Van der Waerden/Schrijver-Valiant like Conjectures and Stable (aka Hyperbolic) Homogeneous Polynomials: One Theorem for all

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Abstract

Let p be a homogeneous polynomial of degree n in n variables, $p(z_1, \ldots, z_n) = p(Z)$, $Z \in C^n$. We call such a polynomial p **H-Stable** if $p(z_1, \ldots, z_n) \neq 0$ provided the real parts $Re(z_i) > 0, 1 \leq i \leq n$. This notion from *Control Theory* is closely related to the notion of *Hyperbolicity* used intensively in the *PDE* theory.

The main theorem in this paper states that if $p(x_1, \ldots, x_n)$ is a homogeneous **H-Stable** polynomial of degree n with nonnegative coefficients; $deg_p(i)$ is the maximum degree of the variable x_i, $C_i = \min(deg_p(i), i)$ and

$$Cap(p) = \inf_{x_i > 0, 1 \leq i \leq n} \frac{p(x_1, \ldots, x_n)}{x_1 \cdots x_n}$$

then the following inequality holds

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} p(0, \ldots, 0) \geq Cap(p) \prod_{2 \leq i \leq n} \left(\frac{C_i - 1}{C_i} \right)^{C_i - 1}.$$

This inequality is a vast (and unifying) generalization of the Van der Waerden conjecture on the permanents of doubly stochastic matrices as well as the Schrijver-Valiant conjecture on the number of perfect matchings in k-regular bipartite graphs. These two famous results correspond to the **H-Stable** polynomials which are products of linear forms.

Our proof is relatively simple and "noncomputational"; it uses just very basic properties of complex numbers and the AM/GM inequality.

1 The permanent, the mixed discriminant, the Van Der Waerden conjecture(s) and homogeneous polynomials

Recall that an $n \times n$ matrix A is called doubly stochastic if it is nonnegative entry-wise and its every column and row sum to one. The set of $n \times n$ doubly stochastic matrices is denoted by Ω_n. Let $\Lambda(k,n)$ denote the set of $n \times n$ matrices with nonnegative integer entries and row and column sums all equal to k. We define the following subset of rational doubly stochastic matrices: $\Omega_{k,n} = \{k^{-1}A : A \in \Lambda(k,n)\}$. In a 1989 paper [2] R.B. Bapat defined the set D_n of doubly stochastic n-tuples of $n \times n$ matrices.
An n-tuple $\mathbf{A} = (A_1, \dots, A_n)$ belongs to D_n iff $A_i \succeq 0$, i.e. A_i is a positive semi-definite matrix, $1 \le i \le n$; $trA_i = 1$ for $1 \le i \le n$; $\sum_{i=1}^n A_i = I$, where I, as usual, stands for the identity matrix. Recall that the permanent of a square matrix A is defined by

$$per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A(i, \sigma(i)).$$

Let us consider an n-tuple $\mathbf{A} = (A_1, A_2, \dots A_n)$, where $A_i = (A_i(k,l) : 1 \le k, l \le n)$ is a complex $n \times n$ matrix ($1 \le i \le n$). Then

$$Det_{\mathbf{A}}(t_1, \dots, t_n) = \det(\sum_{1 \le i \le n} t_i A_i)$$

is a homogeneous polynomial of degree n in $t_1, t_2, \dots, t_n$. The number

$$D(\mathbf{A}) := D(A_1, A_2, \dots, A_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} Det_{\mathbf{A}}(0, \dots, 0) \tag{1}$$

is called the mixed discriminant of $A_1, A_2, \dots, A_n$.

The mixed discriminant is just another name, introduced by A.D. Alexandrov, for 3-dimensional Pascal's hyperdeterminant. The permanent is a particular (diagonal) case of the mixed discriminant. I.e. define the following homogeneous polynomial

$$Prod_A(t_1, \dots, t_n) = \prod_{1 \le i \le n} \sum_{1 \le j \le n} A(i,j) t_j. \tag{2}$$

Then the next identity holds:

$$per(A) = \frac{\partial^n}{\partial t_1, \dots, \partial t_n} Prod_A(0, \dots, 0). \tag{3}$$

Let us recall two famous results and one recent result by the author.

1. **Van der Waerden Conjecture**
The famous Van der Waerden Conjecture [23] states that

$$\min_{A \in \Omega_n} per(A) = \frac{n!}{n^n} =: vdw(n) \quad \textbf{(VDW-bound)}$$

and the minimum is attained uniquely at the matrix J_n in which every entry equals $\frac{1}{n}$. The Van der Waerden Conjecture was posed in 1926 and proved in 1981: D.I. Falikman proved in [5] the lower bound $\frac{n!}{n^n}$; the full conjecture, i.e. the uniqueness part, was proved by G.P. Egorychev in [4].

2. **Schrijver-Valiant Conjecture**
Define

$$\lambda(k,n) = \min\{per(A) : A \in \Omega_{k,n}\} = k^{-n} \min\{per(A) : A \in \Lambda(k,n)\};$$
$$\theta(k) = \lim_{n \to \infty} (\lambda(k,n))^{\frac{1}{n}}.$$

It was proved in [26] that, using our notations, $\theta(k) \le G(k) =: (\frac{k-1}{k})^{k-1}$ and conjectured that $\theta(k) = G(k)$. Though the case of $k=3$ was proved by M. Voorhoeve in 1979 [28], this conjecture was settled only in 1998 [27] (17 years after the published proof of the Van der Waerden Conjecture). The main result of [27] is the remarkable **(Schrijver-bound)**:

$$\min\{per(A) : A \in \Omega_{k,n}\} \ge \left(\frac{k-1}{k}\right)^{(k-1)n} \tag{4}$$

The proof of **(Schrijver-bound)** in [27] is, in the words of its author, "highly complicated".

Remark 1.1: The dynamics of research which led to **(Schrijver-bound)** is quite fascinating. If $k=2$ then $min_{A \in \Lambda(2,n)} per(A) = 2$. Erdos and Renyi conjectured in 1968 paper that 3-regular case already has exponential growth:

$$\min_{A \in \Lambda(3,n)} per(A) \ge a^n, a > 1.$$

This conjecture is implied by **(VDW-bound)**, this connection was another important motivation for the Van der Waerden Conjecture. The Erdos-Renyi conjecture was answered by M. Voorhoeve in 1979 [28]:

$$\min_{A \in \Lambda(3,n)} per(A) \ge 6 \left(\frac{4}{3}\right)^{n-3}. \tag{5}$$

Amazingly, the Voorhoeve's bound (5) is asymptotically sharp and the proof of this fact is probabilistic. In 1981 paper [26], A.Schrijver and W.G.Valiant found a sequence $\mu_{k,n}$ of probabilistic distributions on $\Lambda(k,n)$ such that

$$\lim_{n \to \infty} \left(\min_{A \in \Lambda(k,n)} per(A)\right)^{\frac{1}{n}} \le \lim_{n \to \infty} \left(E_{\mu_{k,n}} per(A)\right)^{\frac{1}{n}} = k \left(\frac{k-1}{k}\right)^{k-1} \tag{6}$$

(*I.M. Wanless recently extended in [30] the upper bound (6) to the boolean matrices in* $\Lambda(k,n)$.)

It follows from the Voorhoeve's bound (5) that

$$\lim_{n\to\infty}\left(E_{\mu_{k,n}}per(A)\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\min_{A\in\Lambda(k,n)} per(A)\right)^{\frac{1}{n}} \quad \text{for} \quad k=2,3.$$

This was the rather bald intuition that gave rise to the *Schrijver-Valiant* 1981 conjecture.
The number $k\left(\frac{k-1}{k}\right)^{k-1}$ in *Schrijver-Valiant* conjecture came up via combinatorics followed by the standard Stirling's formula manipulations. On the other hand $G(k) = (\frac{k-1}{k})^{k-1} = \frac{vdw(k)}{vdw(k-1)}$.
∎

3. **Bapat's Conjecture (Van der Waerden Conjecture for mixed discriminants)**
One of the problems posed in [2] is to determine the minimum of mixed discriminants of doubly stochastic tuples: $min_{A\in D_n} D(A) =?$
Quite naturally, R.V.Bapat conjectured that $\min_{A\in D_n} D(A) = \frac{n!}{n^n}$ **(Bapat-bound)** and that it is attained uniquely at $\mathbf{J}_n =: (\frac{1}{n}I,\ldots,\frac{1}{n}I)$.
In [2] this conjecture was formulated for real matrices. The author had proved it [13] for the complex case, i.e. when matrices A_i above are complex positive semidefinite and, thus, hermitian.

1.1 The Ultimate Unification (and Simplification)

Falikman/Egorychev proofs of the Van Der Waerden conjecture as well our proof of Bapat's conjecture are based on the Alexandrov inequalities for mixed discriminants [1] and some optimization theory, which is rather advanced in the case of the Bapat's conjecture. They all rely heavily on the matrix structure and essentially of non-inductive nature.
(D. I. Falikman independently rediscovered in [5] the diagonal case of the Alexandrov inequalities and used a clever penalty functional. The very short paper [5] is supremely original, it cites only three references and uses none of them.)
The Schrijver's proof has nothing in common with these analytic proofs; it is based on the finely tuned combinatorial arguments and multi-level induction. It heavily relies on the fact that the entries of matrices $A\in\Lambda(k,n)$ are integers.

The main result of this paper is one, easily stated and proved by easy induction, theorem which unifies, generalizes and, in the case of **(Schrijver-bound)**, improves the results described above. This theorem is formulated in terms of the mixed derivative $\frac{\partial^n}{\partial x_1\ldots\partial x_n}p(0,\ldots,0)$ (rewind to the formula (3)) of **H-Stable** (or positive hyperbolic) homogeneous polynomials p.
The next two completely self-contained sections introduce the basics of stable homogeneous polynomials and proofs of the theorem and its corollaries. We have tried to simplify

everything to the undergraduate level, making the paper longer than a dry technical note of 4-5 pages. Our proof of the uniqueness in the generalized Van der Waerden Conjecture is a bit more involved, as it uses Garding's result on the convexity of the hyperbolic cone.

2 Homogeneous Polynomials

The next definition introduces key notations and notions.

Definition 2.1:

1. The linear space of homogeneous polynomials with real (complex) coefficients of degree n and in m variables is denoted $Hom_R(m,n)$ ($Hom_C(m,n)$).
   We denote as $Hom_+(m,n)$ ($Hom_{++}(n,m)$) the closed convex cone of polynomials $p \in Hom_R(m,n)$ with nonnegative (positive) coefficients.

2. For a polynomial $p \in Hom_+(n,n)$ we define its **Capacity** as

$$Cap(p) = \inf_{x_i > 0, \prod_{1 \le i \le n} x_i = 1} p(x_1, \dots, x_n) = \inf_{x_i > 0} \frac{p(x_1, \dots, x_n)}{\prod_{1 \le i \le n} x_i}. \tag{7}$$

3. Consider a polynomial $p \in Hom_C(m,n)$,

$$p(x_1, \dots, x_m) = \sum_{(r_1, \dots, r_m)} a_{r_1, \dots, r_m} \prod_{1 \le i \le m} x_i^{r_i}.$$

   We define $Rank_p(S)$ as the maximal joint degree attained on the subset $S \subset \{1, \dots, m\}$:

$$Rank_p(S) = \max_{a_{r_1, \dots, r_m} \neq 0} \sum_{j \in S} r_j. \tag{8}$$

   If $S = \{i\}$ is a singleton, we define $deg_p(i) = Rank_p(S)$.

4. Let $p \in Hom_+(n,n)$,

$$p(x_1, \dots, x_n) = \sum_{r_1 + \dots + r_n = 1} a_{r_1, \dots, r_n} \prod_{1 \le i \le n} x_i^{r_i}.$$

   Such a homogeneous polynomial p with nonnegative coefficients is called doubly-stochastic if

$$\frac{\partial}{\partial x_i} p(1, 1, \dots, 1) = 1 : 1 \le i \le n.$$

   In other words, $p \in Hom_+(n,n)$ is doubly-stochastic if

$$\sum_{r_1 + \dots + r_n = 1} a_{r_1, \dots, r_n} r_j = 1 : 1 \le j \le n. \tag{9}$$

It follows from the Euler's identity that $p(1,1,\dots,1)=1$:

$$\sum_{r_1+\cdots+r_n=1} a_{r_1,\dots,r_n} = 1 \tag{10}$$

Using the concavity of the logarithm on R_{++} we get that

$$\log\left(p(x_1,\dots,x_n)\right) \geq \sum_{r_1+\cdots+r_n=1} a_{r_1,\dots,r_n} r_i \log(x_i) = \log(x_1\cdots x_n).$$

Therefore

Fact 2.2: If $p \in Hom_+(n,n)$ is doubly-stochastic then $Cap(p)=1$.

5. A polynomial $p \in Hom_C(m,n)$ is called **H-Stable** if $p(Z) \neq 0$ provided $Re(Z) > 0$; is called **H-SStable** if $p(Z) \neq 0$ provided $Re(Z) \geq 0$ and $\sum_{1\leq i\leq m} Re(z_i) > 0$.
*We coined the term "***H-Stable***" to stress two things: Homogeniety and Hurwitz' stability. Other terms are used in the same context:* **Wide Sense Stable** *in [15],* **Half-Plane Property** *in [3].*

6. We define

$$vdw(i) = \frac{i!}{i^i}; G(i) = \frac{vdw(i)}{vdw(i-1)} = \left(\frac{i-1}{i}\right)^{i-1}, i > 1; G(1)=1. \tag{11}$$

Notice that $vdw(i)$ as well as $G(i)$ are strictly decreasing sequences.

∎

Example 2.3:

1. Let $p \in Hom_+(2,2), p(x_1,x_2) = \frac{A}{2}x_1^2 + Cx_1x_2 + \frac{B}{2}x_2^2; A,B,C \geq 0$. Then $Cap(p) = C + \sqrt{AB}$ and the polynomial p is **H-Stable** iff $C \geq \sqrt{AB}$.

2. Let $A \in \Omega_n$ be a doubly stochastic matrix. Then the polynomial $Prod_A$ is doubly-stochastic. Therefore $Cap(Prod_A) = 1$. In the same way, if $\mathbf{A} \in D_n$ is a doubly stochastic n-tuple then the polynomial $Det_{\mathbf{A}}$ is doubly-stochastic and $Cap(Det_{\mathbf{A}}) = 1$.

3. Let $\mathbf{A} = (A_1, A_2, \dots A_m)$ be an m-tuple of PSD hermitian $n \times n$ matrices, and $\sum_{1\leq i\leq m} A_i \succ 0$ (the sum is positive-definite). Then the determinantal polynomial $Det_{\mathbf{A}}(t_1,\dots,t_m) = \det(\sum_{1\leq i\leq m} t_i A_i)$ is **H-Stable** and

$$Rank_{Det_{\mathbf{A}}}(S) = Rank(\sum_{i\in S} A_i). \tag{12}$$

∎

The main result in this paper is the following Theorem.

Theorem 2.4: *Let $p \in Hom_+(n,n)$ be* **H-Stable** *polynomial. Then the following inequality holds*

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0,\dots,0) \geq \prod_{2 \leq i \leq n} G\big(\min(i, deg_p(i))\big) Cap(p). \tag{13}$$

Note that

$$\prod_{2 \leq i \leq n} G\big(\min(i, deg_p(i))\big) \geq \prod_{2 \leq i \leq n} G(i) = vdw(n),$$

which gives the next generalized **Van Der Waerden Inequality**:

Corollary 2.5: *Let $p \in Hom_+(n,n)$ be* **H-Stable** *polynomial. Then*

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0,\dots,0) \geq \frac{n!}{n^n} Cap(p). \tag{14}$$

Corollary (2.5) was conjectured by the author in [10], where it was proved that $\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0,\dots,0) \geq C(n)Cap(p)$ for some constant $C(n)$.

2.1 Three Conjectures/Inequalities

The fundamental nature of Theorem (2.4) is illustrated in the following Example.

Example 2.6:

1. Let $A \in \Omega_n$ be $n \times n$ doubly stochastic matrix. It is easy to show that the polynomial $Prod_A$ is **H-Stable** and doubly-stochastic. Therefore $Cap(Prod_A) = 1$. Applying Corollary (2.5) we get the celebrated Falikman's result [5]:

$$\min_{A \in \Omega_n} per(A) = \frac{n!}{n^n}.$$

(*The complementary uniqueness statement for Corollary (2.5) will be considered in Section(5).*)

2. Let $(A_1, \dots, A_n) = \mathbf{A} \in D_n$ be a doubly stochastic n-tuple. Then the determinantal polynomial $Det_{\mathbf{A}}$ is **H-Stable** and doubly-stochastic. Thus $Cap(Det_{\mathbf{A}}) = 1$ and we get the **(Bapat-bound)**, proved by the author:

$$\min_{\mathbf{A} \in D_n} D(\mathbf{A}) = \frac{n!}{n^n}.$$

3. Important for what follows is the next observation, which is a diagonal case of (12): $deg_{Prod_A}(j)$ **is equal to the number of nonzero entries in the jth column of the matrix A.**
The next Corrolary combines this observation with Theorem(2.4).

Corollary 2.7:

(a) *Let C_j be the number of nonzero entries in the jth column of A, where A is an $n \times n$ matrix with non-negative* **real** *entries. Then*

$$per(A) \geq \prod_{2 \leq j \leq n} G\left(\min(j, C_j)\right) Cap(Prod_A). \tag{15}$$

(b) *Suppose that $C_j \leq k : k+1 \leq j \leq n$. Then*

$$per(A) \geq \left(\left(\frac{k-1}{k}\right)^{k-1}\right)^{n-k} \frac{k!}{k^k} Cap(Prod_A). \tag{16}$$

Let $\Lambda(k,n)$ denote the set of $n \times n$ matrices with nonnegative integer entries and row and column sums all equal to k. The matrices in $\Lambda(k,n)$ correspond to the k-regular bipartite graphs with multiple edges.
Recall the **(Schrijver-bound)**:

$$\min_{A \in \Lambda(k,n)} per(A) \geq k^n G(k)^n = \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n. \tag{17}$$

The Falikman's inequality gives that

$$\min_{A \in \Lambda(k,n)} per(A) \geq k^n vdw(n) > k^n G(k)^n \quad \text{if} \quad k \geq n.$$

Therefore the inequality (17) is interesting only if $k < n$.
Note that if $A \in \Lambda(k,n), k < n$ then all columns of A have at most k nonzero entries. If $A \in \Lambda(k,n)$ then the matrix $\frac{1}{k}A \in \Omega_n$, thus $Cap(Prod_A) = k^n$. As we observed above, $deg_{Prod_A}(j) \leq k$. Applying the inequality (16) to the polynomial $Prod_A$ we get for $k < n$ an improved **(Schrijver-bound)**:

$$\min_{A \in \Lambda(k,n)} per(A) \geq k^n \left(\left(\frac{k-1}{k}\right)^{k-1}\right)^{n-k} \frac{k!}{k^k} > \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n. \tag{18}$$

Interestingly, the inequality (18) recovers for $k = 3$ the Voorhoeve's inequality (5).

4. The inequality (15) is sharp if $C_i = \cdots = C_{n-1} = n; C_n = k : 1 < k \leq n-1$. To see this, consider the doubly stochastic matrix

$$D=\begin{pmatrix} a & \dots & a & b \\ . & \dots & . & . \\ a & \dots & a & b \\ c & \dots & c & 0 \\ . & \dots & . & . \\ c & \dots & c & 0 \end{pmatrix};a=\frac{1-b}{n-1}=\frac{k-1}{k(n-1)},b=\frac{1}{k},c=\frac{1}{n-1}, \tag{19}$$

and the associated polynomial

$$Prod_D(x_1,\dots,x_n)=\left((\sum_{1\le i\le n-1} ax_i)+bx_n \right)^k (\sum_{1\le i\le n-1} cx_i)^{n-k}.$$

Since the matrix D is doubly stochastic, $Cap(Prod_D)=1$. Direct inspection shows that

$$per(D)=(n-1)!(kb)a^{k-1}c^{n-k}=G(k)\frac{(n-1)!}{(n-1)^{n-1}}.$$

Which gives the equality

$$per(D)=Cap(Prod_D)\prod_{2\le j\le n} G\left(\min(j,C_j)\right).$$

It follows that $\min\{per(A): A\in \Omega_n^{(0)}\}=\frac{(n-1)!}{(n-1)^{n-1}}\left(\frac{n-2}{n-1}\right)^{n-2}$, where $\Omega_n^{(0)}$ is the set of $n\times n$ doubly stochastic matrices with at least one zero entry.

∎

2.2 The Main Idea

Let $p\in Hom_+(n,n)$. Define the following polynomials $q_i\in Hom_+(i,i)$:

$$q_n=p, q_i(x_1,\dots,x_i)=\frac{\partial^{n-i}}{\partial x_{i+1}\dots\partial x_n}p(x_1,\dots,x_i,0,\dots,0);1\le i\le n-1.$$

Notice that $q_1(x_1)=\frac{\partial^n}{\partial x_1\dots\partial x_n}p(0)x_1$ and

$$q_2(x_1,x_2)=\frac{\partial^n}{\partial x_1\dots\partial x_n}p(0)x_1x_2+\frac{1}{2}\left(\frac{\partial^n}{\partial x_1\partial x_1\dots\partial x_n}p(0)x_1^2+\frac{\partial^n}{\partial x_2\partial x_2\dots\partial x_n}p(0)x_2^2\right). \tag{20}$$

Therefore, $Cap(q_1)=\frac{\partial^n}{\partial x_1\dots\partial x_n}p(0)$ and

$$Cap(q_2)=\frac{\partial^n}{\partial x_1\dots\partial x_n}p(0)+\sqrt{\frac{\partial^n}{\partial x_1\partial x_1\dots\partial x_n}p(0)\quad\frac{\partial^n}{\partial x_2\partial x_2\dots\partial x_n}p(0)}. \tag{21}$$

Define the univariate polynomial $R(t) = p(x_1, \dots, x_{n-1}, t)$. Then its derivative at zero is

$$R'(0) = q_{n-1}(x_1, \dots, x_{n-1}). \tag{22}$$

Another simple but important observation is the next inequality:

$$deg_{q_i}(i) \le \min\left(i, deg_p(i)\right) \Longleftrightarrow G\left(deg_{q_i}(i)\right) \ge G\left(\min(i, deg_p(i))\right) : 1 \le i \le n. \tag{23}$$

Recall that $vdw(i) = \frac{i!}{i^i}$. Suppose that the next inequalities hold

$$Cap(q_{i-1}) \ge Cap(q_i) \frac{vdw(i)}{vdw(i-1)} = Cap(q_i) G(i) : 2 \le i \le n. \tag{24}$$

Or better, the next stronger ones hold

$$Cap(q_{i-1}) \ge Cap(q_i) G\left(deg_{q_i}(i)\right) : 2 \le i \le n, \tag{25}$$

where

$$G(m) = \frac{vdw(m)}{vdw(m-1)} = \left(\frac{m-1}{m}\right)^{m-1}. \tag{26}$$

The next result, proved by the straigthforward induction, summarizes the main idea of our approach.

Theorem 2.8:

1. *If the inequalities (24) hold then the next generalized Van Der Waerden inequality holds:*

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) = Cap(q_1) \ge vdw(n) Cap(p). \tag{27}$$

In the same way, the next inequality holds for $Cap(q_2)$:

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0) + \sqrt{\frac{\partial^n}{\partial x_1 \partial x_1 \dots \partial x_n} p(0) \frac{\partial^n}{\partial x_2 \partial x_2 \dots \partial x_n} p(0)} \ge 2 vdw(n) Cap(p). \tag{28}$$

2. *If the inequalities (25) hold then the next generalized* **(Schrijver-bound)** *holds:*

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0, \dots, 0) = Cap(q_1) \ge Cap(p) \prod_{2 \le i \le n} G\big(\min(i, deg_p(i))\big). \tag{29}$$

What is left is to prove that the inequalities (25) hold for **H-Stable** polynomials. We break the proof of this statement in two steps.

1. Prove that if $p \in Hom_+(n,n)$ is **H-Stable** then q_{n-1} is either zero or **H-Stable**. Using equation (22), this implication follows from Gauss-Lukas Theorem. Gauss-Lukas Theorem states that if $z_1, \dots, z_n \in C$ are the roots of an univariate polynomial Q then the roots of its derivative Q' belong to the convex hull $CO(\{z_1, \dots, z_n\})$. *This step is, up to minor perturbation arguments, known. See, for instance, [16]. The result in [16] is stated in terms of hyperbolic polynomials, see Remark (5.2) for the connection between* **H-Stable** *and hyperbolic polynomials. Our treatment, described in Section(4), is self-contained, short and elementary.*

2. Prove that $Cap(q_{n-1}) \geq G(deg_p(n))Cap(p)$. This inequality boils down to the next inequality for the univariate polynomial R from (22):

$$R'(0) \geq G(deg(R)) \left(\inf_{t>0} \frac{R(t)}{t} \right).$$

We prove it using AM/GM inequality and the fact that the roots of the polynomial R are real.

It is instructive to see what is going on in the "permanental case": we start with the polynomial $Prod_A$ which is a product of nonnegative linear forms. The very first polynomial in the induction, q_{n-1}, is not of this type in the generic case. I.e. there is no one matrix/graph associated with q_{n-1}. We gave up the matrix structure but had won the game.

In the rest of the paper **Facts** are statements which are quite simple and (most likely) known. We included them having in mind the undergraduate student reader.

3 Univariate Polynomials

Proposition 3.1:

1. *(Gauss-Lukas Theorem)*
   Let $R(z) = \sum_{0 \leq i \leq n} a_i z^i$ be a Hurwitz polynomial with complex coefficients, i.e. all the roots of R have negative real parts.
   Then its derivative R' is Hurwitz.

2. *Let $R(z) = \sum_{0 \leq i \leq n} a_i z^i$ be a Hurwitz polynomial with real coefficients and $a_n > 0$. Then all the coefficients are positive real numbers.*

Proof:

1. Recall that

$$\frac{R'(z)}{R(z)} = \sum_{1 \leq j \leq n} \frac{1}{z - z_j}.$$

Let μ be a root of R'. Consider two cases. First: μ is a root of R. Then clearly $Re(\mu) < 0$. Second: μ is not a root of R. Then

$$L =: \sum_{1 \leq j \leq n} \frac{1}{\mu - z_j} = 0.$$

Suppose that $Re(\mu) \geq 0$. As $(a + ib)^{-1} = \frac{a - ib}{a^2 + b^2}$ we get that

$$Re \left(\frac{1}{\mu - z_j} \right) = \frac{Re(\mu) - Re(z_j)}{(Re(\mu) - Re(z_j))^2 + (Im(\mu) - Im(z_j))^2} > 0.$$

Therefore $Re(L) > 0$ which leads to a contradiction. Thus $Re(\mu) < 0$ and the derivative R' is Hurwitz.

2. This part is easy and well known.

∎

The next simple result binds together all the small pieces of our approach.

Lemma 3.2: *Let $Q(t) = \sum_{0 \le i \le k} a_i t^i; a_k > 0, k \ge 2$ be a polynomial with non-negative coefficients and real (non-positive) roots. Define $C = \inf_{t>0} \frac{Q(t)}{t}$. Then the next inequlity holds:*

$$a_1 = Q'(0) \ge \left(\frac{k-1}{k}\right)^{k-1} C. \tag{30}$$

The equality holds if and only if all the roots of Q are equal negative numbers, i.e. $Q(t) = b(t+a)^k$ for some $a, b > 0$.

Proof: If $Q(0) = 0$ then $Q'(0) \ge C > \left(\frac{k-1}{k}\right)^{k-1} C$.
Let $Q(0) > 0$. We then can assume WLOG that $Q(0) = 1$. In this case all the roots of Q are negative real numbers. Thus

$$Q(t) := \prod_{i=1}^{k} (a_i t + 1) : a_i > 0, 1 \le i \le k,$$

and $Q'(0) = a_1 + \cdots + a_k$.

Using the AM/GM inequality we get that

$$Ct \le Q(t) \le P(t) =: \left(1 + \frac{Q'(0)}{k} t\right)^k, t \ge 0. \tag{31}$$

It follows from basic calculus that

$$\inf_{t>0} \frac{P(t)}{t} = P(s) = Q'(0) \left(\frac{k}{k-1}\right)^{k-1}, \quad \text{where} \quad s = \frac{k}{Q'(0)(k-1)}.$$

Therefore

$$C \le \inf_{t>0} \frac{P(t)}{t} = Q'(0) \left(\frac{k}{k-1}\right)^{k-1},$$

which finally yields the desired inequality

$$Q'(0) \ge \left(\frac{k-1}{k}\right)^{k-1} C, k \ge 2.$$

It follows from the uniqueness condition in the AM/GM inequality that the equality in (30) holds if and only if $0 < a_1 = \cdots = a_k$. ∎

Remark 3.3: The condition that the roots of Q are real can be relaxed in several ways. For instance the statement of Lemma (3.2) holds for any map $f : R_+ \rightarrow R_+$ such that the derivative $f'(0)$ exists and $f^{\frac{1}{k}}$ is concave.
If such map is log-concave, i.e $\log(f)$ is concave, then $f'(0) \geq \frac{1}{e} \inf_{t>0} \frac{f(t)}{t}$.
Notice that the right inequality in (31) is essentially equivalent to the concavity of the function $(Q(t))^{\frac{1}{k}}$ on R_+.
It was shown in [11] that the inequality (30) is equivalent to the **(VDW-bound)** for doubly stochastic matrices $A \in \Omega_n : A = [a|b| \dots |b]$ with two distinct columns. ∎

4 Stable homogeneous polynomials

4.1 Basics

Definition 4.1: A polynomial $p \in Hom_C(m,n)$ is called **H-Stable** if $p(Z) \neq 0$ provided $Re(Z) > 0$; is called **H-SStable** if $p(Z) \neq 0$ provided $Re(Z) \geq 0$ and $\sum_{1 \leq i \leq m} Re(z_i) > 0$. ∎

Fact 4.2: Let $p \in Hom_C(m,n)$ be **H-Stable** and A is $m \times m$ matrix with nonnegative real entries without zero rows. Then the polynomial p_A, defined as $p_A(Z) = p(AZ)$ is also **H-Stable**. If all entries of A are positive then p_A is **H-SStable**.

Fact 4.3: Let $p \in Hom_C(m,n)$, $Y \in C^m, p(Y) \neq 0$. Define the following univariate polynomial of degree n:

$$L_{X,Y}(t) = p(tY - X) = p(Y) \prod_{1 \leq i \leq n} (t - \lambda_{i;Y}(X)) : X \in C^m.$$

Then

$$\lambda_{i;Y}(bX + aY) = b\lambda_{i;Y}(X) + a; p(X) = p(Y) \prod_{1 \leq i \leq n} \lambda_{i;Y}(X). \tag{32}$$

The following simple result substantially simplifies the proofs below. Proposition (4.4) connects the notion of **H-Stability** with the notion of **Hyperbolicity**, see more on this connection in Subsection(5.1).

Proposition 4.4: *A polynomial $p \in Hom_C(m,n)$ is* **H-Stable** *if and only if $p(X) \neq 0 : X \in R^m_{++}$ and the roots of univariate polynomials $P(tX - Y) : X, Y \in R^m_{++}$ are real positive numbers.*

Proof:

1. Suppose that $p(X) \neq 0 : X \in R^m_{++}$ and the roots of univariate polynomials $p(tX - Y) : X, Y \in R^m_{++}$ are real positive numbers. It follows from identities (32) (shift $L \rightarrow L + aX > 0$) that the roots of $P(tX - L) : X \in R^m_{++}, L \in R^m$ are real numbers. We want to prove that this property implies that $p \in Hom_C(m,n)$ is

H-Stable. Let $Z = Re(Z) + iIm(Z) \in C^m : Im(Z) \in R^m, 0 < Re(Z) \in R^m_{++}$. If $p(Z) = 0$ then also $p(-iRe(Z) + Im(Z)) = 0$, which contradicts the real rootedness of $p(tX - Y) : X > 0, Y \in R^m$.

2. Suppose that $p \in Hom_C(m, n)$ is **H-Stable**. Let $X, Y \in R^m_{++}$ and $p(zX - Y) = 0, z = a+bi$. We need to prove that $b = 0$ and $a > 0$. If $b \neq 0$ then $p(aX-Y+biX) = (bi)^n p(X - b^{-1}i(aX - y)) \neq 0$ as the real part $Re(X - b^{-1}i(aX - y)) = X > 0$. Therefore $b = 0$. If $a \leq 0$ then $-(aX-Y) \in R^m_{++}$. Which implies that $p(aX-Y) = (-1)^n p(-(aX - Y)) \neq 0$. Thus $a > 0$.

∎

We will use the following corollaries:

Corollary 4.5: *If $Re(Z) \in R^m_+$ and a polynomial p is* **H-Stable** *then*

$$|p(Z)| \geq |p\,(Re(Z))\,|. \tag{33}$$

Proof: Since p is continuous on C^m hence it is sufficient to assume that $Re(Z) \in R^m_{++}$. It follows from identities (32) that

$$p(Z) = p\,(Re(Z) + iIm(Z)) = p\,(Re(Z)) \prod_{1 \leq j \leq n} (1 + i\lambda_j), \tag{34}$$

where $(\lambda_1, \ldots, \lambda_n)$ are the roots of the univariate polynomial $p(tRe(Z)-Im(Z))$. Because $Re(Z) \in R^m_{++}$, all these roots are real numbers.
Therefore $|p(Z)| = |p(Re(Z))| \prod_{1 \leq j \leq n} |1 + i\lambda_j| \geq |p(Re(Z))|$. ∎

Corollary 4.6: *Let $p \in Hom_C(m, n)$ be* **H-Stable***; $X, Y \in R^m$ and $0 < X + Y \in R^m_{++}$. Then all the roots of the univariate polynomial equation $p(tX + Y) = 0$ are real numbers.*

Proof: Let $p(tX + Y) = 0$, then also $p((t-1)X + (X + Y)) = 0$. Since $X + Y > 0$ hence $t - 1 \neq 0$. As the polynomial p is homogeneous therefore $p(X + (1-t)^{-1}(X + Y)) = 0$. It follows that $(1-t)^{-1}$ is real, thus t is also a real number. ∎

Fact 4.7: Let $p \in Hom_C(m, n)$ be **H-SStable** (**H-Stable**). Then for all $X \in R^m_{++}$ the coefficients of the polynomial $q = \frac{p}{p(X)}$ are positive (nonnegative) real numbers.

Proof: We prove first the case of **H-SStable** polynomials.
Since $q(X) = 1$ we get from (32) that $q(Y)$ is a positive real number for all vectors $Y \in R^m_{++}$. Therefore, by a standard interpolation argument, the coefficients of q are real. We will prove by induction the following equivalent statement: *if $q \in Hom_R(m, n)$ is* **H-SStable** *and $q(Y) > 0$ for all $Y \in R^m_{++}$ then the coefficients of q are all positive.* Write $q(t; Z) = \sum_{0 \leq i \leq n} t^i q_i(Z)$, where $Z \in C^{m-1}$, the polynomials $q_i \in Hom_R(m-1, n-i)$, $0 \leq i \leq n-1$ and $q_n(Z)$ is a real number. Let us fix the complex vector Z such that $Re(Z) \in R^{m-1}_+$ and $Re(Z) \neq 0$. Since q is **H-SStable** hence all roots of the univariate polynomial $q(t; Z)$ have negative real parts. Therefore, using the first part of Proposition

(3.1), we get that polynomials $q_i : 0 \leq i \leq n$ are all **H-SStable**. Since the degree of q is n hence $q_n(Z)$ is a constant, $q_n(Z) = q(1;0) > 0$. Using now the second part of Proposition (3.1), we see that $q_i(Y) > 0$ for all $Y \in R^m_{++}$ and $0 \leq i \leq n$. Continuing this process we will end up with either $m = 1$ or $n = 1$. Both those cases have positive coefficients.

Let $p \in Hom_C(m,n)$ be **H-Stable** and $A > 0$ is $m \times m$ matrix with positive entries such that $AX = X$. Then for all $\epsilon > 0$ the polynomials $q_{I+\epsilon A} \in Hom_R(m,n)$, defined as in Fact(4.2), are **H-SStable** and $\lim_{\epsilon \to 0} q_{I+\epsilon A} = q$. Therefore the coefficients of q are nonnegative real numbers. ∎

From now on we will deal only with the polynomials with nonnegative coefficients.

Corollary 4.8: *Let $p_i \in Hom_+(m,n)$ be a sequence of* **H-Stable** *polynomials and $p = \lim_{i \to \infty} p_i$. Then p is either zero or* **H-Stable**.

Some readers might recognize Corollary (4.8) as a particular case of A. Hurwitz's theorem on limits of sequences of nowhere zero analytical functions. Our proof below is elementary.

Proof: Suppose that p is not zero. Since $p \in Hom_+(m,n)$ hence $p(x_1, \ldots, x_m) > 0$ if $x_j > 0 : 1 \leq j \leq m$. As the polynomials p_i are **H-Stable** therefore $|p_i(Z)| \geq |p_i\left(Re(Z)\right)| : Re(Z) \in R^m_{++}$. Taking the limits we get that $|p(Z)| \geq |p\left(Re(Z)\right)| > 0 : Re(Z) \in R^m_{++}$, which means that p is **H-Stable**. ∎

Fact 4.9: For a polynomial $p \in Hom_C(m,n)$ we define a polynomial $q \in Hom_C(m-1, n-1)$ as

$$q(x_1, \ldots, x_{m-1}) = \frac{\partial}{\partial x_m} p(x_1, \ldots, x_{m-1}, 0).$$

Then the next two statements hold:

1. Let $p \in Hom_+(m,n)$ be **H-SStable**. Then the polynomial q is also **H-SStable**.
2. Let $p \in Hom_+(m,n)$ be **H-Stable**. Then the polynomial q is either zero or **H-Stable**.

Proof:

1. Let $p \in Hom_+(m,n)$ be **H-SStable** and consider an univariate polynomial

$$R(z) = p(Y; z) : z \in C, Y \in C^{m-1}.$$

Suppose that $0 \neq Re(Y) \geq 0$. It follows from the definition of **H-SStability** that $R(z) \neq 0$ if $Re(z) \geq 0$. In other words, the univariate polynomial R is Hurwitz. It follows from Gauss-Lukas Theorem that

$$q(Y) = R'(0) \neq 0,$$

which means that q is **H-SStable**.

2. Let $p \in Hom_+(m,n)$ be **H-Stable** and $q \neq 0$. Take an $m \times m$ matrix $A > 0$. Then the polynomial $p_{I+\epsilon A}$, $p_{I+\epsilon A}(Z) = p\left((I+\epsilon A)Z\right)$ is **H-SStable** for all $\epsilon > 0$. Therefore, using the first part, $q_{I+\epsilon A}$ is **H-SStable**. Clearly $\lim_{\epsilon \to 0} q_{I+\epsilon A} = q$. Since $q \neq 0$, it follows from Corollary (4.8) that q is **H-Stable**.

∎

Theorem 4.10: *Let $p \in Hom_+(n,n)$ be* **H-Stable**, *and*

$$q_{n-1}(x_1,\ldots,x_{n-1}) = \frac{\partial}{\partial x_n} p(x_1,\ldots,x_{n-1},0).$$

Then

$$Cap(q_{n-1}) \geq Cap(p) G\left(deg_p(n)\right). \tag{35}$$

Proof: We need to prove that

$$\frac{\partial}{\partial x_n} p(x_1,\ldots,x_{n-1},0) \geq Cap(p) G\left(deg_p(n)\right), x_1,\ldots,x_{n-1} > 0, \prod_{1 \leq i \leq n-1} x_i = 1.$$

Fix a positive vector $(x_1,\ldots,x_{n-1})$, $\prod_{1 \leq i \leq n-1} x_i = 1$ and define, as in proof of Fact (4.9), the polynomial $R(t) = p(x_1,\ldots,x_{n-1},t)$. It follows from Corollary(4.6) that all the roots of R are real. Since the coefficients of the polynomial R are non-negative hence its roots are non-positive real numbers. It follows from a definition of $Cap(p)$ that $R(t) \geq Cap(p)t$, therefore

$$\inf_{t>0} \frac{R(t)}{t} \geq Cap(p).$$

The degree of the polynomial R is equal to $deg_p(n)$. It finally follows from Lemma(3.2) that

$$q_{n-1}(x_1,\ldots,x_{n-1},0) = R'(0) \geq Cap(p) G\left(deg_p(n)\right).$$

∎

5 Uniqueness in Generalized Van Der Waerden Inequality

5.1 Hyperbolic Polynomials

The following concept of hyperbolic polynomials arose from the theory of partial differential equations [6], [14]. A recent paper [24] gives nice and concise introduction to the area (with simplified proofs of the key theorems) and describes connections to convex optimization.

Definition 5.1:

1. A homogeneous polynomial $p : C^m \to C$ of degree n ($p \in Hom_C(m, n)$) is called hyperbolic in the direction $e \in R^m$ (or e-hyperbolic) if $p(e) \neq 0$ and for each vector $X \in R^m$ the univariate (in λ) polynomial $p(X - \lambda e)$ has exactly n real roots counting their multiplicities.

2. Denote an ordered vector of roots of $p(x - \lambda e)$ as
$$\lambda_e(X) = (\lambda_n(X) \geq \lambda_{n-1}(X) \geq \ldots \lambda_1(X)).$$
Call $X \in R^m$ e-positive (e-nonnegative) if $\lambda_1(X) > 0$ ($\lambda_n(X) \geq 0$). We denote the closed set of e-nonnegative vectors as $N_e(p)$, and the open set of e-positive vectors as $C_e(p)$.

∎

Remark 5.2: Proposition (4.4) essentially says that a polynomial $p \in Hom_C(m, n)$ is **H-Stable** iff p is hyperbolic in some direction $e \in R^m_{++}$ and the inclusion $R^m_+ \subset N_e(p)$ holds. If $p \in Hom_C(m, n)$ is **H-SStable** then any non-zero vector $0 \leq X \in R^m_+$ belongs to the (open) *hyperbolic cone* $C_e(p)$. ∎

We need the next fundamental fact due to L. Garding [6] (we recommend the very readable treatment in [24]):

Theorem 5.3: *Let $p \in Hom_C(m, n)$ be e-hyperbolic polynomial and $d \in C_e(p) \subset R^m$. Then p is also d-hyperbolic and $C_d(p) = C_e(p), N_d(p) = N_e(p)$. Moreover cone $C_e(p)$, called hyperbolic cone, is convex.*

Corollary 5.4:

1. *For any two vectors in the hyperbolic cone $d_1, d_2 \in C_e(p)$ the following set equality holds:*
$$N_{d_1}(p) \bigcap (-N_{d_1}(p)) = N_{d_2}(p) \bigcap (-N_{d_2}(p)) = Null_p. \tag{36}$$
Thus $Null_p \subset R^m$ is a linear subspace.

2. $$Null_p = \{X \in R^m : p(Y + X) = p(Y) \quad \text{for all} \quad Y \in C^m\} \tag{37}$$
Let $Pr(Null_p)$ be orthogonal projector on the linear subspace $Null_p$. It follows from (37) that
$$p(Y) = p\left((I - Pr(Null_p))Y\right) \tag{38}$$

3. *Let $p \in Hom_+(n, n)$ be a doubly-stochastic* **H-Stable** *polynomial. If $(y_1, \ldots, y_n) \in Null_p$ then*
$$y_1 + \cdots + y_n = 0. \tag{39}$$

Proof:

1. It is well known and obvious that if K is a convex cone in some linear space L over reals then the intersection $K \bigcap (-K)$ is a linear subspace of L.

2. Let $T \in C_e(p)$ and $X \in Null_p$. Then all the roots of the equation $p(xT+X) = 0$ are equal to zero. Since $p(T) \neq 0$ and the polynomial $p \in Hom_C(m,n)$ is homogeneous, hence $p(xT+X) = x^n p(T)$. Therefore $p(T+X) = p(T)$ for all $T \in C_e(p)$. As $C_e(p)$ is a non-empty open subset of R^m, equality (37) follows from the analyticity of p.

3. Consider the vector of all ones $e = (1, \dots, 1) \in R^n$ and a vector $Y = (y_1, \dots, y_n) \in Null_p$. Then $d(t) = p(e+tY) = p(e)$ for all $t \in R$. Therefore

$$0 = d'(0) = \sum_{1 \leq i \leq n} y_i \frac{\partial}{\partial x_i} p(1, 1, \dots, 1) = y_1 + \dots + y_n.$$

∎

Example 5.5:

1. Consider the power polynomial $q \in Hom_+(n,n)$, $q(x_1, \dots, x_n) = (a_1x_1 + \dots + a_nx_n)^n$. If the non-zero vector $\mathbf{a} = (a_1, \dots, a_n) \in R^n_+$ then the power polynomial q is **H-Stable**. The correspondind linear subspace

$$Null_p = \mathbf{a}^{\perp} =: \{(y_1, \dots, y_n) \in R^n : \sum_{1 \leq i \leq n} a_i y_i = 0\}, \quad dim(Null_p) = n-1;$$

and $Cap(p) = n^n \prod_{1 \leq i \leq n} a_i$. Therefore $Cap(q) \neq 0$ iff $\mathbf{a} \in R^n_{++}$.
It is easy to see that

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} (a_1x_1 + \dots + a_nx_n)^n = n! a_1 \dots a_n.$$

Therefore

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} q(0, \dots, 0) = Cap(q) \frac{n!}{n^n}. \qquad (40)$$

If $dim(Null_p) = n-1$ and a polynomial $p \in Hom_+(n,n)$ is **H-Stable** then $p(x_1, \dots, x_n) = (b_1x_1 + \dots + b_nx_n)^n$ for some non-zero vector $\mathbf{b} = (b_1, \dots, b_n) \in R^n_+$. The power polynomial p is doubly-stochastic iff $b_i = \frac{1}{n}, 1 \leq i \leq n$.

2. Let $p \in Hom_C(m,n)$ be an e-hyperbolic polynomial, $D \in C_e(p) \subset R^m$ and $X \in R^m$. Suppose that the univariate polynomial $R(t) = p(tD+X) = a(t+b)^n, b \in R$. Define the next real vector $Y = -bD + X$. Then all the roots of the equation $p(Y - \lambda D) = 0$ are equal to zero. Therefore $Y \in N_D(p) \bigcap (-N_D(p)) = Null_p$.

∎

5.2 Uniqueness

Definition 5.6: We call a **H-Stable** polynomial $p \in Hom_+(n,n)$ **extremal** if $Cap(p) > 0$ and

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0,\dots,0) = \frac{n!}{n^n} Cap(p). \qquad (41)$$

▮

Our goal is the next theorem

Theorem 5.7: *A* **H-Stable** *polynomial* $p \in Hom_+(n,n)$ *is* **extremal** *if and only if*

$$p(x_1,\dots,x_n) = (a_1 x_1 + \dots + a_n x_n)^n$$

for some positive real numbers $a_1,\dots,a_n > 0$.
In other words, the equality (41) holds iff $dim(Null_p) = n-1$.

Notice that the "if" part is simple and follows from the equality(40).

We collect the basic properties of **extremal** polynomials in the next proposition.

Proposition 5.8:

1. *If a* **H-Stable** *polynomial* $p \in Hom_+(n,n)$ *is* **extremal** *then all its coefficients are positive real numbers.*
2. *Let* $\mathbf{c} = (c_1,\dots,c_n) \in R^n_{++}$ *and* $p \in Hom_+(n,n)$*. Define the scaled polynomial* $p_{\mathbf{c}}$ *as* $p_{\mathbf{c}}(x_1,\dots,x_n) = p(c_1 x_1,\dots,c_n x_n)$*. If* $p \in Hom_+(n,n)$ *is* **H-Stable extremal** *polynomial then also the scaled polynomial* $p_{\mathbf{c}}$ *is.*

Proof:

1. Our goal in this step is to show that if equation (41) holds then the **H-Stable** polynomial p is, in fact, **H-SStable** and therefore has all positive coefficients.

   Since $G(2)\cdots G(n) = \frac{n!}{n^n}$ and the function G is strictly decreasing on $[0,\infty)$, hence it follows from (13) that $deg_p(n) = n$. Since the inequality (13) is invariant with respect to permutations of variables hence $deg_p(i) = n : 1 \le i \le n$. Which means that $p(e_i) > 0 : 1 \le i \le n$, where $\{e_1,\dots,e_n\}$ is the standard orthonormal basis in R^n. Therefore the polynomial p is **H-SStable**. Thus its coefficients are strictly positive real numbers and all non-zero vectors $Y \in R^n_+$ belong to its open hyperbolic cone $C_e(p)$.

2. First, if $p \in Hom_+(n,n)$ is **H-Stable** then clearly the scaled polynomial $p_{\mathbf{c}}$ is also **H-Stable**.
   It follows from the definition of the **Capacity** that $Cap(p_{\mathbf{c}}) = c_1 \cdots c_n Cap(p)$.
   We get, by a direct computation, that

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} p_{\mathbf{c}}(0,\dots,0) = c_1 \cdots c_n \frac{\partial^n}{\partial x_1 \dots \partial x_n} p(0,\dots,0).$$

This proves that the set of **H-Stable extremal** polynomials is invariant with respect to the scaling.

∎

We need the following simple result (it was essentially proved in Lemma 3.8 from [8]).

Fact 5.9: Consider $p \in Hom_{++}(n,n), p(x_1, \dots x_n) = \sum_{r_1,\dots,r_n} a_{r_1,\dots,r_n} \prod_{1 \leq i \leq n} x_i^{r_i}$. Then there exists a positive vector $\mathbf{t} =: (t_1, \dots, t_n) \in R^n_{++}, t_1 \cdots t_n = 1$ such that

$$p(t_1, \dots, t_n) = Cap(p) = \inf_{x_i > 0, 1 \leq i \leq n; x_1 \cdots x_n = 1} p(x_1, \dots, x_n). \tag{42}$$

Consider the corresponding scaled polynomial $p_{\mathbf{t}}$. Then the polynomial $q = \frac{p_{\mathbf{t}}}{p_{\mathbf{t}}(1,\dots,1)}$ is doubly-stochastic. I.e. $\frac{\partial}{\partial x_i} q(1,1,\dots,1) = 1, 1 \leq i \leq n.$.

Proof: Consider a subset

$$T = \{(x_1, \dots, x_n) \in R^n_{++} : x_1 \cdots x_n = 1 \quad \text{and} \quad p(x_1, \dots, x_n) \leq p(1, \dots, 1)\}.$$

In order to prove that the infimum is attained, it is sufficient to show that the subset T is compact. Clearly, T is closed, and we need to prove that T is bounded. Let $(x_1, \dots, x_n) \in T$ and assume WLOG that $\max_{1 \leq i \leq n} x_i = x_1$. Then

$$a_{n,0,\dots,0} x_1^n \leq p(x_1, \dots, x_n) \leq p(1, \dots, 1) \Rightarrow \max_{1 \leq i \leq n} x_i \leq \frac{p(1, \dots, 1)}{a_{n,0,\dots,0}} < \infty.$$

This shows the desired boundness of T and the existence of the minimum.
Consider a positive vector $(t_1, \dots, t_n)$ such that $p(t_1, \dots, t_n) = Cap(p)$. Define $\alpha_i = \log(t_i), 1 \leq i \leq n$. Then

$$p(exp(\alpha_1), \dots, exp(\alpha_n)) = \min_{\beta_1 + \dots \beta_n = 0} p(exp(\beta_1), \dots, exp(\beta_n)).$$

Therefore there exists the Lagrange multiplier γ such that

$$\frac{\partial}{\partial \alpha_i} p(exp(\alpha_1), \dots, exp(\alpha_n)) = t_i \frac{\partial}{\partial t_i} p(t_1, \dots, t_n) = \gamma, 1 \leq i \leq n.$$

It follows from the Euler's identity that $\gamma = p(t_1, \dots, t_n)$ and

$$\frac{\partial}{\partial x_i} q(1, \dots, 1) = (p(t_1, \dots, t_n))^{-1} t_i \frac{\partial}{\partial t_i} p(t_1, \dots, t_n) = 1.$$

∎

Remark 5.10: It is easy to prove that, in fact, the minimum in (42) is attained uniquely. It was proved in [12] that if $p \in Hom_+(n,n)$ is **H-Stable** then the minimum in (42) exists and attained uniquely iff

$$\frac{\partial^n}{\partial x_i \partial x_j \prod_{m \neq (i,j)} \partial x_m} q(0, \dots, 0) > 0 : 1 \leq i \neq j \leq n.$$

∎

Proof: (**Proof of Theorem (5.7).**)
It follows from Proposition(5.8) and Fact(5.9) that we can assume without loss of generality that the **H-Stable extremal** polynomial $p \in Hom_{++}(n,n)$ is doubly-stochastic and all its coefficients are positive real numbers.

1. **Using uniqueness part of Lemma(3.2)**
   Let $\{e_1, \ldots, e_n\}$ be the standard basis in R^n and p is now a **H-SStable** doubly-stochastic polynomial with positive coefficients, p satisfies the equality (41). We need to look at the case of equality in (35). Recall the polynomial q_{n-1} is given by $q_{n-1}(x_1, \ldots, x_{n-1}) = \frac{\partial}{\partial x_n} p(x_1, \ldots, x_{n-1}, 0)$.

   Clearly, the polynomial $q_{n-1} \in Hom_{++}(n-1, n-1)$ also has positive coefficients. Let
   $$q_{n-1}(t_{1,n}, \ldots, t_{n-1,n}) = \min_{x_1,\ldots,x_{n-1}>0, \prod_{1\le i\le n-1} x_i = 1} q_{n-1}(x_1, \ldots, x_{n-1}).$$
   The existence of such a vector was proved in Proposition (5.9).
   It follows from the uniqueness part of Lemma(3.2) that the univariate polynomial $R(t) = p(t_{1,n}, \ldots, t_{n-1,n}, t) = p(\sum_{1\le i\le n-1} t_{i,n-1} e_i + te_n)$ has n equal negative roots: $R(t) = b(t + a_n)^n; a_n, b > 0$. This fact implies, as in the second part of Example (5.5), that
   $$K_n =: e_n - \sum_{j\neq n} a_{j,n} e_j \in Null_p, a_{j,n} = \frac{t_{j,n}}{a_n} > 0.$$
   Since p is doubly-stochastic, hence it follows from (39) that the coordinates of K_n sum to zero. Which gives that $\sum_{j\neq n} a_{j,n} = 1$.

   In the same way, we get that there exists an $n \times n$ column stochastic matrix A with the zero diagonal and the positive off-diagonal part such that the vectors
   $$K_i =: e_i - \sum_{j\neq i} a_{j,i} e_j \in Null_p, i \le i \le n.$$

2. Recall that our goal is to prove that $dim(Null_p) = n-1$. It follows that
   $$dim(Null_p) \ge dim(L(K_1, \ldots, K_n)) = Rank(I - A),$$
   where $L(K_1, \ldots, K_n)$ is the minimal linear subspace containing the set $\{K_1, \ldots, K_n\}$. Since the polynomial p is non-zero thus $dim(Null_p) \le n-1$. It is easy to see that $Rank(I - A) = n-1$. Indeed, any principal $n-1 \times n-1$ submatrix of $I - A$ is *strictly diagonally dominant* and, therefore, is nonsingular.
   We finally conclude that $dim(Null_p) = n-1$.

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6 Comments

1. Falikman [5] and Egorychev [4] publications were followed by a flurry of expository papers, which clarified and popularized the proofs. The author learned the Egorychev's proof from [17]. It is our guess that many scientists first learned about Alexandrov inequalities for mixed discriminants and Alexandrov-Fenchel inequalities for mixed volumes [1] in one of those expository papers. We would like to distinguish the following two papers: [16] and [25]. They both explicitly connected Alexandrov inequalities for mixed discriminants with homogeneous hyperbolic polynomials. The paper [16] was, essentially a rediscovery of Garding's theory [6]. Still, as the author had read [16] before reading [6], the paper [16] gave us the first hint for the possibility of our approach.
   The paper [25], apparently written as a technical report in 1981 and published only in 2006 in an obscure book, is technically very similar to [16]. Besides, it implicitly introduced the Bapat's conjecture.
   Other related publications are D. London's (univariate) papers [21],[20],[19].

   As far as we know, there were no previously published connections between Shrijver-Valiant conjecture, which was thought to be of purely combinatorial nature, and stable/hyperbolic polynomials.

2. Two main ingredients of our approach, which make the proofs simple, are the usage of the notion of **Capacity** and Lemma (3.2). They together allowed the simple induction. The induction, used in this paper, is by partial differentiation. It is very similar to the inductive proofs of hyperbolic polynomials analogues of Alexandrov inequalities for mixed discriminants in [16],[25]. Using our terminology, these analogues correspond to the fact that the polynomial $q_2(x_1, x_2) = \frac{\partial^{n-2}}{\partial x_3 \ldots \partial x_n} p(x_1, x_2, 0, \ldots, 0)$ is either zero or **H-Stable** provided the polynomial p is **H-Stable**.
   The idea to use **Capacity** in the context of permanents is implicit in [22]. The notion of **Capacity** was crucial for algorithmic results in [7], [8] as $\log(Cap(p)) = \inf_{y_1+\cdots+y_n=0} \log(p(e^{y_1}, \ldots, e^{y_n}))$ and the functional $\log(p(e^{y_1}, \ldots, e^{y_n}))$ is convex for any polynomial with non-negative coefficients.
   Probably, the papers [7], [8] were the first to reformulate Van der Waerden/Bapat conjectures as in inequality (27). Although quite simple, it happened to be a very enlighting observation.

3. Our, inductive by the partial differentiation, approach was initiated in [10]. The main tool there was Vinnikov-Dubrovin determinantal representation [29] of hyperbolic homogeneous polynomials in 3 variables. The paper [10] proved the implication $Cap(p) > 0 \Longrightarrow \frac{\partial^n}{\partial x_1 \ldots \partial x_n} p(0, \ldots, 0) > 0$ for **H-Stable** polynomials $p \in Hom_+(n, n)$. Additionally, it was proved that in this **H-Stable** case the functional $Rank_p(S) =$

$\max_{a_{r_1,\ldots,r_n}\neq 0}\sum_{j\in S} r_j$ is submodular and

$$a_{r_1,\ldots,r_n} > 0 \Longleftrightarrow \sum_{j\in S} r_j \leq Rank_p(S) : S \subset \{1,\ldots,n\}. \quad (43)$$

The characterization (43) is a far reaching generalization of the Hall-Rado theorem. The paper [12] provides algorithmic applications of these results: strongly polynomial deterministic algorithms for the membership problem as for the support as well for the Newton polytope of **H-Stable** polynomials $p \in Hom_+(m,n)$, given as oracles.

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