Equitable matroids

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Submitted: Apr 3, 2006; Accepted: Apr 19, 2006; Published: Apr 24, 2006
Mathematics Subject Classification: 05B35, 05A05

Abstract

One way to choose a basis of a matroid at random is to choose an ordering of the ground set uniformly at random and then use the greedy algorithm to find a basis. We investigate the class of matroids having the property that this procedure yields a basis uniformly at random. We show how this class is related to some other naturally-defined families of matroids and consider how it behaves under well-known matroid operations.

1 Introduction

Counting the bases of a matroid is a well-studied problem with many applications. In some cases, counting the bases exactly is known to be a computationally intractable problem [?, ?]. Therefore a considerable amount of attention has been paid to producing approximations to the number of bases [?, ?]. Work by Jerrum, Valiant, and Vazirani [?] shows that this task is intimately connected to the problem of choosing a basis uniformly at random.

Because of this connection between counting bases and choosing them uniformly at random, a great deal of effort has been spent in the search for efficient ways to randomly select a basis of a matroid [?, ?]. Perhaps the most obvious way to choose a basis of a matroid at random is to use an implementation of the greedy algorithm. Let M be a matroid on the ground set E and let ρ be a linear order on E. Then ρ induces a natural lexicographical order on the subsets of E. The greedy algorithm finds the basis that is minimum in this lexicographical order.

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In general though, if ρ is chosen uniformly at random from all linear orderings of E, some bases of M have a greater chance than others of being the output of this algorithm. In this paper we investigate the class of matroids such that if ρ is chosen uniformly at random, then the output of the greedy algorithm is uniformly distributed over the bases of the matroid.

Let M be a matroid on a ground set E and suppose that ρ is a linear order on E. Let $B(\rho)$ be the minimum basis in the lexicographical order induced by ρ. Suppose that B is a basis of M. Define $p(B, M)$ to be the probability that $B(\rho) = B$, given that ρ is chosen uniformly at random from the set of linear orders of E.

Definition 1.1. Let M be a matroid. If $p(B, M) = p(B', M)$ for any two bases B and B' of M, then M is *equitable.*

Let $O(B, M)$ be the number of linear orders ρ of E such that $B = B(\rho)$. Clearly $p(B, M) = O(B, M)/n!$, where $n = |E|$, so we may equivalently say that M is equitable if $O(B, M) = O(B', M)$ for any two bases B and B' of M.

In this article we describe some properties of equitable matroids and consider their behaviour under some basic matroid operations. We note that a few previously-studied classes of matroids, such as perfect matroid designs and basis-transitive matroids, are contained in the set of equitable matroids. Like those two classes, the class of equitable matroids contains some well-known families of matroids as well as several sporadic members, a few of which we exhibit. We refer to Oxley [**?**] for the basic concepts of matroid theory. Terminology and notation will follow that source.

2 Basic properties of equitable matroids

If M is a uniform matroid, then $O(B, M) = r(M)!r(M^*)!$ for any basis B, so the next result follows.

Proposition 2.1. *All uniform matroids are equitable.*

A simple necessary condition for a matroid to be equitable follows almost immediately from the definition.

Proposition 2.2. *The number of bases of an equitable matroid on n elements divides $n!$.*

The necessary condition in Proposition 2.2 is not sufficient:

Example 1. Let M be the rank-2 matroid on the ground set $\{1, \ldots, 6\}$ where the only non-trivial parallel class of M is $\{3, 4, 5, 6\}$. Then the number of bases of M is 9, which divides 6!. However, if B_1 is the basis $\{1, 2\}$ and B_2 is the basis $\{1, 3\}$, then it is easy to confirm that $O(B_1, M) = 48$ while $O(B_2, M) = 84$.

Let M be a matroid on the ground set E and let B be a basis of M. If $e \in E - B$, then the unique circuit in $B \cup e$ is denoted by $C(e, B)$. Let the partial order $\tau(B, M)$ on E be defined so that $x \leq_{\tau(B, M)} y$ if and only if (i) $x = y$ or (ii) $x \in B$, $y \in E - B$, and $x \in C(y, B)$. A linear order ρ of E is a *linear extension* of $\tau(B, M)$ if $x \leq_{\tau(B, M)} y$ implies $x \leq_\rho y$. The next result is not difficult.

Proposition 2.3. *Let M be a matroid on E, and let B be a basis of M. Suppose ρ is a linear order of E. Then $B(\rho) = B$ if and only if ρ is a linear extension of $\tau(B, M)$.*

It is elementary to demonstrate that $x \leq_{\tau(B,\, M)} y$ if and only if $y \leq_{\tau(E-B,\, M^*)} x$. The next proposition follows immediately from this fact and from Proposition 2.3.

Proposition 2.4. *If M is equitable, then so is M^*.*

Suppose that (X_1, X_2) is a separation of a matroid M. Let $x_i = |X_i|$ and let ρ_i be a linear order of X_i for $i = 1, 2$. There are $\binom{x_1+x_2}{x_1}$ linear orders ρ of $E(M)$ such that, for all $i \in \{1, 2\}$ and all $x, y \in X_i$, $x \leq_{\rho_i} y$ implies $x \leq_\rho y$. This, combined with the fact that the basis B of M is minimum in the lexicographic order induced by ρ if and only if $B \cap X_i$ is a minimum basis of $M|X_i$ for all $i \in \{1, 2\}$, yields the fact that

$$O(B, M) = \binom{x_1 + x_2}{x_1} O(B \cap X_1, M|X_1) O(B \cap X_2, M|X_2).$$

The next result follows.

Proposition 2.5. *A matroid is equitable if and only if all of its connected components are.*

We have shown that the class of equitable matroids is closed under duality and direct sums. We shall see later that a minor of an equitable matroid need not be equitable.

3 Super-equitable matroids

In this section we describe a family of matroids that is properly contained in the class of equitable matroids and which contains many of the most familar equitable matroids. Let M be a matroid on the ground set E and let B be a basis of M. Let $G(B, M)$ be the bipartite graph which has E as its vertex set and $\{\{x, y\} \mid x \in B,\ y \in E - B,\ x \in C(y, B)\}$ as its edge set. Note that $G(B, M)$ is isomorphic to the graph underlying the Hasse diagram of the partial order $\tau(B, M)$.

Definition 3.1. A matroid M is *super-equitable* if $G(B, M)$ and $G(B', M)$ are isomorphic whenever B and B' are bases of M.

Clearly every uniform matroid is super-equitable. The next result is obvious.

Proposition 3.2. *If M is super-equitable, then so is M^*.*

We shall prove that every super-equitable matroid is equitable after some intermediary results. The first follows from [**?**, Lemma 10.2.8, Corollary 10.2.9].

Lemma 3.3. *The vertex sets of the connected components of $G(B, M)$ are the ground sets of connected components of M.*

The 'if' direction of the next result is very easy. To prove the converse we note that any isomorphism between $G(B, M)$ and $G(B', M)$, where B and B' are two bases of the matroid M, must take connected components of M to other connected components by Lemma 3.3. Now it is not difficult to show that if B and B' differ only in a single connected component M' of M, then there must be an isomorphism between $G(B \cap E(M'), M')$ and $G(B' \cap E(M'), M')$.

Proposition 3.4. *A matroid is super-equitable if and only if all of its connected components are.*

Proposition 3.5. *Every super-equitable matroid is equitable.*

Proof. By Propositions 2.5 and 3.4, it will suffice to prove the result for connected super-equitable matroids. Let us therefore suppose that M is a connected super-equitable matroid on the ground set E. Let B and B' be two bases of M. Since $G(B, M)$ and $G(B', M)$ are connected bipartite graphs, it follows that the isomorphism that takes $G(B, M)$ to $G(B', M)$ is either an isomorphism or an anti-isomorphism between the partial orders $\tau(B, M)$ and $\tau(B', M)$. In either case, the number of linear extensions of $\tau(B, M)$ must be the number of linear extensions of $\tau(B', M)$. Therefore M is equitable. □

The converse of Proposition 3.5 is not true: the class of super-equitable matroids is properly contained in the class of equitable matroids. However, examples of matroids that are equitable without being super-equitable are somewhat hard to find. One can check that the truncation of $U_{2,20} \oplus U_{1,8} \oplus U_{1,8}$ is equitable. It is easy to verify that it is not super-equitable.

4 Basis-transitive matroids and PMDs

In this section we discuss two important classes of super-equitable matroids.

Observe that if B and B' are bases of a matroid M, then any automorphism of M that takes B to B' is also an isomorphism between $G(B, M)$ and $G(B', M)$.

Definition 4.1. A matroid M is *basis-transitive* if for any two bases B and B' there exists an automorphism of M that takes B to B'.

The next result follows from our discussion above.

Proposition 4.2. *Every basis-transitive matroid is super-equitable.*

Basis-transitive matroids have been studied in, for example, [?] and [?]. It is easy to see that uniform matroids are basis-transitive, and it is well known that the projective and affine geometries are basis-transitive and so are their truncations.

We now describe another class of super-equitable matroids.

Definition 4.3. If M is a matroid and, for $0 \le i \le r(M)$, all rank-i flats of M have the same cardinality, then M is a *perfect matroid design (PMD)*.

A proof that every PMD is equitable can be found in [**?**, Proposition 3.2.2], although the terminology used is different.

Proposition 4.4. *Every PMD is super-equitable.*

Proof. Let M be a PMD on the set E. For $0 \leq i \leq r(M)$ let α_i be the size of the rank-i flats of M. If B is a basis of M and X is a subset of B, then let $f_B(X)$ be the set $\{e \in E - B \mid C(e,\, B) = X \cup e\}$.

We claim that for any integer $0 \leq i \leq r(M)$ there is an integer β_i, such that if B is any basis of M and X is any subset of B of size i, then $|f_B(X)| = \beta_i$. Clearly β_0 is equal to the number of loops of M. Suppose that the claim is true when $i < k$, where $k \geq 1$. Let X be a subset of size k of a basis B. Note that

$$\mathrm{cl}_M(X) = X \cup \bigcup_{X' \subsetneq X} f_B(X').$$

Since $f_B(Y)$ and $f_B(Y')$ are disjoint if $Y \neq Y'$, it follows from the inductive assumption that

$$\alpha_k = |\,\mathrm{cl}_M(X)| = |X| + |f_B(X)| + \sum_{j=0}^{k-1} \binom{k}{j} \beta_j.$$

Thus $\beta_k = \alpha_k - k - \sum_{j=0}^{k-1} \binom{k}{j} \beta_j$. The result follows easily. □

Perfect matroid designs have been investigated in [**?**] and [**?**]. The class of PMDs includes the uniform matroids, the projective and affine geometries and their truncations. Although this indicates that the class of basis-transitive matroids has a large intersection with the class of PMDs, neither class is contained in the other, as shown by the next two examples.

Example 2. Consider a projective plane that is not a projective geometry, that is, a plane in which Desargues' Theorem does not hold. Considered as a matroid, such a plane is a PMD. However, Li's characterisation of rank-3 basis-transitive matroids [**?**] shows that a non-desarguesian plane cannot be basis-transitive. Therefore not every PMD is basis-transitive.

Example 3. Let r, n, and t be integers such that $1 < r < n$ and $t \geq 2$. Let M be the truncation of the direct sum of t copies of $U_{r,n}$. It is easy to show that M is basis-transitive but not a PMD.

Nor is it true that every super-equitable matroid is either basis-transitive or a PMD, even if we restrict our attention to connected super-equitable matroids.

Example 4. Let M_8 be the rank-4 matroid on the ground set $A \cup B$, where A and B are disjoint sets of size five and three respectively, such that the only non-spanning circuits of M are B and any set of four elements from A. Thus M_8 is isomorphic to the truncation of the direct sum of $U_{3,5}$ and $U_{2,3}$.

There are two types of bases of M_8. The first contains three elements of A and one element of B, while the second contains two elements each from A and B. Clearly M_8 is neither basis-transitive nor a PMD. However, it is not difficult to see that if B_1 is a basis of the first type and B_2 a basis of the second type, then both $G(B_1, M_8)$ and $G(B_2, M_8)$ are isomorphic to the graph that is produced by deleting two adjacent edges from $K_{4,4}$. Thus M_8 is super-equitable.

The next results are easy.

Proposition 4.5. *If M is a basis-transitive matroid then so is M^*.*

Proposition 4.6. *A matroid is basis-transitive if and only if all its connected components are.*

In contrast, the dual of a PMD need not be a PMD [**?**, Section 12.6] and the class of PMDs is not closed under direct sums.

5 Basic Operations

We conclude by considering how the classes we have discussed behave under certain basic matroid operations.

The next example shows that the classes of basis-transitive matroids, PMDs, super-equitable matroids, and equitable matroids are not closed under taking minors.

Example 5. Consider the Fano plane, F_7. We have already noted that projective geometries belong to the intersection of basis-transitive matroids and PMDs, so F_7 is super-equitable, and hence equitable. However, by deleting two points from F_7 we obtain a matroid which is easily seen to be not basis-transitive, super-equitable, equitable, nor a PMD.

The truncation of a PMD is also a PMD, but the next example shows that the classes of basis-transitive matroids, super-equitable matroids, and equitable matroids are not closed under truncation.

Example 6. Let M_9 be the truncation of the direct sum of three copies of $U_{2,3}$. We noted in Example 3 that M_9 is basis-transitive, and hence super-equitable and equitable. The truncation of M_9 is a rank-4 matroid containing three disjoint non-trivial lines. It is easy to see that $T(M_9)$ contains two types of bases. The first type avoids one of the non-trivial lines of $T(M_9)$, while the second type has a non-empty intersection with each non-trivial line. It follows that $T(M_9)$ is not basis-transitive. Furthermore, if B_1 is a basis of the first type and B_2 a basis of the second type, then $G(B_1, M_9)$ has 16 edges and $G(B_2, M_9)$ has 18. Also, $O(B_1, M_9) = 3648$, while $O(B_2, M_9) = 3264$. Thus $T(M_9)$ is neither super-equitable nor equitable.