Connected τ-critical hypergraphs of minimal size

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A hypergraph $\mathscr{H}$ is τ-critical if $\tau(\mathscr{H}-E)<\tau(\mathscr{H})$ for every edge $E\in\mathscr{H}$, where $\tau(\mathscr{H})$ denotes the transversal number of $\mathscr{H}$. It can be shown that a connected τ-critical hypergraph $\mathscr{H}$ has at least $2\tau(\mathscr{H})-1$ edges; this generalises a classical theorem of Gallai on χ-vertex-critical graphs with connected complements. In this paper we study connected τ-critical hypergraphs $\mathscr{H}$ with exactly $2\tau(\mathscr{H})-1$ edges. We prove that such hypergraphs have at least $2\tau(\mathscr{H})-1$ vertices, and characterise those with $2\tau(\mathscr{H})-1$ vertices using a directed odd ear decomposition of an associated digraph. Using Seymour's characterisation of χ-critical 3-chromatic square hypergraphs, we also show that a connected square hypergraph $\mathscr{H}$ with fewer than $2\tau(\mathscr{H})$ edges is τ-critical if and only if it is χ-critical 3-chromatic. Finally, we deduce some new results on χ-vertex-critical graphs with connected complements.

Keywords: τ-critical hypergraph, χ-critical 3-chromatic hypergraph

1 Introduction

A *hypergraph* $\mathscr{H}$ is a finite set of finite non-empty sets called the *edges* of $\mathscr{H}$. The *vertices* of $\mathscr{H}$ are the elements of the set $V(\mathscr{H})=\bigcup_{E\in\mathscr{H}}E$. A set $T\subseteq V(\mathscr{H})$ is a *transversal* (also *vertex cover* or *blocking set*) of $\mathscr{H}$ if $T\cap E\neq\emptyset$ for every $E\in\mathscr{H}$. The smallest cardinality of a transversal of $\mathscr{H}$ is the *transversal number* $\tau(\mathscr{H})$. A *k-colouring* of a hypergraph $\mathscr{H}$ is an assignment of at most k colours to $V(\mathscr{H})$ such that no edge is monochromatic. The chromatic number $\chi(\mathscr{H})$ is the smallest k such that $\mathscr{H}$ admits a k-colouring. A hypergraph $\mathscr{H}$ is *τ-critical* (resp. *χ-critical*) if $\tau(\mathscr{H}-E)<\tau(\mathscr{H})$ (resp. $\chi(\mathscr{H}-E)<\chi(\mathscr{H})$) for every $E\in\mathscr{H}$.

A number of authors have studied τ-critical hypergraphs; see for example [1, 2, 3]. It is trivial to verify that a hypergraph is τ-critical if and only if all its components are τ-critical. So what can be said about *connected* τ-critical hypergraphs? In particular, it seems natural to ask what is the minimal possible number of edges in a connected τ-critical hypergraph.

We first present a sharp lower bound on the number of edges in a connected τ-critical hypergraph, and then investigate the cases where equality is attained. We exhibit a surprising connection with χ-critical 3-chromatic square hypergraphs studied by Seymour [7], and show how our results relate to the work of Gallai [4] on χ-vertex-critical graphs with connected complements.

2 Main results

The following two results were proved in [9]. (A hypergraph is a *star* if all its edges have a common vertex.)

Theorem 1 *If $\mathscr{H}$ is a connected τ-critical hypergraph, then for every $E \in \mathscr{H}$ the edges of $\mathscr{H} - E$ can be partitioned into $\tau(\mathscr{H}) - 1$ stars of size at least two.*

Corollary 2 *If $\mathscr{H}$ is a connected τ-critical hypergraph, then $|\mathscr{H}| \geqslant 2\tau(\mathscr{H}) - 1$.*

The bound in Corollary 2 is sharp, as can be seen by considering odd cycles. Hypergraphs attaining equality in Corollary 2 are called *minimal* connected τ-critical hypergraphs. We might hope that such hypergraphs would be of an analysable form. Indeed, since a partition into 2-stars of a hypergraph corresponds to a matching of its line graph, Theorem 1 implies the following useful result. (A graph G *factor-critical* if $G - x$ has a perfect matching, for every $x \in V(G)$.)

Corollary 3 *If $\mathscr{H}$ is a minimal connected τ-critical hypergraph, then $L(\mathscr{H})$ is factor-critical.*

Lovász [5] proved that every factor-critical graph has an *odd ear decomposition*: it can be built up from a single vertex by successively attaching the end vertices of odd paths. So by Corollary 3 the line graph of a minimal connected τ-critical hypergraph has an odd ear decomposition. This fact can be used to prove the following two results.

Theorem 4 *If $\mathscr{H}$ is a minimal connected τ-critical hypergraph, then $\mathscr{H}$ has a system of distinct representatives.*

Corollary 5 *If $\mathscr{H}$ is a minimal connected τ-critical hypergraph, then $|V(\mathscr{H})| \geqslant |\mathscr{H}|$.*

Again, considering odd cycles shows that the bound in Corollary 5 is sharp. A hypergraph with an equal number of edges and vertices is said to be *square*. As might be expected, the minimal connected τ-critical hypergraphs which are square have particularly nice properties. Indeed, they can be characterised in terms of an odd ear decomposition of an associated digraph.

With any digraph D we can associate the hypergraph $\mathscr{H}_D = \{\{x\} \cup N^+(x) \mid x \in V(D)\}$ where $N^+(x)$ denotes the set of outneighbours of x in D. Note that $\mathscr{H}_D$ is square and has a system of distinct representatives. Conversely, if $\mathscr{H}$ is a square hypergraph with a system of distinct representatives $f : \mathscr{H} \to V(\mathscr{H})$, then $\mathscr{H} = \mathscr{H}_D$, where D is the digraph with vertex set $V(\mathscr{H})$ and arc set $\{(x, y) \mid x \in V(\mathscr{H}), y \in f^{-1}(x) \setminus \{x\}\}$.

A *directed odd ear* with respect to a digraph D consists of a directed odd path such that the two end vertices are in $V(D)$ but no internal vertices belong to $V(D)$. A *directed odd ear decomposition* of a digraph D is a sequence $D_0, \ldots, D_p$ of digraphs such that D_0 is a single vertex, $D_p = D$, and for $i = 1, \ldots, p$, D_i is obtained from D_{i-1} by adding a directed odd ear joining two not necessarily distinct vertices of D_{i-1}.

Seymour [7] proved that a square hypergraph $\mathscr{H}$ is χ-critical 3-chromatic if and only if $\mathscr{H} = \mathscr{H}_D$, where D is a strongly connected digraph with no directed even circuits. The following result can be proved using Seymour's theorem, Corollary 3 and Theorem 4. The *absorption number* $\beta(D)$ of a digraph D is the minimal size of a set $S \subseteq V(D)$ such that every $x \in V(D) \setminus S$ has an outneighbour in S.

Theorem 6 *For any square hypergraph $\mathscr{H}$, the following conditions are equivalent:*

1. *$\mathscr{H}$ is minimal connected τ-critical;*
2. *$\mathscr{H}$ is χ-critical 3-chromatic and $|\mathscr{H}| < 2\tau(\mathscr{H})$;*
3. *$\mathscr{H} = \mathscr{H}_D$, where D has a directed odd ear decomposition, contains no directed even circuits and $|V(D)| < 2\beta(D)$.*

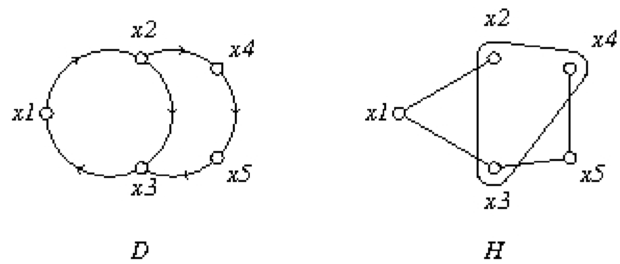

Fig. 1: $\mathscr{H} = \mathscr{H}_D$, where D has a directed odd ear decomposition and contains no directed even circuits; the associated square hypergraph $\mathscr{H}_D$ is minimal connected τ-critical by Theorem 6.

Let $\mathscr{H}^*$ denote the vertex-edge dual of $\mathscr{H}$. The following result can be proved using Corollary 5 and Theorem 6.

Corollary 7 *If $\mathscr{H}$ is a minimal connected τ-critical hypergraph, then so is $\mathscr{H}^*$ if and only if $\mathscr{H}$ is square.*

3 Application to χ-vertex-critical graphs

A hypergraph has the *Helly property* if all its intersecting partial hypergraphs are stars. There is a useful link between the chromatic number of graphs and the transversal number of Helly hypergraphs. Namely, given a graph G, let $\mathscr{A}(G)$ be the hypergraph formed with the maximal independent sets of G, and denote its dual by $\mathscr{A}^*(G)$. It is not difficult to check that $\mathscr{A}^*(G)$ has the Helly property and $\chi(G) = \tau(\mathscr{A}^*(G))$. A graph G is χ-*vertex-critical* if $\chi(G - x) < \chi(G)$, for every vertex $x \in V(G)$; note that a graph G is χ-vertex-critical if and only if $\mathscr{A}^*(G)$ is τ-critical. Hence the restriction to Helly hypergraphs of Corollary 2 is equivalent to the following classical result of Gallai [4], also proved in [6, 8].

Theorem 8 (Gallai 1963) *A χ-vertex-critical graph G with a connected complement has at least $2\chi(G)-1$ vertices.*

The restriction to Helly hypergraphs of Corollary 5 is equivalent to the following result.

Theorem 9 *A χ-vertex-critical graph G with a connected complement and $2\chi(G) - 1$ vertices has at least $2\chi(G) - 1$ maximal independent sets.*

Finally, Theorem 6 implies the following.

Theorem 10 *If G is a graph with a connected complement, $2\chi(G) - 1$ vertices and $2\chi(G) - 1$ maximal independent sets, then G is χ-vertex-critical if and only if $\mathscr{A}^*(G)$ is a χ-critical 3-chromatic Helly hypergraph.*

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