The map asymptotics constant t_g

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Submitted: Jan 28, 2008; Accepted: Mar 22, 2008; Published: Mar 27, 2008
Mathematics Subject Classification: 05C30

Abstract

The constant t_g appears in the asymptotic formulas for a variety of rooted maps on the orientable surface of genus g. Heretofore, studying this constant has been difficult. A new recursion derived by Goulden and Jackson for rooted cubic maps provides a much simpler recursion for t_g that leads to estimates for its asymptotics.

1 Introduction

Let Σ_g be the orientable surface of genus g. A *map* on Σ_g is a graph G embedded on Σ_g such that all components of $\Sigma_g - G$ are simply connected regions. These components are called *faces* of the map. A map is rooted by distinguishing an edge, an end vertex of the edge and a side of the edge.

With $M_{n,g}$ the number of rooted maps on Σ_g with n edges, Bender and Canfield [1] showed that

$$M_{n,g} \sim t_g n^{5(g-1)/2} 12^n \quad \text{as} \quad n \to \infty, \tag{1}$$

*Research supported by NSERC
†Research supported by NSERC

where the t_g are positive constants which can be calculated recursively using a complicated recursion involving, in addition to g, many other parameters. The first three values are

$$t_0 = \frac{2}{\sqrt{\pi}}, \quad t_1 = \frac{1}{24} \quad \text{and} \quad t_2 = \frac{7}{4320\sqrt{\pi}}.$$

Gao [3] showed that many other interesting families of maps also satisfy asymptotic formulas of the form

$$\alpha t_g (\beta n)^{5(g-1)/2} \gamma^n \tag{2}$$

and presented a table of α, β and γ for eleven families. Richmond and Wormald [5] showed that many families of unrooted maps have asymptotics that differ from the rooted asymptotics by a factor of four times the number of edges. See Goulden and Jackson [4] for a discussion of connections with mathematical physics.

Although α, β and γ in (2) seem relatively easy to compute, the common factor t_g has been difficult to study. A recursion for rooted "cubic" maps derived by Goulden and Jackson [4] leads to a much simpler recursion for t_g than that in [1]. We will use it to derive the following recursion and asymptotic estimate for t_g.

Theorem 1 *Define* u_g *by* $u_1 = 1/10$ *and*

$$u_g \;=\; u_{g-1} + \sum_{h=1}^{g-1} \frac{1}{R_1(g,h) R_2(g,h)} u_h u_{g-h} \quad \textit{for} \quad g \geq 2, \tag{3}$$

where

$$R_1(g,h) \;=\; \frac{[1/5]_g}{[1/5]_h [1/5]_{g-h}}, \quad R_2(g,h) \;=\; \frac{[4/5]_{g-1}}{[4/5]_{h-1}[4/5]_{g-h-1}}$$

and $[x]_k$ *is the rising factorial* $x(x+1)\cdots(x+k-1)$. *Then*

$$\begin{aligned} t_g \;&=\; 8 \frac{[1/5]_g [4/5]_{g-1}}{\Gamma\left(\frac{5g-1}{2}\right)} \left(\frac{25}{96}\right)^g u_g \\ &\sim\; \frac{40 \sin(\pi/5) K}{\sqrt{2\pi}} \left(\frac{1440 g}{e}\right)^{-g/2} \quad \textit{as} \quad g \to \infty, \end{aligned} \tag{4}$$

where $u_g \sim K \doteq 0.1034$ *is a constant.*

2 Cubic Maps

A map is called cubic if all its vertices have degree 3. The dual of cubic maps are called triangular maps whose faces all have degree 3. Let $T_{n,g}$ be the number of triangular maps on Σ_g with n vertices and let $C_{n,g}$ be the number of cubic maps on Σ_g with $2n$ vertices. It was shown in [2] that

$$T_{n,g} \;\sim\; 3\left(3^7 \times 2^9\right)^{(g-1)/2} t_g n^{5(g-1)/2} (12\sqrt{3})^n \quad \text{as} \quad n \to \infty. \tag{5}$$

Since a triangular map on Σ_g with v vertices has exactly $2(v+2g-2)$ faces,

$$C_{n,g} \;=\; T_{n-2g+2,g} \;\sim\; 3\times 6^{(g-1)/2} t_g n^{5(g-1)/2} (12\sqrt{3})^n \quad \text{as} \quad n\to\infty. \tag{6}$$

Define

$$H_{n,g} \;=\; (3n+2)C_{n,g} \quad \text{for} \quad n\geq 1, \tag{7}$$

$$H_{-1,0}=1/2, \quad H_{0,0}=2 \quad \text{and} \quad H_{-1,g}=H_{0,g}=0 \;\text{ for }\; g\neq 0.$$

Goulden and Jackson [4] derived the following recursion for $(n,g)\neq(-1,0)$:

$$H_{n,g} \;=\; \frac{4(3n+2)}{n+1}\left(n(3n-2)H_{n-2,g-1} + \sum_{i=-1}^{n-1}\sum_{h=0}^{g} H_{i,h}H_{n-2-i,g-h}\right). \tag{8}$$

This is significantly simpler than the recursion derived in [2]. We will use it to derive information about t_g.

3 Generating Functions

Define the generating functions

$$T_g(x)=\sum_{n\geq 0} T_{n,g}x^n, \quad C_g(x)=\sum_{n\geq 0} C_{n,g}x^n, \quad H_g(x)=\sum_{n\geq 0} H_{n,g}x^n \quad \text{and} \quad F_g(x)=x^2H_g(x).$$

It was shown in [2] that $T_g(x)$ is algebraic for each $g\geq 0$, and

$$T_0(x) \;=\; \frac{1}{2}t^3(1-t)(1-4t+2t^2) \quad \text{with} \quad x \;=\; \frac{1}{2}t(1-t)(1-2t), \tag{9}$$

where $t=t(x)$ is a power series in x with non-negative coefficients.

It follows from (6) and (7) that

$$C_g(x) \;=\; x^{2g-2}T_g(x) \quad \text{for} \quad g\geq 0, \tag{10}$$

$$F_g(x) \;=\; 3x^3C_g'(x)+2x^2C_g(x) \quad \text{for} \quad g\geq 1. \tag{11}$$

We also have

$$\begin{aligned} F_0(x) &= H_{0,0}x^2+\sum_{n\geq 1}(3n+2)C_{n,0}x^{n+2} \\ &= 2x^2+3x^3C_0'(x)+2x^2C_0(x) \\ &= 2x^2+3xT_0'(x)-4T_0(x) \\ &= \frac{1}{2}t^2(1-t), \end{aligned} \tag{12}$$

where we have used (9). Hence $C_g(x)$ and $F_g(x)$ are both algebraic for all $g\geq 0$.

In the following we assume $g \geq 1$. From the recursion (8), we have

$$\begin{aligned}\frac{1}{4}\sum_{n\geq 0}\frac{n+1}{3n+2}H_{n,g}x^n &= \sum_{n\geq 1} n(3n-2)H_{n-2,g-1}x^n \\ &\quad + 2\sum_{n\geq 0} H_{-1,0}H_{n-1,g}x^n + x^2\sum_{h=0}^{g} H_h(x)H_{g-h}(x).\end{aligned}$$

Using (7) with a bit manipulation, we can rewrite the above equation as

$$\begin{aligned}\frac{1}{4}\sum_{n\geq 0}(n+1)C_{n,g}x^n &= 3x^2F''_{g-1}(x) + xF'_{g-1}(x) + xH_{-1,g-1} \\ &\quad + x^{-1}F_g(x) + x^{-2}\sum_{h=0}^{g} F_h(x)F_{g-h}(x).\end{aligned}$$

With $\delta_{i,j}$ the Kronecker delta, this becomes

$$\begin{aligned}x^3C'_g(x) + x^2C_g(x) &= 12x^4F''_{g-1}(x) + 4x^3F'_{g-1}(x) + 2x^3\delta_{g,1} \\ &\quad + 4xF_g(x) + 8F_0(x)F_g(x) + 4\sum_{h=1}^{g-1} F_h(x)F_{g-h}(x).\end{aligned}$$

It follows from (11) that

$$\begin{aligned}\left(1-12x-24F_0(x)\right)F_g(x) &= 36x^4F''_{g-1}(x) + 12x^3F'_{g-1}(x) + 6x^3\delta_{g,1} \\ &\quad + 12\sum_{h=1}^{g-1} F_h(x)F_{g-h}(x) - x^2C_g(x). \qquad (13)\end{aligned}$$

Substituting (12) and (9) into (13), we obtain

$$\begin{aligned}F_g(x) &= \frac{1}{1-6t+6t^2}\Big(36x^4F''_{g-1}(x) + 12x^3F'_{g-1}(x) + 6x^3\delta_{g,1} \\ &\quad + 12\sum_{h=1}^{g-1} F_h(x)F_{g-h}(x) - x^2C_g(x)\Big). \qquad (14)\end{aligned}$$

We now show that this equation can be used to calculate $C_g(x)$ more easily than the method in [2]. For this purpose we set $s = 1-6t+6t^2$ and show inductively that $C_g(x)$ is a polynomial in s divided by s^a for some integer $a = a(g) > 0$. (It can be shown that $a = 5g-3$ is the smallest such a, but we do not do so.) The method for calculating $C_g(x)$ follows from the proof. Then we have

$$x^2 = \frac{1}{432}(s-1)^2(2s+1) \qquad \text{and} \qquad \frac{ds}{dx} = \frac{144x}{s(s-1)}. \qquad (15)$$

Thus

$$x\frac{d}{dx} = x\frac{ds}{dx}\frac{d}{ds} = \frac{(s-1)(2s+1)}{3s}\frac{d}{ds},$$

$$\frac{d^2}{dx^2} = \left(\frac{ds}{dx}\right)^2\frac{d^2}{ds^2} + \frac{d(ds/dx)}{dx}\frac{d}{ds} = \frac{48(2s+1)}{s^2}\frac{d^2}{ds^2} - \frac{48(s+1)}{s^3}\frac{d}{ds}.$$

From the above and (11)

$$F_g(x) + \frac{x^2 C_g}{1-6t+6t^2} \;=\; x^2\left(3x\frac{dC_g}{dx} + \frac{(2s+1)C_g}{s}\right) \;=\; \frac{x^2(2s+1)}{s}\,\frac{d((s-1)C_g)}{ds}.$$

With some algebra, (14) can be rewritten as

$$\begin{aligned}\frac{d((s-1)C_g)}{ds} \;&=\; \frac{4(s-1)^2(2s+1)}{s^2}\,\frac{d^2F_{g-1}}{ds^2} + \frac{4(s-1)}{s^3}\,\frac{dF_{g-1}}{ds}\\ &\quad + \frac{5184}{(s-1)^2(2s+1)^2}\sum_{h=1}^{g-1} F_h F_{g-h} \qquad \text{for } \; g \geq 2. \qquad (16)\end{aligned}$$

In what follows $P(s)$ stands for a polynomial in s and a a positive integer, both different at each occurrence. It was shown in [2] that

$$C_1(x) \;=\; T_1(x) \;=\; \frac{1-s}{12s^2}.$$

By (11), (15) and the induction hypothesis, the right hand side of (16) has the form $P(s)/s^a$. Integrating, $(s-1)C_g = P(s)/s^a + K\log s$. Since we know $C_g(x)$ is algebraic, so is $(s-1)C_g$ and hence $K=0$. Since $s=1$ corresponds to $x=0$, C_g is defined there. It follows that $P(s)$ in $(s-1)C_g = P(s)/s^a$ is divisible by $s-1$, completing the proof.

Using Maple, we obtained

$$\begin{aligned}C_2 \;&=\; \frac{1}{2^6\,3^4}\frac{(2s+1)(17s^2+60s+28)(1-s)^3}{s^7},\\ C_3 \;&=\; \frac{1}{2^9\,3^8}\frac{(5052s^4-747s^3-33960s^2-35620s-9800)(2s+1)^2(s-1)^5}{s^{12}},\\ C_4 \;&=\; \frac{1}{2^{14}\,3^{11}}\frac{P_4(s)(2s+1)^3(s-1)^7}{s^{17}},\\ C_5 \;&=\; \frac{1}{2^{17}\,3^{14}}\frac{P_5(s)(2s+1)^4(1-s)^9}{s^{22}},\end{aligned}$$

where

$$\begin{aligned}P_4(s) \;&=\; -12458544 - 63378560s - 103689240s^2 - 42864016s^3\\ &\quad + 31477893s^4 + 20750256s^5 + 417636s^6,\\ P_5(s) \;&=\; 7703740800 + 50294009360s + 117178660480s^2\\ &\quad + 100386081272s^3 - 16827627792s^4 - 67700509763s^5\\ &\quad - 21455389524s^6 + 4711813020s^7 + 1394857272s^8.\end{aligned}$$

4 Generating Function Asymptotics

Suppose $A(x)$ is an algebraic function and has the following asymptotic expansion around its dominant singularity $1/r$:

$$A(x) \;=\; \sum_{j=l}^{k} a_j(1-rx)^{j/2} + O\left((1-rx)^{(k+1)/2}\right),$$

where a_j are not all zero. Then we write

$$A(x) \approx \sum_{j=l}^{k} a_j (1-rx)^{j/2}.$$

The following lemma is proved in [2].

Lemma 1 *For $g \geq 0$, $T_g(x)$ is algebraic,*

$$T_0(x) \approx \frac{\sqrt{3}}{72} - \frac{5}{216} + \frac{1}{54\sqrt{6}}(1-12\sqrt{3}x)^{3/2},$$

$$T_g(x) \approx 3\left(3^7 \times 2^9\right)^{(g-1)/2} t_g \, \Gamma\left(\frac{5g-3}{2}\right)(1-12\sqrt{3}x)^{-(5g-3)/2} \quad \textit{for} \quad g \geq 1.$$

Let

$$f_g = 24^{-3/2} 6^{g/2} \, \Gamma\left(\frac{5g-1}{2}\right) t_g. \tag{17}$$

Using Lemma 1, (10) and (11), we obtain

$$C_g(x) \approx \frac{288}{(5g-3)} f_g (1-12\sqrt{3}x)^{-(5g-3)/2} \quad \text{for} \quad g \geq 1,$$

$$F_g(x) \approx f_g (1-12\sqrt{3}x)^{-(5g-1)/2} \quad \text{for} \quad g \geq 1.$$

As noted in [2], the function $t(x)$ of (9) has the following asymptotic expansion around its dominant singularity $x = \frac{1}{12\sqrt{3}}$:

$$t \approx \frac{3-\sqrt{3}}{6} - \frac{\sqrt{2}}{6}(1-12\sqrt{3}x)^{1/2}.$$

Using this and (12), we obtain

$$F_0(x) \approx \frac{3-\sqrt{3}}{72} + f_0(1-12\sqrt{3}x)^{1/2},$$

$$\frac{1}{1-6t+6t^2} \approx \frac{\sqrt{6}}{2}(1-12\sqrt{3}x)^{-1/2}.$$

Comparing the coefficients of $(1-12\sqrt{3}x)^{(5g-1)/2}$ on both sides of (14), we obtain

$$f_g = \frac{\sqrt{6}}{96}(5g-4)(5g-6) f_{g-1} + 6\sqrt{6} \sum_{h=1}^{g-1} f_h f_{g-h}. \tag{18}$$

Letting

$$u_g = f_g \left(\frac{25\sqrt{6}}{96}\right)^{-g} \frac{6\sqrt{6}}{[1/5]_g [4/5]_{g-1}}.$$

and using (17), the recursion (18) becomes (3).

5 Asymptotics of t_g

It follows immediately from (3) that $u_g \geq u_{g-1}$ for all $g \geq 2$. To show that u_g approaches a limit K as $g \to \infty$, it suffices to show that u_g is bounded above. The value of K is then calculated using (3).

We use induction to prove $u_g \leq 1$ for all $g \geq 1$. Since $u_1 = \frac{1}{10}$ and $u_2 = u_1 + \frac{1}{480}$, we can assume $g \geq 3$ for the induction step. From now on $g \geq 3$.

Note that

$$\begin{aligned}
R_1(g,1)R_2(g,1) &= 5(g-\tfrac{4}{5})(g-\tfrac{6}{5}) \;>\; 5(g-\tfrac{4}{5})(g-\tfrac{9}{5}) \\
R_1(g,2)R_2(g,2) &= \tfrac{25}{24}(g-\tfrac{6}{5})(g-\tfrac{11}{5})\Big(5(g-\tfrac{4}{5})(g-\tfrac{9}{5})\Big) \\
&> \tfrac{25}{24}(g-\tfrac{6}{5}+\tfrac{4}{5})(g-\tfrac{11}{5}-\tfrac{4}{5})\Big(5(g-\tfrac{4}{5})(g-\tfrac{9}{5})\Big) \\
&\geq 2(g-3)\Big(5(g-\tfrac{4}{5})(g-\tfrac{9}{5})\Big).
\end{aligned}$$

Note that $R_i(g,h) = R_i(g,g-h)$ and, for $h < g/2$, $\frac{R_i(g,h+1)}{R_i(g,h)} \geq 1$. Combining all these observations and the induction hypothesis with (3) we have

$$\begin{aligned}
u_g &= u_{g-1} + \sum_{h=1}^{g-1} \frac{u_h u_{g-h}}{R_1(g,h)R_2(g,h)} \\
&< u_{g-1} + \frac{2u_1 u_{g-1}}{5(g-\frac{4}{5})(g-\frac{9}{5})} + \sum_{h=2}^{g-2} \frac{1}{R_1(g,2)R_2(g,2)} \\
&< u_{g-1} + \frac{1/5}{5(g-\frac{4}{5})(g-\frac{9}{5})} + \frac{1/2}{5(g-\frac{4}{5})(g-\frac{9}{5})} \\
&< u_{g-1} + \frac{1}{5g-9} - \frac{1}{5g-4}.
\end{aligned}$$

Hence

$$u_g \;<\; u_2 + \sum_{k=3}^{g} \left(\frac{1}{5k-9} - \frac{1}{5k-4}\right) \;<\; u_2 + \frac{1}{5\times 3 - 9} \;<\; 1.$$

The asymptotic expression for t_g in (4) is obtained by using

$$[x]_k = \frac{\Gamma(x+k)}{\Gamma(x)}, \qquad \Gamma(1/5)\Gamma(4/5) = \frac{\pi}{\sin(\pi/5)},$$

and Stirling's formula

$$\Gamma(ag+b) \sim \sqrt{2\pi}(ag)^{b-1/2}\left(\frac{ag}{e}\right)^{ag} \quad \text{as} \quad g \to \infty,$$

for constants $a > 0$ and b.

6 Open Questions

We list some open questions.

- From (18), we can show that $f(z) = \sum_{g\geq 1} f_g z^g$ satisfies the following differential equation

$$f(z) = 72\sqrt{6}(f(z))^2 + \frac{\sqrt{6}}{96} z \left(25z^2 f''(z) + 25zf'(z) - f(z) - \frac{\sqrt{6}}{72}\right).$$

  The asymptotic expression of f_g implies that $f(z)$ cannot be algebraic. Can one show that $f(z)$ is not D-finite, that is, $f(z)$ does not satisfy a linear differential equation?

- There is a constant p_g that plays a role for maps on non-orientable like t_g plays for maps on orientable surfaces [3]. Is there a recursion for maps on non-orientable surfaces that can be used to derive a theorem akin to Theorem 1 for p_g?

- Find simple recursions akin to (8) for other classes of rooted maps that lead to simple recursive calculations of their generating functions as in (16).

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