is an abelian group of rank greater than 2 and $\langle Sg_i^{-1}\rangle \ncong C_2 \oplus C_2 \oplus C_{2m}$.

First we will show that the theorem holds for $n = 4$. Let $S = abcd$ such that $rank\langle abc\rangle = rank\langle abd\rangle = rank\langle acd\rangle = rank\langle bcd\rangle = 3$, then $a, b, c, a+b, a+c, b+c, a+b+c$ are distinct elements in $\sum(abcd)$ since $rank\langle abc\rangle = 3$. The case that $rank\langle a, b, c, d\rangle = 4$ is trivial since $f(abcd) = 15$ in this case. It is easy to see that $d \notin \{a, b, c, a+b, a+c, b+c\}$ and $a + d \notin \{a, b, c, a + b, a + c, a + b + c\}$.

(i) If $d = a + b + c, d + a = b + c$ and $d + b = a + c$, then $2a = 2b = 0$ and $\langle S\rangle \cong C_2 \oplus C_2 \oplus C_{2m}$, a contradiction.

(ii) If $d = a+b+c, c = a+b+d$ and $b = a+c+d$, then $2(a+b) = 2(a+d) = 2(a+c) = 0$. Let $b = -a + g_1, c = -a + g_2, d = -a + g_3, o(g_1) = o(g_2) = o(g_3) = 2$, then $g_3 = g_1 + g_2$, and thus $\langle S\rangle \cong C_2 \oplus C_2 \oplus C_{2m}$.

(iii) If $d + a = b + c, d + b = a + c$ and $d + c = a + b$, then $2a = 2b = 2c = 2d$. Let $b = a + g_1, c = a + g_2, d = a + g_3, o(g_1) = o(g_2) = o(g_3) = 2$, then $g_3 = g_1 + g_2$ and so $\langle S\rangle \cong C_2 \oplus C_2 \oplus C_{2m}$.

(iv) If If $d = a + b + c, c = a + b + d$ and $b + a = c + d$, then $2c = 2d = 0, 2(a + b) = 0$. Let $b = -a + g_1, c = g_2, d = g_3, o(g_1) = o(g_2) = o(g_3) = 2$, then $g_1 = g_2 + g_3$, and thus $\langle S\rangle \cong C_2 \oplus C_2 \oplus C_{2m}$.

By symmetry, we conclude that $\langle S\rangle \cong C_2 \oplus C_2 \oplus C_{2m}$ whenever there are three relations. If there are precisely two relations, then $f(abcd) = 13$; If there is only one relation, then $f(abcd) = 14$; If there is no relations between a, b, cd , then $f(abcd) = 15$. Therefore the theorem holds for $n = 4$.

Suppose for contradiction that the theorem holds for some abelian group G of minimum size. Choose $S = (g_i)_{i=1}^n$ to be a counterexample sequence of minimum length $n \geqslant 5$, so $f(S) < 4n - 4$. Also S must generate G by minimality of $|G|$, $rank(G) = 3$ and $G \ncong C_2 \oplus C_2 \oplus C_{2m}$. Moreover, by the minimality of $n \geqslant 5$, we have that the theorem holds for Sg_i^{-1}.

Let $A = \sum(S) \subset G$, then $\sum(Sg_i^{-1}) \subset (A - g_i) \cap A$, and thus $|(A - g_i)\backslash A| \leqslant |A| - f(Sg_i^{-1}) \leqslant 4n - 4 - (4n - 7) = 3$. It follows from Lemma 2.6 that $\min\{|A|, |G\backslash A|\} \leqslant 5$. Since $|A| \geqslant 2|S| - 1 \geqslant 9$, then we have

$$|G\backslash A| \leqslant 5.$$

If $|G\backslash A| = 1$, then $n \leqslant D(G) - 1 \leqslant \frac{|G|-2}{4}$ by Lemma 2.5, and hence

$$|G| = |A| + 1 \leqslant 4n - 4 + 1 \leqslant |G| - 5,$$

is a contradiction. Otherwise, there is some nonzero element $y \in G\backslash A$, and X is still zero-sum-free after appending $-y$, so $n \leqslant D(G) - 2 \leqslant \frac{|G|-6}{4}$. Therefore

$$|G| \leqslant |A| + 5 \leqslant 4n + 1 \leqslant |G| - 1,$$

is again a contradiction. $\square$

4 Proof of Theorem 1.4

Now we are in a position to prove Theorem 1.4.

Proof. If $rank\langle S\rangle \geqslant 3$, then $f(S) \geqslant 4|S|-5 = 4(n_r+2)-5 \geqslant 4n_r-1$. If $rank\langle S\rangle = 2$, since $|S| = n_r+2 \leqslant D(\langle S\rangle)-1$, then $\langle S\rangle \ncong C_2\oplus C_{2m}, C_3\oplus C_{3m}$. If $\langle S\rangle \cong C_4\oplus C_{4m}$, then $|S| = D(G)-1$ and thus $f(S) = |\langle S\rangle|-1 = 4n_r-1$. If $\langle S\rangle \ncong C_4\oplus C_{4m}$, then $f(S) \geqslant 4|S|-9 = 4n_r-1$ by Theorem 1.2. We are done. □

Similarly, by Theorem 1.1, we can prove the following theorem in [6].

Theorem 4.1 ([6] Theorem 1.1) *Let $G = C_{n_1}\oplus\ldots\oplus C_{n_r}$ be a finite abelian group with $1 < n_1|\ldots|n_r$. If $r \geqslant 2$ and $n_{r-1} \geqslant 3$, then every zero-sum free sequence S over G of length $|S| = n_r+1$ satisfies $f(S) \geqslant 3n_r-1$.*

Proof. If $rank\langle S\rangle \geqslant 3$, then $f(S) \geqslant 4|S|-5 = 4(n_r+1)-5 \geqslant 3n_r-1$. If $rank\langle S\rangle = 2$, since $|S| = n_r+1 \leqslant D(\langle S\rangle)-1$, then $\langle S\rangle \ncong C_2\oplus C_{2m}$. Therefore $f(S) \geqslant 3|S|-4 \geqslant 3(n_r+1)-4 = 3n_r-1$ by Theorem 1.1. We are done. □

We recall a conjecture by Bollobás and Leader, stated in [2].

Conjecture 4.1 *Let $G = C_n\oplus C_n$ with $n \geqslant 2$ and let (e_1,e_2) be a basis of G. If $k \in [0,n-2]$ and*

$$S = e_1^{n-1}e_2^{k+1} \in \mathcal{F}(G).$$

Then we have $f(G,n+k) = f(S) = (k+2)n-1$.

By a main result of [6] and Theorem 1.4, the conjecture holds for $k \in \{0,1,2,n-2\}$. Moreover, the following general conjecture stated in [6] holds for $k=2$.

Conjecture 4.2 *Let $G = C_{n_1}\oplus\ldots\oplus C_{n_r}$ be a finite abelian group with $r \geqslant 2$ and $1 < n_1|\ldots|n_r$. Let $(e_1,\ldots,e_r)$ be a basis of G with $ord(e_i) = n_i$ for all $i \in [1,r], k \in [0,n_{r-1}-2]$ and*

$$S = e_r^{n_r-1}e_{r-1}^{k+1} \in \mathcal{F}(G).$$

Then we have $f(G,n_r+k) = f(S) = (k+2)n_r-1$.

References

[1] J.D. Bovey, P. Erdős, and I. Niven, *Conditions for zero sum modulo n*, Canad. Math. Bull. **18** (1975), 27 – 29.

[2] B. Bollobás and I. Leader, *The number of k-sums modulo k*, J. Number Theory **78**(1999), 27-35.

[3] S.T. Chapman and W.W. Smith, *A characterization of minimal zero-sequences of index one in finite cyclic groups*, Integers **5(1)** (2005), Paper A27, 5pp.

[4] R.B. Eggleton and P. Erdős, *Two combinatorial problems in group theory*, Acta Arith. **21**(1972), 111-116.

[5] W.D. Gao and I. Leader, *sums and k-sums in an abelian groups of order k*, J. Number Theory **120**(2006), 26-32.

[6] W. Gao, Y. Li, J. Peng and F. Sun, *On subsequence sums of a zero-sum free sequence II*, The Electronic Journal of Combinatorics, **15**(2008), ♯R117.

[7] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, Vol. 278, Chapman & Hall/CRC, 2006.

[8] O. Ordaz and D. Quiroz, *The Erdős-Ginzburg-Ziv theorem in abelian non-cyclic groups* Divulg. Mat. **8(2)**(2000)113-119.

[9] J. E. Olson and E.T.White, *sums from a sequence of group elements*, in : Number Theory and Algebra, Academic Press, New York, 1977, pp. 215-222.

[10] A. Pixton, *sSequences with small subsums sets*, J. Number Theory **129**(2009), 806-817.

[11] F. Sun, *On subsequence sums of a zero-sum free sequence*, The Electronic Journal of Combinatorics. **14**(2007), ♯R52.

[12] P.Z. Yuan, *On the index of minimal zero-sum sequences over finite cyclic groups*, J. Combin. Theory Ser. A **114**(2007), 1545-1551.

[13] P.Z. Yuan, *Subsequence sums of a zero-sumfree sequence*, European Journal of Combinatorics, **30**(2009), 439-446.