Excluded subposets in the Boolean lattice

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We are looking for the maximum number of subsets of an n-element set not containing 4 distinct subsets satisfying $A \subset B, C \subset B, C \subset D$. It is proved that this number is at least the number of the $\lfloor \frac{n}{2} \rfloor$-element sets times $1 + \frac{2}{n}$, on the other hand an upper bound is given with 4 replaced by the value 2.

Keywords: extremal problems, families of subsets

Let $[n] = \{1, 2, \ldots, n\}$ be a finite set, families $\mathcal{F}, \mathcal{G}$, etc. of its subsets will be investigated. $\binom{[n]}{k}$ denotes the family of all k-element subsets of $[n]$. Let P be a poset. The goal of the present investigations is to determine the maximum size of a family $\mathcal{F} \subset 2^{[n]}$ which does not contain P as a (non-necessarily induced) subposet. This maximum is denoted by $\mathrm{La}(n, P)$. In some cases two posets, say P_1, P_2 could be excluded. The maximum number of subsets is denoted by $\mathrm{La}(n, P_1, P_2)$ in this case.

The easiest example is the case when P consist of two comparable elements. Then we are actually looking for the largest family without inclusion that is without two distinct members $F, G \in \mathcal{F}$ such that $F \subset G$. The well-known Sperner theorem ([4]) gives the answer, the maximum is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

We say that the distinct sets $A, B_1 \ldots, B_r$ form an *r-fork* if they satisfy $A \subset B_1, \ldots, B_r$. A is called the *handle*, B_is are called the *prongs* of the fork. On the other hand, the distinct sets $A, B_1 \ldots, B_r$ form an r-*brush* if they satisfy $B_1, \ldots, B_r \subset A$. The r-forks and the r-brush are denoted by $F(r), B(r)$, respectively. An old theorem solves the problem when the 2-fork and the 2-brush are excluded.

Theorem 1 [3]

$$\mathrm{La}(n, F(2), B(2)) = 2\binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

The optimal construction is the family

$$\mathcal{F} = \left\{ F : \ F \in \binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor} \right\} \bigcup \left\{ F \cup \{n\} : \ F \in \binom{[n-1]}{\lfloor \frac{n-1}{2} \rfloor} \right\}.$$

We have proved the following theorem in a paper appearing soon.

Theorem 2 [2] *Let $n \geq 3$. If the family $\mathcal{F} \subseteq 2^{[n]}$ contains no four distinct sets A, B, C, D such that $A \subset C, A \subset D, B \subset C, B \subset D$, then $|\mathcal{F}|$ cannot exceed the sum of the two largest binomial coefficients of order n, i.e., $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$.*

[†]The work was supported by the Hungarian National Foundation for Scientific Research (OTKA), grant numbers T037846 and T034702.

Following the suggestion of J.R. Griggs, such a family could be called a *butterfly-free meadow*. The optimal construction here is obvious, one can take all the subsets of sizes $\lfloor n/2 \rfloor$ and $\lfloor n/2 \rfloor + 1$.

In all of these cases the maximum size of the family is exactly determined. This is not true when the r-fork is excluded. In a paper under preparation A. De Bonis and the present author proved the following theorem.

Theorem 3 [1]

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{r}{n}+O(\frac{1}{n^2})\right) \leq \mathrm{La}(F(r+1)) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+2\frac{r}{n}+O(\frac{\log n}{n^{3/2}})\right).$$

A weaker version of the upper bound in this theorem was obtained in [5]: the constant in the second term was larger. There is still a gap between the lower and upper bounds in the second term: a factor 2. This however seems to be a serious difficulty. The best construction (lower bound) contains all sets in one level and a thinned next level.

Let the poset N consist of 4 elements illustrated here with 4 distinct sets satisfying $A \subset B, C \subset B, C \subset D$. We were not able to determine $\mathrm{La}(n,N)$ for a long time. Recently, a new method jointly developed by J.R. Griggs, helped us to prove the following theorem.

Theorem 4

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{2}{n}+o(\frac{1}{n})\right) \leq \mathrm{La}(n,N) \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1+\frac{4}{n}+o(\frac{1}{n})\right).$$

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