

A note on t -designs with t intersection numbers

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We use the Ray-Chaudhuri and Wilson inequality for a 0-design with t intersection numbers to prove that ‘For a fixed block size k , there exist finitely many parametrically feasible t -designs with t intersection numbers and $\lambda > 1$ ’.

Keywords: t -designs

1 Introduction

Let X be a finite set of v elements, called *points*, and let β be a finite family of distinct k -subsets of X , called *blocks*. Then the pair $D = (X, \beta)$ is called a t -design with parameters (v, k, λ) if any t -subset of X is contained in exactly λ members of β . If λ_i denotes the number of blocks containing i points, $i = 0, 1, 2, \dots, t-1$, then λ_i is independent of the choice of the i points and $\lambda_i \binom{k-i}{t-i} = \lambda \binom{v-i}{t-i}$. In particular, $b = \lambda_0$ is the number of blocks, and $\lambda_1 = r$ is the number of blocks through any point of D . A 0-design is a pair (X, β) where β is a collection of k -subsets of X . For $0 \leq x < k$, x is called *intersection number of D* if there exists $B, B' \in \beta$ such that $|B \cap B'| = x$. A 2-design with two intersection numbers is said to be a *quasi-symmetric design*.

For a 0-design with t intersection numbers, Ray-Chaudhuri and Wilson proved that $b \leq \binom{v}{t}$ (see [1]). This result is used by Sane and Shrikhande [7] to prove that, for a fixed value of the block size k , there exist finitely many quasi-symmetric designs with $\lambda > 1$.

In the literature many finiteness results for quasi-symmetric designs and quasi-symmetric 3-designs are proved by using this result by Sane and Shrikhande (see [5], [6], [7], [8]).

Our main aim in this paper is to extend the result by Sane and Shrikhande to t -designs with t intersection numbers. More specifically, we obtain a relation between the parameters of a t -design with t intersection numbers, and we use it to show that, for a fixed value of the block size k , v takes finitely many values. Finally, we use the result by Ray-Chaudhuri and Wilson to complete the proof.

2 Main Results

Theorem 2.1 (Ray-Chaudhuri and Wilson) *Let D be a 0-design with t ($0 < t \leq k \leq v - t$) intersection numbers x_1, x_2, \dots, x_t ($0 \leq x_1 < x_2 < \dots < x_t < k$). Then $b \leq \binom{v}{t}$.*

Lemma 2.2 *Let D be a t -design with t intersection numbers x_1, x_2, \dots, x_t ($0 \leq x_1 < x_2 < \dots < x_t < k$). Then the following relation holds:*

$$\begin{vmatrix} x_2 - x_1 & \cdots & x_t - x_1 & x_1(b-1) - (r-1) \\ \binom{x_2}{2} - \binom{x_1}{2} & \cdots & \binom{x_t}{2} - \binom{x_1}{2} & \binom{x_1}{2}(b-1) - \binom{k}{2}(\lambda_2 - 1) \\ \vdots & \vdots & \vdots & \vdots \\ \binom{x_2}{t} - \binom{x_1}{t} & \cdots & \binom{x_t}{t} - \binom{x_1}{t} & \binom{x_1}{t}(b-1) - \binom{k}{t}(\lambda_t - 1) \end{vmatrix} = 0. \tag{1}$$

Proof: Let $a_i, 2 \leq i \leq t$, be the number of blocks intersecting B_0 in x_i points. Fix a block B_0 and count in two ways the number of $(j + 1)$ -tuples $(\{p_1, p_2, \dots, p_j\}, B)$, where B is a block of D other than B_0 , and where p_1, p_2, \dots, p_j are distinct points of D contained in $B \cap B_0$. For $j = 1, 2, \dots, t$, we have

$$\sum_{i=2}^t \binom{x_i}{j} a_i + \binom{x_1}{j} \left(b - 1 - \sum_{i=2}^t a_i \right) - \binom{k}{j} (\lambda_j - 1) = 0.$$

We rewrite these equations as follows:

$$\sum_{i=2}^t \left[\binom{x_i}{j} - \binom{x_1}{j} \right] a_i + \binom{x_1}{j} (b - 1) - \binom{k}{j} (\lambda_j - 1) = 0.$$

These are t equations in $t - 1$ unknowns a_2, a_3, \dots, a_t . Since $a_2, a_3, \dots, a_t, 1$ cannot be simultaneously zero, the coefficient matrix is singular. Hence the determinant given in equation (1) must be zero. \square

We would like to point out that equation (1) is an extension of equation (1) of [7]. The latter is found to be useful in the study of quasi-symmetric designs. An immediate application of (1) is given in the following theorem.

Theorem 2.3 *For a fixed block size k , there exist finitely many parametrically feasible t -designs with t intersection numbers x_1, x_2, \dots, x_t ($0 \leq x_1 < x_2 < \dots < x_t < k$) and $\lambda > 1$.*

Proof: By Theorem 2.1, the inequality $b \leq \binom{v}{t}$ holds for any t -design with t intersection numbers. Hence it suffices to show that for a fixed k, v takes finitely many values.

We divide the proof in two parts, by considering the cases $x_1 \neq 0$ and $x_1 = 0$ separately.

For $x_1 \neq 0$, we have the equality

$$\sum_{i=2}^t (x_i - x_1) a_i = k(r - 1) - x_1(b - 1).$$

This implies $k(r - 1) - x_1(b - 1) > 0$. Hence, $x_1 < \frac{k(r-1)}{b-1} < \frac{kr}{b} = \frac{k^2}{v}$. Now it is easy to see that $v < \frac{k^2}{x_1}$.

For $x_1 = 0$, by Lemma 2.2 we have

$$\begin{vmatrix} x_2 & \cdots & x_t & k(r-1) \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} & \binom{k}{2}(\lambda_2-1) \\ \vdots & \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} & \binom{k}{t-1}(\lambda_{t-1}-1) \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} & \binom{k}{t}(\lambda_t-1) \end{vmatrix} = 0.$$

Now we put $\lambda_j = \binom{v-j}{t-j}\lambda/\binom{k-j}{t-j}$, $j = 1, 2, \dots, t-1$, to get

$$A \begin{vmatrix} k \\ \binom{k}{2} \\ \vdots \\ \binom{k}{t-1} \\ \binom{k}{t} \end{vmatrix} = \lambda A \begin{vmatrix} k \binom{v-1}{t-1} / \binom{k-1}{t-1} \\ \binom{k}{2} \binom{v-2}{t-2} / \binom{k}{t-1} \\ \vdots \\ \binom{k}{t-1} \binom{v-t+1}{1} / \binom{k-t+1}{1} \\ \binom{k}{t} \end{vmatrix},$$

where

$$A = \begin{bmatrix} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} \end{bmatrix}.$$

We simplify the above equation as follows:

$$A \begin{vmatrix} k \\ \binom{k}{2} \\ \vdots \\ \binom{k}{t-1} \\ \binom{k}{t} \end{vmatrix} = \lambda A \begin{vmatrix} k \frac{(v-1)(v-2)\cdots(v-t+2)(v-t+1)}{(k-1)(k-2)\cdots(k-t+2)(k-t+1)} \\ \binom{k}{2} \frac{(v-2)(v-3)\cdots(v-t+2)(v-t+1)}{(k-2)(k-3)\cdots(k-t+2)(k-t+1)} \\ \vdots \\ \binom{k}{t-1} \frac{v-t+1}{k-t+1} \\ \binom{k}{t} \end{vmatrix}.$$

Now we multiply both sides by $(k-1)(k-2)\cdots(k-t+2)(k-t+1)$, to get

$$\begin{aligned} & \left| \begin{array}{c} k \\ \binom{k}{2} \\ \vdots \\ \binom{k}{t-1} \\ \binom{k}{t} \end{array} \right| (k-1)(k-2)\cdots(k-t+2)(k-t+1) \\ &= \lambda \left| \begin{array}{c} k(v-1)(v-2)\cdots(v-t+2)(v-t+1) \\ \binom{k}{2}(v-2)(v-3)\cdots(v-t+2)(v-t+1)(k-1) \\ \vdots \\ \binom{k}{t-1}(v-t+1)(k-1)(k-2)\cdots(k-t+2) \\ \binom{k}{t}(k-1)(k-2)\cdots(k-t+2)(k-t+1) \end{array} \right|. \end{aligned}$$

Note that λ divides the left hand side. Hence λ takes finitely many values only. It is clear from the above equation, by expanding the right hand side determinant with respect to the last column of the matrix, that $v-t+1$ divides

$$(k-1)(k-2)\cdots(k-t+1) \left(\left| \begin{array}{c} k \\ \binom{k}{2} \\ \vdots \\ \binom{k}{t-1} \\ \binom{k}{t} \end{array} \right| - \lambda \binom{k}{t} \left| \begin{array}{ccc} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} \end{array} \right| \right).$$

For a fixed k , this is a finite number. If it is non-zero, then v takes only finitely many values, otherwise $v-t+2$ divides

$$\binom{k}{t-1}(k-1)(k-2)\cdots(k-t+2) \left| \begin{array}{ccc} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-2} & \cdots & \binom{x_t}{t-2} \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} \end{array} \right|.$$

This can be simplified to

$$\binom{k}{t-1} (k-1)(k-2)\cdots(k-t+2) \frac{x_2 x_3 \cdots x_t}{2! 3! \cdots (t-2)! t!} \begin{vmatrix} 1 & \cdots & 1 \\ x_2 & \cdots & x_t \\ \vdots & \vdots & \vdots \\ x_2^{t-3} & \cdots & x_t^{t-3} \\ x_2^{t-1} & \cdots & x_t^{t-1} \end{vmatrix}.$$

It is easy to check that the above determinant is non-zero. This completes the proof. □

In [7], Sane and Shrikhande proved analogues of Theorem 2.3 for quasi-symmetric designs. As in our paper, they also divide their proof in two parts, depending on whether or not the smaller intersection number, x , is non-zero or not. The case $x_1 \neq 0$ of Theorem 2.3 is similar to that of $x \neq 0$ of [7]. But to prove their result for the case $x = 0$, they use a rather long argument, whereas our proof is elementary and short.

References

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