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# Standard Relations of Multiple Polylogarithm Values at Roots of Unity

Dedicated to Prof. Keqin Feng on his 70th birthday

Jianqiang Zhao

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Abstract. Let $N$ be a positive integer. In this paper we shall study the special values of multiple polylogarithms at $N$th roots of unity, called multiple polylogarithm values (MPVs) of level $N$. Our primary goal in this paper is to investigate the relations among the MPVs of the same weight and level by using the regularized double shuffle relations, regularized distribution relations, lifted versions of such relations from lower weights, and weight one relations which are produced by relations of weight one MPVs. We call relations from the above four families *standard*. Let $d(w,N)$ be the dimension of the $\mathbb{Q}$-vector space generated by all MPVs of weight $w$ and level $N$. Recently Deligne and Goncharov were able to obtain some lower bound of $d(w,N)$ using the motivic mechanism. We call a level $N$ *standard* if $N=1,2,3$ or $N=p^n$ for prime $p\geq 5$. Our computation suggests the following dichotomy: If $N$ is standard then the standard relations should produce all the linear relations and if further $N>3$ then the bound of $d(w,N)$ by Deligne and Goncharov can be improved; otherwise there should be non-standard relations among MPVs for all sufficiently large weights (depending only on $N$) and the bound by Deligne and Goncharov may be sharp. We write down some of the non-standard relations explicitly with good numerical verification. In two instances ($N=4$, $w=3,4$) we can rigorously prove these relations by using the octahedral symmetry of $\{0,\infty,\pm1,\pm\sqrt{-1}\}$. Throughout the paper we provide many conjectures which are strongly supported by computational evidence.

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Keywords and Phrases: Multiple polylogarithms, double shuffle relations, distribution relations, regularization, standard relations.

Contents



## 1 Introduction

In recent years, there is a revival of interest in multi-valued classical polylogarithms (polylogs) and their generalizations. For any positive integers $s_1,\dots,s_\ell$, multiple polylogs of complex variables are defined as follows (note that our index order is opposite to that of [19]):

$$Li_{s_1,\dots,s_\ell}(x_1,\dots,x_\ell)=\sum_{k_1>\dots>k_\ell>0}\frac{x_1^{k_1}\cdots x_\ell^{k_\ell}}{k_1^{s_1}\cdots k_\ell^{s_\ell}}, \tag{1}$$

where $|x_1\cdots x_j|<1$ for $j=1,\dots,\ell$. It can be analytically continued to a multi-valued meromorhpic function on $\mathbb{C}^\ell$ (see [29]). Conventionally $\ell$ is called the *depth* (or *length*) and $s_1 + \cdots + s_\ell$ the *weight*. When the depth $\ell = 1$ the function is nothing but the classical polylog. When the weight is also 1 one gets the MacLaurin series of $-\log(1-x)$. Moreover, setting $x_1 = \cdots = x_\ell = 1$ and $s_1 > 1$ one obtains the well-known multiple zeta values (MZVs). If one allows $x_j$'s to be $\pm 1$ then one gets the so-called alternating Euler sums.

### 1.1 Multiple polylog values at roots of unity

In this paper, the primary objects of study are the multiple polylog values at roots of unity (MPVs). These special values, MZVs and the alternating Euler sums in particular, have attracted a lot attention in recent years after they were found to be connected to many branches of mathematics and physics (see, for e.g., [7, 8, 10, 11, 15, 19, 28]). Results up to around year 2000 can be found in the comprehensive survey paper [6].
Starting from early 1990s Hoffman [21, 22] has constructed some quasi-shuffle (called *stuffle* in [6]) algebras reflecting the essential combinatorial properties of MZVs. Later he [23] extends this to incorporate MPVs although his definition of $*$-product is different from ours. This approach was then improved in [24] and [26] to study MZVs and MPVs in general, respectively, where the regularized double shuffle relations play prominent roles. One derives these relations by comparing (1) with another expression of the multiple polylogs given by the following iterated integral:

$$Li_{s_1,\dots,s_\ell}(x_1,\dots,x_\ell) = (-1)^\ell \int_0^1 \left(\frac{dt}{t}\right)^{\circ(s_1-1)} \circ \frac{dt}{t-a_1} \circ \cdots \circ \left(\frac{dt}{t}\right)^{\circ(s_\ell-1)} \circ \frac{dt}{t-a_\ell}, \quad (2)$$

where $a_i = 1/(x_1 \dots x_i)$ for $1 \le i \le \ell$. Here, one defines the iterated integrals recursively by $\int_a^b f(t) \circ w(t) = \int_a^b (\int_a^x w(t)) f(x)$ for any 1-form $w(t)$ and concatenation of 1-forms $f(t)$. One may think the path lies in $\mathbb{C}$; however, it is more revealing to use iterated integrals in $\mathbb{C}^\ell$ to find the analytic continuation of this function (see [29]).
The main feature of this paper is a quantitative comparison between the results obtained by Racinet [26] who considers MPVs from the motivic viewpoint of Drinfeld associators, and those by Deligne and Goncharov [17] who study the motivic fundamental groups of $\mathbb{P}^1 - (\{0,\infty\} \cup \boldsymbol{\mu}_N)$ by using the theory of mixed Tate motives over $S$-integers of number fields, where $\boldsymbol{\mu}_N$ is the group of $N$th roots of unity.
Fix an $N$th root of unity $\mu = \mu_N := \exp(2\pi\sqrt{-1}/N)$. An MPV of *level* $N$ is a number of the form

$$L_N(s_1,\dots,s_\ell | i_1,\dots,i_\ell) := Li_{s_1,\dots,s_\ell}(\mu^{i_1},\dots,\mu^{i_\ell}). \quad (3)$$

We will always identify $(i_1,\dots,i_\ell)$ with $(i_1,\dots,i_\ell) \pmod N$. It is easy to see from (1) that an MPV converges if and only if $(s_1,\mu^{i_1}) \ne (1,1)$. Clearly, all

MPVs of level $N$ are automatically of level $Nk$ for every positive integer $k$. For example when $i_1 = \cdots = i_\ell = 0$ or $N = 1$ one gets the MZV $\zeta(s_1, \dots, s_\ell)$. When $N = 2$ one recovers the alternating Euler sums studied in [8, 31]. To save space, if a string $S$ repeats $n$ times then $\{S\}^n$ will be used. For example, $L_N(\{2\}^2|\{0\}^2) = \zeta(2,2) = \pi^4/120$.

Standard conjectures in arithmetic geometry imply that $\mathbb{Q}$-linear relations among MVPs can only exist between those of the same weight. Let $\mathcal{MPV}(w, N)$ be the $\mathbb{Q}$-span of all the MPVs of weight $w$ and level $N$. Let $d(w, N)$ denoted its dimension. In general, it is very difficult to determine $d(w, N)$ because any nontrivial lower bound would provide some nontrivial irrational/transcendental result which is related to a variant of Grothendieck's period conjecture (see [16] or [17, 5.27(c)]). For example, one can show easily that $\mathcal{MPV}(2, 4) = \langle \log^2 2, \pi^2, \pi \log 2\sqrt{-1}, (K-1)\sqrt{-1}\rangle$, where $K = \sum_{n\geq 0}(-1)^n/(2n+1)^2$ is the Catalan's constant. From a variant of Grothendieck's period conjecture we know $d(2, 4) = 4$ (see [16]) but we don't have an unconditional proof yet. Namely, we cannot prove that the four number $\log^2 2, \pi^2, \pi \log 2\sqrt{-1}, (K-1)\sqrt{-1}$ are linearly independent over $\mathbb{Q}$. Thus, nontrivial lower bound of $d(w, N)$ is hard to come by.

On the other hand, one may obtain upper bound of $d(w, N)$ by finding as many linear relations in $\mathcal{MPV}(w, N)$ as possible. As in the cases of MZVs and the alternating Euler sums the double shuffle relations play important roles in revealing the relations among MPVs. In such a relation if all the MPVs involved are convergent it is called a *finite double shuffle relation* (FDS). In general one needs to use regularization to obtain *regularized double shuffle* relations (RDS) involving divergent MPVs. We shall recall this theory in §2 building on the results of [24, 26].

From the point of view of Lyndon words and quasi-symmetric functions Bigotte et al. [3, 4] have studied MPVs (they call them *colored MZVs*) primarily by using double shuffle relations and monodromy argument (cf. [4, Thm. 5.1]). However, when the level $N \geq 2$, these double shuffle relations often are not complete, as we shall see in this paper (for level two, see also [5]).

### 1.2 Standard relations of MPVs

If the level $N > 3$ then there are many non-trivial linear relations in $\mathcal{MPV}(1, N)$ of weight one whose structure is clear to us. Multiplied by MPVs of weight $w-1$ these relations can produce non-trivial linear relations among MPVs of weight $w$ which are called the *weight one relations.* Similar to these relations one may produce new relations by multiplying MPVs on all of the other types of relations among MPVs of lower weights. We call such relations *lifted relations.*

It is well-known that among MPVs there are the so-called *finite distribution relations* (FDT), see (14). Racinet [26] further considers the regularization of these relations by regarding MPVs as the coefficients of some group-like element in a suitably defined pro-Lie-algebra of motivic origin (see §4). Our computation shows that the *regularized distribution relations* (RDT) do contribute to new relations not covered by RDS and FDT. But they are not enough yet to produce all the lifted RDS.

Definition 1.1. We call a $\mathbb{Q}$-linear relation between MPVs *standard*[1] if it can be produced by some $\mathbb{Q}$-linear combinations of the following four families of relations: regularized double shuffle relations (RDS), regularized distribution relations (RDT), weight one relations, and lifted relations from the above. Otherwise, it is called a *non-standard* relation.

It is commonly believed that all linear relations among MPVs (i.e. levels one MPVs) are consequences of RDS. When level $N = 2$ we believe that all linear relations among the alternating Euler sums are consequences of RDS and RDT. Further, in this case, the RDT should correspond to the doubling and generalized doubling relations of [5].

### 1.3 Main results

The main goal of this paper is to provide some extensive numerical evidence concerning the (in)completeness of the standard relations. Namely, these relations in general are not enough to produce all the $\mathbb{Q}$-linear relation between MPVs (see Remark 8.2 and Conjecture 8.5); however, we have the following result (see Thm. 8.6 and Thm. 8.3).

Theorem 1.2. *Let $p \geq 5$ be a prime. Then $d(2,p) \leq (5p+7)(p+1)/24$ and $d(2,p^2) < (p^2-p+2)^2/4$. If a variant of Grothendieck's period conjecture* [17, 5.27(c)] *is true then the equality holds for $d(2,p)$ and the standard relations in $\mathcal{MPV}(2,p)$ imply all the others.*

If weight $w = 2$ and $N = 5^2, 7^2, 11^2, 13^2$ or $5^3$, then our computation (see Table 1) shows that the standard relations are very likely to be complete. However, if $N > 3$ is a 2-power or 3-power or has at least two distinct prime factors then the standard relations are often incomplete. Moreover, we don't know how to obtain the non-standard relations rigorously except that when the level $N = 4$ we get (see Thm. 9.1)

Theorem 1.3. *If the conjecture in* [17, 5.27(c)] *is true then all the linear relations among MPVs of level four and weight three (resp. weight four) are the consequences of the standard relations and the octahedral relation* (53) *(resp. the five octahedral relations* (54)-(58)*).*

Most of the MPV identities in this paper are discovered with the help of MAPLE using symbolic computations. We have verified all relations numerically by GiNaC [27] with an error bound $< 10^{-90}$. Some results contained in this paper were announced in [30].

[1] This term was suggested by P. Deligne in a letter to Goncharov and Racinet dated Feb. 25, 2008.

### 1.4 Acknowledgment

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## 2 The double shuffle relations and the algebra $\mathfrak{A}$

In this section we recall the procedure to transform the shuffle relations among MPVs into some pure algebra structures. This is a rather straight-forward variation of a theme first studied by Hoffman for MZVs (see, for e.g., [22, 23]) and then further developed by Ihara et al. in [24] and by Racinet in [26]. Most of the results in this section are well-known but we include them for the convenience of the reader.
It is Kontsevich [25] who first noticed that MZVs can be represented by iterated integrals. One can easily extend this to MPVs [26]. Set

$$a = \frac{dt}{t}, \qquad b_i = \frac{\mu^i dt}{1-\mu^i t} \quad \text{for } i = 0, 1, \dots, N-1.$$

For every positive integer $n$ define the word of length $n$

$$y_{n,i} := a^{n-1} b_i.$$

Then it is straight-forward to verify using (2) that if $(s_1, \mu^{i_1}) \neq (1, 1)$ then (cf. [26, (2.5)])

$$L_N(s_1, \dots, s_n | i_1, i_2, \dots, i_n) = \int_0^1 y_{s_1,i_1} y_{s_2,i_1+i_2} \cdots y_{s_n,i_1+i_2+\cdots+i_n}. \qquad (4)$$

One can now define an algebra of words as follows:

Definition 2.1. Set $A_0 = \{\mathbf{1}\}$ to be the set of the empty word. Define $\mathfrak{A} = \mathbb{Q}\langle A \rangle$ to be the graded noncommutative polynomial $\mathbb{Q}$-algebra generated by letters $a$ and $b_i$ for $i \equiv 0, \dots, N-1 \pmod N$, where $A$ is a locally finite set of generators whose degree $n$ part $A_n$ consists of words (i.e., a monomial in the letters) of depth $n$. Let $\mathfrak{A}^0$ be the subalgebra of $\mathfrak{A}$ generated by words not beginning with $b_0$ and not ending with $a$. The words in $\mathfrak{A}^0$ are called *admissible words.*

Observe that every MPV can be expressed as an iterated integral over the closed interval $[0,1]$ of an admissible word $w$ in $\mathfrak{A}^0$. This is denoted by

$$Z(w) := \int_0^1 w. \tag{5}$$

We remark that the length $\lg(w)$ of $w$ is equal to the weight of $Z(w)$. Therefore in general one has (cf. [26, (2.5) and (2.6)])

$$L_N(s_1,\dots,s_n|i_1,i_2,\dots,i_n) = Z(y_{s_1,i_1}y_{s_2,i_1+i_2}\cdots y_{s_n,i_1+i_2+\dots+i_n}), \tag{6}$$

$$Z(y_{s_1,i_1}y_{s_2,i_2}\cdots y_{s_n,i_n}) = L_N(s_1,\dots,s_n|i_1,i_2-i_1,\dots,i_n-i_{n-1}). \tag{7}$$

For example $L_3(1,2,2|1,0,2) = Z(y_{1,1}y_{2,1}y_{2,0})$. On the other hand, during 1960s Chen developed a theory of iterated integral which can be applied in our situation.

Lemma 2.2. ([12, (1.5.1)]) *Let $\omega_i$ $(i\ge 1)$ be $\mathbb{C}$-valued 1-forms on a manifold $M$. For every path $p$,*

$$\int_p \omega_1\cdots\omega_r \int_p \omega_{r+1}\cdots\omega_{r+s} = \int_p (\omega_1\cdots\omega_r)ш(\omega_{r+1}\cdots\omega_{r+s})$$

*where ш is the shuffle product defined by*

$$(\omega_1\cdots\omega_r)ш(\omega_{r+1}\cdots\omega_{r+s}) := \sum_{\substack{\sigma\in S_{r+s},\sigma^{-1}(1)<\dots<\sigma^{-1}(r)\\ \sigma^{-1}(r+1)<\dots<\sigma^{-1}(r+s)}} \omega_{\sigma(1)}\cdots\omega_{\sigma(r+s)}.$$

For example, one has

$$\begin{aligned}
&L_N(1|1)L_N(2,3|1,2) = Z(y_{1,1})Z(y_{2,1}y_{3,3}) = Z(b_1ш(ab_1a^2b_3))\\
=&Z(b_1ab_1a^2b_3 + 2ab_1^2a^2b_3 + (ab_1)^2ab_3 + ab_1a^2b_1b_3 + ab_1a^2b_3b_1)\\
=&Z(y_{1,1}y_{2,1}y_{3,3} + 2y_{2,1}y_{1,1}y_{3,3} + y_{2,1}^2y_{2,3} + y_{2,1}y_{3,1}y_{1,3} + y_{2,1}y_{3,3}y_{1,1})\\
=&L_N(1,2,3|1,0,2) + 2L_N(2,1,3|1,0,2) + L_N(2,2,2|1,0,2)\\
&\qquad + L_N(2,3,1|1,0,2) + L_N(2,3,1|1,2,N-2).
\end{aligned}$$

Let $\mathfrak{A}_ш$ be the algebra of $\mathfrak{A}$ together with the multiplication defined by shuffle product ш. Denote the subalgebra $\mathfrak{A}^0$ by $\mathfrak{A}^0_ш$ when one considers the shuffle product. Then one can easily prove

Proposition 2.3. *The map $Z : \mathfrak{A}^0_ш \longrightarrow \mathbb{C}$ is an algebra homomorphism.*

On the other hand, MPVs are known to satisfy the series stuffle relations. For example

$$L_N(2|5)L_N(3|4) = L_N(2,3|5,4) + L_N(3,2|4,5) + L_N(5|9).$$

To study such relations in general one has the following definition.

Definition 2.4. Denote by $\mathfrak{A}^1$ the subalgebra of $\mathfrak{A}$ which is generated by words $y_{s,i}$ with $s \in \mathbb{N}$ and $i \equiv 0, \dots, N-1 \pmod N$. Equivalently, $\mathfrak{A}^1$ is the subalgebra of $\mathfrak{A}$ generated by words not ending with $a$. For any word $w = y_{s_1,i_1} y_{s_2,i_2} \cdots y_{s_n,i_n} \in \mathfrak{A}^1$ and positive integer $j$ one defines the exponent shifting operator $\tau_j$ by

$$\tau_j(w) = y_{s_1,j+i_1} y_{s_2,j+i_2} \cdots y_{s_n,j+i_n}.$$

For convenience, on the empty word we adopt the convention that $\tau_j(\mathbf{1}) = \mathbf{1}$. We then define another multiplication $*$ on $\mathfrak{A}^1$ by requiring that $*$ distribute over addition, that $\mathbf{1} * w = w * \mathbf{1} = w$ for any word $w$, and that, for any words $\omega_1, \omega_2$,

$$y_{s,j}\omega_1 * y_{t,k}\omega_2 = y_{s,j}\Big(\tau_j\big(\tau_{-j}(\omega_1) * y_{t,k}\omega_2\big)\Big) + y_{t,k}\Big(\tau_k\big(y_{s,j}\omega_1 * \tau_{-k}(\omega_2)\big)\Big) + y_{s+t,j+k}\Big(\tau_{j+k}\big(\tau_{-j}(\omega_1) * \tau_{-k}(\omega_2)\big)\Big). \quad (8)$$

This multiplication is called the *stuffle product* in [6].

If one denotes by $\mathfrak{A}^1_*$ the algebra $(\mathfrak{A}^1, *)$ then it is not hard to show that

Proposition 2.5. (cf. [22, Thm. 2.1]) *The polynomial algebra $\mathfrak{A}^1_*$ is a commutative graded $\mathbb{Q}$-algebra.*

Now one can define the subalgebra $\mathfrak{A}^0_*$ similar to $\mathfrak{A}^0_{ш}$ by replacing the shuffle product by the stuffle product. Then by induction on the lengths and using the series definition one can quickly check that for any $\omega_1, \omega_2 \in \mathfrak{A}^0_*$

$$Z(\omega_1)Z(\omega_2) = Z(\omega_1 * \omega_2).$$

This implies that

Proposition 2.6. *The map $Z : \mathfrak{A}^0_* \longrightarrow \mathbb{C}$ is an algebra homomorphism.*

Definition 2.7. Let $w$ be a positive integer such that $w \geq 2$. For nontrivial $\omega_1, \omega_2 \in \mathfrak{A}^0$ with $\lg(\omega_1) + \lg(\omega_2) = w$ one says that

$$Z(\omega_1 ш \omega_2 - \omega_1 * \omega_2) = 0 \quad (9)$$

is a *finite double shuffle* relation (FDS) of weight $w$.

It is known that even in level one these relations are not enough to provide all the relations among MZVs. However, it is believed that one can remedy this by considering *regularized double shuffle* relation (RDS) produced by the following mechanism. This is explained in detail in [24] when Ihara, Kaneko and Zagier consider MZVs where they call these *extended double shuffle* relations or EDS. It is also contained in [26] with a different formulation.

To produce RDS, first, combining Propositions 2.6 and 2.3 one can easily prove the following algebraic result (cf. [24, Prop. 1]):

Proposition 2.8. *One has two algebra homomorphisms:*

$$Z^* : (\mathfrak{A}^1_*, *) \longrightarrow \mathbb{C}[T], \quad \text{and} \quad Z^{ш} : (\mathfrak{A}^1_{ш}, ш) \longrightarrow \mathbb{C}[T]$$

*which are uniquely determined by the properties that they both extend the evaluation map $Z : \mathfrak{A}^0 \longrightarrow \mathbb{C}$ by sending $b_0 = y_{1,0}$ to $T$.*

Second, in order to establish the crucial relation between $Z^*$ and $Z^{ш}$ one can adopt the machinery in [24] as follows. For any $(\mathbf{s}|\mathbf{i}) = (s_1, \dots, s_n|i_1, \dots, i_n)$ where $i_j$'s are integers and $s_j$'s are positive integers, let the image of the corresponding words in $\mathfrak{A}^1$ under $Z^*$ and $Z^{ш}$ be denoted by $Z^*_{(\mathbf{s}|\mathbf{i})}(T)$ and $Z^{ш}_{(\mathbf{s}|\mathbf{i})}(T)$ respectively.

Theorem 2.9. (cf. [26, Cor. 2.24]) *Define a $\mathbb{C}$-linear map $\rho : \mathbb{C}[T] \to \mathbb{C}[T]$ by*

$$\rho(e^{Tu}) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right) e^{Tu}, \qquad |u| < 1.$$

*Then for any index set $(\mathbf{s}|\mathbf{i})$ one has*

$$Z^{ш}_{(\mathbf{s}|\mathbf{i})}(T) = \rho(Z^*_{(\mathbf{s}|\mathbf{i})}(T)). \tag{10}$$

Definition 2.10. Let $w$ be a positive integer such that $w \geq 2$. Let $(\mathbf{s}|\mathbf{i})$ be any index set with the weight of $\mathbf{s}$ equal to $w$. Then every weight $w$ MPV relation produced by (10) is called a *regularized double shuffle* relation (RDS) of weight $w$. This is obtained by formally setting $T = 0$ in (10).

Theorem 2.9 is a generalization of [24, Thm. 1] to the higher level MPV cases. The proof is essentially the same. The above steps can be easily transformed to computer codes which are used in our MAPLE programs. For example, one gets by stuffle product

$$\begin{aligned} TL_N(2|3) =& Z^*_{(1|0)}(T) Z^*_{(2|3)}(T) = Z^*(y_{1,0} * y_{2,3}) \\ =& Z^*_{(1,2|0,3)}(T) + Z^*_{(2,1|3,3)}(T) + Z^*_{(3|3)}(T), \end{aligned}$$

while using shuffle product one has

$$\begin{aligned} TL_N(2|3) =& Z^{ш}_{(1|0)}(T) Z^{ш}_{(2|3)}(T) = Z^{ш}(y_{1,0} ш y_{2,3}) = Z^{ш}(b_0 ш a b_3) \\ =& Z^{ш}_{(1,2|0,3)}(T) + Z^{ш}_{(2,1|0,3)}(T) + Z^{ш}_{(2,1|3,0)}(T). \end{aligned}$$

Hence one discovers the following RDS by comparing the above two expressions using Thm. 2.9:

$$L_N(2,1|3,0) + L_N(3|3) = L_N(2,1|3, N-3) + L_N(2,1|0,3).$$

## 3 Weight one relations

When $N \geq 4$ there exist linear relations among MPVs of weight one by a theorem of Bass [1]. These relations are important because by multiplying any MPV of weight $w-1$ by such a relation one can get a relation between MPVs of weight $w$ which is called a *weight one relation.* This is one of the key ideas in finding the formula in [17, 5.25] concerning $d(w,N)$.
Clearly, there are $N-1$ MPVs of weight 1 and level $N$:

$$L_N(1|j) = -\log(1-\mu^j), \qquad 0 < j < N,$$

where $\mu = \mu_N = \exp(2\pi\sqrt{-1}/N)$ as before. Here one can take $\mathbb{C} - (-\infty, 0]$ as the principle branch of the logarithm. Further, it follows from the motivic theory of classical polylogs developed by Beilinson and Deligne in [2] and the Borel's theorem (see [20, Thm. 2.1]) that the $\mathbb{Q}$-dimension of $\mathcal{MPV}(1,N)$ is

$$d(1,N) = \dim K_1(\mathbb{Z}[\mu_N][1/N]) \otimes \mathbb{Q} + 1 = \varphi(N)/2 + \nu(N),$$

where $\varphi$ is the Euler's totient function and $\nu(N)$ is the number of distinct prime factors of $N$. Hence there are many linear relations among $L_N(1|j)$. For instance, if $j < N/2$ then one has the symmetric relation

$$-\log(1-\mu^j) = -\log(1-\mu^{N-j}) - \log(-\mu^j) = -\log(1-\mu^{N-j}) + \frac{N-2j}{N}\pi\sqrt{-1}.$$

Thus for all $1 < j < N/2$

$$(N-2)(L_N(1|j) - L_N(1|N-j)) = (N-2j)(L_N(1|1) - L_N(1|N-1)). \quad (11)$$

Further, from [1, (B)] for any divisor $d$ of $N$ and $1 \leq a < N/d$ one has the distribution relation

$$\sum_{0\leq j<d} L_N(1|a+jN/d) = L_N(1|ad). \quad (12)$$

It follows from the main result of Bass [1] (corrected by Ennola [18]) that all the linear relations between $L_N(1|j)$ are consequences of (11) and (12). Hence the weight one relations have the following forms in words: for all $w \in \mathfrak{A}^0$

$$\begin{cases} (N-2)Z(y_{1,j} * w - y_{1,-j} * w) = (N-2j)Z(y_{1,1} * w - y_{1,-1} * w), \\ \displaystyle\sum_{0\leq j<d} Z(y_{1,a+jN/d} * w) = Z(y_{1,ad} * w). \end{cases} \quad (13)$$

## 4 Regularized distribution relations

It is well-known that multiple polylogs satisfy the following distribution formula (cf. [26, Prop. 2.25]):

$$Li_{s_1,\dots,s_n}(x_1,\dots,x_n) = d^{s_1+\dots+s_n-n} \sum_{y_j^d = x_j, 1\leq j\leq n} Li_{s_1,\dots,s_n}(y_1,\dots,y_n), \quad (14)$$

for all positive integer $d$. When $s_1 = 1$ one has to exclude the divergent case $x_1 = 1$. We call these *finite distribution relations* (FDT). Racinet further considers the regularized version of these relations, which we now recall briefly. Fix an embedding $\boldsymbol{\mu}_N \hookrightarrow \mathbb{C}$ and denote by $\Gamma$ its image. Define two sets of words

$$\mathbf{X} := \mathbf{X}_\Gamma = \{x_\sigma : \sigma \in \Gamma \cup \{0\}\}, \quad \text{and} \quad \mathbf{Y} := \mathbf{Y}_\Gamma = \{x_0^{n-1} x_\sigma : n \in \mathbb{N}, \sigma \in \Gamma\}.$$

Then one may consider the coproduct $\Delta$ of $\mathbb{Q}\langle\mathbf{X}\rangle$ defined by $\Delta x_\sigma = 1 \otimes x_\sigma + x_\sigma \otimes 1$ for all $\sigma \in \Gamma \cup \{0\}$. For every path $\gamma \in \mathbb{P}^1(\mathbb{C}) - (\{0,\infty\} \cup \Gamma)$ Racinet defines the group-like element $\mathcal{I}_\gamma \in \mathbb{C}\langle\langle\mathbf{X}\rangle\rangle$ by

$$\mathcal{I}_\gamma := \sum_{p\in\mathbb{N},\sigma_1,\dots,\sigma_p\in\Gamma\cup\{0\}} \mathcal{I}_\gamma(\sigma_1,\dots,\sigma_p) x_{\sigma_1}\cdots x_{\sigma_p},$$

where $\mathcal{I}_\gamma(\sigma_1,\dots,\sigma_p)$ is the iterated integral $\int_\gamma \omega(\sigma_1)\cdots\omega(\sigma_p)$ with

$$\omega(\sigma)(t) = \begin{cases} \sigma\, dt/(1-\sigma t), & \text{if } \sigma \neq 0; \\ dt/t, & \text{if } \sigma = 0. \end{cases}$$

(One has to correct the obvious typo in the displayed formula just before Prop. 2.8 in [26] by changing $a_j$ to $\alpha_j$.) This $I_\gamma$ is essentially the same element denoted by dch in [17]. Note that $\mathbb{Q}\langle\mathbf{Y}\rangle$ is the sub-algebra of $\mathbb{Q}\langle\mathbf{X}\rangle$ generated by words not ending with $x_0$. Let $\pi_\mathbf{Y} : \mathbb{Q}\langle\mathbf{X}\rangle \to \mathbb{Q}\langle\mathbf{Y}\rangle$ be the projection. As $x_0$ is a primitive element one quickly deduces that $(\mathbb{Q}\langle\mathbf{Y}\rangle, \Delta)$ has a graded co-algebra structure.

Let $\mathbb{Q}\langle\mathbf{X}\rangle_{\mathsf{cv}}$ be the sub-algebra of $\mathbb{Q}\langle\mathbf{X}\rangle$ not beginning with $x_1$ and not ending with $x_0$. Let $\pi_{\mathsf{cv}} : \mathbb{Q}\langle\mathbf{X}\rangle \to \mathbb{Q}\langle\mathbf{X}\rangle_{\mathsf{cv}}$ be the projection. Passing to the limit one gets:

Proposition 4.1. ([26, Prop.2.11]) *The series* $\mathcal{I}_{\mathsf{cv}} := \lim_{a\to 0^+, b\to 1^-} \pi_{\mathsf{cv}}(\mathcal{I}_{[a,b]})$ *is group-like in* $(\mathbb{C}\langle\langle\mathbf{X}\rangle\rangle_{\mathsf{cv}}, \Delta)$.

*Remark* 4.2. The algebras $\mathfrak{A}$, $\mathfrak{A}^0$ and $\mathfrak{A}^1$ in §2 are essentially equal to $\mathbb{Q}\langle\mathbf{X}\rangle$, $\mathbb{Q}\langle\mathbf{X}\rangle_{\mathsf{cv}}$ and $\mathbb{Q}\langle\mathbf{Y}\rangle$, respectively, after setting $a = x_0$ and $b_j = x_{\mu^j}$.

Let $\mathcal{I}$ be the unique group-like element in $(\mathbb{C}\langle\langle\mathbf{X}\rangle\rangle, \Delta)$ whose coefficients of $x_0$ and $x_1$ are 0 such that $\pi_{\mathsf{cv}}(\mathcal{I}) = \mathcal{I}_{\mathsf{cv}}$. In order to do the numerical computation one needs to determine explicitly the coefficients for $\mathcal{I}$. Put

$$\mathcal{I} = \sum_{p\in\mathbb{N},,\sigma_1,\dots,\sigma_p\in\Gamma\cup\{0\}} C(\sigma_1,\dots,\sigma_p) x_{\sigma_1}\cdots x_{\sigma_p}. \tag{15}$$

Proposition 4.3. *Let $p$, $m$ and $n$ be three non-negative integers. If $p > 0$ then* *we assume* $\sigma_1 \neq 1$ *and* $\sigma_p \neq 0$*. Set* $(\sigma_1, \ldots, \sigma_p, \{0\}^n) = (\sigma_1, \ldots, \sigma_q)$*. Then*

$$C(\{1\}^m, \sigma_1, \ldots, \sigma_p, \{0\}^n)$$
$$= \begin{cases} 0, & \text{if } mn = p = 0; \\ Z\big(\pi_{\mathsf{cv}}(x_{\sigma_1} \cdots x_{\sigma_p})\big), & \text{if } m = n = 0; \\ -\dfrac{1}{m} \displaystyle\sum_{i=1}^{q} C(\{1\}^{m-1}, \sigma_1, \ldots, \sigma_i, 1, \sigma_{i+1}, \ldots, \sigma_q), & \text{if } m > 0; \\ -\dfrac{1}{n} \displaystyle\sum_{i=1}^{p} C(\sigma_1, \ldots, \sigma_{i-1}, 0, \sigma_i, \ldots, \sigma_p, \{0\}^{n-1}), & \text{if } m = 0, n > 0. \end{cases} \tag{16}$$

*Here* $Z$ *is defined by* (5) *after using the identification given by Remark* 4.2.

*Remark* 4.4. This proposition provides the recursive relations one may use to compute all the coefficients of $\mathcal{I}$.

*Proof.* Since $\mathcal{I}$ is group-like one has

$$\Delta \mathcal{I} = \mathcal{I} \otimes \mathcal{I}. \tag{17}$$

The first case follows from this immediately since $C(0) = C(1) = 0$. The second case is essentially the definition (5) of $Z$. If $m > 0$ then one can compare the coefficient of $x_1 \otimes x_1^{m-1} x_{\sigma_1} \cdots x_{\sigma_q}$ of the two sides of (17) and find the relation (16). Finally, if $m = 0$ and $n > 0$ then one may similarly consider the coefficient of $x_{\sigma_1} \cdots x_{\sigma_p} x_0^{n-1} \otimes x_0$ in (17). This finishes the proof of the proposition. □

For any divisor $d$ of $N$ let $\Gamma^d = \{\sigma^d : \sigma \in \Gamma\}$, $i_d : \Gamma^d \hookrightarrow \Gamma$ the embedding, and $p^d : \Gamma \twoheadrightarrow \Gamma^d$ the $d$th power map. They induce two algebra homomorphisms:

$$p^d_* : \mathbb{Q}\langle \mathbf{X}_\Gamma \rangle \longrightarrow \mathbb{Q}\langle \mathbf{X}_{\Gamma^d} \rangle$$
$$x_\sigma \longmapsto \begin{cases} dx_0, & \text{if } \sigma = 0, \\ x_{\sigma^d}, & \text{if } \sigma \in \Gamma, \end{cases}$$

and

$$i^*_d : \mathbb{Q}\langle \mathbf{X}_\Gamma \rangle \longrightarrow \mathbb{Q}\langle \mathbf{X}_{\Gamma^d} \rangle$$
$$x_\sigma \longmapsto \begin{cases} x_0, & \text{if } \sigma = 0, \\ x_\sigma, & \text{if } \sigma \in \Gamma^d, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that both $i^*_d$ and $p^d_*$ are $\Delta$-coalgebra morphisms such that $i^*_d(\mathcal{I})$ and $p^d_*(\mathcal{I})$ have the same image under the map $\pi_{\mathsf{cv}}$. By the standard Lie-algebra mechanism one has

PROPOSITION 4.5. ([26, Prop.2.26]) *For every divisor $d$ of $N$*

$$p^d_*(\mathcal{I}) = \exp\left(\sum_{\sigma^d=1,\sigma\neq 1} Li_1(\sigma)x_1\right) i^*_d(\mathcal{I}). \tag{18}$$

Combined with Proposition 4.3 the above result provides the so-called *regularized distribution relations* (RDT) which of course include all the FDT of MPVs given by (14).
However, sometimes FDT are not independent of the other relations. In the next theorem one sees that when the weight $w = 2$ and level $N$ is a prime, all the distribution relations in (14), where $x_j = 1$ for all $j$, are consequences of RDS of MPVs of level $N$.

THEOREM 4.6. *For any prime $p$ write $L(i,j) = L_p(1,1|i,j)$ and $D(i) = L_p(2|i)$. Define for $1 \le i,j < p$:*

$$\text{FDT} := -D(0) + p\sum_{j=0}^{p-1} D(j), \qquad \text{RDS}(i) := D(i) + L(i,0) - L(i,-i),$$
$$\text{FDS}(i,j) := D(i+j) + L(i,j) + L(j,i) - L(i,j-i) - L(j,i-j).$$

*Then one has*

$$\text{FDT} = \sum_{1\le i<p} \text{FDS}(i,i) + 2\sum_{1\le j<i<p} \text{FDS}(i,j) + 2\sum_{i=1}^{p-1}\text{RDS}(i). \tag{19}$$

*Proof.* When $p = 2$ the second term on the right hand side of (19) is vacuous. Then it is easy to see that both sides of (19) are equal to $D(0) + 2D(1)$.
We now assume $p \ge 3$. Changing the order of summation yields that

$$\begin{aligned}
&2\sum_{1\le j<i<p} D(i+j) = \sum_{i=2}^{p-1}\sum_{j=1}^{i-1} D(i+j) + \sum_{j=1}^{p-2}\sum_{i=j+1}^{p-1} D(i+j)\\
=&\sum_{i=2}^{p-2}\sum_{i\neq j=1}^{p-1} D(i+j) + \sum_{j=1}^{p-2} D(j-1) + \sum_{i=2}^{p-1} D(i+1)\\
=&(p-3)\sum_{j=0}^{p-1} D(j) - \sum_{i=2}^{p-2} D(i) - \sum_{i=1}^{p-1} D(2i) + \sum_{j=1}^{p-2} D(j) + \sum_{j=2}^{p-1} D(j) + 2D(0)\\
=&(p-1)D(0) + (p-3)\sum_{j=1}^{p-1} D(j)
\end{aligned}$$

since $\sum_{j=0}^{p-1} D(i+j) = \sum_{j=0}^{p-1} D(j)$ for all $i$ and $\sum_{i=1}^{p-1} D(2i) = \sum_{i=1}^{p-1} D(i)$. This implies that the dilogarithms on the right hand side of (19) exactly add up to

FDT. Thus one only needs to show that all the double logarithms on the right hand side of (19) cancel out.

First observe that $L(i,0)$ in $\text{FDS}(i,i)$ and $\text{RDS}(i)$ cancel out each other. Now let us consider the lattice points $(i,j)$ of $\mathbb{Z}^2$ corresponding to $L(i,j)$. The points $(i,j)$ corresponding to $L(i,j)$ with positive signs fill in exactly the area inside the square $[1,p-1]\times[1,p-1]$ (including boundary): $L(i,i)$ in $\text{FDS}(i,i)$ provides the diagonal $y=x$, $\sum_{1\le j<i<p} L(i,j)$ (resp. $\sum_{1\le j<i<p} L(j,i)$) form the lower right (resp. upper left) triangular region.

For the negative terms of the double logs, $L(i,-i)$ in $\text{RDS}(i)$ provides the diagonal $x+y=p$, $\sum_{1\le j<i<p} L(i,j-i)=\sum_{i=2}^{p-1}\sum_{j=p+1-i}^{p-1} L(i,j)$ form the upper right triangular region. Similarly, by changing the order of summation $\sum_{1\le j<i<p} L(j,i-j)=\sum_{i=1}^{p-2}\sum_{j=i+1}^{p-1} L(i,j-i)=\sum_{i=1}^{p-2}\sum_{j=1}^{p-1-i} L(i,j)$ fills the lower left region. □

To conclude this section we remark that numerical evidence up to level $N=169$ supports the following

Conjecture 4.7. *In weight two, all RDT are consequences of the weight one relations, RDS and depth two FDT.*

## 5 Lifted relations from lower weights

Note that when $N=3$ there are no weight one relations nor (regularized) distribution relations. When we deal with MZVs (resp. alternating Euler sums) we expect that all the linear relations come from RDS (resp. RDS and RDT). Since there is no weight one relation when level $N\le 3$ it is natural to ask if RDS and RDT are enough when $N=3$. Surprisingly, the answer is no.

The first counterexample is in weight four, i.e., $(w,N)=(4,3)$. Easy computation shows that there are 144 MPVs in this case among which there are 239 nontrivial RDS of weight four which include 191 FDS of weight four (see (9) and (10)). Furthermore, it is easy to verify that all the seven RDT (including four FDT) can be derived from RDS. Using these relations we get 127 independent linear relations among the 144 MPVs. But we have $d(4,3)\le 16$ by [17, 5.25], so there must be at least one more linearly independent relation. Where else can we find it? The answer is the so-called *lifted relations.*.

We know that a product of two weight two MPVs is of weight four. So on each of the five RDS (including two FDS) of weight two in $\mathcal{MPV}(2,3)$ we can multiply any one of the nine MPVs of $(w,N)=(2,3)$ to get a relation in $\mathcal{MPV}(4,3)$. For instance, we have a FDS

$$Z(y_{1,1}*y_{1,1}-y_{1,1}ш y_{1,1})=L_3(2|2)+2L_3(1,1|1,1)-L_3(1,1|1,0)=0.$$

Multiplying by $L_3(1,1|1,1)=Z(y_{1,1}y_{1,2})$ we obtain a new relation which is linearly independent from RDS of weight four in $\mathcal{MPV}(4,3)$:

$$\begin{aligned}
&Z\big(y_{1,1}y_{1,2}ш(y_{2,0}+2y_{1,1}y_{1,2}-2y_{1,1}y_{1,0})\big)\\
=&L_3(1,1,2|1,1,0)+2L_3(1,2,1|1,1,0)+2L_3(2,1,1|1,1,0)\\
+&L_3(2,1,1|2,2,1)+4L_3(\{1\}^4|1,1,2,1)+8L_3(\{1\}^4|1,0,1,0)\\
-&6L_3(\{1\}^4|1,0,0,1)-4L_3(\{1\}^4|1,0,1,2)-2L_3(\{1\}^4|1,1,2,0)=0.
\end{aligned}$$

Such relations coming from the lower weights are called *lifted relations.* In this way, when $(w,N)=(4,3)$ we can produce 45 lifted RDS relations from weight two, 58 from weight three. We may also lift RDT and obtain nine and six relations from weight two and three, respectively. However, all the lifted relations together only produce one new linearly independent relation, as expected. Hence we find totally 128 linearly independent relations among the 144 MPVs with $(w,N)=(4,3)$. This implies that $d(4,3)\le 16$ which is the same bound obtained by [17, 5.25] and is proved to be exact under a variant of Grothendieck's period conjecture by Deligne [16].
For levels $N\ge 4$ one may lift not only RDS and RDT but also the weight one relations. But by a moment reflection one sees that the lifted weight one relations are still weight one relations by themselves so one doesn't really need to consider them after all.

Definition 5.1. We call a $\mathbb{Q}$-linear relation among MPVs *standard* if it can be produced by some $\mathbb{Q}$-linear combinations of the following four families of relations: regularized double shuffle relations, regularized distribution relations, weight one relations, and lifted relations from the above. Otherwise, it is called a *non-standard* relation.

In general, there are no inclusion relations among the four families of the standard relations.
Computation in small weight cases supports the following

Conjecture 5.2. *Suppose $N=3$ or 4. Every MPV of level $N$ is a linear combination of MPVs of the form $L(\{1\}^w|t_1,\dots,t_w)$ with $t_j\in\{1,2\}$. Consequently, the $\mathbb{Q}$-dimension of the MPVs of weight $w$ and level $N$ is given by $d(w,N)=2^w$ for all $w\ge 1$.*

*Remark* 5.3. The data in Table 2 in §7 shows that one cannot produce enough relations by using only the standard relations when $(w,N)=(3,4)$. In fact, even though one has $d(3,4)\le 8$ and $d(4,4)\le 16$ by [17, 5.25], one can only show that $d(3,4)\le 9$ and $d(4,4)\le 21$ by using only the standard relations. However, thanks to the octahedral symmetry of $\mathbb{P}^1-(\{0,\infty\}\cup\boldsymbol{\mu}_4)$ one can find (presumably all) the non-standard relations in these two cases (see Thm. 9.1).
*Remark* 5.4. Let $N=2,3,4$ or 8. Assuming a variant of Grothendieck's period conjecture, Deligne [16] constructed explicitly a set of basis for $\mathcal{MPV}(w,N)$. His results would also imply that $d(w,2)$ is given by the Fibonacci numbers, $d(w,3)=d(w,4)=2^w$, and $d(w,8)=3^w$ under Grothendieck's period conjecture.

## 6 Some conjectures of FDS and RDS

Fix a level $N$. Let $R$ be a commutative $\mathbb{Q}$-algebra with 1 and a homomorphism $Z_R : \mathfrak{A}^0 \longrightarrow R$ such that the *finite double shuffle* (FDS) property holds:

$$Z_R(\omega_1 ш \omega_2) = Z_R(\omega_1 * \omega_2) = Z_R(\omega_1)Z_R(\omega_2).$$

We then extend $Z_R$ to $Z_R^{ш}$ and $Z_R^*$ as before. Define an $R$-module automorphism $\rho_R$ of $R[T]$ by

$$\rho_R(e^{Tu}) = A_R(u)e^{Tu}$$

where

$$A_R(u) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} Z_R(a^{n-1}b_0)u^n\right) \in R[\![u]\!].$$

If a map $Z_R : \mathfrak{A}^0 \longrightarrow R$ satisfies the FDS and $(Z_R^{ш} - \rho_R \circ Z_R^*)(\omega) = 0$ for all $\omega \in \mathfrak{A}^1$ then we say that $Z_R$ has the *regularized double shuffle* (RDS) property. Let $R_{RDS}$ be the universal algebra (together with a map $Z_{RDS} : \mathfrak{A}^0 \longrightarrow R_{RDS}$) such that for every $\mathbb{Q}$-algebra $R$ and a map $Z_R : \mathfrak{A}^0 \longrightarrow R$ satisfying RDS there always exists a map $\varphi_R$ to make the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{A}^0 & \xrightarrow{Z_{RDS}} & R_{RDS} \\ & {\scriptstyle Z_R}\searrow & \downarrow{\scriptstyle \varphi_R} \\ & & R \end{array}$$

When $N = 2$ computation by Blümlein, Broadhurst and Vermaseren [5] shows that the finite distribution relations and the regularized distribution relations (18) contribute non-trivially when the weight $w = 8$ and $w = 11$, respectively. When $N = 3$ computation shows that the lifted relations contribute non-trivially when the weight $w = 4$ (see §5) and $w = 5$: we can only get $d(5, 3) \leq 33$ instead of the conjecturally correct dimension 32 without using the lifted relations. Note that in this case there are 612 FDS of weight five, 191 RDS of weight five, 8 FDT and 7 RDT.

One may use the fact that $Z_R$ is an algebra homomorphism to produce *lifted finite double shuffle* and *lifted regularized double shuffle* relations as follows: for all $\omega_1 \in \mathfrak{A}^1$, $\omega_0, \omega_0', \omega_0'' \in \mathfrak{A}^0$ with $\lg(\omega_1) + \lg(\omega_0) = \lg(\omega_0) + \lg(\omega_0') + \lg(\omega_0'') = w$

$$Z_R^{ш}(\omega_1 ш \omega_0) - \rho_R \circ Z_R^*(\omega_1)Z_R^{ш}(\omega_0) = 0, \quad Z_R\big((\omega_0 * \omega_0') * \omega_0'' - (\omega_0 ш \omega_0') * \omega_0''\big) = 0.$$

In general, one can define the universal objects $Z_{SR}$ and $R_{SR}$ corresponding to the standard relations similar to $Z_{RDS}$ and $R_{RDS}$ such that for every $\mathbb{Q}$-algebra $R$ and a map $Z_R : \mathfrak{A}^0 \longrightarrow R$ satisfying the standard relations there always exists a map $\varphi_R$ to make the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{A}^0 & \xrightarrow{Z_{SR}} & R_{SR} \\ & {\scriptstyle Z_R}\searrow & \downarrow{\scriptstyle \varphi_R} \\ & & R \end{array} \qquad (20)$$

Recall that one has the evaluation map $Z : \mathfrak{A}^0 \longrightarrow \mathbb{C}$ by Prop. 2.8 which extends (5).

Conjecture 6.1. *Let $(R, Z_R) = (\mathbb{R}, Z)$ if $N = 1$, and $(R, Z_R) = (\mathbb{C}, Z)$ if $N = 2, 3$ or $N = p^n$ with prime $p \geq 5$. If $N = 1$ (resp. $N = 2$) then the map $\varphi_{\mathbb{R}}$ is injective, namely, the algebra of MPVs of level one or two is isomorphic to $R_{RDS}$ (resp. $R_{SR}$). If $N = 3$ or $N = p^n$ $(p \geq 5)$ then the map $\varphi_{\mathbb{C}}$ is injective so the algebra of MPVs of level $N$ is isomorphic to $R_{SR}$.*

The above conjecture generalizes [24, Conjecture 1]. It means that all the linear relations among MPVs can be produced by RDS when $N = 1$ or 2, and by the standard ones when $N = 3$ or $p^n$ with prime $p \geq 5$. When $N = p \geq 5$, $p$ a prime, this is proved in Thm. 8.6 under the assumption of a variant of Grothendieck's period conjecture.
Computation in many cases such as those listed in Remark 8.2 and Conjecture 8.5 show that MPVs must satisfy some other relations apart from the standard ones when $N$ has at least two distinct prime factors, so a naive generalization of Conjecture 6.1 to all levels does not exist at present. However, when $N = 4$ one can show that octahedral symmetry of $\mathbb{P}^1 - (\{0, \infty\} \cup \boldsymbol{\mu}_4)$ provide all the non-standard relations under the standard assumption (see Thm. 9.1). But since we only have numerical evidence in weight three and weight four it may be a little premature to form a conjecture for level four.

## 7 The structure of MPVs and some examples

In this section we concentrate on RDS between MPVs of small weights. Most of the computations in this section are carried out by MAPLE. We have checked the consistency of these relations with many known ones and verified our results numerically using GiNac [27] and EZ-face [9].
By considering all the admissible words we see easily that the number of distinct MPVs of weight $w \geq 2$ and level $N$ is $N^2(N+1)^{w-2}$ and there are at most $N(N+1)^{w-2}$ RDS (but not FDS). If $w \geq 4$ then the number of FDS is given by

$$(N-1)N^2(N+1)^{w-3} + \Big(\Big[\frac{w}{2}\Big] - 1\Big)N^4(N+1)^{w-4} = \Big(N^2\Big[\frac{w}{2}\Big] - 1\Big)N^2(N+1)^{w-4}.$$

If $w = 2$ (resp. $w = 3$) then the number of FDS is $(N-1)^2$ (resp. $N^2(N-1)$). Therefore, it is not hard to see that the number of standard relations grow polynomially with the level $N$ but exponentially with the weight $w$.

### 7.1 Weight one

From §3 we know that all relations in weight one follow from (11) and (12), and no RDS exists. The relations in weight one are crucial for higher level cases because they provide the weight one relations considered in §3. Moreover, easy computation by (11) and (12) shows that there is a hidden integral structure, namely, in each level there exists a $\mathbb{Q}$-basis consisting of MPVs such that every other MPV is a $\mathbb{Z}$-linear combination of the basis elements. This fact is proved by Conrad [13, Theorem 4.6]. Similar results should hold for higher weight cases and we hope to return to this in a future publication [14].

### 7.2 Weight two

There are $N^2$ MPVs of weight two and level $N$:

$$L_N(1,1|i,j), \quad L_N(2|j), \qquad 1 \le i \le N-1, 0 \le j \le N-1.$$

For $1 \le i,j < N$ the FDS $Z^*(y_{1,i} * y_{1,j}) = Z^{\text{Ш}}(y_{1,i} ш y_{1,j})$ yields

$$L_N(2|i+j) + L_N(1,1|i,j) + L_N(1,1|j,i) = L_N(1,1|i,j-i) + L_N(1,1|j,i-j). \tag{21}$$

Now from RDS $\rho(Z^*(y_{1,0} * y_{1,i})) = Z^{\text{Ш}}(y_{1,0} ш y_{1,i})$ we get for $1 \le i < N$

$$L_N(1,1|i,0) + L_N(2|i) = L_N(1,1|i,-i). \tag{22}$$

The FDT in (14) yields: for every divisor $d$ of $N$, and $1 \le a,b < d' := N/d$

$$L_N(2|ad) = d \sum_{j=0}^{d-1} L_N(2|a+jd'), \tag{23}$$

$$L_N(1,1|ad,bd) = \sum_{j,k=0}^{d-1} L_N(1,1|a+jd',b+kd'). \tag{24}$$

To derive the RDT we can compare the coefficients of $x_1 x_{\mu^{ad}}$ in (18) and use Prop. 4.3 to get: for every divisor $d$ of $N$, and $1 \le a < d'$

$$\begin{aligned} L_N(1|ad) \sum_{j=1}^{d-1} L_N(1|jd') = & \sum_{j=1}^{d-1}\sum_{k=0}^{d-1} L_N(1,1|jd',a+kd') \\ & - \sum_{k=0}^{d-1} L_N(1,1|a+kd',-a-kd') - L_N(1,1|ad,-ad). \end{aligned} \tag{25}$$

By definition, the weight one relations are obtained from (11) and (12). For example, if $N=p$ is a prime then (12) is trivial and (11) is equivalent to the following: for all $1 \le j < h$ ($h := (p-1)/2$)

$$L_N(1|j) - L_N(1|-j) = (p-2j)(L_N(1|h) - L_N(1|h+1)). \tag{26}$$

Thus multiplying by $L_N(1|i)$ ($1 \le i < p$) and applying the shuffle relation $L_N(1|a)L_N(1|b) = L_N(1^2|a,b-a) + L_N(1^2|b,a-b)$ (here we put $L_N(1^2|-) = L_N(1,1|-)$ to save space) we get:

$$\begin{aligned} & L_N(1^2|i,j-i) + L_N(1^2|j,i-j) - L_N(1^2|i,-j-i) - L_N(1^2|-j,i+j) \\ = & (p-2j)\big(L_N(1^2|i,h-i) + L_N(1^2|h,i-h) - L_N(1^2|i,-i-h) - L_N(1^2|-h,i+h)\big). \end{aligned} \tag{27}$$

Computation shows that the following conjecture should hold.

Conjecture 7.1. *The RDT* (25) *follows from the combination of the following relations: the weight one relations, the RDS* (21) *and* (22)*, and the FDT* (23) *and* (24)*.*

## 7.3 Weight three

Apparently there are $N^2(N+1)$ MPVs of weight three and level $N$: for each choice $(i,j,k)$ with $1 \le i \le N-1, 0 \le j,k \le N-1$ we have four MPVs of level $N$:

$$L_N(1^3|i,j,k) := L_N(1,1,1|i,j,k), \quad L_N(1,2|i,j), \quad L_N(2,1|j,k), \quad L_N(3|k).$$

For $1 \le i,j,k < N$ the FDS $Z^*\big(y_{1,i} * (y_{1,j}y_{1,k})\big) = Z^{Ш}\big(y_{1,i}Ш(y_{1,j}y_{1,k})\big)$ yields

$$\begin{aligned} L_N(1^3|i,j-i,k) &+ L_N(1^3|j,i-j,k+j-i) + L_N(1^3|j,k,i-k-j) \\ &= L_N(2,1|i+j,k) + L_N(1,2|j,i+k) \\ &\qquad + L_N(1^3|i,j,k) + L_N(1^3|j,i,k) + L_N(1^3|j,k,i). \end{aligned} \tag{28}$$

For $1 \le i,j < N$ the FDS $Z^*(y_{1,i} * y_{2,j}) = Z^{Ш}(y_{1,i}Ш y_{2,j})$ yields

$$\begin{aligned} L_N(3|i+j) &+ L_N(1,2|i,j) + L_N(2,1|j,i) \\ &= L_N(1,2|i,j-i) + L_N(2,1|i,j-i) + L_N(2,1|j,i-j). \end{aligned} \tag{29}$$

Moreover, there are three ways to produce RDS. Since $\rho(T) = T$ the first family of RDS come from $Z^*\big(y_{1,0} * (y_{1,i}y_{1,i+j})\big) = Z^{Ш}\big(y_{1,0}Ш(y_{1,i}y_{1,i+j})\big)$ for $1 \le i \le N-1, 0 \le j \le N-1$:

$$\begin{aligned} y_{1,0} * (y_{1,i}y_{1,i+j}) &= y_{1,0}y_{1,i}y_{1,i+j} + y_{1,i}\tau_i(y_{1,0} * y_{1,j}) + y_{2,i}y_{1,i+j} \\ =& y_{1,0}y_{1,i}y_{1,i+j} + y_{1,i}y_{1,i}y_{1,i+j} + y_{1,i}y_{1,i+j}y_{1,i+j} + y_{1,i}y_{2,i+j} + y_{2,i}y_{1,i+j} \end{aligned}$$

On the other hand,

$$y_{1,0}Ш y_{1,i}y_{1,i+j} = y_{1,0}y_{1,i}y_{1,i+j} + y_{1,i}y_{1,0}y_{1,i+j} + y_{1,i}y_{1,i+j}y_{1,0}.$$

Hence

$$\begin{aligned} L_N(1^3|i,0,j) &+ L_N(1^3|i,j,0) + L_N(1,2|i,j) + L_N(2,1|i,j) \\ &= L_N(1^3|i,-i,i+j) + L_N(1^3|i,j,-i-j). \end{aligned} \tag{30}$$

The second family of RDS follow from $\rho(Z^*(y_{1,0} * y_{2,i})) = Z^{Ш}(y_{1,0}Ш y_{2,i})$:

$$y_{1,0}y_{2,i} + y_{2,i}y_{1,i} + y_{3,i} = y_{1,0}y_{2,i} + y_{2,0}y_{1,i} + y_{2,i}y_{1,0}$$

which implies that

$$L_N(2,1,i,0)+L_N(3,i)=L_N(2,1,i,-i)+L_N(2,1,0,i). \tag{31}$$

Now we consider the last family of RDS. By the definition of stuffle product:

$$\begin{aligned}y_{1,0}*y_{1,0}*y_{1,i}=&(2y_{1,0}^2+y_{2,0})*y_{1,i}\\ =&2y_{1,0}(y_{1,0}*y_{1,i})+2y_{1,i}^3+2y_{2,i}y_{1,i}+y_{2,0}*y_{1,i}\\ =&2y_{1,0}^2y_{1,i}+2y_{1,0}y_{1,i}^2+2y_{1,0}y_{2,i}+2y_{1,i}^3+2y_{2,i}y_{1,i}+y_{2,0}*y_{1,i}.\end{aligned}$$

Applying $\rho\circ Z^*$ and noticing that $Z^{ш}_{(2|0)}(T)=\zeta(2)$ we get

$$\begin{aligned}(T^2+\zeta(2))Z^{ш}_{(1|i)}(T)=2Z^{ш}_{(1^3|0,0,i)}(T)+2Z^{ш}_{(1^3|0,i,i)}(T)+2Z^{ш}_{(1,2|0,i)}(T)\\ +2Z^{ш}_{(1^3|i,i,i)}(T)+2Z^{ш}_{(2,1|i,i)}(T)+Z^{ш}_{(2|0)}(T)Z^{ш}_{(1|i)}(T).\end{aligned} \tag{32}$$

On the other hand by the definition of shuffle product

$$y_{1,0}шy_{1,0}шy_{1,i}=2y_{1,0}^2шy_{1,i}=2y_{1,0}^2y_{1,i}+2y_{1,0}y_{1,i}y_{1,0}+2y_{1,i}y_{1,0}^2$$

Applying $Z^{ш}$ we get

$$T^2Z^{ш}_{(1|i)}(T)=2Z^{ш}_{(1^3|0,0,i)}(T)+2Z^{ш}_{(1^3|0,i,0)}(T)+2Z^{ш}_{(1^3|i,0,0)}(T). \tag{33}$$

We further have

$$\begin{aligned}&Z^{ш}(y_{1,0}y_{1,i}^2+y_{1,0}y_{2,i}-y_{1,0}y_{1,i}y_{1,0})\\ =&Z^{ш}(1^3|0,i,i)(T)+Z^{ш}_{(1,2|0,i)}(T)-Z^{ш}_{(1^3|0,i,0)}(T)\\ =&2Z^{ш}_{(1^3|i,0,0)}(T)-Z^{ш}_{(2,1|i,0)}(T)-Z^{ш}_{(2,1|0,i)}(T)-Z^{ш}_{(1^3|i,0,i)}(T)-Z^{ш}_{(1^3|i,i,0)}(T)\end{aligned}$$

where we have used the facts that

$$\begin{aligned}Z^{ш}_{(1,2|0,i)}(T)=&TZ^{ш}_{(2|i)}(T)-Z^{ш}_{(2,1|i,0)}(T)-Z^{ш}_{(2,1|0,i)}(T)\\ Z^{ш}_{(1^3|0,i,i)}(T)=&TZ^{ш}_{(1,1|i,i)}(T)-Z^{ш}_{(1^3|i,0,i)}(T)-Z^{ш}_{(1^3|i,i,0)}(T)\\ Z^{ш}_{(1^3|0,i,0)}(T)=&TZ^{ш}_{(1,1|i,0)}-2Z^{ш}_{(1^3|i,0,0)}(T)\\ Z^{ш}_{(1,1|i,0)}(T)=&Z^{ш}_{(2|i)}(T)+Z^{ш}_{(1,1|i,i)}(T).\end{aligned}$$

Hence for $1\le i<N$ we have by subtracting (33) from (32)

$$\begin{aligned}L_N(1^3|i,0,0)+L_N(2,1|i,0)+L_N(1^3|i,-i,0)=\\ L_N(2,1|i,-i)+L_N(2,1|0,i)+L_N(1^3|i,-i,i)+L_N(1^3|i,0,-i).\end{aligned} \tag{34}$$

Setting $j=0$ in (30) and subtracting from (34) we get

$$L_N(1^3|i,-i,0)=L_N(2,1|i,-i)+L_N(2,1|0,i)+L_N(1^3|i,0,0)+L_N(1,2|i,0). \tag{35}$$

### 7.4 Upper bound of $d(w, N)$ by Deligne and Goncharov

By using the theory of motivic fundamental groups of $\mathbb{P}^1 - (\{0, \infty\} \cup \boldsymbol{\mu}_N)$ Deligne and Goncharov [17, 5.25] show that $d(w, N) \le D(w, N)$ where $D(w, N)$ are defined by the formal power series

$$1 + \sum_{w=1}^{\infty} D(w, N) t^w = \begin{cases} (1 - t^2 - t^3)^{-1}, & \text{if } N = 1; \\ (1 - t - t^2)^{-1}, & \text{if } N = 2; \\ (1 - at + bt^2)^{-1}, & \text{if } N \ge 3, \end{cases} \tag{36}$$

where $a = a(N) := \varphi(N)/2 + \nu(N)$, $b = b(N) := \nu(N) - 1$, $\varphi$ is the Euler's totient function and $\nu(N)$ is the number of distinct prime factors of $N$. If $N > 2$ then we have

$$\sum_{w=1}^{\infty} D(w, N) t^w = at + (a^2 - b)t^2 + (a^3 - 2ab)t^3 + (a^4 - 3a^2 b + b^2)t^4 + \cdots$$

In particular, if $p$ is a prime then for any positive integer $n$

$$D(w, p^n) = a(p^n)^w = \left( \frac{p^{n-1}(p-1)}{2} + 1 \right)^w . \tag{37}$$

We will compare the bound obtained by the standard relations to the bound $D(w, N)$ in the next two sections.

## 8 Computational results in weight two

In this section we combine the analysis in the previous sections and the theory developed by Deligne and Goncharov [17] to present a detailed computation in weight two and level $N \le 169$.
Let $\mathcal{G} := \iota(\operatorname{Lie} U_\omega)$ be the motivic fundamental Lie algebra (see [17, (5.12.2)]) associated to the motivic fundamental group of $\mathbb{P}^1 - (\{0, \infty\} \cup \boldsymbol{\mu}_N)$. As pointed out in §6.13 of op. cit. one may safely replace $\mathcal{G}(\boldsymbol{\mu}_N)^{(\ell)}$ by $\mathcal{G}$ throughout [20]. Then it follows from the proof of [17, 5.25] that if conjecture [17, 5.27(c)] is true, which we assume in the following, then

$$d(2, N) = D(2, N) - \dim \ker(\beta_N), \tag{38}$$

where $\beta_N : \bigwedge^2 \mathcal{G}_{-1,-1} \longrightarrow \mathcal{G}_{-2,-2}$ is given by Ihara's bracket $\beta_N(a \wedge b) = \{a, b\}$ defined by (5.13.6) of op. cit. Here $\mathcal{G}_{\bullet,\bullet}$ is the associated graded of the weight and depth gradings of $\mathcal{G}$ (see [20, §2.1]). Let $k(N) := \dim \ker(\beta_N)$. Then

$$\delta_1(N) := \dim \mathcal{G}_{-1,-1} = \begin{cases} 1, & \text{if } N = 1 \text{ or } 2; \\ \varphi(N)/2 + \nu(N) - 1, & \text{if } N \ge 3, \end{cases} \tag{39}$$

by [20, Thm. 2.1]. Thus

$$i(N) := \dim \operatorname{Im}(\beta_N) = \delta_1(N)(\delta_1(N) - 1)/2 - k(N). \tag{40}$$

| $N$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta_1$ | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 2 | 3 | 3 | 5 | 3 | 6 | 4 | 5 | 4 |
| $i$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 3 | 3 | 5 | 3 | 8 | 6 | 10 | 6 |
| $k$ | 0 | 0 | 0 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 5 | 0 | 7 | 0 | 0 | 0 |
| $\delta_2$ | 0 | 0 | 1 | 1 | 2 | 2 | 4 | 3 | 6 | 5 | 10 | 5 | 14 | 9 | 14 | 10 |
| $sr$ | 0 | 0 | 1 | 1 | 2 | 2 | 4 | 4 | 6 | 6 | 10 | 8 | 14 | 12 | 16 | 16 |
| $D$ | 1 | 2 | 4 | 4 | 9 | 8 | 16 | 9 | 16 | 15 | 36 | 15 | 49 | 24 | 35 | 25 |
| $SR$ | 1 | 2 | 4 | 4 | 8 | 8 | 14 | 10 | 16 | 16 | 31 | 18 | 42 | 27 | 37 | 31 |
| $d$ | 1 | 2 | 4 | 4 | 8 | 8 | 14 | 9 | 16 | 15 | 31 | 15 | 42 | 24 | 35 | 25 |

| $N$ | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta_1$ | 8 | 4 | 9 | 5 | 7 | 6 | 11 | 5 | 10 | 7 | 9 | 7 | 14 |
| $i$ | 16 | 6 | 21 | 10 | 21 | 15 | 33 | 10 | 40 | 21 | 36 | 21 | 56 |
| $k$ | 12 | 0 | 15 | 0 | 0 | 0 | 22 | 0 | 5 | 0 | 0 | 0 | 35 |
| $\delta_2$ | 24 | 9 | 30 | 14 | 27 | 20 | 44 | 14 | 50 | 27 | 45 | 27 | 70 |
| $sr$ | 24 | 18 | 30 | 24 | 32 | 30 | 44 | 32 | 50 | 42 | 54 | 48 | 70 |
| $D$ | 81 | 24 | 100 | 35 | 63 | 48 | 144 | 35 | 121 | 63 | 100 | 63 | 225 |
| $SR$ | 69 | 33 | 85 | 45 | 68 | 58 | 122 | 53 | 116 | 78 | 109 | 84 | 190 |
| $d$ | 69 | 24 | 85 | 35 | 63 | 48 | 122 | 35 | 116 | 63 | 100 | 63 | 190 |

| $N$ | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta_1$ | 6 | 15 | 8 | 11 | 9 | 13 | 7 | 18 | 10 | 13 | 9 | 20 |
| $i$ | 15 | 65 | 28 | 55 | 36 | 78 | 21 | 96 | 45 | 78 | 36 | 120 |
| $k$ | 0 | 40 | 0 | 0 | 0 | 0 | 0 | 57 | 0 | 0 | 0 | 70 |
| $\delta_2$ | 19 | 80 | 36 | 65 | 44 | 90 | 27 | 114 | 54 | 90 | 44 | 140 |
| $sr$ | 48 | 80 | 64 | 80 | 72 | 96 | 72 | 114 | 90 | 112 | 96 | 140 |
| $D$ | 47 | 256 | 81 | 143 | 99 | 195 | 63 | 361 | 120 | 195 | 99 | 441 |
| $SR$ | 76 | 216 | 109 | 158 | 127 | 201 | 108 | 304 | 156 | 217 | 151 | 371 |
| $d$ | 47 | 216 | 81 | 143 | 99 | 195 | 63 | 304 | 120 | 195 | 99 | 371 |

| $N$ | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 121 | 125 | 169 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta_1$ | 8 | 21 | 11 | 13 | 12 | 23 | 9 | 21 | 55 | 50 | 78 |
| $i$ | 28 | 133 | 55 | 78 | 66 | 161 | 36 | 175 | 1155 | 1200 | 2288 |
| $k$ | 0 | 77 | 0 | 0 | 0 | 92 | 0 | 35 | 330 | 25 | 715 |
| $\delta_2$ | 34 | 154 | 65 | 90 | 77 | 184 | 44 | 196 | 1210 | 1250 | 2366 |
| $sr$ | 96 | 154 | 120 | 144 | 132 | 184 | 128 | 196 | 1210 | 1250 | 2366 |
| $D$ | 79 | 484 | 143 | 195 | 168 | 576 | 99 | 484 | 3136 | 2601 | 6241 |
| $SR$ | 141 | 407 | 198 | 249 | 223 | 484 | 183 | 449 | 2806 | 2576 | 5526 |
| $d$ | 79 | 407 | 143 | 195 | 168 | 484 | 99 | 449 | 2806 | 2576 | 5526 |

Table 1: Upper bounds of $d(2, N)$ by the standard relations and [17, 5.25].

Since $\dim \mathcal{G}_{-2,-1} = \varphi(N)/2$ if $N > 2$ and 0 otherwise the dimension of the degree two part of $\mathcal{G}$ is

$$\delta_2(N) := \dim \mathcal{G}_{-2,-1} + \dim \mathcal{G}_{-2,-2} = \begin{cases} i(N), & \text{if } N = 1 \text{ or } 2; \\ \varphi(N)/2 + i(N), & \text{if } N \geq 3. \end{cases} \tag{41}$$

Let $sr(N)$ be the upper bound of $\delta_2(N)$ obtained by the standard relations. This can be computed by the method described in [30, §2]. Let $SR(N)$ be the upper bound of $d(2, N)$ similarly obtained by the standard relations. In Table 1 we use MAPLE to provide the following data: $k(N)$, $sr(N)$, and $SR(N)$. Then we can calculate $\delta_1(N)$, $i(N)$ and $\delta_2(N)$ by (39), (40) (41), respectively. From (38) we can check the consistency by verifying

$$sr(N) - \delta_2(N) = SR(N) - d(2, N) = SR(N) - D(2, N) + k(N)$$

which gives the number of linearly independent non-standard relations (assuming the conjecture in [17, 5.27(c)]). In Table 1 we provide some computational data of the above quantities. To save space we write $D = D(2, N)$ and $d = d(2, N)$.

Definition 8.1. We call the level $N$ *standard* if either (i) $N = 1, 2$ or 3, or (ii) $N$ is a prime power $p^n$ ($p \geq 5$). Otherwise $N$ is called *non-standard*.

*Remark* 8.2. We now make the following comments in the weight two case from Table 1.
(a) When $p \geq 11$ the vector space $\ker \beta_p$ contains a subspace isomorphic to the space of cusp forms of weight two on $X_1(p)$ which has dimension $(p-5)(p-7)/24$ (see [20, Lemma 2.3 & Theorem 7.8]). So it must contain another piece which has dimension $(p-3)/2$ since $\dim(\ker \beta_p) = (p^2-1)/24$ by [30, (6)]. One may wonder if this missing piece has any significance in geometry and/or number theory.
(b) If $N$ is a 2-power or a 3-power then $D(2, N)$ should be sharp. See Remark 5.4.
(c) If $N$ has at least two distinct prime factors then $D(2, N)$ seems to be sharp, though we don't have any theory to support it.
(d) Suppose the conjecture in [17, 5.27(c)] is true. Then by [17, 5.27], (b) and (c) is equivalent to saying that the kernel of $\beta_N$ is trivial if the level $N$ is non-standard. We believe this is also a necessary condition on $N$ for $\beta_N$ to be trivial.
(e) If the level $N > 3$ is standard then $\beta_N$ is *unlikely* to be injective. We conjecture that non-standard relation doesn't exist (i.e., $SR(N)$ is sharp), though for prime power levels we only have verified this for the first four prime square levels $N = 5^2, 7^2, 11^2, 13^2$, and the first cubic power level $N = 5^3$.

The equation $\dim \beta_p = (p^2 - 1)/24$ (see [30, (6)]) together with Theorem 8.6 confirms Remark 8.2(e) for prime levels if we assume a variant of Grothendieck's period conjecture [17, 5.27(c)]. The next result partially confirms Remark 8.2(e) in the case when the level is a prime square.

Theorem 8.3. *If $p \geq 5$ is a prime then* $\ker \beta_{p^2} \neq 0$ *and*

$$d(2,p^2) < D(2,p^2) = (p^2-p+2)^2/4.$$

*Proof.* By the proof of Delign-Goncharov's bound $D(2,p^2)$ in [17, 5.25] we only need to show $\ker \beta_{p^2} \neq 0$. In the following we adopt the same notation as in [17] and [30].

Fix a primitive $p^2$th root of unity $\mu$. Put $e(a) = e_{\mu^a}$ for all integer $a$. Define

$$g_{k,j} = e(pk+j) + e(p^2-pk-j) + e(pj) + e(p^2-pj)$$

for $0 \leq k < (p-1)/2$, $1 \leq j \leq p-1$, and for $k=(p-1)/2$, $1 \leq j \leq (p-1)/2$. One only needs to prove the following

Claim. Let $h = (p-3)/2$. Then one has

$$\begin{aligned}
&\sum_{k=0}^{h}\sum_{l=k}^{h}\sum_{j=2}^{p-2}\{g_{k,1}, g_{l,j}\} + \sum_{k=0}^{h+1}\sum_{j=2}^{h+1}\{g_{k,1}, g_{h+1,j}\}\\
&+\sum_{k=0}^{h}\sum_{l=k+1}^{h}\sum_{j=2}^{p-2}\{g_{k,p-1}, g_{l,j}\} + \sum_{k=0}^{h}\sum_{j=2}^{h+1}\{g_{k,p-1}, g_{h+1,j}\}\\
&-\sum_{k=0}^{h}\sum_{l=k}^{h}\sum_{j=2}^{p-2}\{g_{k,j}, g_{l,p-1}\} - \sum_{k=0}^{h}\sum_{l=k}^{h}\sum_{j=2}^{p-2}\{g_{k,j}, g_{l+1,1}\} = 0.
\end{aligned}$$

There are $h(2h+3)^2 = hp^2$ distinct terms on the left, each with coefficient $\pm 1$.

The proof of the claim is straight-forward by a little tedious change of indices and regrouping.

$$-\sum_{k=0}^{h}\sum_{l=k}^{h}\sum_{j=2}^{p-2}\{g_{k,j}, g_{l+1,1}\} = \sum_{k=0}^{h}\sum_{l=0}^{k}\sum_{j=2}^{p-2}\{g_{k+1,1}, g_{l,j}\} = \sum_{k=1}^{h+1}\sum_{l=0}^{k-1}\sum_{j=2}^{p-2}\{g_{k,1}, g_{l,j}\}.$$

Then the expression in the claim becomes

$$\begin{aligned}
&\sum_{k=1}^{h}\sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{k,1}, g_{l,j}\} + \sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{0,1}, g_{l,j}\} + \sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{h+1,1}, g_{l,j}\}\\
&+\sum_{k=0}^{h+1}\sum_{j=2}^{h+1}\{g_{k,1}, g_{h+1,j}\} + \sum_{k=0}^{h}\sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{k,p-1}, g_{l,j}\} + \sum_{k=0}^{h}\sum_{j=2}^{h+1}\{g_{k,p-1}, g_{h+1,j}\}\\
&=\sum_{k=0}^{h+1}\sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{k,1}, g_{l,j}\} + \sum_{k=0}^{h+1}\sum_{j=2}^{h+1}\{g_{k,1}, g_{h+1,j}\}\\
&+\sum_{k=0}^{h}\sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{k,p-1}, g_{l,j}\} + \sum_{k=0}^{h}\sum_{j=2}^{h+1}\{g_{k,p-1}, g_{h+1,j}\}.
\end{aligned}$$

Let us write $\{a,b\} = \{e(a), e(b)\}$. By definition

$$
\begin{aligned}
&\{g_{k,1}, g_{l,j}\}\\
=&\{pk+1, pl+j\} + \{-pk-1, pl+j\} + \{p, pl+j\} + \{-p, pl+j\}\\
+&\{pk+1, -pl-j\} + \{-pk-1, -pl-j\} + \{p, -pl-j\} + \{-p, -pl-j\}\\
+&\{pk+1, pj\} + \{-pk-1, pj\} + \{p, pj\} + \{-p, pj\}\\
+&\{pk+1, -pj\} + \{-pk-1, -pj\} + \{p, -pj\} + \{-p, -pj\}\\
=&\{pk+1, pl+j\} + \{p(p-k)-1, pl+j\} + \{p, pl+j\} + \{-p, pl+j\}\\
+&\{pk+1, p(p-1-l)+p-j\} + \{p(p-k)-1, p(p-1-l)+p-j\}\\
&\qquad\qquad + \{p, p(p-1-l)+p-j\} + \{-p, p(p-1-l)+p-j\}\\
+&\{pk+1, pj\} + \{p(p-k)-1, pj\} + \{p, pj\} + \{-p, pj\}\\
+&\{pk+1, p(p-j)\} + \{p(p-k)-1, p(p-j)\} + \{p, p(p-j)\} + \{-p, p(p-j)\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{k=0}^{h+1}\sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{k,1}, g_{l,j}\} = \sum_{k=0}^{h+1}\sum_{l=0}^{h}\sum_{j=2}^{p-2}\{pk+1, pl+j\} + \{p(p-k)-1, pl+j\}\\
+&\{pk+1, p(p-1-l)+j\} + \{p(p-k)-1, p(p-1-l)+j\}\\
+&\{p, pl+j\} + \{-p, pl+j\} + \{p, p(p-1-l)+j\} + \{-p, p(p-1-l)+j\}\\
+&2\{pk+1, pj\} + 2\{p(p-k)-1, pj\} + 2\{p, pj\} + 2\{-p, pj\}\\
=&\sum_{k=0}^{h+1}\sum_{l=0, l\neq h+1}^{p-1}\sum_{j=2}^{p-2}\{pk+1, pl+j\} + \sum_{k=h+2}^{p}\sum_{l=0, l\neq h+1}^{p-1}\sum_{j=2}^{p-2}\{pk-1, pl+j\}\\
+&\sum_{k=0}^{h+1}\sum_{l=0, l\neq h+1}^{p-1}\sum_{j=2}^{p-2}\Big(\{p, pl+j\} + \{-p, pl+j\}\Big) + 2(h+1)\sum_{k=0}^{h+1}\sum_{j=2}^{p-2}\{pk+1, pj\}\\
+&2(h+1)\sum_{k=h+2}^{p}\sum_{j=2}^{p-2}\{pk-1, pj\} + 2(h+2)(h+1)\sum_{j=2}^{p-2}\Big(\{p, pj\} + \{-p, pj\}\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sum_{k=0}^{h+1}\sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{k,1}, g_{l,j}\} + \sum_{k=0}^{h+1}\sum_{j=2}^{h+1}\{g_{k,1}, g_{h+1,j}\}\\
=&\sum_{k=0}^{h+1}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\{pk+1, pl+j\} + \sum_{k=h+2}^{p}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\{pk-1, pl+j\}\\
+&\frac{p+1}{2}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\Big(\{p, pl+j\} + \{-p, pl+j\}\Big) + p\sum_{k=0}^{h+1}\sum_{j=2}^{p-2}\{pk+1, pj\}\\
+&p\sum_{k=h+2}^{p}\sum_{j=2}^{p-2}\{pk-1, pj\} + \frac{p(p+1)}{2}\sum_{j=2}^{p-2}\Big(\{p, pj\} + \{-p, pj\}\Big).
\end{aligned}
$$

Similarly,

$$\begin{aligned}
&\sum_{k=0}^{h}\sum_{l=0}^{h}\sum_{j=2}^{p-2}\{g_{k,p-1},g_{l,j}\}+\sum_{k=0}^{h}\sum_{j=2}^{h+1}\{g_{k,p-1},g_{h+1,j}\}\\
=&\sum_{k=1}^{h+1}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\{pk-1,pl+j\}+\sum_{k=h+2}^{p-1}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\{pk+1,pl+j\}\\
+&\frac{p-1}{2}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\Big(\{p,pl+j\}+\{-p,pl+j\}\Big)+p\sum_{k=1}^{h+1}\sum_{j=2}^{p-2}\{pk-1,pj\}\\
+&p\sum_{k=h+2}^{p-1}\sum_{j=2}^{p-2}\{pk+1,pj\}+\frac{p(p-1)}{2}\sum_{j=2}^{p-2}\Big(\{p,pj\}+\{-p,pj\}\Big).
\end{aligned}$$

Altogether the expression in the claim is reduced to

$$\begin{aligned}
X:=&\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\{pk+1,pl+j\}+\sum_{k=1}^{p}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\{pk-1,pl+j\}\\
&+p\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\Big(\{p,pl+j\}+\{-p,pl+j\}\Big)+p\sum_{k=0}^{p-1}\sum_{j=2}^{p-2}\{pk+1,pj\}\\
&+p\sum_{k=1}^{p}\sum_{j=2}^{p-2}\{pk-1,pj\}+p^2\sum_{j=2}^{p-2}\Big(\{p,pj\}+\{-p,pj\}\Big).
\end{aligned}$$

To see this last expression can be reduced to 0 we recall that by definition [17, (5.13.6)]

$$\{a,b\}=\{e_a,e_b\}=[e_a,e_b]+\partial_a(e_b)-\partial_b(e_a),$$

where $\partial_a$ is the derivation defined by $\partial_a(e_0)=0$ and $\partial_a(e_\zeta)=[-[\zeta](e_a),e_\zeta]$ for any $p^2$th root of unity $\zeta$ (see [17, (5.13.4)]). Thus by abuse of notation $[x,y]=[e(x),e(y)]$ we get

$$\begin{aligned}
X=&\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\Big([pk+1,pl+j]-[p(k+l)+j+1,pl+j]\\
&\qquad\qquad+[p(k+l)+j+1,pk+1]\Big) \qquad (42)\\
&+\sum_{k=1}^{p}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\Big([pk-1,pl+j]-[p(k+l)+j-1,pl+j]\\
&\qquad\qquad+[p(k+l)+j-1,pk-1]\Big) \qquad (43)\\
&+p\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}\Big([p,pl+j]-[p(l+1)+j,pl+j]+[p(l+1)+j,p]
\end{aligned}$$

$$+[-p,pl+j]-[p(l-1)+j,pl+j]+[p(l-1)+j,-p]\Big) \quad (44)$$

$$+p\sum_{k=0}^{p-1}\sum_{j=2}^{p-2}\Big([pk+1,pj]-[p(j+k)+1,pj]+[p(j+k)+1,pk+1]\Big) \quad (45)$$

$$+p\sum_{k=1}^{p}\sum_{j=2}^{p-2}\Big([pk-1,pj]-[p(j+k)-1,pj]+[p(j+k)-1,pk-1]\Big) \quad (46)$$

$$+p^2\sum_{j=2}^{p-2}\Big([p,pj]-[p(j+1),pj]+[p(j+1),p]+[-p,pj] - [p(j-1),pj]+[p(j-1),-p]\Big). \quad (47)$$

Now by skew-symmetry of Lie bracket

$$\begin{aligned}
&(42)+(43)\\
=&\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}[pk+1,pl+j]+\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}[pk+j,pl+j+1]\\
-&\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}\sum_{j=3}^{p-1}[pk+1,pl+j]+\sum_{k=1}^{p}\sum_{l=0}^{p-1}\sum_{j=2}^{p-2}[pk-1,pl+j]\\
-&\sum_{k=1}^{p}\sum_{l=0}^{p-1}\sum_{j=1}^{p-3}[p(k+l)+j,pl+j+1]+\sum_{k=1}^{p}\sum_{l=0}^{p-1}\sum_{j=1}^{p-3}[pl+j,pk-1]\\
=&\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}[pk+1,pl+2]+\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}[pk+p-2,pl+p-1]\\
-&\sum_{k=0}^{p-1}\sum_{l=0}^{p-1}[pk+1,pl+p-1]+\sum_{k=1}^{p}\sum_{l=0}^{p-1}[pk-1,pl+p-2]\\
-&\sum_{k=1}^{p}\sum_{l=0}^{p-1}[pk+1,pl+2]+\sum_{k=1}^{p}\sum_{l=0}^{p-1}[pl+1,pk-1]=0.
\end{aligned}$$

Similarly we can easily find that (44) = (45) = (46) = (47) = 0. This finishes the proof of the theorem. □

*Remark* 8.4. The theorem corrects a misprint in the statement of [30, Thm. 2].

In the three cases $(w,N)=(2,8),(2,10)$ and $(2,12)$ we see that $SR(N)>d(w;N)=D(w;N)$. By numerical computation we have

Conjecture 8.5. *We have*

$$d(2,8)=9,\quad d(2,10)=d(2,12)=15,$$

*and the following relations are the linearly independent non-standard relations: let* $L_N(-)=L_N(1,1|-)$ *and* $L_N^{(2)}(-)=L_N(2|-)$*, then*

$$\begin{aligned}
37L_8(1,1)=&34L_8^{(2)}(5)+112L_8(3,1)+11L_8(3,0)+37L_8^{(2)}(1)-2L_8(2,6)\\
&+3L_8(7,3)-111L_8(5,7)+38L_8(7,7)-8L_8(5,5), \qquad (48)\\
7L_{10}(5,2)=&72L_{10}^{(2)}(1)+265L_{10}^{(2)}(7)-7L_{10}(2,5)+64L_{10}(9,8)+14L_{10}(5,6)\\
&-467L_{10}(4,2)+467L_{10}(8,6)-164L_{10}(9,4)+166L_{10}(7,9)\\
&-260L_{10}(8,1)-66L_{10}(3,9)-7L_{10}(6,9)+7L_{10}(6,5). \qquad (49)\\
L_{12}(8,7)=&5L_{12}^{(2)}(5)+8L_{12}(8,10)-6L_{12}(10,11)-8L_{12}(9,11)+L_{12}(10,9)\\
&-15L_{12}(8,1)+5L_{12}(9,10)+5L_{12}(6,1)-L_{12}(1,1)\\
&+6L_{12}(8,11)-11L_{12}(6,11)+8L_{12}(8,3)-L_{12}(11,8), \qquad (50)\\
60L_{12}(8,11)=&38L_{12}(8,7)+348L_{12}(10,11)+502L_{12}(9,11)\\
&-492L_{12}(10,9)+600L_{12}(8,1)-552L_{12}(9,10)\\
&-154L_{12}(11,10)+20L_{12}(6,1)+261L_{12}(6,11)\\
&-502L_{12}(8,3)+221L_{12}(11,8)-319L_{12}(8,10), \qquad (51)\\
221L_{12}(1,1)=&1854L_{12}(8,10)+562L_{12}(8,7)-1018L_{12}(10,11)\\
&-2416L_{12}(9,11)+319L_{12}(10,9)-4270L_{12}(8,1)\\
&+2293L_{12}(9,10)+956L_{12}(11,10)+1110L_{12}(6,1)\\
&+2416L_{12}(8,11)-3305L_{12}(6,11)+2416L_{12}(8,3). \qquad (52)
\end{aligned}$$

When $N$ is a non-standard level we find that very often there are non-standard relations among MPVs. For examples, the five relations in Conjecture 8.5 are discovered only through numerical computation. On the other hand, we expect that the standard relations are enough to produce all the linear relations when $N$ *is* standard. In weight two, when $N$ is a prime the answer is confirmed by the next theorem if one assumes a variant of Grothendieck's period conjecture. Computations above provided the primary motivation of this result at the initial stage of this work.

Theorem 8.6. ([30]) *Let* $p\ge 5$ *be a prime. Then*

$$d(2,p)\le \frac{(5p+7)(p+1)}{24}.$$

*If the conjecture in* [17, 5.27(c)] *is true then the equality holds and the standard relations in* $\mathcal{MPV}(2,p)$ *imply all the others.*

*Proof.* See the proof of [30, Thm. 1]. □

It follows from [30, (6)] that the kernel $\beta_p$ has dimension

$$k(p)=\frac{p^2-1}{24}$$

for all prime $p\ge 5$. From the data in Table 1 we have

Conjecture 8.7. *(a) For all prime $p \geq 5$ kernel $\beta_{p^2}$ has dimension*

$$k(p^2) = \frac{p(p-1)(p-2)(p-3)}{24}.$$

*As a consequence, the upper bound of $d(2,p^2)$ produced by the standard relations is*

$$d(2,p^2) \leq \frac{5p^4 - 6p^3 + 19p^2 - 18p + 24}{24}.$$

*(b) The standard relations produce all the linear relations and the upper bound in (a) is sharp.*

Conjecture 8.8. *(a) For all prime $p \geq 5$ kernel $\beta_{p^3}$ has dimension*

$$k(p^3) = \frac{p^2(p-1)(p-2)(p-3)(p-4)}{24}.$$

*As a consequence, the upper bound of $d(2,p^3)$ produced by the standard relations is*

$$d(2,p^3) \leq \frac{5p^6 - 2p^5 - 29p^4 + 74p^3 - 48p^2 + 24}{24}.$$

*(b) The standard relations produce all the linear relations and the upper bound in (a) is sharp.*

## 9 Computational results in weight three, four and five

In this last section we briefly discuss our results in weight three, four and five. Since the computational complexity increases exponentially with the weight we cannot do as many cases as we have done in weight two.

Combining the FDS (28), (29), RDS (30)-(35), and the weight one relations (13) and using MAPLE we have verified that $d(3,1) = 1, d(3,2) \leq 3, d(3,3) \leq 8$....

| $N$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $SR(3)$ | 1 | 3 | 8 | 9 | 22 | 23 | 50 |
| $D(3)$ | 1 | 3 | 8 | 8 | 27 | 21 | 64 |
| $SR(4)$ | 1 | 5 | 16 | 21 | 61 | 69 | |
| $D(4)$ | 1 | 5 | 16 | 16 | 81 | 55 | 256 |
| $SR(5)$ | 2 | 8 | 32 | | | | |
| $D(5)$ | 2 | 8 | 32 | 32 | 243 | 144 | 1024 |
| $N$ | 8 | 9 | 10 | 11 | 12 | 13 | |
| $SR(3)$ | 38 | 67 | 70 | 157 | 94 | 246 | |
| $D(3)$ | 27 | 64 | 56 | 216 | 56 | 343 | |

Table 2: Upper bounds of $d(w,N)$ by the standard relations and [17, 5.25].

We have done similar computation in other small weight and low level cases and listed the results in Table 2. The values of Deligne and Goncharov's bound $D(w) = D(w,N)$ in Table 2 should be compared with the bound $SR(w) = SR(w,N)$ obtained by the standard relations.
Note that $SR(3,4) = D(3,4)+1$. By numerical computation using EZface [9] and GiNac [27] we find the following non-standard relation in weight 3:

$$\begin{aligned} 5L_4(1,2|2,3) =& 46L_4(1,1,1|1,0,0) - 7L_4(1,1,1|2,2,1) - 13L_4(1,1,1|1,1,1) \\ &+ 13L_4(1,2|3,1) - L_4(1,1,1|3,2,0) + 25L_4(1,1,1|3,0,0) \\ &- 8L_4(1,1,1|1,1,2) + 18L_4(2,1|3,0), \end{aligned} \tag{53}$$

and five non-standard relations in weight 4:

$$\begin{aligned} 0 =& -255608l_1 - 265360l_2 - 219216l_3 - 19306179l_4 - 214008l_5 + 45560l_6 \\ &- 148296l_7 - 1117280l_8 - 677152l_9 + 86512l_{10} - 239320l_{11} - 50032l_{12} \\ &- 121008l_{13} - 96944l_{14} + 202328l_{15} - 1178499l_{16} + 98944l_{17} \\ &+ 1565754l_{18} + 23071580l_{19} + 363568l_{20} - 3310177l_{21}, \end{aligned} \tag{54}$$

$$\begin{aligned} 0 =& 29752l_1 + 23312l_2 + 10960l_3 + 6123413l_4 + 16440l_5 - 12408l_6 \\ &+ 7144l_7 + 58272l_8 + 86976l_9 - 15952l_{10} + 41144l_{11} + 13552l_{12} \\ &+ 29552l_{13} + 9840l_{14} - 36696l_{15} + 375805l_{16} - 41760l_{17} \\ &- 477366l_{18} - 7196900l_{19} - 62128l_{20} + 1048983l_{21}, \end{aligned} \tag{55}$$

$$\begin{aligned} 0 =& 477444l_1 + 431352l_2 + 268168l_3 + 98404710l_4 + 308964l_5 - 233140l_6 \\ &+ 130028l_7 + 1563872l_8 + 1516032l_9 - 296664l_{10} + 702308l_{11} + 190136l_{12} \\ &+ 506440l_{13} + 141592l_{14} - 636468l_{15} + 6027441l_{16} - 701600l_{17} \\ &- 7683609l_{18} - 115803282l_{19} - 1063768l_{20} + 16877562l_{21}, \end{aligned} \tag{56}$$

$$\begin{aligned} 0 =& -5976l_1 + 1776l_2 + 8496l_3 - 2132671l_4 + 3176l_5 + 1752l_6 \\ &+ 3832l_7 + 50976l_8 - 2688l_9 + 2320l_{10} - 10264l_{11} - 5808l_{12} \\ &- 6128l_{13} + 2320l_{14} + 8120l_{15} - 132307l_{16} + 13856l_{17} \\ &+ 162614l_{18} + 2487604l_{19} + 12720l_{20} - 368485l_{21}, \end{aligned} \tag{57}$$

$$\begin{aligned} 0 =& -474064l_1 - 405248l_2 - 243520l_3 - 54556373l_4 - 283952l_5 + 84368l_6 \\ &- 170640l_7 - 1033056l_8 - 994784l_9 + 174880l_{10} - 540432l_{11} - 156544l_{12} \\ &- 240512l_{13} - 49344l_{14} + 411152l_{15} - 3357683l_{16} + 292256l_{17} \\ &+ 4291792l_{18} + 64572648l_{19} + 743136l_{20} - 9470695l_{21}. \end{aligned} \tag{58}$$

where by setting $L = L_4$, $1^4 = \{1\}^4$, ...

$$\begin{aligned} &l_1 = L(1^4|2,1,0,1), && l_2 = L(1^4|2,1^2,0), && l_3 = L(1^4|2,0,3,1), \\ &l_4 = L(1^4|2,0^3), && l_5 = L(1^4|1,2,0,3), && l_6 = L(1^4|3^2,0,3), \\ &l_7 = L(1^4|3,1,3,2), && l_8 = L(1^4|3,0^3), && l_9 = L(1^4|3,0,1,0), \\ &l_{10} = L(1^4|3,0,1^2), && l_{11} = L(2,1^2|0,3,0), && l_{12} = L(3,1|0,3), \\ &l_{13} = L(1^4|2,2,3,0), && l_{14} = L(2,1^2|3,1^2), && l_{15} = L(2,1^2|3,0,3), \\ &l_{16} = L(1^2,2|2^3), && l_{17} = L(1^4|2,0,1,0), && l_{18} = L(2,1^2|2^2,0), \\ &l_{19} = L(1^4|\{2,0\}^2), && l_{20} = L(2^2|3,0), && l_{21} = L(1^4|2^4). \end{aligned}$$

We now can prove this by using the octahedral symmetry of $\mathbb{P}^1 - (\{0, \infty\} \cup \boldsymbol{\mu}_4)$ (see Remark 5.3). This idea was suggested to the author by Deligne in a letter dated Feb. 14, 2008.

THEOREM 9.1. ([30]) *If the conjecture in* [17, 5.27(c)] *is true then all the linear relations among MPVs of level four and weight three (resp. weight four) are the consequences of the standard relations and the octahedral relation* (53) *(resp. the five octahedral relations* (54)-(58)*).*

*Proof.* For the proof please see[30, §3]. □

From the available data in Table 2 we can formulate the following conjecture.

CONJECTURE 9.2. *Suppose the level $N = p$ is a prime $\geq 5$. Then*

$$d(3, p) \leq \frac{p^3 + 4p^2 + 5p + 14}{12}.$$

*Moreover, equality holds if standard relations produce all the linear relations.*

We formulated this conjecture under the belief that the upper bound of $d(3, p)$ produced by the standard relations should be a polynomial of $p$ of degree 3. Then we find the coefficients by the bounds of $d(3, p)$ for $p = 5, 7, 11, 13$ in Table 2.
When $w > 2$ it's not too hard to improve the bound of $d(w, p)$ given in [17, 5.25] by the same idea as used in the proof of [17, 5.24] (for example, decrease the bound by $(p^2 - 1)/24$). But they are often not the best. We conclude our paper with the following conjecture.

CONJECTURE 9.3. *If $N$ is a standard level then the standard relations always provide the sharp bounds of $d(w, N)$, namely, all linear relations can be derived from the standard ones, if further $N > 3$ then the bound $D(w, N)$ in* (36) *by Deligne and Goncharov can be lowered. If $N$ is a non-standard level then the bound $D(w, N)$ is sharp and there exists a positive integer $w_0(N)$ so that at least one non-standard relation exists in $\mathcal{MPV}(w, N)$ for each $w \geq w_0(N)$.*

It is likely that one can take $w_0(4) = w_0(6) = w_0(9) = 3$ and $w_0(N) = 2$ for all the other non-standard levels $N$.

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# The Line Bundles on Moduli Stacks of Principal Bundles on a Curve

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Abstract. Let $G$ be an affine reductive algebraic group over an algebraically closed field $k$. We determine the Picard group of the moduli stacks of principal $G$–bundles on any smooth projective curve over $k$.

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## 1. Introduction

As long as moduli spaces of bundles on a smooth projective algebraic curve $C$ have been studied, their Picard groups have attracted some interest. The first case was the coarse moduli scheme of semistable vector bundles with fixed determinant over a curve $C$ of genus $g_C \geq 2$. Seshadri proved that its Picard group is infinite cyclic in the coprime case [28]; Drézet and Narasimhan showed that this remains valid in the non–coprime case also [9].
The case of principal $G$–bundles over $C$ for simply connected, almost simple groups $G$ over the complex numbers has been studied intensively, motivated also by the relation to conformal field theory and the Verlinde formula [1, 12, 20]. Here Kumar and Narasimhan [19] showed that the Picard group of the coarse moduli scheme of semistable $G$–principal bundles over a curve $C$ of genus $g_C \geq 2$ embeds as a subgroup of finite index into the Picard group of the affine Grassmannian, which is canonically isomorphic to $\mathbb{Z}$; this finite index was determined recently in [6]. Concerning the Picard group of the moduli stack $\mathcal{M}_G$ of principal $G$–bundles over a curve $C$ of any genus $g_C \geq 0$, Laszlo and Sorger [23, 30] showed that its canonical map to the Picard group $\mathbb{Z}$ of the affine Grassmannian is actually an isomorphism. Faltings [13] has generalised this result to positive characteristic, and in fact to arbitrary noetherian base scheme.

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If $G$ is not simply connected, then the moduli stack $\mathcal{M}_G$ has several connected components which are indexed by $\pi_1(G)$. For any $d \in \pi_1(G)$, let $\mathcal{M}_G^d$ be the corresponding connected component of $\mathcal{M}_G$. For semisimple, almost simple groups $G$ over $\mathbb{C}$, the Picard group $\mathrm{Pic}(\mathcal{M}_G^d)$ has been determined case by case by Beauville, Laszlo and Sorger [2, 22]. It is finitely generated, and its torsion part is a direct sum of $2g_C$ copies of $\pi_1(G)$. Furthermore, its torsion–free part again embeds as a subgroup of finite index into the Picard group $\mathbb{Z}$ of the affine Grassmannian. Together with a general expression for this index, Teleman [31] also proved these statements, using topological and analytic methods.

In this paper, we determine the Picard group $\mathrm{Pic}(\mathcal{M}_G^d)$ for any reductive group $G$, working over an algebraically closed ground field $k$ without any restriction on the characteristic of $k$ (for all $g_C \geq 0$). Endowing this group with a natural scheme structure, we prove that the resulting group scheme $\underline{\mathrm{Pic}}(\mathcal{M}_G^d)$ over $k$ contains, as an open subgroup, the scheme of homomorphisms from $\pi_1(G)$ to the Jacobian $J_C$, with the quotient being a finitely generated free abelian group which we denote by $\mathrm{NS}(\mathcal{M}_G^d)$ and call it the Néron–Severi group (see Theorem 5.3.1). We introduce this Néron–Severi group combinatorially in § 5.2; in particular, Proposition 5.2.11 describes it as follows: the group $\mathrm{NS}(\mathcal{M}_G^d)$ contains a subgroup $\mathrm{NS}(\mathcal{M}_{G^{\mathrm{ab}}})$ which depends only on the torus $G^{\mathrm{ab}} = G/[G,G]$; the quotient is a group of Weyl–invariant symmetric bilinear forms on the root system of the semisimple part $[G,G]$, subject to certain integrality conditions that generalise Teleman's result in [31].

We also describe the maps of Picard groups induced by group homomorphisms $G \longrightarrow H$. An interesting effect appears for the inclusion $\iota_G : T_G \hookrightarrow G$ of a maximal torus, say for semisimple $G$: Here the induced map $\mathrm{NS}(\mathcal{M}_G^d) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_G}^\delta)$ for a lift $\delta \in \pi_1(T_G)$ of $d$ involves contracting each bilinear form in $\mathrm{NS}(\mathcal{M}_G^d)$ to a linear form by means of $\delta$ (cf. Definition 4.3.5). In general, the map of Picard groups induced by a group homomorphism $G \longrightarrow H$ is a combination of this effect and of more straightforward induced maps (cf. Definition 5.2.7 and Theorem 5.3.1.iv). In particular, these induced maps are different on different components of $\mathcal{M}_G$, whereas the Picard groups $\mathrm{Pic}(\mathcal{M}_G^d)$ themselves are essentially independent of $d$.

Our proof is based on Faltings' result in the simply connected case. To deduce the general case, the strategy of [2] and [22] is followed, meaning we "cover" the moduli stack $\mathcal{M}_G^d$ by a moduli stack of "twisted" bundles as in [2] under the universal cover of $G$, more precisely under an appropriate torus times the universal cover of the semisimple part $[G,G]$. To this "covering", we apply Laszlo's [22] method of descent for torsors under a group stack. To understand the relevant descent data, it turns out that we may restrict to the maximal torus $T_G$ in $G$, roughly speaking because the pullback $\iota_G^*$ is injective on the Picard groups of the moduli stacks.

We briefly describe the structure of this paper. In Section 2, we recall the relevant moduli stacks and collect some basic facts. Section 3 deals with the case that $G = T$ is a torus. Section 4 treats the "twisted" simply connected case as indicated above. In the final Section 5, we put everything together to

prove our main theorem, namely Theorem 5.3.1. Each section begins with a slightly more detailed description of its contents.
Our motivation for this work was to understand the existence of Poincaré families on the corresponding coarse moduli schemes, or in other words to decide whether these moduli stacks are neutral as gerbes over their coarse moduli schemes. The consequences for this question are spelled out in [4].

## 2. The stack of $G$–bundles and its Picard functor

Here we introduce the basic objects of this paper, namely the moduli stack of principal $G$–bundles on an algebraic curve and its Picard functor. The main purpose of this section is to fix some notation and terminology; along the way, we record a few basic facts for later use.

2.1. A Picard functor for algebraic stacks. Throughout this paper, we work over an algebraically closed field $k$. There is no restriction on the characteristic of $k$. We say that a stack $\mathcal{X}$ over $k$ is *algebraic* if it is an Artin stack and also locally of finite type over $k$. Every algebraic stack $\mathcal{X} \neq \emptyset$ admits a point $x_0 : \operatorname{Spec}(k) \longrightarrow \mathcal{X}$ according to Hilbert's Nullstellensatz.
A 1–morphism $\Phi : \mathcal{X} \longrightarrow \mathcal{Y}$ of stacks is an *equivalence* if some 1–morphism $\Psi : \mathcal{Y} \longrightarrow \mathcal{X}$ admits 2–isomorphisms $\Psi \circ \Phi \cong \mathrm{id}_{\mathcal{X}}$ and $\Phi \circ \Psi \cong \mathrm{id}_{\mathcal{Y}}$. A diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{A} & \mathcal{X}' \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi'} \\ \mathcal{Y} & \xrightarrow{B} & \mathcal{Y}' \end{array}$$

of stacks and 1–morphisms is 2*–commutative* if a 2–isomorphism $\Phi' \circ A \cong B \circ \Phi$ is given. Such a 2–commutative diagram is 2*–cartesian* if the induced 1–morphism from $\mathcal{X}$ to the fibre product of stacks $\mathcal{X}' \times_{\mathcal{Y}'} \mathcal{Y}$ is an equivalence.
Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks over $k$. As usual, we denote by $\operatorname{Pic}(\mathcal{X})$ the abelian group of isomorphism classes of line bundles $\mathcal{L}$ on $\mathcal{X}$. If $\mathcal{X} \neq \emptyset$, then

$$\mathrm{pr}_2^* : \operatorname{Pic}(\mathcal{Y}) \longrightarrow \operatorname{Pic}(\mathcal{X} \times \mathcal{Y})$$

is injective because $x_0^* : \operatorname{Pic}(\mathcal{X} \times \mathcal{Y}) \longrightarrow \operatorname{Pic}(\mathcal{Y})$ is a left inverse of $\mathrm{pr}_2^*$.

Definition 2.1.1. The *Picard functor* $\underline{\operatorname{Pic}}(\mathcal{X})$ is the functor from schemes $S$ of finite type over $k$ to abelian groups that sends $S$ to $\operatorname{Pic}(\mathcal{X} \times S)/ \mathrm{pr}_2^* \operatorname{Pic}(S)$.

If $\underline{\operatorname{Pic}}(\mathcal{X})$ is representable, then we denote the representing scheme again by $\underline{\operatorname{Pic}}(\mathcal{X})$. If $\underline{\operatorname{Pic}}(\mathcal{X})$ is the constant sheaf given by an abelian group $\Lambda$, then we say that $\underline{\operatorname{Pic}}(\mathcal{X})$ is *discrete* and simply write $\underline{\operatorname{Pic}}(\mathcal{X}) \cong \Lambda$. (Since the constant Zariski sheaf $\Lambda$ is already an fppf sheaf, it is not necessary to specify the topology here.)

Lemma 2.1.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be algebraic stacks over $k$ with $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = k$.*
i) *The canonical map*

$$\mathrm{pr}_2^* : \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \longrightarrow \Gamma(\mathcal{X} \times \mathcal{Y}, \mathcal{O}_{\mathcal{X} \times \mathcal{Y}})$$

*is an isomorphism.*

ii) *Let* $\mathcal{L} \in \mathrm{Pic}(\mathcal{X} \times \mathcal{Y})$ *be given. If there is an atlas* $u : U \longrightarrow \mathcal{Y}$ *for which* $u^*\mathcal{L} \in \mathrm{Pic}(\mathcal{X} \times U)$ *is trivial, then* $\mathcal{L} \in \mathrm{pr}_2^* \mathrm{Pic}(\mathcal{Y})$.

*Proof.* i) Since the question is local in $\mathcal{Y}$, we may assume that $\mathcal{Y} = \mathrm{Spec}(A)$ is an affine scheme over $k$. In this case, we have

$$\Gamma(\mathcal{X} \times \mathcal{Y}, \mathcal{O}_{\mathcal{X}\times\mathcal{Y}}) = \Gamma(\mathcal{X}, (\mathrm{pr}_1)_*\mathcal{O}_{\mathcal{X}\times\mathcal{Y}}) = \Gamma(\mathcal{X}, A \otimes_k \mathcal{O}_{\mathcal{X}}) = A = \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}).$$

ii) Choose a point $x_0 : \mathrm{Spec}(k) \longrightarrow \mathcal{X}$. We claim that $\mathcal{L}$ is isomorphic to $\mathrm{pr}_2^* \mathcal{L}_{x_0}$ for $\mathcal{L}_{x_0} := x_0^*\mathcal{L} \in \mathrm{Pic}(\mathcal{Y})$. More precisely there is a unique isomorphism $\mathcal{L} \cong \mathrm{pr}_2^* \mathcal{L}_{x_0}$ whose restriction to $\{x_0\} \times \mathcal{Y} \cong \mathcal{Y}$ is the identity. To prove this, due to the uniqueness involved, this claim is local in $\mathcal{Y}$. Hence we may assume $\mathcal{Y} = U$, which by assumption means that $\mathcal{L}$ is trivial. In this case, statement (i) implies the claim. □

Corollary 2.1.3. *For* $\nu = 1, 2$, *let* $\mathcal{X}_\nu$ *be an algebraic stack over* $k$ *with* $\Gamma(\mathcal{X}_\nu, \mathcal{O}_{\mathcal{X}_\nu}) = k$. *Let* $\Phi : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ *be a* 1*–morphism such that the induced morphism of functors* $\Phi^* : \underline{\mathrm{Pic}}(\mathcal{X}_2) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{X}_1)$ *is injective. Then*

$$\Phi^* : \mathrm{Pic}(\mathcal{X}_2 \times \mathcal{Y}) \longrightarrow \mathrm{Pic}(\mathcal{X}_1 \times \mathcal{Y})$$

*is injective for every algebraic stack* $\mathcal{Y}$ *over* $k$.

*Proof.* Since $\mathcal{Y}$ is assumed to be locally of finite type over $k$, we can choose an atlas $u : U \longrightarrow \mathcal{Y}$ such that $U$ is a disjoint union of schemes of finite type over $k$. Suppose that $\mathcal{L} \in \mathrm{Pic}(\mathcal{X}_2 \times \mathcal{Y})$ has trivial pullback $\Phi^*\mathcal{L} \in \mathrm{Pic}(\mathcal{X}_1 \times \mathcal{Y})$. Then $(\Phi \times u)^*\mathcal{L} \in \mathrm{Pic}(\mathcal{X}_1 \times U)$ is also trivial. Using the assumption on $\Phi^*$ it follows that $u^*\mathcal{L} \in \mathrm{Pic}(\mathcal{X}_2 \times U)$ is trivial. Now apply Lemma 2.1.2(ii). □

We will also need the following stacky version of the standard see–saw principle.

Lemma 2.1.4. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be two nonempty algebraic stacks over* $k$. *If* $\underline{\mathrm{Pic}}(\mathcal{X})$ *is discrete, and* $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = k$, *then*

$$\mathrm{pr}_1^* \oplus \mathrm{pr}_2^* : \underline{\mathrm{Pic}}(\mathcal{X}) \oplus \underline{\mathrm{Pic}}(\mathcal{Y}) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{X} \times \mathcal{Y})$$

*is an isomorphism of functors.*

*Proof.* Choose points $x_0 : \mathrm{Spec}(k) \longrightarrow \mathcal{X}$ and $y_0 : \mathrm{Spec}(k) \longrightarrow \mathcal{Y}$. The morphism of functors $\mathrm{pr}_1^* \oplus \mathrm{pr}_2^*$ in question is injective, because

$$y_0^* \oplus x_0^* : \underline{\mathrm{Pic}}(\mathcal{X} \times \mathcal{Y}) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{X}) \oplus \underline{\mathrm{Pic}}(\mathcal{Y})$$

is a left inverse of it. Therefore, to prove the lemma it suffices to show that $y_0^* \oplus x_0^*$ is also injective.

So let a scheme $S$ of finite type over $k$ be given, as well as a line bundle $\mathcal{L}$ on $\mathcal{X} \times \mathcal{Y} \times S$ such that $y_0^*\mathcal{L}$ is trivial in $\underline{\mathrm{Pic}}(\mathcal{X})$. We claim that $\mathcal{L}$ is isomorphic to the pullback of a line bundle on $\mathcal{Y} \times S$.

To prove the claim, tensoring $\mathcal{L}$ with an appropriate line bundle on $S$ if necessary, we may assume that $y_0^*\mathcal{L}$ is trivial in $\mathrm{Pic}(\mathcal{X} \times S)$. By assumption,

$\underline{\mathrm{Pic}}(\mathcal{X}) \cong \Lambda$ for some abelian group $\Lambda$. Sending any $(y, s) : \mathrm{Spec}(k) \longrightarrow \mathcal{Y} \times S$ to the isomorphism class of

$$(y, s)^*(\mathcal{L}) \in \mathrm{Pic}(\mathcal{X})$$

we obtain a Zariski–locally constant map from the set of $k$–points in $\mathcal{Y} \times S$ to $\Lambda$. This map vanishes on $\{y_0\} \times S$, and hence it vanishes identically on $\mathcal{Y} \times S$ because $\mathcal{Y}$ is connected. This means that $u^*\mathcal{L} \in \mathrm{Pic}(\mathcal{X} \times U)$ is trivial for any atlas $u : U \longrightarrow \mathcal{Y} \times S$. Now Lemma 2.1.2(ii) completes the proof of the claim.
If moreover $x_0^*\mathcal{L}$ is trivial in $\underline{\mathrm{Pic}}(\mathcal{Y})$, then $\mathcal{L}$ is even isomorphic to the pullback of a line bundle on $S$, and hence trivial in $\underline{\mathrm{Pic}}(\mathcal{X} \times \mathcal{Y})$. This proves the injectivity of $y_0^* \oplus x_0^*$, and hence the lemma follows. $\square$

2.2. Principal $G$–bundles over a curve. We fix an irreducible smooth projective curve $C$ over the algebraically closed base field $k$. The genus of $C$ will be denoted by $g_C$. Given a linear algebraic group $G \hookrightarrow \mathrm{GL}_n$, we denote by

$$\mathcal{M}_G$$

the moduli stack of principal $G$–bundles $E$ on $C$. More precisely, $\mathcal{M}_G$ is given by the groupoid $\mathcal{M}_G(S)$ of principal $G$–bundles on $S \times C$ for every $k$–scheme $S$. The stack $\mathcal{M}_G$ is known to be algebraic over $k$ (see for example [23, Proposition 3.4], or [24, Théorème 4.6.2.1] together with [29, Lemma 4.8.1]).
Given a morphism of linear algebraic groups $\varphi : G \longrightarrow H$, the extension of the structure group by $\varphi$ defines a canonical 1–morphism

$$\varphi_* : \mathcal{M}_G \longrightarrow \mathcal{M}_H$$

which more precisely sends a principal $G$–bundle $E$ to the principal $H$–bundle

$$\varphi_* E := E \times^G H := (E \times G)/H,$$

following the convention that principal bundles carry a *right* group action. One has a canonical 2–isomorphism $(\psi \circ \varphi)_* \cong \psi_* \circ \varphi_*$ whenever $\psi : H \longrightarrow K$ is another morphism of linear algebraic groups.

Lemma 2.2.1. *Suppose that the diagram of linear algebraic groups*

$$\begin{CD} H @>{\psi_2}>> G_2 \\ @V{\psi_1}VV @VV{\varphi_2}V \\ G_1 @>>{\varphi_1}> G \end{CD}$$

*is cartesian. Then the induced 2–commutative diagram of moduli stacks*

$$\begin{CD} \mathcal{M}_H @>{(\psi_2)_*}>> \mathcal{M}_{G_2} \\ @V{(\psi_1)_*}VV @VV{(\varphi_2)_*}V \\ \mathcal{M}_{G_1} @>>{(\varphi_1)_*}> \mathcal{M}_G \end{CD}$$

*is 2–cartesian.*

*Proof.* The above 2–commutative diagram defines a 1–morphism

$$\mathcal{M}_H \longrightarrow \mathcal{M}_{G_1} \times_{\mathcal{M}_G} \mathcal{M}_{G_2}.$$

To construct an inverse, let $E$ be a principal $G$–bundle on some $k$–scheme $X$. For $\nu = 1, 2$, let $E_\nu$ be a principal $G_\nu$–bundle on $X$ together with an isomorphism $E_\nu \times^{G_\nu} G \cong E$; note that the latter defines a map $E_\nu \longrightarrow E$ of schemes over $X$. Then $G_1 \times G_2$ acts on $E_1 \times_X E_2$, and the closed subgroup $H \subseteq G_1 \times G_2$ preserves the closed subscheme

$$F := E_1 \times_E E_2 \subseteq E_1 \times_X E_2 .$$

This action turns $F$ into a principal $H$–bundle. Thus we obtain in particular a 1–morphism

$$\mathcal{M}_{G_1} \times_{\mathcal{M}_G} \mathcal{M}_{G_2} \longrightarrow \mathcal{M}_H.$$

It is easy to check that this is the required inverse. □

Let $Z$ be a closed subgroup in the center of $G$. Then the multiplication $Z \times G \longrightarrow G$ is a group homomorphism; we denote the induced 1–morphism by

$$\_ \otimes \_ : \mathcal{M}_Z \times \mathcal{M}_G \longrightarrow \mathcal{M}_G$$

and call it *tensor product*. In particular, tensoring with a principal $Z$–bundle $\xi$ on $C$ defines a 1–morphism which we denote by

$$t_\xi : \mathcal{M}_G \longrightarrow \mathcal{M}_G. \tag{1}$$

For commutative $G$, this tensor product makes $\mathcal{M}_G$ a group stack.

Suppose now that $G$ is reductive. We follow the convention that all reductive groups are smooth and connected. In particular, $\mathcal{M}_G$ is also smooth [3, 4.5.1], so its connected components and its irreducible components coincide; we denote this set of components by $\pi_0(\mathcal{M}_G)$. This set $\pi_0(\mathcal{M}_G)$ can be described as follows; cf. for example [15] or [16].

Let $\iota_G : T_G \hookrightarrow G$ be the inclusion of a maximal torus, with cocharacter group $\Lambda_{T_G} := \mathrm{Hom}(\mathbb{G}_m, T_G)$. Let $\Lambda_{\mathrm{coroots}} \subseteq \Lambda_{T_G}$ be the subgroup generated by the coroots of $G$. The Weyl group $W$ of $(G, T_G)$ acts on $\Lambda_{T_G}$. For every root $\alpha$ with corresponding coroot $\alpha^\vee$, the reflection $s_\alpha \in W$ acts on $\lambda \in \Lambda_{T_G}$ by the formula $s_\alpha(\lambda) = \lambda - \langle \alpha, \lambda \rangle \alpha^\vee$. As the $s_\alpha$ generate $W$, this implies $w(\lambda) - \lambda \in \Lambda_{\mathrm{coroots}}$ for all $w \in W$ and all $\lambda \in \Lambda_{T_G}$. Thus $W$ acts trivially on $\Lambda_{T_G}/\Lambda_{\mathrm{coroots}}$, so this quotient is, up to a *canonical* isomorphism, independent of the choice of $T_G$. We denote this quotient by $\pi_1(G)$; if $\pi_1(G)$ is trivial, then $G$ is called simply connected. For $k = \mathbb{C}$, these definitions coincide with the usual notions for the topological space $G(\mathbb{C})$.

Sending each line bundle on $C$ to its degree we define an isomorphism $\pi_0(\mathcal{M}_{\mathbb{G}_m}) \longrightarrow \mathbb{Z}$, which induces an isomorphism $\pi_0(\mathcal{M}_{T_G}) \longrightarrow \Lambda_{T_G}$. Its inverse, composed with the map

$$(\iota_G)_* : \pi_0(\mathcal{M}_{T_G}) \longrightarrow \pi_0(\mathcal{M}_G)\,,$$

is known to induce a canonical bijection

$$\pi_1(G) = \Lambda_{T_G}/\Lambda_{\mathrm{coroots}} \xrightarrow{\sim} \pi_0(\mathcal{M}_G),$$

cf. [10] and [16]. We denote by $\mathcal{M}_G^d$ the component of $\mathcal{M}_G$ given by $d \in \pi_1(G)$.

Lemma 2.2.2. *Let $\varphi : G \twoheadrightarrow H$ be an epimorphism of reductive groups over $k$ whose kernel is contained in the center of $G$. For each $d \in \pi_1(G)$, the 1–morphism*

$$\varphi_* : \mathcal{M}_G^d \longrightarrow \mathcal{M}_H^e, \qquad e := \varphi_*(d) \in \pi_1(H),$$

*is faithfully flat.*

*Proof.* Let $T_H \subseteq H$ be the image of the maximal torus $T_G \subseteq G$. Let $B_G \subseteq G$ be a Borel subgroup containing $T_G$; then

$$B_H := \varphi(B_G) \subset H$$

is a Borel subgroup of $H$ containing $T_H$. For the moment, we denote

- by $\mathcal{M}_{T_G}^d \subseteq \mathcal{M}_{T_G}$ and $\mathcal{M}_{B_G}^d \subseteq \mathcal{M}_{B_G}$ the inverse images of $\mathcal{M}_G^d \subseteq \mathcal{M}_G$, and
- by $\mathcal{M}_{T_H}^e \subseteq \mathcal{M}_{T_H}$ and $\mathcal{M}_{B_H}^e \subseteq \mathcal{M}_{B_H}$ the inverse images of $\mathcal{M}_H^e \subseteq \mathcal{M}_H$.

Let $\pi_G : B_G \twoheadrightarrow T_G$ and $\pi_H : B_H \twoheadrightarrow T_H$ denote the canonical surjections. Then

$$\mathcal{M}_{B_G}^d = (\pi_G)_*^{-1}(\mathcal{M}_{T_G}^d) \qquad \text{and} \qquad \mathcal{M}_{B_H}^e = (\pi_H)_*^{-1}(\mathcal{M}_{T_H}^e),$$

because $\pi_0(\mathcal{M}_{T_G}) = \pi_0(\mathcal{M}_{B_G})$ and $\pi_0(\mathcal{M}_{T_H}) = \pi_0(\mathcal{M}_{B_H})$ according to the proof of [10, Proposition 5]. Applying Lemma 2.2.1 to the two cartesian squares

$$\begin{array}{ccccc}
T_G & \xleftarrow{\pi_G} & B_G & \hookrightarrow & G \\
\downarrow{\scriptstyle \varphi_T} & & \downarrow{\scriptstyle \varphi_B} & & \downarrow{\scriptstyle \varphi} \\
T_H & \xleftarrow{\pi_H} & B_H & \hookrightarrow & H
\end{array}$$

of groups, we get two 2–cartesian squares

$$\begin{array}{ccccc}
\mathcal{M}_{T_G}^d & \longleftarrow & \mathcal{M}_{B_G}^d & \longrightarrow & \mathcal{M}_G^d \\
\downarrow{\scriptstyle (\varphi_T)_*} & & \downarrow{\scriptstyle (\varphi_B)_*} & & \downarrow{\scriptstyle \varphi_*} \\
\mathcal{M}_{T_H}^e & \longleftarrow & \mathcal{M}_{B_H}^e & \longrightarrow & \mathcal{M}_H^e
\end{array}$$

of moduli stacks. Since $(\varphi_T)_*$ is faithfully flat, its pullback $(\varphi_B)_*$ is so as well. This implies that $\varphi_*$ is also faithfully flat, as some open substack of $\mathcal{M}_{B_H}^e$ maps smoothly and surjectively onto $\mathcal{M}_H^e$, according to [10, Propositions 1 and 2]. □

## 3. The case of torus

This section deals with the Picard functor of the moduli stack $\mathcal{M}_G^0$ in the special case where $G = T$ is a torus. We explain in the second subsection that its description involves the character group $\mathrm{Hom}(T, \mathbb{G}_m)$ and the Picard functor of its coarse moduli scheme, which is isomorphic to a product of copies of the Jacobian $J_C$. As a preparation, the first subsection deals with the Néron–Severi group of such products of principally polarised abelian varieties. A little care is required to keep everything functorial in $T$, since this functoriality will be needed later.

3.1. On principally polarised abelian varieties. Let $A$ be an abelian variety over $k$, with dual abelian variety $A^\vee$ and Néron–Severi group

$$\mathrm{NS}(A) := \mathrm{Pic}(A)/A^\vee(k)\,.$$

For a line bundle $L$ on $A$, the standard morphism

$$\phi_L : A \longrightarrow A^\vee$$

sends $a \in A(k)$ to $\tau_a^*(L) \otimes L^{\mathrm{dual}}$ where $\tau_a : A \longrightarrow A$ is the translation by $a$. $\phi_L$ is a homomorphism by the theorem of the cube [27, §6]. Let a principal polarisation

$$\phi : A \xrightarrow{\sim} A^\vee$$

be given. Let

$$c^\phi : \mathrm{NS}(A) \longrightarrow \mathrm{End}\, A$$

be the injective homomorphism that sends the class of $L$ to $\phi^{-1} \circ \phi_L$. We denote by $\dagger : \mathrm{End}\, A \longrightarrow \mathrm{End}\, A$ the Rosati involution associated to $\phi$; so by definition, it sends $\alpha : A \longrightarrow A$ to $\alpha^\dagger := \phi^{-1} \circ \alpha^\vee \circ \phi$.

Lemma 3.1.1. *An endomorphism $\alpha \in \mathrm{End}(A)$ is in the image of $c^\phi$ if and only if $\alpha^\dagger = \alpha$.*

*Proof.* If $k = \mathbb{C}$, this is contained in [21, Chapter 5, Proposition 2.1]. For polarisations of arbitrary degree, the analogous statement about $\mathrm{End}(A) \otimes \mathbb{Q}$ is shown in [27, p. 190]; its proof carries over to the situation of this lemma as follows.
Let $l$ be a prime number, $l \neq \mathrm{char}(k)$, and let

$$e_l : T_l(A) \times T_l(A^\vee) \longrightarrow \mathbb{Z}_l(1)$$

be the standard pairing between the Tate modules of $A$ and $A^\vee$, cf. [27, §20]. According to [27, §20, Theorem 2 and §23, Theorem 3], a homomorphism $\psi : A \longrightarrow A^\vee$ is of the form $\psi = \phi_L$ for some line bundle $L$ on $A$ if and only if

$$e_l(x, \psi_* y) = -e_l(y, \psi_* x) \quad \text{for all} \quad x, y \in T_l(A)\,.$$

In particular, this holds for $\phi$. Hence the right hand side equals

$$-e_l(y, \psi_* x) = -e_l(y, \phi_* \phi_*^{-1} \psi_* x) = e_l(\phi_*^{-1} \psi_* x, \phi_* y) = e_l(x, \psi_*^\vee (\phi^{-1})_*^\vee \phi_* y)\,,$$

where the last equality follows from [27, p. 186, equation (I)]. Since the pairing $e_l$ is nondegenerate, it follows that $\psi = \phi_L$ holds for some $L$ if and only if

$$\psi_* y = \psi_*^\vee (\phi^{-1})_*^\vee \phi_* y \quad \text{for all} \quad y \in T_l(A)\,,$$

hence if and only if $\psi = \psi^\vee \circ (\phi^{-1})^\vee \circ \phi$. By definition of the Rosati involution $\dagger$, the latter is equivalent to $(\phi^{-1} \circ \psi)^\dagger = \phi^{-1} \circ \psi$. □

Let $\Lambda$ be a finitely generated free abelian group. Let $\Lambda \otimes A$ denote the abelian variety over $k$ with group of $S$–valued points $\Lambda \otimes A(S)$ for any $k$–scheme $S$.

Definition 3.1.2. The subgroup

$$\mathrm{Hom}^s(\Lambda \otimes \Lambda, \mathrm{End}\, A) \subseteq \mathrm{Hom}(\Lambda \otimes \Lambda, \mathrm{End}\, A)$$

consists of all $b : \Lambda \otimes \Lambda \longrightarrow \mathrm{End}\, A$ with $b(\lambda_1 \otimes \lambda_2)^\dagger = b(\lambda_2 \otimes \lambda_1)$ for all $\lambda_1, \lambda_2 \in \Lambda$.

Corollary 3.1.3. *There is a unique isomorphism*

$$c_\Lambda^\phi : \mathrm{NS}(\Lambda \otimes A) \xrightarrow{\sim} \mathrm{Hom}^s(\Lambda \otimes \Lambda, \mathrm{End}\, A)$$

*which sends the class of each line bundle $L$ on $\Lambda \otimes A$ to the linear map*

$$c_\Lambda^\phi(L) : \Lambda \otimes \Lambda \longrightarrow \mathrm{End}\, A$$

*defined by sending $\lambda_1 \otimes \lambda_2$ for $\lambda_1, \lambda_2 \in \Lambda$ to the composition*

$$A \xrightarrow{\lambda_1 \otimes \_} \Lambda \otimes A \xrightarrow{\phi_L} (\Lambda \otimes A)^\vee \xrightarrow{(\lambda_2 \otimes \_)^\vee} A^\vee \xrightarrow{\phi^{-1}} A.$$

*Proof.* The uniqueness is clear. For the existence, we may then choose an isomorphism $\Lambda \cong \mathbb{Z}^r$; it yields an isomorphism $\Lambda \otimes A \cong A^r$. Let

$$\phi^r = \underbrace{\phi \times \cdots \times \phi}_{r \text{ factors}} : A^r \xrightarrow{\sim} (A^\vee)^r = (A^r)^\vee$$

be the diagonal principal polarisation on $A^r$. According to Lemma 3.1.1,

$$c^{\phi^r} : \mathrm{NS}(A^r) \longrightarrow \mathrm{End}(A^r)$$

is an isomorphism onto the Rosati–invariants. Under the standard isomorphisms

$$\mathrm{End}(A^r) = \mathrm{Mat}_{r \times r}(\mathrm{End}\, A) = \mathrm{Hom}(\mathbb{Z}^r \otimes \mathbb{Z}^r, \mathrm{End}\, A),$$

the Rosati involution on $\mathrm{End}(A^r)$ corresponds to the involution $(\alpha_{ij}) \longmapsto (\alpha_{ji}^\dagger)$ on $\mathrm{Mat}_{r \times r}(\mathrm{End}\, A)$, and hence the Rosati–invariant part of $\mathrm{End}(A^r)$ corresponds to $\mathrm{Hom}^s(\mathbb{Z}^r \otimes \mathbb{Z}^r, \mathrm{End}\, A)$. Thus we obtain an isomorphism

$$\mathrm{NS}(\Lambda \otimes A) \cong \mathrm{NS}(A^r) \xrightarrow{c^{\phi^r}} \mathrm{Hom}^s(\mathbb{Z}^r \otimes \mathbb{Z}^r, \mathrm{End}\, A) \cong \mathrm{Hom}^s(\Lambda \otimes \Lambda, \mathrm{End}\, A).$$

By construction, it maps the class of each line bundle $L$ on $\Lambda \otimes A$ to the map $c_\Lambda^\phi(L) : \Lambda \otimes \Lambda \longrightarrow \mathrm{End}\, A$ prescribed above. □

3.2. Line bundles on $\mathcal{M}_T^0$. Let $T \cong \mathbb{G}_m^r$ be a torus over $k$. We will always denote by

$$\Lambda_T := \mathrm{Hom}(\mathbb{G}_m, T)$$

the cocharacter lattice. We set in the previous subsection this finitely generated free abelian group and the Jacobian variety $J_C$, endowed with the principal polarisation $\phi_\Theta : J_C \xrightarrow{\sim} J_C^\vee$ given by the autoduality of $J_C$. Recall that $\phi_\Theta$ comes from a line bundle $\mathcal{O}(\Theta)$ on $J_C$ corresponding to a theta divisor $\Theta \subseteq J_C$.

Definition 3.2.1. The finitely generated free abelian group

$$\mathrm{NS}(\mathcal{M}_T) := \mathrm{Hom}(\Lambda_T, \mathbb{Z}) \oplus \mathrm{Hom}^s(\Lambda_T \otimes \Lambda_T, \mathrm{End}\, J_C)$$

is the *Néron–Severi group* of $\mathcal{M}_T$.

For each finitely generated abelian group $\Lambda$, we denote by $\underline{\mathrm{Hom}}(\Lambda, J_C)$ the $k$–scheme of homomorphisms from $\Lambda$ to $J_C$. If $\Lambda \cong \mathbb{Z}^r \times \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_s$, then

$$\underline{\mathrm{Hom}}(\Lambda, J_C) \cong J_C^r \times J_C[n_1] \times \cdots \times J_C[n_s]$$

where $J_C[n]$ denotes the kernel of the map $J_C \longrightarrow J_C$ defined by multiplication with $n$.

Proposition 3.2.2. i) *The Picard functor* $\underline{\mathrm{Pic}}(\mathcal{M}_T^0)$ *is representable by a scheme locally of finite type over* $k$.

ii) *There is a canonical exact sequence of commutative group schemes*

$$0 \longrightarrow \underline{\mathrm{Hom}}(\Lambda_T, J_C) \xrightarrow{j_T} \underline{\mathrm{Pic}}(\mathcal{M}_T^0) \xrightarrow{c_T} \mathrm{NS}(\mathcal{M}_T) \longrightarrow 0.$$

iii) *Let* $\xi$ *be a principal* $T$*–bundle of degree* $0 \in \Lambda_T$ *on* $C$*. Then the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda_T, J_C) & \xrightarrow{j_T} & \underline{\mathrm{Pic}}(\mathcal{M}_T^0) & \xrightarrow{c_T} & \mathrm{NS}(\mathcal{M}_T) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle t_\xi^*} & & \| & & \\ 0 & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda_T, J_C) & \xrightarrow{j_T} & \underline{\mathrm{Pic}}(\mathcal{M}_T^0) & \xrightarrow{c_T} & \mathrm{NS}(\mathcal{M}_T) & \longrightarrow & 0 \end{array}$$

*commutes.*

*Proof.* Let a line bundle $\mathcal{L}$ on $\mathcal{M}_T^0$ be given. Consider the point in $\mathcal{M}_T^0$ given by a principal $T$–bundle $\xi$ on $C$ of degree $0 \in \Lambda_T$. The fiber of $\mathcal{L}_\xi$ at this point is a 1-dimensional vector space $\mathcal{L}_\xi$, endowed with a group homomorphism

$$w(\mathcal{L})_\xi : T = \mathrm{Aut}(\xi) \longrightarrow \mathrm{Aut}(\mathcal{L}_\xi) = \mathbb{G}_m$$

since $\mathcal{L}$ is a line bundle on the stack. As $\mathcal{M}_T^0$ is connected, the character $w(\mathcal{L})_\xi$ is independent of $\xi$; we denote it by

$$w(\mathcal{L}) : T \longrightarrow \mathbb{G}_m$$

and call it the *weight* $w(\mathcal{L})$ of $\mathcal{L}$. Let

$$q : \mathcal{M}_T^0 \longrightarrow \mathfrak{M}_T^0$$

be the canonical morphism to the coarse moduli scheme $\mathfrak{M}_T^0$, which is an abelian variety canonically isomorphic to $\underline{\mathrm{Hom}}(\Lambda_T, J_C)$. Line bundles of weight 0 on $\mathcal{M}_T^0$ descend to $\mathfrak{M}_T^0$, so the sequence

$$0 \longrightarrow \mathrm{Pic}(\mathfrak{M}_T^0) \xrightarrow{q^*} \mathrm{Pic}(\mathcal{M}_T^0) \xrightarrow{w} \mathrm{Hom}(\Lambda_T, \mathbb{Z})$$

is exact. This extends for families. Since $\underline{\mathrm{Pic}}(A)$ is representable for any abelian variety $A$, the proof of (i) is now complete.

Standard theory of abelian varieties and Corollary 3.1.3 together yield another short exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(\Lambda_T, J_C) \longrightarrow \underline{\mathrm{Pic}}(\mathfrak{M}_T^0) \longrightarrow \mathrm{Hom}^s(\Lambda_T \otimes \Lambda_T, \mathrm{End}\, J_C) \longrightarrow 0\,.$$

Given a character $\chi : T \longrightarrow \mathbb{G}_m$ and $p \in C(k)$, we denote by $\chi_* \mathcal{L}_p^{\mathrm{univ}}$ the line bundle on $\mathcal{M}_T^0$ that associates to each $T$–bundle $L$ on $C$ the $\mathbb{G}_m$–bundle $\chi_* L_p$.

Clearly, $\chi_* \mathcal{L}_p^{\mathrm{univ}}$ has weight $\chi$; in particular, it follows that $w$ is surjective, so we get an exact sequence of discrete abelian groups

$$0 \longrightarrow \mathrm{Hom}^s(\Lambda_T \otimes \Lambda_T, \mathrm{End}\, J_C) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{M}_T^0)/\underline{\mathrm{Hom}}(\Lambda_T, J_C) \longrightarrow \mathrm{Hom}(\Lambda_T, \mathbb{Z}) \longrightarrow 0\,.$$

Since $C$ is connected, the algebraic equivalence class of $\chi_* \mathcal{L}_p^{\mathrm{univ}}$ does not depend on the choice of $p$; sending $\chi$ to the class of $\chi_* \mathcal{L}_p^{\mathrm{univ}}$ thus defines a canonical splitting of the latter exact sequence. This proves (ii).
Finally, it is standard that $t_\xi^*$ (see (1)) is the identity map on $\underline{\mathrm{Pic}}^0(\mathfrak{M}_T^0) = \underline{\mathrm{Hom}}(\Lambda, J_C)$ (see [26, Proposition 9.2]), and $t_\xi^*$ induces the identity map on the discrete quotient $\underline{\mathrm{Pic}}(\mathcal{M}_T^0)/\underline{\mathrm{Pic}}^0(\mathfrak{M}_T^0)$ because $\xi$ can be connected to the trivial $T$–bundle in $\mathcal{M}_T^0$. □

*Remark* 3.2.3. The exact sequence in Proposition 3.2.2(ii) is functorial in $T$. More precisely, each homomorphism of tori $\varphi : T \longrightarrow T'$ induces a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda_{T'}, J_C) & \xrightarrow{j_{T'}} & \underline{\mathrm{Pic}}(\mathcal{M}_{T'}^0) & \xrightarrow{c_{T'}} & \mathrm{NS}(\mathcal{M}_{T'}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} & & \\ 0 & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda_T, J_C) & \xrightarrow{j_T} & \underline{\mathrm{Pic}}(\mathcal{M}_T^0) & \xrightarrow{c_T} & \mathrm{NS}(\mathcal{M}_T) & \longrightarrow & 0. \end{array}$$

COROLLARY 3.2.4. *Let $T_1$ and $T_2$ be tori over $k$. Then*

$$\mathrm{pr}_1^* \oplus \mathrm{pr}_2^* : \underline{\mathrm{Pic}}(\mathcal{M}_{T_1}^0) \oplus \underline{\mathrm{Pic}}(\mathcal{M}_{T_2}^0) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{M}_{T_1 \times T_2}^0)$$

*is a closed immersion of commutative group schemes over $k$.*

*Proof.* As before, let $\Lambda_{T_1}$, $\Lambda_{T_2}$ and $\Lambda_{T_1 \times T_2}$ denote the cocharacter lattices. Then

$$\mathrm{pr}_1^* \oplus \mathrm{pr}_2^* : \underline{\mathrm{Hom}}(\Lambda_{T_1}, J_C) \oplus \underline{\mathrm{Hom}}(\Lambda_{T_2}, J_C) \longrightarrow \underline{\mathrm{Hom}}(\Lambda_{T_1 \times T_2}, J_C)$$

is an isomorphism, and the homomorphism of discrete abelian groups

$$\mathrm{pr}_1^* \oplus \mathrm{pr}_2^* : \mathrm{NS}(\mathcal{M}_{T_1}) \oplus \mathrm{NS}(\mathcal{M}_{T_2}) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_1 \times T_2})$$

is injective by Definition 3.2.1. □

## 4. THE TWISTED SIMPLY CONNECTED CASE

Throughout most of this section, the reductive group $G$ over $k$ will be simply connected. Using the work of Faltings [13] on the Picard functor of $\mathcal{M}_G$, we describe here the Picard functor of the twisted moduli stacks $\mathcal{M}_{\widehat{G},L}$ introduced in [2]. In the case $G = \mathrm{SL}_n$, these are moduli stacks of vector bundles with fixed determinant; their construction in general is recalled in Subsection 4.2 below.
The result, proved in that subsection as Proposition 4.2.3, is essentially the same: for almost simple $G$, line bundles on $\mathcal{M}_{\widehat{G},L}$ are classified by an integer, their so–called central charge. The main tool for that are as usual algebraic loop groups; what we need about them is collected in Subsection 4.1.

For later use, we need to keep track of the functoriality in $G$, in particular of the pullback to a maximal torus $T_G$ in $G$. To state this more easily, we translate the central charge into a Weyl–invariant symmetric bilinear form on the cocharacter lattice of $T_G$, replacing each integer by the corresponding multiple of the basic inner product. This allows to describe the pullback to $T_G$ in Proposition 4.4.7(iii). Along the way, we also consider the pullback along representations of $G$; these just correspond to the pullback of bilinear forms, which reformulates — and generalises to arbitrary characteristic — the usual multiplication by the Dynkin index [20]. Subsection 4.3 describes these pullback maps combinatorially in terms of the root system, and Subsection 4.4 proves that these combinatorial maps actually give the pullback of line bundles on these moduli stacks.

4.1. Loop groups. Let $G$ be a reductive group over $k$. We denote

- by $LG$ the algebraic loop group of $G$, meaning the group ind–scheme over $k$ whose group of $A$–valued points for any $k$–algebra $A$ is $G(A((t)))$,
- by $L^+G \subseteq LG$ the subgroup with $A$–valued points $G(A[[t]]) \subseteq G(A((t)))$,
- and for $n \geq 1$, by $L^{\geq n}G \subseteq L^+G$ the kernel of the reduction modulo $t^n$.

Note that $L^+G$ and $L^{\geq n}G$ are affine group schemes over $k$. The $k$-algebra corresponding to $L^{\geq n}G$ is the inductive limit over all $N > n$ of the $k$–algebras corresponding to $L^{\geq n}G/L^{\geq N}$. A similar statement holds for $L^+G$.
If $X$ is anything defined over $k$, let $X_S$ denote its pullback to a $k$–scheme $S$.

Lemma 4.1.1. *Let $S$ be a reduced scheme over $k$. For $n \geq 1$, every morphism $\varphi : (L^{\geq n}G)_S \longrightarrow (\mathbb{G}_m)_S$ of group schemes over $S$ is trivial.*

*Proof.* This follows from the fact that $L^{\geq n}G$ is pro–unipotent; more precisely: As $S$ is reduced, the claim can be checked on geometric points $\mathrm{Spec}(k') \longrightarrow S$. Replacing $k$ by the larger algebraically closed field $k'$ if necessary, we may thus assume $S = \mathrm{Spec}(k)$; then $\varphi$ is a morphism $L^{\geq n}G \longrightarrow \mathbb{G}_m$.
Since the $k$–algebra corresponding to $\mathbb{G}_m$ is finitely generated, it follows that $\varphi$ factors through $L^{\geq n}G/L^{\geq N}G$ for some $N > n$. Denoting by $\mathfrak{g}$ the Lie algebra of $G$, [8, II, §4, Theorem 3.5] provides an exact sequence

$$1 \longrightarrow L^{\geq N}G \longrightarrow L^{\geq N-1}G \longrightarrow \mathfrak{g} \longrightarrow 1.$$

Thus the restriction of $\varphi$ to $L^{\geq N-1}G$ induces a character on the additive group scheme underlying $\mathfrak{g}$. Hence this restriction has to vanish, so $\varphi$ also factors through $L^{\geq n}G/L^{\geq N-1}G$. Iterating this argument shows that $\varphi$ is trivial. □

Lemma 4.1.2. *Suppose that the reductive group $G$ is simply connected, in particular semisimple. If a central extension of group schemes over $k$*

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{H} \xrightarrow{\pi} L^+G \longrightarrow 1 \tag{2}$$

*splits over $L^{\geq n}G$ for some $n \geq 1$, then it splits over $L^+G$.*

*Proof.* Let a splitting over $L^{\geq n}G$ be given, i.e. a homomorphism of group schemes $\sigma : L^{\geq n}G \longrightarrow \mathcal{H}$ such that $\pi \circ \sigma = \mathrm{id}$. Given points $h \in \mathcal{H}(S)$ and $g \in L^{\geq n}G(S)$ for some $k$-scheme $S$, the two elements

$$h \cdot \sigma(g) \cdot h^{-1} \qquad \text{and} \qquad \sigma(\pi(h) \cdot g \cdot \pi(h)^{-1})$$

in $\mathcal{H}(S)$ have the same image under $\pi$, so their difference is an element in $\mathbb{G}_m(S)$, which we denote by $\varphi_h(g)$. Sending $h$ and $g$ to $h$ and $\varphi_h(g)$ defines a morphism

$$\varphi : (L^{\geq n}G)_{\mathcal{H}} \longrightarrow (\mathbb{G}_m)_{\mathcal{H}}$$

of group schemes over $\mathcal{H}$. Since $L^+G/L^{\geq 1}G \cong G$ and $L^{\geq N-1}G/L^{\geq N} \cong \mathfrak{g}$ for $N \geq 2$ are smooth, their successive extension $L^+G/L^{\geq N}G$ is also smooth. Thus the limit $L^+G$ is reduced, so $\mathcal{H}$ is reduced as well. Using the previous lemma, it follows that $\varphi$ is the constant map 1; in other words, $\sigma$ commutes with conjugation. $\sigma$ is a closed immersion because $\pi \circ \sigma$ is, so $\sigma$ is an isomorphism onto a closed normal subgroup, and the quotient is a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{H}/\sigma(L^{\geq n}G) \longrightarrow L^+G/L^{\geq n}G \longrightarrow 1.$$

If $n \geq 2$, then this restricts to a central extension of $L^{\geq n-1}G/L^{\geq n}G \cong \mathfrak{g}$ by $\mathbb{G}_m$. It can be shown that any such extension splits.
(Indeed, the unipotent radical of the extension projects isomorphically to the quotient $\mathfrak{g}$. Note that the unipotent radical does not intersect the subgroup $\mathbb{G}_m$, and the quotient by the subgroup generated by the unipotent radical and $\mathbb{G}_m$ is reductive, so this this reductive quotient being a quotient of $\mathfrak{g}$ is in fact trivial.)
Therefore, the image of a section $\mathfrak{g} \longrightarrow \mathcal{H}/\sigma(L^{\geq n}G)$ has an inverse image in $\mathcal{H}$ which $\pi$ maps isomorphically onto $L^{\geq n-1}G \subseteq L^+G$. Hence the given central extension (2) splits over $L^{\geq n-1}G$ as well. Repeating this argument, we get a splitting over $L^{\geq 1}G$, and finally also over $L^+G$, because every central extension of $L^+G/L^{\geq 1}G \cong G$ by $\mathbb{G}_m$ splits as well, $G$ being simply connected.
(To prove the last assertion, for any extension $\widetilde{G}$ of $G$ by $\mathbb{G}_m$, consider the commutator subgroup $[\widetilde{G}, \widetilde{G}]$ of $\widetilde{G}$. It projects surjectively to the commutator subgroup of $G$ which is $G$ itself. Since $[\widetilde{G}, \widetilde{G}]$ is connected and reduced, and $G$ is simply connected, this surjective morphism must be an isomorphism.) □

4.2. Descent from the affine Grassmannian. Let $G$ be a reductive group over $k$. We denote by $\mathrm{Gr}_G$ the affine Grassmannian of $G$, i.e. the quotient $LG/L^+G$ in the category of fppf–sheaves. Let $\widehat{\mathcal{O}}_{C,p}$ denote the completion of the local ring $\mathcal{O}_{C,p}$ of the scheme $C$ in a point $p \in C(k)$. Given a uniformising element $z \in \widehat{\mathcal{O}}_{C,p}$, there is a standard 1–morphism

$$\mathrm{glue}_{p,z} : \mathrm{Gr}_G \longrightarrow \mathcal{M}_G$$

that sends each coset $f \cdot L^+G$ to the trivial $G$–bundles over $C \backslash \{p\}$ and over $\widehat{\mathcal{O}}_{C,p}$, glued by the automorphism $f(z)$ of the trivial $G$–bundle over the intersection; cf. for example [23, Section 3], [13, Corollary 16], or [14, Proposition 3].

For the rest of this subsection, we assume that $G$ is simply connected, hence semisimple. In this case, $\mathrm{Gr}_G$ is known to be an ind–scheme over $k$. More precisely, [13, Theorem 8] implies that $\mathrm{Gr}_G$ is an inductive limit of projective Schubert varieties over $k$, which are reduced and irreducible. Thus the canonical map

$$\mathrm{pr}_2^* : \Gamma(S, \mathcal{O}_S) \longrightarrow \Gamma(\mathrm{Gr}_G \times S, \mathcal{O}_{\mathrm{Gr}_G \times S}) \tag{3}$$

is an isomorphism for every scheme $S$ of finite type over $k$.
Define the Picard functor $\underline{\mathrm{Pic}}(\mathrm{Gr}_G)$ from schemes of finite type over $k$ to abelian groups as in definition 2.1.1. The following theorem about it is proved in full generality in [13]. Over $k = \mathbb{C}$, the group $\mathrm{Pic}(\mathrm{Gr}_G)$ is also determined in [25] as well as in [20], and $\mathrm{Pic}(\mathcal{M}_G)$ is determined in [23] together with [30].

THEOREM 4.2.1 (Faltings). *Let $G$ be simply connected and almost simple.*

i) $\underline{\mathrm{Pic}}(\mathrm{Gr}_G) \cong \mathbb{Z}$.
ii) $\mathrm{glue}_{p,z}^* : \underline{\mathrm{Pic}}(\mathcal{M}_G) \longrightarrow \underline{\mathrm{Pic}}(\mathrm{Gr}_G)$ *is an isomorphism of functors.*

The purpose of this subsection is to carry part (ii) over to twisted moduli stacks in the sense of [2]; cf. also the first remark on page 67 of [13]. More precisely, let an exact sequence of reductive groups

$$1 \longrightarrow G \longrightarrow \widehat{G} \xrightarrow{\mathrm{dt}} \mathbb{G}_m \longrightarrow 1 \tag{4}$$

be given, and a line bundle $L$ on $C$. We denote by $\mathcal{M}_{\widehat{G},L}$ the moduli stack of principal $\widehat{G}$–bundles $E$ on $C$ together with an isomorphism $\mathrm{dt}_* E \cong L$; cf. section 2 of [2]. If for example the given exact sequence is

$$1 \longrightarrow \mathrm{SL}_n \longrightarrow \mathrm{GL}_n \xrightarrow{\det} \mathbb{G}_m \longrightarrow 1,$$

then $\mathcal{M}_{\mathrm{GL}_n,L}$ is the moduli stack of vector bundles with fixed determinant $L$. In general, the stack $\mathcal{M}_{\widehat{G},L}$ comes with a 2–cartesian diagram

$$\begin{array}{ccc} \mathcal{M}_{\widehat{G},L} & \longrightarrow & \mathcal{M}_{\widehat{G}} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{dt}_*} \\ \mathrm{Spec}(k) & \xrightarrow{L} & \mathcal{M}_{\mathbb{G}_m} \end{array}$$

from which we see in particular that $\mathcal{M}_{\widehat{G},L}$ is algebraic. It satisfies the following variant of the Drinfeld–Simpson uniformisation theorem [10, Theorem 3].

LEMMA 4.2.2. *Let a point $p \in C(k)$ and a principal $\widehat{G}$–bundle $\mathcal{E}$ on $C \times S$ for some $k$–scheme $S$ be given. Every trivialisation of the line bundle $\mathrm{dt}_* \mathcal{E}$ over $(C \setminus \{p\}) \times S$ can étale–locally in $S$ be lifted to a trivialisation of $\mathcal{E}$ over $(C \setminus \{p\}) \times S$.*

*Proof.* The proof in [10] carries over to this situation as follows. Choose a maximal torus $T_{\widehat{G}} \subseteq \widehat{G}$. Using [10, Theorem 1], we may assume that $\mathcal{E}$ comes from a principal $T_{\widehat{G}}$–bundle; cf. the first paragraph in the proof of [10, Theorem 3]. Arguing as in the third paragraph of that proof, we may change this principal

$T_{\widehat{G}}$–bundle by the extension of $\mathbb{G}_m$–bundles along coroots $\mathbb{G}_m \longrightarrow T_{\widehat{G}}$. Since simple coroots freely generate the kernel $T_G$ of $T_{\widehat{G}} \twoheadrightarrow \mathbb{G}_m$, we can thus achieve that this $T_{\widehat{G}}$–bundle is trivial over $(C \setminus \{p\}) \times S$. Because $\mathbb{G}_m$ is a direct factor of $T_{\widehat{G}}$, we can hence lift the given trivialisation to the $T_{\widehat{G}}$–bundle, and hence also to $\mathcal{E}$. □

Let $d \in \mathbb{Z}$ be the degree of $L$. Since dt in (4) maps the (reduced) identity component $Z^0 \cong \mathbb{G}_m$ of the center in $\widehat{G}$ surjectively onto $\mathbb{G}_m$, there is a $Z^0$–bundle $\xi$ (of degree 0) on $C$ with $\mathrm{dt}_*(\xi) \otimes \mathcal{O}_C(dp) \cong L$; tensoring with it defines an equivalence

$$t_\xi : \mathcal{M}_{\widehat{G},\mathcal{O}_C(dp)} \xrightarrow{\sim} \mathcal{M}_{\widehat{G},L}.$$

Choose a homomorphism $\delta : \mathbb{G}_m \longrightarrow \widehat{G}$ with $\mathrm{dt} \circ \delta = d \in \mathbb{Z} = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$. We denote by $t^\delta \in L\widehat{G}(k)$ the image of the tautological loop $t \in L\mathbb{G}_m(k)$ under $\delta_* : L\mathbb{G}_m \longrightarrow L\widehat{G}$. The map

$$t^\delta \cdot \_ : \mathrm{Gr}_G \longrightarrow \mathrm{Gr}_{\widehat{G}}$$

sends, for each point $f$ in $LG$, the coset $f \cdot L^+G$ to the coset $t^\delta f \cdot L^+\widehat{G}$. Its composition $\mathrm{Gr}_G \longrightarrow \mathcal{M}_{\widehat{G}}$ with $\mathrm{glue}_{p,z}$ factors naturally through a 1–morphism

$$\mathrm{glue}_{p,z,\delta} : \mathrm{Gr}_G \longrightarrow \mathcal{M}_{\widehat{G},\mathcal{O}_C(dp)},$$

because $\mathrm{dt}_* \circ (t^\delta \cdot \_) : LG \longrightarrow L\widehat{G} \longrightarrow L\mathbb{G}_m$ is the constant map $t^d$, which via gluing yields the line bundle $\mathcal{O}_C(dp)$. Lemma 4.2.2 provides local sections of $\mathrm{glue}_{p,z,\delta}$. These show in particular that

$$\mathrm{glue}^*_{p,z,\delta} : \Gamma(\mathcal{M}_{\widehat{G},\mathcal{O}_C(dp)}, \mathcal{O}_{\mathcal{M}_{\widehat{G},\mathcal{O}_C(dp)}}) \longrightarrow \Gamma(\mathrm{Gr}_G, \mathcal{O}_{\mathrm{Gr}_G})$$

is injective. Hence both spaces of sections contain only the constants, since $\Gamma(\mathrm{Gr}_G, \mathcal{O}_{\mathrm{Gr}_G}) = k$ by equation (3). Using the above equivalence $t_\xi$, this implies

$$\Gamma(\mathcal{M}_{\widehat{G},L}, \mathcal{O}_{\mathcal{M}_{\widehat{G},L}}) = k\,. \tag{5}$$

Proposition 4.2.3. *Let $G$ be simply connected and almost simple. Then*

$$\mathrm{glue}^*_{p,z,\delta} : \underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},\mathcal{O}_C(dp)}) \longrightarrow \underline{\mathrm{Pic}}(\mathrm{Gr}_G)$$

*is an isomorphism of functors.*

*Proof.* $LG$ acts on $\mathrm{Gr}_G$ by multiplication from the left. Embedding the $k$–algebra $\mathcal{O}_{C\setminus p} := \Gamma(C \setminus \{p\}, \mathcal{O}_C)$ into $k((t))$ via the Laurent development at $p$ in the variable $t = z$, we denote by $L_{C\setminus p}G \subseteq LG$ the subgroup with $A$–valued points $G(A \otimes_k \mathcal{O}_{C\setminus p}) \subseteq G(A((t)))$ for any $k$–algebra $A$. Consider the stack quotient $L_{C\setminus p}G\backslash \mathrm{Gr}_G$. The map $\mathrm{glue}_{p,z}$ descends to an equivalence

$$L_{C\setminus p}G\backslash \mathrm{Gr}_G \xrightarrow{\sim} \mathcal{M}_G$$

because the action of $L_{C\setminus p}G$ on $\mathrm{Gr}_G$ corresponds to changing trivialisations over $C \setminus \{p\}$; cf. for example [23, Theorem 1.3] or [13, Corollary 16].

More generally, consider the conjugate

$$L^\delta_{C\setminus p}G := t^{-\delta} \cdot L_{C\setminus p}G \cdot t^\delta \subseteq L\widehat{G},$$

which is actually contained in $LG$ since $LG$ is normal in $L\widehat{G}$. Using Lemma 4.2.2, we see that the map $\mathrm{glue}_{p,z,\delta}$ descends to an equivalence

$$L^{\delta}_{C\setminus p}G\backslash \mathrm{Gr}_G \xrightarrow{\sim} \mathcal{M}_{\widehat{G},\mathcal{O}_C(dp)},$$

because the action of $L^{\delta}_{C\setminus p}G$ on $\mathrm{Gr}_G$ again corresponds to changing trivialisations over $C \setminus \{p\}$.

Let $S$ be a scheme of finite type over $k$. Each line bundle on $S \times \mathcal{M}_{\widehat{G},L}$ with trivial pullback to $S \times \mathrm{Gr}_G$ comes from a character $(L^{\delta}_{C\setminus p}G)_S \longrightarrow (\mathbb{G}_m)_S$, since the map (3) is bijective. But $L^{\delta}_{C\setminus p}G$ is isomorphic to $L_{C\setminus p}G$, and every character $(L_{C\setminus p}G)_S \longrightarrow (\mathbb{G}_m)_S$ is trivial according to [13, p. 66f.]. This already shows that the morphism of Picard functors $\mathrm{glue}^*_{p,z,\delta}$ is injective.

The action of $LG$ on $\mathrm{Gr}_G$ induces the trivial action on $\underline{\mathrm{Pic}}(\mathrm{Gr}_G) \cong \mathbb{Z}$, for example because it preserves ampleness, or alternatively because $LG$ is connected. Let a line bundle $\mathcal{L}$ on $\mathrm{Gr}_G$ be given. We denote by $\mathrm{Mum}_{LG}(\mathcal{L})$ the Mumford group. So $\mathrm{Mum}_{LG}(\mathcal{L})$ is the functor from schemes of finite type over $k$ to groups that sends $S$ to the group of pairs $(f, g)$ consisting of an element $f \in LG(S)$ and an isomorphism $g : f^*\mathcal{L}_S \xrightarrow{\sim} \mathcal{L}_S$ of line bundles on $\mathrm{Gr}_G \times S$. If $f = 1$, then $g \in \mathbb{G}_m(S)$ due to the bijectivity of (3), while for arbitrary $f \in LG(S)$, the line bundles $\mathcal{L}_S$ and $f^*\mathcal{L}_S$ have the same image in $\underline{\mathrm{Pic}}(\mathrm{Gr}_G)(S)$, implying that $\mathcal{L}_S$ and $f^*\mathcal{L}_S$ are Zariski–locally in $S$ isomorphic. Consequently, we have a short exact sequence of sheaves in the Zariski topology

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Mum}_{LG}(\mathcal{L}) \xrightarrow{q} LG \longrightarrow 1\,. \tag{6}$$

This central extension splits over $L^+G \subseteq LG$, because the restricted action of $L^+G$ on $\mathrm{Gr}_G$ has a fixed point. We have to show that it also splits over $L^{\delta}_{C\setminus p}G \subseteq LG$.

Note that $L^{\delta}_{C\setminus p}G = \gamma(L_{C\setminus p}G)$ for the automorphism $\gamma$ of $LG$ given by conjugation with $t^{\delta}$. Hence it is equivalent to show that the central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Mum}_{LG}(\mathcal{L}) \xrightarrow{\gamma^{-1}\circ q} LG \longrightarrow 1 \tag{7}$$

splits over $L_{C\setminus p}G$. We know already that it splits over $\gamma^{-1}(L^+G)$, in particular over $L^{\geq n}G$ for some $n \geq 1$. Thus it also splits over $L^+G$, due to Lemma 4.1.2. Hence it comes from a line bundle on $LG/L^+G = \mathrm{Gr}_G$ (whose associated $\mathbb{G}_m$–bundle has total space $\mathrm{Mum}_{LG}(\mathcal{L})/L^+G$, where $L^+G$ acts from the right via the splitting). According to Theorem 4.2.1(ii), this line bundle admits a $L_{C\setminus p}G$–linearisation, and hence the extension (7) splits indeed over $L_{C\setminus p}G$.

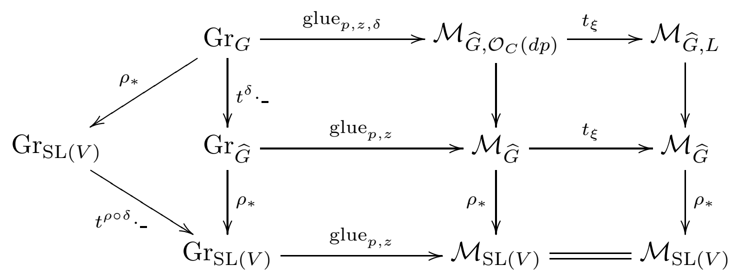
Thus the extension (6) splits over $L^{\delta}_{C\setminus p}G$, so $\mathcal{L}$ admits an $L^{\delta}_{C\setminus p}G$–linearisation and consequently descends to $\mathcal{M}_{\widehat{G},\mathcal{O}_C(dp)}$. This proves that $\mathrm{glue}^*_{p,z,\delta}$ is surjective as a homomorphism of Picard groups. Hence it is also surjective as a morphism of Picard functors, because $\underline{\mathrm{Pic}}(\mathrm{Gr}_G) \cong \mathbb{Z}$ is discrete by Theorem 4.2.1(i). □

*Remark* 4.2.4. Put $G^{\mathrm{ad}} := G/Z$, where $Z \subseteq G$ denotes the center. Given a representation $\rho : G^{\mathrm{ad}} \longrightarrow \mathrm{SL}(V)$, we denote its compositions with the canonical epimorphisms $G \twoheadrightarrow G^{\mathrm{ad}}$ and $\widehat{G} \twoheadrightarrow G^{\mathrm{ad}}$ also by $\rho$. Then the diagram

$$\begin{array}{ccc} \underline{\mathrm{Pic}}(\mathcal{M}_{\mathrm{SL}(V)}) & \xrightarrow{\mathrm{glue}^*_{p,z}} & \underline{\mathrm{Pic}}(\mathrm{Gr}_{\mathrm{SL}(V)}) \\ \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \rho^*} \\ \underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},L}) & \xrightarrow{(t_\xi \circ \mathrm{glue}_{p,z,\delta})^*} & \underline{\mathrm{Pic}}(\mathrm{Gr}_G) \end{array}$$

commutes.

*Proof.* Let $t^{\rho\circ\delta} \in L\,\mathrm{SL}(V)$ denote the image of the canonical loop $t \in L\mathbb{G}_m$ under the composition $\rho\circ\delta : \mathbb{G}_m \longrightarrow \mathrm{SL}(V)$. Then the left part of the diagram




commutes. The four remaining squares are 2–commutative by construction of the 1–morphisms $\mathrm{glue}_{p,z,\delta}$, $\mathrm{glue}_{p,z}$ and $t_\xi$. Applying $\underline{\mathrm{Pic}}$ to the exterior pentagon yields the required commutative square, as $L\,\mathrm{SL}(V)$ acts trivially on $\underline{\mathrm{Pic}}(\mathrm{Gr}_{\mathrm{SL}(V)})$. □

4.3. Néron–Severi groups $\mathrm{NS}(\mathcal{M}_G)$ for simply connected $G$. Let $G$ be a reductive group over $k$; later in this subsection, we will assume that $G$ is simply connected. Choose a maximal torus $T_G \subseteq G$, and let

$$\mathrm{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^W \tag{8}$$

denote the abelian group of bilinear forms $b : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathbb{Z}$ that are invariant under the Weyl group $W = W_G$ of $(G, T_G)$.

Up to a *canonical* isomorphism, the group (8) does not depend on the choice of $T_G$. More precisely, let $T'_G \subseteq G$ be another maximal torus; then the conjugation $\gamma_g : G \longrightarrow G$ with some $g \in G(k)$ provides an isomorphism from $T_G$ to $T'_G$, and the induced isomorphism from $\mathrm{Hom}(\Lambda_{T'_G} \otimes \Lambda_{T'_G}, \mathbb{Z})^W$ to $\mathrm{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^W$ does not depend on the choice of $g$.

The group (8) is also functorial in $G$. More precisely, let $\varphi : G \longrightarrow H$ be a homomorphism of reductive groups over $k$. Choose a maximal torus $T_H \subseteq H$ containing $\varphi(T_G)$.

Lemma 4.3.1. *Let $T'_G \subseteq G$ be another maximal torus, and let $T'_H \subseteq H$ be a maximal torus containing $\varphi(T'_G)$. For every $g \in G(k)$ with $T'_G = \gamma_g(T_G)$, there* *is an $h \in H(k)$ with $T'_H = \gamma_h(T_H)$ such that the following diagram commutes:*

$$\begin{array}{ccc} T_G & \xrightarrow{\gamma_g} & T'_G \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ T_H & \xrightarrow{\gamma_h} & T'_H \end{array}$$

*Proof.* The diagram

$$\begin{array}{ccccc} T_G & = & T_G & \xrightarrow{\gamma_g} & T'_G \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\ T_H & \dashrightarrow & \gamma_{\varphi(g)}^{-1}(T'_H) & \xrightarrow{\gamma_{\varphi(g)}} & T'_H \end{array}$$

allows us to assume $T'_G = T_G$ and $g = 1$ without loss of generality. Then $T_H$ and $T'_H$ are maximal tori in the centraliser of $\varphi(T_G)$, which is reductive according to [17, 26.2. Corollary A]. Thus $T'_H = \gamma_h(T_H)$ for an appropriate $k$–point $h$ of this centraliser, and $\gamma_h \circ \varphi = \varphi$ on $T_G$ by definition of the centraliser. □

Applying the lemma with $T'_G = T_G$ and $T'_H = T_H$, we see that the pullback along $\varphi_* : \Lambda_{T_G} \longrightarrow \Lambda_{T_H}$ of a $W_H$–invariant form $\Lambda_{T_H} \otimes \Lambda_{T_H} \longrightarrow \mathbb{Z}$ is $W_G$–invariant, so we get an induced map

(9) $$\varphi^* : \mathrm{Hom}(\Lambda_{T_H} \otimes \Lambda_{T_H}, \mathbb{Z})^{W_H} \longrightarrow \mathrm{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^{W_G}$$

which does not depend on the choice of $T_G$ and $T_H$ by the above lemma again. *For the rest of this subsection, we assume that $G$ and $H$ are simply connected.*

Definition 4.3.2. i) The *Néron–Severi group* $\mathrm{NS}(\mathcal{M}_G)$ is the subgroup

$$\mathrm{NS}(\mathcal{M}_G) \subseteq \mathrm{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^W$$

of symmetric forms $b : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathbb{Z}$ with $b(\lambda \otimes \lambda) \in 2\mathbb{Z}$ for all $\lambda \in \Lambda_{T_G}$.

ii) Given a homomorphism $\varphi : G \longrightarrow H$, we denote by

$$\varphi^* : \mathrm{NS}(\mathcal{M}_H) \longrightarrow \mathrm{NS}(\mathcal{M}_G)$$

the restriction of the induced map $\varphi^*$ in (9).

*Remarks* 4.3.3. i) If $G = G_1 \times G_2$ for simply connected groups $G_1$ and $G_2$, then

$$\mathrm{NS}(\mathcal{M}_G) = \mathrm{NS}(\mathcal{M}_{G_1}) \oplus \mathrm{NS}(\mathcal{M}_{G_2}),$$

since each element of $\mathrm{Hom}(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathbb{Z})^{W_G}$ vanishes on $\Lambda_{T_{G_1}} \otimes \Lambda_{T_{G_2}} + \Lambda_{T_{G_2}} \otimes \Lambda_{T_{G_1}}$.

ii) If on the other hand $G$ is almost simple, then

$$\mathrm{NS}(\mathcal{M}_G) = \mathbb{Z} \cdot b_G$$

where the *basic inner product* $b_G$ is the unique element of $\mathrm{NS}(\mathcal{M}_G)$ that satisfies $b_G(\alpha^\vee, \alpha^\vee) = 2$ for all short coroots $\alpha^\vee \in \Lambda_{T_G}$ of $G$.

iii) Let $G$ and $H$ be almost simple. The *Dynkin index* $d_\varphi \in \mathbb{Z}$ of a homomorphism $\varphi : G \longrightarrow H$ is defined by $\varphi^*(b_H) = d_\varphi \cdot b_G$, cf. [11, §2]. If $\varphi$ is nontrivial, then $d_\varphi > 0$, since $b_G$ and $b_H$ are positive definite.

Let $Z \subseteq G$ be the center. Then $G^{\mathrm{ad}} := G/Z$ contains $T_{G^{\mathrm{ad}}} := T_G/Z$ as a maximal torus, with cocharacter lattice $\Lambda_{T_{G^{\mathrm{ad}}}} \subseteq \Lambda_{T_G} \otimes \mathbb{Q}$.
We say that a homomorphism $l : \Lambda \longrightarrow \Lambda'$ between finitely generated free abelian groups $\Lambda$ and $\Lambda'$ is *integral* on a subgroup $\widetilde{\Lambda} \subseteq \Lambda \otimes \mathbb{Q}$ if its restriction to $\Lambda \cap \widetilde{\Lambda}$ admits a linear extension $\widetilde{l} : \widetilde{\Lambda} \longrightarrow \Lambda'$. By abuse of language, we will not distinguish between $l$ and its unique linear extension $\widetilde{l}$.

Lemma 4.3.4. *Every element $b : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathbb{Z}$ of $\mathrm{NS}(\mathcal{M}_G)$ is integral on $\Lambda_{T_{G^{ad}}} \otimes \Lambda_{T_G}$ and on $\Lambda_{T_G} \otimes \Lambda_{T_{G^{ad}}}$.*

*Proof.* Let $\alpha : \Lambda_{T_G} \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$ be a root of $G$, with corresponding coroot $\alpha^\vee \in \Lambda_{T_G}$. Lemme 2 in [5, Chapitre VI, §1] implies the formula

$$b(\lambda \otimes \alpha^\vee) = \alpha(\lambda) \cdot b(\alpha^\vee \otimes \alpha^\vee)/2$$

for all $\lambda \in \Lambda_{T_G}$. Thus $b(\_ \otimes \alpha^\vee) : \Lambda_{T_G} \longrightarrow \mathbb{Z}$ is an integer multiple of $\alpha$; hence it is integral on $\Lambda_{T_{G^{\mathrm{ad}}}}$, the largest subgroup of $\Lambda_{T_G} \otimes \mathbb{Q}$ on which all roots are integral. But the coroots $\alpha^\vee$ generate $\Lambda_{T_G}$, as $G$ is simply connected. □

Now let $\iota_G : T_G \hookrightarrow G$ denote the inclusion of the chosen maximal torus.

Definition 4.3.5. Given $\delta \in \Lambda_{T_{G^{\mathrm{ad}}}}$, the homomorphism

$$(\iota_G)^{\mathrm{NS},\delta} : \mathrm{NS}(\mathcal{M}_G) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_G})$$

sends $b : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathbb{Z}$ to

$$b(-\delta \otimes \_) : \Lambda_{T_G} \longrightarrow \mathbb{Z} \qquad \text{and} \qquad \mathrm{id}_{J_C} \cdot b : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathrm{End}\, J_C.$$

This map $(\iota_G)^{\mathrm{NS},\delta}$ is injective if $g_C \geq 1$, because all multiples of $\mathrm{id}_{J_C}$ are then nonzero in $\mathrm{End}\, J_C$. If $g_C = 0$, then $\mathrm{End}\, J_C = 0$, but we still have the following

Lemma 4.3.6. *Every coset $d \in \Lambda_{T_{G^{ad}}}/\Lambda_{T_G} = \pi_1(G^{ad})$ admits a lift $\delta \in \Lambda_{T_{G^{ad}}}$ such that the map $(\iota_G)^{\mathrm{NS},\delta} : \mathrm{NS}(\mathcal{M}_G) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_G})$ is injective.*

*Proof.* Using Remark 4.3.3, we may assume that $G$ is almost simple. In this case, $(\iota_G)^{\mathrm{NS},\delta}$ is injective whenever $\delta \neq 0$, because $\mathrm{NS}(\mathcal{M}_G)$ is cyclic and its generator $b_G : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathbb{Z}$ is as a bilinear form nondegenerate. □

*Remark* 4.3.7. Given $\varphi : G \longrightarrow H$, let $\iota_H : T_H \hookrightarrow H$ be a maximal torus with $\varphi(T_G) \subseteq T_H$. If $\delta \in \Lambda_{T_G}$, or if more generally $\delta \in \Lambda_{T_{G^{\mathrm{ad}}}}$ is mapped to $\Lambda_{H^{\mathrm{ad}}}$ by $\varphi_* : \Lambda_{T_G} \otimes \mathbb{Q} \longrightarrow \Lambda_{T_H} \otimes \mathbb{Q}$, then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{NS}(\mathcal{M}_H) & \xrightarrow{(\iota_H)^{\mathrm{NS},\varphi_*\delta}} & \mathrm{NS}(\mathcal{M}_{T_H}) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{NS}(\mathcal{M}_G) & \xrightarrow{(\iota_G)^{\mathrm{NS},\delta}} & \mathrm{NS}(\mathcal{M}_{T_G}) \end{array}$$

4.4. The pullback to torus bundles. Let $\mathcal{L}^{\det} = \mathcal{L}_n^{\det}$ be determinant of cohomology line bundle [18] on $\mathcal{M}_{\mathrm{GL}_n}$, whose fibre at a vector bundle $E$ on $C$ is $\det \mathrm{H}^*(E) = \det \mathrm{H}^0(E) \otimes \det \mathrm{H}^1(E)^{\mathrm{dual}}$.

Lemma 4.4.1. *Let $\xi$ be a line bundle of degree $d$ on $C$. Then the composition*

$$\mathrm{Pic}(\mathcal{M}_{\mathbb{G}_m}) \xrightarrow{t_\xi^*} \mathrm{Pic}(\mathcal{M}_{\mathbb{G}_m}^0) \xrightarrow{c_{\mathbb{G}_m}} \mathrm{NS}(\mathcal{M}_{\mathbb{G}_m}) = \mathbb{Z} \oplus \mathrm{End}_{J_C}$$

*maps $\mathcal{L}^{\det}$ to $1 - g_C + d \in \mathbb{Z}$ and $-\mathrm{id}_{J_C} \in \mathrm{End}_{J_C}$.*

*Proof.* For any line bundle $L$ on $C$ and any point $p \in C(k)$, we have a canonical exact sequence

$$0 \longrightarrow L(-p) \longrightarrow L \longrightarrow L_p \longrightarrow 0$$

of coherent sheaves on $C$. Varying $L$ and taking the determinant of cohomology, we see that the two line bundles $\mathcal{L}^{\det}$ and $t_{\mathcal{O}(-p)}^*\mathcal{L}^{\det}$ on $\mathcal{M}_{\mathbb{G}_m}^0$ have the same image in the second summand $\mathrm{End}\, J_C$ of $\mathrm{NS}(\mathcal{M}_{\mathbb{G}_m})$. Thus the image of $t_\xi^*\mathcal{L}^{\det}$ in $\mathrm{End}\, J_C$ does not depend on $\xi$; this image is $-\mathrm{id}_{J_C}$ because the principal polarisation $\phi_\Theta : J_C \longrightarrow J_C^\vee$ is essentially given by the dual of the line bundle $\mathcal{L}^{\det}$.
The weight of $t_\xi^*\mathcal{L}^{\det}$ at a line bundle $L$ of degree 0 on $C$ is the Euler characteristic of $L \otimes \xi$, which is indeed $1 - g_C + d$ by Riemann–Roch theorem. □

Let $\iota : T_{\mathrm{SL}_n} \hookrightarrow \mathrm{SL}_n$ be the inclusion of the maximal torus $T_{\mathrm{SL}_n} := \mathbb{G}_m^n \cap \mathrm{SL}_n$, where $\mathbb{G}_m^n \subseteq \mathrm{GL}_n$ as diagonal matrices. Then the cocharacter lattice $\Lambda_{T_{\mathrm{SL}_n}}$ is the group of all $d = (d_1, \dots, d_n) \in \mathbb{Z}^n$ with $d_1 + \dots + d_n = 0$. The basic inner product $b_{\mathrm{SL}_n} : \Lambda_{T_{\mathrm{SL}_n}} \otimes \Lambda_{T_{\mathrm{SL}_n}} \longrightarrow \mathbb{Z}$ is the restriction of the standard scalar product on $\mathbb{Z}^n$.

Corollary 4.4.2. *Let $\xi$ be a principal $T_{\mathrm{SL}_n}$–bundle of degree $d \in \Lambda_{T_{\mathrm{SL}_n}}$ on $C$. Then the composition*

$$\mathrm{Pic}(\mathcal{M}_{\mathrm{SL}_n}) \xrightarrow{\iota^*} \mathrm{Pic}(\mathcal{M}_{T_{\mathrm{SL}_n}}) \xrightarrow{t_\xi^*} \mathrm{Pic}(\mathcal{M}_{T_{\mathrm{SL}_n}}^0) \xrightarrow{c_{T_{\mathrm{SL}_n}}} \mathrm{NS}(\mathcal{M}_{T_{\mathrm{SL}_n}})$$

*maps $\mathcal{L}^{\det}$ to $b_{\mathrm{SL}_n}(d \otimes \_) : \Lambda_{T_{\mathrm{SL}_n}} \longrightarrow \mathbb{Z}$ and $-\mathrm{id}_{J_C} \cdot b_{\mathrm{SL}_n} : \Lambda_{T_{\mathrm{SL}_n}} \otimes \Lambda_{T_{\mathrm{SL}_n}} \longrightarrow \mathrm{End}\, J_C$.*

*Proof.* Since the determinant of cohomology takes direct sums to tensor products, the pullback of $\mathcal{L}_n^{\det}$ to $\mathcal{M}_{\mathbb{G}_m^n}$ is isomorphic to $\mathrm{pr}_1^* \mathcal{L}_1^{\det} \otimes \dots \otimes \mathrm{pr}_n^* \mathcal{L}_1^{\det}$, where $\mathrm{pr}_\nu : \mathbb{G}_m^n \twoheadrightarrow \mathbb{G}_m$ is the projection onto the $\nu$th factor. Now use the previous lemma to compute the image of $\mathcal{L}_n^{\det}$ in $\mathrm{NS}(\mathcal{M}_{\mathbb{G}_m^n})$ and then restrict to $\mathrm{NS}(\mathcal{M}_{T_{\mathrm{SL}_n}})$. □

Corollary 4.4.3. *If $\rho : \mathrm{SL}_2 \longrightarrow \mathrm{SL}(V)$ has Dynkin index $d_\rho$, then the pullback $\rho^* : \mathrm{Pic}(\mathcal{M}_{\mathrm{SL}(V)}) \longrightarrow \mathrm{Pic}(\mathcal{M}_{\mathrm{SL}_2})$ maps $\mathcal{L}^{\det}$ to $(\mathcal{L}_2^{\det})^{\otimes d_\rho}$.*

*Proof.* Let $\iota : T_{\mathrm{SL}(V)} \hookrightarrow \mathrm{SL}(V)$ be the inclusion of a maximal torus that contains the image of the standard torus $T_{\mathrm{SL}_2} \hookrightarrow \mathrm{SL}_2$. The diagram

$$\begin{array}{ccccccc}
\mathrm{Pic}(\mathcal{M}_{\mathrm{SL}(V)}) & \xrightarrow{\iota^*} & \mathrm{Pic}(\mathcal{M}_{T_{\mathrm{SL}(V)}}) & \xrightarrow{t^*_{\rho_*(\xi)}} & \mathrm{Pic}(\mathcal{M}^0_{T_{\mathrm{SL}(V)}}) & \xrightarrow{c_{T_{\mathrm{SL}(V)}}} & \mathrm{NS}(\mathcal{M}_{T_{\mathrm{SL}(V)}}) \\
\downarrow{\rho^*} & & \downarrow{\rho^*} & & \downarrow{\rho^*} & & \downarrow{\rho^*} \\
\mathrm{Pic}(\mathcal{M}_{\mathrm{SL}_2}) & \xrightarrow{\iota^*} & \mathrm{Pic}(\mathcal{M}_{T_{\mathrm{SL}_2}}) & \xrightarrow{t^*_{\xi}} & \mathrm{Pic}(\mathcal{M}^0_{T_{\mathrm{SL}_2}}) & \xrightarrow{c_{T_{\mathrm{SL}_2}}} & \mathrm{NS}(\mathcal{M}_{T_{\mathrm{SL}_2}})
\end{array}$$

commutes for each principal $T_{\mathrm{SL}_2}$–bundle $\xi$ on $C$. We choose $\xi$ in such a way that $\deg(\xi) \in \Lambda_{T_{\mathrm{SL}_2}} \cong \mathbb{Z}$ is nonzero if $g_C = 0$. Then the composition

$$c_{T_{\mathrm{SL}_2}} \circ t^*_\xi \circ \iota^* : \mathrm{Pic}(\mathcal{M}_{\mathrm{SL}_2}) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_{\mathrm{SL}_2}})$$

of the lower row is injective according to Theorem 4.2.1 and Corollary 4.4.2. The latter moreover implies that the two elements $\rho^*(\mathcal{L}^{\det})$ and $(\mathcal{L}_2^{\det})^{\otimes d_\rho}$ in $\mathrm{Pic}(\mathcal{M}_{\mathrm{SL}_2})$ have the same image in $\mathrm{NS}(\mathcal{M}_{T_{\mathrm{SL}_2}})$. □

Now suppose that the reductive group $G$ is simply connected and almost simple. We denote by $\mathcal{O}_{\mathrm{Gr}_G}(1)$ the unique generator of $\mathrm{Pic}(\mathrm{Gr}_G)$ that is ample on every closed subscheme, and by $\mathcal{O}_{\mathrm{Gr}_G}(n)$ its $n$th tensor power for $n \in \mathbb{Z}$.
Over $k = \mathbb{C}$, the following is proved by a different method in section 5 of [20].

Proposition 4.4.4 (Kumar-Narasimhan-Ramanathan). *If $\rho : G \longrightarrow \mathrm{SL}(V)$ has Dynkin index $d_\rho$, then $\rho^* : \mathrm{Pic}(\mathrm{Gr}_{\mathrm{SL}(V)}) \longrightarrow \mathrm{Pic}(\mathrm{Gr}_G)$ maps $\mathcal{O}(1)$ to $\mathcal{O}_{\mathrm{Gr}_G}(d_\rho)$.*

*Proof.* Let $\varphi : \mathrm{SL}_2 \longrightarrow G$ be given by a short coroot. Then $d_\varphi = 1$ by definition, and [13] implies that $\varphi^* : \mathrm{Pic}(\mathrm{Gr}_G) \longrightarrow \mathrm{Pic}(\mathrm{Gr}_{\mathrm{SL}_2})$ maps $\mathcal{O}(1)$ to $\mathcal{O}(1)$, for example because $\varphi^* : \mathrm{Pic}(\mathcal{M}_G) \longrightarrow \mathrm{Pic}(\mathcal{M}_{\mathrm{SL}_2})$ preserves central charges according to their definition [13, p. 59]. Hence it suffices to prove the claim for $\rho \circ \varphi$ instead of $\rho$. This case follows from Corollary 4.4.3, since $\mathrm{glue}^*_{p,z}(\mathcal{L}^{\det}_n) \cong \mathcal{O}_{\mathrm{Gr}_{\mathrm{SL}_n}}(-1)$. □

As in Subsection 4.2, we assume given an exact sequence of reductive groups

$$1 \longrightarrow G \longrightarrow \widehat{G} \xrightarrow{\mathrm{dt}} \mathbb{G}_m \longrightarrow 1$$

with $G$ simply connected, and a line bundle $L$ on $C$.

Corollary 4.4.5. *Suppose that $G$ is almost simple. Then the isomorphism*

$$(t_\xi \circ \mathrm{glue}_{p,z,\delta})^* : \underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},L}) \xrightarrow{\sim} \underline{\mathrm{Pic}}(\mathrm{Gr}_G)$$

*constructed in Subsection 4.2 does not depend on the choice of $p$, $z$, $\xi$ or $\delta$.*

We say that a line bundle on $\mathcal{M}_{\widehat{G},L}$ has *central charge* $n \in \mathbb{Z}$ if this isomorphism maps it to $\mathcal{O}_{\mathrm{Gr}_G}(n)$; this is consistent with the standard central charge of line bundles on $\mathcal{M}_G$, as defined for example in [13].

*Proof.* If $\rho : G \longrightarrow \mathrm{SL}(V)$ is a nontrivial representation, then $d_\rho > 0$, as explained in Remark 4.3.3(iii). Using Proposition 4.4.4, this implies that

$$\rho^* : \mathrm{Pic}(\mathrm{Gr}_{\mathrm{SL}(V)}) \longrightarrow \mathrm{Pic}(\mathrm{Gr}_G)$$

is injective. Due to Remark 4.2.4, it thus suffices to check that

$$\mathrm{glue}^*_{p,z} : \underline{\mathrm{Pic}}(\mathcal{M}_{\mathrm{SL}(V)}) \xrightarrow{\sim} \underline{\mathrm{Pic}}(\mathrm{Gr}_{\mathrm{SL}(V)})$$

does not depend on $p$ or $z$. This is clear, since it maps $\mathcal{L}^{\det}$ to $\mathcal{O}_{\mathrm{Gr}_{\mathrm{SL}(V)}}(-1)$. □

The chosen maximal torus $\iota_G : T_G \hookrightarrow G$ induces maximal tori $\iota_{\widehat{G}} : T_{\widehat{G}} \hookrightarrow \widehat{G}$ and $\iota_{G^{\mathrm{ad}}} : T_{G^{\mathrm{ad}}} \hookrightarrow G^{\mathrm{ad}}$ compatible with the canonical maps $G \hookrightarrow \widehat{G} \twoheadrightarrow G^{\mathrm{ad}}$. Given a principal $T_{\widehat{G}}$–bundle $\widehat{\xi}$ on $C$ and an isomorphism $\mathrm{dt}_* \widehat{\xi} \cong L$, the composition

$$\mathcal{M}^0_{T_G} \xrightarrow{t_{\widehat{\xi}}} \mathcal{M}_{T_{\widehat{G}}} \xrightarrow{(\iota_{\widehat{G}})_*} \mathcal{M}_{\widehat{G}}$$

factors naturally through a 1–morphism

$$\iota_{\widehat{\xi}} : \mathcal{M}^0_{T_G} \longrightarrow \mathcal{M}_{\widehat{G},L}. \tag{10}$$

*Remark* 4.4.6. Given a representation $\rho : G^{\mathrm{ad}} \longrightarrow \mathrm{SL}(V)$, let $\iota : T_{\mathrm{SL}(V)} \hookrightarrow \mathrm{SL}(V)$ be a maximal torus containing $\rho(T_{G^{\mathrm{ad}}})$. Then the diagram

$$\begin{array}{ccccc}
\mathcal{M}^0_{T_G} & & \xrightarrow{\quad\iota_{\widehat{\xi}}\quad} & & \mathcal{M}_{\widehat{G},L} \\
\downarrow{\rho_*} & & & & \downarrow{\rho_*} \\
\mathcal{M}^0_{T_{\mathrm{SL}(V)}} & \xrightarrow{t_{\rho_*\widehat{\xi}}} & \mathcal{M}_{T_{\mathrm{SL}(V)}} & \xrightarrow{\iota_*} & \mathcal{M}_{\mathrm{SL}(V)}
\end{array}$$

is 2–commutative, by construction of $\iota_{\widehat{\xi}}$.

Proposition 4.4.7. i) $\Gamma(\mathcal{M}_{\widehat{G},L}, \mathcal{O}_{\mathcal{M}_{\widehat{G},L}}) = k$.

ii) *There is a canonical isomorphism*

$$c_G : \underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},L}) \xrightarrow{\sim} \mathrm{NS}(\mathcal{M}_G).$$

iii) *For all choices of $\iota_G : T_G \hookrightarrow G$ and of $\widehat{\xi}$, the diagram*

$$\begin{array}{ccc}
\underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},L}) & \xrightarrow{\iota^*_{\widehat{\xi}}} & \underline{\mathrm{Pic}}(\mathcal{M}^0_{T_G}) \\
\downarrow{c_G} & & \downarrow{c_{T_G}} \\
\mathrm{NS}(\mathcal{M}_G) & \xrightarrow{(\iota_G)^{\mathrm{NS},\bar{\delta}}} & \mathrm{NS}(\mathcal{M}_{T_G})
\end{array}$$

*commutes; here $\bar{\delta} \in \Lambda_{T_{G^{ad}}}$ denotes the image of $\widehat{\delta} := \deg \widehat{\xi} \in \Lambda_{T_{\widehat{G}}}$.*

*Proof.* We start with the special case that $G$ is almost simple. Here part (i) of the proposition is just equation (5) from Subsection 4.2.

We let $c_G$ send the line bundle of central charge 1 to the basic inner product $b_G \in \mathrm{NS}(\mathcal{M}_G)$. Due to Theorem 4.2.1(i), Proposition 4.2.3, Corollary 4.4.5

and Remark 4.3.3(ii), this defines a canonical isomorphism, and hence proves (ii).
To see that the diagram in (iii) then commutes, we choose a nontrivial representation $\rho : G^{\mathrm{ad}} \longrightarrow \mathrm{SL}(V)$. We note the functorialities, with respect to $\rho$, according to Remark 4.4.6, Remark 4.2.4, Proposition 4.4.4, Remark 4.3.7 and Remark 3.2.3. In view of these, comparing Corollary 4.4.2 and Definition 4.3.5 shows that the two images of $\rho^*\mathcal{L}^{\det} \in \mathrm{Pic}(\mathcal{M}_{\widehat{G},L})$ in $\mathrm{NS}(\mathcal{M}_{T_G})$ coincide. Since the former generates a subgroup of finite index and the latter is torsionfree, the diagram in (iii) commutes.
For the general case, we use the unique decomposition

$$G = G_1 \times \cdots \times G_r$$

into simply connected and almost simple factors $G_i$. As $\widehat{G}$ is generated by its center and $G$, every normal subgroup in $G$ is still normal in $\widehat{G}$. Let $\widehat{G}_i$ denote the quotient of $\widehat{G}$ modulo the closed normal subgroup $\prod_{j \neq i} G_j$; then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & \widehat{G} & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{pr}_i} & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & G_i & \longrightarrow & \widehat{G}_i & \xrightarrow{\mathrm{dt}_i} & \mathbb{G}_m & \longrightarrow & 0 \end{array}$$

is a morphism of short exact sequences. Since the resulting diagram

$$\begin{array}{ccc} \widehat{G} & \longrightarrow & \prod_i \widehat{G}_i \\ {\scriptstyle \mathrm{dt}}\big\downarrow & & \big\downarrow{\scriptstyle \prod \mathrm{dt}_i} \\ \mathbb{G}_m & \xrightarrow{\mathrm{diag}} & \mathbb{G}_m^r \end{array}$$

is cartesian, it induces an equivalence of moduli stacks

$$\mathcal{M}_{\widehat{G},L} \xrightarrow{\sim} \mathcal{M}_{\widehat{G}_1,L} \times \cdots \times \mathcal{M}_{\widehat{G}_r,L} \tag{11}$$

due to Lemma 2.2.1. We note that equation (5), Lemma 2.1.2(i), Lemma 2.1.4, Remark 4.3.3(i) and Corollary 3.2.4 ensure that various constructions are compatible with the products in (11). Therefore, the general case follows from the already treated almost simple case. □

## 5. The reductive case

In this section, we finally describe the Picard functor $\underline{\mathrm{Pic}}(\mathcal{M}_G^d)$ for any reductive group $G$ over $k$ and any $d \in \pi_1(G)$. We denote

- by $\zeta : Z^0 \hookrightarrow G$ the (reduced) identity component of the center $Z \subseteq G$, and
- by $\pi : \widetilde{G} \longrightarrow G$ the universal cover of $G' := [G, G] \subseteq G$.

Our strategy is to descend along the central isogeny

$$\zeta \cdot \pi : Z^0 \times \widetilde{G} \longrightarrow G,$$

applying the previous two sections to $Z^0$ and to $\widetilde{G}$, respectively. The 1–morphism of moduli stacks given by such a central isogeny is a torsor under a group stack; Subsection 5.1 explains descent of line bundles along such torsors, generalising the method introduced by Laszlo [22] for quotients of $\mathrm{SL}_n$. In Subsection 5.2, we define combinatorially what will be the discrete torsionfree part of $\underline{\mathrm{Pic}}(\mathcal{M}_G^d)$; finally, these Picard functors and their functoriality in $G$ are described in Subsection 5.3.
The following notation is used throughout this section. The reductive group $G$ yields semisimple groups and central isogenies

$$\widetilde{G} \twoheadrightarrow G' \twoheadrightarrow \bar{G} := G/Z^0 \twoheadrightarrow G^{\mathrm{ad}} := G/Z.$$

We denote by $\bar{d} \in \pi_1(\bar{G}) \subseteq \pi_1(G^{\mathrm{ad}})$ the image of $d \in \pi_1(G)$. The choice of a maximal torus $\iota_G : T_G \hookrightarrow G$ induces maximal tori and isogenies

$$T_{\widetilde{G}} \twoheadrightarrow T_{G'} \twoheadrightarrow T_{\bar{G}} \twoheadrightarrow T_{G^{\mathrm{ad}}}.$$

Their cocharacter lattices are hence subgroups of finite index

$$\Lambda_{T_{\widetilde{G}}} \hookrightarrow \Lambda_{T_{G'}} \hookrightarrow \Lambda_{T_{\bar{G}}} \hookrightarrow \Lambda_{T_{G^{\mathrm{ad}}}}.$$

The central isogeny $\zeta \cdot \pi$ makes $\Lambda_{Z^0} \oplus \Lambda_{T_{\widetilde{G}}}$ a subgroup of finite index in $\Lambda_{T_G}$.

5.1. Torsors under a group stack. All stacks in this subsection are stacks over $k$, and all morphisms are over $k$. Following [7, 22], we recall the notion of a torsor under a group stack.
Let $\mathcal{G}$ be a group stack. We denote by 1 the unit object in $\mathcal{G}$, and by $g_1 \cdot g_2$ the image of two objects $g_1$ and $g_2$ under the multiplication 1–morphism $\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$.

Definition 5.1.1. An *action* of $\mathcal{G}$ on a 1–morphism of stacks $\Phi : \mathcal{X} \longrightarrow \mathcal{Y}$ consists of a 1–morphism

$$\mathcal{G} \times \mathcal{X} \longrightarrow \mathcal{X}, \qquad (g, x) \longmapsto g \cdot x,$$

and of three 2–morphisms, which assign to each $k$–scheme $S$ and each object

| | |
|---|---|
| $x$ in $\mathcal{X}(S)$ | an isomorphism $1 \cdot x \xrightarrow{\sim} x$ in $\mathcal{X}(S)$, |
| $(g, x)$ in $(\mathcal{G} \times \mathcal{X})(S)$ | an isomorphism $\Phi(g \cdot x) \xrightarrow{\sim} \Phi(x)$ in $\mathcal{Y}(S)$, |
| $(g_1, g_2, x)$ in $(\mathcal{G} \times \mathcal{G} \times \mathcal{X})(S)$ | an isomorphism $(g_1 \cdot g_2) \cdot x \xrightarrow{\sim} g_1 \cdot (g_2 \cdot x)$ in $\mathcal{X}(S)$. |

These morphisms are required to satisfy the following five compatibility conditions: the two resulting isomorphisms

$$(g \cdot 1) \cdot x \xrightarrow{\sim} g \cdot x \text{ in } \mathcal{X}(S),$$
$$(1 \cdot g) \cdot x \xrightarrow{\sim} g \cdot x \text{ in } \mathcal{X}(S),$$
$$\Phi(1 \cdot x) \xrightarrow{\sim} \Phi(x) \text{ in } \mathcal{Y}(S),$$
$$\Phi((g_1 \cdot g_2) \cdot x) \xrightarrow{\sim} \Phi(x) \text{ in } \mathcal{Y}(S),$$
$$\text{and } (g_1 \cdot g_2 \cdot g_3) \cdot x \xrightarrow{\sim} g_1 \cdot (g_2 \cdot (g_3 \cdot x)) \text{ in } \mathcal{X}(S),$$

coincide for all $k$–schemes $S$ and all objects $g, g_1, g_2, g_3$ in $\mathcal{G}(S)$ and $x$ in $\mathcal{X}(S)$.

*Example* 5.1.2. Let $\varphi : G \longrightarrow H$ be a homomorphism of linear algebraic groups over $k$, and let $Z$ be a closed subgroup in the center of $G$ with $Z \subseteq \ker(\varphi)$. Then the group stack $\mathcal{M}_Z$ acts on the 1–morphism $\varphi_* : \mathcal{M}_G \longrightarrow \mathcal{M}_H$ via the tensor product $\_ \otimes \_ : \mathcal{M}_Z \times \mathcal{M}_G \longrightarrow \mathcal{M}_G$.

From now on, we assume that the group stack $\mathcal{G}$ is algebraic.

Definition 5.1.3. A *$\mathcal{G}$–torsor* is a faithfully flat 1–morphism of algebraic stacks $\Phi : \mathcal{X} \longrightarrow \mathcal{Y}$ together with an action of $\mathcal{G}$ on $\Phi$ such that the resulting 1–morphism

$$\mathcal{G} \times \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}, \quad (g, x) \longmapsto (g \cdot x, x)$$

is an isomorphism.

*Example* 5.1.4. Suppose that $\varphi : G \twoheadrightarrow H$ is a central isogeny of reductive groups with kernel $\mu$. For each $d \in \pi_1(G)$, the 1–morphism

$$\varphi_* : \mathcal{M}_G^d \longrightarrow \mathcal{M}_H^e \qquad e := \varphi_*(d) \in \pi_1(H) \tag{12}$$

is a torsor under the group stack $\mathcal{M}_\mu$, for the action described in example 5.1.2.

*Proof.* The 1–morphism $\varphi_*$ is faithfully flat by Lemma 2.2.2. The 1–morphism

$$\mathcal{M}_\mu \times \mathcal{M}_G \longrightarrow \mathcal{M}_G \times_{\mathcal{M}_H} \mathcal{M}_G, \quad (L, E) \longmapsto (L \otimes E, E)$$

is an isomorphism due to Lemma 2.2.1. Since $\varphi_* : \pi_1(G) \longrightarrow \pi_1(H)$ is injective, $\mathcal{M}_G^d \subseteq \mathcal{M}_G$ is the inverse image of $\mathcal{M}_H^e \subseteq \mathcal{M}_H$ under $\varphi_*$; hence the restriction

$$\mathcal{M}_\mu \times \mathcal{M}_G^d \longrightarrow \mathcal{M}_G^d \times_{\mathcal{M}_H^e} \mathcal{M}_G^d$$

is an isomorphism as well. □

Definition 5.1.5. Let $\Phi_\nu : \mathcal{X}_\nu \longrightarrow \mathcal{Y}_\nu$ be a $\mathcal{G}$–torsor for $\nu = 1, 2$. A *morphism of $\mathcal{G}$–torsors* from $\Phi_1$ to $\Phi_2$ consists of two 1–morphisms

$$A : \mathcal{X}_1 \longrightarrow \mathcal{X}_2 \qquad \text{and} \qquad B : \mathcal{Y}_1 \longrightarrow \mathcal{Y}_2$$

and of two 2–morphisms, which assign to each $k$–scheme $S$ and each object

$$\begin{array}{ll} x \text{ in } \mathcal{X}_1(S) & \text{an isomorphism } \Phi_2 A(x) \xrightarrow{\sim} B\Phi_1(x) \text{ in } \mathcal{Y}_2(S), \\ (g, x) \text{ in } (\mathcal{G} \times \mathcal{X}_1)(S) & \text{an isomorphism } A(g \cdot x) \xrightarrow{\sim} g \cdot A(x) \text{ in } \mathcal{X}_2(S). \end{array}$$

These morphisms are required to satisfy the following three compatibility conditions: the two resulting isomorphisms

$$A(1 \cdot x) \xrightarrow{\sim} A(x) \text{ in } \mathcal{X}_2(S),$$
$$\Phi_2 A(g \cdot x) \xrightarrow{\sim} B\Phi_1(x) \text{ in } \mathcal{Y}_2(S)$$
$$\text{and } A((g_1 \cdot g_2) \cdot x) \xrightarrow{\sim} g_1 \cdot (g_2 \cdot A(x)) \text{ in } \mathcal{X}_2(S)$$

coincide for all $k$–schemes $S$ and all objects $g, g_1, g_2$ in $\mathcal{G}(S)$ and $x$ in $\mathcal{X}_1(S)$.

*Example* 5.1.6. Let a cartesian square of reductive groups over $k$

$$\begin{array}{ccc} G_1 & \xrightarrow{\alpha} & G_2 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ H_1 & \xrightarrow{\beta} & H_2 \end{array}$$

be given. Suppose that $\varphi_1$ and $\varphi_2$ are central isogenies, and denote their common kernel by $\mu$. For each $d_1 \in \pi_1(G_1)$, the diagram

$$\begin{array}{ccc} \mathcal{M}_{G_1}^{d_1} & \xrightarrow{\alpha_*} & \mathcal{M}_{G_2}^{d_2} \\ {\scriptstyle (\varphi_1)_*}\downarrow & & \downarrow{\scriptstyle (\varphi_2)_*} \\ \mathcal{M}_{H_1}^{e_1} & \xrightarrow{\beta_*} & \mathcal{M}_{H_2}^{e_2} \end{array} \qquad \begin{array}{c} d_2 := \alpha_*(d_1) \in \pi_1(G_2) \\ \\ e_\nu := (\varphi_\nu)_*(d_\nu) \in \pi_1(H_\nu) \end{array}$$

is then a morphism of torsors under the group stack $\mathcal{M}_\mu$.

Proposition 5.1.7. *Let a $\mathcal{G}$–torsor $\Phi_\nu : \mathcal{X}_\nu \longrightarrow \mathcal{Y}_\nu$ with $\Gamma(\mathcal{X}_\nu, \mathcal{O}_{\mathcal{X}_\nu}) = k$ be given for $\nu = 1, 2$, together with a morphism of $\mathcal{G}$–torsors*

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{A} & \mathcal{X}_2 \\ {\scriptstyle \Phi_1}\downarrow & & \downarrow{\scriptstyle \Phi_2} \\ \mathcal{Y}_1 & \xrightarrow{B} & \mathcal{Y}_2 \end{array}$$

*such that the induced morphism of Picard functors $A^* : \underline{\mathrm{Pic}}(\mathcal{X}_2) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{X}_1)$ is injective. Then the diagram of Picard functors*

$$\begin{array}{ccc} \underline{\mathrm{Pic}}(\mathcal{X}_1) & \xleftarrow{A^*} & \underline{\mathrm{Pic}}(\mathcal{X}_2) \\ {\scriptstyle \Phi_1^*}\uparrow & & \uparrow{\scriptstyle \Phi_2^*} \\ \underline{\mathrm{Pic}}(\mathcal{Y}_1) & \xleftarrow{B^*} & \underline{\mathrm{Pic}}(\mathcal{Y}_2) \end{array}$$

*is a pullback square.*

*Proof.* The proof of [22, Theorem 5.7] generalises to this situation as follows. Let $S$ be a scheme of finite type over $k$. For a line bundle $\mathcal{L}$ on $S \times \mathcal{X}_\nu$, we denote by $\mathrm{Lin}^{\mathcal{G}}(\mathcal{L})$ the set of its $\mathcal{G}$–linearisations, cf. [22, Definition 2.8]. According to Lemma 2.1.2(i), each automorphism of $\mathcal{L}$ comes from $\Gamma(S, \mathcal{O}_S^*)$ and hence respects each linearisation of $\mathcal{L}$. Thus [22, Theorem 4.1] provides a canonical bijection between the set $\mathrm{Lin}^{\mathcal{G}}(\mathcal{L})$ and the fibre of

$$\Phi_\nu^* : \mathrm{Pic}(S \times \mathcal{Y}_\nu) \longrightarrow \mathrm{Pic}(S \times \mathcal{X}_\nu)$$

over the isomorphism class of $\mathcal{L}$.

Let $\mathcal{T}$ be an algebraic stack over $k$. We denote for the moment by $\mathcal{P}ic(\mathcal{T})$ the groupoid of line bundles on $\mathcal{T}$ and their isomorphisms. Lemma 2.1.2(i) and Corollary 2.1.3 show that the functor

$$A^* : \mathcal{P}ic(\mathcal{T} \times \mathcal{X}_2) \longrightarrow \mathcal{P}ic(\mathcal{T} \times \mathcal{X}_1)$$

is fully faithful for every $\mathcal{T}$. We recall that an element in $\mathrm{Lin}^{\mathcal{G}}(\mathcal{L})$ is an isomorphism in $\mathcal{P}ic(\mathcal{G} \times S \times \mathcal{X}_\nu)$ between two pullbacks of $\mathcal{L}$ such that certain induced diagrams in $\mathcal{P}ic(S \times \mathcal{X}_\nu)$ and in $\mathcal{P}ic(\mathcal{G} \times \mathcal{G} \times S \times \mathcal{X}_\nu)$ commute. Thus it follows for all $\mathcal{L} \in \mathrm{Pic}(S \times \mathcal{X}_2)$ that the canonical map

$$A^* : \mathrm{Lin}^{\mathcal{G}}(\mathcal{L}) \longrightarrow \mathrm{Lin}^{\mathcal{G}}(A^*\mathcal{L})$$

is bijective. Hence the diagram of abelian groups

$$\begin{array}{ccc} \mathrm{Pic}(S \times \mathcal{X}_1) & \xleftarrow{A^*} & \mathrm{Pic}(S \times \mathcal{X}_2) \\ \Phi_1^* \big\uparrow & & \big\uparrow \Phi_2^* \\ \mathrm{Pic}(S \times \mathcal{Y}_1) & \xleftarrow{B^*} & \mathrm{Pic}(S \times \mathcal{Y}_2) \end{array}$$

is a pullback square, as required. □

5.2. Néron–Severi groups $\mathrm{NS}(\mathcal{M}_G^d)$ for reductive $G$.

Definition 5.2.1. The *Néron–Severi group* $\mathrm{NS}(\mathcal{M}_G^d)$ is the subgroup

$$\mathrm{NS}(\mathcal{M}_G^d) \subseteq \mathrm{NS}(\mathcal{M}_{Z^0}) \oplus \mathrm{NS}(\mathcal{M}_{\tilde{G}})$$

of all triples $l_Z : \Lambda_{Z^0} \longrightarrow \mathbb{Z}$, $b_Z : \Lambda_{Z^0} \otimes \Lambda_{Z^0} \longrightarrow \mathrm{End}\, J_C$ and $b : \Lambda_{T_{\tilde{G}}} \otimes \Lambda_{T_{\tilde{G}}} \longrightarrow \mathbb{Z}$ with the following properties:

(1) For every lift $\bar{\delta} \in \Lambda_{T_{\tilde{G}}}$ of $\bar{d} \in \pi_1(\bar{G})$, the direct sum

$$l_Z \oplus b(-\bar{\delta} \otimes \_) : \Lambda_{Z^0} \oplus \Lambda_{T_{\tilde{G}}} \longrightarrow \mathbb{Z}$$

is integral on $\Lambda_{T_G}$.

(2) The orthogonal direct sum

$$b_Z \perp (\mathrm{id}_{J_C} \cdot b) : (\Lambda_{Z^0} \oplus \Lambda_{T_{\tilde{G}}}) \otimes (\Lambda_{Z^0} \oplus \Lambda_{T_{\tilde{G}}}) \longrightarrow \mathrm{End}\, J_C$$

is integral on $\Lambda_{T_G} \otimes \Lambda_{T_G}$.

Lemma 5.2.2. *If condition 1 above holds for one lift $\bar{\delta} \in \Lambda_{T_{\tilde{G}}}$ of $\bar{d} \in \pi_1(\bar{G})$, then it holds for every lift $\bar{\delta} \in \Lambda_{T_{\tilde{G}}}$ of the same element $\bar{d} \in \pi_1(\bar{G})$.*

*Proof.* Any two lifts $\bar{\delta}$ of $\bar{d}$ differ by some element $\lambda \in \Lambda_{T_{\tilde{G}}}$. Lemma 4.3.4 states in particular that

$$b(-\lambda \otimes \_) : \Lambda_{T_{\tilde{G}}} \longrightarrow \mathbb{Z}$$

is integral on $\Lambda_{T_{\tilde{G}}}$, and hence admits an extension $\Lambda_{T_G} \longrightarrow \mathbb{Z}$ that vanishes on $\Lambda_{Z^0}$. □

*Remark* 5.2.3. If $G$ is simply connected, then $\mathrm{NS}(\mathcal{M}_G^0)$ coincides with the group $\mathrm{NS}(\mathcal{M}_G)$ of definition 4.3.2. If $G = T$ is a torus, then $\mathrm{NS}(\mathcal{M}_T^d)$ coincides for all $d \in \pi_1(T)$ with the group $\mathrm{NS}(\mathcal{M}_T)$ of definition 3.2.1.

*Remark* 5.2.4. The Weyl group $W$ of $(G, T_G)$ acts trivially on $\mathrm{NS}(\mathcal{M}_G^d)$. Hence the group $\mathrm{NS}(\mathcal{M}_G^d)$ does not depend on the choice of $T_G$; cf. Subsection 4.3.

Definition 5.2.5. Given a lift $\delta \in \Lambda_{T_G}$ of $d \in \pi_1(G)$, the homomorphism

$$(\iota_G)^{\mathrm{NS},\delta} : \mathrm{NS}(\mathcal{M}_G^d) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_G})$$

sends $(l_Z, b_Z) \in \mathrm{NS}(\mathcal{M}_{Z^0})$ and $b \in \mathrm{NS}(\mathcal{M}_{\widetilde{G}})$ to the pair

$$l_Z \oplus b(-\bar{\delta} \otimes \_) : \Lambda_G \longrightarrow \mathbb{Z} \quad \text{and} \quad b_Z \perp (\mathrm{id}_{J_C} \cdot b) : \Lambda_{T_G} \otimes \Lambda_{T_G} \longrightarrow \mathrm{End}\, J_C$$

where $\bar{\delta} \in \Lambda_{T_{\widetilde{G}}}$ denotes the image of $\delta$.

Note that this definition agrees with the earlier definition 4.3.5 in the cases covered by both, namely $G$ simply connected and $\delta \in \Lambda_{T_G}$.

Lemma 5.2.6. *Given a lift $\delta \in \Lambda_{T_G}$ of $d \in \pi_1(G)$, the diagram*

$$\begin{array}{ccc}
\mathrm{NS}(\mathcal{M}_G^d) & \xrightarrow{\qquad (\iota_G)^{\mathrm{NS},\delta} \qquad} & \mathrm{NS}(\mathcal{M}_{T_G}) \\
\big\downarrow & & \big\downarrow {\scriptstyle (\zeta\cdot\pi)^*} \\
\mathrm{NS}(\mathcal{M}_{Z^0}) \oplus \mathrm{NS}(\mathcal{M}_{\widetilde{G}}) \xrightarrow{\mathrm{id} \oplus (\iota_{\widetilde{G}})^{\mathrm{NS},\bar{\delta}}} & \mathrm{NS}(\mathcal{M}_{Z^0}) \oplus \mathrm{NS}(\mathcal{M}_{T_{\widetilde{G}}}) \hookrightarrow & \mathrm{NS}(\mathcal{M}_{Z^0 \times T_{\widetilde{G}}})
\end{array}$$

*is a pullback square; here $\bar{\delta} \in \Lambda_{T_{G^{ad}}}$ again denotes the image of $\delta$.*

*Proof.* This follows directly from the definitions. □

Let $e \in \pi_1(H)$ be the image of $d \in \pi_1(G)$ under a homomorphism of reductive groups $\varphi : G \longrightarrow H$. $\varphi$ induces a map $\varphi : \widetilde{G} \longrightarrow \widetilde{H}$ between the universal covers of their commutator subgroups. If $\varphi$ maps the identity component $Z_G^0$ in the center $Z_G$ of $G$ to the center $Z_H$ of $H$, then it induces an obvious pullback map

$$\varphi^* : \mathrm{NS}(\mathcal{M}_H^e) \longrightarrow \mathrm{NS}(\mathcal{M}_G^d)$$

which sends $l_Z$, $b_Z$ and $b$ simply to $\varphi^* l_Z$, $\varphi^* b_Z$ and $\varphi^* b$. This is a special case of the following map, which $\varphi$ induces even without the hypothesis on the centers, and which also generalises the previous definition 5.2.5.

Definition 5.2.7. Choose a maximal torus $\iota_H : T_H \hookrightarrow H$ containing $\varphi(T_G)$, and a lift $\delta \in \Lambda_{T_G}$ of $d \in \pi_1(G)$; let $\eta \in \Lambda_{T_H}$ be the image of $\delta$. Then the map

$$\varphi^{\mathrm{NS},d} : \mathrm{NS}(\mathcal{M}_H^e) \longrightarrow \mathrm{NS}(\mathcal{M}_G^d)$$

sends $(l_Z, b_Z) \in \mathrm{NS}(\mathcal{M}_{Z_H^0})$ and $b \in \mathrm{NS}(\mathcal{M}_{\widetilde{H}})$ to the pullback along $\varphi : Z_G^0 \longrightarrow T_H$ of $(\iota_H)^{\mathrm{NS},\eta}(l_Z, b_Z, b) \in \mathrm{NS}(\mathcal{M}_{T_H})$, together with $\varphi^* b \in \mathrm{NS}(\mathcal{M}_{\widetilde{G}})$.

Lemma 5.2.8. *The map $\varphi^{\mathrm{NS},d}$ does not depend on the choice of $T_G$, $T_H$ or $\delta$.*

*Proof.* Let $W_G$ denote the Weyl group of $(G, T_G)$. It acts trivially on $\Lambda_{Z_G^0}$, and without nontrivial coinvariants on $\Lambda_{T_{\widetilde{G}}}$; these two observations imply

$$\mathrm{Hom}(\Lambda_{T_{\widetilde{G}}} \otimes \Lambda_{Z_G^0}, \mathbb{Z})^{W_G} = 0. \tag{13}$$

Lemma 4.3.4 states that $b$ is integral on $\Lambda_{T_{\widetilde{H}}} \otimes \Lambda_{T_{\widetilde{H}}}$; its composition with the canonical projection $\Lambda_{T_H} \twoheadrightarrow \Lambda_{T_{\widetilde{H}}}$ is a Weyl–invariant map $b_r : \Lambda_{T_{\widetilde{H}}} \otimes \Lambda_{T_H} \longrightarrow \mathbb{Z}$. As explained in Subsection 4.3, Lemma 4.3.1 implies that $\varphi^* b_r : \Lambda_{T_{\widetilde{G}}} \otimes \Lambda_{T_G} \longrightarrow \mathbb{Z}$ is still Weyl–invariant; hence it vanishes on $\Lambda_{T_{\widetilde{G}}} \otimes \Lambda_{Z_G^0}$ by (13).

Any two lifts $\delta$ of $d$ differ by some element $\lambda \in \Lambda_{T_{\widetilde{G}}}$; then the two images of $(l_Z, b_Z, b) \in \mathrm{NS}(\mathcal{M}_H^e)$ in $\mathrm{NS}(\mathcal{M}_{T_H})$ differ, according to the proof of Lemma 5.2.2, only by $b_r(-\lambda \otimes \_) : \Lambda_{T_H} \longrightarrow \mathbb{Z}$. Thus their compositions with $\varphi : \Lambda_{Z_G^0} \longrightarrow \Lambda_{T_H}$ coincide by the previous paragraph. This shows that the two images of $(l_Z, b_Z, b)$ have the same component in the direct summand $\mathrm{Hom}(\Lambda_{Z_G^0}, \mathbb{Z})$ of $\mathrm{NS}(\mathcal{M}_G^d)$; since the other two components do not involve $\delta$ at all, the independence on $\delta$ follows.
The independence on $T_G$ and $T_H$ is then a consequence of Lemma 4.3.1, since the Weyl groups $W_G$ and $W_H$ act trivially on $\mathrm{NS}(\mathcal{M}_G^d)$ and on $\mathrm{NS}(\mathcal{M}_H^e)$. □

Lemma 5.2.9. *For all maximal tori $\iota_G : T_G \hookrightarrow G$ and $\iota_H : T_H \hookrightarrow H$ with $\varphi(T_G) \subseteq T_H$, and all lifts $\delta \in \Lambda_{T_G}$ of $d \in \pi_1(G)$, the diagram*

$$\begin{array}{ccc} \mathrm{NS}(\mathcal{M}_H^e) & \xrightarrow{(\iota_H)^{\mathrm{NS},\eta}} & \mathrm{NS}(\mathcal{M}_{T_H}) \\ \downarrow{\scriptstyle \varphi^{\mathrm{NS},d}} & & \downarrow{\scriptstyle \varphi^*} \\ \mathrm{NS}(\mathcal{M}_G^d) & \xrightarrow{(\iota_G)^{\mathrm{NS},\delta}} & \mathrm{NS}(\mathcal{M}_{T_G}) \end{array}$$

*commutes, with $\eta := \varphi_* \delta \in \Lambda_{T_H}$ and $e := \varphi_* d \in \pi_1(H)$ as in definition 5.2.7.*

*Proof.* Given an element in $\mathrm{NS}(\mathcal{M}_H^e)$, we have to compare its two images in $\mathrm{NS}(\mathcal{M}_{T_G})$. The definition 5.2.7 of $\varphi^{\mathrm{NS},d}$ directly implies that both have the same pullback to $\mathrm{NS}(\mathcal{M}_{Z_G^0})$ and to $\mathrm{NS}(\mathcal{M}_{T_{\widetilde{G}}})$. Moreover, their components in the direct summand $\mathrm{Hom}^s(\Lambda_{T_G} \otimes \Lambda_{T_G}, \mathrm{End}\, J_C)$ of $\mathrm{NS}(\mathcal{M}_{T_G})$ are both Weyl–invariant due to Lemma 4.3.1; thus equation (13) above shows that these components vanish on $\Lambda_{T_{\widetilde{G}}} \otimes \Lambda_{Z_G^0}$ and on $\Lambda_{Z_G^0} \otimes \Lambda_{T_{\widetilde{G}}}$. Hence two images in question even have the same pullback to $\mathrm{NS}(\mathcal{M}_{Z_G^0 \times T_{\widetilde{G}}})$. But $\Lambda_{Z_G^0} \oplus \Lambda_{T_{\widetilde{G}}}$ has finite index in $\Lambda_{T_G}$. □

Corollary 5.2.10. *Let $\psi : H \longrightarrow K$ be another homomorphism of reductive groups, and put $f := \psi_* e \in \pi_1(K)$. Then*

$$\varphi^{\mathrm{NS},d} \circ \psi^{\mathrm{NS},e} = (\psi \circ \varphi)^{\mathrm{NS},d} : \mathrm{NS}(\mathcal{M}_K^f) \longrightarrow \mathrm{NS}(\mathcal{M}_G^d).$$

*Proof.* According to the previous lemma, this equality holds after composition with $(\iota_G)^{\mathrm{NS},\delta} : \mathrm{NS}(\mathcal{M}_G^d) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_G})$ for any lift $\delta \in \Lambda_{T_G}$ of $d$. Due to the Lemma 4.3.6 and Lemma 5.2.6, there is a lift $\delta$ of $d$ such that $(\iota_G)^{\mathrm{NS},\delta}$ is injective. □

We conclude this subsection with a more explicit description of $\mathrm{NS}(\mathcal{M}_G^d)$. It turns out that genus $g_C = 0$ is special. This generalises the description obtained for $k = \mathbb{C}$ and $G$ semisimple by different methods in [31, Section V].

Proposition 5.2.11. *Let $q : G \twoheadrightarrow G/G' =: G^{ab}$ denote the maximal abelian quotient of $G$. Then the sequence of abelian groups*

$$0 \longrightarrow \mathrm{NS}(\mathcal{M}_{G^{ab}}) \xrightarrow{q^*} \mathrm{NS}(\mathcal{M}_G^d) \xrightarrow{\mathrm{pr}_2} \mathrm{NS}(\mathcal{M}_{\widetilde{G}})$$

*is exact, and the image of the map* $\mathrm{pr}_2$ *in it consists of all forms* $b : \Lambda_{T_{\tilde{G}}} \otimes \Lambda_{T_{\tilde{G}}} \longrightarrow \mathbb{Z}$ *in* $\mathrm{NS}(\mathcal{M}_{\tilde{G}})$ *that are integral*

- *on* $\Lambda_{T_{\tilde{G}}} \otimes \Lambda_{T_{G'}}$*, if* $g_C \geq 1$*;*
- *on* $(\mathbb{Z}\bar{\delta}) \otimes \Lambda_{T_{G'}}$ *for a lift* $\bar{\delta} \in \Lambda_{T_{\tilde{G}}}$ *of* $\bar{d} \in \pi_1(\bar{G})$*, if* $g_C = 0$*.*

*The condition does not depend on the choice of this lift* $\bar{\delta}$*, due to Lemma 4.3.4.*

*Proof.* Since $q : Z^0 \longrightarrow G^{\mathrm{ab}}$ is an isogeny, $q^*$ is injective; it clearly maps into the kernel of $\mathrm{pr}_2$. Conversely, let $(l_Z, b_Z, b) \in \mathrm{NS}(\mathcal{M}_G^d)$ be in the kernel of $\mathrm{pr}_2$; this means $b = 0$. Then condition 1 in the definition 5.2.1 of $\mathrm{NS}(\mathcal{M}_G^d)$ provides a map

$$l_Z \oplus 0 : \Lambda_{T_G} \longrightarrow \mathbb{Z}$$

which vanishes on $\Lambda_{T_{\tilde{G}}}$, and hence also on $\Lambda_{T_{G'}}$; thus it is induced from a map on $\Lambda_{T_G}/\Lambda_{T_{G'}} = \Lambda_{G^{\mathrm{ab}}}$. Similarly, condition 2 in the same definition provides a map $b_Z \perp 0$ on $\Lambda_{T_G} \otimes \Lambda_{T_G}$ which vanishes on $\Lambda_{T_{\tilde{G}}} \otimes \Lambda_{T_G} + \Lambda_{T_G} \otimes \Lambda_{T_{\tilde{G}}}$, and hence also on $\Lambda_{T_{G'}} \otimes \Lambda_{T_G} + \Lambda_{T_G} \otimes \Lambda_{T_{G'}}$; thus it is induced from a map on the quotient $\Lambda_{G^{\mathrm{ab}}} \otimes \Lambda_{G^{\mathrm{ab}}}$. This proves the exactness.
Now let $b \in \mathrm{NS}(\mathcal{M}_{\tilde{G}}^d)$ be in the image of $\mathrm{pr}_2$. Then $b$ is integral on $(\mathbb{Z}\bar{\delta}) \otimes \Lambda_{G'}$ by condition 1 in definition 5.2.1. If $g_C \geq 1$, then

$$\_ \cdot \mathrm{id}_{J_C} : \mathbb{Z} \longrightarrow \mathrm{End}\, J_C$$

is injective with torsionfree cokernel; thus condition 2 in definition 5.2.1 implies that

$$0 \oplus b : (\Lambda_{Z^0} \oplus \Lambda_{T_{\tilde{G}}}) \otimes \Lambda_{T_{\tilde{G}}} \longrightarrow \mathbb{Z}$$

is integral on $\Lambda_{T_G} \otimes \Lambda_{T_{G'}}$ and hence, vanishing on $\Lambda_{Z^0} \subseteq \Lambda_{T_G}$, comes from a map on the quotient $\Lambda_{T_{\tilde{G}}} \otimes \Lambda_{T_{G'}}$. This shows that $b$ satisfies the stated condition.
Conversely, suppose that $b \in \mathrm{NS}(\mathcal{M}_{\tilde{G}}^d)$ satisfies the stated condition. Then $b$ is integral on $(\mathbb{Z}\bar{\delta}) \otimes \Lambda_{T_{G'}}$; since $\Lambda_{T_{G'}} \subseteq \Lambda_{T_G}$ is a direct summand,

$$b(-\bar{\delta} \otimes \_) : \Lambda_{T_{G'}} \longrightarrow \mathbb{Z}$$

can thus be extended to $\Lambda_{T_G}$. We restrict it to a map $l_Z : \Lambda_{Z^0} \longrightarrow \mathbb{Z}$. In the case $g_C = 0$, the triple $(l_Z, 0, b)$ is in $\mathrm{NS}(\mathcal{M}_G^d)$ and hence an inverse image of $b$. It remains to consider $g_C \geq 1$. Then $b$ is by assumption integral on $\Lambda_{T_{\tilde{G}}} \otimes \Lambda_{T_{G'}}$, so composing it with the canonical subjection $\Lambda_{T_G} \twoheadrightarrow \Lambda_{T_{\tilde{G}}}$ defines a linear map $\Lambda_{T_G} \otimes \Lambda_{T_{G'}} \longrightarrow \mathbb{Z}$. Since $b$ is symmetric, this extends canonically to a symmetric linear map from

$$\Lambda_{T_G} \otimes \Lambda_{T_{G'}} + \Lambda_{T_{G'}} \otimes \Lambda_{T_G} \subseteq \Lambda_{T_G} \otimes \Lambda_{T_G}$$

to $\mathbb{Z}$. It can be extended further to a symmetric linear map from $\Lambda_{T_G} \otimes \Lambda_{T_G}$ to $\mathbb{Z}$, because $\Lambda_{T_{G'}} \subseteq \Lambda_{T_G}$ is a direct summand. Multiplying it with $\mathrm{id}_{J_C}$ and restricting to $\Lambda_{Z^0}$ defines an element $b_Z \in \mathrm{Hom}^s(\Lambda_{Z^0} \otimes \Lambda_{Z^0}, \mathrm{End}\, J_C)$. By construction, the triple $(l_Z, b_Z, b)$ is in $\mathrm{NS}(\mathcal{M}_G^d)$ and hence an inverse image of $b$. □

In particular, the free abelian group $\mathrm{NS}(\mathcal{M}_G^d)$ has rank

$$\operatorname{rk}\mathrm{NS}(\mathcal{M}_G^d) = r + r \cdot \operatorname{rk}\mathrm{NS}(J_C) + \frac{r(r-1)}{2} \cdot \operatorname{rk}\operatorname{End}(J_C) + s$$

if $G^{\mathrm{ab}} \cong \mathbb{G}_m^r$ is a torus of rank $r$, and $G^{\mathrm{ad}}$ contains $s$ simple factors.

5.3. Proof of the main result.

Theorem 5.3.1. i) *$\Gamma(\mathcal{M}_G^d, \mathcal{O}_{\mathcal{M}_G^d}) = k$.*

ii) *The functor $\underline{\mathrm{Pic}}(\mathcal{M}_G^d)$ is representable by a $k$–scheme locally of finite type.*

iii) *There is a canonical exact sequence*

$$0 \longrightarrow \underline{\mathrm{Hom}}(\pi_1(G), J_C) \xrightarrow{j_G} \underline{\mathrm{Pic}}(\mathcal{M}_G^d) \xrightarrow{c_G} \mathrm{NS}(\mathcal{M}_G^d) \longrightarrow 0$$

*of commutative group schemes over $k$.*

iv) *For every homomorphism of reductive groups $\varphi : G \longrightarrow H$, the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathrm{Hom}}(\pi_1(H), J_C) & \xrightarrow{j_H} & \underline{\mathrm{Pic}}(\mathcal{M}_H^e) & \xrightarrow{c_H} & \mathrm{NS}(\mathcal{M}_H^e) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^{\mathrm{NS},d}} & & \\ 0 & \longrightarrow & \underline{\mathrm{Hom}}(\pi_1(G), J_C) & \xrightarrow{j_G} & \underline{\mathrm{Pic}}(\mathcal{M}_G^d) & \xrightarrow{c_G} & \mathrm{NS}(\mathcal{M}_G^d) & \longrightarrow & 0 \end{array}$$

*commutes; here $e := \varphi_*(d) \in \pi_1(H)$.*

*Proof.* We record for later use the commutative square of abelian groups

$$\begin{array}{ccc} \pi_1(G) & \xleftarrow{\mathrm{pr}} & \Lambda_{T_G} \\ \uparrow{\scriptstyle \zeta_*} & & \uparrow{\scriptstyle (\zeta\cdot\pi)_*} \\ \Lambda_{Z^0} & \xleftarrow{\mathrm{pr}_1} & \Lambda_{Z^0 \times T_{\widetilde{G}}}. \end{array}$$

The mapping cone of this commutative square

$$0 \longrightarrow \Lambda_{Z^0} \oplus \Lambda_{T_{\widetilde{G}}} \longrightarrow \Lambda_{Z^0} \oplus \Lambda_{T_G} \longrightarrow \pi_1(G) \longrightarrow 0 \tag{14}$$

is exact, because its subsequence $0 \longrightarrow \Lambda_{T_{\widetilde{G}}} \longrightarrow \Lambda_{T_G} \longrightarrow \pi_1(G) \longrightarrow 0$ is exact, and the resulting sequence of quotients $0 \longrightarrow \Lambda_{Z^0} = \Lambda_{Z^0} \longrightarrow 0 \longrightarrow 0$ is also exact.

Lemma 5.3.2. *There is an exact sequence of reductive groups*

$$1 \longrightarrow \widetilde{G} \longrightarrow \widehat{G} \xrightarrow{\mathrm{dt}} \mathbb{G}_m \longrightarrow 1 \tag{15}$$

*and an extension $\widehat{\pi} : \widehat{G} \longrightarrow G$ of $\pi : \widetilde{G} \longrightarrow G$ such that $\widehat{\pi}_* : \pi_1(\widehat{G}) \longrightarrow \pi_1(G)$ maps $1 \in \mathbb{Z} = \pi_1(\mathbb{G}_m) = \pi_1(\widehat{G})$ to the given element $d \in \pi_1(G)$.*

*Proof.* We view the given $d \in \pi_1(G)$ as a coset $d \subseteq \Lambda_{T_G}$ modulo $\Lambda_{\mathrm{coroots}}$. Let

$$\Lambda_{T_{\widehat{G}}} \subseteq \Lambda_{T_G} \oplus \mathbb{Z}$$

be generated by $\Lambda_{\mathrm{coroots}} \oplus 0$ and $(d, 1)$, and let

$$(\widehat{\pi}, \mathrm{dt}) : \widehat{G} \longrightarrow G \times \mathbb{G}_m$$

be the reductive group with the same root system as $G$, whose maximal torus $T_{\widehat{G}} = \widehat{\pi}^{-1}(T_G)$ has cocharacter lattice $\mathrm{Hom}(\mathbb{G}_m, T_{\widehat{G}}) = \Lambda_{T_{\widehat{G}}}$. As $\pi_*$ maps $\Lambda_{T_{\widetilde{G}}}$ isomorphically onto $\Lambda_{\mathrm{coroots}}$, we obtain an exact sequence

$$0 \longrightarrow \Lambda_{T_{\widetilde{G}}} \xrightarrow{\pi_*} \Lambda_{T_{\widehat{G}}} \xrightarrow{\mathrm{pr}_2} \mathbb{Z} \longrightarrow 0,$$

which yields the required exact sequence (15) of groups. By its construction, $\widehat{\pi}_*$ maps the canonical generator $1 \in \pi_1(\mathbb{G}_m) = \pi_1(\widehat{G})$ to $d \in \pi_1(G)$. □

Let $\mu$ denote the kernel of the central isogeny $\zeta \cdot \pi : Z^0 \times \widetilde{G} \twoheadrightarrow G$. Then

$$\psi : Z^0 \times \widehat{G} \longrightarrow G \times \mathbb{G}_m, \qquad (z^0, \widehat{g}) \longmapsto \big(\zeta(z^0) \cdot \widehat{\pi}(\widehat{g}), \mathrm{dt}(\widehat{g})\big)$$

is by construction a central isogeny with kernel $\mu$. Hence the induced 1–morphism

$$\psi_* : \mathcal{M}^0_{Z^0} \times \mathcal{M}^1_{\widehat{G}} \longrightarrow \mathcal{M}^d_G \times \mathcal{M}^1_{\mathbb{G}_m}$$

is faithfully flat by Lemma 2.2.2. Restricting to the point $\mathrm{Spec}(k) \longrightarrow \mathcal{M}^1_{\mathbb{G}_m}$ given by a line bundle $L$ of degree 1 on $C$, we get a faithfully flat 1–morphism

$$(\psi_*)_L : \mathcal{M}^0_{Z^0} \times \mathcal{M}_{\widehat{G}, L} \longrightarrow \mathcal{M}^d_G.$$

Since $\Gamma(\mathcal{M}^0_{Z^0} \times \mathcal{M}_{\widehat{G},L}, \mathcal{O}) = k$ by Proposition 4.4.7(i) and Lemma 2.1.2(i), part (i) of the theorem follows. The group stack $\mathcal{M}_\mu$ acts by tensor product on these two 1–morphisms $\psi_*$ and $(\psi_*)_L$, turning both into $\mathcal{M}_\mu$–torsors; cf. Example 5.1.4. The idea is to descend line bundles along the torsor $(\psi_*)_L$.
We choose a principal $T_{\widehat{G}}$–bundle $\widehat{\xi}$ on $C$ together with an isomorphism of line bundles $\mathrm{dt}_* \widehat{\xi} \cong L$. Then $\xi := \widehat{\pi}_*(\widehat{\xi})$ is a principal $T_G$–bundle on $C$; their degrees $\widehat{\delta} := \deg(\widehat{\xi}) \in \Lambda_{T_{\widehat{G}}}$ and $\delta := \deg(\xi) \in \Lambda_{T_G}$ are lifts of $d \in \pi_1(G)$. The diagram

$$(16) \qquad \begin{array}{ccc} Z^0 \times T_{\widehat{G}} & \xrightarrow{\mathrm{id} \times \iota_{\widehat{G}}} & Z^0 \times \widehat{G} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ T_G \times \mathbb{G}_m & \xrightarrow{\iota_G \times \mathrm{id}} & G \times \mathbb{G}_m \end{array}$$

of groups induces the right square in the 2–commutative diagram

$$(17) \qquad \begin{array}{ccccc} \mathcal{M}^0_{Z^0} \times \mathcal{M}^0_{T_{\widehat{G}}} & \xrightarrow{\mathrm{id} \times t_{\widehat{\xi}}} & \mathcal{M}^0_{Z^0} \times \mathcal{M}^{\widehat{\delta}}_{T_{\widehat{G}}} & \xrightarrow{(\mathrm{id} \times \iota_{\widehat{G}})_*} & \mathcal{M}^0_{Z^0} \times \mathcal{M}^1_{\widehat{G}} \\ \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} \\ \mathcal{M}^0_{T_G} \times \mathcal{M}^1_{\mathbb{G}_m} & \xrightarrow{t_\xi \times \mathrm{id}} & \mathcal{M}^\delta_{T_G} \times \mathcal{M}^1_{\mathbb{G}_m} & \xrightarrow{(\iota_G \times \mathrm{id})_*} & \mathcal{M}^d_G \times \mathcal{M}^1_{\mathbb{G}_m} \end{array}$$

of moduli stacks; note that $t_{\widehat{\xi}}$ and $t_\xi$ are equivalences. Restricting the outer rectangle again to the point $\operatorname{Spec}(k) \longrightarrow \mathcal{M}^1_{\mathbb{G}_m}$ given by $L$, we get the diagram

$$(18)\qquad \begin{array}{ccccc} \mathcal{M}^0_{Z^0\times T_{\widetilde{G}}} & \overset{\sim}{=\!=} & \mathcal{M}^0_{Z^0}\times\mathcal{M}^0_{T_{\widetilde{G}}} & \xrightarrow{\ \mathrm{id}\times\iota_{\widehat{\xi}}\ } & \mathcal{M}^0_{Z^0}\times\mathcal{M}_{\widehat{G},L} \\ & \searrow^{(\zeta\cdot\pi)_*} & \downarrow{\scriptstyle \zeta_*\otimes\pi_*} & & \downarrow{\scriptstyle (\psi_*)_L} \\ & & \mathcal{M}^0_{T_G} & \xrightarrow{\ (\iota_G)_*\circ t_\xi\ } & \mathcal{M}^d_G \end{array}$$

containing an instance $\iota_{\widehat{\xi}}$ of the 1–morphism (10) defined in Subsection 4.4. According to the Proposition 3.2.2 and Proposition 4.4.7,

$$\iota^*_{\widehat{\xi}} : \underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},L}) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{M}^0_{T_{\widetilde{G}}})$$

is a morphism of group schemes over $k$. This morphism is a closed immersion, according to Proposition 4.4.7(iii), if $g_C \geq 1$ or if $\widehat{\xi}$ is chosen appropriately, as explained in Lemma 4.3.6; we assume this in the sequel. Using Lemma 2.1.4 and Corollary 3.2.4, it follows that

$$(\mathrm{id}\times\iota_{\widehat{\xi}})^* : \underline{\mathrm{Pic}}(\mathcal{M}^0_{Z^0}) \oplus \underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},L}) \cong \underline{\mathrm{Pic}}(\mathcal{M}^0_{Z^0}\times\mathcal{M}_{\widehat{G},L}) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{M}^0_{Z^0\times T_{\widetilde{G}}})$$

is a closed immersion of group schemes over $k$ as well.
The group stack $\mathcal{M}_\mu$ still acts by tensor product on the vertical 1–morphisms in (17) and in (18). Since the diagram (16) of groups is cartesian, (17) and (18) are morphisms of $\mathcal{G}$–torsors; cf. Example 5.1.6. Proposition 5.1.7 applies to the latter morphism of torsors, yielding a cartesian square

$$(19)\qquad \begin{array}{ccc} \underline{\mathrm{Pic}}(\mathcal{M}^d_G) & \xrightarrow{\ t^*_\xi\circ\iota^*_G\ } & \underline{\mathrm{Pic}}(\mathcal{M}^0_{T_G}) \\ \downarrow{\scriptstyle \psi^*_L} & & \downarrow{\scriptstyle (\zeta\cdot\pi)^*} \\ \underline{\mathrm{Pic}}(\mathcal{M}^0_{Z^0}) \oplus \underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},L}) & \xrightarrow{\ (\mathrm{id}\times\iota_{\widehat{\xi}})^*\ } & \underline{\mathrm{Pic}}(\mathcal{M}^0_{Z^0\times T_{\widetilde{G}}}) \end{array}$$

of Picard functors. Thus $\underline{\mathrm{Pic}}(\mathcal{M}^d_G)$ is representable, and $t^*_\xi \circ \iota^*_G$ is a closed immersion; this proves part (ii) of the theorem.
The image of the mapping cone (14) under the exact functor $\underline{\mathrm{Hom}}(\_, J_C)$, and the mapping cones of the two cartesian squares given by diagram (19) and Lemma 5.2.6, are the columns of the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \underline{\mathrm{Hom}}(\pi_1(G), J_C) & & \underline{\mathrm{Pic}}(\mathcal{M}_G^d) & & \mathrm{NS}(\mathcal{M}_G^d) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & \underline{\mathrm{Pic}}(\mathcal{M}_{Z^0}^0) & & \mathrm{NS}(\mathcal{M}_{Z^0}) & & \\
 & & \underline{\mathrm{Hom}}(\Lambda_{Z^0}, J_C) & & \oplus & & \oplus & & \\
0 & \longrightarrow & \oplus & \xrightarrow{j_{Z^0} \oplus j_{T_G}} & \underline{\mathrm{Pic}}(\mathcal{M}_{\widehat{G},L}) & \xrightarrow{c_{Z^0} \oplus c_{\widetilde{G}} \oplus c_{T_G}} & \mathrm{NS}(\mathcal{M}_{\widetilde{G}}) & \longrightarrow & 0 \\
 & & \underline{\mathrm{Hom}}(\Lambda_{T_G}, J_C) & & \oplus & & \oplus & & \\
 & & & & \underline{\mathrm{Pic}}(\mathcal{M}_{T_G}^0) & & \mathrm{NS}(\mathcal{M}_{T_G}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda_{Z^0 \times T_{\widetilde{G}}}, J_C) & \xrightarrow{j_{Z^0 \times T_{\widetilde{G}}}} & \underline{\mathrm{Pic}}(\mathcal{M}_{Z^0 \times T_{\widetilde{G}}}^0) & \xrightarrow{c_{Z^0 \times T_{\widetilde{G}}}} & \mathrm{NS}(\mathcal{M}_{Z^0 \times T_{\widetilde{G}}}) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

whose two rows are exact due to Proposition 3.2.2(ii) and Proposition 4.4.7(ii). Applying the snake lemma to this diagram, we get an exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(\pi_1(G), J_C) \xrightarrow{j_G(\iota_G, \delta)} \underline{\mathrm{Pic}}(\mathcal{M}_G^d) \xrightarrow{c_G(\iota_G, \delta)} \mathrm{NS}(\mathcal{M}_G^d) \longrightarrow 0.$$

The image of $j_G(\iota_G, \delta)$ and the kernel of $c_G(\iota_G, \delta)$ are a priori independent of the choices made, since both are the largest quasicompact open subgroup in $\underline{\mathrm{Pic}}(\mathcal{M}_G^d)$. If $G$ is a torus and $d = 0$, then this is the exact sequence of Proposition 3.2.2; in general, the construction provides a morphism of exact sequences

(20)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \underline{\mathrm{Hom}}(\pi_1(G), J_C) & \xrightarrow{j_G(\iota_G, \delta)} & \underline{\mathrm{Pic}}(\mathcal{M}_G^d) & \xrightarrow{c_G(\iota_G, \delta)} & \mathrm{NS}(\mathcal{M}_G^d) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{pr}^*} & & \downarrow{\scriptstyle t_\xi^* \circ \iota_G^*} & & \downarrow{\scriptstyle (\iota_G)^{\mathrm{NS}, \delta}} & & \\
0 & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda_{T_G}, J_C) & \xrightarrow{j_{T_G}} & \underline{\mathrm{Pic}}(\mathcal{M}_{T_G}^0) & \xrightarrow{c_{T_G}} & \mathrm{NS}(\mathcal{M}_{T_G}) & \longrightarrow & 0
\end{array}$$

whose three vertical maps are all injective. Using Proposition 3.2.2(iii), this implies that $j_G(\iota_G, \delta)$ and $c_G(\iota_G, \delta)$ depend at most on the choice of $\iota_G : T_G \hookrightarrow G$ and of $\delta$, but not on the choice of $\widehat{G}$, $L$ or $\widehat{\xi}$; thus the notation. Together with the following two lemmas, this proves the remaining parts (iii) and (iv) of the theorem. □

Lemma 5.3.3. *The above map* $j_G(\iota_G, \delta) : \underline{\mathrm{Hom}}(\pi_1(G), J_C) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{M}_G^d)$

i) *does not depend on the lift* $\delta \in \Lambda_{T_G}$ *of* $d \in \pi_1(G)$,
ii) *does not depend on the maximal torus* $\iota_G : T_G \hookrightarrow G$, *and*
iii) *satisfies* $\varphi^* \circ j_H = j_G \circ \varphi^* : \underline{\mathrm{Hom}}(\pi_1(H), J_C) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{M}_G^d)$ *for all* $\varphi : G \longrightarrow H$.

*Proof.* If $G$ is a torus, then $\delta$ and $\iota_G$ are unique, so (i) and (ii) hold trivially. The claim is empty for $g_C = 0$, so we assume $g_C \geq 1$. Then the above construction works for all lifts $\delta$ of $d$, because $\iota^*_{\widehat{\xi}}$ is a closed immersion for all $\widehat{\xi}$.

Given $\varphi : G \longrightarrow H$ and a maximal torus $\iota_H : T_H \hookrightarrow H$ with $\varphi(T_G) \subseteq T_H$, we again put $e := \varphi_* d \in \pi_1(H)$ and $\eta := \varphi_* \delta \in \Lambda_{T_H}$. Then the diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\pi_1(H), J_C) & \xrightarrow{j_H(\iota_H, \eta)} & \underline{\mathrm{Pic}}(\mathcal{M}^e_H) \\ \big\downarrow \varphi^* & & \big\downarrow \varphi^* \\ \underline{\mathrm{Hom}}(\pi_1(G), J_C) & \xrightarrow{j_G(\iota_G, \delta)} & \underline{\mathrm{Pic}}(\mathcal{M}^d_G) \end{array} \tag{21}$$

commutes, because it commutes after composition with the closed immersion

$$t^*_\xi \circ \iota^*_G : \underline{\mathrm{Pic}}(\mathcal{M}^d_G) \longrightarrow \underline{\mathrm{Pic}}(\mathcal{M}^0_{T_G})$$

from diagram (20), using Remark 3.2.3. In particular, (iii) follows from (i) and (ii).

i) For $G = \mathrm{GL}_2$, it suffices to take $\varphi = \det : \mathrm{GL}_2 \longrightarrow \mathbb{G}_m$ in the above diagram (21), since $\det_* : \pi_1(\mathrm{GL}_2) \longrightarrow \pi_1(\mathbb{G}_m)$ is an isomorphism.

For $G = \mathrm{PGL}_2$, it then suffices to take $\varphi = \mathrm{pr} : \mathrm{GL}_2 \twoheadrightarrow \mathrm{PGL}_2$ in the same diagram (21), since $\mathrm{pr}_* : \pi_1(\mathrm{GL}_2) \longrightarrow \pi_1(\mathrm{PGL}_2)$ is surjective.

As (i) holds trivially for $G = \mathrm{SL}_2$, and clearly holds for $G \times \mathbb{G}_m$ if it holds for $G$, this proves (i) for all groups $G$ of semisimple rank one.

In the general case, let $\alpha^\vee \in \Lambda_{T_G}$ be a coroot, and let $\varphi : G_\alpha \hookrightarrow G$ be the corresponding subgroup of semisimple rank one. Then the diagram (21) shows $j_G(\iota_G, \delta) = j_G(\iota_G, \delta + \alpha^\vee)$, since $\varphi_* : \pi_1(G_\alpha) \longrightarrow \pi_1(G)$ is surjective. This completes the proof of i, because any two lifts $\delta$ of $d$ differ by a sum of coroots.

ii) now follows from Weyl–invariance; cf. Subsection 4.3. $\square$

Lemma 5.3.4. *The above map $c_G(\iota_G, \delta) : \underline{\mathrm{Pic}}(\mathcal{M}^d_G) \longrightarrow \mathrm{NS}(\mathcal{M}^d_G)$*

i) *does not depend on the lift $\delta \in \Lambda_{T_G}$ of $d \in \pi_1(G)$,*
ii) *does not depend on the maximal torus $\iota_G : T_G \hookrightarrow G$, and*
iii) *satisfies $\varphi^{\mathrm{NS},d} \circ c_H = c_G \circ \varphi^* : \underline{\mathrm{Pic}}(\mathcal{M}^e_H) \longrightarrow \mathrm{NS}(\mathcal{M}^d_G)$ for all $\varphi : G \longrightarrow H$.*

*Proof.* If $G$ is a torus, then $\delta$ and $\iota_G$ are unique; if $G$ is simply connected, then $c_G(\iota_G, \delta)$ coincides by construction with the isomorphism $c_G$ of Proposition 4.4.7(ii). In both cases, (i) and (ii) follow, and we can use the notation $c_G$ without ambiguity.

Given a representation $\rho : G \longrightarrow \mathrm{SL}(V)$, the diagram

$$\begin{array}{ccc} \underline{\mathrm{Pic}}(\mathcal{M}_{\mathrm{SL}(V)}) & \xrightarrow{c_{\mathrm{SL}(V)}} & \mathrm{NS}(\mathcal{M}_{\mathrm{SL}(V)}) \\ \big\downarrow \rho^* & & \big\downarrow \rho^{\mathrm{NS},d} \\ \underline{\mathrm{Pic}}(\mathcal{M}^d_G) & \xrightarrow{c_G(\iota_G, \delta)} & \mathrm{NS}(\mathcal{M}^d_G) \end{array} \tag{22}$$

commutes, because it commutes after composition with the injective map

$$(\iota_G)^{\mathrm{NS},\delta} : \mathrm{NS}(\mathcal{M}_G^d) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_G})$$

from diagram (20), using Lemma 5.2.9, Corollary 4.4.2, Remark 3.2.3, and the 2–commutative squares

$$\begin{array}{ccccc}
\mathcal{M}^0_{T_G} & \xrightarrow{t_\xi} & \mathcal{M}^\delta_{T_G} & \xrightarrow{(\iota_G)_*} & \mathcal{M}^d_G \\
\downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} & & \downarrow{\scriptstyle \rho_*} \\
\mathcal{M}^0_{T_{\mathrm{SL}(V)}} & \xrightarrow{t_{\rho_*\xi}} & \mathcal{M}^{\rho_*\delta}_{T_{\mathrm{SL}(V)}} & \xrightarrow{\iota_*} & \mathcal{M}_{\mathrm{SL}(V)})
\end{array}$$

in which $\iota : T_{\mathrm{SL}(V)} \hookrightarrow \mathrm{SL}(V)$ is a maximal torus containing $\rho(T_G)$.
Similarly, given a homomorphism $\chi : G \longrightarrow T$ to a torus $T$, the diagram

$$\begin{array}{ccc}
\underline{\mathrm{Pic}}(\mathcal{M}_T^{\chi_* d}) & \xrightarrow{c_T} & \mathrm{NS}(\mathcal{M}_T) \\
\downarrow{\scriptstyle \chi^*} & & \downarrow{\scriptstyle \chi^*} \\
\underline{\mathrm{Pic}}(\mathcal{M}_G^d) & \xrightarrow{c_G(\iota_G,\delta)} & \mathrm{NS}(\mathcal{M}_G^d)
\end{array} \tag{23}$$

commutes, again because it commutes after composition with the same injective map $(\iota_G)^{\mathrm{NS},\delta}$ from diagram (20), using Lemma 5.2.9, Remark 3.2.3, and the 2–commutative squares

$$\begin{array}{ccccc}
\mathcal{M}^0_{T_G} & \xrightarrow{t_\xi} & \mathcal{M}^\delta_{T_G} & \xrightarrow{(\iota_G)_*} & \mathcal{M}^d_G \\
\downarrow{\scriptstyle \chi_*} & & \downarrow{\scriptstyle \chi_*} & & \downarrow{\scriptstyle \chi_*} \\
\mathcal{M}^0_T & \xrightarrow{t_{\chi_*\xi}} & \mathcal{M}^{\chi_*\delta}_T & = & \mathcal{M}^{\chi_*\delta}_T.
\end{array}$$

The two commutative diagrams (22) and (23) show that the restriction of $c_G(\iota_G, \delta)$ to the images of all $\rho^*$ and all $\chi^*$ in $\underline{\mathrm{Pic}}(\mathcal{M}_G^d)$ modulo $\underline{\mathrm{Hom}}(\pi_1(G), J_C)$ does not depend on the choice of $\delta$ or $\iota_G$. But these images generate a subgroup of finite index, according to Proposition 5.2.11 and Remark 4.3.3. Thus (i) and (ii) follow. The functoriality in (iii) is proved similarly; it suffices to apply these arguments to homomorphisms $\rho : H \longrightarrow \mathrm{SL}(V)$, $\chi : H \longrightarrow T$ and their compositions with $\varphi : G \longrightarrow H$, using Corollary 5.2.10. □

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# Singular Bott-Chern Classes and the Arithmetic Grothendieck Riemann Roch Theorem for Closed Immersions

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Abstract. We study the singular Bott-Chern classes introduced by Bismut, Gillet and Soulé. Singular Bott-Chern classes are the main ingredient to define direct images for closed immersions in arithmetic $K$-theory. In this paper we give an axiomatic definition of a theory of singular Bott-Chern classes, study their properties, and classify all possible theories of this kind. We identify the theory defined by Bismut, Gillet and Soulé as the only one that satisfies the additional condition of being homogeneous. We include a proof of the arithmetic Grothendieck-Riemann-Roch theorem for closed immersions that generalizes a result of Bismut, Gillet and Soulé and was already proved by Zha. This result can be combined with the arithmetic Grothendieck-Riemann-Roch theorem for submersions to extend this theorem to arbitrary projective morphisms. As a byproduct of this study we obtain two results of independent interest. First, we prove a Poincaré lemma for the complex of currents with fixed wave front set, and second we prove that certain direct images of Bott-Chern classes are closed.

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Contents



## 0 Introduction

Chern-Weil theory associates to each hermitian vector bundle a family of closed characteristic forms that represent the characteristic classes of the vector bundle. The characteristic classes are compatible with exact sequences. But this is not true for the characteristic forms. The Bott-Chern classes measure the lack of compatibility of the characteristic forms with exact sequences.

The Grothendieck-Riemann-Roch theorem gives a formula that relates direct images and characteristic classes. In general this formula is not valid for the characteristic forms. The singular Bott-Chern classes measure, in a functorial way, the failure of an exact Grothendieck-Riemann-Roch theorem for closed immersions at the level of characteristic forms. In the same spirit, the analytic torsion forms measure the failure of an exact Grothendieck-Riemann-Roch theorem for submersions at the level of characteristic forms. Hence singular Bott-Chern classes and analytic torsion forms are analogous objects, the first for closed immersions and the second for submersions.

Let us give a more precise description of Bott-Chern classes and singular Bott-Chern classes. Let $X$ be a complex manifold and let $\varphi$ be a symmetric power series in $r$ variables with real coefficients. Let $\overline{E}=(E,h)$ be a rank $r$ holomorphic vector bundle provided with a hermitian metric. Using Chern-Weil theory, we can associate to $\overline{E}$ a differential form $\varphi(\overline{E})=\varphi(-K)$, where $K$ is

the curvature tensor of $E$ viewed as a matrix of 2-forms. The differential form $\varphi(\overline{E})$ is closed and is a sum of components of bidegree $(p,p)$ for $p \geq 0$.
If

$$\overline{\xi}\colon 0 \longrightarrow \overline{E}' \longrightarrow \overline{E} \longrightarrow \overline{E}'' \longrightarrow 0$$

is a short exact sequence of holomorphic vector bundles provided with hermitian metrics, then the differential forms $\varphi(\overline{E})$ and $\varphi(\overline{E}' \oplus \overline{E}')$ may be different, but they represent the same cohomology class.
The Bott-Chern form associated to $\overline{\xi}$ is a solution of the differential equation

$$-2\partial\bar{\partial}\varphi(\overline{\xi}) = \varphi(\overline{E}' \oplus \overline{E}') - \varphi(\overline{E}) \tag{0.1}$$

obtained in a functorial way. The class of a Bott-Chern form modulo the image of $\partial$ and $\overline{\partial}$ is called a Bott-Chern class and is denoted by $\widetilde{\varphi}(\overline{\xi})$.
There are three ways of defining the Bott-Chern classes. The first one is the original definition of Bott and Chern [7]. It is based on a deformation between the connection associated to $\overline{E}$ and the connection associated to $\overline{E}' \oplus \overline{E}''$. This deformation is parameterized by a real variable.
In [17] Gillet and Soulé introduced a second definition of Bott-Chern classes that is based on a deformation between $\overline{E}$ and $\overline{E}' \oplus \overline{E}''$ parameterized by a projective line. This second definition is used in [4] to prove that the Bott-Chern classes are characterized by three properties

(i) The differential equation (0.1).

(ii) Functoriality (i.e. compatibility with pull-backs via holomorphic maps).

(iii) The vanishing of the Bott-Chern class of a orthogonally split exact sequence.

In [4] Bismut, Gillet and Soulé have a third definition of Bott-Chern classes based on the theory of superconnections. This definition is useful to link Bott-Chern classes with analytic torsion forms.
The definition of Bott-Chern classes can be generalized to any bounded exact sequence of hermitian vector bundles (see section 2 for details). Let

$$\overline{\xi}\colon 0 \longrightarrow (E_n, h_n) \longrightarrow \dots \longrightarrow (E_1, h_1) \longrightarrow (E_0, h_0) \longrightarrow 0$$

be a bounded acyclic complex of hermitian vector bundles; by this we mean a bounded acyclic complex of vector bundles, where each vector bundle is equipped with an arbitrarily chosen hermitian metric. Let

$$r = \sum_{i \text{ even}} \operatorname{rk}(E_i) = \sum_{i \text{ odd}} \operatorname{rk}(E_i).$$

As before, let $\varphi$ be a symmetric power series in $r$ variables. A Bott-Chern class associated to $\overline{\xi}$ satisfies the differential equation

$$-2\partial\bar{\partial}\widetilde{\varphi}(\overline{\xi}) = \varphi(\bigoplus_k \overline{E}_{2k}) - \varphi(\bigoplus_k \overline{E}_{2k+1}).$$

In particular, let "ch" denote the power series associated to the Chern character class. The Chern character class has the advantage of being additive for direct sums. Then, the Bott-Chern class associated to the long exact sequence $\overline{\xi}$ and to the Chern character class satisfies the differential equation

$$-2\partial\bar{\partial}\widetilde{\mathrm{ch}}(\overline{\xi}) = -\sum_{k=0}^{n}(-1)^{i}\,\mathrm{ch}(\overline{E}_k).$$

Let now $i\colon Y \longrightarrow X$ be a closed immersion of complex manifolds. Let $\overline{F}$ be a holomorphic vector bundle on $Y$ provided with a hermitian metric. Let $\overline{N}$ be the normal bundle to $Y$ in $X$ provided also with a hermitian metric. Let

$$0 \longrightarrow \overline{E}_n \longrightarrow \overline{E}_{n-1} \longrightarrow \dots \longrightarrow \overline{E}_0 \longrightarrow i_*F \longrightarrow 0$$

be a resolution of the coherent sheaf $i_*F$ by locally free sheaves, provided with hermitian metrics (following Zha [32] we shall call such a sequence a metric on the coherent sheaf $i_*F$). Let Td denote the Todd characteristic class. Then the Grothendieck-Riemann-Roch theorem for the closed immersion $i$ implies that the current $i_*(\mathrm{Td}(\overline{N})^{-1}\,\mathrm{ch}(\overline{F}))$ and the differential form $\sum_k(-1)^k\,\mathrm{ch}(\overline{E}_k)$ represent the same class in cohomology. We denote $\overline{\xi}$ the data consisting in the closed embedding $i$, the hermitian bundle $\overline{N}$, the hermitian bundle $\overline{F}$ and the resolution $\overline{E}_* \longrightarrow i_*F$.

In the paper [5], Bismut, Gillet and Soulé introduced a current associated to the above situation. These currents are called singular Bott-Chern currents and denoted in [5] by $T(\overline{\xi})$. When the hermitian metrics satisfy a certain technical condition (condition A of Bismut) then the singular Bott-Chern current $T(\overline{\xi})$ satisfies the differential equation

$$-2\partial\bar{\partial}T(\overline{\xi}) = i_*(\mathrm{Td}(\overline{N})^{-1}\,\mathrm{ch}(\overline{F})) - \sum_{i=0}^{n}(-1)^{i}\,\mathrm{ch}(\overline{E}_i).$$

These singular Bott-Chern currents are among the main ingredients of the proof of Gillet and Soulé's arithmetic Riemann-Roch theorem. In fact it is the main ingredient of the arithmetic Riemann-Roch theorem for closed immersions [6]. This definition of singular Bott-Chern classes is based on the formalism of superconnections, like the third definition of ordinary Bott-Chern classes.

In his thesis [32], Zha gave another definition of singular Bott-Chern currents and used it to give a proof of a different version of the arithmetic Riemann-Roch theorem. This second definition is analogous to Bott and Chern's original definition. Nevertheless there is no explicit comparison between the two definitions of singular Bott-Chern currents.

One of the purposes of this note is to give a third construction of singular Bott-Chern currents, in fact of their classes modulo the image of $\partial$ and $\overline{\partial}$, which could be seen as analogous to the second definition of Bott-Chern classes. Moreover we will use this third construction to give an axiomatic definition of a theory of singular Bott-Chern classes. A theory of singular Bott-Chern classes is an assignment that, to each data $\overline{\xi}$ as above, associates a class of currents $T(\overline{\xi})$, that satisfies the analogue of conditions (i), (ii) and (iii). The main technical point of this axiomatic definition is that the conditions analogous to (i), (ii) and (iii) above are not enough to characterize the singular Bott-Chern classes. Thus we are led to the problem of classifying the possible theories of Bott-Chern classes, which is the other purpose of this paper.

We fix a theory $T$ of singular Bott-Chern classes. Let $Y$ be a complex manifold and let $\overline{N}$ and $\overline{F}$ be two hermitian holomorphic vector bundles on $Y$. We write $P = \mathbb{P}(N \oplus 1)$ for the projective completion of $N$. Let $s\colon Y \longrightarrow P$ be the inclusion as the zero section and let $\pi_P\colon P \longrightarrow Y$ be the projection. Let $\overline{K}_*$ be the Koszul resolution of $s_*\mathcal{O}_Y$ endowed with the metric induced by $\overline{N}$. Then we have a resolution by hermitian vector bundles

$$K(\overline{F}, \overline{N})\colon \overline{K}_* \otimes \pi_P^*\overline{F} \longrightarrow s_*F.$$

To these data we associate a singular Bott-Chern class $T(K(\overline{F}, \overline{N}))$. It turns out that the current

$$\frac{1}{(2\pi i)^{\operatorname{rk} N}} \int_{\pi_P} T(K(\overline{F}, \overline{N})) = (\pi_P)_* T(K(\overline{F}, \overline{N}))$$

is closed (see section 3 for general properties of the Bott-Chern classes that imply this property) and determines a characteristic class $C_T(F, N)$ on $Y$ for the vector bundles $N$ and $F$. Conversely, any arbitrary characteristic class for pairs of vector bundles can be obtained in this way. This allows us to classify the possible theories of singular Bott-Chern classes:

Claim (theorem 7.1). The assignment that sends a singular Bott-Chern class $T$ to the characteristic class $C_T$ is a bijection between the set of theories of singular Bott-Chern classes and the set of characteristic classes.

The next objective of this note is to study the properties of the different theories of singular Bott-Chern classes and of the corresponding characteristic classes. We mention, in the first place, that for the functoriality condition to make sense, we have to study the wave front sets of the currents representing the singular Bott-Chern classes. In particular we use a Poincaré Lemma for currents with fixed wave front set. This result implies that, in each singular Bott-Chern class, we can find a representative with controlled wave front set that can be pulled back with respect certain morphisms.

We also investigate how different properties of the singular Bott-Chern classes $T$ are reflected in properties of the characteristic classes $C_T$. We thus characterize the compatibility of the singular Bott-Chern classes with the projection formula, by the property of $C_T$ of being compatible with the projection formula. We also relate the compatibility of the singular Bott-Chern classes with the composition of successive closed immersions to an additivity property of the associated characteristic class.

Furthermore, we show that we can add a natural fourth axiom to the conditions analogue to (i), (ii) and (iii), namely the condition of being homogeneous (see section 9 for the precise definition).

Claim (theorem 9.11). There exists a unique homogeneous theory of singular Bott-Chern classes.

Thanks to this axiomatic characterization, we prove that this theory agrees with the theories of singular Bott-Chern classes introduced by Bismut, Gillet and Soulé [6], and by Zha [32]. In particular this provides us a comparison between the two definitions. We will also characterize the characteristic class $C_{T^h}$ for the theory of homogeneous singular Bott-Chern classes.
The last objective of this paper is to give a proof of the arithmetic Riemann-Roch theorem for closed immersions. A version of this theorem was proved by Bismut, Gillet and Soulé and by Zha.
Next we will discuss the contents of the different sections of this paper. In section §1 we recall the properties of characteristic classes in analytic Deligne cohomology. A characteristic class is just a functorial assignment that associates a cohomology class to each vector bundle. The main result of this section is that any characteristic class is given by a power series on the Chern classes, with appropriate coefficients.
In section §2 we recall the theory of Bott-Chern forms and its main properties. The contents of this section are standard although the presentation is slightly different to the ones published in the literature.
In section §3 we study certain direct images of Bott-Chern forms. The main result of this section is that, even if the Bott-Chern classes are not closed, certain direct images of Bott-Chern classes are closed. This result generalizes previous results of Bismut, Gillet and Soulé and of Mourougane. This result is used to prove that the class $C_T$ mentioned previously is indeed a cohomology class, but it can be of independent interest because it implies that several identities in characteristic classes are valid at the level of differential forms.
In section §4 we study the cohomology of the complex of currents with a fixed wave front set. The main result of this section is a Poincaré lemma for currents of this kind. This implies in particular a $\partial\bar{\partial}$-lemma. The results of this section are necessary to state the functorial properties of singular Bott-Chern classes.
In section §5 we recall the deformation of resolutions, that is a generalization of the deformation to the normal cone, and we also recall the construction of the Koszul resolution. These are the main geometric tools used to study singular Bott-Chern classes.
Sections §6 to §9 are devoted to the definition and study of the theories of singular Bott-Chern classes. Section §6 contains the definition and first properties. Section §7 is devoted to the classification theorem of such theories. In section §8 we study how properties of the theory of singular Bott-Chern classes and of the associated characteristic class are related. And in section §9 we define the theory of homogeneous singular Bott-Chern classes and we prove that it agrees with the theories defined by Bismut, Gillet and Soulé and by Zha.

Finally in section §10 we define arithmetic $K$-groups associated to a $\mathcal{D}_{\log}$-arithmetic variety $(\mathcal{X}, \mathcal{C})$ (in the sense of [13]) and push-forward maps for closed immersions of metrized arithmetic varieties, at the level of the arithmetic $K$-groups. After studying the compatibility of these maps with the projection formula and with the push-forward map at the level of currents, we prove a general Riemann-Roch theorem for closed immersions (theorem 10.28) that compares the direct images in the arithmetic $K$-groups with the direct images in the arithmetic Chow groups. This theorem is compatible, if we choose the theory of homogeneous singular Bott-Chern classes, with the arithmetic Riemann-Roch theorem for closed immersions proved by Bismut, Gillet and Soulé [6] and it agrees with the theorem proved by Zha [32]. Theorem 10.28, together with the arithmetic Grothendieck-Riemann-Roch theorem for submersions proved in [16], can be used to obtain an arithmetic Grothendieck-Riemann-Roch theorem for projective morphisms of regular arithmetic varieties.
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## 1 Characteristic classes in analytic Deligne cohomology

A characteristic class for complex vector bundles is a functorial assignment which, to each complex continuous vector bundle on a paracompact topological space $X$, assigns a class in a suitable cohomology theory of $X$. For example, if the cohomology theory is singular cohomology, it is well known that each characteristic class can be expressed as a power series in the Chern classes. This can be seen for instance, showing that continuous complex vector bundles on a paracompact space $X$ can be classified by homotopy classes of maps from $X$ to the classifying space $BGL_\infty(\mathbb{C})$ and that the cohomology of $BGL_\infty(\mathbb{C})$ is generated by the Chern classes (see for instance [28]).
The aim of this section is to show that a similar result is true if we restrict the class of spaces to the class of quasi-projective smooth complex manifolds, the class of maps to the class of algebraic maps and the class of vector bundles to the class of algebraic vector bundles and we choose analytic Deligne cohomology as our cohomology theory.
This result and the techniques used to prove it are standard. We will use the splitting principle to reduce to the case of line bundles and will then use the projective spaces as a model of the classifying space $BGL_1(\mathbb{C})$. In this section

we also recall the definition of Chern classes in analytic Deligne cohomology and we fix some notations that will be used through the paper.

Definition 1.1. Let $X$ be a complex manifold. For each integer $p$, *the analytic real Deligne complex* of $X$ is

$$\mathbb{R}_{X,\mathcal{D}}(p) = (\underline{\mathbb{R}}(p) \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \dots \longrightarrow \Omega^{p-1}_X)$$
$$\cong s(\underline{\mathbb{R}}(p) \oplus F^p\Omega^*_X \longrightarrow \Omega^*_X),$$

where $\underline{\mathbb{R}}(p)$ is the constant sheaf $(2\pi i)^p\mathbb{R} \subseteq \underline{\mathbb{C}}$. The *analytic real Deligne cohomology of* $X$, denoted $H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(p))$, is the hyper-cohomology of the above complex.

Analytic Deligne cohomology satisfies the following result.

Theorem 1.2. *The assignment* $X \longmapsto H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(*)) = \bigoplus_p H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(p))$ *is a contravariant functor between the category of complex manifolds and holomorphic maps and the category of unitary bigraded rings that are graded commutative (with respect to the first degree) and associative. Moreover there exists a functorial map*

$$c\colon \operatorname{Pic}(X) = H^1(X, \mathcal{O}^*_X) \longrightarrow H^2_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(1))$$

*and, for each closed immersion of complex manifolds* $i\colon Y \longrightarrow X$ *of codimension* $p$*, there exists a morphism*

$$i_*\colon H^*_{\mathcal{D}^{\mathrm{an}}}(Y, \mathbb{R}(*)) \longrightarrow H^{*+2p}_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(* + p))$$

*satisfying the properties*

*A1 Let* $X$ *be a complex manifold and let* $E$ *be a holomorphic vector bundle of rank* $r$*. Let* $\mathbb{P}(E)$ *be the associated projective bundle and let* $\mathcal{O}(-1)$ *the tautological line bundle. The map*

$$\pi^*\colon H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(*)) \longrightarrow H^*_{\mathcal{D}^{\mathrm{an}}}(\mathbb{P}(E), \mathbb{R}(*))$$

*induced by the projection* $\pi\colon \mathbb{P}(E) \longrightarrow X$ *gives to the second ring a structure of left module over the first. Then the elements* $c(\mathrm{cl}(\mathcal{O}(-1)))^i$*,* $i = 0, \dots, r-1$ *form a basis of this module.*

*A2 If* $X$ *is a complex manifold,* $L$ *a line bundle,* $s$ *a holomorphic section of* $L$ *that is transverse to the zero section,* $Y$ *is the zero locus of* $s$ *and* $i\colon Y \longrightarrow X$ *the inclusion, then*

$$c(\mathrm{cl}(L)) = i_*(1_Y).$$

*A3 If* $j\colon Z \longrightarrow Y$ *and* $i\colon Y \longrightarrow X$ *are closed immersions of complex manifolds then* $(ij)_* = i_*j_*$.

*A4 If $i\colon Y \longrightarrow X$ is a closed immersion of complex manifolds then, for every $a \in H^*_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(*))$ and $b \in H^*_{\mathcal{D}^{\mathrm{an}}}(Y, \mathbb{R}(*))$*

$$i_*(bi^*a) = (i_*b)a.$$

*Proof.* The functoriality is clear. The product structure is described, for instance, in [15]. The morphism $c$ is defined by the morphism in the derived category

$$\mathcal{O}^*_X[1] \xleftarrow{\cong} s(\underline{\mathbb{Z}}(1) \to \mathcal{O}_X) \longrightarrow s(\underline{\mathbb{R}}(1) \to \mathcal{O}_X) = \mathbb{R}_{\mathcal{D}}(1).$$

The morphism $i_*$ can be constructed by resolving the sheaves $\mathbb{R}_{\mathcal{D}}(p)$ by means of currents (see [26] for a related construction). Properties A3 and A4 follow easily from this construction.

By abuse of notation, we will denote by $c_1(\mathcal{O}(-1))$ the first Chern class of $\mathcal{O}(-1)$ with the algebro-geometric twist, in any of the groups $H^2(\mathbb{P}(E), \underline{\mathbb{R}}(1))$, $H^2(\mathbb{P}(E), \underline{\mathbb{C}})$, $H^1(\mathbb{P}(E), \Omega^1_{\mathbb{P}(E)})$. Then, we have sheaf isomorphisms (see for instance [22] for a related result),

$$\bigoplus_{i=0}^{r-1} \underline{\mathbb{R}}_X(p-i)[-2i] \longrightarrow R\pi_*\underline{\mathbb{R}}_{\mathbb{P}(E)}(p)$$

$$\bigoplus_{i=0}^{r-1} \Omega^*_X[-2i] \longrightarrow R\pi_*\Omega^*_{\mathbb{P}(E)}$$

$$\bigoplus_{i=0}^{r-1} F^{p-i}\Omega^*_X[-2i] \longrightarrow R\pi_*F^p\Omega^*_{\mathbb{P}(E)}$$

given, all of them, by $(a_0, \dots, a_{r-1}) \longmapsto \sum a_i c_1(\mathcal{O}(-1))^i$. Hence we obtain a sheaf isomorphism

$$\bigoplus_{i=0}^{r-1} \mathbb{R}_{X,\mathcal{D}}(p-i)[-2i] \longrightarrow R\pi_*\mathbb{R}_{\mathbb{P}(E),\mathcal{D}}(p)$$

from which property A1 follows. Finally property A2 in this context is given by the Poincare-Lelong formula (see [13] proposition 5.64). □

Notation 1.3. For the convenience of the reader, we gather here together several notations and conventions regarding the differential forms, currents and Deligne cohomology that will be used through the paper.

Throughout this paper we will use consistently the algebro-geometric twist. In particular the Chern classes $c_i$, $i = 0, \dots$ in Betti cohomology will live in $c_i \in H^{2i}(X, \mathbb{R}(i))$; hence our normalizations differ from the ones in [18] where real forms and currents are used.

Moreover we will use the following notations. We will denote by $\mathscr{E}^*_X$ the sheaf of Dolbeault algebras of differential forms on $X$ and by $\mathscr{D}^*_X$ the sheaf of Dolbeault complexes of currents on $X$ (see [13] §5.4 for the structure of Dolbeault complex of $\mathscr{D}^*_X$). We will denote by $E^*(X)$ and by $D^*(X)$ the complexes of global sections of $\mathscr{E}^*_X$ and $\mathscr{D}^*_X$ respectively. Following [9] and [13] definition 5.10, we denote by $(\mathcal{D}^*(\_,*), \mathrm{d}_{\mathcal{D}})$ the functor that associates to a Dolbeault complex its corresponding Deligne complex. For shorthand, we will denote

$$\mathcal{D}^*(X,p) = \mathcal{D}^*(E^*(X),p),$$
$$\mathcal{D}^*_D(X,p) = \mathcal{D}^*(D^*(X),p).$$

To keep track of the algebro-geometric twist we will use the conventions of [13] §5.4 regarding the current associated to a locally integrable differential form

$$[\omega](\eta) = \frac{1}{(2\pi i)^{\dim X}} \int_X \eta \wedge \omega$$

and the current associated with a subvariety $Y$

$$\delta_Y(\eta) = \frac{1}{(2\pi i)^{\dim Y}} \int_Y \eta.$$

With these conventions, we have a bigraded morphism $\mathcal{D}^*(X,*) \to \mathcal{D}^*_D(X,*)$ and, if $Y$ has codimension $p$, the current $\delta_Y$ belongs to $\mathcal{D}^{2p}_D(X,p)$. Then $\mathcal{D}^*(X,p)$ and $\mathcal{D}^*_D(X,p)$ are the complex of global sections of an acyclic resolution of $\mathbb{R}_{X,\mathcal{D}}(p)$. Therefore

$$H^*_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(p)) = H^*(\mathcal{D}(X,p)) = H^*(\mathcal{D}_D(X,p)).$$

If $f : X \to Y$ is a proper smooth morphism of complex manifolds of relative dimension $e$, then the integral along the fibre morphism

$$f_* : \mathcal{D}^k(X,p) \longrightarrow \mathcal{D}^{k-2e}(X,p-e)$$

is given by

$$f_*\omega = \frac{1}{(2\pi i)^e} \int_f \omega. \tag{1.4}$$

If $(\mathcal{D}^*(*), \mathrm{d}_{\mathcal{D}})$ is a Deligne complex associated to a Dolbeault complex, we will write

$$\widetilde{\mathcal{D}}^k(X,p) := \mathcal{D}^k(X,p)/\,\mathrm{d}_{\mathcal{D}}\,\mathcal{D}^{k-1}(X,p).$$

Finally, following [13] 5.14 we denote by $\bullet$ the product in the Deligne complex that induces the usual product in Deligne cohomology. Note that, if $\omega \in \bigoplus_p \mathcal{D}^{2p}(X,p)$, then for any $\eta \in \mathcal{D}^*(X,*)$ we have $\omega \bullet \eta = \eta \bullet \omega = \eta \wedge \omega$. Sometimes, in this case we will just write $\eta\omega := \eta \bullet \omega$.

We denote by $*$ the complex manifold consisting on one single point. Then

$$H^n_{\mathcal{D}^{\mathrm{an}}}(*,p) = \begin{cases} \mathbb{R}(p) := (2\pi i)^p\mathbb{R}, & \text{if } n=0,\ p \le 0, \\ \mathbb{R}(p-1) := (2\pi i)^{p-1}\mathbb{R}, & \text{if } n=1,\ p>0. \\ \{0\}, & \text{otherwise.} \end{cases}$$

The product structure in this case is the bigraded product that is given by complex number multiplication when the degrees allow the product to be non zero. We will denote by $\mathbb{D}$ this ring. This is the base ring for analytic Deligne cohomology. Note that, in particular, $H^1_{\mathcal{D}^{\mathrm{an}}}(*,1) = \mathbb{R} = \mathbb{C}/\mathbb{R}(1)$. We will denote by $\mathbf{1}_1$ the image of 1 in $H^1_{\mathcal{D}^{\mathrm{an}}}(*,1)$.
Following [23], theorem 1.2 implies the existence of a theory of Chern classes for holomorphic vector bundles in analytic Deligne cohomology. That is, to every vector bundle $E$, we can associate a collection of Chern classes $c_i(E) \in H^{2i}_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(i))$, $i \geq 1$ in a functorial way.
We want to see that all possible characteristic classes in analytic Deligne cohomology can be derived from the Chern classes.

Definition 1.5. Let $n \geq 1$ be an integer and let $r_1 \geq 1, \dots, r_n \geq 1$ be a collection of integers. A *theory of characteristic classes for $n$-tuples of vector bundles of rank* $r_1, \dots, r_n$ is an assignment that, to each $n$-tuple of isomorphism classes of vector bundles $(E_1, \dots, E_n)$ over a complex manifold $X$, with $\operatorname{rk}(E_i) = r_i$, assigns a class

$$\operatorname{cl}(E_1, \dots, E_n) \in \bigoplus_{k,p} H^k_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(p))$$

in a functorial way. That is, for every morphism $f\colon X \longrightarrow Y$ of complex manifolds, the equality

$$f^*(\operatorname{cl}(E_1, \dots, E_n)) = \operatorname{cl}(f^*E_1, \dots, f^*E_n)$$

holds

The first consequence of the functoriality and certain homotopy property of analytic Deligne cohomology classes is the following.

Proposition 1.6. *Let* cl *be a theory of characteristic classes for $n$-tuples of vector bundles of rank $r_1, \dots, r_n$. Let $X$ be a complex manifold and let $(E_1, \dots, E_n)$ be a $n$-tuple of vector bundles over $X$ with* $\operatorname{rk}(E_i) = r_i$ *for all $i$. Let $1 \leq j \leq n$ and let*

$$0 \longrightarrow E'_j \longrightarrow E_j \longrightarrow E''_j \longrightarrow 0,$$

*be a short exact sequence. Then the equality*

$$\operatorname{cl}(E_1, \dots, E_j, \dots, E_n) = \operatorname{cl}(E_1, \dots, E'_j \oplus E''_j, \dots, E_n)$$

*holds.*

*Proof.* Let $\iota_0, \iota_\infty \colon X \longrightarrow X \times \mathbb{P}^1$ be the inclusion as the fiber over 0 and the fiber over $\infty$ respectively. Then there exists a vector bundle $\widetilde{E}_j$ on $X \times \mathbb{P}^1$ (see for instance [19] (1.2.3.1) or definition 2.5 below) such that $\iota_0^*\widetilde{E}_j \cong E_j$ and $\iota_\infty^*\widetilde{E}_j \cong E'_j \oplus E''_j$. Let $p_1\colon X \times \mathbb{P}^1 \longrightarrow X$ be the first projection. Let $\omega \in \bigoplus_{k,p} \mathcal{D}^k(X,p)$ be any $\mathrm{d}_{\mathcal{D}}$-closed form that represents

$\operatorname{cl}(p_1^*E_1,\dots,\widetilde{E}_j,\dots,p_1^*E_n)$. Then, by functoriality we know that $\iota_0^*\omega$ represents $\operatorname{cl}(E_1,\dots,E_j,\dots,E_n)$ and $\iota_\infty^*\omega$ represents $\operatorname{cl}(E_1,\dots,E_j'\oplus E_j'',\dots,E_n)$. We write

$$\beta=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log t\bar{t}\bullet\omega,$$

where $t$ is the absolute coordinate of $\mathbb{P}^1$. Then

$$\mathrm{d}_{\mathcal{D}}\,\beta=\iota_\infty^*\omega-\iota_0^*\omega$$

which implies the result. □

A standard method to produce characteristic classes for vector bundles is to choose hermitian metrics on the vector bundles and to construct closed differential forms out of them. The following result shows that functoriality implies that the cohomology classes represented by these forms are independent from the hermitian metrics and therefore are characteristic classes. When working with hermitian vector bundles we will use the convention that, if $E$ denotes the vector bundle, then $\overline{E}=(E,h)$ will denote the vector bundle together with the hermitian metric.

Proposition 1.7. *Let $n\geq 1$ be an integer and let $r_1\geq 1,\dots,r_n\geq 1$ be a collection of integers. Let* cl *be an assignment that, to each $n$-tuple $(\overline{E}_1,\dots,\overline{E}_n)=((E_1,h_1),\dots,(E_n,h_n))$ of isometry classes of hermitian vector bundles of rank $r_1,\dots,r_n$ over a complex manifold $X$, associates a cohomology class*

$$\operatorname{cl}(\overline{E}_1,\dots,\overline{E}_n)\in\bigoplus_{k,p}H^k_{\mathcal{D}}(X,\mathbb{R}(p))$$

*such that, for each morphism $f:Y\to X$,*

$$\operatorname{cl}(f^*\overline{E}_1,\dots,f^*\overline{E}_n)=f^*\operatorname{cl}(\overline{E}_1,\dots,\overline{E}_n).$$

*Then the cohomology class $\operatorname{cl}(\overline{E}_1,\dots,\overline{E}_n)$ is independent from the hermitian metrics. Therefore it is a well defined characteristic class.*

*Proof.* Let $1\leq j\leq n$ be an integer and let $\overline{E}_j'=(E_j,h_j')$ be the vector bundle underlying $\overline{E}_j$ with a different choice of metric. Let $\iota_0$, $\iota_\infty$ and $p_1$ be as in the proof of proposition 1.6. Then we can choose a hermitian metric $h$ on $p_1^*E_j$, such that $\iota_0^*(p_1^*E_j,h)=\overline{E}_j$ and $\iota_\infty^*(p_1^*E_j,h)=\overline{E}_j'$. Let $\omega$ be any smooth closed differential form on $X\times\mathbb{P}^1$ that represents $\operatorname{cl}(p_1^*\overline{E}_1,\dots,(p_1^*E_1,h),\dots,p_1^*\overline{E}_n)$. Then,

$$\beta=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log t\bar{t}\bullet\omega$$

satisfies

$$\mathrm{d}_{\mathcal{D}}\,\beta=\iota_\infty^*\omega-\iota_0^*\omega$$

which implies the result. □

We are interested in vector bundles that can be extended to a projective variety. Therefore we will restrict ourselves to the algebraic category. So, by a complex algebraic manifold we will mean the complex manifold associated to a smooth quasi-projective variety over $\mathbb{C}$. When working with an algebraic manifold, by a vector bundle we will mean the holomorphic vector bundle associated to an algebraic vector bundle.
We will denote by $\mathbb{D}[[x_1,\dots,x_r]]$ the ring of commutative formal power series. That is, the unknowns $x_1,\dots,x_r$ commute with each other and with $\mathbb{D}$. We turn it into a commutative bigraded ring by declaring that the unknowns $x_i$ have bidegree $(2,1)$. The symmetric group in $r$ elements, $\mathfrak{S}_r$ acts on $\mathbb{D}[[x_1,\dots,x_r]]$. The subalgebra of invariant elements is generated over $\mathbb{D}$ by the elementary symmetric functions. The main result of this section is the following

Theorem 1.8. *Let* cl *be a theory of characteristic classes for $n$-tuples of vector bundles of rank $r_1,\dots,r_n$. Then, there is a power series $\varphi\in\mathbb{D}[[x_1,\dots,x_r]]$ in $r=r_1+\dots+r_n$ variables with coefficients in the ring $\mathbb{D}$, such that, for each complex algebraic manifold $X$ and each $n$-tuple of algebraic vector bundles $(E_1,\dots,E_n)$ over $X$ with $\operatorname{rk}(E_i)=r_i$ this equality holds:*

$$\operatorname{cl}(E_1,\dots,E_n)=\varphi(c_1(E_1),\dots,c_{r_1}(E_1),\dots,c_1(E_n),\dots,c_{r_n}(E_n)). \tag{1.9}$$

*Conversely, any power series $\varphi$ as before determines a theory of characteristic classes for $n$-tuples of vector bundles of rank $r_1,\dots,r_n$, by equation* (1.9).

*Proof.* The second statement is obvious from the properties of Chern classes. Since we are assuming $X$ quasi-projective, given $n$ algebraic vector bundles $E_1,\dots,E_n$ on $X$, there is a smooth projective compactification $\widetilde{X}$ and vector bundles $\widetilde{E}_1,\dots,\widetilde{E}_n$ on $\widetilde{X}$, such that $E_i=\widetilde{E}_i|_X$ (see for instance [14] proposition 2.2), we are reduced to the case when $X$ is projective. In this case, analytic Deligne cohomology agrees with ordinary Deligne cohomology.
Let us assume first that $r_1=\dots=r_n=1$ and that we have a characteristic class cl for $n$ line bundles. Then, for each $n$-tuple of positive integers $m_1,\dots,m_n$ we consider the space $\mathbb{P}^{m_1,\dots,m_n}=\mathbb{P}^{m_1}_{\mathbb{C}}\times\dots\times\mathbb{P}^{m_n}_{\mathbb{C}}$ and we denote by $p_i$ the projection over the $i$-th factor. Then

$$\bigoplus_{k,p}H^k_{\mathcal{D}}(\mathbb{P}^{m_1,\dots,m_n},\mathbb{R}(p))=\mathbb{D}[x_1,\dots,x_n]\Big/(x_1^{m_1},\dots,x_n^{m_n})$$

is a quotient of the polynomial ring generated by the classes $x_i=c_1(p_i^*\mathcal{O}(1))$ with coefficients in the ring $\mathbb{D}$. Therefore, there is a polynomial $\varphi_{m_1,\dots,m_n}$ in $n$ variables such that

$$\operatorname{cl}(p_1^*\mathcal{O}(1),\dots,p_1^*\mathcal{O}(1))=\varphi_{m_1,\dots,m_n}(x_1,\dots,x_n).$$

If $m_1\le m_1'$, ..., $m_n\le m_n'$ then, by functoriality, the polynomial $\varphi_{m_1,\dots,m_n}$ is the truncation of the polynomial $\varphi_{m_1',\dots,m_n'}$. Therefore there is a power series

in $n$ variables, $\varphi$ such that $\varphi_{m_1,\dots,m_n}$ is the truncation of $\varphi$ in the appropriate quotient of the polynomial ring.
Let $L_1,\dots,L_n$ be line bundles on a projective algebraic manifold that are generated by global sections. Then they determine a morphism $f\colon X \longrightarrow \mathbb{P}^{m_1,\dots,m_n}$ such that $L_i = f^*p_i^*\mathcal{O}(1)$. Therefore, again by functoriality, we obtain

$$\operatorname{cl}(L_1,\dots,L_n) = \varphi(c_1(L_1),\dots,c_1(L_n)).$$

From the class cl we can define a new characteristic class for $n+1$ line bundles by the formula

$$\operatorname{cl}'(L_1,\dots,L_n,M) = \operatorname{cl}(L_1\otimes M^\vee,\dots,L_n\otimes M^\vee).$$

When $L_1,\dots,L_n$ and $M$ are generated by global sections we have that there is a power series $\psi$ such that

$$\operatorname{cl}'(L_1,\dots,L_n,M) = \psi(c_1(L_1),\dots,c_1(L_n),c_1(M)).$$

Moreover, when the line bundles $L_i\otimes M^\vee$ are also generated by global sections the following holds

$$\begin{aligned}\psi(c_1(L_1),\dots,c_1(L_n),c_1(M)) &= \varphi(c_1(L_1\otimes M^\vee),\dots,c_1(L_n\otimes M^\vee))\\ &= \varphi(c_1(L_1)-c_1(M),\dots,c_1(L_n)-c_1(M)).\end{aligned}$$

Considering the system of spaces $\mathbb{P}^{m_1,\dots,m_n,m_{n+1}}$ with line bundles

$$L_i = p_i^*\mathcal{O}(1)\otimes p_{n+1}^*\mathcal{O}(1),\ i=1,\dots,n,\quad M = p_{n+1}^*\mathcal{O}(1),$$

we see that there is an identity of power series

$$\varphi(x_1-y,\dots,x_n-y) = \psi(x_1,\dots,x_n,y).$$

Now let $X$ be a projective complex manifold and let $L_1,\dots,L_n$ be arbitrary line bundles. Then there is a line bundle $M$ such that $M$ and $L_i' = L_i\otimes M$, $i=1,\dots,n$ are generated by global sections. Then we have

$$\begin{aligned}\operatorname{cl}(L_1,\dots,L_n) &= \operatorname{cl}(L_1'\otimes M^\vee,\dots,L_n'\otimes M^\vee)\\ &= \operatorname{cl}'(L_1',\dots,L_n',M)\\ &= \psi(c_1(L_1'),\dots,c_1(L_n'),c_1(M))\\ &= \varphi((c_1(L_1')-c_1(M),\dots,c_1(L_n')-c_1(M)))\\ &= \varphi(c_1(L_1),\dots,c_1(L_n)).\end{aligned}$$

The case of arbitrary rank vector bundles follows from the case of rank one vector bundles by proposition 1.6 and the splitting principle. We next recall the argument. Given a projective complex manifold $X$ and vector bundles $E_1,\dots,E_n$ of rank $r_1,\dots,r_n$, we can find a proper morphism $\pi\colon \widetilde{X} \longrightarrow X$, with $\widetilde{X}$ a complex projective manifold, and such that the induced morphism

$$\pi^*\colon H^*_{\mathcal{D}}(X,\mathbb{R}(*)) \longrightarrow H^*_{\mathcal{D}}(\widetilde{X},\mathbb{R}(*))$$

is injective and every bundle $\pi^*(E_i)$ admits a holomorphic filtration

$$0 = K_{i,0} \subset K_{i,1} \subset \cdots \subset K_{i,r_i-1} \subset K_{i,r_i} = \pi^*(E_i),$$

with $L_{i,j} = K_{i,j}/K_{i,j-1}$ a line bundle. If cl is a characteristic class for $n$-tuples of vector bundles of rank $r_1, \dots, r_n$, we define a characteristic class for $r_1 + \cdots + r_n$-tuples of line bundles by the formula

$$\operatorname{cl}'(L_{1,1}, \dots, L_{1,r_1}, \dots, L_{n,1}, \dots, L_{n,r_n}) = \operatorname{cl}(L_{1,1} \oplus \cdots \oplus L_{1,r_1}, \dots, L_{n,1} \oplus \cdots \oplus, L_{n,r_n}).$$

By the case of line bundles we know that there is a power series in $r_1 + \cdots + r_n$ variables $\psi$ such that

$$\operatorname{cl}'(L_{1,1}, \dots, L_{1,r_1}, \dots, L_{n,1}, \dots, L_{n,r_n}) = \psi(c_1(L_{1,1}), \dots, c_1(L_{n,r_n})).$$

Since the class cl$'$ is symmetric under the group $\mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_n}$, the same is true for the power series $\psi$. Therefore $\psi$ can be written in terms of symmetric elementary functions. That is, there is another power series in $r_1 + \cdots + r_n$ variables $\varphi$, such that

$$\psi(x_{1,1}, \dots, x_{n,r_n}) = \varphi(s_1(x_{1,1}, \dots, x_{1,r_1}), \dots, s_{r_1}(x_{1,1}, \dots, x_{1,r_1}), \dots \\ \dots, s_1(x_{n,1}, \dots, x_{n,r_n}), \dots, s_{r_n}(x_{n,1}, \dots, x_{n,r_n})),$$

where $s_i$ is the $i$-th elementary symmetric function of the appropriate number of variables. Then

$$\begin{aligned}\pi^*(\operatorname{cl}(E_1, \dots, E_n)) &= \operatorname{cl}(\pi^* E_1, \dots, \pi^* E_n)) \\ &= \operatorname{cl}'(L_{1,1}, \dots, L_{n,r_n}) \\ &= \psi(c_1(L_{1,1}), \dots, c_1(L_{n,r_n})) \\ &= \varphi(c_1(\pi^* E_1), \dots, c_{r_1}(\pi^* E_1), \dots, c_1(\pi^* E_n), \dots, c_{r_n}(\pi^* E_n)) \\ &= \pi^* \varphi(c_1(E_1), \dots, c_{r_1}(E_1), \dots, c_1(E_n), \dots, c_{r_n}(E_n)).\end{aligned}$$

Therefore, the result follows from the injectivity of $\pi^*$. □

Remark 1.10. It would be interesting to know if the functoriality of a characteristic class in enough to imply that it is a power series in the Chern classes for arbitrary complex manifolds and holomorphic vector bundles.

## 2 Bott-Chern classes

The aim of this section is to recall the theory of Bott-Chern classes. For more details we refer the reader to [7], [4], [19], [31], [14], [10] and [12]. Note however that the theory we present here is equivalent, although not identical, to the different versions that appear in the literature.

Let $X$ be a complex manifold and let $\overline{E} = (E, h)$ be a rank $r$ holomorphic vector bundle provided with a hermitian metric. Let $\phi \in \mathbb{D}[[x_1, \dots, x_r]]$ be a formal power series in $r$ variables that is symmetric under the action of $\mathfrak{S}_r$. Let $s_i$, $i = 1, \dots, r$ be the elementary symmetric functions in $r$ variables. Then $\phi(x_1, \dots, x_r) = \varphi(s_1, \dots, s_r)$ for certain power series $\varphi$. By Chern-Weil theory we can obtain a representative of the class

$$\phi(E) := \varphi(c_1(E), \dots, c_r(E)) \in \bigoplus_{k,p} H^k_{\mathcal{D}^{\mathrm{an}}}(X, \mathbb{R}(p))$$

as follows.
We denote also by $\phi$ the invariant power series in $r \times r$ matrices defined by $\phi$. Let $K$ be the curvature matrix of the hermitian holomorphic connection of $(E, h)$. The entries of $K$ in a particular trivialization of $E$ are local sections of $\mathcal{D}^2(X, 1)$. Then we write

$$\phi(E, h) = \phi(-K) \in \bigoplus_{k,p} \mathcal{D}^k(X, p).$$

The form $\phi(E, h)$ is well defined, closed, and it represents the class $\phi(E)$.
Now let

$$\overline{E}_* = (\dots \xrightarrow{f_{n+1}} \overline{E}_n \xrightarrow{f_n} \overline{E}_{n-1} \xrightarrow{f_{n-1}} \dots)$$

be a bounded acyclic complex of hermitian vector bundles; by this we mean a bounded acyclic complex of vector bundles, where each vector bundle is equipped with an arbitrarily chosen hermitian metric.
Write

$$r = \sum_{i \text{ even}} \mathrm{rk}(E_i) = \sum_{i \text{ odd}} \mathrm{rk}(E_i).$$

and let $\phi$ be a symmetric power series in $r$ variables.
As before, we can define the Chern forms

$$\phi(\bigoplus_{i \text{ even}} (E_i, h_i)) \text{ and } \phi(\bigoplus_{i \text{ odd}} (E_i, h_i)),$$

that represent the Chern classes $\phi(\bigoplus_{i \text{ even}} E_i)$ and $\phi(\bigoplus_{i \text{ odd}} E_i)$. The Chern classes are compatible with respect to exact sequences, that is,

$$\phi(\bigoplus_{i \text{ even}} E_i) = \phi(\bigoplus_{i \text{ odd}} E_i).$$

But, in general, this is not true for the Chern forms. This lack of compatibility with exact sequences on the level of Chern forms is measured by the Bott-Chern classes.

DEFINITION 2.1. Let

$$\overline{E}_* = (\dots \xrightarrow{f_{n+1}} \overline{E}_n \xrightarrow{f_n} \overline{E}_{n-1} \xrightarrow{f_{n-1}} \dots)$$

be an acyclic complex of hermitian vector bundles, we will say that $\overline{E}_*$ is an *orthogonally split complex* of vector bundles if, for any integer $n$, the exact sequence

$$0 \longrightarrow \operatorname{Ker} f_n \longrightarrow \overline{E}_n \longrightarrow \operatorname{Ker} f_{n-1} \longrightarrow 0$$

is split, there is a splitting section $s_n\colon \operatorname{Ker} f_{n-1} \to E_n$ such that $\overline{E}_n$ is the orthogonal direct sum of $\operatorname{Ker} f_n$ and $\operatorname{Im} s_n$ and the metrics induced in the subbundle $\operatorname{Ker} f_{n-1}$ by the inclusion $\operatorname{Ker} f_{n-1} \subset \overline{E}_{n-1}$ and by the section $s_n$ agree.

Notation 2.2. Let $(x : y)$ be homogeneous coordinates of $\mathbb{P}^1$ and let $t = x/y$ be the absolute coordinate. In order to make certain choices of metrics in a functorial way, we fix once and for all a partition of unity $\{\sigma_0, \sigma_\infty\}$, over $\mathbb{P}^1$ subordinated to the open cover of $\mathbb{P}^1$ given by the open subsets $\{\{|y| > 1/2|x|\}, \{|x| > 1/2|y|\}\}$. As usual we will write $\infty = (1 : 0)$, $0 = (0 : 1)$.

The fundamental result of the theory of Bott-Chern classes is the following theorem (see [7], [4], [19]).

Theorem 2.3. *There is a unique way to attach to each bounded exact complex $\overline{E}_*$ as above, a class $\widetilde{\phi}(\overline{E}_*)$ in*

$$\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(X,k) = \bigoplus_k \mathcal{D}^{2k-1}(X,k)/\operatorname{Im}(\mathrm{d}_{\mathcal{D}})$$

*satisfying the following properties*

*(i) (Differential equation)*

$$\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(\overline{E}_*) = \phi\Big(\bigoplus_{i \text{ even}} (E_i, h_i)\Big) - \phi\Big(\bigoplus_{i \text{ odd}} (E_i, h_i)\Big). \tag{2.4}$$

*(ii) (Functoriality) $f^*\widetilde{\phi}(\overline{E}_*) = \widetilde{\phi}(f^*\overline{E}_*)$, for every holomorphic map $f\colon X' \longrightarrow X$.*

*(iii) (Normalization) If $\overline{E}_*$ is orthogonally split, then $\widetilde{\phi}(\overline{E}_*) = 0$.*

*Proof.* We first recall how to prove the uniqueness.
Let $\overline{K}_i = (K_i, g_i)$, where $K_i = \operatorname{Ker} f_i$ and $g_i$ is the metric induced by the inclusion $K_i \subset E_i$. Consider the complex manifold $X \times \mathbb{P}^1$ with projections $p_1$ and $p_2$. For every vector bundle $F$ on $X$ we will denote $F(i) = p_1^*F \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(i)$. Let $\widetilde{C}_* = \widetilde{C}(E_*)_*$ be the complex of vector bundles on $X \times \mathbb{P}^1$ given by $\widetilde{C}_i = E_i(i) \oplus E_{i-1}(i-1)$ with differential $d(s,t) = (t,0)$. Let $\widetilde{D}_* = \widetilde{D}(E_*)_*$ be the complex of vector bundles with $\widetilde{D}_i = E_{i-1}(i) \oplus E_{i-2}(i-1)$ and differential $d(s,t) = (t,0)$. Using notation 2.2 we define the map $\psi\colon \widetilde{C}(E_*)_i \longrightarrow \widetilde{D}(E_*)_i$ given by $\psi(s,t) = (f_i(s) - t \otimes y, f_{i-1}(t))$. It is a morphism of complexes.

Definition 2.5. The *first transgression exact sequence* of $E_*$ is given by

$$\operatorname{tr}_1(E_*)_* = \operatorname{Ker}\psi.$$

On $X \times \mathbb{A}^1$, the map $p_1^* E_i \longrightarrow \widetilde{C}(E_*)_i$ given by $s \longmapsto (s \otimes y^i, f_i(s) \otimes y^{i-1})$ induces an isomorphism of complexes

$$p_1^* E_* \longrightarrow \operatorname{tr}_1(E_*)_*|_{X \times \mathbb{A}^1}, \tag{2.6}$$

and in particular isomorphisms

$$\operatorname{tr}_1(E_*)_i|_{X \times \{0\}} \cong E_i. \tag{2.7}$$

Moreover, we have isomorphisms

$$\operatorname{tr}_1(E_*)_i|_{X \times \{\infty\}} \cong K_i \oplus K_{i-1}. \tag{2.8}$$

Definition 2.9. We will denote by $\operatorname{tr}_1(\overline{E}_*)_*$ the complex $\operatorname{tr}_1(E_*)_*$ provided with any hermitian metric such that the isomorphisms (2.7) and (2.8) are isometries. If we need a functorial choice of metric, we proceed as follows. On $X \times (\mathbb{P}^1 \setminus \{0\})$ we consider the metric induced by $\widetilde{C}$ on $\operatorname{tr}_1(E_*)_*$. On $X \times (\mathbb{P}^1 \setminus \{\infty\})$ we consider the metric induced by the isomorphism (2.6). We glue both metrics by means of the partition of unity of notation 2.2.

In particular, we have that $\operatorname{tr}_1(\overline{E}_*)|_{X \times \{\infty\}}$ is orthogonally split. We assume that there exists a theory of Bott-Chern classes satisfying the above properties. Thus, there exists a class of differential forms $\widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*)$ with the following properties. By (i) this class satisfies

$$\mathrm{d}_{\mathcal{D}} \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*) = \phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i)) - \phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i).$$

By (ii), it satisfies

$$\widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*) \mid_{X \times \{0\}} = \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_* \mid_{X \times \{0\}}) = \widetilde{\phi}(\overline{E}_*).$$

Finally, by (ii) and (iii) it satisfies

$$\widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*) \mid_{X \times \{\infty\}} = \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_* \mid_{X \times \{\infty\}}) = 0.$$

Let $\phi(\operatorname{tr}_1(\overline{E}_*)_*)$ be any representative of the class $\widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*)$.
Then, in the group $\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(X, k)$, we have

$$\begin{aligned}
0 &= \mathrm{d}_{\mathcal{D}} \frac{1}{2\pi i} \int_{\mathbb{P}^1} \frac{-1}{2} \log(t\bar{t}) \bullet \phi(\operatorname{tr}_1(\overline{E}_*)_*) \\
&= \frac{1}{2\pi i} \int_{\mathbb{P}^1} \left( \mathrm{d}_{\mathcal{D}} \frac{-1}{2} \log(t\bar{t}) \bullet \phi(\operatorname{tr}_1(\overline{E}_*)_*) - \frac{-1}{2} \log(t\bar{t}) \bullet \mathrm{d}_{\mathcal{D}} \phi(\operatorname{tr}_1(\overline{E}_*)_*) \right) \\
&= \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*)|_{X \times \{\infty\}} - \widetilde{\phi}(\operatorname{tr}_1(\overline{E}_*)_*)|_{X \times \{0\}} \\
&\quad - \frac{1}{2\pi i} \int_{\mathbb{P}^1} \frac{-1}{2} \log(t\bar{t}) \bullet (\phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i) - \phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i)) \\
&= -\widetilde{\phi}(\overline{E}_*) - \frac{1}{2\pi i} \int_{\mathbb{P}^1} \frac{-1}{2} \log(t\bar{t}) \bullet (\phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i) - \phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i)).
\end{aligned}$$

Hence, if such a theory exists, it should satisfy the formula

$$\widetilde{\phi}(\overline{E}_*) = \frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{-1}{2}\log(t\bar{t}) \bullet (\phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i) - \phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i)). \quad (2.10)$$

Therefore $\widetilde{\phi}(\overline{E}_*)$ is determined by properties (i), (ii) and (iii).
In order to prove the existence of a theory of functorial Bott-Chern forms, we have to see that the right hand side of equation (2.10) is independent from the choice of the metric on $\operatorname{tr}_1(\overline{E}_*)_*$ and that it satisfies the properties (i), (ii) and (iii). For this the reader can follow the proof of [4] theorem 1.29.

□

In view of the proof of theorem 2.3, we can define the Bott-Chern classes as follows.

Definition 2.11. Let

$$\overline{E}_*\colon 0 \longrightarrow (E_n, h_n) \longrightarrow \dots \longrightarrow (E_1, h_1) \longrightarrow (E_0, h_0) \longrightarrow 0$$

be a bounded acyclic complex of hermitian vector bundles. Let

$$r = \sum_{i \text{ even}} \operatorname{rk}(E_i) = \sum_{i \text{ odd}} \operatorname{rk}(E_i).$$

Let $\phi \in \mathbb{D}[[x_1, \dots, x_r]]^{\mathfrak{S}_r}$ be a symmetric power series in $r$ variables. Then the *Bott-Chern class* associated to $\phi$ and $\overline{E}_*$ is the element of $\bigoplus_{k,p} \widetilde{\mathcal{D}}^k(E_X, p)$ given by

$$\widetilde{\phi}(\overline{E}_*) = \frac{1}{2\pi i}\int_{\mathbb{P}^1} \frac{-1}{2}\log(t\bar{t}) \bullet (\phi(\bigoplus_{i \text{ odd}} \operatorname{tr}_1(\overline{E}_*)_i) - \phi(\bigoplus_{i \text{ even}} \operatorname{tr}_1(\overline{E}_*)_i)).$$

The following property is obvious from the definition.

Lemma 2.12. *Let $\overline{E}_*$ be an acyclic complex of hermitian vector bundles. Then, for any integer $k$,*

$$\widetilde{\phi}(\overline{E}_*[k]) = (-1)^k \widetilde{\phi}(\overline{E}_*).$$

□

Particular cases of Bott-Chern classes are obtained when we consider a single vector bundle with two different hermitian metrics or a short exact sequence of vector bundles. Note however that, in order to fix the sign of the Bott-Chern classes on these cases, one has to choose the degree of the vector bundles involved, for instance as in the next definition.

Definition 2.13. Let $E$ be a holomorphic vector bundle of rank $r$, let $h_0$ and $h_1$ be two hermitian metrics and let $\phi$ be an invariant power series of $r$ variables. We will denote by $\widetilde{\phi}(E, h_0, h_1)$ the Bott-Chern class associated to the complex

$$\overline{\xi}\colon 0 \longrightarrow (E, h_1) \longrightarrow (E, h_0) \longrightarrow 0,$$

where $(E, h_0)$ sits in degree zero.

Therefore, this class satisfies

$$\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(E,h_0,h_1)=\phi(E,h_0)-\phi(E,h_1).$$

In fact we can characterize $\widetilde{\phi}(E,h_0,h_1)$ axiomatically as follows.

Proposition 2.14. *Given $\phi$, a symmetric power series in $r$ variables, there is a unique way to attach, to each rank $r$ vector bundle $E$ on a complex manifold $X$ and metrics $h_0$ and $h_1$, a class $\widetilde{\phi}(E,h_0,h_1)$ satisfying*

*(i)* $\mathrm{d}_{\mathcal{D}}\,\widetilde{\phi}(E,h_0,h_1)=\phi(E,h_0)-\phi(E,h_1)$.

*(ii)* $f^*\widetilde{\phi}(E,h_0,h_1)=\widetilde{\phi}(f^*(E,h_0,h_1))$ *for every holomorphic map* $f\colon Y\longrightarrow X$.

*(iii)* $\widetilde{\phi}(E,h,h)=0$.

*Moreover, if we denote* $\widetilde{E}:=\mathrm{tr}_1(\overline{\xi})_1$, *then it satisfies*

$$\widetilde{E}|_{X\times\{\infty\}}\cong(E,h_0),\quad \widetilde{E}|_{X\times\{0\}}\cong(E,h_1)$$

*and*

$$\widetilde{\phi}(E,h_0,h_1)=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log(t\bar{t})\bullet\phi(\widetilde{E}). \tag{2.15}$$

*Proof.* The axiomatic characterization is proved as in theorem 2.3. In order to prove equation (2.15), if we follow the notations of the proof of theorem 2.3 we have $K_0=(E,h_0)$ and $K_1=0$. Therefore $\mathrm{tr}_1(\overline{\xi})_0=p_1^*(E,h_0)$, while $\widetilde{E}:=\mathrm{tr}_1(\overline{\xi})_1$ satisfies $\widetilde{E}|_{X\times\{0\}}=(E,h_1)$ and $\widetilde{E}|_{X\times\{\infty\}}=(E,h_0)$. Using the antisymmetry of $\log t\bar{t}$ under the involution $t\mapsto 1/t$ we obtain

$$\widetilde{\phi}(E,h_0,h_1)=\widetilde{\phi}(\overline{\xi})=\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log(t\bar{t})\bullet\phi(\widetilde{E}).$$

□

We can also treat the case of short exact sequences. If

$$\overline{\varepsilon}\colon 0\longrightarrow\overline{E}_2\longrightarrow\overline{E}_1\longrightarrow\overline{E}_0\longrightarrow 0$$

is a short exact sequence of hermitian vector bundles, by convention, we will assume that $\overline{E}_0$ sits in degree zero. This fixs the sign of $\widetilde{\phi}(\overline{\varepsilon})$.

Proposition 2.16. *Given $\phi$, a symmetric power series in $r$ variables, there is a unique way to attach, to each short exact sequence of hermitian vector bundles on a complex manifold $X$*

$$\overline{\varepsilon}\colon 0\longrightarrow\overline{E}_2\longrightarrow\overline{E}_1\longrightarrow\overline{E}_0\longrightarrow 0,$$

*where $\overline{E}_1$ has rank $r$, a class $\widetilde{\phi}(\overline{\varepsilon})$ satisfying* *(i)* $\mathrm{d}_{\mathcal{D}}\, \widetilde{\phi}(\overline{\varepsilon}) = \phi(\overline{E}_0 \oplus \overline{E}_2) - \phi(\overline{E}_1)$.

*(ii)* $f^*\widetilde{\phi}(\overline{\varepsilon}) = \widetilde{\phi}(f^*(\overline{\epsilon}))$ *for every holomorphic map* $f\colon Y \longrightarrow X$.

*(iii)* $\widetilde{\phi}(\overline{\varepsilon}) = 0$ *whenever* $\overline{\varepsilon}$ *is orthogonally split.*

□

The following additivity result of Bott-Chern classes will be useful later.

Lemma 2.17. *Let* $\overline{A}_{*,*}$ *be a bounded exact sequence of bounded exact sequences of hermitian vector bundles. Let*

$$r = \sum_{i,j\ even} \mathrm{rk}(A_{i,j}) = \sum_{i,j\ odd} \mathrm{rk}(A_{i,j}) = \sum_{\substack{i\ odd\\ j\ even}} \mathrm{rk}(A_{i,j}) = \sum_{\substack{i\ even\\ j\ odd}} \mathrm{rk}(A_{i,j}).$$

*Let* $\phi$ *be a symmetric power series in* $r$ *variables. Then*

$$\widetilde{\phi}(\bigoplus_{k\ even} \overline{A}_{k,*}) - \widetilde{\phi}(\bigoplus_{k\ odd} \overline{A}_{k,*}) = \widetilde{\phi}(\bigoplus_{k\ even} \overline{A}_{*,k}) - \widetilde{\phi}(\bigoplus_{k\ odd} \overline{A}_{*,k}).$$

*Proof.* The proof is analogous to the proof of proposition 6.13 and is left to the reader. □

Corollary 2.18. *Let* $\overline{A}_{*,*}$ *be a bounded double complex of hermitian vector bundles with exact rows, let*

$$r = \sum_{i+j\ even} \mathrm{rk}(A_{i,j}) = \sum_{i+j\ odd} \mathrm{rk}(A_{i,j})$$

*and let* $\phi$ *be a symmetric power series in* $r$ *variables. Then*

$$\widetilde{\phi}(\mathrm{Tot}\, \overline{A}_{*,*}) = \widetilde{\phi}(\bigoplus_k \overline{A}_{*,k}[-k]).$$

*Proof.* Let $k_0$ be an integer such that $\overline{A}_{k,l} = 0$ for $k < k_0$. For any integer $n$ we denote by $\mathrm{Tot}_n = \mathrm{Tot}((\overline{A}_{k,l})_{k\geq n})$ the total complex of the exact complex formed by the rows with index greater or equal than $n$. Then $\mathrm{Tot}_{k_0} = \mathrm{Tot}(\overline{A}_{*,*})$. For each $k$ there is an exact sequence of complexes

$$0 \longrightarrow \mathrm{Tot}_{k+1} \longrightarrow \mathrm{Tot}_k \oplus \bigoplus_{l<k} \overline{A}_{l,*}[-l] \longrightarrow \bigoplus_{l\leq k} \overline{A}_{l,*}[-l] \longrightarrow 0,$$

which is orthogonally split in each degree. Therefore by lemma 2.17 we obtain

$$\widetilde{\phi}(\mathrm{Tot}_k \oplus \bigoplus_{l<k} \overline{A}_{l,*}[-l]) = \widetilde{\phi}(\mathrm{Tot}_{k-1} \oplus \bigoplus_{l\leq k} \overline{A}_{l,*}[-l]).$$

Hence the result follows by induction. □

A particularly important characteristic class is the Chern character. This class is additive for exact sequences. Specializing lemma 2.17 and corollary 2.18 to the Chern character we obtain

Corollary 2.19. *With the hypothesis of lemma 2.17, the following equality holds:*

$$\sum_k (-1)^k \widetilde{\mathrm{ch}}(\overline{A}_{k,*}) = \sum_k (-1)^k \widetilde{\mathrm{ch}}(\overline{A}_{*,k}) = \widetilde{\mathrm{ch}}(\mathrm{Tot}\,\overline{A}_{*,*}).$$

□

Our next aim is to extend the Bott-Chern classes associated to the Chern character to metrized coherent sheaves. This extension is due to Zha [32], although it is still unpublished.

Definition 2.20. A *metrized coherent sheaf* $\overline{\mathcal{F}}$ on $X$ is a pair $(\mathcal{F}, \overline{E}_* \to \mathcal{F})$ where $\mathcal{F}$ is a coherent sheaf on $X$ and

$$0 \to \overline{E}_n \to \overline{E}_{n-1} \to \cdots \to \overline{E}_0 \to \mathcal{F} \to 0$$

is a finite resolution by hermitian vector bundles of the coherent sheaf $\mathcal{F}$. This resolution is also called the metric of $\overline{\mathcal{F}}$.
If $\overline{E}$ is a hermitian vector bundle, we will also denote by $\overline{E}$ the metrized coherent sheaf $(E, \overline{E} \xrightarrow{\mathrm{id}} E)$.

Note that the coherent sheaf 0 may have non trivial metrics. In fact, any exact sequence of hermitian vector bundles

$$0 \to \overline{A}_n \to \cdots \to \overline{A}_0 \to 0 \to 0$$

can be seen as a metric on 0. It will be denoted $\overline{0}_{A_*}$. A metric on 0 is said to be *orthogonally split* if the exact sequence is orthogonally split.
A morphism of metrized coherent sheaves $\overline{\mathcal{F}}_1 \to \overline{\mathcal{F}}_2$ is just a morphism of sheaves $\mathcal{F}_1 \to \mathcal{F}_2$. A sequence of metrized coherent sheaves

$$\overline{\varepsilon}\colon \qquad \ldots \longrightarrow \overline{\mathcal{F}}_{n+1} \longrightarrow \overline{\mathcal{F}}_n \longrightarrow \overline{\mathcal{F}}_{n-1} \longrightarrow \ldots$$

is said to be exact if it is exact as a sequence of coherent sheaves.

Definition 2.21. Let $\overline{\mathcal{F}} = (\mathcal{F}, \overline{E}_* \to \mathcal{F})$ be a metrized coherent sheaf. Then the *Chern character form* associated to $\overline{\mathcal{F}}$ is given by

$$\mathrm{ch}(\overline{\mathcal{F}}) = \sum_i (-1)^i \,\mathrm{ch}(\overline{E}_i).$$

Definition 2.22. *An exact sequence of metrized coherent sheaves with com-*

*patible metrics* is a commutative diagram

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{E}_{n,1} & \to & \dots & \to & \overline{E}_{0,1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{E}_{n,0} & \to & \dots & \to & \overline{E}_{0,0} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{F}_n & \to & \dots & \to & \mathcal{F}_0 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad (2.23)$$

where all the rows and columns are exact. The columns of this diagram are the individual metrics of each coherent sheaf. We will say that an exact sequence with compatible metrics is *orthogonally split* if each row of vector bundles is an orthogonally split exact sequence of hermitian vector bundles.

As in the case of exact sequences of hermitian vector bundles, the Chern character form is not compatible with exact sequences of metrized coherent sheaves and we can define a secondary Bott-Chern character which measures the lack of compatibility between the metrics.

Theorem 2.24. *1) There is a unique way to attach to every finite exact sequence of metrized coherent sheaves with compatible metrics*

$$\overline{\varepsilon}\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \dots \to \overline{\mathcal{F}}_0 \to 0$$

*on a complex manifold $X$ a Bott-Chern secondary character*

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$$

*such that the following axioms are satisfied:*

*(i) (Differential equation)*

$$\mathrm{d}_{\mathcal{D}}\,\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = \sum_k (-1)^k \,\mathrm{ch}(\overline{\mathcal{F}_k}).$$

*(ii) (Functoriality) If $f\colon X' \longrightarrow X$ is a morphism of complex manifolds, that is tor-independent from the coherent sheaves $\mathcal{F}_k$, then*

$$f^*(\widetilde{\mathrm{ch}})(\overline{\varepsilon}) = \widetilde{\mathrm{ch}}(f^*\overline{\varepsilon}),$$

*where the exact sequence $f^*\overline{\varepsilon}$ exists thanks to the tor-independence.*

*(iii) (Horizontal normalization) If $\overline{\varepsilon}$ is orthogonally split then*

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = 0.$$

*2) There is a unique way to attach to every finite exact sequence of metrized coherent sheaves*

$$\overline{\varepsilon}\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \cdots \to \overline{\mathcal{F}}_0 \to 0$$

*on a complex manifold $X$ a Bott-Chern secondary character*

$$\widetilde{\operatorname{ch}}(\overline{\varepsilon}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$$

*such that the axioms (i), (ii) and (iii) above and the axiom (iv) below are satisfied:*

*(iv) (Vertical normalization) For every bounded complex of hermitian vector bundles*

$$\cdots \to \overline{A}_k \to \cdots \to \overline{A}_0 \to 0$$

*that is orthogonally split, and every bounded complex of metrized coherent sheaves*

$$\overline{\varepsilon}\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \cdots \to \overline{\mathcal{F}}_0 \to 0$$

*where the metrics are given by $\overline{E}_{i,*} \to \mathcal{F}_i$, if, for some $i_0$ we denote*

$$\overline{\mathcal{F}}'_{i_0} = (\mathcal{F}_{i_0}, \overline{E}_{i_0,*} \oplus \overline{A}_* \to \mathcal{F}_{i_0})$$

*and*

$$\overline{\varepsilon}'\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \cdots \to \overline{\mathcal{F}}'_{i_0} \to \cdots \to \overline{\mathcal{F}}_0 \to 0,$$

*then $\widetilde{\operatorname{ch}}(\overline{\varepsilon}') = \widetilde{\operatorname{ch}}(\overline{\varepsilon})$.*

*Proof. 1)* The uniqueness is proved using the standard deformation argument. By definition, the metrics of the coherent sheaves form a diagram like (2.23). On $X \times \mathbb{P}^1$, for each $j \geq 0$ we consider the exact sequences $\widetilde{E}_{*,j} = \operatorname{tr}_1(E_{*,j})$ associated to the rows of the diagram with the hermitian metrics of definition 2.9. Then, for each $i,j$ there are maps $\mathrm{d}\colon \widetilde{E}_{i,j} \to \widetilde{E}_{i-1,j}$, and $\delta\colon \widetilde{E}_{i,j} \to \widetilde{E}_{i,j-1}$. We denote

$$\widetilde{\mathcal{F}}_i = \operatorname{Coker}(\delta\colon \widetilde{E}_{i,1} \to \widetilde{E}_{i,0}).$$

Using the definition of $\operatorname{tr}_1$ and diagram chasing one can prove that there is a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{E}_{n,1} & \to & \dots & \to & \widetilde{E}_{0,1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{E}_{n,0} & \to & \dots & \to & \widetilde{E}_{0,0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{\mathcal{F}}_n & \to & \dots & \to & \widetilde{\mathcal{F}}_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.25}$$

where all the rows and columns are exact. In particular this implies that the inclusions $i_0\colon X \to X \times \{0\} \to X \times \mathbb{P}^1$ and $i_\infty\colon X \to X \times \{\infty\} \to X \times \mathbb{P}^1$ are tor-independent from the sheaves $\widetilde{\mathcal{F}}_i$. But $i_0^*\widetilde{\mathcal{F}}_*$ is isometric with $\overline{\mathcal{F}}_*$ and $i_\infty^*\widetilde{\mathcal{F}}_*$ is orthogonally split. Hence, by the standard argument, axioms (i), (ii) and (iii) imply that

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = \sum_j (-1)^j \widetilde{\mathrm{ch}}(\overline{E}_{*,j}). \tag{2.26}$$

To prove the existence we use equation (2.26) as definition. Then the properties of the Bott-Chern classes of exact sequences of hermitian vector bundles imply that axioms (i), (ii) and (iii) are satisfied.

*Proof of 2).* We first assume that such theory exists. Let

$$\dots \to \overline{A}_k \to \dots \to \overline{A}_0 \to 0$$

be a bounded complex of hermitian vector bundles, non necessarily orthogonally split, and

$$\overline{\varepsilon}\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \dots \to \overline{\mathcal{F}}_0 \to 0$$

a bounded complex of metrized coherent sheaves where the metrics are given by $\overline{E}_{i,*} \to \mathcal{F}_i$. As in axiom (iv), for some $i_0$ we denote

$$\overline{\mathcal{F}}'_{i_0} = (\mathcal{F}_{i_0}, \overline{E}_{i,*} \oplus \overline{A}_* \to \mathcal{F}_{i_0})$$

and

$$\overline{\varepsilon}'\colon \qquad 0 \to \overline{\mathcal{F}}_n \to \dots \to \overline{\mathcal{F}}'_{i_0} \to \dots \to \overline{\mathcal{F}}_0 \to 0.$$

By axioms (i), (ii) and (iv), the class $(-1)^{i_0}(\widetilde{\mathrm{ch}}(\overline{\varepsilon}') - \widetilde{\mathrm{ch}}(\overline{\varepsilon}))$ satisfies the properties that characterize $\widetilde{\mathrm{ch}}(A_*)$. Therefore $\widetilde{\mathrm{ch}}(\overline{\varepsilon}') = \widetilde{\mathrm{ch}}(\overline{\varepsilon}) + (-1)^{i_0}\widetilde{\mathrm{ch}}(A_*)$.

Fix again a number $i_0$ and assume that there is an exact sequence of resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{A}_\bullet & \longrightarrow & \overline{E}'_{i_0,*} & \longrightarrow & \overline{E}_{i_0,*} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathcal{F}_{i_0} & = & \mathcal{F}_{i_0} & & \end{array} \tag{2.27}$$

Let now $\overline{\varepsilon}'$ denote the exact sequence $\overline{\varepsilon}$ but with the metric $\overline{E}'_{i_0,*}$ in the position $i_0$. Let $\overline{\eta}_j$ denote the $j$-th row of the diagram (2.27). Again using a deformation argument one sees that

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}') - \widetilde{\mathrm{ch}}(\overline{\varepsilon}) = (-1)^{i_0}\left(\widetilde{\mathrm{ch}}(\overline{A}_*) - \sum_j (-1)^j\, \widetilde{\mathrm{ch}}(\eta_j)\right). \tag{2.28}$$

Choose now a compatible system of metrics

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \overline{D}_{n,1} & \rightarrow & \dots & \rightarrow & \overline{D}_{0,1} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \overline{D}_{n,0} & \rightarrow & \dots & \rightarrow & \overline{D}_{0,0} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{F}_n & \rightarrow & \dots & \rightarrow & \mathcal{F}_0 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.29}$$

we denote by $\overline{\lambda}_j$ each row of the above diagram. For each $i$, choose a resolution $\overline{E}'_{i,*} \longrightarrow \mathcal{F}_i$ such that there exist exact sequences of resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{A}_{i,*} & \longrightarrow & \overline{E}'_{i,*} & \longrightarrow & \overline{E}_{i,*} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathcal{F}_i & = & \mathcal{F}_i & & \end{array} \tag{2.30}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{B}_{i,*} & \longrightarrow & \overline{E}'_{i,*} & \longrightarrow & \overline{D}_{i,*} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & \mathcal{F}_i & = & \mathcal{F}_i & & \end{array} \tag{2.31}$$

We denote by $\overline{\eta}_{i,j}$ each row of the diagram (2.30) and by $\overline{\mu}_{i,j}$ each row of the diagram (2.31). Then, by (2.28) and (2.26), we have

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = \sum_j (-1)^j \, \widetilde{\mathrm{ch}}(\overline{\lambda}_j) + \sum_i (-1)^i (\widetilde{\mathrm{ch}}(\overline{B}_{i,*}) - \widetilde{\mathrm{ch}}(\overline{A}_{i,*})) \\ + \sum_{i,j} (-1)^{i+j} (\widetilde{\mathrm{ch}}(\overline{\eta}_{i,j}) - \widetilde{\mathrm{ch}}(\overline{\mu}_{i,j})) \tag{2.32}$$

Thus, $\widetilde{\mathrm{ch}}(\overline{\varepsilon})$ is uniquely determined by axioms (i) to (iv). To prove the existence we use equation (2.32) as definition. We have to show that this definition is independent of the choices of the new resolutions. This independence follows from corollary 2.19. Once we know that the Bott-Chern classes are well defined, it is clear that they satisfy axioms (i), (ii), (iii) and (iv). □

Proposition 2.33. *(Compatibility with exact squares) If*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & \overline{\mathcal{F}}_{n+1,m+1} & \to & \overline{\mathcal{F}}_{n+1,m} & \to & \overline{\mathcal{F}}_{n+1,m-1} & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & \overline{\mathcal{F}}_{n,m+1} & \to & \overline{\mathcal{F}}_{n,m} & \to & \overline{\mathcal{F}}_{n,m-1} & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \to & \overline{\mathcal{F}}_{n-1,m+1} & \to & \overline{\mathcal{F}}_{n-1,m} & \to & \overline{\mathcal{F}}_{n-1,m-1} & \to & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

*is a bounded commutative diagram of metrized coherent sheaves, where all the rows* $\dots(\overline{\varepsilon}_{n-1})$, $(\overline{\varepsilon}_n)$, $(\overline{\varepsilon}_{n+1})$, $\dots$ *and all the columns* $(\overline{\eta}_{m-1})$, $(\overline{\eta}_m)$, $(\overline{\eta}_{m+1})$ *are exact, then*

$$\sum_n (-1)^n \widetilde{\mathrm{ch}}(\overline{\varepsilon}_n) = \sum_m (-1)^m \widetilde{\mathrm{ch}}(\overline{\eta}_m).$$

*Proof.* This follows from equation (2.32) and corollary 2.19. □

We will use the notation of definition 2.13 also in the case of metrized coherent sheaves.

It is easy to verify the following result.

Proposition 2.34. *Let*

$$(\overline{\varepsilon}) \qquad \dots \longrightarrow \overline{E}_{n+1} \longrightarrow \overline{E}_n \longrightarrow \overline{E}_{n-1} \longrightarrow \dots$$

*be a finite exact sequence of hermitian vector bundles. Then the Bott-Chern classes obtained by theorem 2.24 and by theorem 2.3 agree.* □

Proposition 2.35. *Let* $\overline{\mathcal{F}} = (\mathcal{F}, \overline{E}_* \to \mathcal{F})$ *be a metrized coherent sheaf. We consider the exact sequence of metrized coherent sheaves*

$$\overline{\varepsilon}\colon \qquad 0 \longrightarrow \overline{E}_n \to \dots \to \overline{E}_0 \to \overline{\mathcal{F}} \to 0,$$

*where, by abuse of notation,* $\overline{E}_i = (E_i, \overline{E}_i \xrightarrow{=} E_i)$. *Then* $\widetilde{\mathrm{ch}}(\overline{\varepsilon}) = 0$.

*Proof.* Define $\mathcal{K}_i = \mathrm{Ker}(E_i \to E_{i-1})$, $i = 1, \dots, n$ and $\mathcal{K}_0 = \mathrm{Ker}(E_0 \to \mathcal{F})$. Write

$$\overline{\mathcal{K}_i} = (\mathcal{K}_i, 0 \to \overline{E}_n \to \dots \to \overline{E}_{i+1} \to \mathcal{K}_i),\ i = 0, \dots, n,$$

and $\overline{\mathcal{K}}_{-1} = \overline{\mathcal{F}}$. If we prove that

$$\widetilde{\mathrm{ch}}(0 \to \overline{\mathcal{K}_i} \to \overline{E}_i \to \overline{\mathcal{K}}_{i-1} \to 0) = 0, \tag{2.36}$$

then we obtain the result by induction using proposition 2.33. In order to prove equation (2.36) we apply equation (2.32). To this end consider resolutions

$$\begin{aligned} \overline{D}_{0,*} &\longrightarrow \mathcal{K}_{i-1}, & \overline{D}_{0,k} &= \overline{E}_{k+i} \\ \overline{D}_{1,*} &\longrightarrow E_{i}, & \overline{D}_{1,k} &= \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} \\ \overline{D}_{2,*} &\longrightarrow \mathcal{K}_{i}, & \overline{D}_{2,k} &= \overline{E}_{k+i+1} \end{aligned}$$

with the map $D_{2,k} \overset{\Delta}{\to} D_{1,k}$ given by $s \mapsto (s, \mathrm{d}\, s)$ and the map $D_{1,k} \overset{\nabla}{\to} D_{0,k}$ given by $(s,t) \mapsto t - \mathrm{d}\, s$. The differential of the complex $D_{1,k}$ is given by $(s,t) \mapsto (t,0)$. Using equations (2.32) and (2.26) we write the left hand side of equation (2.36) in terms of Bott-Chern classes of vector bundles. All the exact sequences involved are orthogonally split except maybe the sequences

$$\overline{\lambda}_k\colon \qquad 0 \to \overline{D}_{2,k} \to \overline{D}_{1,k} \to \overline{D}_{0,k} \to 0.$$

But now we consider the diagrams

$$\begin{array}{ccccc} \overline{E}_{k+i+1} & \xrightarrow{i_1} & \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} & \xrightarrow{p_2} & \overline{E}_{k+i} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} \\ \overline{E}_{k+i+1} & \xrightarrow{\Delta} & \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} & \xrightarrow{\nabla} & \overline{E}_{k+i} \end{array}$$

and

$$\begin{array}{ccccc} \overline{E}_{k+i} & \xrightarrow{i_2} & \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} & \xrightarrow{p_1} & \overline{E}_{k+i+1} \ , \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} \\ \overline{E}_{k+i} & \xrightarrow{i_2} & \overline{E}_{k+i+1} \oplus \overline{E}_{k+i} & \xrightarrow{p_1} & \overline{E}_{k+i+1} \end{array}$$

where $i_i$, $i_2$ are the natural inclusions, $p_1$ and $p_2$ are the projections and $f(s,t) = (s, t + f(s))$. These diagrams and corollary 2.19 imply that $\widetilde{\mathrm{ch}}(\overline{\lambda}_k) = 0$. □

Remark 2.37. In [32], Zha shows that the Bott-Chern classes associated to exact sequences of metrized coherent sheaves are characterized by proposition 2.34, proposition 2.35 and proposition 2.33. We prefer the characterization in terms of the differential equation, the functoriality and the normalization, because it relies on natural extensions of the corresponding axioms that define the Bott-Chern classes for exact sequences of hermitian vector bundles. Moreover, this approach will be used in a subsequent paper where we will study singular Bott-Chern classes associated to arbitrary proper morphisms.

The following generalization of proposition 2.35 will be useful later. Let

$$\varepsilon\colon 0 \to \mathcal{G}_n \to \mathcal{G}_{n-1} \to \cdots \to \mathcal{G}_0 \to \mathcal{F} \to 0$$

be a finite resolution of a coherent sheaf by coherent sheaves. Assume that we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & \overline{E}_{1,n} & \rightarrow & \dots & \rightarrow & \overline{E}_{1,0} & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & \overline{E}_{0,n} & \rightarrow & \dots & \rightarrow & \overline{E}_{0,0} & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \overline{\mathcal{G}}_n & \rightarrow & \dots & \rightarrow & \overline{\mathcal{G}}_0 & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & 0 & & & &
\end{array}$$

where the columns are exact, the rows are complexes and the $\overline{E}_{i,j}$ are hermitian vector bundles. The columns of this diagram define metrized coherent sheaves $\overline{\mathcal{G}}_i$. Let $\overline{\mathcal{F}}$ be the metrized coherent sheaf defined by the resolution $\operatorname{Tot}(\overline{E}_{*,*}) \longrightarrow \mathcal{F}$.

Proposition 2.38. *With the notations above, let $\overline{\varepsilon}$ be the exact sequence of metrized coherent sheaves*

$$\overline{\varepsilon}\colon 0 \to \overline{\mathcal{G}}_n \to \overline{\mathcal{G}}_{n-1} \to \dots \to \overline{\mathcal{G}}_0 \to \overline{\mathcal{F}} \to 0$$

*Then* $\widetilde{\operatorname{ch}}(\overline{\varepsilon}) = 0$.

*Proof.* For each $k$, let $\operatorname{Tot}_k = \operatorname{Tot}((E_{*,j})_{j \geq k})$. There are inclusions $\operatorname{Tot}_k \longrightarrow \operatorname{Tot}_{k-1}$. Let $\overline{D}_{*,j} = s(\operatorname{Tot}_{j+1} \to \operatorname{Tot}_j)$ with the hermitian metric induced by $\overline{E}_{*,*}$. There are exact sequences of complexes

$$0 \longrightarrow \overline{E}_{*,j} \longrightarrow \overline{D}_{*,j} \longrightarrow s(\operatorname{Tot}_{j+1} \to \operatorname{Tot}_{j+1}) \longrightarrow 0 \tag{2.39}$$

that are orthogonally split at each degree. The third complex is orthogonally split. Therefore, if we denote by $h_E$ and $h_D$ the metric structures of $\mathcal{G}_j$ induced respectively by the first and second column of diagram (2.39), then

$$\widetilde{\operatorname{ch}}(\mathcal{G}_j, h_E, h_D) = 0. \tag{2.40}$$

There is a commutative diagram of resolutions

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \overline{D}_{1,n} & \rightarrow & \dots & \rightarrow & \overline{D}_{1,0} & \rightarrow & (\operatorname{Tot}_0)_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \overline{D}_{0,n} & \rightarrow & \dots & \rightarrow & \overline{D}_{0,0} & \rightarrow & (\operatorname{Tot}_0)_0 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{G}_n & \rightarrow & \dots & \rightarrow & \mathcal{G}_0 & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows of degree greater or equal than zero are orthogonally split. Hence the result follows from equation (2.26), equation (2.40) and proposition 2.33. □

Remark 2.41. We have only defined the Bott-Chern classes associated to the Chern character. Everything applies without change to any additive characteristic class. The reader will find no difficulty to adapt the previous results to any multiplicative characteristic class like the Todd genus or the total Chern class.

## 3 Direct images of Bott-Chern classes

The aim of this section is to show that certain direct images of Bott-Chern classes are closed. This result is a generalization of results of Bismut, Gillet and Soulé [6] page 325 and of Mourougane [29] proposition 6. The fact that these direct images of Bott-Chern classes are closed implies that certain relations between characteristic classes are true at the level of differential forms (see corollary 3.7 and corollary 3.8).

In the first part of this section we deal with differential geometry. Thus all the varieties will be differentiable manifolds.

Let $G_1$ be a Lie group and let $\pi\colon N_2 \longrightarrow M_2$ be a principal bundle with structure group $G_2$ and connection $\omega_2$. Assume that there is a left action of $G_1$ over $N_2$ that commutes with the right action of $G_2$ and such that the connection $\omega_2$ is $G_1$-invariant.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be the Lie algebras of $G_1$ and $G_2$. Every element $\gamma \in \mathfrak{g}_1$ defines a tangent vector field $\gamma^*$ over $N_2$ given by

$$\gamma^*_p = \left.\frac{d}{dt}\right|_{t=0} \exp(t\gamma)p.$$

Let $(\gamma^*)^V$ be the vertical component of $\gamma^*$ with respect to the connection $\omega_2$. For every point $p \in N_2$, we denote by $\varphi(\gamma, p) \in \mathfrak{g}_2$ the element characterized by $(\gamma^*)^V_p = \varphi(\gamma, p)^*_p$, where $\varphi(\gamma, p)^*$ is the fundamental vector field associated to $\varphi(\gamma, p)$.

The commutativity of the actions of $G_1$ and $G_2$ and the invariance of the connection $\omega_2$ implies that, for $g \in G_1$ and $\gamma \in \mathfrak{g}_1$, the following equalities hold

$$L_{g*}(\gamma^*) = (\operatorname{ad}(g)\gamma^*), \tag{3.1}$$

$$L_{g*}(\gamma^*)^V = (\operatorname{ad}(g)\gamma^*)^V, \tag{3.2}$$

$$\varphi(\operatorname{ad}(g)\gamma, p) = \varphi(\gamma, g^{-1}p). \tag{3.3}$$

Let $\mathcal{G}_2$ be the vector bundle over $M_2$ associated to $N_2$ and the adjoint representation of $G_2$. That is,

$$\mathcal{G}_2 = N_2 \times \mathfrak{g}_2 \left/ \left\langle (pg, v) \sim (p, \operatorname{ad}(g)v) \right\rangle \right. .$$

Thus, we can identify smooth sections of $\mathcal{G}_2$ with $\mathfrak{g}_2$-valued functions on $N_2$ that are invariant under the action of $G_2$. In this way, $\varphi(\gamma, p)$ determines a section

$$\varphi(\gamma) \in C^{\infty}(N_2, \mathfrak{g}_2)^{G_2} = C^{\infty}(M_2, \mathcal{G}_2).$$

Equation (3.3) implies that, for $g \in G_1$ and $\gamma \in \mathfrak{g}_1$,

$$\varphi(\operatorname{ad}(g)\gamma) = L^*_{g^{-1}}\varphi(\gamma).$$

We denote by $\Omega^{\omega_2}$ the curvature of the connection $\omega_2$. Let $P$ be an invariant function on $\mathfrak{g}_2$, then $P(\Omega^{\omega_2} + \varphi(\gamma))$ is a well defined differential form on $M_2$.

Proposition 3.4. *Let $P$ be an invariant function on $\mathfrak{g}_2$ and let $\mu$ be a current on $M_2$ invariant under the action of $G_1$. Then $\mu(P(\Omega^{\omega_2}+\varphi(\gamma)))$ is an invariant function on $\mathfrak{g}_1$.*

*Proof.* Let $g \in G_1$. Then,

$$\begin{aligned}
\mu(P(\Omega^{\omega_2} + \varphi(\operatorname{ad}(g)\gamma))) &= \mu(P(\Omega^{\omega_2} + L^*_{g^{-1}}\varphi(\gamma))) \\
&= \mu(P(L^*_{g^{-1}}\Omega^{\omega_2} + L^*_{g^{-1}}\varphi(\gamma))) \\
&= L_{g^{-1}*}(\mu)(P(\Omega^{\omega_2} + \varphi(\gamma))) \\
&= \mu(P(\Omega^{\omega_2} + \varphi(\gamma)))
\end{aligned}$$

□

Let now $N_1 \longrightarrow M_1$ be a principal bundle with structure group $G_1$ and provided with a connection $\omega_1$. Then we can form the diagram

$$\begin{array}{ccc}
N_1 \times N_2 & \xrightarrow{\pi_1} & N_1 \underset{G_1}{\times} N_2 \\
\big\downarrow \pi' & & \big\downarrow \pi \\
N_1 \times M_2 & \xrightarrow{\pi_2} & N_1 \underset{G_1}{\times} M_2 \\
& & \big\downarrow q \\
& & M_1
\end{array}$$

Then $\pi$ is a principal bundle with structure group $G_2$. The connections $\omega_1$ and $\omega_2$ induce a connection on the principal bundle $\pi$. The subbundle of horizontal vectors with respect to this connection is given by $\pi_{1*}(T^H N_1 \oplus T^H N_2)$. We will denote this connection by $\omega_{1,2}$. We are interested in computing the curvature $\omega_{1,2}$.

In fact, all the maps in the above diagram are fiber bundles provided with a connection. When applicable, given a vector field $U$ in any of these spaces, we will denote by $U^{H,1}$ the horizontal lifting to $N_1 \times N_2$, by $U^{H,2}$ the horizontal lifting to $N_1 \underset{G_1}{\times} N_2$ and by $U^{H,3}$ the horizontal lifting to $N_1 \underset{G_1}{\times} M_2$.

The tangent space $T(N_1 \times N_2)$ can be decomposed as direct sum in the following ways

$$\begin{aligned} T(N_1 \times N_2) &= T^H N_1 \oplus T^V N_1 \oplus T^H N_2 \oplus T^V N_2 \\ &= T^H N_1 \oplus T^V N_1 \oplus T^H N_2 \oplus \operatorname{Ker} \pi_{1*}, \end{aligned} \tag{3.5}$$

For every point $(x, y) \in N_1 \times N_2$ we have that $(\operatorname{Ker} \pi_{1*})_{(x,y)} \subset T^V_x N_1 \oplus T_y N_2$. Moreover, there is an isomorphism $\mathfrak{g}_1 \longrightarrow (\operatorname{Ker} \pi_{1*})_{(x,y)}$ that sends an element $\gamma \in \mathfrak{g}_1$ to the element $(\gamma^*_x, -\gamma^*_y) \in T^V_x N_1 \oplus T_y N_2$.
The tangent space to $N_1 \underset{G_1}{\times} M_2$ can be decomposed as the sum of the subbundle of vertical vectors with respect to $q$ and the subbundle of horizontal vectors defined by the connection $\omega_1$. The horizontal lifting to $N_1 \times N_2$ of a vertical vector lies in $T^H N_2$ and the horizontal lifting of a horizontal vector lies in $T^H N_1$.
Let $U$, $V$ be two vector fields on $M_1$ and let $U^{H,3}$, $V^{H,3}$ be the horizontal liftings to $N_1 \underset{G_1}{\times} M_2$. Then

$$\begin{aligned} \Omega^{\omega_{1,2}}(U^{H,3}, V^{H,3}) &= [U^{H,3}, V^{H,3}]^{H,2} - [U^{H,2}, V^{H,2}] \\ &= \pi_{1*}([U^{H,3}, V^{H,3}]^{H,1} - [U^{H,1}, V^{H,1}]) \\ &= \pi_{1*}([U^{H,3}, V^{H,3}]^{H,1} - [U, V]^{H,1} + [U, V]^{H,1} - [U^{H,1}, V^{H,1}]) \\ &= \pi_{1,*}([U^{H,3}, V^{H,3}]^{H,1} - [U, V]^{H,1} + \Omega^{\omega_1}(U, V)). \end{aligned}$$

But, we have

$$\begin{aligned} \Omega^{\omega_{1,2}}(U^{H,3}, V^{H,3}) &\in T^V N_2, \\ \Omega^{\omega_1}(U, V) &\in T^V N_1, \\ [U^{H,3}, V^{H,3}]^{H,1} - [U, V]^{H,1} &\in T^H N_2. \end{aligned}$$

Therefore, by the direct sum decomposition (3.5) we obtain that

$$\Omega^{\omega_{1,2}}(U^{H,3}, V^{H,3}) = ((\pi_{1*}\Omega^{\omega_1}(U, V)))^V,$$

where the vertical part is taken with respect to the fib re bundle $\pi$.
If $U$ is a horizontal vector field over $N_1 \underset{G_1}{\times} M_2$ and $V$ is a vertical vector field, a similar argument shows that $\Omega^{\omega_{1,2}}(U, V) = 0$. Finally, if $U$ and $V$ are vector fields on $M_2$, they determine vertical vector fields on $N_1 \underset{G_1}{\times} M_2$. Then the horizontal liftings $U^{H,1}$ and $V^{H,1}$ are induced by horizontal liftings of $U$ and $V$ to $N_2$. Therefore, reasoning as before we see that

$$\Omega^{\omega_{1,2}}(U, V) = \Omega^{\omega_2}(U, V).$$

Proposition 3.6. *Let $G_1$ and $G_2$ be Lie groups, with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. For $i = 1, 2$, let $N_i \longrightarrow M_i$ be a principal bundle with structure group*

*$G_i$, provided with a connection $\omega_i$. Assume that there is a left action of $G_1$ over $N_2$ that commutes with the right action of $G_2$ and that the connection $\omega_2$ is invariant under the $G_1$-action. We form the $G_2$-principal bundle $\pi\colon N_1 \underset{G_1}{\times} N_2 \longrightarrow N_1 \underset{G_1}{\times} M_2$ with the induced connection $\omega_{1,2}$ and curvature $\Omega^{\omega_{1,2}}$. Let $P$ be any invariant function on $\mathfrak{g}_2$. Thus $P(\Omega^{\omega_{1,2}})$ is a well defined closed differential form on $N_1 \underset{G_1}{\times} M_2$. Let $\mu$ be a current on $M_2$ invariant under the $G_1$-action. Being $G_1$ invariant, the current $\mu$ induces a current on $N_1 \underset{G_1}{\times} M_2$, that we denote also by $\mu$. Let $q\colon N_1 \underset{G_1}{\times} M_2 \longrightarrow M_1$ be the projection. Then $q_*(P(\Omega^{\omega_{1,2}}) \wedge \mu)$ is a closed differential form on $M_1$.*

*Proof.* Let $U \subset M_1$ be a trivializing open subset for $N_1$ and choose a trivialization of $N_1 \mid_U \cong U \times G_1$. With this trivialization, we can identify $\Omega^{\omega_1} \mid_U$ with a 2-form on $U$ with values in $\mathfrak{g}_1$.
For $\gamma \in \mathfrak{g}_1$, we denote by

$$\psi_\mu(\gamma) = \mu(P(\Omega^{\omega_2} + \varphi(\gamma)))$$

the invariant function provided by proposition 3.4.
Then

$$q_*(P(\Omega^{\omega_{1,2}}) \wedge \mu) = \psi_\mu(\Omega^{\omega_1}).$$

Therefore, the result follows from the usual Chern-Weil theory. □

We go back now to complex geometry and analytic real Deligne cohomology and to the notations 1.3, in particular (1.4).

Corollary 3.7. *Let $X$ be a complex manifold and let $\overline{E} = (E, h^E)$ be a rank $r$ hermitian holomorphic vector bundle on $X$. Let $\pi\colon \mathbb{P}(E) \longrightarrow X$ be the associated projective bundle. On $\mathbb{P}(E)$ we consider the tautological exact sequence*

$$\overline{\xi}\colon 0 \longrightarrow \overline{\mathcal{O}(-1)} \longrightarrow \pi^*\overline{E} \longrightarrow \overline{Q} \longrightarrow 0$$

*where all the vector bundles have the induced metric. Let $P_1$, $P_2$ and $P_3$ be invariant power series in $1$, $r-1$ and $r$ variables respectively with coefficients in $\mathbb{D}$. Let $P_1(\overline{\mathcal{O}(-1)})$ and $P_2(\overline{Q})$ be the associated Chern forms and let $\widetilde{P}_3(\overline{\xi})$ the associated Bott-Chern class. Then*

$$\pi_*(P_1(\overline{\mathcal{O}(-1)}) \bullet P_2(\overline{Q}) \bullet \widetilde{P}_3(\overline{\xi})) \in \bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(X,k)$$

*is closed. Hence it defines a class in analytic real Deligne cohomology. This class does not depend on the hermitian metric of $E$.*

*Proof.* We consider $\mathbb{C}^r$ with the standard hermitian metric. On the space $\mathbb{P}(\mathbb{C}^r)$ we have the tautological exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1) \xrightarrow{f} \mathbb{C}^r \longrightarrow Q \longrightarrow 0.$$

Let $(x:y)$ be homogeneous coordinates on $\mathbb{P}^1$ and let $t=x/y$ be the absolute coordinate. Let $p_1$ and $p_2$ be the two projections of $M_2=\mathbb{P}(\mathbb{C}^r)\times\mathbb{P}^1$. Let $\widetilde{E}$ be the cokernel of the map

$$\begin{array}{ccc} p_1^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1) & \longrightarrow & p_1^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1)\otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)\oplus p_1^*\mathbb{C}^r\otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1) \\ s & \longmapsto & s\otimes y+f(s)\otimes x \end{array}$$

with the metric induced by the standard metric of $\mathbb{C}^r$ and the Fubini-Study metric of $\mathcal{O}_{\mathbb{P}(1)}(1)$.

Let $N_2$ be the principal bundle over $M_2$ formed by the triples $(e_1,e_2,e_3)$, where $e_1$, $e_2$ and $e_3$ are unitary frames of $p_1^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1)$, $p_1^*Q$ and $\widetilde{E}$ respectively. The structure group of this principal bundle is $G_2=U(1)\times U(r-1)\times U(r)$. Let $\omega_2$ be the connection induced by the hermitian holomorphic connections on the vector bundles $p_1^*\mathcal{O}_{\mathbb{P}(\mathbb{C}^r)}(-1)$, $p_1^*Q$ and $\widetilde{E}$.

Now we denote $M_1=X$, and let $N_1$ be the bundle of unitary frames of $\overline{E}$. This is a principal bundle over $M_1$ with structure group $G_1=U(r)$.

The group $G_1$ acts on the left on $N_2$. This action commutes with the right action of $G_2$ and the connection $\omega_2$ is invariant under this action.

Let $\mu=[-\log(|t|)]$ be the current on $M_2$ associated to the locally integrable function $-\log(|t|)$. This current is invariant under the action of $G_1$ because this group acts trivially on the factor $\mathbb{P}^1$.

The invariant power series $P_1$, $P_2$ and $P_3$ determine an invariant function $P$ on $\mathfrak{g}_2$, the Lie algebra of $G_2$.

Let $\omega_1$ be the connection induced in $N_1$ by the holomorphic hermitian connection on $\overline{E}$. As before let $\omega_{1,2}$ be the connection on $N_1\underset{G_1}{\times}N_2$ induced by $\omega_1$ and $\omega_2$ and let $q\colon N_1\underset{G_1}{\times}M_2\longrightarrow M_1$ be the projection. Observe that $N_1\underset{G_1}{\times}M_2=\mathbb{P}(E)\times\mathbb{P}^1$ and $q=\pi\circ p_1$.

By the projection formula and the definition of Bott-Chern classes we have

$$\pi_*(P_1(\overline{\mathcal{O}(-1)})\wedge P_2(\overline{Q})\wedge\widetilde{P}_3(\overline{\xi}))=q_*(\mu\bullet P(\Omega^{\omega_{1,2}})),$$

Therefore the fact that it is closed follows from 3.6. Since, for fixed $P_1$, $P_2$ and $P_3$, the construction is functorial on $(X.\overline{E})$, the fact that the class in analytic real Deligne cohomology does not depend on the choice of the hermitian metric follows from proposition 1.7. □

Corollary 3.8. *Let $\overline{E}=(E,h^E)$ be a hermitian holomorphic vector bundle on a complex manifold $X$. We consider the projective bundle $\pi\colon\mathbb{P}(E\oplus\mathbb{C})\longrightarrow X$. Let $\overline{Q}$ be the universal quotient bundle on the space $\mathbb{P}(E\oplus\mathbb{C})$ with the induced metric. Then the following equality of differential forms holds*

$$\pi_*\sum_i(-1)^i\operatorname{ch}(\bigwedge^i\overline{Q}^\vee)=\pi_*(c_r(\overline{Q})\operatorname{Td}^{-1}(\overline{Q}))=\operatorname{Td}^{-1}(\overline{E}).$$

*Proof.* Let $\overline{\xi}$ be the tautological exact sequence with induced metrics. We first prove that

$$\pi_*(c_r(\overline{Q})\operatorname{Td}(\overline{\mathcal{O}(-1)})) = 1.$$

We can write $\operatorname{Td}(\overline{\mathcal{O}(-1)}) = 1 + c_1(\overline{\mathcal{O}(-1)})\phi(\overline{\mathcal{O}(-1)})$ for certain power series $\phi$. Since $c_{r+1}(\overline{E}\oplus\mathbb{C}) = 0$ we have

$$c_r(\overline{Q})c_1(\overline{\mathcal{O}(-1)}) = \mathrm{d}_{\mathcal{D}}\,\widetilde{c}_{r+1}(\overline{\xi}).$$

Therefore, by corollary 3.7, we have

$$\begin{aligned}\pi_*(c_r(\overline{Q})\operatorname{Td}(\overline{\mathcal{O}(-1)})) &= \pi_*(c_r(\overline{Q})) + \pi_*(c_r(\overline{Q})c_1(\overline{\mathcal{O}(-1)})\phi(\overline{\mathcal{O}(-1)}))\\ &= 1 + \mathrm{d}_{\mathcal{D}}\,\pi_*(\widetilde{c}_{r+1}(\overline{\xi})\phi(\overline{\mathcal{O}(-1)}))\\ &= 1.\end{aligned}$$

Then the corollary follows from corollary 3.7 by using the identity

$$\begin{aligned}\pi_*(c_r(\overline{Q})\operatorname{Td}^{-1}(\overline{Q})) = \pi_*(c_r(\overline{Q})\operatorname{Td}(\overline{\mathcal{O}(-1)})\pi^*\operatorname{Td}^{-1}(\overline{E}))\\ + \mathrm{d}_{\mathcal{D}}\,\pi_*(c_r(\overline{Q})\operatorname{Td}(\overline{\mathcal{O}(-1)})\widetilde{\operatorname{Td}^{-1}}(\overline{\xi})).\end{aligned}$$

□

The following generalization of corollary 3.7 provides many relations between integrals of Bott-Chern classes and is left to the reader.

Corollary 3.9. *Let $X$ be a complex manifold and let $\overline{E} = (E, h^E)$ be a rank $r$ hermitian holomorphic vector bundle on $X$. Let $\pi\colon \mathbb{P}(E) \longrightarrow X$ be the associated projective bundle. On $\mathbb{P}(E)$ we consider the tautological exact sequence*

$$\overline{\xi}\colon 0 \longrightarrow \overline{\mathcal{O}(-1)} \longrightarrow \pi^*\overline{E} \longrightarrow \overline{Q} \longrightarrow 0$$

*where all the vector bundles have the induced metric. Let $P_1$ and $P_2$ be invariant power series in $1$ and $r-1$ variables respectively with coefficients in $\mathbb{D}$ and let $P_3,\dots,P_k$ be invariant power series in $r$ variables with coefficients in $\mathbb{D}$. Let $P_1(\overline{\mathcal{O}(-1)})$ and $P_2(\overline{Q})$ be the associated Chern forms and let $\widetilde{P}_3(\overline{\xi}),\dots,\widetilde{P}_k(\overline{\xi})$ be the associated Bott-Chern classes. Then*

$$\pi_*(P_1(\overline{\mathcal{O}(-1)}) \bullet P_2(\overline{Q}) \bullet \widetilde{P}_3(\overline{\xi}) \bullet \cdots \bullet \widetilde{P}_k(\overline{\xi}))$$

*is a closed differential form on $X$ for any choice of the ordering in computing the non associative product under the integral.*

## 4 Cohomology of currents and wave front sets

The aim of this section is to prove the Poincaré lemma for the complex of currents with fixed wave front set. This implies in particular a certain $\partial\bar{\partial}$-lemma (corollary 4.7) that will allow us to control the singularities of singular Bott-Chern classes.

Let $X$ be a complex manifold of dimension $n$. Following notation 1.3 recall that there is a canonical isomorphism

$$H^*_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(p)) \cong H^*(\mathcal{D}^*_D(X,p)).$$

A current $\eta$ can be viewed as a generalized section of a vector bundle and, as such, has a wave front set that is denoted by $\mathrm{WF}(\eta)$. The theory of wave front sets of distributions is developed in [25] chap. VIII. For the theory of wave front sets of generalized sections, the reader can consult [24] chap. VI. Although we will work with currents and hence with generalized sections of vector bundles, we will follow [25].

The wave front set of $\eta$ is a closed conical subset of the cotangent bundle of $X$ minus the zero section $T^*X_0 = T^*X \setminus \{0\}$. This set describes the points and directions of the singularities of $\eta$ and it allows us to define certain products and inverse images of currents.

Let $S \subset T^*X_0$ be a closed conical subset, we will denote by $\mathscr{D}^*_{X,S}$ the subsheaf of currents whose wave front set is contained in $S$. We will denote by $D^*(X,S)$ its complex of global sections.

For every open set $U \subset X$ there is an appropriate notion of convergence in $\mathscr{D}^*_{X,S}(U)$ (see [25] VIII Definition 8.2.2). All references to continuity below are with respect to this notion of convergence.

We next summarize the basic properties of wave front sets.

Proposition 4.1. *Let $u$ be a generalized section of a vector bundle and let $P$ be a differential operator with smooth coefficients. Then*

$$\mathrm{WF}(Pu) \subseteq \mathrm{WF}(u).$$

*Proof.* This is [25] VIII (8.1.11). □

Corollary 4.2. *The sheaf $\mathscr{D}^*_{X,S}$ is closed under $\partial$ and $\bar{\partial}$. Therefore it is a sheaf of Dolbeault complexes.*

Let $f\colon X \longrightarrow Y$ be a morphism of complex manifolds. The *set of normal directions* of $f$ is

$$N_f = \{(f(x),v) \in T^*Y \mid df(x)^t v = 0\}.$$

This set measures the singularities of $f$. For instance, if $f$ is a smooth map then $N_f = 0$ whereas, if $f$ is a closed immersion, $N_f$ is the conormal bundle of $f(X)$. Let $S \subset T^*Y_0$ be a closed conical subset. We will say that $f$ is transverse to $S$ if $N_f \cap S = \emptyset$. We will denote

$$f^*S = \{(x, df(x)^t v) \in T^*X_0 \mid (f(x),v) \in S\}.$$

Theorem 4.3. *Let $f\colon X \longrightarrow Y$ be a morphism of complex manifolds that is transverse to $S$. Then there exists one and only one extension of the pull-back morphism $f^*\colon \mathscr{E}^*_Y \longrightarrow \mathscr{E}^*_X$ to a continuous morphism*

$$f^*\colon \mathscr{D}^*_{Y,S} \longrightarrow \mathscr{D}^*_{X,f^*S}.$$

*In particular there is a continuous morphism of complexes*

$$D^*(Y,S) \longrightarrow D^*(X,f^*S).$$

*Proof.* This follows from [25] theorem 8.2.4. □

We now recall the effect of correspondences on the wave front sets.
Let $K \in D^*(X\times Y)$, and let $S$ be a conical subset of $T^*Y_0$. We will write

$$\begin{aligned}\mathrm{WF}(K)_X &= \{(x,\xi)\in T^*X_0 \mid \exists y\in Y,(x,y,\xi,0)\in \mathrm{WF}(K)\}\\ \mathrm{WF}'(K)_Y &= \{(y,\eta)\in T^*Y_0 \mid \exists x\in X,(x,y,0,-\eta)\in \mathrm{WF}(K)\}\\ \mathrm{WF}'(K)\circ S &= \{(x,\xi)\in T^*X_0 \mid \exists (y,\eta)\in S,(x,y,\xi,-\eta)\in \mathrm{WF}(K)\}.\end{aligned}$$

Theorem 4.4. *The image of the correspondence map*

$$\begin{array}{ccc} E^*_c(Y) & \longrightarrow & D^*(X)\\ \eta & \longmapsto & p_{1*}(K\wedge p_2^*(\eta))\end{array}$$

*is contained in $D^*(X,WF(K)_X)$. Moreover, if $S\cap \mathrm{WF}'(K)_Y=\emptyset$, then there exists one and only one extension to a continuous map*

$$D^*_c(Y,S)\longrightarrow D^*(X,S'),$$

*where $S'=\mathrm{WF}(K)_X\cup \mathrm{WF}'(K)\circ S$.*

*Proof.* This is [25] theorem 8.2.13. □

We are now in a position to state and prove the Poincaré lemma for currents with fixed wave front set. As usual, we will denote by $F$ the Hodge filtration of any Dolbeault complex.

Theorem 4.5 (Poincaré lemma). *Let $S$ be any conical subset of $T^*X_0$. Then the natural morphism*

$$\iota\colon (E^*(X),F)\longrightarrow (D^*(X,S),F)$$

*is a filtered quasi-isomorphism.*

*Proof.* Let $K$ be the Bochner-Martinelli integral operator on $\mathbb{C}^n\times\mathbb{C}^n$. It is the operator

$$\begin{array}{ccc} E^{p,q}_c(\mathbb{C}^n) & \longrightarrow & E^{p,q-1}(\mathbb{C}^n)\\ \varphi & \longmapsto & \int_{w\in\mathbb{C}^n} k(z,w)\wedge\varphi(w),\end{array}$$

where $k$ is the Bochner-Martinelli kernel ([21] pag. 383). Thus $k$ is a differential form on $\mathbb{C}^n\times\mathbb{C}^n$ with singularities only along the diagonal.
Using the explicit description of $k$ in [21], it can be seen that $WF(k)=N^*\Delta_0$, the conormal bundle of the diagonal. By theorem 4.4, the operator $K$ defines a continuous linear map from $\Gamma_c(\mathbb{C}^n,\mathscr{D}^*_{\mathbb{C}^n,S})$ to $\Gamma(\mathbb{C}^n,\mathscr{D}^*_{\mathbb{C}^n,S})$. This is the key

fact that allows us to adapt the proof of the Poincaré Lemma for arbitrary currents to the case of currents with fixed wave front set.
We will prove that the sheaf inclusion

$$(\mathscr{E}_X, F) \longrightarrow (\mathscr{D}_{X,S}, F)$$

is a filtered quasi-isomorphism. Then the theorem will follow from the fact that both are fine sheaves.
The previous statement is equivalent to the fact that, for any integer $p \geq 0$, the inclusion

$$\iota \colon \mathscr{E}_X^{p,*} \longrightarrow \mathscr{D}_{X,S}^{p,*}$$

is a quasi-isomorphism.
Let $x \in X$, since exactness can be checked at the level of stalks, we need to show that

$$\iota_x \colon \mathscr{E}_{X,x}^{p,*} \longrightarrow \mathscr{D}_{X,S,x}^{p,*}$$

is a quasi-isomorphism. let $U$ be a coordinate neighborhood around $x$ and let $x \in V \subset U$ be a relatively compact open subset.
Let $\rho \in C_c^{\infty}(U)$ be a function with compact support such that $\rho \mid_V = 1$. We define an operator

$$K\rho \colon \mathscr{D}_{X,S}^{p,q}(U) \longrightarrow \mathscr{D}_{X,S}^{p,q-1}(V).$$

If $T \in \mathscr{D}_{X,S}^{p,q}(U)$ and $\varphi \in E_c^*(V)$ is a test form, then

$$K\rho(T)(\varphi) = (-1)^{p+q} T(\rho K(\varphi)).$$

Hence, using that $\bar{\partial} K(\varphi) + K(\bar{\partial}\varphi) = \varphi$, and that $\varphi = \rho\varphi$, we have

$$(\bar{\partial} K\rho T + K\rho\bar{\partial} T + T)(\varphi) = -T(\bar{\partial}(\rho) \wedge K(\varphi)).$$

Observe that, even if the support of $\varphi$ is contained in $V$, the support of $K(\varphi)$ can be $\mathbb{C}^n$; therefore the right hand side of the above equation may be non zero.
We compute

$$\begin{aligned} T(\bar{\partial}(\rho) \wedge K(\varphi)) &= T\left(\bar{\partial}(\rho) \wedge \int_{w \in \mathbb{C}^n} k(w,z) \wedge \varphi(w)\right) \\ &= T\left(\int_{w \in \mathbb{C}^n} \bar{\partial}(\rho) \wedge k(w,z) \wedge \varphi(w)\right). \end{aligned}$$

Since $\operatorname{supp}(\varphi) \subset V$ and $\bar{\partial}(\rho)|_V \equiv 0$, we can find a number $\epsilon > 0$ such that, if $\|z - w\| < \epsilon$, then $\bar{\partial}(\rho) \wedge k(w,z) \wedge \varphi(w) = 0$. Since the singularities of $k(w,z)$ are concentrated on the diagonal, it follows that the differential form $\bar{\partial}(\rho) \wedge k(w,z) \wedge \varphi(w)$ is smooth. Therefore, the current in $V$ given by

$$\varphi \longmapsto T\left(\int_{w \in \mathbb{C}^n} \bar{\partial}(\rho) \wedge k(w,z) \wedge \varphi(w)\right),$$

is the current associated to the smooth differential form $T_z\left(\bar{\partial}(\rho)\wedge k(w,z)\right)$, where the subindex $z$ means that $T$ only acts on the $z$ variable, being $w\in V$ a parameter. This smooth form will be denoted by $\Psi(T)$.
Summing up, we have shown that, for any current $T\in\mathscr{D}^{p,q}_{X,S}(U)$ there exists a smooth differential form $\Psi(T)\in\mathscr{E}^{p,q}_X(V)$ such that

$$T\mid_V=-\bar{\partial}K\rho T-K\rho\bar{\partial}T-\Psi(T).$$

Observe that we can not say that $\Psi$ is a quasi-inverse of $\iota_x$ because it depends on the choice of $\rho$ and it is not possible to choose a single $\rho$ that can be applied to all $T$. Hence it is not a well defined operator at the level of stalks. Let now $T\in\mathscr{D}^{p,*}_{X,S,x}$ be closed. It is defined in some neighborhood of $x$, say $U'$. Applying the above procedure we find a smooth differential form $\Psi(T)$ defined on a relatively compact subset of $U'$, say $V'$, that is cohomologous to $T$. Hence the map induced by $\iota_x$ in cohomology is surjective. Let $\omega\in\mathscr{E}^{p,*}_{X,x}$ be closed and such that $\iota_x\omega=\bar{\partial}T$ for some $T\in\mathscr{D}^{p,*-1}_{X,S,x}$. We may assume that $\omega$ and $T$ are defined is some neighborhood $U''$ of $x$. Then, on some relatively compact subset $V''\subset U''$, we have

$$\omega\mid_{V''}=\bar{\partial}T\mid_{V''}=-\bar{\partial}K\rho\omega-\bar{\partial}\Psi(T).$$

Since $K\rho\omega$ and $\Psi(T)$ are smooth differential forms we conclude that the map induced by $\iota_x$ in cohomology is injective. □

We will denote by $\mathcal{D}^*_D(X,S,p)$ the Deligne complex associated to $D^*(X,S)$. The following two results are direct consequences of theorem 4.5.

Corollary 4.6. *The inclusion $\mathcal{D}^*_D(X,S,p)\longrightarrow\mathcal{D}^*_D(X,p)$ induces an isomorphism*

$$H^*(\mathcal{D}^*_D(X,S,p))\cong H^*_{\mathcal{D}^{\mathrm{an}}}(X,\mathbb{R}(p)).$$

Corollary 4.7. *(i) Let $\eta\in\mathcal{D}^n_D(X,p)$ be a current such that*

$$\mathrm{d}_{\mathcal{D}}\,\eta\in\mathcal{D}^{n+1}_D(X,S,p),$$

*then there is a current $a\in\mathcal{D}^{n-1}_D(X,p)$ such that $\eta+\mathrm{d}_{\mathcal{D}}\,a\in\mathcal{D}^n_D(X,S,p)$.*

*(ii) Let $\eta\in\mathcal{D}^n_D(X,S,p)$ be a current such that there is a current $a\in\mathcal{D}^{n-1}_D(X,p)$ with $\eta=\mathrm{d}_{\mathcal{D}}\,a$, then there is a current $b\in\mathcal{D}^{n-1}_D(X,S,p)$ such that $\eta=\mathrm{d}_{\mathcal{D}}\,b$.*

□

## 5 Deformation of resolutions

In this section we will recall the deformation of resolutions based on the Grassmannian graph construction of [1]. We will also recall the Koszul resolution associated to a section of a vector bundle.

The main theme is that given a bounded complex $E_*$ of locally free sheaves (with some properties) on a complex manifold $X$, one can construct a bounded complex $\operatorname{tr}_1(E_*)_*$ over a certain manifold $W$. This new manifold has a birational map $\pi\colon W \longrightarrow X\times\mathbb{P}^1$, that is an isomorphism over $X\times\mathbb{P}^1\setminus\{\infty\}$. The complex $\operatorname{tr}_1(E_*)_*$ agrees with the original complex over $X\times\{0\}$ and is particularly simple over $\pi^{-1}(X\times\{\infty\})$. Thus $\operatorname{tr}_1(E_*)_*$ is a deformation of the original complex to a simpler one. The two examples we are interested in are: first, when the original complex is exact, then $W$ agrees with $X\times\mathbb{P}^1$ and $\operatorname{tr}_1(E_*)_*$ was defined in 2.5. Its restriction to $\pi^{-1}(X\times\{\infty\})$ is split; second, when $i\colon Y\longrightarrow X$ is a closed immersion of complex manifolds, and $E_*$ is a bounded resolution of $i_*\mathcal{O}_Y$, then $W$ agrees with the deformation to the normal cone of $Y$ and the restriction of $\operatorname{tr}_1(E_*)_*$ to $\pi^{-1}(X\times\{\infty\})$ is an extension of a Koszul resolution by a split complex. Note that, if we allow singularities, then the Grassmannian graph construction is much more general.

The deformation of resolutions is based on the Grassmannian graph construction of [1], and, in the form that we present here, has been developed in [6] and [20].

In order to fix notations we first recall the deformation to the normal cone and the Koszul resolution associated to the zero section of a vector bundle.

Let $Y\hookrightarrow X$ be a closed immersion of complex manifolds, with $Y$ of pure codimension $n$. In the sequel we will use notation 2.2. Let $W=W_{Y/X}$ be the blow-up of $X\times\mathbb{P}^1$ along $Y\times\{\infty\}$. Since $Y$ and $X\times\mathbb{P}^1$ are manifolds, $W$ is also a manifold. The map $\pi\colon W\longrightarrow X\times\mathbb{P}^1$ is an isomorphism away from $Y\times\{\infty\}$; we will write $P$ for the exceptional divisor of the blow-up. Then

$$P=\mathbb{P}(N_{Y/X}\otimes N_{\infty/\mathbb{P}^1}^{-1}\oplus\mathbb{C}).$$

Thus $P$ can be seen as the projective completion of the vector bundle $N_{Y/X}\otimes N_{\infty/\mathbb{P}^1}^{-1}$. Note that $N_{\infty/\mathbb{P}^1}$ is trivial although not canonically trivial. Nevertheless we can choose to trivialize it by means of the section $y\in\mathcal{O}_{\mathbb{P}^1}(1)$. Sometimes we will tacitly assume this trivialization and omit $N_{\infty/\mathbb{P}^1}$ from the formulae.

The map $q_W\colon W\longrightarrow\mathbb{P}^1$, obtained by composing $\pi$ with the projection $q\colon X\times\mathbb{P}^1\longrightarrow\mathbb{P}^1$, is flat and, for $t\in\mathbb{P}^1$, we have

$$q_W^{-1}(t)\cong\begin{cases}X\times\{t\}, & \text{if } t\neq\infty,\\ P\cup\widetilde{X}, & \text{if } t=\infty,\end{cases}$$

where $\widetilde{X}$ is the blow-up of $X$ along $Y$, and $P\cap\widetilde{X}$ is, at the same time, the divisor at $\infty$ of $P$ and the exceptional divisor of $\widetilde{X}$.

Following [6] we will use the following notations

$$\begin{array}{ccc} P & \xrightarrow{\ f\ } & W\\ {\scriptstyle\pi_P}\big\downarrow & & \big\downarrow{\scriptstyle\pi}\\ Y\times\{\infty\} & \xrightarrow{\ i_\infty\ } & X\times\mathbb{P}^1\end{array}$$

$$\begin{aligned}
i\colon Y &\longrightarrow X, && \\
W_\infty &= \pi^{-1}(\infty) = P \cup \widetilde{X}, && \\
q\colon X \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1, && \text{the projection,}\\
p\colon X \times \mathbb{P}^1 &\longrightarrow X, && \text{the projection,}\\
q_W &= q \circ \pi && \\
p_W &= p \circ \pi && \\
q_Y\colon Y \times \mathbb{P}^1 &\longrightarrow \mathbb{P}^1, && \text{the projection,}\\
p_Y\colon Y \times \mathbb{P}^1 &\longrightarrow Y, && \text{the projection,}\\
j\colon Y \times \mathbb{P}^1 &\longrightarrow W && \text{the induced map,}\\
j_\infty\colon Y \times \{\infty\} &\longrightarrow P. &&
\end{aligned}$$

Given any map $g\colon Z \longrightarrow X \times \mathbb{P}^1$, we will denote $p_Z = p \circ g$ and $q_Z = q \circ g$. For instance $p_P = p \circ \pi \circ f = p_W \circ f = i \circ \pi_P$, where, in the last equality, we are identifying $Y$ with $Y \times \{\infty\}$.

We next recall the construction of the Koszul resolution. Let $Y$ be a complex manifold and let $N$ be a rank $n$ vector bundle. Let $P = \mathbb{P}(N \oplus \mathbb{C})$ be the projective bundle of lines in $N \oplus \mathbb{C}$. It is obtained by completing $N$ with the divisor at infinity. Let $\pi_P\colon P \longrightarrow Y$ be the projection and let $s\colon Y \longrightarrow P$ be the zero section. On $P$ there is a tautological short exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi_P^*(N \oplus \mathbb{C}) \longrightarrow Q \longrightarrow 0. \tag{5.1}$$

The above exact sequence and the inclusion $\mathbb{C} \longrightarrow \pi_P^*(N \oplus \mathbb{C})$ induce a section $\sigma\colon \mathcal{O}_P \longrightarrow Q$ that vanishes along the zero section $s(Y)$. By duality we obtain a morphism $Q^\vee \longrightarrow \mathcal{O}_P$ that induces a long exact sequence

$$0 \longrightarrow \bigwedge^n Q^\vee \longrightarrow \dots \longrightarrow \bigwedge^1 Q^\vee \longrightarrow \mathcal{O}_P \longrightarrow s_*\mathcal{O}_Y \longrightarrow 0.$$

If $F$ is another vector bundle over $Y$, we obtain an exact sequence,

$$0 \longrightarrow \bigwedge^n Q^\vee \otimes \pi_P^* F \longrightarrow \dots \longrightarrow \bigwedge^1 Q^\vee \otimes \pi_P^* F \longrightarrow \pi_P^* F \longrightarrow s_* F \longrightarrow 0. \tag{5.2}$$

Definition 5.3. The *Koszul resolution* of $s_*(F)$ is the resolution (5.2). The complex

$$0 \longrightarrow \bigwedge^n Q^\vee \otimes \pi_P^* F \longrightarrow \dots \longrightarrow \bigwedge^1 Q^\vee \otimes \pi_P^* F \longrightarrow \pi_P^* F \longrightarrow 0$$

will be denoted by $K(F, N)$. When $\overline{N}$ is a hermitian vector bundle, the exact sequence (5.1) induces a hermitian metric on $Q$. If, moreover, $\overline{F}$ is also a hermitian vector bundle, all the vector bundles that appear in the Koszul resolution have an induced hermitian metric. We will denote by $K(\overline{F}, \overline{N})$ the corresponding complex of hermitian vector bundles.

In particular, we shall write $K(\overline{\mathcal{O}_Y}, \overline{N})$ if $F = \mathcal{O}_Y$ is endowed with the trivial metric $\|1\| = 1$, unless expressly stated otherwise.

We finish this section by recalling the results about deformation of resolutions that will be used in the sequel. For more details see [1] II.1, [6] Section 4 (c) and [20] Section 1.

Theorem 5.4. *Let $i : Y \hookrightarrow X$ be a closed immersion of complex manifolds, where $Y$ may be empty. Let $U = X \setminus Y$. Let $F$ be a vector bundle over $Y$ and $E_* \longrightarrow i_*F \longrightarrow 0$ be a resolution of $i_*F$. Then there exists a complex manifold $W = W(E_*)$, called the Grassmannian graph construction, with a birational map $\pi\colon W \longrightarrow X \times \mathbb{P}^1$ and a complex of vector bundles, $\operatorname{tr}_1(E_*)_*$, over $W$ such that*

*(i) The map $\pi$ is an isomorphism away from $Y \times \{\infty\}$. The restriction of $\operatorname{tr}_1(E_*)_*$ to $X \times (\mathbb{P}^1 \setminus \{\infty\})$ is isomorphic to $p_W^* E_*$ restricted to $X \times (\mathbb{P}^1 \setminus \{\infty\})$. Moreover, If $\widetilde{X}$ is the Zariski closure of $U \times \{\infty\}$ inside $W$, the restriction of $\operatorname{tr}_1(E_*)_*$ to $\widetilde{X}$ is split acyclic. In particular, if $Y$ is empty or $F$ is the zero vector bundle, hence $E_*$ is acyclic in the whole $X$, then $W = X \times \mathbb{P}^1$ and $\operatorname{tr}_1(E_*)_*$ is the first transgression exact sequence introduced in 2.5.*

*(ii) When $Y$ is non-empty and $F$ is a non-zero vector bundle over $Y$, then $W(E_*)$ agrees with $W_{Y/X}$, the deformation to the normal cone of $Y$. Moreover, there is an exact sequence of resolutions on $P$*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & \operatorname{tr}_1(E_*)_* \mid_P & \longrightarrow & K(F, N_{Y/X} \otimes N^{-1}_{\infty/\mathbb{P}^1}) & \longrightarrow & 0\,, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & (j_\infty)_*F & \xrightarrow{=} & (j_\infty)_*F & & \end{array}$$

*where $A_*$ is split acyclic and $K(F, N_{Y/X} \otimes N^{-1}_{\infty/\mathbb{P}^1})$ is the Koszul resolution.*

*(iii) Let $f\colon X' \longrightarrow X$ be a morphism of complex manifolds and assume that we are in one of the following cases:*

*(a) The map $f$ is smooth.*

*(b) The map $f$ is arbitrary and $E_*$ is acyclic.*

*(c) $f$ is transverse to $Y$.*

*Then $E'_* := f^*(E_*)$ is exact over $f^{-1}(U)$,*

$$W' := W(E'_*) = W \underset{X}{\times} X',$$

*with $f_W\colon W' \longrightarrow W$ the induced map, and we have $f_W^*(\operatorname{tr}_1(E_*)_*) = \operatorname{tr}_1(f^*(E_*))_*$.*

(iv) *If the vector bundles $E_i$ are provided with hermitian metrics, then one can choose a hermitian metric on* $\operatorname{tr}_1(E_*)_*$ *such that its restriction to* $X \times \{0\}$ *is isometric to* $E_*$ *and the restriction to* $U \times \{\infty\}$ *is orthogonally split. We will denote by* $\operatorname{tr}_1(\overline{E}_*)_*$ *the complex* $\operatorname{tr}_1(E_*)_*$ *with such a choice of hermitian metrics. Moreover, this choice of metrics can be made functorial. That is, if $f$ is a map as in item (iii), then*

$$f_W^*(\operatorname{tr}_1(\overline{E}_*)_*) = \operatorname{tr}_1(f^*(\overline{E}_*))_*$$

*Proof.* The case when $E_*$ is acyclic has already been treated. For the case when $Y$ is non-empty and $F$ is non zero, we first recall the construction of the Grassmannian graph of an arbitrary complex from [20], which is more general than what we need here. If $E$ is a vector bundle over $X$ we will denote by $E(i)$ the vector bundle over $X \times \mathbb{P}^1$ given by $E(i) = p^*E \otimes q^*\mathcal{O}(i)$.

Let $\widetilde{C}_*$ be the complex of locally free sheaves given by $\widetilde{C}_i = E_i(i) \oplus E_{i-1}(i-1)$ with differential given by $\mathrm{d}(a,b) = (b,0)$. On $X \times (\mathbb{P}^1 \setminus \{\infty\})$ we consider, for each $i$, the inclusion of vector bundles $\gamma_i \colon E_i \hookrightarrow \widetilde{C}_i$ given by $s \longmapsto (s \otimes y^i, \mathrm{d}\, s \otimes y^{i-1})$. Let $G$ be the product of the Grassmann bundles $Gr(n_i, \widetilde{C}_i)$ that parametrize rank $n_i = \operatorname{rk} E_i$ subbundles of $\widetilde{C}_i$ over $X \times \mathbb{P}^1$. The inclusion $\gamma_* \colon \bigoplus E_i \longrightarrow \bigoplus \widetilde{C}_i$ induces a section $s$ of $G$ over $X \times \mathbb{A}^1$.

Then $W(E_*)$ is defined to be the closure of $s(X \times \mathbb{A}^1)$ in $G$. Since the projection from $G$ to $X \times \mathbb{P}^1$ is proper, the same is true for the induced map $\pi \colon W \longrightarrow X \times \mathbb{P}^1$. For each $i$, the induced map $W \longrightarrow Gr(n_i, \widetilde{C}_i)$ defines a subbundle $\operatorname{tr}_1(E_*)_i$ of $\pi^*\widetilde{C}_i$. This subbundle agrees with $E_i$ over $X \times \mathbb{A}^1$. The differential of $\widetilde{C}_*$ induces a differential on $\operatorname{tr}_1(E_*)_*$.

Assume now that the bundles $E_i$ are provided with hermitian metrics. Using the Fubini-Study metric of $\mathcal{O}(1)$ we obtain induced metrics on $\widetilde{C}_i$. Over $\pi^{-1}(X \times (\mathbb{P}^1 \setminus \{\infty\}))$ we induce a metric on $\operatorname{tr}_1(E_*)_i$ by means of the identification with $E_i$. Over $\pi^{-1}(X \times (\mathbb{P}^1 \setminus \{0\}))$ we consider on $\operatorname{tr}_1(E_*)_i$ the metric induced by $\widetilde{C}_i$. We glue together both metrics with the partition of unity $\{\sigma_0, \sigma_\infty\}$ of notation 2.2.

In the case we are interested there is a more explicit description of $\operatorname{tr}_1(E_*)_*$ given in [6] Section 4 (c). Namely, $\operatorname{tr}_1(E_*)_i$ is the kernel of the morphism

$$\phi \colon p_W^*\widetilde{C}_i = p_W^*E_i(i) \oplus p_W^*E_{i-1}(i-1) \longrightarrow p_W^*E_{i-1}(i) \oplus p_W^*E_{i-2}(i-1) \quad (5.5)$$

given by $\phi(s,t) = (\mathrm{d}\, s - t \otimes y, \mathrm{d}\, t)$.

The only statements that are not explicitly proved in [6] Section 4 (c) or [20] Section 1 are the functoriality when $f$ is not smooth and the properties of the explicit choice of metrics.

If the complex $E_*$ is acyclic, then the same is true for $E'_* = f^*E_*$. In this case $W = X \times \mathbb{P}^1$ and $W' = X' \times \mathbb{P}^1$. Then the functoriality follows from the definition of $\operatorname{tr}_1(E_*)_*$.

Assume now that we are in case (iii)c. We can form the Cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array}$$

where $i'$ is also a closed immersion of complex manifolds. Then we have that $E'_*$ is a resolution of $i'_* g^* F$. Hence $W' = W(E'_*)$ is the deformation to the normal cone of $Y'$ and therefore $W' = W \underset{X}{\times} X'$. Again the functoriality of $\operatorname{tr}_1(E_*)_*$ can be checked using the explicit construction of [20] Section 1 that we have recalled above. □

Remark 5.6. (i) The definition of $\operatorname{tr}_1(E_*)$ can be extended to any bounded chain complex over a integral scheme (see [20]).

(ii) There is a sign difference in the definition of the inclusion $\gamma$ used in [20] and the one used in [6]. We have followed the signs of the first reference.

## 6 Singular Bott-Chern classes

Throughout this section we will use notation 1.3. In particular we will write

$$\widetilde{\mathcal{D}}^n_D(X,p) = \mathcal{D}^n_D(X,p)/\operatorname{d}_{\mathcal{D}} \mathcal{D}^{n-1}_D(X,p),$$
$$\widetilde{\mathcal{D}}^n_D(X,S,p) = \mathcal{D}^n_D(X,S,p)/\operatorname{d}_{\mathcal{D}} \mathcal{D}^{n-1}_D(X,S,p).$$

A particularly important current is $W_1 \in \mathcal{D}^1_D(\mathbb{P}^1,1)$ given by

$$W_1 = [\frac{-1}{2}\log\|t\|^2]. \tag{6.1}$$

With the above convention, this means that

$$W_1(\eta) = \frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{-1}{2}\log\|t\|^2 \bullet \eta. \tag{6.2}$$

By the Poincaré-Lelong equation

$$\operatorname{d}_{\mathcal{D}} W_1 = \delta_\infty - \delta_0. \tag{6.3}$$

Note that the current $W_1$ was used in the construction of Bott-Chern classes (definition 2.11) and will also have a role in the definition of singular Bott-Chern classes.

Before defining singular Bott-Chern classes we need to define the objects that give rise to them.

Definition 6.4. Let $i\colon Y \longrightarrow X$ be a closed immersion of complex manifolds. Let $N$ be the normal bundle of $Y$ and let $h_N$ be a hermitian metric on $N$. We denote $\overline{N} = (N, h_N)$. Let $r_N$ be the rank of $N$, that agrees with the codimension of $Y$ in $X$. Let $\overline{F} = (F, h_F)$ be a hermitian vector bundle on $Y$ of rank $r_F$. Let $\overline{E}_* \to i_*F$ be a metric on the coherent sheaf $i_*F$. The four-tuple

$$\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*). \tag{6.5}$$

is called a *hermitian embedded vector bundle*. The number $r_F$ will be called the *rank* of $\overline{\xi}$ and the number $r_N$ will be called the *codimension* of $\overline{\xi}$.
By convention, any exact complex of hermitian vector bundles on $X$ will be considered a hermitian embedded vector bundle of any rank and codimension.

Obviously, to any hermitian embedded vector bundle we can associate the metrized coherent sheaf $(i_*F, \overline{E}_* \to i_*F)$.

Definition 6.6. A *singular Bott-Chern class* for a hermitian embedded vector bundle $\overline{\xi}$ is a class $\widetilde{\eta} \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_D(X,p)$ such that

$$\mathrm{d}_{\mathcal{D}}\,\eta = \sum_{i=0}^{n}(-1)^i[\operatorname{ch}(\overline{E}_i)] - i_*([\operatorname{Td}^{-1}(\overline{N})\operatorname{ch}(\overline{F})]) \tag{6.7}$$

for any current $\eta \in \tilde{\eta}$.

The existence of this class is guaranteed by the Grothendieck-Riemann-Roch theorem, which implies that the two currents in the right hand side of equation (6.7) are cohomologous.
Even if we have defined singular Bott-Chern classes as classes of currents with arbitrary singularities, it is an important observation that in each singular Bott-Chern class we can find representatives with controlled singularities. Let $N^*_{Y,0}$ be the conormal bundle of $Y$ with the zero section deleted. It is a closed conical subset of $T^*_0(X)$. Since the current

$$\sum_{i=0}^{n}(-1)^i[\operatorname{ch}(\overline{E}_i)] - i_*([\operatorname{Td}^{-1}(\overline{N})\operatorname{ch}(\overline{F})]) = \sum_{i=0}^{n}(-1)^i[\operatorname{ch}(\overline{E}_i)] - \operatorname{Td}^{-1}(\overline{N})\operatorname{ch}(\overline{F})\delta_Y$$

belongs to $\mathcal{D}^*_D(X, N^*_{Y,0}, p)$, by corollary 4.7, we obtain

Proposition 6.8. *Let $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ be a hermitian embedded vector bundle as before. Then any singular Bott-Chern class for $\overline{\xi}$ belongs to the subset*

$$\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_D(X, N^*_{Y,0}, p) \subset \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_D(X,p).$$

□

This result will allow us to define inverse images of singular Bott-Chern classes for certain maps.
Let $f\colon X' \longrightarrow X$ be a morphism of complex manifolds that is transverse to $Y$. We form the Cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array} .$$

Observe that, by the transversality hypothesis, the normal bundle to $Y'$ on $X'$ is the inverse image of the normal bundle to $Y$ on $X$ and $f^*E_*$ is a resolution of $i'_*g^*F$. Thus we write $f^*\overline{\xi} = (i', f^*\overline{N}, g^*\overline{F}, f^*\overline{E}_*)$, which is a hermitian embedded vector bundle.
By proposition 6.8, given any singular Bott-Chern class $\widetilde{\eta}$ for $\xi$, we can find a representative $\eta \in \bigoplus_p \mathcal{D}_D^{2p-1}(X, N^*_{Y,0}, p)$. By theorem 4.3, there is a well defined current $f^*\eta$ and it is a singular Bott-Chern current for $f^*\xi$. Therefore we can define $f^*(\widetilde{\eta}) = \widetilde{f^*(\eta)}$. Again by theorem 4.3, this class does not depend on the choice of the representative $\eta$.
Our next objective is to study the possible definitions of functorial singular Bott-Chern classes.

Definition 6.9. Let $r_F$ and $r_N$ be two integers. A *theory of singular Bott-Chern classes of rank $r_F$ and codimension $r_N$* is an assignment which, to each hermitian embedded vector bundle $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ of rank $r_F$ and codimension $r_N$, assigns a class of currents

$$T(\overline{\xi}) \in \bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, p)$$

satisfying the following properties

(i) (Differential equation) The following equality holds

$$\mathrm{d}_{\mathcal{D}}\, T(\overline{\xi}) = \sum_i (-1)^i [\mathrm{ch}(\overline{E}_i)] - i_*([\mathrm{Td}^{-1}(\overline{N})\,\mathrm{ch}(\overline{F})]). \tag{6.10}$$

(ii) (Functoriality) For every morphism $f\colon X' \longrightarrow X$ of complex manifolds that is transverse to $Y$, then

$$f^*T(\overline{\xi}) = T(f^*\overline{\xi}).$$

(iii) (Normalization) Let $\overline{A} = (A_*, g_*)$ be a non-negatively graded orthogonally split complex of vector bundles. Write $\overline{\xi} \oplus \overline{A} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_* \oplus \overline{A}_*)$. Then $T(\overline{\xi}) = T(\overline{\xi} \oplus \overline{A})$. Moreover, if $X = \operatorname{Spec}\mathbb{C}$ is one point, $Y = \emptyset$ and $\overline{E}_* = 0$, then $T(\overline{\xi}) = 0$.

A *theory of singular Bott-Chern classes* is an assignment as before, for all positive integers $r_F$ and $r_M$. When the inclusion $i$ and the bundles $F$ and $N$ are clear from the context, we will denote $T(\overline{\xi})$ by $T(\overline{E}_*)$. Sometimes we will have to restrict ourselves to complex algebraic manifolds and algebraic vector bundles. In this case we will talk of *theory of singular Bott-Chern classes for algebraic vector bundles*.

Remark 6.11. (i) Recall that the case when $Y = \emptyset$ and $\overline{E}_*$ is any bounded exact sequence of hermitian vector bundles is considered a hermitian embedded vector bundle of arbitrary rank. In this case, the properties above imply that

$$T(\overline{\xi}) = [\widetilde{\mathrm{ch}}(\overline{E}_*)],$$

where $\widetilde{\mathrm{ch}}$ is the Bott-Chern class associated to the Chern character. That is, for acyclic complexes, any theory of singular Bott-Chern classes agrees with the Bott-Chern classes associated to the Chern character.

(ii) If the map $f$ is transverse to $Y$, then either $f^{-1}(Y)$ is empty or it has the same codimension as $Y$. Moreover, it is clear that $f^*F$ has the same rank as $F$. Therefore, the properties of singular Bott-Chern classes do not mix rank or codimension. This is why we have defined singular Bott-Chern classes for a particular rank and codimension.

(iii) By contrast with the case of Bott-Chern classes, the properties above are not enough to characterize singular Bott-Chern classes.

For the rest of this section we will assume the existence of a theory of singular Bott-Chern classes and we will obtain some consequences of the definition.
We start with the compatibility of singular Bott-Chern classes with exact sequences and Bott-Chern classes.
Let

$$\overline{\chi}\colon 0 \longrightarrow \overline{F}_n \longrightarrow \dots \longrightarrow \overline{F}_1 \longrightarrow \overline{F}_0 \longrightarrow 0 \qquad (6.12)$$

be a bounded exact sequence of hermitian vector bundles on $Y$. For $j = 0, \dots, n$, let $\overline{E}_{j,*} \longrightarrow i_*F_j$ be a resolution, and assume that they fit in a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \overline{E}_{n,*} & \longrightarrow & \cdots & \longrightarrow & \overline{E}_{1,*} & \longrightarrow & \overline{E}_{0,*} & \longrightarrow & 0\ , \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_*F_n & \longrightarrow & \cdots & \longrightarrow & i_*F_1 & \longrightarrow & i_*F_0 & \longrightarrow & 0 \end{array}$$

with exact rows. We write $\overline{\xi_j} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}_j, \overline{E}_{j,*})$. For each $k$, we denote by $\overline{\eta}_k$ the exact sequence

$$0 \longrightarrow \overline{E}_{n,k} \longrightarrow \dots \longrightarrow \overline{E}_{1,k} \longrightarrow \overline{E}_{0,k} \longrightarrow 0.$$

Proposition 6.13. *With the above notations, the following equation holds:*

$$T(\bigoplus_{j\ even}\overline{\xi}_j)-T(\bigoplus_{j\ odd}\overline{\xi}_j)=\sum_k(-1)^k[\widetilde{\mathrm{ch}}(\overline{\eta}_k)]-i_*([\mathrm{Td}^{-1}(\overline{N})\widetilde{\mathrm{ch}}(\overline{\chi})]).$$

Here the direct sum of hermitian embedded vector bundles, involving the same embedding and the same hermitian normal bundle, is defined in the obvious manner.

*Proof.* We consider the construction of theorem 5.4 for each of the exact sequences $\overline{\eta}_k$ and the exact sequence $\overline{\chi}$. For each $k$, we have $W_X := W(\overline{\eta}_k) = X \times \mathbb{P}^1$ and we denote $W_Y := W(\overline{\chi}) = Y \times \mathbb{P}^1$. On $W_Y$ we consider the transgression exact sequence $\mathrm{tr}_1(\overline{\chi})_*$ and on $W_X$ we consider the transgression exact sequences $\mathrm{tr}_1(\overline{\eta_k})_*$. We denote by $j\colon W_Y \longrightarrow W_X$ the induced morphism. Then there is an exact sequence (of exact sequences)

$$\dots \longrightarrow \mathrm{tr}_1(\overline{\eta}_1)_* \longrightarrow \mathrm{tr}_1(\overline{\eta}_0)_* \longrightarrow j_*\,\mathrm{tr}_1(\overline{\chi})_* \longrightarrow 0.$$

We denote

$$\mathrm{tr}_1(\overline{\chi})_+ = \bigoplus_{j\text{ even}} \mathrm{tr}_1(\overline{\chi})_j, \qquad \mathrm{tr}_1(\overline{\chi})_- = \bigoplus_{j\text{ odd}} \mathrm{tr}_1(\overline{\chi})_j,$$
$$\mathrm{tr}_1(\overline{\eta}_k)_+ = \bigoplus_{j\text{ even}} \mathrm{tr}_1(\overline{\eta}_k)_j, \quad \mathrm{tr}_1(\overline{\eta}_k)_- = \bigoplus_{j\text{ odd}} \mathrm{tr}_1(\overline{\eta}_k)_j,$$

and

$$\mathrm{tr}_1(\overline{\xi})_+ = (j\colon W_Y \longrightarrow W_X, p_Y^*\overline{N}, \mathrm{tr}_1(\overline{\chi})_+, \mathrm{tr}_1(\overline{\eta}_*)_+),$$
$$\mathrm{tr}_1(\overline{\xi})_- = (j\colon W_Y \longrightarrow W_X, p_Y^*\overline{N}, \mathrm{tr}_1(\overline{\chi})_-, \mathrm{tr}_1(\overline{\eta}_*)_-),$$

where here $p_Y\colon W_Y \longrightarrow Y$ denotes the projection.
We consider the current on $X\times\mathbb{P}^1$ given by $W_1 \bullet \big(T(\mathrm{tr}_1(\overline{\xi})_+) - T(\mathrm{tr}_1(\overline{\xi})_-)\big)$. This current is well defined because the wave front set of $W_1$ is the conormal bundle of $(X\times\{0\})\cup(X\times\{\infty\})$, whereas the wave front set of $T(\mathrm{tr}_1(\overline{\xi})_\pm)$ is the conormal bundle of $Y\times\mathbb{P}^1$.
By the functoriality of the transgression exact sequences, we obtain that

$$\mathrm{tr}_1(\overline{\xi})_+ \mid_{X\times\{0\}} = \bigoplus_{j\text{ even}} \overline{\xi}_j, \quad \mathrm{tr}_1(\overline{\xi})_- \mid_{X\times\{0\}} = \bigoplus_{j\text{ odd}} \overline{\xi}_j.$$

Moreover, using the fact that, for any bounded acyclic complex of hermitian vector bundles $\overline{E}_*$, the exact sequence $\mathrm{tr}_1(\overline{E}_*)\mid_{X\times\{\infty\}}$ is orthogonally split, we have an isometry

$$\mathrm{tr}_1(\overline{\xi})_+ \mid_{X\times\{\infty\}} \cong \mathrm{tr}_1(\overline{\xi})_- \mid_{X\times\{\infty\}}.$$

We now denote by $p_X\colon W_X \longrightarrow X$ the projection. Using the properties that define a theory of singular Bott-Chern classes, in the group $\bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, N^*_{Y,0}, p)$, the following holds

$$\begin{aligned}
0 &= \mathrm{d}_{\mathcal{D}}(p_X)_* \left(W_1 \bullet T(\mathrm{tr}_1(\overline{\xi})_+) - W_1 \bullet T(\mathrm{tr}_1(\overline{\xi})_-)\right) \\
&= \left(T(\mathrm{tr}_1(\overline{\xi})_+) - T(\mathrm{tr}_1(\overline{\xi})_-)\right) |_{X\times\{\infty\}} - \left(T(\mathrm{tr}_1(\overline{\xi})_+) - T(\mathrm{tr}_1(\overline{\xi})_-)\right) |_{X\times\{0\}} \\
&- (p_X)_* \sum_k (-1)^k W_1 \bullet (\mathrm{ch}(\mathrm{tr}_1(\overline{\eta}_k)_+) - \mathrm{ch}(\mathrm{tr}_1(\overline{\eta}_k)_-)) \\
&+ (p_X)_* \left(W_1 \bullet j_* \left[\mathrm{Td}^{-1}(p_Y^*\overline{N})\,\mathrm{ch}(\mathrm{tr}_1(\overline{\chi})_+) - \mathrm{Td}^{-1}(p_Y^*\overline{N})\,\mathrm{ch}(\mathrm{tr}_1(\overline{\chi})_-)\right]\right) \\
&= -T(\bigoplus_{j \text{ even}} \overline{\xi}_j) + T(\bigoplus_{j \text{ odd}} \overline{\xi}_j) + \sum (-1)^k [\widetilde{\mathrm{ch}}(\overline{\eta}_k)] - i_*[\mathrm{Td}^{-1}(\overline{N}) \bullet \widetilde{\mathrm{ch}}(\overline{\chi})],
\end{aligned}$$

which implies the proposition. □

The following result is a consequence of proposition 6.13 and theorem 2.24.

Corollary 6.14. *Let $Y \longrightarrow X$ be a closed immersion of complex manifolds. Let $\overline{\chi}$ be an exact sequence of hermitian vector bundles on $Y$ as* (6.12). *For each $j$, let $\xi_j = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}_j, \overline{E}_{j,*})$ be a hermitian embedded vector bundle. We denote by $\overline{\varepsilon}$ the induced exact sequence of metrized coherent sheaves. Then*

$$T(\bigoplus_{j \text{ even}} \overline{\xi}_j) - T(\bigoplus_{j \text{ odd}} \overline{\xi}_j) = [\widetilde{\mathrm{ch}}(\overline{\varepsilon})] - i_*([\mathrm{Td}^{-1}(\overline{N})\widetilde{\mathrm{ch}}(\overline{\chi})]).$$

□

We now study the effect of changing the metric of the normal bundle $N$.

Proposition 6.15. *Let $\overline{\xi}_0 = (i, \overline{N}_0, \overline{F}, \overline{E}_*)$ be a hermitian embedded vector bundle, where $\overline{N}_0 = (N, h_0)$. Let $h_1$ be another metric in the vector bundle $N$ and write $\overline{N}_1 = (N, h_1)$, $\overline{\xi}_1 = (i, \overline{N}_1, \overline{F}, \overline{E}_*)$. Then*

$$T(\overline{\xi}_0) - T(\overline{\xi}_1) = -i_*[\widetilde{\mathrm{Td}^{-1}}(N, h_0, h_1)\,\mathrm{ch}(\overline{F})].$$

*Proof.* The proof is completely analogous to the proof of proposition 6.13. □

We now study the case when $Y$ is the zero section of a completed vector bundle. Let $\overline{F}$ and $\overline{N}$ be hermitian vector bundles over $Y$. We denote $P = \mathbb{P}(N \oplus \mathbb{C})$, the projective bundle of lines in $N \oplus \mathcal{O}_Y$. Let $s\colon Y \longrightarrow P$ denote the zero section and let $\pi_P\colon P \longrightarrow Y$ denote the projection. Let $K(\overline{F}, \overline{N})$ be the Koszul resolution of definition 5.3. We will use the notations before this definition.
The following result is due to Bismut, Gillet and Soulé for the particular choice of singular Bott-Chern classes defined in [6].

Theorem 6.16. *Let $T$ be a theory of singular Bott-Chern classes of rank $r_F$ and codimension $r_N$. Let $Y$ be a complex manifold and let $\overline{F}$ and $\overline{N}$ be hermitian vector bundles of rank $r_F$ and $r_N$ respectively. Then the current $(\pi_P)_*(T(K(\overline{F}, \overline{N})))$ is closed. Moreover the cohomology class that it represents does not depend on the metric of $N$ and $F$ and determines a characteristic class for pairs of vector bundles of rank $r_F$ and $r_N$. We denote this class by $C_T(F, N)$.*

*Proof.* We have that

$$\begin{aligned}
&\mathrm{d}_{\mathcal{D}}(\pi_P)_*(T(K(\overline{F},\overline{N})))\\
&\qquad=(\pi_P)_*(\mathrm{d}_{\mathcal{D}}\,T(K(\overline{F},\overline{N})))\\
&\qquad=(\pi_P)_*\left(\sum_{k=0}^{r}(-1)^k[\mathrm{ch}(\bigwedge^k\overline{Q}^\vee)\pi_P^*\,\mathrm{ch}(\overline{F})]-s_*[\mathrm{Td}^{-1}(\overline{N})\,\mathrm{ch}(\overline{F})]\right)\\
&\qquad=\big((\pi_P)_*[c_r(\overline{Q})\,\mathrm{Td}^{-1}(\overline{Q})]-[\mathrm{Td}^{-1}(\overline{N})]\big)\,\mathrm{ch}(\overline{F})).
\end{aligned}$$

Therefore, the fact that the current $(\pi_P)_*(T(K(\overline{F},\overline{N})))$ is closed follows from corollary 3.8. The fact that this class is functorial on $(Y,\overline{N},\overline{F})$ is clear from the construction Thus, the fact that it does not depend on the hermitian metrics of $N$ and $F$ follows from proposition 1.7. □

Remark 6.17. By theorem 1.8 we know that, if we restrict ourselves to the algebraic category, $C_T(F,N)$ is given by a power series on the Chern classes with coefficients in $\mathbb{D}$. By degree reasons

$$C_T(F,N)\in\bigoplus_p H^{2p-1}_{\mathcal{D}^{\mathrm{an}}}(Y,\mathbb{R}(p)).$$

Let $\mathbf{1}_1\in H^1_{\mathcal{D}}(*,\mathbb{R}(1))$ be the element determined by the constant function with value 1 in $\mathcal{D}^1(*,1)$. Then $C_T(F,N)/\mathbf{1}_1$ is a power series in the Chern classes of $N$ and $F$ with real coefficients.

## 7 Classification of theories of singular Bott-Chern classes

The aim of this section is to give a complete classification of the possible theories of singular Bott-Chern classes. This classification is given in terms of the characteristic class $C_T$ introduced in the previous section.

Theorem 7.1. *Let $r_F$ and $r_N$ be two positive integers. Let $C$ be a characteristic class for pairs of vector bundles of rank $r_F$ and $r_N$. Then there exists a unique theory $T_C$ of singular Bott-Chern classes of rank $r_F$ and codimension $r_N$ such that $C_{T_C}=C$.*

*Proof.* We first prove the uniqueness. Assume that $T$ is a theory of singular Bott-Chern classes such that $C_T=C$. Let $\overline{\xi}=(i\colon Y\longrightarrow X,\overline{N},\overline{F},\overline{E}_*)$ be a hermitian embedded vector bundle as in section 6. Let $W$ be the deformation to the normal cone of $Y$. We will use all the notations of section 5. In particular, we will denote by $p_{\widetilde{X}}\colon\widetilde{X}\longrightarrow X$ and $p_P\colon P\longrightarrow X$ the morphisms induced by restricting $p_W$. Recall that $p_P$ can be factored as

$$P\xrightarrow{\pi_P}Y\xrightarrow{i}X.$$

The normal vector bundle to the inclusion $j\colon Y\times\mathbb{P}^1\longrightarrow W$ is isomorphic to $p_Y^*N\otimes q_Y^*\mathcal{O}(-1)$. We provide it with the hermitian metric induced by the metric of $N$ and the Fubini-Study metric of $\mathcal{O}(-1)$ and we denote it by $\overline{N}'$.

By theorem 5.4 we have a complex of hermitian vector bundles, $\mathrm{tr}_1(E_*)_*$ such that the restriction $\mathrm{tr}_1(E_*)_*|_{X\times\{0\}}$ is isometric to $E_*$, the restriction $\mathrm{tr}_1(E_*)_*|_{\widetilde{X}}$ is orthogonally split and there is an exact sequence on $P$

$$0 \longrightarrow A_* \longrightarrow \mathrm{tr}_1(E_*)_*|_P \longrightarrow K(F,N) \longrightarrow 0,$$

where $A_*$ is split acyclic and $K(F,N)$ is the Koszul resolution. Recall that we have trivialized $N^{-1}_{\infty/\mathbb{P}^1}$ by means of the section $y$ of $\mathcal{O}_{\mathbb{P}^1}(1)$. We choose a hermitian metric in every bundle of $A_*$ such that it becomes orthogonally split. For each $k$ we will denote by $\overline{\eta}_k$ the exact sequence of hermitian vector bundles

$$0 \longrightarrow \overline{A}_k \longrightarrow \mathrm{tr}_1(\overline{E}_*)_k|_P \longrightarrow K(\overline{F},\overline{N})_k \longrightarrow 0. \tag{7.2}$$

Observe that the current $W_1$ is defined as the current associated to a locally integrable differential form. The pull-back of this form to $W$ is also locally integrable. Therefore it defines a current on $W$ that we also denote by $W_1$. Moreover, since the wave front sets of $W_1$ and of $T(\mathrm{tr}_1(\overline{E}_*)_*)$ are disjoint, there is a well defined current $W_1 \bullet T(\mathrm{tr}_1(\overline{E}_*)_*)$. Then, using the properties of singular Bott-Chern classes in definition 6.9, the equality

$$\begin{aligned}0 &= \mathrm{d}_{\mathcal{D}}(p_W)_*\left(W_1 \bullet T(\mathrm{tr}_1(\overline{E}_*)_*)\right)\\ &= (p_{\widetilde{X}})_*(T(\mathrm{tr}_1(\overline{E}_*)_*)|_{\widetilde{X}}) + (p_P)_*(T(\mathrm{tr}_1(\overline{E}_*)_*)|_P) - T(\overline{\xi})\\ &\quad - (p_W)_*\left(W_1 \bullet \left(\sum_k (-1)^k \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_*) - (j_*(\mathrm{ch}(p_Y^*\overline{F})\,\mathrm{Td}^{-1}(\overline{N}'))\right)\right)\end{aligned}$$

holds in the group $\bigoplus_k \widetilde{\mathcal{D}}^{2k-1}(X,k)$. By properties 6.9(ii) and 6.9(iii), $T(\mathrm{tr}_1(\overline{E}_*)_*)|_{\widetilde{X}} = T(\mathrm{tr}_1(\overline{E}_*)_*|_{\widetilde{X}}) = 0$.
By proposition 6.13 we have

$$T(\mathrm{tr}_1(\overline{E}_*)_*|_P) = T(K(\overline{F},\overline{N})) - \sum_k (-1)^k[\widetilde{\mathrm{ch}}(\overline{\eta}_k)].$$

Moreover, we have

$$(p_P)_*(T(K(\overline{F},\overline{N}))) = i_*(\pi_P)_*(T(K(\overline{F},\overline{N}))) = i_*C_T(F,N).$$

By the definition of $N'$ and the choice of its metric, there are two differential forms $a, b$ on $Y$, such that

$$\mathrm{ch}(p_Y^*\overline{F})\,\mathrm{Td}^{-1}(\overline{N}') = p_Y^*(a) + p_Y^*(b) \wedge q_Y^*(c_1(\mathcal{O}(-1))).$$

We denote $\omega = -c_1(\mathcal{O}(-1))$. By the properties of the Fubini-Study metric, $\omega$ is invariant under the involution of $\mathbb{P}^1$ that sends $t$ to $1/t$. Then

$$(p_W)_*\left(W_1 \bullet (j_*(\mathrm{ch}(p_Y^*\overline{F})\,\mathrm{Td}^{-1}(\overline{N}'))\right) = i_*(p_Y)_*(W_1 \bullet (p_Y^*a + p_Y^*b\omega)) = 0$$

because the current $W_1$ changes sign under the involution $t \longmapsto 1/t$.
Summing up, we have obtained the equation

$$T(\overline{\xi}) = -(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_k)\right) - \sum_k (-1)^k (p_P)_*[\widetilde{\mathrm{ch}}(\overline{\eta}_k)] + i_* C_T(F,N). \quad (7.3)$$

Hence the singular Bott-Chern class is characterized by the properties of definition 6.9 and the characteristic class $C_T$.
In order to prove the existence of a theory of singular Bott-Chern classes, we use equation (7.3) to define a class $T_C(\xi)$ as follows.

Definition 7.4. Let $C$ be a characteristic class for pairs of vector bundles of rank $r_F$ and $r_N$ as in theorem 7.1. Let $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ be as in definition 6.9. Let $\overline{A}_*$, $\mathrm{tr}_1(\overline{E}_*)_*$ and $\overline{\eta}_*$ be as in (7.2). Then we define

$$T_C(\overline{\xi}) = -(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_k)\right) - \sum_k (-1)^k (p_P)_*[\widetilde{\mathrm{ch}}(\overline{\eta}_k)] + i_* C(F,N). \quad (7.5)$$

We have to prove that this definition does not depend on the choice of the metric of $\mathrm{tr}_1(\overline{E}_*)_*$ or the metric of $\overline{A}_*$, that $T_C$ satisfies the properties of definition 6.9 and that the characteristic class $C_{T_C}$ agrees with $C$.
First we prove the independence from the metrics. We denote by $h_k$ the hermitian metric on $\mathrm{tr}_1(\overline{E}_*)_k$ and by $g_k$ the hermitian metric on $A_k$. Let $h'_k$ and $g'_k$ be another choice of metrics satisfying also that $(A_*, g'_*)$ is orthogonally split, that $(\mathrm{tr}_1(E_*)_k, h'_k)|_{X\times\{0\}}$ is isometric to $\overline{E}_k$ and that $(\mathrm{tr}_1(E_*)_k, h'_k)|_{\widetilde{X}}$ is orthogonally split. We denote by $\overline{\eta}'_k$ the exact sequence $\eta_k$ provided with the metrics $g'$ and $h'$. Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$, we have

$$\sum_k (-1)^k (p_P)_*[\widetilde{\mathrm{ch}}(\overline{\eta}_k)] - \sum_k (-1)^k (p_P)_*[\widetilde{\mathrm{ch}}(\overline{\eta}'_k)] = \sum_k (-1)^k (p_P)_*\left[\widetilde{\mathrm{ch}}(A_k, g_k, g'_k)\right] - \sum_k (-1)^k (p_P)_*\left[\widetilde{\mathrm{ch}}(\mathrm{tr}_1(E_*)_k|_P, h_k, h'_k)\right]. \quad (7.6)$$

Observe that the first term of the right hand side vanishes due to the hypothesis of $A_*$ being orthogonally split for both metrics.

Moreover, we also have,

$$(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(E_*)_k, h_k)\right) - (p_W)_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(E_*)_k, h'_k)\right) = (p_W)_*\left(\sum_k (-1)^k W_1 \bullet \mathrm{d}_{\mathcal{D}}\, \widetilde{\operatorname{ch}}(\operatorname{tr}_1(E_*)_k, h_k, h'_k)\right). \quad (7.7)$$

But, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)$,

$$(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \mathrm{d}_{\mathcal{D}}\, \widetilde{\operatorname{ch}}(\operatorname{tr}_1(E_*)_k, h_k, h'_k)\right) = \sum_k (-1)^k (p_{\widetilde{X}})_*[\widetilde{\operatorname{ch}}(\operatorname{tr}_1(E_*)_k, h_k, h'_k)]|_{\widetilde{X}} + \sum_k (-1)^k (p_P)_*[\widetilde{\operatorname{ch}}(\operatorname{tr}_1(E_*)_k, h_k, h'_k)]|_P) - \sum_k (-1)^k [\widetilde{\operatorname{ch}}(\operatorname{tr}_1(E_*)_k, h_k, h'_k)]|_{X\times\{0\}}. \quad (7.8)$$

The last term of the right hand side vanishes because the metrics $h_k$ and $h'_k$ agree when restricted to $X \times \{0\}$ and the first term vanishes by the hypothesis that $\operatorname{tr}_1(E_*)_*|_{\widetilde{X}}$ is orthogonally split with both metrics. Combining equations (7.6), (7.7) and (7.8) we obtain that the right hand side of equation (7.5) does not depend on the choice of metrics.

We next prove the property (i) of definition 6.9. We compute

$$\begin{aligned}\mathrm{d}_{\mathcal{D}}\, T_C(\overline{\xi}) = -\sum_k (-1)^k \left((p_{\widetilde{X}})_* \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_k|_{\widetilde{X}}) + (p_P)_* \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_k|_P)\right) \\ + \sum_k (-1)^k \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_k|_{X\times\{0\}}) \\ - \sum_k (-1)^k (p_P)_* \left(\operatorname{ch}(\overline{A}_k) + \operatorname{ch}(K(\overline{F},\overline{N})_k) - \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_k|_P)\right).\end{aligned}$$

Using that $\overline{A}_*$ and that $\operatorname{tr}_1(\overline{E}_*)_*|_{\widetilde{X}}$ are orthogonally split and corollary 3.8 we obtain

$$\begin{aligned}\mathrm{d}_{\mathcal{D}}\, T_C(\overline{\xi}) &= \sum_k (-1)^k \operatorname{ch}(\overline{E}_k) - \sum_k (-1)^k (p_P)_* \operatorname{ch}(K(\overline{F},\overline{N})_k) \\ &= \sum_k (-1)^k [\operatorname{ch}(\overline{E}_k)] - (p_P)_*[c_r(\overline{Q}) \operatorname{Td}^{-1}(\overline{Q})] \\ &= \sum_k (-1)^k [\operatorname{ch}(\overline{E}_k)] - i_*[\operatorname{ch}(\overline{F}) \operatorname{Td}^{-1}(\overline{N})].\end{aligned}$$

We now prove the normalization property. We consider first the case when $Y=\emptyset$ and $\overline{E}_*$ is a non-negatively graded orthogonally split complex. We denote by

$$\overline{K}_i = \operatorname{Ker}(\mathrm{d}_i\colon E_i \longrightarrow E_{i-1})$$

with the induced metric. By hypothesis there are isometries

$$\overline{E}_i = \overline{K}_i \oplus \overline{K}_{i-1}.$$

Under these isometries, the differential is $\mathrm{d}(s,t)=(t,0)$. Following the explicit construction of $\operatorname{tr}_1(E_*)$ given in [20], recalled in definition 2.5, we see that

$$\operatorname{tr}_1(E_*)_i = p^*K_i \otimes q^*\mathcal{O}(i) \oplus p^*K_{i-1}\otimes q^*\mathcal{O}(i-1) = K_i(i)\oplus K_{i-1}(i-1).$$

Moreover, we can induce a metric on $\operatorname{tr}_1(E_*)_*$ satisfying the hypothesis of definition 2.9 by means of the metric of the bundles $K_i$ and the Fubini-Study metric on the bundles $\mathcal{O}(i)$. It is clear that the second and third terms of the right hand side of equation (7.3) are zero. For the first term we have

$$\begin{aligned}
\sum_k (-1)^k (p_W)_* W_1 \bullet \big(\operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_k)\big) \\
&= (p_W)_* \left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\overline{K}_k(k) \overset{\perp}{\oplus} \overline{K}_{k-1}(k-1))\right)\\
&= (p_W)_* \left(W_1 \bullet (a + b\wedge\omega)\right),
\end{aligned}$$

where $\omega$ is the Fubini-Study $(1,1)$-form on $\mathbb{P}^1$ and $a,b$ are inverse images of differential forms on $X$. Therefore we obtain that $T_C(\overline{E}_*)=0$.

Now let $\overline{\xi}=(i\colon Y\longrightarrow X,\overline{N},\overline{F},\overline{E}_*)$ and let $\overline{B}_*$ be a non-negatively graded orthogonally split complex of vector bundles. By [20] section 1.1, we have that $W(E_*\oplus B_*)=W(E_*)$ and that

$$\operatorname{tr}_1(E_*\oplus B_*) = \operatorname{tr}_1(E_*) \oplus \pi^* \operatorname{tr}_1(B_*).$$

In order to compute $T_C(\overline{\xi})$, we have to consider the exact sequences of hermitian vector bundles over $P$

$$\overline{\eta}_k\colon 0 \longrightarrow \overline{A}_k \longrightarrow \operatorname{tr}_1(\overline{E}_*)_k|_P \longrightarrow K(\overline{F},\overline{N})_k \longrightarrow 0,$$

whereas, in order to compute $T_C(\overline{\xi}\oplus\overline{B}_*)$, we consider the sequences

$$\overline{\eta}'_k\colon$$
$$0 \longrightarrow \overline{A}_k \oplus \pi^*(\operatorname{tr}_1(\overline{B})_k)|_P \longrightarrow \operatorname{tr}_1(\overline{E}_*)_k \oplus \pi^*(\operatorname{tr}_1(\overline{B})_k)|_P \longrightarrow K(\overline{F},\overline{N})_k \longrightarrow 0.$$

By the additivity of Bott-Chern classes, we have that $\widetilde{\mathrm{ch}}(\overline{\eta}_k) = \widetilde{\mathrm{ch}}(\overline{\eta}'_k)$. Therefore

$$\begin{aligned} T_C(\overline{\xi} \oplus \bar{B}_*) - T_C(\overline{\xi}) &= -(p_W)_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(\overline{E}_* \oplus \overline{B}_*)_k) \right) \\ &\quad + (p_W)_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(\overline{E}_*)_k) \right) \\ &= -(p_W)_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\mathrm{tr}_1(\overline{B}_*)_k) \right) \\ &= 0. \end{aligned}$$

The proof of the functoriality is left to the reader.
Finally we prove that $C_{T_C} = C$. Let $Y$ be a complex manifold and let $\overline{F}$ and $\overline{N}$ be two hermitian vector bundles. We write $X = \mathbb{P}(N \oplus \mathbb{C})$. Let $i\colon Y \longrightarrow X$ be the inclusion given by the zero section and let $\pi_X : X \longrightarrow Y$ be the projection. On $X$ we have the tautological exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi_X^*(N \oplus \mathbb{C}) \longrightarrow Q \longrightarrow 0$$

and the Koszul resolution, denoted $K(\overline{F}, \overline{N})$. We denote

$$\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, K(\overline{F}, \overline{N})).$$

Using the definition of $T_C$, that is, equation (7.5), and the fact that $T_C$ satisfies the properties of definition 6.9, hence equation (7.3) is satisfied, we obtain that

$$i_* C(F, N) = i_* C_{T_C}(F, N)$$

Applying $(\pi_X)_*$ we obtain that $C(F, N) = C_{T_C}(F, N)$ which finishes the proof of theorem 7.1. □

## 8 Transitivity and projection formula

We now investigate how different properties of the characteristic class $C_T$ are reflected in the corresponding theory of singular Bott-Chern classes.

Proposition 8.1. *Let $i\colon Y \hookrightarrow X$ be a closed immersion of complex manifolds. Let $\overline{F}$ be a hermitian vector bundle on $Y$ and $\overline{G}$ a hermitian vector bundle on $X$. Let $\overline{N}$ denote the normal bundle to $Y$ provided with a hermitian metric. Let $\overline{E}_*$ be a finite resolution of $i_*F$ by hermitian vector bundles. We denote $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ and $\overline{\xi} \otimes \overline{G} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F} \otimes i^*\overline{G}, \overline{E}_* \otimes \overline{G})$. Then*

$$T(\overline{\xi} \otimes \overline{G}) - T(\overline{\xi}) \bullet \mathrm{ch}(\overline{G}) = i_*(C_T(F \otimes i^*G, N)) - i_*(C_T(F, N)) \bullet \mathrm{ch}(\overline{G}).$$

*Proof.* Since the construction of $\operatorname{tr}_1(E_*)_*$ is local on $X$ and $Y$ and compatible with finite sums, we have that

$$W(E_*) = W(E_* \otimes G), \qquad \operatorname{tr}_1(\overline{E}_* \otimes \overline{G})_* = \operatorname{tr}_1(\overline{E}_*)_* \otimes p_W^* \overline{G}.$$

We first compute

$$\begin{aligned}(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E}_* \otimes \overline{G})_*)\right) \\ = (p_W)_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_*) p_W^* \operatorname{ch}(\overline{G})\right) \\ = (p_W)_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E}_*)_*)\right) \operatorname{ch}(\overline{G}). \quad (8.2)\end{aligned}$$

The Koszul resolution of $i_*(F \otimes i^* G)$ is given by

$$K(F \otimes i^* G, N) = K(F, N) \otimes p_P^* G.$$

For each $k \geq 0$, we will denote by $\overline{\eta}_k \otimes p_P^* \overline{G}$ the exact sequence

$$0 \longrightarrow \overline{A}_k \otimes p_P^* \overline{G} \longrightarrow \operatorname{tr}_1(\overline{E}_* \otimes \overline{G})_k|_P \longrightarrow K(\overline{F}, \overline{N})_k \otimes p_P^* \overline{G} \longrightarrow 0.$$

Then, we have

$$(p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k \otimes p_P^* \overline{G})] = (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k) \bullet p_P^* \operatorname{ch}(\overline{G})] = (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k)] \bullet \operatorname{ch}(\overline{G}) \quad (8.3)$$

Thus the proposition follows from equation (8.2), equation (8.3) and formula (7.3). □

Definition 8.4. We will say that a theory of singular Bott-Chern classes is *compatible with the projection formula* if, whenever we are in the situation of proposition 8.1, the following equality holds:

$$T(\overline{\xi} \otimes \overline{G}) = T(\overline{\xi}) \bullet \operatorname{ch}(\overline{G}).$$

We will say that a characteristic class $C$ (of pairs of vector bundles) is *compatible with the projection formula* if it satisfies

$$C(F, N) = C(\mathcal{O}_Y, N) \bullet \operatorname{ch}(F).$$

Corollary 8.5. *A theory of singular Bott-Chern classes $T$ is compatible with the projection formula if and only if it is the case for the associated characteristic class $C_T$.*

*Proof.* Assume that $C_T$ is compatible with the projection formula and that we are in the situation of proposition 8.1. Then

$$\begin{aligned} i_* C_T(F \otimes i^* G, N)) &= i_*(C_T(\mathcal{O}_Y, N) \bullet \operatorname{ch}(F \otimes i^* G)) \\ &= i_*(C_T(\mathcal{O}_Y, N) \bullet \operatorname{ch}(F) i^* \operatorname{ch}(G)) \\ &= i_*(C_T(\mathcal{O}_Y, N) \bullet \operatorname{ch}(F)) \operatorname{ch}(G) \\ &= i_*(C_T(F, N)) \bullet \operatorname{ch}(G). \end{aligned}$$

Thus, by proposition 8.1, $T$ is compatible with the projection formula.
Assume that $T$ is compatible with the projection formula. Let $\overline{F}$ and $\overline{N}$ be hermitian vector bundles over a complex manifold $Y$. Let $s\colon Y \hookrightarrow P := \mathbb{P}(N \oplus \mathbb{C})$ be the zero section and let $\pi\colon P \longrightarrow Y$ be the projection. Then

$$\begin{aligned} C_T(F, N) &= \pi_*(T(K(\overline{F}, \overline{N}))) \\ &= \pi_*(T(K(\overline{\mathcal{O}}_Y, \overline{N}) \otimes \pi^* \overline{F})) \\ &= \pi_*(T(K(\overline{\mathcal{O}}_Y, \overline{N})) \bullet \pi^* \operatorname{ch}(F)) \\ &= \pi_*(T(K(\overline{\mathcal{O}}_Y, \overline{N}))) \bullet \operatorname{ch}(F) \\ &= C_T(\mathcal{O}_Y, N) \bullet \operatorname{ch}(F). \end{aligned}$$

□

We will next investigate the relationship between singular Bott-Chern classes and compositions of closed immersions. Thus, let

$$Y \overset{i_{Y/X}}{\hookrightarrow} X \overset{i_{X/M}}{\hookrightarrow} M, \qquad i_{Y/M}\colon Y \hookrightarrow M$$

be a composition of closed immersions. Assume that the normal bundles $N_{Y/X}$, $N_{X/M}$ and $N_{Y/M}$ are provided with hermitian metrics. We will denote by $\overline{\varepsilon}$ the exact sequence

$$\overline{\varepsilon}\colon 0 \to \overline{N}_{Y/X} \to \overline{N}_{Y/M} \to i^*_{Y/X} \overline{N}_{X/M} \to 0. \tag{8.6}$$

Let $P_{X/M} = \mathbb{P}(N_{X/M} \oplus \mathbb{C})$ be the projective completion of the normal cone to $X$ in $M$. Then there is an isomorphism

$$N_{Y/P_{X/M}} \cong N_{Y/X} \oplus i^*_{Y/X} N_{X/M}. \tag{8.7}$$

We denote by $\overline{N}_{Y/P_{X/M}}$ the vector bundle on the left hand side with the hermitian metric induced by the isomorphism (8.7).
Let $\overline{F}$ be a hermitian vector bundle over $Y$, let $\overline{E}_* \longrightarrow (i_{Y/X})_* F$ be a resolution by hermitian vector bundles. Let $\overline{E}'_{*,*}$ be a complex of complexes of vector

bundles over $M$, such that, for each $k \geq 0$, $\overline{E}'_{k,*} \longrightarrow (i_{X/M})_* E_k$ is a resolution, and there is a commutative diagram of resolutions

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & E'_{k+1,*} & \longrightarrow & E'_{k,*} & \longrightarrow & E'_{k-1,*} & \longrightarrow & \cdots . \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & (i_{X/M})_* E_{k+1} & \longrightarrow & (i_{X/M})_* E_k & \longrightarrow & (i_{X/M})_* E_{k-1} & \longrightarrow & \cdots
\end{array}$$

It follows that we have a resolution $\mathrm{Tot}(\overline{E}'_{*,*}) \longrightarrow (i_{Y/M})_* F$ of $(i_{Y/M})_* F$ by hermitian vector bundles.

Notation 8.8. We will denote

$$\begin{aligned}
\overline{\xi}_{Y \hookrightarrow X} &= (i_{Y/X}, \overline{N}_{Y/X}, \overline{F}, \overline{E}_*),\\
\overline{\xi}_{Y \hookrightarrow M} &= (i_{Y/M}, \overline{N}_{Y/M}, \overline{F}, \mathrm{Tot}(\overline{E}'_{*,*})),\\
\overline{\xi}_{X \hookrightarrow M,k} &= (i_{X/M}, \overline{N}_{X/M}, \overline{E}_k, \overline{E}'_{k,*}).
\end{aligned}$$

We will also denote by $\overline{\xi}_{Y \hookrightarrow P_{X/M}}$ the hermitian embedded vector bundle

$$\left(Y \hookrightarrow P_{X/M}, \overline{N}_{Y/P_{X/M}}, \overline{F}, \mathrm{Tot}(\pi^*_{P_{X/M}} \overline{E}_* \otimes K(\mathcal{O}_X, \overline{N}_{X/M}))\right).$$

Let $T$ be a theory of singular Bott-Chern classes, and let $C_T$ be its associated characteristic class. Our aim now is to relate $T(\overline{\xi}_{Y \hookrightarrow X})$, $T(\overline{\xi}_{Y \hookrightarrow M})$ and $T(\overline{\xi}_{X \hookrightarrow M,k})$.
Let $W_X$ be the deformation to the normal cone of $X$ in $M$. As before we denote by $j_X \colon X \times \mathbb{P}^1 \longrightarrow W_X$ the inclusion.
We denote by $W$ the deformation to the normal cone of $j_X(Y \times \mathbb{P}^1)$ in $W_X$. This double deformation is represented in figure 1. There is a proper map $q_W \colon W \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. The fibers of $q_W$ over the corners of $\mathbb{P}^1 \times \mathbb{P}^1$ are as follows:

$$\begin{aligned}
q_W^{-1}(0,0) &= M,\\
q_W^{-1}(\infty,0) &= \widetilde{M}_X \times \{0\} \cup P_{X/M},\\
q_W^{-1}(0,\infty) &= \widetilde{M}_Y \cup P_{Y/M},\\
q_W^{-1}(\infty,\infty) &= \widetilde{M}_X \times \{\infty\} \cup \widetilde{P}_{X/M} \cup P_{Y/P_{X/M}},
\end{aligned}$$

where $\widetilde{M}_X$ and $\widetilde{M}_Y$ are the blow-up of $M$ along $X$ and $Y$ respectively, $P_{Y/M} = \mathbb{P}(N_{Y/M} \oplus \mathbb{C})$ is the projective completion of the normal cone to $Y$ in $M$, $P_{Y/P_{X/M}}$ of the normal cone to $Y$ in $P_{X/M}$ and $\widetilde{P}_{X/M}$ is the blow-up of $P_{X/M}$

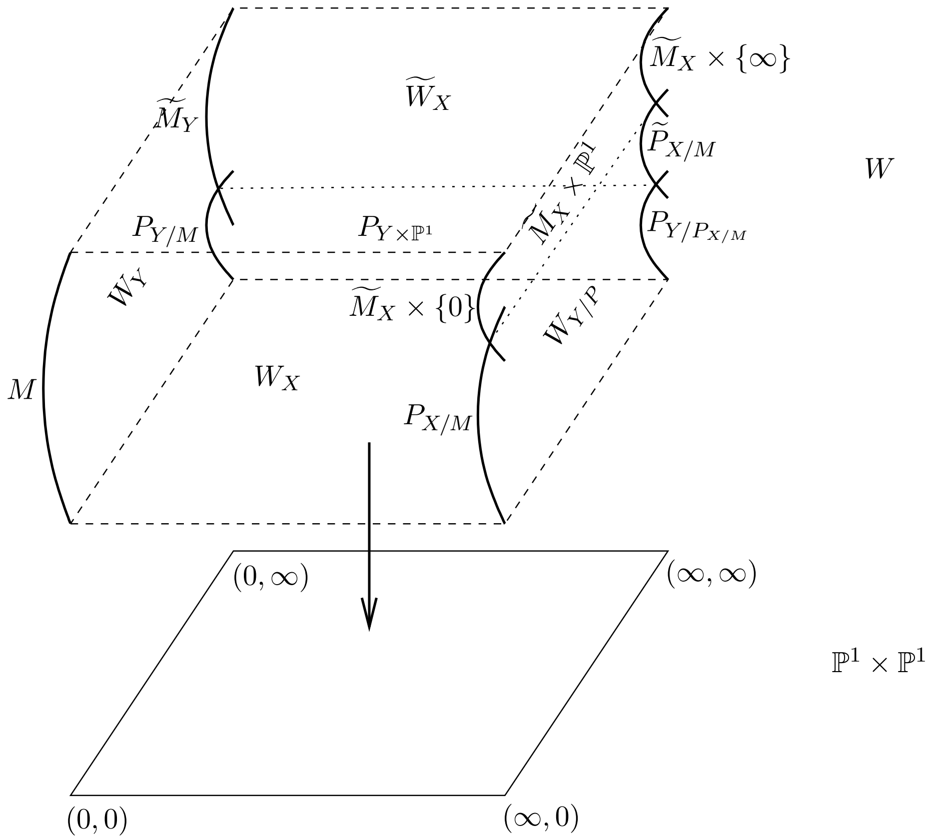


Figure 1: Double deformation

along $Y$. The preimages by $\pi$ of the different faces of $\mathbb{P}^1 \times \mathbb{P}^1$ are as follows:

$$\begin{aligned}
q_W^{-1}(\mathbb{P}^1 \times \{0\}) &= W_X,\\
q_W^{-1}(\{0\} \times \mathbb{P}^1) &= W_Y,\\
q_W^{-1}(\mathbb{P}^1 \times \{\infty\}) &= \widetilde{W}_X \cup P_{Y\times\mathbb{P}^1},\\
q_W^{-1}(\{\infty\} \times \mathbb{P}^1) &= \widetilde{M}_X \times \mathbb{P}^1 \cup W_{Y/P},
\end{aligned}$$

where $W_Y$ is the deformation to the normal cone of $Y$ in $M$, the component $\widetilde{W}_X$ is the blow-up of $W_X$ along $j_X(Y \times \mathbb{P}^1)$, while $P_{Y\times\mathbb{P}^1} = \mathbb{P}(N_{Y\times\mathbb{P}^1/W_X} \oplus \mathbb{C})$ is the projective completion of the normal cone to $j_X(Y \times \mathbb{P}^1)$ in $W_X$ and $W_{Y/P}$ is the deformation to the normal cone of $Y$ inside $P_{X/M}$. All the above subvarieties will be called boundary components of $W$.

We will use the following notations for the different maps.

$$\begin{aligned}
&p_X\colon X\times\mathbb{P}^1\longrightarrow X && p_Y\colon Y\times\mathbb{P}^1\longrightarrow Y\\
&p_{Y\times\mathbb{P}^1}\colon Y\times\mathbb{P}^1\times\mathbb{P}^1\longrightarrow Y\times\mathbb{P}^1 && p_{\widetilde{M}_X\times\mathbb{P}^1}\colon \widetilde{M}_X\times\mathbb{P}^1\longrightarrow M\\
&p_{W_{Y/P}}\colon W_{Y/P}\longrightarrow M && p_{W_Y}\colon W_Y\longrightarrow M\\
&p_{W_X}\colon W_X\longrightarrow M && p_{P_{Y\times\mathbb{P}^1}}\colon P_{Y\times\mathbb{P}^1}\longrightarrow M\\
&p_{\widetilde{W}_X}\colon \widetilde{W}_X\longrightarrow M && p_{P_{Y/P_{X/M}}}\colon P_{Y/P_{X/M}}\longrightarrow M\\
&p_{P_{X/M}}\colon P_{X/M}\longrightarrow M && p_{\widetilde{P}_{X/M}}\colon \widetilde{P}_{X/M}\longrightarrow M\\
&p_{P_{Y/M}}\colon P_{Y/M}\longrightarrow M && p_W\colon W\longrightarrow M\\
&j_Y\colon Y\times\mathbb{P}^1\longrightarrow W_Y && j'_Y\colon Y\times\mathbb{P}^1\longrightarrow W_X\\
&j_{Y\times\mathbb{P}^1}\colon Y\times\mathbb{P}^1\times\mathbb{P}^1\longrightarrow W && i_{Y/P_{X/M}}\colon Y\longrightarrow P_{X/M}\\
&\pi_{P_{X/M}}\colon P_{X/M}\longrightarrow X && \pi_{P_{Y/M}}\colon P_{Y/M}\longrightarrow Y\\
&\pi_{P_{Y/P}}\colon P_{Y/P_{X/M}}\longrightarrow Y && \pi_{P_{Y\times\mathbb{P}^1}}\colon P_{Y\times\mathbb{P}^1}\longrightarrow Y\times\mathbb{P}^1\\
&\pi_{\widetilde{M}_X}\colon \widetilde{M}_X\longrightarrow M && \pi_{\widetilde{M}_Y}\colon \widetilde{M}_Y\longrightarrow M
\end{aligned}$$

Note that the map $p_{\widetilde{M}_X\times\mathbb{P}^1}$ factors through the blow-up $\widetilde{M}_X\longrightarrow M$ and the map $p_{\widetilde{W}_X}$ factors through the blow-up $\widetilde{M}_Y\longrightarrow M$, whereas the maps $p_{W_{Y/P}}$, $p_{P_{X/M}}$ and $p_{\widetilde{P}_{X/M}}$ factor through the inclusion $X\hookrightarrow M$ and the maps $p_{P_{Y\times\mathbb{P}^1}}$, $p_{P_{Y/M}}$ and $p_{P_{Y/P_{X/M}}}$ factor through the inclusion $Y\hookrightarrow M$.
The normal bundle to $X\times\mathbb{P}^1$ in $W_X$ is isomorphic to $p_X^*N_{X/M}\otimes q_X^*\mathcal{O}(-1)$ and we consider on it the metric induced by the metric on $\overline{N}_{X/M}$ and the Fubini-Study metric on $\mathcal{O}(-1)$. We denote it by $\overline{N}_{X\times\mathbb{P}^1/W_X}$. The normal bundle to $Y\times\mathbb{P}^1$ in $W_X$ satisfies

$$\begin{aligned}
N_{Y\times\mathbb{P}^1/W_X}|_{Y\times\{0\}}&\cong N_{Y/M}\\
N_{Y\times\mathbb{P}^1/W_X}|_{Y\times\{\infty\}}&\cong N_{Y/X}\oplus i_{Y/X}^*N_{X/M}.
\end{aligned}$$

On $N_{Y\times\mathbb{P}^1/W_X}$ we choose a hermitian metric such that the above isomorphisms are isometries. Finally, on the normal bundle to $Y\times\mathbb{P}^1\times\mathbb{P}^1$ in $W$, we define a metric using the same procedure as the definition of the metric of $\overline{N}_{X\times\mathbb{P}^1/W_X}$.
On $W_X$ we obtain a sequence of resolutions $\operatorname{tr}_1(\overline{E}')_{n,*}\longrightarrow(j_X)_*p_X^*E_n$. They form a complex of complexes $\operatorname{tr}_1(\overline{E}')_{*,*}$ and the associated total complex $\operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*})$ provides us with a resolution

$$\operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*})_*\longrightarrow(j'_Y)_*p_Y^*F. \tag{8.9}$$

The restriction of $\operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*})$ to $M$ is $\operatorname{Tot}(\overline{E}'_{*,*})$. The restriction of each complex $\operatorname{tr}_1(\overline{E}')_{n,*}$ to $\widetilde{M}_X\times\{0\}$ is orthogonally split. Therefore the restriction of $\operatorname{Tot}(\operatorname{tr}_1(\overline{E}'))$ to $\widetilde{M}_X\times\{0\}$ is the total complex of a complex of orthogonally split complexes. So it is acyclic although not necessarily orthogonally split. The restriction of each complex $\operatorname{tr}_1(\overline{E}')_{n,*}$ to $P_{X/M}$ fits in an exact sequence

$$0 \longrightarrow \overline{A}_{n,*} \longrightarrow \operatorname{tr}_1(\overline{E}')_{n,*}|_{P_{X/M}} \longrightarrow \pi^*_{P_{X/M}}\overline{E}_n \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_* \longrightarrow 0.$$

These exact sequences glue together giving a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Tot}(\overline{A}_{*,*}) & \hookrightarrow & \operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*}|_{P_{X/M}}) & \twoheadrightarrow & \operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E}_* \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_*) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \hookrightarrow & (i_{Y/P_{X/M}})_*F & \twoheadrightarrow & (i_{Y/P_{X/M}})_*F
\end{array}$$

where the rows are short exact sequences. Even if the complexes $(\overline{A}_n)_*$ are orthogonally split, this is not necessarily the case for $\operatorname{Tot}(\overline{A}_{*,*})$. To ease the notation we will denote $\overline{A}_* = \operatorname{Tot}(\overline{A}_{*,*})$.
Applying theorem 5.4 to the resolution (8.9), we obtain a complex of hermitian vector bundles $\widetilde{E}'_* = \operatorname{tr}_1(\operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*}))$ which is a resolution of the coherent sheaf $(j_{Y\times\mathbb{P}^1})_* p^*_{Y\times\mathbb{P}^1} p^*_Y F$.
We now study the restriction of $\widetilde{E}'_*$ to each of the boundary components of $W$.

- The restriction of $\widetilde{E}'_*$ to $W_X$ is just $\operatorname{Tot}(\operatorname{tr}_1(\overline{E}'))$ which has already been described. For each $k \geq 0$, we will denote by $\eta^1_k$ the short exact sequence of hermitian vector bundles on $P_{X/M}$

  $$\overline{A}_k \hookrightarrow \operatorname{Tot}(\operatorname{tr}_1(\overline{E}')_{*,*}|_{P_{X/M}})_k \twoheadrightarrow \operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E} \otimes K(\mathcal{O}_X, \overline{N}_{X/M}))_k\ ,$$

  whereas, for each $n, k \geq 0$ we will denote by $\eta^1_{n,k}$ the short exact sequence

  $$\overline{A}_{n,k} \hookrightarrow \operatorname{tr}_1(\overline{E}')_{n,k}|_{P_{X/M}} \twoheadrightarrow \pi^*_{P_{X/M}}\overline{E}_n \otimes K(\mathcal{O}_X, \overline{N}_{X/M})_k\ .$$

- Its restriction to $W_Y$ is $\operatorname{tr}_1(\operatorname{Tot}(\overline{E}'))$. It is a resolution of $(j_Y)_* p^*_Y F$. Its restriction to $\widetilde{M}_Y$ is orthogonally split, whereas its restriction to $P_{Y/M}$ fits in an exact sequence

  $$0 \longrightarrow \overline{B}_* \longrightarrow \operatorname{tr}_1(\operatorname{Tot}(\overline{E}'))_*|_{P_{Y/M}} \longrightarrow \pi^*_{P_{Y/M}}\overline{F} \otimes K(\overline{\mathcal{O}}_Y, \overline{N}_{Y/M}) \longrightarrow 0.$$

  For each $k \geq 0$ we will denote by $\eta^2_k$ the degree $k$ piece of the above exact sequence.

- Its restriction to $\widetilde{M}_X \times \mathbb{P}^1$ is an acyclic complex, such that its further restriction to $\widetilde{M}_X \times \{0\}$ is acyclic and its restriction to $\widetilde{M}_X \times \{\infty\}$ is orthogonally split.

- Its restriction to $W_{Y/P}$ fits in a short exact sequence

$$0 \to \operatorname{tr}_1(\overline{A}_*) \to \widetilde{E}'_*|_{W_{Y/P}} \to \operatorname{tr}_1(\operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E} \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M}))) \to 0.$$

For each $k \geq 0$, we will denote by $\mu^1_k$ the exact sequence of hermitian vector bundles over $W_{Y/P}$ given by the piece of degree $k$ of this exact sequence. The three terms of the above exact sequence become orthogonally split when restricted to $\widetilde{P}_{X/M}$. By contrast, when restricted to $P_{Y/P_{X/M}}$ they fit in a commutative diagram

$$\begin{array}{ccccc}
\overline{C}^1_* & \hookrightarrow & \overline{C}^2_* & \twoheadrightarrow & \overline{C}^3 \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{tr}_1(\overline{A})_*|_{P_{Y/P_{X/M}}} & \hookrightarrow & \widetilde{E}'_*|_{P_{Y/P_{X/M}}} & \twoheadrightarrow & \overline{D}^2_* \\
\downarrow & & \downarrow & & \downarrow \\
0 & \hookrightarrow & \overline{D}^1_* & \twoheadrightarrow & \overline{D}^1_*
\end{array}$$

where the complexes $\overline{C}^i_*$ are orthogonally split, and

$$\begin{aligned}
\overline{D}^1_* &= \pi^*_{P_{Y/P}}\overline{F} \otimes K(\overline{\mathcal{O}}_Y, \overline{N}_{Y/P_{X/M}}),\\
\overline{D}^2_* &= \operatorname{tr}_1(\operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E} \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})))|_{P_{Y/P_{X/M}}}.
\end{aligned}$$

For each $k \geq 0$, we will denote by $\eta^3_k$ the exact sequence corresponding to the piece of degree $k$ of the second row of the above diagram, by $\eta^4_k$ that of the second column and by $\eta^5_k$ that of the third column. Notice that the map in the third row is an isometry. We assume that the metric on $C^1_*$ is chosen in such a way that the first column is an isometry. Since the complexes $\overline{C}^i_*$ are orthogonally split, by lemma 2.17 we obtain

$$\sum_k (-1)^k \left(\widetilde{\operatorname{ch}}(\eta^3_k) - \widetilde{\operatorname{ch}}(\eta^4_k) + \widetilde{\operatorname{ch}}(\eta^5_k)\right) = 0. \tag{8.10}$$

Note that the restriction of $\mu^1_k$ to $P_{X/M}$ agrees with $\eta^1_k$, whereas its restriction to $P_{Y/P_{X/M}}$ agrees with $\eta^3_k$.

- Its restriction to $\widetilde{W}_X$ is orthogonally split.

- Finally its restriction to $P_{Y\times\mathbb{P}^1}$ fits in an exact sequence

$$\overline{D}_* \hookrightarrow \widetilde{E}'_*|_{P_{Y\times\mathbb{P}^1}} \twoheadrightarrow \pi^*_{P_{Y\times\mathbb{P}^1}} p^*_{Y\times\mathbb{P}^1}\overline{F} \otimes K(\mathcal{O}_{Y\times\mathbb{P}^1}, \overline{N}_{Y\times\mathbb{P}^1/W_X}),$$

where $\overline{D}_*$ is orthogonally split. For each $k \geq 0$ we will denote by $\mu^2_k$ the piece of degree $k$ of this exact sequence. Note that the restriction of $\mu^2_k$ to $P_{Y/M}$ agrees with $\eta^2_k$ and the restriction of $\mu^2_k$ to $P_{Y/P_{X/M}}$ agrees with $\eta^4_k$.

On $\mathbb{P}^1 \times \mathbb{P}^1$ we denote the two projections by $p_1$ and $p_2$. Since the currents $p_1^* W_1$ and $p_2^* W_1$ have disjoint wave front sets we can define the current $W_2 = p_1^* W_1 \bullet p_2^* W_1 \in \mathcal{D}^2_D(\mathbb{P}^1 \times \mathbb{P}^1, 2)$ which satisfies

$$\mathrm{d}_{\mathcal{D}} W_2 = (\delta_{\{\infty\}\times\mathbb{P}^1} - \delta_{\{0\}\times\mathbb{P}^1}) \bullet p_2^* W_1 - p_1^* W_1 \bullet (\delta_{\mathbb{P}^1\times\{\infty\}} - \delta_{\mathbb{P}^1\times\{0\}}). \quad (8.11)$$

The key point in order to study the compatibility of singular Bott-Chern classes and composition of closed immersions is that, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(M,p)$, we have

$$\mathrm{d}_{\mathcal{D}} (p_W)_* \left( \sum_k (-1)^k W_2 \bullet \mathrm{ch}(\widetilde{E}'_k) \right) = 0.$$

We compute this class using the equation (8.11). It can be decomposed as follows.

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}} (p_W)_* \left( \sum_k (-1)^k W_2 \bullet \mathrm{ch}(\widetilde{E}'_k) \right) = \\
(p_{\widetilde{M}_X \times \mathbb{P}^1})_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{\widetilde{M}_X \times \mathbb{P}^1}) \right) && \text{(a)}\\
+ (p_{W_{Y/P}})_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{W_{Y/P}}) \right) && \text{(b)}\\
- (p_{W_Y})_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{W_Y}) \right) && \text{(c)}\\
- (p_{\widetilde{W}_X})_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{\widetilde{W}_X}) \right) && \text{(d)}\\
- (p_{P_{Y\times\mathbb{P}^1}})_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{P_{Y\times\mathbb{P}^1}}) \right) && \text{(e)}\\
+ (p_{W_X})_* \left( \sum_k (-1)^k W_1 \bullet \mathrm{ch}(\widetilde{E}'_k|_{W_X}) \right) && \text{(f)}\\
=: I_a + I_b - I_c - I_d - I_e + I_f
\end{aligned}$$

We compute each of the above terms.
(a) Since the restriction $\widetilde{E}'|_{\widetilde{M}_X \times \{\infty\}}$ is orthogonally split, we have

$$I_a = -(\pi_{\widetilde{M}_X})_* \widetilde{\mathrm{ch}}(\widetilde{E}'|_{\widetilde{M}_X \times \{0\}}).$$

But, using lemma 2.17 and the fact, for each $k$, the complexes $\operatorname{tr}_1(\overline{E}')_{k,*}|_{\widetilde{M}_X}$ are orthogonally split, we obtain that $I_a = 0$.
(b) We compute

$$\begin{aligned}
I_b =&(p_{W_{Y/P}})_* \left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\widetilde{E}'_k|_{W_{Y/P}})\right)\\
=&(p_{W_{Y/P}})_* \Bigg(W_1 \bullet \sum_k (-1)^k (-\operatorname{d}_{\mathcal{D}} \widetilde{\operatorname{ch}}(\mu^1_k) + \operatorname{ch}(\operatorname{tr}_1(\overline{A}_*)_k)\\
&\qquad + \operatorname{ch}(\operatorname{tr}_1(\operatorname{Tot}(\pi^*_{P_{X/M}}\overline{E} \otimes K(\mathcal{O}_X, \overline{N}_{X/M})))_k))\Bigg)\\
=&\sum_k (-1)^k(-(p_{P_{Y/P_{X/M}}})_* \widetilde{\operatorname{ch}}(\eta^3_k) - (p_{\widetilde{P}_{X/M}})_* \widetilde{\operatorname{ch}}(\mu^1_k|_{\widetilde{P}_{X/M}}) + (p_{P_{X/M}})_* \widetilde{\operatorname{ch}}(\eta^1_k))\\
&- \widetilde{\operatorname{ch}}(\overline{A})\\
&- (i_{X/M})_*(\pi_{P_{X/M}})_* T(\overline{\xi}_{Y\hookrightarrow P_{X/M}}) + (i_{Y/M})_* C_T(F, N_{Y/P_{X/M}})\\
&- \sum_k (-1)^k (p_{P_{Y/P_{X/M}}}) \widetilde{\operatorname{ch}}(\eta^5_k),
\end{aligned}$$

where $\xi_{Y\hookrightarrow P_{X/M}}$ is as in notation 8.8.
By corollary 2.19 and the fact that the exact sequences $\overline{A}_{k,*}$ are orthogonally split, the term $\widetilde{\operatorname{ch}}(\overline{A})$ vanishes.
Also by corollary 2.19 we can see that

$$\sum_k (-1)^k (p_{\widetilde{P}_{X/M}})_* \widetilde{\operatorname{ch}}(\mu^1_k|_{\widetilde{P}_{X/M}})$$

vanishes.
Therefore we conclude

$$\begin{aligned}
I_b =&\sum_k (-1)^k(-(p_{P_{Y/P_{X/M}}})_* \widetilde{\operatorname{ch}}(\eta^3_k) + (p_{P_{X/M}})_* \widetilde{\operatorname{ch}}(\eta^1_k)) - (p_{P_{Y/P_{X/M}}}) \widetilde{\operatorname{ch}}(\eta^5_k)\\
&- (i_{X/M})_*(\pi_{P_{X/M}})_* T(\overline{\xi}_{Y\hookrightarrow P_{X/M}}) + (i_{Y/M})_* C_T(F, N_{Y/P_{X/M}}).
\end{aligned}$$

(c) By the definition of singular Bott-Chern forms we have

$$I_c = -T(\overline{\xi}_{Y\hookrightarrow M}) + (i_{Y/M})_* C_T(F, N_{Y/M}) - \sum_k (-1)^k (p_{P_{Y/M}})_* \widetilde{\operatorname{ch}}(\eta^2_k),$$

(d) Since the restriction of $\widetilde{E}'_*$ to $\widetilde{W}_X$ is orthogonally split, we have $I_d = 0$.

(e) We compute

$$\begin{aligned}I_e =&(p_{P_{Y\times\mathbb{P}^1}})_*\left(\sum_k(-1)^k W_1\bullet \mathrm{ch}(\widetilde{E}'_k|_{P_{Y\times\mathbb{P}^1}})\right)\\ =&(p_{P_{Y\times\mathbb{P}^1}})_*\Bigg(W_1\bullet\sum_k(-1)^k\big(-\mathrm{d}_{\mathcal{D}}\,\widetilde{ch}(\mu_k^2)+\mathrm{ch}(\overline{D}_k)\\ &\qquad\qquad +\mathrm{ch}(\pi^*_{P_{Y\times\mathbb{P}^1}}p_Y^*\overline{F}\otimes K(\mathcal{O}_{Y\times\mathbb{P}^1},\overline{N}_{Y\times\mathbb{P}^1/W_X})_k)\big)\Bigg).\end{aligned}$$

The term $\sum(-1)^k\,\mathrm{ch}(\overline{D}_k)$ vanishes because the complex $D_*$ is orthogonally split. We have

$$\begin{aligned}&\sum_k(-1)^k(p_{P_{Y\times\mathbb{P}^1}})_*(W_1\bullet\mathrm{ch}(\pi^*_{P_{Y\times\mathbb{P}^1}}p_Y^*\overline{F}\otimes K(\overline{\mathcal{O}}_{Y\times\mathbb{P}^1},\overline{N}_{Y\times\mathbb{P}^1/W_X})_k))\\ &=(i_{Y/M})_*\,\mathrm{ch}(\overline{F})\bullet(p_Y)_*\left(W_1\bullet\pi^*_{P_{Y\times\mathbb{P}^1}}\sum_k(-1)^k\,\mathrm{ch}(K(\overline{\mathcal{O}}_{Y\times\mathbb{P}^1},\overline{N}_{Y\times\mathbb{P}^1/W_X})_k)\right)\\ &\qquad=(i_{Y/M})_*\,\mathrm{ch}(\overline{F})\bullet(p_Y)_*\left(W_1\bullet\mathrm{Td}^{-1}(\overline{N}_{Y\times\mathbb{P}^1/W_X})\right)\\ &\qquad\qquad=(i_{Y/M})_*\,\mathrm{ch}(\overline{F})\bullet\widetilde{\mathrm{Td}^{-1}}(\overline{\varepsilon}_N),\end{aligned}\tag{8.12}$$

where $\overline{\varepsilon}_N$ is the exact sequence (8.6).
Therefore we obtain

$$\begin{aligned}I_e=&-\sum_k(-1)^k(p_{P_{Y/P_{X/M}}})_*\widetilde{\mathrm{ch}}(\eta_k^4)+\sum_k(-1)^k(p_{P_{Y/M}})_*\widetilde{\mathrm{ch}}(\eta_k^2)\\ &+(i_{Y/M})_*\,\mathrm{ch}(\overline{F})\bullet\widetilde{\mathrm{Td}^{-1}}(\overline{\varepsilon}_N).\end{aligned}$$

(f) Finally we have

$$\begin{aligned}I_f=&-\sum_k(-1)^kT(\overline{\xi}_{X\hookrightarrow M,k})+\sum_k(-1)^k(i_{X/M})_*C_T(E_k,N_{X/M})\\ &-\sum_{k,l}(-1)^{k+l}(p_{P_{X/M}})_*\widetilde{\mathrm{ch}}(\eta^1_{k,l}).\end{aligned}$$

By corollary 2.19 we have that

$$\sum_{m,l}(-1)^{m+l}(p_{P_{X/M}})_*\widetilde{\mathrm{ch}}(\eta^1_{m,l})=\sum_k(-1)^k(p_{P_{X/M}})_*\widetilde{\mathrm{ch}}(\eta^1_k).$$

Thus

$$\begin{aligned}I_f=&-\sum_k(-1)^kT(\overline{\xi}_{X\hookrightarrow M,k})+\sum_k(-1)^k(i_{X/M})_*C_T(E_k,N_{X/M})\\ &-\sum_k(-1)^k(p_{P_{X/M}})_*\widetilde{\mathrm{ch}}(\eta^1_k).\end{aligned}$$

Summing up all the terms we have computed, and taking into account equation (8.10) and the fact that

$$C_T(F, N_{Y/M}) = C_T(F, N_{Y/P_{X/M}})$$

we have obtained the following partial result.

Lemma 8.13. *Let $i_{Y/M} = i_{X/M} \circ i_{Y/X}$ be a composition of closed immersions of complex manifolds. Let $T$ be a theory of singular Bott-Chern classes with $C_T$ its associated characteristic class. Let $\overline{\xi}_{Y\hookrightarrow M}$, $\overline{\xi}_{X\hookrightarrow M,k}$ and $\overline{\xi}_{Y\hookrightarrow P_{X/M}}$ be as in notation 8.8, and let $\overline{\varepsilon}$ be as in* (8.6)*. Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(M,p)$, the equation*

$$\begin{aligned} T(\overline{\xi}_{Y\hookrightarrow M}) = \sum_k (-1)^k T(\overline{\xi}_{X\hookrightarrow M,k}) - \sum_k (-1)^k (i_{X/M})_* C_T(E_k, N_{X/M}) \\ + (i_{X/M})_*(\pi_{P_{X/M}})_* T(\overline{\xi}_{Y\hookrightarrow P_{X/M}}) + (i_{Y/M})_* \operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}^{-1}}(\overline{\varepsilon}_N) \quad (8.14) \end{aligned}$$

*holds.*

In order to compute the third term of the right hand side of equation (8.14) we consider the following situation

$$\begin{array}{ccc} Y \times_X P_{X/M} & \xrightarrow{\ j\ } & P_{X/M} \\ \pi \downarrow \uparrow s & & \pi \downarrow \uparrow s \\ Y & \xrightarrow{\ i\ } & X \end{array}.$$

To ease the notation, we denote $P_{X/M}$ by $P$, $Y \underset{X}{\times} P_{X/M}$ by $X'$ and we denote by $P'$ the projective completion of the normal cone to $X'$ in $P$ and by $\pi_{P'}\colon P' \longrightarrow X'$, $\pi_{X'/Y}\colon X' \longrightarrow Y$ and $\pi_{P'/Y}\colon P' \longrightarrow Y$ the projections. Observe that $X$ and $X'$ intersect transversely along $Y$. Moreover, $N_{Y/X'} = i^*_{Y/X} N_{X/M}$, $N_{X'/P} = \pi^*_{X'/Y} N_{Y/X}$ and $N_{Y/P} = N_{Y/X} \oplus N_{Y/X'}$. We use these identifications to define metrics on $N_{Y/X'}$, $N_{X'/P}$ and $N_{Y/P}$. Therefore the exact sequence

$$0 \longrightarrow \overline{N}_{Y/X'} \longrightarrow \overline{N}_{Y/P} \longrightarrow i^*_{Y/X'} \overline{N}_{X'/P} \longrightarrow 0$$

is orthogonally split.
We apply the previous lemma to the composition of closed inclusions

$$Y \hookrightarrow X' \hookrightarrow P,$$

the vector bundle $\overline{F}$ over $Y$ and the resolutions

$$\begin{aligned} \pi^*\overline{F} \otimes j^* K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_* \longrightarrow s_* F \\ \pi^*\overline{E}_* \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_k \longrightarrow j_*(\pi^* F \otimes j^* K(\mathcal{O}_X, N_{X/M})_k). \end{aligned}$$

We denote by $\overline{\xi}_{Y\hookrightarrow P}$ and $\overline{\xi}_{X'\hookrightarrow P,k}$ the hermitian embedded vector bundles corresponding to the above resolutions. If $i_{Y/P'}: Y \hookrightarrow P'$ is the induced inclusion, we denote by $\overline{\xi}_{Y\hookrightarrow P'}$ the hermitian embedded vector bundle

$$\left(i_{Y/P'}, \overline{N}_{Y/P'}, \overline{F}, \operatorname{Tot}(\pi^*_{P'} j^* K(\overline{\mathcal{O}}_X, \overline{N}_{X/M}) \otimes K(\overline{\mathcal{O}}_{X'}, \overline{N}_{X'/P}) \otimes (\pi_{P'/Y})^* \overline{F})\right).$$

Note that the hermitian embedded vector bundle $\overline{\xi}_{Y\hookrightarrow P}$ agrees with the hermitian embedded vector bundle denoted $\overline{\xi}_{Y\hookrightarrow P_{X/M}}$ in lemma 8.13. Moreover, we have that

$$\overline{\xi}_{X'\hookrightarrow P,k} = \pi^* \overline{\xi}_{Y\hookrightarrow X} \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_k.$$

Applying lemma 8.13, we obtain

$$\begin{aligned} T(\overline{\xi}_{Y\hookrightarrow P_{X/M}}) = \sum_k (-1)^k T(\overline{\xi}_{X'\hookrightarrow P_{X/M},k}) \\ - \sum_k (-1)^k j_* C_T(\pi^* F \otimes j^* K(\mathcal{O}_X, N_{X/M})_k, N_{X'/P}) \\ + j_*(\pi_{P'})_* T(\overline{\xi}_{Y\hookrightarrow P'}) \end{aligned} \tag{8.15}$$

By proposition 8.1,

$$\begin{aligned} \sum_k (-1)^k T(\overline{\xi}_{X'\hookrightarrow P_{X/M},k}) &= \sum_k (-1)^k T(\pi^* \overline{\xi}_{Y\hookrightarrow X} \otimes K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_k) \\ &= T(\pi^* \overline{\xi}_{Y\hookrightarrow X}) \bullet \sum_k (-1)^k \operatorname{ch}(K(\overline{\mathcal{O}}_X, \overline{N}_{X/M})_k) \\ &+ \sum_k (-1)^k j_* C_T(\pi^* F \otimes j^* K(\mathcal{O}_X, N_{X/M})_k, N_{X'/P}) \\ &- \sum_k (-1)^k j_* C_T(\pi^* F, N_{X'/P}) \bullet \operatorname{ch}(K(\mathcal{O}_X, N_{X/M})_k) \end{aligned} \tag{8.16}$$

We now want to compute the term $(i_{X/M})_*(\pi_{P_{X/M}})_* j_*(\pi_{P'})_* T(\overline{\xi}_{Y\hookrightarrow P'})$.
Observe that we can identify

$$P' = \mathbb{P}(i^*_{Y/X} N_{X/M} \oplus \mathbb{C}) \underset{Y}{\times} \mathbb{P}(s^* N_{X'/P} \oplus \mathbb{C}),$$

where $s^* N_{X'/P}$ is canonically isomorphic to $N_{Y/X}$.
Moreover

$$(i_{X/M})_*(\pi_{P_{X/M}})_* j_*(\pi_{P'})_* T(\overline{\xi}_{Y\hookrightarrow P'}) = (i_{Y/M})_*(\pi_{P'/Y})_* T(\overline{\xi}_{Y\hookrightarrow P'}).$$

Definition 8.17. We denote

$$C^{\mathrm{ad}}_T(F, N_{Y/X}, i^*_{Y/X} N_{X/M}) = (\pi_{P'/Y})_* T(\overline{\xi}_{Y\hookrightarrow P'})$$

and we define

$$\rho(F, N_{Y/X}, i^*_{Y/X} N_{X/M}) = C_T(F, N_{Y/M}) - C^{\mathrm{ad}}_T(F, N_{Y/X}, i^*_{Y/X} N_{X/M}). \tag{8.18}$$

Lemma 8.19. *The current $C_T^{\mathrm{ad}}(F, N_{Y/X}, i_{Y/X}^{*}N_{X/M})$ is closed and defines a characteristic class of triples of vector bundles. Therefore $\rho$ is also a characteristic class. Moreover the class $\rho$ does not depend on the theory of singular Bott-Chern classes $T$.*

*Proof.* The fact that $C_T^{\mathrm{ad}}(F, N_{Y/X}, i_{Y/X}^{*}N_{X/M})$ is closed and determines a characteristic class is proved as in 6.16. The independence of $\rho$ from to $T$ is seen as follows. We denote by $\overline{K}'_{*}$ the complex

$$\operatorname{Tot}(\pi_{P'}^{*}j^{*}K(\overline{\mathcal{O}}_X, \overline{N}_{X/M}) \otimes K(\overline{\mathcal{O}}_{X'}, \overline{N}_{X'/P})) \otimes (\pi_{P'/Y})^{*}\overline{F}.$$

This complex is a resolution of $(i_{Y/P'})_{*}\overline{F}$
Let $W$ be the blow-up of $P' \times \mathbb{P}^1$ along $Y \times \infty$, and let $\operatorname{tr}_1(\overline{K}')_{*}$ be the deformation of complexes on $W$ given by theorem 5.4. Just by looking at the rank of the different vector bundles we see that the restriction of $\operatorname{tr}_1(\overline{K}')_{*}$ to $P_{Y/P'}$, the exceptional divisor of this blow-up, is isomorphic (although not necessarily isometric) to the Koszul complex $K(\overline{F}, \overline{N}_{X/M})_{*}$. Then, by equation (7.3)

$$\begin{aligned}T(\overline{\xi}_{Y\hookrightarrow P'}) - (i_{Y/P'})_{*}C_T(F, N_{Y/M}) = \\ -(p_W)_{*}\left(W_1 \bullet \sum_k (-1)^k \operatorname{ch}(\operatorname{tr}_1(\overline{K}')_k)\right) \\ - \sum_k (-1)^k (p_P)_{*}\widetilde{\operatorname{ch}}(\operatorname{tr}_1(\overline{K}')_k|_{P_{Y/P'}}, K(\overline{F}, \overline{N}_{X/M})_k).\end{aligned}$$

Since the right hand side of this equation does not depend on the theory $T$, the result is proved. □

Using equations (8.15), (8.16), lemma 8.19 and the projection formula, we obtain

$$\begin{aligned}(\pi_{P_{X/M}})_{*}T(\overline{\xi}_{Y\hookrightarrow P_{X/M}}) &= \big(T(\overline{\xi}_{Y\hookrightarrow X}) - (i_{Y/X})_{*}C_T(F, N_{Y/X})\big) \\ &\quad \bullet (\pi_{P_{X/M}})_{*}\sum_k (-1)^k \operatorname{ch}(K(\mathcal{O}_X, \overline{N}_{X/M})_k) \\ &\quad + (\pi_{P_{X/M}})_{*}j_{*}(\pi_{P'})_{*}T(\overline{\xi}_{Y\hookrightarrow P'}) \\ &= \big(T(\overline{\xi}_{Y\hookrightarrow X}) - (i_{Y/X})_{*}C_T(F, N_{Y/X})\big) \bullet \operatorname{Td}^{-1}(\overline{N}_{X/M}) \\ &\quad + (i_{Y/X})_{*}C_T^{\mathrm{ad}}(F, N_{Y/X}, i_{Y/X}^{*}N_{X/M}) \\ &= \big(T(\overline{\xi}_{Y\hookrightarrow X}) - (i_{Y/X})_{*}C_T(F, N_{Y/X})\big) \bullet \operatorname{Td}^{-1}(\overline{N}_{X/M}) \\ &\quad + (i_{Y/X})_{*}C_T(F, N_{Y/M}) - \rho(F, N_{Y/X}, i_{Y/X}^{*}N_{X/M}).\end{aligned} \tag{8.20}$$

Joining this equation and lemma 8.13 we obtain the main relationship between singular Bott-Chern classes and composition of closed immersions.

Proposition 8.21. *Let $i_{Y/M} = i_{X/M} \circ i_{Y/X}$ be a composition of closed immersions of complex manifolds. Let $T$ be a theory of singular Bott-Chern classes with $C_T$ its associated characteristic class. Let $\overline{\xi}_{Y\hookrightarrow M}$, $\overline{\xi}_{X\hookrightarrow M,k}$ and $\overline{\xi}_{Y\hookrightarrow P_{X/M}}$ be as in notation 8.8 and let $\overline{\varepsilon}$ be as in* (8.6). *Then, in the group $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(M,p)$, we have the equation*

$$\begin{aligned} T(\overline{\xi}_{Y\hookrightarrow M}) = \sum_k (-1)^k T(\overline{\xi}_{X\hookrightarrow M,k}) + (i_{X/M})_*(T(\overline{\xi}_{Y\hookrightarrow X}) \bullet \operatorname{Td}^{-1}(\overline{N}_{X/M})) \\ + (i_{Y/M})_* \operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}^{-1}}(\overline{\varepsilon}_N) \\ + (i_{Y/M})_* C_T^{\mathrm{ad}}(F, N_{Y/X}, i^*_{Y/X} N_{X/M}) \\ - (i_{X/M})_*((i_{Y/X})_* C_T(F, N_{Y/X}) \bullet \operatorname{Td}^{-1}(N_{X/M})) \\ - (i_{X/M})_* \sum_k (-1)^k C_T(E_k, N_{X/M}) \end{aligned}$$

We can simplify the formula of proposition 8.21 if we assume that our theory of singular Bott-Chern classes is compatible with the projection formula.

Corollary 8.22. *With the hypothesis of proposition 8.21, assume furthermore that $T$ is compatible with the projection formula. Then*

$$\begin{aligned} T(\overline{\xi}_{Y\hookrightarrow M}) = \sum_k (-1)^k T(\overline{\xi}_{X\hookrightarrow M,k}) + (i_{X/M})_*(T(\overline{\xi}_{Y\hookrightarrow X}) \bullet \operatorname{Td}^{-1}(\overline{N}_{X/M})) \\ + (i_{Y/M})_* \operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}^{-1}}(\overline{\varepsilon}_N) \\ + (i_{Y/M})_* \Big[ C_T^{\mathrm{ad}}(F, N_{Y/X}, i^*_{Y/X} N_{X/M}) - C_T(F, N_{Y/X}) \bullet \operatorname{Td}^{-1}(i^*_{Y/X} N_{X/M})) \\ - C_T(F, i^*_{Y/X} N_{X/M}) \bullet \operatorname{Td}^{-1}(N_{Y/X}) \Big] \end{aligned}$$

*Proof.* Since $T$ is compatible with the projection formula, then $C_T$ is also. Therefore, using the Grothendieck-Riemann-Roch theorem for closed immersions at the level of analytic Deligne cohomology classes, we have

$$\begin{aligned} \sum_k (-1)^k C_T(E_k, N_{X/M}) &= C_T(\mathcal{O}_X, N_{X/M}) \bullet \sum_k (-1)^k \operatorname{ch}(E_k) \\ &= C_T(\mathcal{O}_X, N_{X/M}) \bullet (i_{Y/X})_*(\operatorname{ch}(F) \bullet \operatorname{Td}^{-1}(N_{Y/X})) \\ &= (i_{Y/X})_*(i^*_{Y/X} C_T(\mathcal{O}_X, N_{X/M}) \bullet \operatorname{ch}(F) \bullet \operatorname{Td}^{-1}(N_{Y/X})) \\ &= (i_{Y/X})_*(C_T(F, i^*_{Y/X} N_{X/M}) \bullet \operatorname{Td}^{-1}(N_{Y/X})), \end{aligned}$$

which implies the result. □

Definition 8.23. Let $T$ be a theory of singular Bott-Chern classes. We will say that $T$ is *transitive* if the equation

$$T(\overline{\xi}_{Y\hookrightarrow M}) = \sum_k (-1)^k T(\overline{\xi}_{X\hookrightarrow M,k}) + (i_{X/M})_*(T(\overline{\xi}_{Y\hookrightarrow X}) \bullet \operatorname{Td}^{-1}(\overline{N}_{X/M})) \\ + (i_{Y/M})_* \operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}^{-1}}(\overline{\varepsilon}_N) \quad (8.24)$$

holds. When equation (8.24) is satisfied for a particular choice of complex immersions and resolutions, we say that the theory $T$ is *transitive with respect to this particular choice.*

We now introduce an abstract version of definition 8.17.

Definition 8.25. Given any characteristic class $C$ of pairs of vector bundles, we will denote

$$C^{\rho}(F, N_1, N_2) := C(F, N_1 \oplus N_2) - \rho(F, N_1, N_2),$$

where $\rho$ is the characteristic class of definition 8.17.

Note that, when $T$ is a theory of singular Bott-Chern classes we have

$$C^{\rho}_T(F, N_1, N_2) = C^{\mathrm{ad}}_T(F, N_1, N_2).$$

Definition 8.26. We will say that a characteristic class $C$ (of pairs of vector bundles) is *ρ-Todd additive* (in the second variable) if it satisfies

$$C(F, N_1 \oplus N_2) = C(F, N_1) \bullet \operatorname{Td}^{-1}(N_2) + C(F, N_2) \bullet \operatorname{Td}^{-1}(N_1) + \rho(F, N_1, N_2)$$

or, equivalently,

$$C^{\rho}(F, N_1, N_2) = C(F, N_1) \bullet \operatorname{Td}^{-1}(N_2) + C(F, N_2) \bullet \operatorname{Td}^{-1}(N_1).$$

A direct consequence of corollary 8.22 is

Corollary 8.27. *Let $T$ be a theory of singular Bott-Chern classes that is compatible with the projection formula. Then it is transitive if and only if the associated characteristic class $C_T$ is ρ-Todd additive.*

Since we are mainly interested in singular Bott-Chern classes that are transitive and compatible with the projection formula, we will study characteristic classes that are compatible with the projection formula and $\rho$-Todd-additive in the second variable. Since we want to express any characteristic class in terms of a power series we will restrict ourselves to the algebraic category.

Proposition 8.28. *Let $C$ be a class that is compatible with the projection formula and ρ-Todd additive in the second variable. Then $C$ determines a power series $\phi_C(x)$ given by*

$$C(\mathcal{O}_Y, L) = \phi_C(c_1(L)), \quad (8.29)$$

*for every complex algebraic manifold $Y$ and algebraic line bundle $Y$. Conversely, given any power series in one variable $\phi(x)$, there exists a unique characteristic class for algebraic vector bundles that is compatible with the projection formula and $\rho$-Todd additive in the second variable such that equation* (8.29) *holds.*

*Proof.* This result follows directly from the splitting principle and theorem 1.8. ☐

Remark 8.30. The utility of corollary 8.27 and proposition 8.28 is limited by the fact that we do not know an explicit formula for the class $\rho(\mathcal{O}_Y, N_1, N_2)$. This class is related with the arithmetic difference between $\mathbb{P}_Y(N_1 \oplus N_2 \oplus \mathbb{C})$ and $\mathbb{P}_Y(N_1 \oplus \mathbb{C}) \underset{Y}{\times} \mathbb{P}_Y(N_2 \oplus \mathbb{C})$, the second space being simpler than the first. The main ingredients needed to compute this class are the Bott-Chern classes of the tautological exact sequence. Therefore the work of Mourougane [29] might be useful for computing this class.

Recall that an additive genus is a characteristic class for algebraic vector bundles $S$ such that

$$S(N_1 \oplus N_2) = S(N_1) + S(N_2).$$

Let $\phi(x) = \sum_{i=0}^{\infty} a_i x^i$ be a power series in one variable. There is a one to one correspondence between additive genus and power series characterized by the condition that $S(L) = \phi(c_1(L))$, for each line bundle $L$.
Since the class $\rho$ does not depend on the theory $T$ it cancels out when considering the difference between two different theories of singular Bott-Chern classes.

Proposition 8.31. *Let $C_1$ and $C_2$ be two characteristic classes for pairs of algebraic vector bundles that are compatible with the projection formula and $\rho$-Todd-additive in the second variable. Then there is a unique additive genus $S_{12}$ such that*

$$C_1(F, N) - C_2(F, N) = \operatorname{ch}(F) \bullet \operatorname{Td}(N)^{-1} \bullet S_{12}(N). \tag{8.32}$$

We can summarize the results of this section in the following theorem.

Theorem 8.33. *There is a one to one correspondence between theories of singular Bott-Chern classes for complex algebraic manifolds that are transitive and compatible with the projection formula, and formal power series $\phi(x) \in \mathbb{R}[[x]]$. To each theory of singular Bott-Chern classes corresponds the power series $\phi$ such that*

$$C_T(\mathcal{O}_Y, L) = \mathbf{1}_1 \bullet \phi(c_1(L)), \tag{8.34}$$

*for every complex algebraic manifold $Y$ and every algebraic line bundle $L$. To each power series $\phi$ it corresponds a unique class $C$, compatible with the projection formula and $\rho$-Todd-additive in the second variable, characterized by equation* (8.34) *and a theory of singular Bott-Chern given by definition 7.4.*

Even if we do not know the exact value of the class $\rho$ another consequence of corollary 8.27 is that, in order to prove the transitivity of a theory of singular Bott-Chern classes it is enough to check it for a particular class of compositions.

Corollary 8.35. *Let $T$ be a theory of singular Bott-Chern classes compatible with the projection formula. Then $T$ is transitive if and only if for any compact complex manifold $Y$ and vector bundles $N_1$, $N_2$, the theory $T$ is transitive with respect to the composition of inclusions*

$$Y \hookrightarrow \mathbb{P}_Y(N_1 \oplus \mathbb{C}) \hookrightarrow \mathbb{P}_Y(N_1 \oplus \mathbb{C}) \times_Y \mathbb{P}_Y(N_2 \oplus \mathbb{C})$$

*and the Koszul resolutions.* □

We can make the previous corollary a little more explicit. Let $\pi_1$ and $\pi_2$ be the projections from $P := \mathbb{P}_Y(N_1 \oplus \mathbb{C}) \times_Y \mathbb{P}_Y(N_2 \oplus \mathbb{C})$ to $P_1 := \mathbb{P}_Y(N_1 \oplus \mathbb{C})$ and $P_2 := \mathbb{P}_Y(N_2 \oplus \mathbb{C})$ respectively. Let $\overline{K}_1 = K(\overline{\mathcal{O}}_Y, \overline{N}_1)$ and $\overline{K}_2 = K(\overline{\mathcal{O}}_Y, \overline{N}_2)$ be the Koszul resolutions in $P_1$ and $P_2$ respectively. Then,

$$\overline{K} = \pi_1^* K_1 \otimes \pi_2^* K_2$$

is a resolution of $\mathcal{O}_Y$ in $P$. Then the theory $T$ is transitive in this case if

$$T(\overline{K}) = \pi_2^* T(\overline{K}_2) \bullet \pi_1^*(c_{r_1}(\overline{Q}_1) \bullet \operatorname{Td}^{-1}(\overline{Q}_1)) + (i_1)_*(T(\overline{K}_1) \bullet p_1^* \operatorname{Td}^{-1}(\overline{N}_2)),$$

where $r_1$ is the rank of $N_1$, $\overline{Q}_1$ is the tautological quotient bundle in $P_1$ with the induced metric, $i_1 \colon P_1 \longrightarrow P$ is the inclusion and $p_1 \colon P_1 \longrightarrow Y$ is the projection.

The singular Bott-Chern classes that we have defined depend on the choice of a hermitian metric on the normal bundle and behave well with respect inverse images. Nevertheless, when one is interested in covariant functorial properties and, in particular, in a composition of closed immersions, it might be interesting to consider a variant of singular Bott-Chern classes that depend on the choice of metrics on the tangent bundles to $Y$ and $X$.

Notation 8.36. Let $\overline{\xi} = (i \colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_* \to i_* F)$ be a hermitian embedded vector bundle. Let $\overline{T}_X$ and $\overline{T}_Y$ be the tangent bundles to $X$ and $Y$ provided with hermitian metrics. As usual we write $\operatorname{Td}(Y) = \operatorname{Td}(\overline{T}_Y)$ and $\operatorname{Td}(X) = \operatorname{Td}(\overline{T}_X)$. We put

$$\overline{\xi}_c = (i \colon Y \longrightarrow X, \overline{T}_X, \overline{T}_Y, \overline{F}, \overline{E}_* \to i_* F).$$

By abuse of notation we will also say that $\overline{\xi}_c$ is a hermitian embedded vector bundle. In this situation we denote by $\overline{\xi}_{N_{Y/X}}$ the exact sequence of hermitian vector bundles

$$\overline{\xi}_{N_{Y/X}} \colon 0 \longrightarrow \overline{T}_Y \longrightarrow i^* \overline{T}_X \longrightarrow \overline{N}_{Y/X} \longrightarrow 0.$$

If there is no danger of confusion we will denote $\overline{N} = \overline{N}_{Y/X}$ and therefore $\overline{\xi}_N = \overline{\xi}_{N_{Y/X}}$.

Definition 8.37. Let $T$ be a theory of singular Bott-Chern classes. Then the *covariant singular Bott-Chern class* associated to $T$ is given by

$$T_c(\overline{\xi}_c) = T(\overline{\xi}) + i_*(\operatorname{ch}(\overline{F}) \bullet \widetilde{\operatorname{Td}^{-1}}(\overline{\xi}_{N_{Y/X}}) \operatorname{Td}(Y)) \tag{8.38}$$

Proposition 8.39. *The covariant singular Bott-Chern classes satisfy the following properties*

*(i) The class $T_c(\overline{\xi}_c)$ does not depend on the choice of the metric on $N_{Y/X}$.*

*(ii) The differential equation*

$$\mathrm{d}_{\mathcal{D}} T_c(\overline{\xi}_c) = \sum_k (-1)^k \operatorname{ch}(\overline{E}_k) - i_*(\operatorname{ch}(\overline{F}) \bullet \operatorname{Td}(Y)) \bullet \operatorname{Td}^{-1}(X) \tag{8.40}$$

*holds.*

*(iii) If the theory $T$ is compatible with the projection formula, then*

$$T_c(\overline{\xi}_c \otimes \overline{G}) = T_c(\overline{\xi}_c) \bullet \operatorname{ch}(\overline{G}).$$

*(iv) If, moreover, the theory $T$ is transitive, then, using notation 8.8 adapted to the current setting, we have*

$$\begin{aligned} T_c(\overline{\xi}_{Y\hookrightarrow M,c}) = \sum_k (-1)^k T_c(\overline{\xi}_{X\hookrightarrow M,k,c}) \\ + (i_{X/M})_*(T_c(\overline{\xi}_{Y\hookrightarrow X,c}) \bullet \operatorname{Td}(X)) \bullet \operatorname{Td}^{-1}(M). \end{aligned} \tag{8.41}$$

*(v) With the hypothesis of corollary 6.14, we have*

$$T_c(\bigoplus_{j\ even} \overline{\xi}_{j,c}) - T_c(\bigoplus_{j\ odd} \overline{\xi}_{j,c}) = [\widetilde{\operatorname{ch}}(\overline{\varepsilon})] - i_*([\widetilde{\operatorname{ch}}(\overline{\chi}) \bullet \operatorname{Td}(Y)]) \bullet \operatorname{Td}^{-1}(X). \tag{8.42}$$

*Proof.* All the statements follow from straightforward computations. □

## 9 Homogeneous singular Bott-Chern classes

In this section we will show that, by adding a natural fourth axiom to definition 6.9, we obtain a unique theory of singular Bott-Chern classes that we call homogeneous singular Bott-Chern classes, and we will compare it with the classes previously defined by Bismut, Gillet and Soulé and by Zha.
In the paper [6], Bismut, Gillet and Soulé introduced a theory of singular Bott-Chern classes that is the main ingredient in their construction of direct images for closed immersions.
Strictly speaking, the construction of [6] only produces a theory of singular Bott-Chern classes in the sense of this paper when the metrics involved satisfy

a technical condition, called Condition (A) of Bismut. Nevertheless, there is a unique way to extend the definition of [6] from metrics satisfying Bismut's condition (A) to general metrics in such a way that one obtains a theory of singular Bott-Chern classes in the sense of this paper.
In his thesis [32], Zha gave another definition of singular Bott-Chern classes, and he also used them to define direct images for closed immersions in Arakelov theory.
We will recall the construction of both theories of singular Bott-Chern classes and we will show that they agree with the theory of homogeneous singular Bott-Chern classes.
We warn the reader that the normalizations we use differ from the normalizations in [6] and [32]. The main difference is that we insist on using the algebro-geometric twist in cohomology, whereas in the other two papers the authors use cohomology with real coefficients.
Let $r_F$ and $r_N$ be two positive integers. Let $Y$ be a complex manifold and let $\overline{F}$ and $\overline{N}$ be two hermitian vector bundles of rank $r_F$ and $r_N$ respectively. Let $P = \mathbb{P}(N \oplus \mathbb{C})$ and let $s$ be the zero section. We will follow the notations of definition 5.3. Then $T(K(\overline{F},\overline{N}))$ satisfies the differential equation

$$\mathrm{d}_{\mathcal{D}} T(K(\overline{F},\overline{N})) = c_{r_N}(\overline{Q}) \operatorname{Td}^{-1}(\overline{Q}) \operatorname{ch}(\pi_P^* \overline{F}) - s_*(\operatorname{ch}(\overline{F}) \operatorname{Td}^{-1}(\overline{N})).$$

Therefore, the class

$$\widetilde{e}_T(\overline{F},\overline{N}) := T(K(\overline{F},\overline{N})) \bullet \operatorname{Td}(\overline{Q}) \bullet \operatorname{ch}^{-1}(\pi_P^* \overline{F})$$

satisfies the simpler equation

$$\mathrm{d}_{\mathcal{D}} \widetilde{e}_T(\overline{F},\overline{N}) = [c_{r_N}(\overline{Q})] - \delta_Y. \tag{9.1}$$

Observe that the right hand side of this equation belongs to $\mathcal{D}_D^{2r_N}(P, r_N)$. Thus it seems natural to introduce the following definition.

Definition 9.2. Let $T$ be a theory of singular Bott-Chern classes of rank $r_F > 0$ and codimension $r_N$. Then the class

$$\widetilde{e}_T(\overline{F},\overline{N}) = T(K(\overline{F},\overline{N})) \bullet \operatorname{Td}(\overline{Q}) \bullet \operatorname{ch}^{-1}(\pi_P^* \overline{F})$$

is called the *Euler-Green class associated to $T$*. The class $T(K(\overline{F},\overline{N}))$ is said to be *homogeneous* if

$$\widetilde{e}_T(\overline{F},\overline{N}) \in \widetilde{\mathcal{D}}_D^{2r_N-1}(P, r_N).$$

A theory of singular Bott-Chern classes of rank 0 is said to be *homogeneous* if it agrees with the theory of Bott-Chern classes associated to the Chern character. Finally, a theory of singular Bott-Chern classes is said to be *homogeneous* if its restrictions to all ranks and codimensions are homogeneous.

The main interest of the above definition is the following result.

Theorem 9.3. *Given two positive integers $r_F$ and $r_N$ there exists a unique theory of homogeneous singular Bott-Chern classes of rank $r_F$ and codimension $r_N$.*

*Proof.* The proof of this result is based on the theory of Euler-Green classes. Let $P = \mathbb{P}(N \oplus \mathbb{C})$ be as before, and let $s$ denote the zero section of $P$. Let $D_\infty$ be the subvariety of $P$ that parametrizes the lines contained in $N$. Then $D_\infty = \mathbb{P}(N)$.

Lemma 9.4. *There exists a unique class $\widetilde{e}(P, \overline{Q}, s) \in \mathcal{D}_D^{2r_N-1}(P, r_N)$ such that*

*(i) It satisfies*

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{e}(P, \overline{Q}, s) = [c_{r_N}(\overline{Q})] - \delta_Y. \tag{9.5}$$

*(ii) The restriction $\widetilde{e}(P, \overline{Q}, s)|_{D_\infty} = 0$.*

*Proof.* We first show the uniqueness. Assume that $\widetilde{e}$ and $\widetilde{e}'$ are two classes that satisfy the hypothesis of the theorem. Then $\widetilde{e}' - \widetilde{e}$ is closed. Hence it determines a cohomology class in $H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(P, r_N)$. Since, by theorem 1.2, the restriction

$$H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(P, r_N) \longrightarrow H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(D_\infty, r_N) \tag{9.6}$$

is an isomorphism, condition (ii) implies that $\widetilde{e}' = \widetilde{e}$. Now we prove the existence. Since $Y$ is the zero locus of the section $s$, that is transversal to the zero section of $Q$, we know that the currents $[c_{r_N}]$ and $\delta_Y$ are cohomologous. Therefore there exists an element $\widetilde{a} \in \widetilde{\mathcal{D}}_D^{2r_N-1}(P, r_N)$ such that $\mathrm{d}_{\mathcal{D}}\, \widetilde{a} = [c_{r_N}(\overline{Q})] - \delta_Y$. Since $\overline{Q}$ restricted to $D_\infty$ splits as an orthogonal direct sum

$$\overline{Q}|_{D_\infty} = \overline{S} \oplus \overline{\mathbb{C}} \tag{9.7}$$

where the metric on the factor $\mathbb{C}$ is trivial, and the section $s$ restricts to the constant section 1, we obtain that $([c_{r_N}(\overline{Q})] - \delta_Y)|_{D_\infty} = 0$. Therefore $\widetilde{a}$ determines a class in $H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(P, r_N)$. Using again that (9.6) is an isomorphism, we find an element $\widetilde{b} \in H^{2r_N-1}_{\mathcal{D}^{\mathrm{an}}}(P, r_N)$, such that $\widetilde{e} = \widetilde{a} - \widetilde{b}$ satisfies the conditions of the lemma. $\square$

We continue with the proof of theorem 9.3. We first prove the uniqueness. Let $T$ be a theory of homogeneous singular Bott-Chern classes. The splitting (9.7) implies easily that the restriction of the Koszul resolution $K(\overline{F}, \overline{N})$ to $D_\infty$ is orthogonally split. By the functoriality of singular Bott-Chern classes, $T(K(\overline{F}, \overline{N}))|_{D_\infty} = 0$. Thus the class

$$\widetilde{e}_T(\overline{F}, \overline{N}) := T(K(\overline{F}, \overline{N})) \bullet \mathrm{Td}(\overline{Q}) \bullet \mathrm{ch}^{-1}(\pi_P^* \overline{F}) \in \widetilde{\mathcal{D}}_D^{2r_N-1}(P, r_N)$$

satisfies the two conditions of lemma 9.4. Therefore $\widetilde{e}_T(\overline{F}, \overline{N}) = \widetilde{e}(P, \overline{Q}, s)$ and

$$T(K(\overline{F}, \overline{N})) = \widetilde{e}(P, \overline{Q}, s) \bullet \mathrm{Td}^{-1}(\overline{Q}) \bullet \mathrm{ch}(\pi_P^* \overline{F}), \tag{9.8}$$

where the right hand side does not depend on the theory $T$. In consequence we have that

$$C_T(F,N) = (\pi_P)_* T(K(\overline{F},\overline{N})) \tag{9.9}$$

does not depend on the theory $T$. Thus by the uniqueness in theorem 7.1 we obtain the uniqueness here.
For the existence we observe

Lemma 9.10. *The current*

$$C(F,N) = (\pi_P)_*(\widetilde{e}(P,\overline{Q},s) \bullet \mathrm{Td}^{-1}(\overline{Q})) \bullet \mathrm{ch}(\overline{F})$$

*is a characteristic class for pairs of vector bundles of rank $r_F$ and $r_N$.*

*Proof.* We first compute, using equation (9.5) and corollary 3.8,

$$\begin{aligned}
\mathrm{d}_{\mathcal{D}}\, C(F,N) &= (\pi_P)_* \left(\mathrm{d}_{\mathcal{D}}\, \widetilde{e}(P,\overline{Q},s) \bullet \mathrm{Td}^{-1}(\overline{Q})\right) \bullet \mathrm{ch}(\overline{F}) \\
&= (\pi_P)_* \left(([c_{r_N}(\overline{Q})] - \delta_Y) \bullet \mathrm{Td}^{-1}(\overline{Q})\right) \bullet \mathrm{ch}(\overline{F}) \\
&= (\pi_P)_* \left(c_{r_N}(\overline{Q}) \bullet \mathrm{Td}^{-1}(\overline{Q})\right) \bullet \mathrm{ch}(\overline{F}) - \mathrm{Td}^{-1}(\overline{N}) \bullet \mathrm{ch}(\overline{F}) \\
&= 0.
\end{aligned}$$

Thus $C(F,N)$ determines a cohomology class. This class is functorial by construction. By proposition 1.7 this class does not depend on the metric and defines a characteristic class. □

By the existence in theorem 7.1 we obtain a theory of singular Bott-Chern classes $T_C$ that is easily seen to be homogeneous. □

A reformulation of theorem 9.3 is

Theorem 9.11. *There exists a unique way to associate to each hermitian embedded vector bundle $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ a class of currents*

$$T^h(\overline{\xi}) \in \bigoplus_p \widetilde{\mathcal{D}}_D^{2p-1}(X, N^*_{Y,0}, p)$$

*that we call homogeneous singular Bott-Chern class, satisfying the following properties*

*(i) (Differential equation) The equality*

$$\mathrm{d}_{\mathcal{D}}\, T^h(\overline{\xi}) = \sum_i (-1)^i [\mathrm{ch}(\overline{E}_i)] - i_*([\mathrm{Td}^{-1}(\overline{N})\, \mathrm{ch}(\overline{F})]) \tag{9.12}$$

*holds.*

*(ii) (Functoriality) For every morphism $f\colon X' \longrightarrow X$ of complex manifolds that is transverse to $Y$,*

$$f^* T^h(\overline{\xi}) = T^h(f^*\overline{\xi}).$$

(iii) *(Normalization) Let* $\overline{A} = (A_*, g_*)$ *be a non-negatively graded orthogonally split complex of vector bundles. Write* $\overline{\xi} \oplus \overline{A} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_* \oplus \overline{A}_*)$*. Then* $T^h(\overline{\xi}) = T^h(\overline{\xi} \oplus \overline{A})$*. Moreover, if* $X = \operatorname{Spec}\mathbb{C}$ *is one point,* $Y = \emptyset$ *and* $\overline{E}_* = 0$*, then* $T^h(\overline{\xi}) = 0$.

(iv) *(Homogeneity) If* $r_F = \operatorname{rk}(F) > 0$ *and* $r_N = \operatorname{rk}(N) > 0$*, then, with the notations of definition 9.2,*

$$T^h(K(\overline{F}, \overline{N})) \bullet \operatorname{Td}(\overline{Q}) \bullet \operatorname{ch}^{-1}(\pi_P^* \overline{F}) \in \widetilde{\mathcal{D}}_D^{2r_N - 1}(P, r_N).$$

□

The class $\widetilde{e}(P, \overline{Q}, s)$ of lemma 9.4 is a particular case of the Euler-Green classes introduced by Bismut, Gillet and Soulé in [6]. The basic properties of the Euler-Green classes are summarized in the following results.

Proposition 9.13. *Let* $X$ *be a complex manifold, let* $\overline{E}$ *be a hermitian holomorphic vector bundle of rank* $r$ *and let* $s$ *be a holomorphic section of* $E$ *that is transverse to the zero section. Denote by* $Y$ *the zero locus of* $s$*. There is a unique way to assign to each* $(X, \overline{E}, s)$ *as before a class of currents*

$$\widetilde{e}(X, \overline{E}, s) \in \widetilde{\mathcal{D}}_D^{2r-1}(X, N^*_{Y,0}, r)$$

*satisfying the following properties*

(i) *(Differential equation)*

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{e}(X, \overline{E}, s) = c_r(\overline{E}) - \delta_Y. \tag{9.14}$$

(ii) *(Functoriality) If* $f\colon X' \longrightarrow X$ *is a morphism transverse to* $Y$ *then*

$$\widetilde{e}(X', f^*\overline{E}, f^*s) = f^*\widetilde{e}(X, \overline{E}, s). \tag{9.15}$$

(iii) *(Multiplicativity) Let* $\overline{E}_1$ *and* $\overline{E}_2$ *be hermitian holomorphic vector bundles, and let* $s_1$ *and* $s_2$ *be holomorphic sections of* $\overline{E}_1$ *and* $\overline{E}_2$ *respectively that are transverse to the zero section and with zero locus* $Y_1$ *and* $Y_2$*. We write* $\overline{E} = \overline{E}_1 \oplus \overline{E}_2$ *and* $s = s_1 \oplus s_2$*. Assume that* $s$ *is transverse to the zero section; hence* $Y_1$ *and* $Y_2$ *meet transversely. With this hypothesis we have*

$$\begin{aligned}\widetilde{e}(X, \overline{E}, s) &= \widetilde{e}(X, \overline{E}_1, s_1) \wedge c_{r_2}(\overline{E}_2) + \delta_{Y_1} \wedge \widetilde{e}(X, \overline{E}_2, s_2)\\ &= \widetilde{e}(X, \overline{E}_1, s_1) \wedge \delta_{Y_2} + c_{r_1}(\overline{E}_1) \wedge \widetilde{e}(X, \overline{E}_2, s_2).\end{aligned}$$

(iv) *(Line bundles) If* $\overline{L}$ *is a hermitian line bundle and* $s$ *is a section of* $L$*, then*

$$\widetilde{e}(X, \overline{L}, s) = -\log \|s\|. \tag{9.16}$$

*Proof.* Bismut, Gillet and Soulé prove the existence by constructing explicitly an Euler-Green current in the total space of $E$ and pulling it back to $X$ by the section $s$. For the uniqueness, first we see that properties (i) and (ii) imply that, if $h_0$ and $h_1$ are two hermitian metrics in $E$, then

$$\widetilde{e}(X,(E,h_0),s)-\widetilde{e}(X,(E,h_1),s)=\widetilde{c}_r(E,h_0,h_1). \tag{9.17}$$

We now consider $\pi\colon P=\mathbb{P}(E\oplus\mathbb{C})\longrightarrow X$, with the tautological exact sequence

$$0\longrightarrow \mathcal{O}(-1)\longrightarrow \pi^*E\oplus\mathbb{C}\longrightarrow Q\longrightarrow 0$$

On $Q$ we consider the metric induced by the metric of $\overline{E}$ and the trivial metric on the factor $\mathbb{C}$, and let $s_Q$ the section of $Q$ induced by the section 1 of $\mathbb{C}$. Let $D_\infty$ be as in lemma 9.4. Then properties (ii) to (iv) imply that $\widetilde{e}(P,\overline{Q},s_Q)|_{D_\infty}=0$. Hence by lemma 9.4 $\widetilde{e}$ is uniquely determined. Finally, let $f\colon X\longrightarrow P$ be the map given by $x\longmapsto(s(x):-1)$. Then $f^*Q\cong E$, although they are not necessarily isometric, and $f^*s_Q=s$. Therefore, the functoriality and equation (9.17) determine $\widetilde{e}(X,\overline{E},s)$.

To prove the existence, we use lemma 9.4, functoriality and equation (9.17) to define the Euler-Green classes. It is easy to show that they are well defined and satisfy properties (i) to (iv). □

Equation (9.8) relating homogeneous singular Bott-Chern classes and Euler-Green classes in a particular case can be generalized to arbitrary vector bundles.

Proposition 9.18. *Let $X$ be a complex manifold, $\overline{E}$ a hermitian vector bundle over $X$, $s$ a section of $E$ transversal to the zero section and $i\colon Y\longrightarrow X$ the zero locus of $s$. Let $K(\overline{E})$ be the Koszul resolution of $i_*\mathcal{O}_Y$ determined by $\overline{E}$ and $s$. We can identify $N_{Y/X}$ with $i^*E$. We denote by $\overline{N}_{Y/X}$ the vector bundle with the metric induced by the above identification. Then*

$$T^h(i,\overline{\mathcal{O}}_Y,\overline{N}_{Y/X},K(\overline{E}))=\widetilde{e}(X,\overline{E},s)\bullet\operatorname{Td}^{-1}(\overline{E}).$$

*Proof.* Let $P=\mathbb{P}(E\oplus\mathbb{C})$. We follow the notation of proposition 9.13. We denote by $h_0$ the original metric of $\overline{E}$ and by $h_1$ the metric induced by the isomorphism $E\cong f^*Q$. Observe that $h_0$ and $h_1$ agree when restricted to $Y$, because the preimage of $\overline{Q}$ by the zero section agrees with $\overline{E}$. Hence there is an isometry $\overline{N}_{Y/X}\cong i^*f^*\overline{Q}$. We denote $T^h(K(\overline{E}))=T^h(i,\overline{\mathcal{O}}_Y,\overline{N}_{Y/X},K(\overline{E}))$.

Then we have

$$\begin{aligned}
T^h(K(\overline{E})) &= f^*T^h(K(\overline{\mathcal{O}_X},\overline{E})) + \sum_i (-1)^i \widetilde{\mathrm{ch}}(\bigwedge^i E^\vee, h_0, h_1)\\
&= f^*(\widetilde{e}(P,\overline{Q},s_Q)\bullet \mathrm{Td}^{-1}(\overline{Q})) + \widetilde{c}_r(E,h_0,h_1)\bullet \mathrm{Td}^{-1}(E,h_1)\\
&\qquad + c_r(E,h_0)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(E,h_1) - \widetilde{c}_r(E,h_0,h_1)\bullet \mathrm{Td}^{-1}(E,h_1)\\
&\qquad + \widetilde{c}_r(E,h_0,h_1)\bullet \mathrm{Td}^{-1}(E,h_1) + c_r(E,h_0)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(E,h_0) - \widetilde{e}(X,\overline{E},s)\bullet \mathrm{d}_{\mathcal{D}}\widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&\qquad + c_r(E,h_0)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(E,h_0) - \mathrm{d}_{\mathcal{D}}\widetilde{e}(X,\overline{E},s)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&\qquad + c_r(E,h_0)\bullet \widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(E,h_0) + i_*\widetilde{\mathrm{Td}^{-1}}(E,h_0,h_1)|_Y\\
&= \widetilde{e}(X,\overline{E},s)\bullet \mathrm{Td}^{-1}(\overline{E}),
\end{aligned}$$

which concludes the proof. □

Theorem 9.19. *The theory of homogeneous singular Bott-Chern classes is compatible with the projection formula and transitive.*

*Proof.* We have

$$\begin{aligned}
C_{T^h}(F,N) &= (\pi_P)_*T^h(K(\overline{F},\overline{N}))\\
&= (\pi_P)_*(\widetilde{e}(P,\overline{Q},s)\bullet \mathrm{Td}^{-1}(\overline{Q})\bullet \mathrm{ch}(\pi_P^*\overline{F}))\\
&= (\pi_P)_*(\widetilde{e}(P,\overline{Q},s)\bullet \mathrm{Td}^{-1}(\overline{Q}))\bullet \mathrm{ch}(\overline{F})\\
&= C_{T^h}(\mathcal{O}_Y,N)\bullet \mathrm{ch}(F).
\end{aligned}$$

Thus $C_{T^h}$ is compatible with the projection formula.
We now prove the transitivity. Let $Y$, $N_1$ and $N_2$ be as in corollary 8.35. We follow the notation after this corollary. Then applying proposition 9.18 we obtain

$$T^h(\overline{K}) = \widetilde{e}(P,\pi_1^*\overline{Q}_1\oplus\pi_2^*\overline{Q}_2, s_1+s_2)\bullet \mathrm{Td}^{-1}(\pi_1^*\overline{Q}_1\oplus\pi_2^*\overline{Q}_2), \tag{9.20}$$

where $s_i$ denote the tautological section of $\overline{Q}_i$ or its preimage by $\pi_i$.
Then, by proposition 9.13 (iii), taking into account that $Y_1 = P_2$,

$$\begin{aligned}
T^h(\overline{K}) = \pi_1^*(c_{r_1}(\overline{Q}_1)\,\mathrm{Td}^{-1}(\overline{Q}_1))\bullet \pi_2^*(\widetilde{e}(P_2,\overline{Q}_2,s_2)\,\mathrm{Td}^{-1}(\overline{Q}_2))&\\
+ (i_1)_*(\widetilde{e}(P_1,\overline{Q}_1,s_1)\,\mathrm{Td}^{-1}(\overline{Q}_1)\bullet p_1^*\,\mathrm{Td}^{-1}(\overline{N}_2)).&
\end{aligned} \tag{9.21}$$

Applying again proposition 9.18 we obtain

$$T^h(\overline{K}) = \pi_1^*(c_{r_1}(\overline{Q}_1)\operatorname{Td}^{-1}(\overline{Q}_1)) \bullet \pi_2^*(T^h(\overline{K}_2)) + (i_1)_*(T^h(\overline{K}_1) \bullet p_1^* \operatorname{Td}^{-1}(\overline{N}_2)). \tag{9.22}$$

Thus, by corollary 8.35 the theory of homogeneous singular Bott-Chern classes is transitive. □

We next recall the construction of singular Bott-Chern classes of Bismut, Gillet and Soulé. Let $i\colon Y \longrightarrow X$ be a closed immersion of complex manifolds and let $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ be a hermitian embedded vector bundle. We consider the associated complex of sheaves

$$0 \to E_n \xrightarrow{v} \dots \xrightarrow{v} E_0 \to 0,$$

where we denote by $v$ the differential of this complex.
This complex is exact for all $x \in X \setminus Y$. The cohomology sheaves of this complex are holomorphic vector bundles on $Y$ which we denote by

$$H_n = \mathcal{H}_n(E_*|_Y), \quad H = \bigoplus_n H_n.$$

For each $x \in Y$ and $U \in T_x X$ we denote by $\partial_U v(x)$ the derivative of the map $v$ calculated in any holomorphic trivialization of $E$ near $x$. Then $\partial_U v(x)$ acts on $H_x$. Moreover, this action only depends on the class $y$ of $U$ in $N_x$. We denote it by $\partial_y v(x)$. Moreover $(\partial_y v(x))^2 = 0$; therefore the pull-back of $H$ to the total space of $N$ together with $\partial_y v$ is a complex that we denote by $(H, \partial_y v)$.
On the total space of $N$, the interior multiplication by $y \in N$ turns $\bigwedge N^\vee$ into a Koszul complex. By abuse of notation we denote also by $\iota_y$ the operator $\iota_y \otimes 1$ acting on $\bigwedge N^\vee \otimes F$. There is a canonical isomorphism between the complexes $(H, \partial_y v)$ and $(\bigwedge N^\vee \otimes F, \iota_y)$. An explicit description of this isomorphism can be found in [3] §1.
Let $v^*$ be the adjoint of the operator $v$ with respect to the metrics of $\overline{E}_*$. Then we have an identification of vector bundles over $Y$

$$H_k = \{f \in E_k \mid vf = v^* f = 0\}.$$

This identification induces a hermitian metric on $H_k$, and hence on $H$. Note that the metrics on $N$ and $F$ also induce a hermitian metric on $\bigwedge N^\vee \otimes F$.

Definition 9.23. We say that $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ satisfies Bismut assumption (A) if the canonical isomorphism between $(H, \partial_y v)$ and $(\bigwedge N^\vee \otimes F, \iota_y)$ is an isometry.

Proposition 9.24. *Let $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ be as before, with $\overline{N} = (N, h_N)$ and $\overline{F} = (F, h_F)$. Then there exist metrics $h'_{E_k}$ over $E_k$ such that the hermitian embedded vector bundle $\overline{\xi}' = (i, \overline{N}, \overline{F}, (E_*, h'_{E_*}))$ satisfies Bismut assumption (A).*

*Proof.* This is [3] proposition 1.6. □

Let $\nabla^E$ be the canonical hermitian holomorphic connection on $E$ and let $V = v + v^*$. Then

$$A_u = \nabla^E + \sqrt{u}V$$

is a superconnection on $E$.
Let $\nabla^H$ be the canonical hermitian connection on $H$. Then

$$B = \nabla^H + \partial_y v + (\partial_y v)^*$$

is a superconnection on $H$.
Let $N_H$ be the number operator on the complex $(E, v)$, that is, $N_H$ acts on $E_k$ by multiplication by $k$, and let $\mathrm{Tr}_s$ denote the supertrace. Recall that here we are using the symbol [ ] to denote the current associated to a locally integrable differential form and the symbol $\delta_Y$ to denote the current integration along a subvariety, both with the normalizations of notation 1.3.
For $0 < Re(s) \leq 1/2$ let $\zeta_E(s)$ be the current on $X$ given by the formula

$$\zeta_E(s) = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \Big\{ \left[\mathrm{Tr}_s\left(N_H \exp(-A_u^2)\right)\right] - i_* \left[\int_N \mathrm{Tr}_s\left(N_H \exp(-B^2)\right)\right] \Big\} \,\mathrm{d}\, u. \quad (9.25)$$

This current is well defined and extends to a current that depends holomorphically on $s$ near 0.

Definition 9.26. Assume that $\overline{\xi} = (i, \overline{N}, \overline{F}, \overline{E}_*)$ satisfies Bismut assumption (A). Then we denote

$$T^{BGS}(\overline{\xi}) = -\frac{1}{2}\zeta'_E(0).$$

By abuse of notation we will denote also by $T^{BGS}(\overline{\xi})$ its class in $\widetilde{\bigoplus}_p \widetilde{\mathcal{D}}_D^{2p-1}(X, p)$.
Let now $\overline{\xi} = (i, \overline{N}, \overline{F}, (E_*, h_{E_*}))$ be general and let $\overline{\xi}' = (i, \overline{N}, \overline{F}, (E_*, h'_{E_*}))$ be any hermitian embedded vector bundle satisfying assumption (A) provided by proposition 9.24. Then we denote

$$T^{BGS}(\overline{\xi}) = T^{BGS}(\overline{\xi}') + \sum_i (-1)^i \widetilde{\mathrm{ch}}(E_i, h_{E_i}, h'_{E_i}),$$

where $\widetilde{\mathrm{ch}}(E_i, h_{E_i}, h'_{E_i})$ is as in definition 2.13.

Remark 9.27. This definition only agrees (up to a normalization factor) with the definition in [6] for hermitian embedded vector bundles that satisfy assumption (A).

Theorem 9.28. *The assignment that, to each hermitian embedded vector bundle $\overline{\xi}$, associates the current $T^{BGS}(\overline{\xi})$, is a theory of singular Bott-Chern classes that agrees with $T^h$.*

*Proof.* First we have to show that, when $\overline{\xi}$ does not satisfy assumption (A) then $T^{BGS}(\overline{\xi})$ is well defined. Assume that $\overline{\xi}'' = (i, \overline{N}, \overline{F}, (E_*, h'_{E_*}))$ is another choice of hermitian embedded vector bundle satisfying assumption (A). By lemma 2.17 we have that

$$\widetilde{\mathrm{ch}}(E_i, h_i, h'_i) + \widetilde{\mathrm{ch}}(E_i, h'_i, h''_i) + \widetilde{\mathrm{ch}}(E_i, h''_i, h_i) = 0.$$

By [6] theorem 2.5 we have that

$$T^{BGS}(\overline{\xi}') - T^{BGS}(\overline{\xi}'') = \sum_i (-1)^i \widetilde{\mathrm{ch}}(E_i, h'_{E_i}, h''_{E_i}).$$

Summing up we obtain that $T^{BGS}(\overline{\xi})$ is well defined.
If the hermitian embedded vector bundle $\overline{\xi}$ satisfies Bismut assumption (A) then, by [6] theorem 1.9, $T^{BGS}(\overline{\xi})$ satisfies equation (6.10). If $\overline{\xi}$ does not satisfy assumption (A) then, combining [6] theorem 1.9 and equation (2.4), we also obtain that $T^{BGS}(\overline{\xi})$ satisfies equation (6.10).
The functoriality property is [6] theorem 1.10.
In order to prove the normalization property, let $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ be a hermitian embedded vector bundle that satisfies assumption (A) and let $\overline{A}$ be a non-negatively graded orthogonally split complex of vector bundles on $X$. Observe that $\overline{A}$ is also a (trivial) hermitian embedded vector bundle. Then $\overline{A}$ and $\overline{\xi} \oplus \overline{A}$ also satisfy assumption (A). By [6] theorem 2.9

$$T^{BGS}(\overline{\xi} \oplus \overline{A}) = T^{BGS}(\overline{\xi}) + T^{BGS}(\overline{A}).$$

But by [5] remark 2.3, $T^{BGS}(\overline{A})$ agrees with the Bott-Chern class associated to the Chern character and the exact complex $\overline{A}$. Since $A$ is orthogonally split we have $T^{BGS}(\overline{A}) = 0$. Now the case when $\xi$ does not satisfy assumption (A) follows from the definition.
By [6] theorem 3.17, with the hypothesis of proposition 9.18, we have that

$$\begin{aligned} T^{BGS}(i, \overline{\mathcal{O}}_Y, \overline{N}_{Y/X}, K(\overline{E})) &= \widetilde{e}(X, \overline{E}, s) \bullet \mathrm{Td}^{-1}(\overline{E}) \\ &= T^h(i, \overline{\mathcal{O}}_Y, \overline{N}_{Y/X}, K(\overline{E})). \end{aligned}$$

From this it follows that $C_{T^{BGS}} = C_{T^h}$ and by theorem 7.1, $T^{BGS} = T^h$. □

We now recall Zha's construction. Note that, in order to obtain a theory of singular Bott-Chern classes, we have changed the normalization convention from the one used by Zha. Note also that Zha does not define explicitly a singular Bott-Chern class, but such a definition is implicit in his definition of direct images for closed immersions. Let $Y$ be a complex manifold and let $\overline{N} = (N, h)$ be a hermitian vector bundle. We denote $P = \mathbb{P}(N \oplus \mathbb{C})$. Let

$\pi\colon P \longrightarrow Y$ denote the projection and let $\iota\colon Y \longrightarrow P$ denote the inclusion as the zero section. On $P$ we consider the tautological exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi^* N \oplus \mathcal{O}_P \longrightarrow Q \longrightarrow 0.$$

Let $h_1$ denote the hermitian metric on $Q^\vee$ induced by the metric of $N$ and the trivial metric on $\mathcal{O}_P$ and let $h_0$ denote the semi-definite hermitian form on $Q^\vee$ induced by the map $Q^\vee \longrightarrow \mathcal{O}_P$ obtained from the above exact sequence and the trivial metric on $\mathcal{O}_P$. Let $h_t = (1-t^2)h_0 + t^2 h_1$. It is a hermitian metric on $Q^\vee$. We will denote $\overline{Q}_t^\vee = (Q^\vee, h_t)$. Let $\nabla_t$ be the associated hermitian holomorphic connection and let $N_t$ denote the endomorphism defined by

$$\frac{\mathrm{d}}{\mathrm{d}\, t} \langle v, w\rangle_t = \langle N_t v, w\rangle .$$

For each $n \geq 1$, let Det denote the alternate $n$-linear form on the space of $n$ by $n$ matrices such that

$$\det(A) = \operatorname{Det}(A, \dots, A).$$

We denote $\det(B; A) = \operatorname{Det}(B, A, \dots, A)$.
Zha introduced the differential form

$$\widetilde{e}_Z(\overline{Q}^\vee) = \frac{-1}{2} \lim_{s\to 0} \int_s^1 \det(N_t, \nabla_t^2)\,\mathrm{d}\, t \tag{9.29}$$

which is a smooth form on $P \setminus \iota(Y)$, locally integrable on $P$. Hence it defines a current, also denoted by $\widetilde{e}_Z(\overline{Q}^\vee)$ on $P$. The important property of this current is that it satisfies

$$\mathrm{d}_{\mathcal{D}}\, \overline{e}_Z(Q^\vee) = c_n(\overline{Q}_1) - \delta_Y. \tag{9.30}$$

In [32], Zha denotes by $C(\overline{Q}^\vee)$ a form that differs from $\widetilde{e}_Z$ by the normalization factor and the sign. We denote it by $\widetilde{e}_Z$ because it agrees with the Euler-Green current introduced in [6].

Proposition 9.31. *The equality*

$$\widetilde{e}_Z(Q^\vee) = \widetilde{e}(P, \overline{Q}_1, s_Q)$$

*holds.*

*Proof.* With the notations of lemma 9.4, both classes satisfy equation (9.30) and their restriction to $D_\infty$ is zero. By lemma 9.4 they agree. □

Definition 9.32. Let $\overline{\xi} = (i\colon Y \longrightarrow X, \overline{N}, \overline{F}, \overline{E}_*)$ be as in definition 6.9. Let $\overline{A}_*$, $\operatorname{tr}_1(\overline{E})_*$ and $\overline{\eta}_*$ be as in (7.2). Then we define

$$\begin{aligned} T^Z(\overline{\xi}) = &-(p_W)_*\left(\sum_k (-1)^k W_1 \bullet \operatorname{ch}(\operatorname{tr}_1(\overline{E})_k)\right) \\ &- \sum_k (-1)^k (p_P)_*[\widetilde{\operatorname{ch}}(\overline{\eta}_k)] + (p_P)_*(\operatorname{ch}(\pi_p^* \overline{F}) \operatorname{Td}^{-1}(\overline{Q}_1) \widetilde{e}_Z(\overline{Q}_1^\vee)). \end{aligned} \tag{9.33}$$

It follows directly from the definition that $T^Z$ is the theory of singular Bott-Chern classes associated to the class

$$C_Z(F,N) = (p_P)_*(\mathrm{ch}(\pi_p^*\overline{F})\,\mathrm{Td}^{-1}(\overline{Q}_1)\widetilde{e}_Z(\overline{Q}_1^\vee)). \tag{9.34}$$

Theorem 9.35. *The theory of singular Bott-Chern classes $T^Z$ agrees with the theory of homogeneous singular Bott-Chern classes $T^h$.*

*Proof.* The result follows directly from theorem 7.1, equation (9.34) and proposition 9.18. □

Next we want to use 8.33 to give another characterization of $T^h$. To this end we only need to compute the characteristic class $C_{T^h}(\mathcal{O}_Y, L)$ for a line bundle $L$ as a power series in $c_1(L)$.

Theorem 9.36. *The theory of homogeneous singular Bott-Chern classes of algebraic vector bundles is the unique theory of singular Bott-Chern classes of algebraic vector bundles that is compatible with the projection formula and transitive and that satisfies*

$$C_{T^h}(\mathcal{O}_Y, L) = \mathbf{1}_1 \bullet \phi(c_1(L)),$$

*where $\phi$ is the power series*

$$\phi(x) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}H_{n+1}}{(n+2)!}x^n,$$

*and where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, $n \geq 1$ are the harmonic numbers.*

We already know that $T^h$ is compatible with the projection formula and transitive. Thus it only remains to compute the power series $\phi$.

Let $\overline{L} = (L, h_L)$ be a hermitian line bundle over a complex manifold $Y$. Let $z$ be a system of holomorphic coordinates of $Y$. Let $e$ be a local section of $L$ and let $h(z) = h(e_z, e_z)$. Let $P = \mathbb{P}(L \oplus \mathbb{C})$, with $\pi\colon P \longrightarrow Y$ the projection and $\iota\colon Y \longrightarrow P$ the zero section. We choose homogeneous coordinates on $P$ given by $(z, (x:y))$, here $(x:y)$ represents the line of $L_z \oplus \mathbb{C}$ generated by $xe(z) + y\mathbf{1}$, where $\mathbf{1}$ is a generator of $\mathbb{C}$ of norm 1. On the open set $y \neq 0$ we will use the absolute coordinate $t = x/y$. Let

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \pi^*(L \oplus \mathbb{C}) \longrightarrow Q \longrightarrow 0$$

be the tautological exact sequence. The section $s = \{\mathbf{1}\}$ is a global section of $Q$ that vanishes along the zero section. Moreover we have

$$\|s\|^2_{(z,(x:y))} = \frac{x\bar{x}h(z)}{y\bar{y} + x\bar{x}h(z)} = \frac{t\bar{t}h}{1 + t\bar{t}h}.$$

Then (recall that we are using the algebro-geometric normalization)

$$\begin{aligned}
c_1(\overline{Q}) &= \partial\bar\partial \log \|s\|^2 && (9.37)\\
&= \partial\bar\partial \log \frac{t\bar t h}{1+t\bar t h} && (9.38)\\
&= \partial\left(\frac{1+t\bar t h}{t\bar t h}\frac{t\bar\partial(\bar t h)(1+t\bar t h)-t^2\bar t h\bar\partial(\bar t h)}{(1+t\bar t h)^2}\right) && (9.39)\\
&= \partial\left(\frac{t\bar\partial(\bar t h)}{t\bar t h(1+t\bar t h)}\right) && (9.40)\\
&= \partial\left(\frac{\bar\partial(\bar t h)}{\bar t h}\right)\frac{1}{1+t\bar t h}-\frac{\bar t\partial(ht)\wedge\bar\partial(\bar t h)}{\bar t h(1+t\bar t h)^2} && (9.41)\\
&= \frac{\pi^* c_1(\overline{L})}{1+t\bar t h}-\frac{\partial(th)\wedge\bar\partial(\bar t h)}{h(1+t\bar t h)^2}. && (9.42)
\end{aligned}$$

We now consider the Koszul resolution

$$\overline{K}\colon 0 \longrightarrow Q^\vee \xrightarrow{\ s\ } \mathcal{O}_p \longrightarrow \iota_*\mathcal{O}_X \longrightarrow 0.$$

We denote by $T^h(\overline{K})$ the singular Bott-Chern class associated to this Koszul complex. Then, by proposition 9.13 and proposition 9.18,

$$T^h(\overline{K}) = -\frac{1}{2}\operatorname{Td}^{-1}(\overline{Q})\log\|s\|^2.$$

In order to compute $\pi_* T^h(\overline{K})$ we have to compute first $\pi_* c_1(\overline{Q})^n \log\|s\|^2$. But

$$c_1(\overline{Q})^n = \frac{\pi^* c_1(\overline{L})^n}{(1+t\bar t h)^n} - n\left(\frac{\pi^* c_1(\overline{L})}{(1+t\bar t h)}\right)^{n-1}\frac{\partial(th)\wedge\bar\partial(\bar t h)}{h(1+t\bar t h)^2}.$$

Therefore

$$\begin{aligned}
\pi_* c_1(\overline{Q})^n\log\|s\|^2 &= -nc_1(\overline{L})^{n-1}\frac{1}{2\pi i}\int_{\mathbb{P}^1}\frac{\partial(th)\wedge\bar\partial(\bar t h)}{h(1+t\bar t h)^{n+1}}\log\frac{t\bar t h}{1+t\bar t h}\\
&= -nc_1(\overline{L})^{n-1}\frac{1}{2\pi i}\int_0^{2\pi}\int_0^\infty \log\frac{r^2}{1+r^2}\frac{-2ir\,\mathrm{d}\,\theta\,\mathrm{d}\,r}{(1+r^2)^{n+1}}\\
&= nc_1(\overline{L})^{n-1}\int_0^1 \log(1-w)w^{n-1}\,\mathrm{d}\,w\\
&= -c_1(\overline{L})^{n-1}H_n,
\end{aligned}$$

where $H_n$, $n\geq 1$ are the harmonic numbers. Since

$$Td^{-1}(\overline{Q}) = \frac{1-\exp(-c_1(\overline{Q}))}{c_1(\overline{Q})} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}c_1(\overline{Q})^n,$$

we obtain

$$C_{T^h}(\mathcal{O}_Y, L) = \pi_* T^h(\overline{K}) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^{n+1}H_{n+1}}{(n+2)!} c_1(\overline{L})^n \mathbf{1}_1.$$

Then, a reformulation of proposition 8.31 is

Corollary 9.43. *Let $T$ be a theory of singular Bott-Chern classes for algebraic vector bundles that is compatible with the projection formula and transitive. Then there is a unique additive genus $S_T$ such that*

$$C_T(F, N) - C_{T^h}(F, N) = \mathrm{ch}(F) \bullet \mathrm{Td}(N)^{-1} \bullet S_T(N). \tag{9.44}$$

*Conversely, any additive genus determines a theory of singular Bott-Chern classes by the formula (9.44).*

## 10 The arithmetic Riemann-Roch theorem for regular closed immersions

In this section we recall the definition of arithmetic Chow groups and arithmetic $K$-groups. We see that each choice of an additive theory of singular Bott-Chern classes allows us to define direct images for closed immersions in arithmetic $K$-theory. Once the direct images for closed immersions are defined, we prove the arithmetic Grothendieck-Riemann-Roch theorem for closed immersions. A version of this theorem was proved earlier by Bismut, Gillet and Soulé [6] when there is a commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{i} & \mathcal{X} \\ & \searrow^{f} & \downarrow{\scriptstyle g} \\ & & \mathcal{Z} \end{array} ,$$

where $i$ is a closed immersion and $f$ and $g$ are smooth over $\mathbb{C}$. The version of this theorem given in this paper is due to Zha [32], but still unpublished. The theorem of Bismut, Gillet and Soulé compares $g_* \widehat{\mathrm{ch}}(i_*\overline{E})$ with $f_* \widehat{\mathrm{ch}}(\overline{E})$, whereas the theorem of Zha compares directly $\widehat{\mathrm{ch}}(i_*\overline{E})$ with $i_* \widehat{\mathrm{ch}}(\overline{E})$. The main difference between the theorem of Bismut, Gillet and Soulé and that of Zha is the kind of arithmetic Chow groups they use. In the first case these groups are only covariant for proper morphisms that are smooth over $\mathbb{C}$; thus the Grothendieck-Riemann-Roch can only be stated for a diagram as above, while in the second case a version of these groups that are covariant for arbitrary proper morphisms is used.

Since each choice of a theory of singular Bott-Chern classes gives rise to a different definition of direct images for closed immersions, the arithmetic Grothendieck-Riemann-Roch theorem will have a correction term that depends on the theory of singular Bott-Chern classes used. In the particular case of the

homogeneous singular Bott-Chern classes, which are the theories used by Bismut, Gillet and Soulé and by Zha, this correction term vanishes and we obtain the simplest formula. In this case the arithmetic Grothendieck-Riemann-Roch theorem is formally identical to the classical one.
Let $(A, \Sigma, F_\infty)$ be an arithmetic ring [18]. Since we will allow the arithmetic varieties to be non regular and we will use Chow groups indexed by dimension, following [20] we will assume that the ring $A$ is equidimensional and Jacobson. Let $F$ be the field of fractions of A. An *arithmetic variety* $\mathcal{X}$ is a scheme flat and quasi-projective over $A$ such that $\mathcal{X}_F = \mathcal{X} \times \operatorname{Spec} F$ is smooth. Then $X := \mathcal{X}_\Sigma$ is a complex algebraic manifold, which is endowed with an anti-holomorphic automorphism $F_\infty$. One also associates to $\mathcal{X}$ the real variety $X_\mathbb{R} = (X, F_\infty)$.
Following [13], to each regular arithmetic variety we can associate different kinds of arithmetic Chow groups. Concerning arithmetic Chow groups, we shall use the terminology and notation in op. cit. §4 and §6.
Let $\mathcal{D}_{\log}$ be the Deligne complex of sheaves defined in [13] section 5.3; we refer to op. cit. for the precise definition and properties. A $\mathcal{D}_{\log}$-*arithmetic variety* is a pair $(\mathcal{X}, \mathcal{C})$ consisting of an arithmetic variety $\mathcal{X}$ and a complex of sheaves $\mathcal{C}$ on $X_\mathbb{R}$ which is a $\mathcal{D}_{\log}$-complex (see op. cit. section 3.1).
We are interested in the following $\mathcal{D}_{\log}$-complexes of sheaves:

(i) The Deligne complex $\mathcal{D}_{\mathrm{l,a},X}$ of differential forms on $X$ with logarithmic and arbitrary singularities. That is, for every Zariski open subset $U$ of $X$, we write

$$E^*_{\mathrm{l,a},X}(U) = \varinjlim_{\overline{U}} \Gamma(\overline{U}, \mathscr{E}^*_{\overline{U}}(\log B)),$$

where the limit is taken over all diagrams

$$\begin{array}{ccc} U & \xrightarrow{\bar{\iota}} & \overline{U} \\ & \searrow^{\iota} & \downarrow{\scriptstyle\beta} \\ & & X \end{array}$$

such that $\bar{\iota}$ is an open immersion, $\beta$ is a proper morphism, $B = \overline{U} \setminus U$, is a normal crossing divisor and $\mathscr{E}^*_{\overline{U}}(\log B)$ denotes the sheaf of smooth differential forms on $U$ with logarithmic singularities along $B$ introduced in [8] .

For any Zariski open subset $U \subseteq X$, we put

$$\mathcal{D}^*_{\mathrm{l,a},X}(U,p) = (\mathcal{D}^*_{\mathrm{l,a},X}(U,p), \mathrm{d}_\mathcal{D}) = (\mathcal{D}^*(E_{\mathrm{l,a},X}(U), p), \mathrm{d}_\mathcal{D}).$$

If $U$ is now a Zariski open subset of $X_\mathbb{R}$, then we write

$$\mathcal{D}^*_{\mathrm{l,a},X}(U,p) = (\mathcal{D}^*_{\mathrm{l,a},X}(U,p), \mathrm{d}_\mathcal{D}) = (\mathcal{D}^*_{\mathrm{l,a},X}(U_\mathbb{C},p)^\sigma, \mathrm{d}_\mathcal{D}),$$

where $\sigma$ is the involution $\sigma(\eta) = \overline{F^*_\infty \eta}$ as in [13] notation 5.65.

Note that the sections of $\mathcal{D}^*_{\mathrm{l,a},X}$ over an open set $U \subset X$ are differential forms on $U$ with logarithmic singularities along $X \setminus U$ and arbitrary singularities along $\overline{X} \setminus X$, where $\overline{X}$ is an arbitrary compactification of $X$. Therefore the complex of global sections satisfy

$$\mathcal{D}^*_{\mathrm{l,a},X}(X,*) = \mathcal{D}^*(X,*),$$

where the right hand side complex has been introduced in section §1. The complex $\mathcal{D}^*_{\mathrm{l,a},X}$ is a particular case of the construction of [12] section 3.6.

(ii) The Deligne complex $\mathcal{D}_{\mathrm{cur},X}$ of currents on $X$. This is the complex introduced in [13] definition 6.30.

When $\mathcal{X}$ is regular, applying the theory of [13] we can define the arithmetic Chow groups $\widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$ and $\widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$. These groups satisfy the following properties

(i) There are natural morphisms

$$\widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$$

and, when applicable, all properties below will be compatible with these morphisms.

(ii) There is a product structure that turns $\widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})_{\mathbb{Q}}$ into an associative and commutative algebra. Moreover, it turns $\widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})_{\mathbb{Q}}$ into a $\widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})_{\mathbb{Q}}$-module.

(iii) If $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a map of regular arithmetic varieties, there are pull-back morphisms

$$f^*\colon \widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{\mathrm{CH}}^*(\mathcal{Y},\mathcal{D}_{\mathrm{l,a},Y}).$$

If moreover, $f$ is smooth over $F$, there are pull-back morphisms

$$f^*\colon \widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{cur},X}) \longrightarrow \widehat{\mathrm{CH}}^*(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y}).$$

The inverse image is compatible with the product structure.

(iv) If $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a proper map of regular arithmetic varieties of relative dimension $d$, there are push-forward morphisms

$$f_*\colon \widehat{\mathrm{CH}}^*(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{\mathrm{CH}}^{*-d}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X}).$$

If moreover, $f$ is smooth over $F$, there are push-forward morphisms

$$f_*\colon \widehat{\mathrm{CH}}^*(\mathcal{Y},\mathcal{D}_{\mathrm{l,a},Y}) \longrightarrow \widehat{\mathrm{CH}}^{*-d}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X}).$$

The push-forward morphism satisfies the projection formula and is compatible with base change.

(v) The groups $\widehat{\mathrm{CH}}^{*}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})$ are naturally isomorphic to the groups defined by Gillet and Soulé in [18] (see [12] theorem 3.33). When $X$ is generically projective, the groups $\widehat{\mathrm{CH}}^{*}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$ are isomorphic to analogous groups introduced by Kawaguchi and Moriwaki [27] and are very similar to the weak arithmetic Chow groups introduced by Zha (see [11]).

(vi) There are well-defined maps

$$\begin{aligned}\zeta\colon \widehat{\mathrm{CH}}^{p}(\mathcal{X}, \mathcal{C}) &\longrightarrow \mathrm{CH}^{p}(\mathcal{X}),\\ \mathrm{a}\colon \widetilde{\mathcal{C}}^{2p-1}(X_{\mathbb{R}}, p) &\longrightarrow \widehat{\mathrm{CH}}^{p}(\mathcal{X}, \mathcal{C}),\\ \omega\colon \widehat{\mathrm{CH}}^{p}(\mathcal{X}, \mathcal{C}) &\longrightarrow \mathrm{Z}\mathcal{C}^{2p}(X_{\mathbb{R}}, p),\end{aligned}$$

where $\mathcal{C}$ is either $\mathcal{D}_{\mathrm{l,a},X}$ or $\mathcal{D}_{\mathrm{cur},X}$. For the precise definition of these maps see [13] notation 4.12.

When $\mathcal{X}$ is not necessarily regular, following [20] and combining with the definition of [13] we can define the arithmetic Chow groups indexed by dimension $\widehat{\mathrm{CH}}_{*}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X})$ and $\widehat{\mathrm{CH}}_{*}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$ (see [12] section 5.3).
They have the following properties (see [20]).

(i) If $\mathcal{X}$ is regular and equidimensional of dimension $n$ then there are isomorphisms

$$\begin{aligned}\widehat{\mathrm{CH}}_{*}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}) &\cong \widehat{\mathrm{CH}}^{n-*}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}),\\ \widehat{\mathrm{CH}}_{*}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X}) &\cong \widehat{\mathrm{CH}}^{n-*}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X}).\end{aligned}$$

(ii) If $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a proper map between arithmetic varieties then there is a push-forward map

$$f_{*}\colon \widehat{\mathrm{CH}}_{*}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{\mathrm{CH}}_{*}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X}).$$

If $f$ is smooth over $F$ then there is a push-forward map

$$f_{*}\colon \widehat{\mathrm{CH}}_{*}(\mathcal{Y}, \mathcal{D}_{\mathrm{l,a},Y}) \longrightarrow \widehat{\mathrm{CH}}_{*}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}).$$

(iii) If $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a flat map or, more generally, a local complete intersection (l.c.i) map of relative dimension $d$, there are pull-back morphisms

$$f^{*}\colon \widehat{\mathrm{CH}}_{*}(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{\mathrm{CH}}_{*+d}(\mathcal{Y}, \mathcal{D}_{\mathrm{l,a},Y}).$$

If moreover, $f$ is smooth over $F$, there are pull-back morphisms

$$f^{*}\colon \widehat{\mathrm{CH}}_{*}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X}) \longrightarrow \widehat{\mathrm{CH}}_{*+d}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}).$$

(iv) If $f\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a morphism of arithmetic varieties with $\mathcal{X}$ regular, then there is a cap product

$$\widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{D}_{\mathrm{l,a},X}) \otimes \widehat{\mathrm{CH}}_d(\mathcal{Y}, \mathcal{D}_{\mathrm{l,a},Y}) \longrightarrow \widehat{\mathrm{CH}}_{d-p}(\mathcal{Y}, \mathcal{D}_{\mathrm{l,a},Y})_{\mathbb{Q}},$$

and a similar cap-product with the groups $\widehat{\mathrm{CH}}_d(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$. This product is denoted by $y \otimes x \mapsto y._f x$,

For more properties of these groups see [20].
We will define now the arithmetic $K$-groups in this context. As a matter of convention, in the sequel we will use slanted letters to denote a object defined over $A$ and the same letter in roman type for the corresponding object defined over $\mathbb{C}$. For instance we will denote a vector bundle over $\mathcal{X}$ by $\mathcal{E}$ and the corresponding vector bundle over $X$ by $E$.

Definition 10.1. A *hermitian vector bundle* on an arithmetic variety $\mathcal{X}$, $\overline{\mathcal{E}}$, is a locally free sheaf $\mathcal{E}$ with a hermitian metric $h_E$ on the vector bundle $E$ induced on $X$, that is invariant under $F_\infty$. A sequence of hermitian vector bundles on $\mathcal{X}$

$$(\overline{\varepsilon}) \qquad \dots \longrightarrow \overline{\mathcal{E}}_{n+1} \longrightarrow \overline{\mathcal{E}}_n \longrightarrow \overline{\mathcal{E}}_{n-1} \longrightarrow \dots$$

is said to be exact if it is exact as a sequence of vector bundles.
A *metrized coherent sheaf* is a pair $\overline{\mathcal{F}} = (\mathcal{F}, \overline{E}_* \to F)$, where $\mathcal{F}$ is a coherent sheaf on $\mathcal{X}$ and $\overline{E}_* \to F$ is a resolution of the coherent sheaf $F = \mathcal{F}_{\mathbb{C}}$ by hermitian vector bundles, that is defined over $\mathbb{R}$, hence is invariant under $F_\infty$. We assume that the hermitian metrics are also invariant under $F_\infty$.

Recall that to every hermitian vector bundle we can associate a collection of Chern forms, denoted by $c_p$. Moreover, the invariance of the hermitian metric under $F_\infty$ implies that the Chern forms will be invariant under the involution $\sigma$. Thus

$$c_p(\overline{\mathcal{E}}) \in \mathcal{D}^{2p}_{\mathrm{l,a},X}(X_{\mathbb{R}}, p) = \mathcal{D}^{2p}(X,p)^\sigma.$$

We will denote also by $c_p(\overline{\mathcal{E}})$ its image in $\mathcal{D}^{2p}_{\mathrm{cur},X}(X_{\mathbb{R}}, p)$. In particular we have defined the Chern character $\mathrm{ch}(\overline{\mathcal{E}})$ in either of the groups $\bigoplus_p \mathcal{D}^{2p}_{\mathrm{l,a},X}(X_{\mathbb{R}}, p)$ or $\bigoplus_p \mathcal{D}^{2p}_{\mathrm{cur},X}(X_{\mathbb{R}}, p)$. Moreover, to each finite exact sequence $(\overline{\varepsilon})$ of hermitian vector bundles on $\mathcal{X}$ we can attach a secondary Bott-Chern class $\widetilde{\mathrm{ch}}(\overline{\varepsilon})$. Again, the fact that the sequence is defined over $A$ and the invariance of the metrics with respect to $F_\infty$ imply that

$$\widetilde{\mathrm{ch}}(\overline{\varepsilon}) \in \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_{\mathrm{l,a},X}(X_{\mathbb{R}}, p) = \bigoplus_p \widetilde{\mathcal{D}}^{2p-1}(X,p)^\sigma.$$

We will denote also by $\widetilde{\mathrm{ch}}(\overline{\varepsilon})$ its image in $\bigoplus_p \widetilde{\mathcal{D}}^{2p-1}_{\mathrm{cur},X}(X_{\mathbb{R}}, p)$. The Bott-Chern classes associated to exact sequences of metrized coherent sheaves enjoy the same properties.

DEFINITION 10.2. Let $\mathcal{X}$ be an arithmetic variety and let $\mathcal{C}^*(*)$ be one of the two $\mathcal{D}_{\log}$-complexes $\mathcal{D}_{\mathrm{l,a},X}$ or $\mathcal{D}_{\mathrm{cur},X}$. The *arithmetic $K$-group* associated to the $\mathcal{D}_{\log}$-arithmetic variety $(\mathcal{X},\mathcal{C})$ is the abelian group $\widehat{K}(\mathcal{X},\mathcal{C})$ generated by pairs $(\overline{\mathcal{E}},\eta)$, where $\overline{\mathcal{E}}$ is a hermitian vector bundle on $\mathcal{X}$ and $\eta \in \bigoplus_{p\geq 0}\widetilde{\mathcal{C}}^{2p-1}(X_{\mathbb{R}},p)$, modulo relations

$$(\overline{\mathcal{E}}_1,\eta_1)+(\overline{\mathcal{E}}_2,\eta_2)=(\overline{\mathcal{E}},\tilde{\mathrm{ch}}(\overline{\varepsilon})+\eta_1+\eta_2) \tag{10.3}$$

for each short exact sequence

$$(\overline{\varepsilon}) \qquad 0\longrightarrow \overline{\mathcal{E}}_1 \longrightarrow \overline{\mathcal{E}} \longrightarrow \overline{\mathcal{E}}_2 \longrightarrow 0\ .$$

The *arithmetic $K'$-group* associated to the $\mathcal{D}_{\log}$-arithmetic variety $(\mathcal{X},\mathcal{C})$ is the abelian group $\widehat{K}'(\mathcal{X},\mathcal{C})$ generated by pairs $(\overline{\mathcal{F}},\eta)$, where $\overline{\mathcal{F}}$ is a metrized coherent sheaf on $\mathcal{X}$ and $\eta \in \bigoplus_{p\geq 0}\widetilde{\mathcal{C}}^{2p-1}(X_{\mathbb{R}},p)$, modulo relations

$$(\overline{\mathcal{F}}_1,\eta_1)+(\overline{\mathcal{F}}_2,\eta_2)=(\overline{\mathcal{F}},\tilde{\mathrm{ch}}(\overline{\varepsilon})+\eta_1+\eta_2) \tag{10.4}$$

for each short exact sequence of metrized coherent sheaves

$$(\overline{\varepsilon}) \qquad 0\longrightarrow \overline{\mathcal{F}}_1 \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}_2 \longrightarrow 0\ .$$

We now give some properties of the arithmetic $K$-groups. As their proofs are similar, in the essential points, to those of analogous statements in, for example, [18] in the regular case and [20] in the singular case, we omit them.

(i) We have natural morphisms

$$\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})\longrightarrow \widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X}) \text{ and } \widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})\longrightarrow \widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{cur},X}).$$

When applicable, all properties below will be compatible with these morphisms.

(ii) $\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$ is a ring. The product structure is given by

$$(\overline{\mathcal{F}}_1,\eta_1)\cdot(\overline{\mathcal{F}}_2,\eta_2)=(\overline{\mathcal{F}}_1\otimes\overline{\mathcal{F}}_2,\mathrm{ch}(\overline{\mathcal{F}}_1)\bullet\eta_2+\eta_1\bullet\mathrm{ch}(\overline{\mathcal{F}}_2)+\mathrm{d}_{\mathcal{D}}\,\eta_1\bullet\eta_2) \tag{10.5}$$

(iii) $\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$ is a $\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$-module.

(iv) There are natural maps

$$\widehat{K}(\mathcal{X},\mathcal{C})\longrightarrow \widehat{K}'(\mathcal{X},\mathcal{C})$$

that, when $\mathcal{X}$ is regular, are isomorphisms.

(v) The groups $\widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$ and $\widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$ are $\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$-modules.

(vi) There are natural maps

$$\omega\colon \widehat{K}'(\mathcal{X},\mathcal{C}) \longrightarrow \bigoplus_p Z\mathcal{C}^{2p}(p)$$

that send the class of a pair $(\overline{\mathcal{F}},\eta)$ with $\overline{\mathcal{F}}=(\mathcal{F},\overline{E}_*\to\mathcal{F}_{\mathbb{C}})$ to the form (or current)

$$\omega(\overline{\mathcal{F}},\eta)=\sum_i(-1)^i\operatorname{ch}(\overline{E}_i)+\mathrm{d}_{\mathcal{D}}\,\eta.$$

(vii) When $\mathcal{X}$ is regular, there exists a Chern character,

$$\widehat{\operatorname{ch}}\colon \widehat{K}(\mathcal{X},\mathcal{C})_{\mathbb{Q}} \longrightarrow \bigoplus_p \widehat{\operatorname{CH}}^p(\mathcal{X},\mathcal{C})_{\mathbb{Q}},$$

that is an isomorphism. Moreover, if $\mathcal{C}=\mathcal{D}_{\mathrm{l,a},X}$ this isomorphism is compatible with the product structure. If $\mathcal{X}$ is not regular, there is a biadditive pairing

$$\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})\otimes\widehat{\operatorname{CH}}_*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X}) \longrightarrow \widehat{\operatorname{CH}}_*(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})_{\mathbb{Q}},$$

and a similar pairing with the groups $\widehat{\operatorname{CH}}_*(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$, which is denoted in both cases by $\alpha\otimes x\mapsto\widehat{\operatorname{ch}}(\alpha)\cap x$. For the properties of this product see [20] pg. 496.

(viii) If $\mathcal{Y}$ and $\mathcal{X}$ are arithmetic varieties and $f\colon\mathcal{Y}\to\mathcal{X}$ is a morphism of arithmetic varieties, $f$ induces a morphism of rings:

$$f^*\colon \widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})\to\widehat{K}(\mathcal{Y},\mathcal{D}_{\mathrm{l,a},Y}).$$

When $f$ is flat, the inverse image is also defined for the groups $\widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$. Moreover, if $f_{\mathbb{C}}$ is smooth, the inverse image can be defined for the groups $\widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$ and, when in addition $f$ is flat, for the groups $\widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$.

In what follows we will be interested in direct images for closed immersions. Since the direct images in arithmetic $K$-theory will depend on the choice of a metric, we have the following

Definition 10.6. A *metrized arithmetic variety* is a pair $(\mathcal{X},h_X)$ consisting of an arithmetic variety $\mathcal{X}$ and a hermitian metric on the complex tangent bundle $T_X$ that is invariant under $F_\infty$.

Let $(\mathcal{X},h_X)$ and $(\mathcal{Y},h_Y)$ be metrized arithmetic varieties and let $i\colon\mathcal{Y}\longrightarrow\mathcal{X}$ be a closed immersion. Over the complex numbers, we are in the situation of notation 8.36. In particular we have a canonical exact sequence of hermitian vector bundles

$$\overline{\xi}_N\colon 0\longrightarrow\overline{T}_Y\longrightarrow i^*\overline{T}_X\longrightarrow\overline{N}_{Y/X}\longrightarrow 0 \qquad (10.7)$$

where the tangent bundles $T_Y$, $T_X$ are endowed with the hermitian metrics $h_Y$, $h_X$ respectively and the normal bundle $N_{Y/X}$ is endowed with an arbitrary hermitian metric $h_N$. We will follow the conventions of notation 8.36.

We next define push-forward maps, via a closed immersion, for the elements of the arithmetic $K$-group of a metrized arithmetic variety. We will define two kinds of push-forward maps. One will depend only on a metric on the complex normal bundle $N_{Y/X}$. By contrast, the second will depend on the choice of metrics on the complex tangent bundles $T_X$ and $T_Y$. The second definition allows us to see $K'(\_, \mathcal{D}_{\mathrm{cur},Y})$ as a functor from the category whose objects are metrized arithmetic varieties and whose morphisms are closed immersions to the category of abelian groups.

As we deal with hermitian vector bundles and metrized coherent sheaves, both definitions will involve the choice of a theory of singular Bott-Chern classes. In order for the push forward to be well defined in $K$-theory we need a minimal additivity property for the singular Bott-Chern classes.

Definition 10.8. A theory of singular Bott-Chern classes $T$ is called *additive* if for any closed embedding of complex manifolds $i\colon Y \hookrightarrow X$ and any hermitian embedded vector bundles $\overline{\xi}_1 = (i, \overline{N}, \overline{F}_1, \overline{E}_{1,*})$, $\overline{\xi}_2 = (i, \overline{N}, \overline{F}_2, \overline{E}_{2,*})$ the equation

$$T(\overline{\xi}_1 \oplus \overline{\xi}_2) = T(\overline{\xi}_1) + T(\overline{\xi}_2)$$

is satisfied.

Let $C$ be a characteristic class for pairs of vector bundles. We say that it is *additive* (in the first variable) if

$$C(F_1 \oplus F_2, N) = C(F_1, N) + C(F_2, N)$$

for any vector bundles $F_1, F_2, N$ on a complex manifold $X$.

The following statement follows directly from equation 7.5:

Proposition 10.9. *A theory of singular Bott-Chern classes $T$ is additive if and only if the corresponding characteristic class $C_T$ is additive in the first variable.*

Note that a theory of singular Bott-Chern classes consists in joining theories of singular Bott-Chern classes in arbitrary rank and codimension (definition 6.9). The property of being additive gives a compatibility condition for these theories, by respect to the hermitian vector bundles $\overline{F}$ (with the notation used in definition 6.9). Note also that if a theory of singular Bott-Chern classes is compatible with the projection formula then it is additive.

Definition 10.10. Let $T$ be an additive theory of singular Bott-Chern classes, and let $T_c$ be the associated covariant class as in definition 8.37. Let $i\colon (\mathcal{Y}, h_Y) \longrightarrow (\mathcal{X}, h_X)$ be a closed immersion of metrized arithmetic varieties and let $\overline{N} = \overline{N}_{Y/X} = (N_{Y/X}, h_N)$ be a choice of a hermitian metric on the complex normal bundle. The *push-forward maps*

$$i_*^{T_c}, i_*^{T} \colon \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{K}'(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$$

are defined by

$$\begin{aligned} i_*^{T_c}(\overline{\mathcal{F}},\eta) &= [((i_*\mathcal{F},\overline{E}_*\to(i_*\mathcal{F})_{\mathbb{C}}),0)]-[(0,T_c(\overline{\xi}_c))] \\ &\qquad +[(0,i_*(\eta\operatorname{Td}(Y)i^*\operatorname{Td}^{-1}(X)))] \qquad (10.11)\\ i_*^{T}(\overline{\mathcal{F}},\eta) &= [((i_*\mathcal{F},\overline{E}_*\to(i_*\mathcal{F})_{\mathbb{C}}),0)]-[(0,T(\overline{\xi}))] \\ &\qquad +[(0,i_*(\eta\operatorname{Td}^{-1}(\overline{N}_{Y/X})))]. \qquad (10.12)\end{aligned}$$

Here

$$0\to\overline{E}_n\to\dots\to\overline{E}_1\to\overline{E}_0\to(i_*\mathcal{F})_{\mathbb{C}}\to 0$$

is a finite resolution of the coherent sheaf $(i_*\mathcal{F})_{\mathbb{C}}$ by hermitian vector bundles, $\overline{\xi}=(i,\overline{N}_{X/Y},\overline{\mathcal{F}}_{\mathbb{C}},\overline{E}_*)$ is the induced hermitian embedded vector bundle on $X$, and $\overline{\xi}_c=(i,\overline{T}_X,\overline{T}_Y,\overline{\mathcal{F}}_{\mathbb{C}},\overline{E}_*)$ as in definition 8.37.
We can extend this definition to push-forward maps

$$i_*^{T_c},i_*^{T}\colon \widehat{K}'(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y})\longrightarrow\widehat{K}'(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$$

by the rule

$$\begin{aligned} i_*^{T_c}(\overline{\mathcal{F}},\eta) &= [((i_*\mathcal{F},\operatorname{Tot}(\overline{E}_{*,*})\to(i_*\mathcal{F})_{\mathbb{C}}),0)]-\sum_i(-1)^i[(0,T_c(\overline{\xi}_{i,c}))] \\ &\qquad +[(0,i_*(\eta\operatorname{Td}(Y)i^*\operatorname{Td}^{-1}(X)))], \qquad (10.13)\\ i_*^{T}(\overline{\mathcal{F}},\eta) &= [((i_*\mathcal{F},\operatorname{Tot}(\overline{E}_{*,*})\to(i_*\mathcal{F})_{\mathbb{C}}),0)]-\sum_i(-1)^i[(0,T(\overline{\xi}_i))] \\ &\qquad +[(0,i_*(\eta\operatorname{Td}^{-1}(\overline{N}_{Y/X})))], \qquad (10.14)\end{aligned}$$

where $0\to\overline{E}_n\to\dots\to\overline{E}_0\to\mathcal{F}_{\mathbb{C}}\to 0$ is a resolution of $\mathcal{F}_{\mathbb{C}}$ by hermitian vector bundles, $\overline{E}_{*,*}$ is a complex of complexes of vector bundles over $X$, such that, for each $i\geq 0$, $\overline{E}_{i,*}\to i_*E_i$ is also a resolution by hermitian vector bundles and $\overline{\xi}_i=(i,\overline{N}_{X/Y},\overline{E}_i,\overline{E}_{i,*})$ is the induced hermitian embedded vector bundle and $\overline{\xi}_{i,c}$ is as in definition 8.37. We suppose that there is a commutative diagram of resolutions

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & E_{k+1,*} & \longrightarrow & E_{k,*} & \longrightarrow & E_{k-1,*} & \longrightarrow & \cdots\ . \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & i_*E_{k+1} & \longrightarrow & i_*E_k & \longrightarrow & i_*E_{k-1} & \longrightarrow & \cdots \end{array}$$

hence a resolution $\operatorname{Tot}(\overline{E}_{*,*})\longrightarrow(i_*\mathcal{F})_{\mathbb{C}}$ by hermitian vector bundles.

Note that, whenever the push-forward $i_*^T$ appears, we will assume that we have chosen a metric on $N_{Y/X}$.
The two push-forward maps are related by the equation

$$i_*^{T_c}(\overline{\mathcal{F}},\eta)=i_*^{T}(\overline{\mathcal{F}},\eta)-\left[\left(0,i_*\left(\omega(\overline{\mathcal{F}},\eta)\widetilde{\operatorname{Td}^{-1}}(\overline{\xi}_N)\operatorname{Td}(Y)\right)\right)\right], \qquad (10.15)$$

where $\overline{\xi}_N$ is the exact sequence (10.7).

Proposition 10.16. *The push-forward maps $i_*^T$, $i_*^{T_c}$ are well defined. That is, they do not depend on the choice of a representative of a class in $\widehat{K}$, nor on the choice of metrics on the coherent sheaf $(i_*\mathcal{F})_{\mathbb{C}}$. The first one does not depend on the choice of metrics on $T_X$ nor on $T_Y$, whereas the second one does not depend on the choice of a metric on the normal bundle $N_{Y/X}$. Moreover, if $i$ is a regular closed immersion or $\mathcal{X}$ is a regular arithmetic variety, then $i_*^{T_c}$ and $i_*^T$ can be lifted to maps*

$$i_*^{T_c}, i_*^T\colon \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},Y}).$$

*Proof.* The fact that $i_*^T$ only depends on the metric on $\overline{N}$ and not on the metrics on $T_X$ and $T_Y$ and that for $i_*^{T_c}$ is the opposite, follows directly from the definition in the first case and from proposition 8.39 in the second.
We will only prove the other statements for $i_*^{T_c}$, as the other case is analogous. We first prove the independence from the metric chosen on the coherent sheaf $(i_*\mathcal{F})_{\mathbb{C}}$. If $\overline{E}_* \to (i_*\mathcal{F})_{\mathbb{C}}$, $\overline{E}'_* \to (i_*\mathcal{F})_{\mathbb{C}}$ are two such metrics, inducing the hermitian embedded vector bundles $\overline{\xi}$ respectively $\overline{\xi}'$, then, using corollary 6.14

$$T_c(\overline{\xi}'_c) - T_c(\overline{\xi}_c) = T(\overline{\xi}') - T(\overline{\xi}) = \widetilde{\mathrm{ch}}(\overline{\varepsilon}),$$

where $\overline{\varepsilon}$ is the exact complex of hermitian embedded vector bundles

$$\overline{\varepsilon}\colon 0 \longrightarrow \overline{\xi} \longrightarrow \overline{\xi}' \longrightarrow 0,$$

where $\overline{\xi}'$ sits in degree zero.
Therefore, by equation 10.4,

$$\begin{aligned}&[((i_*\mathcal{F}, \overline{E}_* \to (i_*\mathcal{F})_{\mathbb{C}}), 0)] - [(0, T_c(\overline{\xi_c}))]\\ &\qquad\qquad = [((i_*\mathcal{F}, \overline{E}'_* \to (i_*\mathcal{F})_{\mathbb{C}}), 0)] - [(0, T_c(\overline{\xi}'_c))].\end{aligned}$$

Since the last term of equation 10.11 does not depend on the metric on $(i_*\mathcal{F})_{\mathbb{C}}$, we obtain that $i_*^{T_c}$ does not depend on this metric.
For proving that the push-forward map $i_*^{T_c}$ is well defined it remains to show the independence from the choice of a representative of a class in $\widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$. We consider an exact sequence of hermitian vector bundles on $\mathcal{Y}$

$$\overline{\varepsilon}\colon 0 \longrightarrow \overline{\mathcal{F}}_1 \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}_2 \longrightarrow 0$$

and two classes $\eta_1, \eta_2 \in \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-1}_{\mathrm{cur}}(Y, p)$. We also denote $\overline{\varepsilon}$ the induced exact sequence of hermitian vector bundles on $Y$. We have to prove

$$i_*^{T_c}([(\overline{\mathcal{F}}, \eta_1 + \eta_2 + \widetilde{\mathrm{ch}}(\overline{\varepsilon})]) = i_*^{T_c}([(\overline{\mathcal{F}_1}, \eta_1)]) + i_*^{T_c}([(\overline{\mathcal{F}_2}, \eta_2)]). \tag{10.17}$$

Since it is clear that $i_*^{T_c}(0, \eta_1 + \eta_2) = i_*^{T_c}(0, \eta_1) + i_*^{T_c}(0, \eta_2)$, we are led to prove

$$i_*^{T_c}([(\overline{\mathcal{F}}, \widetilde{\mathrm{ch}}(\overline{\varepsilon})]) = i_*^{T_c}([(\overline{\mathcal{F}_1}, 0)]) + i_*^{T_c}([(\overline{\mathcal{F}_2}, 0)]). \tag{10.18}$$

We choose metrics on the coherent sheaves $(i_*\mathcal{F}_1)_{\mathbb{C}}$, $(i_*\mathcal{F}_2)_{\mathbb{C}}$ and $(i_*\mathcal{F})_{\mathbb{C}}$ respectively:

$$\overline{E}_{1,*} \longrightarrow (i_*\mathcal{F}_1)_{\mathbb{C}}\ ,\ \overline{E}_{2,*} \longrightarrow (i_*\mathcal{F}_2)_{\mathbb{C}}\ ,\ \overline{E}_{*} \longrightarrow (i_*\mathcal{F})_{\mathbb{C}}.$$

We denote $\overline{\xi}_1, \overline{\xi}_2, \overline{\xi}$ the induced hermitian embedded vector bundles. We obtain an exact sequence of metrized coherent sheaves on $\mathcal{X}$:

$$\overline{\nu}\colon 0 \longrightarrow \overline{i_*\mathcal{F}_1} \longrightarrow \overline{i_*\mathcal{F}} \longrightarrow \overline{i_*\mathcal{F}_2} \longrightarrow 0.$$

Then, using the fact that the theory $T$ is additive and equation (8.42) we have

$$T_c(\overline{\xi}_{1,c}) + T_c(\overline{\xi}_{2,c}) - T_c(\overline{\xi}_c) = [\widetilde{\operatorname{ch}}(\overline{\nu})] - i_*([\widetilde{\operatorname{ch}}(\overline{\varepsilon}) \bullet \operatorname{Td}(Y)]) \bullet \operatorname{Td}^{-1}(X). \quad (10.19)$$

Moreover, by the relation (10.4),

$$[(\overline{i_*\mathcal{F}_1}, 0)] + [(\overline{i_*\mathcal{F}_2}, 0)] = [(\overline{i_*\mathcal{F}}, \widetilde{\operatorname{ch}}(\overline{\nu}))]. \quad (10.20)$$

Hence, we compute,

$$\begin{aligned} i_*^{T_c}([(\overline{\mathcal{F}}, \widetilde{\operatorname{ch}}(\overline{\varepsilon})]) &- i_*^{T_c}([(\overline{\mathcal{F}_1}, 0)]) - i_*^{T_c}([(\overline{\mathcal{F}_2}, 0)]) \\ &= [(i_*\overline{\mathcal{F}}, 0)] - [(i_*\overline{\mathcal{F}_1}, 0)] - [(i_*\overline{\mathcal{F}_2}, 0)] \\ &\quad - [(0, T_c(\overline{\xi}_c))] + [(0, T_c(\overline{\xi_{1,c}}))] + [(0, T_c(\overline{\xi_{2,c}}))] \\ &\quad + [(0, i_*([\widetilde{\operatorname{ch}}(\overline{\varepsilon})] \bullet \operatorname{Td}(Y) \bullet i^* \operatorname{Td}^{-1}(X)))] \\ &= -[(0, i_*([\widetilde{\operatorname{ch}}(\overline{\varepsilon})] \bullet \operatorname{Td}(Y) \bullet i^* \operatorname{Td}^{-1}(X))))] \\ &\quad + [(0, i_*([\widetilde{\operatorname{ch}}(\overline{\varepsilon})] \bullet \operatorname{Td}(Y) \bullet i^* \operatorname{Td}^{-1}(X))))] \\ &= 0. \end{aligned}$$

The proof that $i_*^{T_c}$ for metrized coherent sheaves is well defined is similar. The proof of its independence from choice of a metric on $N_{Y/X}$ or from the choice of the resolutions and metrics in $X$ is the same as before. Now let

$$0 \longrightarrow \overline{\mathcal{F}}' \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}'' \longrightarrow 0$$

be a short exact sequence of metrized coherent sheaves on $\mathcal{Y}$. This means that we have resolutions $\overline{E}'_* \to \mathcal{F}'_{\mathbb{C}}$, $\overline{E}_* \to \mathcal{F}_{\mathbb{C}}$ and $\overline{E}''_* \to \mathcal{F}''_{\mathbb{C}}$. Using theorem 2.24 we can suppose that there is a commutative diagram of resolutions

$$\begin{array}{ccccccccc} 0 & \to & \overline{E}'_* & \to & \overline{E}_* & \to & \overline{E}''_* & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{F}'_{\mathbb{C}} & \to & \mathcal{F}_{\mathbb{C}} & \to & \mathcal{F}''_{\mathbb{C}} & \to & 0, \end{array} \quad (10.21)$$

with exact rows. Moreover, we can assume that the complexes of complexes $\overline{E}'_{*,*}$, $\overline{E}_{*,*}$, $\overline{E}''_{*,*}$ used in definition 10.10 are chosen compatible with diagram (10.21). Thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Tot}\overline{E}'_{*,*} & \to & \operatorname{Tot}\overline{E}_{*,*} & \to & \operatorname{Tot}\overline{E}''_{*,*} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & i_*\mathcal{F}'_{\mathbb{C}} & \to & i_*\mathcal{F}_{\mathbb{C}} & \to & i_*\mathcal{F}''_{\mathbb{C}} & \to & 0. \end{array} \quad (10.22)$$

We denote by $\overline{\nu}$ the exact sequence of metrized coherent sheaves on $X$ defined by diagram (10.22). We denote $\overline{\chi}_i$ the exact sequence of hermitian vector bundles on $Y$

$$\overline{\chi}_i\colon 0 \longrightarrow \overline{E}'_i \longrightarrow \overline{E}_i \longrightarrow \overline{E}''_i \longrightarrow 0,$$

and by $\overline{\varepsilon}$ the exact sequence of metrized coherent sheaves on $X$

$$\overline{\varepsilon}_i\colon 0 \longrightarrow \overline{i_*E}'_i \longrightarrow \overline{i_*E_i} \longrightarrow \overline{i_*E}''_i \longrightarrow 0.$$

Moreover, let $\overline{\xi}_i$, $\overline{\xi}'_i$ and $\overline{\xi}''_i$ denote the hermitian embedded vector bundles defined by the above resolutions and $\overline{E}_i$, $\overline{E}'_i$ and $\overline{E}''_i$ respectively and let $\overline{\xi}_{i,c}$, $\overline{\xi}'_{i,c}$ and $\overline{\xi}''_{i,c}$ be as in definition 8.37. Then, using proposition 2.38 and equation (8.42) we obtain

$$\begin{aligned}\widetilde{\mathrm{ch}}(\overline{\nu}) &= \sum_i (-1)^i \widetilde{\mathrm{ch}}(\overline{\varepsilon})\\ &= \sum_i (-1)^i (T_c(\overline{\xi}'_{i,c}) + T_c(\overline{\xi}''_{i,c}) - T_c(\overline{\xi}_{i,c}))\\ &\quad + \sum_i (-1)^i i_*(\widetilde{\mathrm{ch}}(\overline{\chi}_i) \bullet \mathrm{Td}(Y)) \bullet \mathrm{Td}^{-1}(X)\end{aligned} \tag{10.23}$$

Now the proof follows as before, but using equation (10.23) instead of equation (10.19).

If $\mathcal{X}$ is a regular arithmetic variety, the lifting property follows from the isomorphism between the $\widehat{K}$-groups and the $\widehat{K}'$-groups.

Suppose now that $i\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a regular closed immersion and let $[\overline{\mathcal{F}}, \eta] \in \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$. Then it follows from [2] III that the coherent sheaf $i_*\mathcal{F}$ can be resolved

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \dots \longrightarrow \mathcal{E}_0 \longrightarrow i_*\mathcal{F} \longrightarrow 0$$

with $\mathcal{E}_i$ locally free sheaves on $\mathcal{X}$. Moreover we endow the vector bundles $E_i$ induced on $X$ with hermitian metrics and so we obtain a metric on the coherent sheaf $i_*\mathcal{F}$ and the corresponding hermitian embedded vector bundle $\overline{\xi}$. Using the independence from the resolutions and on the metrics we see that the equation 10.11 defines an element in $\widehat{K}(\mathcal{X}, \mathcal{D}_{\mathrm{cur},X})$. □

Proposition 10.24. *For any element $\alpha \in \widehat{K}'(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$ we have*

$$\omega(i_*^{T_c}(\alpha))\,\mathrm{Td}(X) = i_*(\omega(\alpha)\,\mathrm{Td}(Y)) \tag{10.25}$$

$$\omega(i_*^{T}(\alpha)) = i_*(\omega(\alpha)\,\mathrm{Td}^{-1}(N_{Y/X})) \tag{10.26}$$

*Proof.* We will prove the statement only for $i_*^{T_c}$. We consider first a class of the form $[\overline{\mathcal{F}}, 0]$. Using equation (8.38) we obtain, after choosing a metric $\overline{E}_i \longrightarrow (i_*\mathcal{F})_{\mathbb{C}}$, and considering the induced hermitian embedded vector bundle

$\overline{\xi_c}$:

$$\begin{aligned}\omega(i_*^{T_c}([\overline{\mathcal{F}},0]))\operatorname{Td}(X) &= \Big(\sum(-1)^i \operatorname{ch}(\overline{E_i}) - \mathrm{d}_{\mathcal{D}}\, T_c(\overline{\xi_c})\Big)\operatorname{Td}(X)\\ &= i_*(\operatorname{ch}(\overline{F}) \bullet \operatorname{Td}(Y) \bullet i^* \operatorname{Td}^{-1}(X) i^*(\operatorname{Td}(X)))\\ &= i_*(\operatorname{ch}(\overline{F}) \bullet \operatorname{Td}(Y))\\ &= i_*(\omega([\overline{\mathcal{F}},0])\operatorname{Td}(Y))\end{aligned}$$

Taking now a class of the form $[0,\eta]$ we obtain:

$$\begin{aligned}\omega(i_*^{T_c}([0,\eta]))\operatorname{Td}(X) &= \mathrm{d}_{\mathcal{D}}\left(i_*(\eta \operatorname{Td}(Y) i^* \operatorname{Td}^{-1}(X))\right)\operatorname{Td}(X)\\ &= i_*\,\mathrm{d}_{\mathcal{D}}(\eta\operatorname{Td}(Y))\\ &= i_*(\omega([0,\eta])\operatorname{Td}(Y))\end{aligned}$$

and hence the equality 10.25 is proved. □

The next proposition explains the terminology "compatible with the projection formula" and "transitive" that we used for theories of singular Bott-Chern classes. The second statement is the main reason to introduce the push-forward $i_*^{T_c}$.

Proposition 10.27. *If the theory of singular Bott-Chern classes is compatible with the projection formula, we have that, for $\alpha \in \widehat{K}'(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y})$ and $\beta \in \widehat{K}(\mathcal{X},\mathcal{D}_{\mathrm{l,a},X})$ the following equalities hold*

$$\begin{aligned}i_*^{T_c}(\alpha i^*\beta) &= i_*^{T_c}(\alpha)\beta,\\ i_*^{T}(\alpha i^*\beta) &= i_*^{T}(\alpha)\beta.\end{aligned}$$

*If moreover the theory of singular Bott-Chern classes is transitive and $j\colon(\mathcal{Z},h_Z)\longrightarrow(\mathcal{Y},h_Y)$ is another closed immersion of metrized arithmetic varieties, then*

$$(i\circ j)_*^{T_c} = i_*^{T_c}\circ j_*^{T_c}.$$

*Proof.* We prove first the projection formula. For simplicity we will treat the case when $\alpha \in \widehat{K}(\mathcal{Y},\mathcal{D}_{\mathrm{cur},Y})$. Let $\alpha = (\overline{\mathcal{F}},\eta)$, let $\overline{\xi_c} = (i,\overline{T}_X,\overline{T}_Y,\overline{\mathcal{F}}_{\mathbb{C}},\overline{E}_*)$ be a hermitian embedded vector bundle and let $\beta = (\overline{\mathcal{E}},\chi)$. Using equations (10.11) and (10.5), we obtain

$$\begin{aligned}i_*^{T_c}(\alpha i^*\beta) - i_*^{T_c}(\alpha)\beta &= -\sum_i(-1)^i\operatorname{ch}(\overline{E}_i)\bullet\chi + \mathrm{d}_{\mathcal{D}}(T_c(\overline{\xi}_c))\bullet\chi\\ &\quad + i_*(\operatorname{ch}((\overline{\mathcal{F}})_{\mathbb{C}})\bullet\operatorname{Td}(Y)))\bullet\operatorname{Td}^{-1}(X)\bullet\chi\\ &\quad + T_c(\overline{\xi}_c)\bullet\operatorname{ch}(\overline{\mathcal{E}}_{\mathbb{C}}) - T_c(\overline{\xi}_c\otimes\overline{\mathcal{E}}_{\mathbb{C}})\\ &= T_c(\overline{\xi}_c\otimes\overline{\mathcal{E}}_{\mathbb{C}}) - T_c(\overline{\xi}_c)\bullet\operatorname{ch}(\overline{\mathcal{E}}_{\mathbb{C}}).\end{aligned}$$

Therefore, if $T$ is compatible with the projection formula, then the projection formula holds.
The fact that, if moreover $T$ is transitive then $(i\circ j)_*^{T_c} = i_*^{T_c}\circ j_*^{T_c}$ follows directly from the definition and equation (8.41). □

If $i\colon \mathcal{Y} \longrightarrow \mathcal{X}$ is a regular closed immersion between arithmetic varieties, then the normal cone $\mathcal{N}_{\mathcal{Y}/\mathcal{X}}$ is a locally free sheaf. The choice of a hermitian metric on $N_{Y/X}$ determines a hermitian vector bundle $\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}}$. If now $i\colon (\mathcal{Y}, h_Y) \longrightarrow (\mathcal{X}, h_X)$ is a closed immersion between regular metrized arithmetic varieties, then the tangent bundles $\mathcal{T}_{\mathcal{Y}}$ and $\mathcal{T}_{\mathcal{X}}$ are virtual vector bundles. Since over $\mathbb{C}$ they define vector bundles, we can provide them with hermitian metrics and denote the hermitian virtual vector bundles by $\overline{\mathcal{T}}_{\mathcal{X}}$ and $\overline{\mathcal{T}}_{\mathcal{Y}}$. There are well defined clases $\widehat{\mathrm{Td}}(\mathcal{Y}) = \widehat{\mathrm{Td}}(\overline{\mathcal{T}}_{\mathcal{Y}})$ and $\widehat{\mathrm{Td}}(\mathcal{X}) = \widehat{\mathrm{Td}}(\overline{\mathcal{T}}_{\mathcal{X}})$.
The arithmetic Grothendieck-Riemann-Roch theorem for closed immersions compares the direct images in the arithmetic $K$-groups with the direct images in the arithmetic Chow groups.

Theorem 10.28 ([6], [32]). *Let $T$ be a theory of singular Bott-Chern classes and let $S_T$ be the additive genus of corollary 9.43.*

*(i) Let $i\colon \mathcal{Y} \longrightarrow \mathcal{X}$ be a regular closed immersion between arithmetic varieties. Assume that we have chosen a hermitian metric on the complex bundle $N_{Y/X}$. Then, for any $\alpha = (\overline{\mathcal{F}}, \eta) \in \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$ the equation*

$$\widehat{\mathrm{ch}}(i_*^T(\alpha)) = i_*(\widehat{\mathrm{ch}}(\alpha)\widehat{\mathrm{Td}}^{-1}(\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}})) - \mathrm{a}(i_*(\mathrm{ch}(\mathcal{F}_{\mathbb{C}})\,\mathrm{Td}^{-1}(N_{Y/X})S_T(N)) \tag{10.29}$$

*holds.*

*(ii) Let $i\colon (\mathcal{Y}, h_Y) \longrightarrow (\mathcal{X}, h_X)$ be a closed immersion between regular metrized arithmetic varieties. Then, for any $\alpha = (\overline{\mathcal{F}}, \eta) \in \widehat{K}(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y})$ the equation*

$$\widehat{\mathrm{ch}}(i_*^{T_c}(\alpha))\widehat{\mathrm{Td}}(\mathcal{X}) = i_*(\widehat{\mathrm{ch}}(\alpha)\widehat{\mathrm{Td}}(\mathcal{Y})) - \mathrm{a}(i_*(\mathrm{ch}(\mathcal{F}_{\mathbb{C}})\,\mathrm{Td}(Y)S_T(N))) \tag{10.30}$$

*holds.*

*Proof.* The proof follows the classical pattern of the deformation to the normal cone as in [6] and [32].
Let $\mathcal{W}$ be the deformation to the normal cone to $\mathcal{Y}$ in $\mathcal{X}$. We will follow the notation of section 5. Since $i$ is a regular closed immersion, there is a finite resolution by locally free sheaves

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to i_*\mathcal{F} \to 0.$$

We choose hermitian metrics on the complex bundles $E_i = (\mathcal{E}_i)_{\mathbb{C}}$. The immersion $j\colon \mathcal{Y} \times \mathbb{P}^1 \longrightarrow \mathcal{W}$ is also a regular immersion. The construction of theorem 5.4 is valid over the arithmetic ring $A$. Therefore we have a resolution by hermitian vector bundles

$$0 \to \widetilde{\mathcal{G}}_n \to \cdots \to \widetilde{\mathcal{G}}_1 \to \widetilde{\mathcal{G}}_0 \to i_*\mathcal{F} \to 0.$$

such that its restriction to $\mathcal{X}\times\{0\}$ is isometric to $\mathcal{E}_*$. Its restriction to $\widetilde{\mathcal{X}}$ is orthogonally split, and its restriction to $\mathcal{P}=\mathbb{P}(\mathcal{N}_{\mathcal{Y}/\mathcal{X}}\oplus\mathcal{O}_{\mathcal{Y}})$ fits in a short exact sequence

$$0\longrightarrow\overline{\mathcal{A}}_*\longrightarrow\widetilde{\mathcal{E}}_*|_{\mathcal{P}}\longrightarrow K(\overline{\mathcal{F}},\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}})\longrightarrow 0,$$

where $\overline{\mathcal{A}}_*$ is orthogonally split and $K(\overline{\mathcal{F}},\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}})$ is the Koszul resolution. We denote by $\overline{\eta}_k$ the piece of degree $k$ of this exact sequence. Let $t$ be the absolute coordinate of $\mathbb{P}^1$. It defines a rational function in $\mathcal{W}$ and

$$\widehat{\operatorname{div}}(t)=(\mathcal{X}_0+\mathcal{P}+\widetilde{\mathcal{X}},(0,-\frac{1}{2}\log t\bar{t}))$$

The key point of the proof of the theorem is that, in the group $\widehat{\mathrm{CH}}^*(\mathcal{X},\mathcal{D}_{\mathrm{cur},X})$, we have

$$(p_{\mathcal{W}})_*(\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*)\widehat{\operatorname{div}}(t))=0.$$

Using the definition of the product in the arithmetic Chow rings we obtain

$$\begin{aligned}(p_{\mathcal{W}})_*(\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*)\widehat{\operatorname{div}}(t))=\widehat{\operatorname{ch}}(\overline{\mathcal{E}}_*)-(p_{\widetilde{\mathcal{X}}})_*\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*|_{\widetilde{\mathcal{X}}})-(p_{\widetilde{\mathcal{P}}})_*\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*|_{\mathcal{P}})\\ +\operatorname{a}((p_W)_*(\operatorname{ch}((\widetilde{\mathcal{E}}_*)_{\mathbb{C}})\bullet W_1)).\end{aligned}\tag{10.31}$$

But we have

$$\widehat{\operatorname{ch}}(\overline{\mathcal{E}}_*)=\widehat{\operatorname{ch}}(i_*^T(\overline{\mathcal{F}}))+\operatorname{a}(T(\overline{\xi})),\tag{10.32}$$

$$(p_{\widetilde{\mathcal{X}}})_*\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*|_{\widetilde{\mathcal{X}}})=0,\tag{10.33}$$

$$(p_{\widetilde{\mathcal{P}}})_*\widehat{\operatorname{ch}}(\widetilde{\mathcal{E}}_*|_{\mathcal{P}})=i_*(\pi_{\mathcal{P}})_*(\widehat{\operatorname{ch}}(K(\overline{\mathcal{F}},\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}}))-\sum_k(-1)^k\operatorname{a}(\widetilde{\operatorname{ch}}(\overline{\eta}_k))).\tag{10.34}$$

Moreover, by equation (7.3),

$$\begin{aligned}\operatorname{a}((p_W)_*(\operatorname{ch}((\widetilde{\mathcal{E}}_*)_{\mathbb{C}})\bullet W_1))=-\operatorname{a}(T(\overline{\xi}))-\sum_k(-1)^k\operatorname{a}(\widetilde{\operatorname{ch}}(\overline{\eta}_k)))\\ +\operatorname{a}(i_*C_T(\mathcal{F}_{\mathbb{C}},\mathcal{N}_{\mathbb{C}})).\end{aligned}\tag{10.35}$$

Thus we are led to compute $i_*(\pi_{\mathcal{P}})_*\widehat{\operatorname{ch}}(K(\overline{\mathcal{F}},\overline{\mathcal{N}}_{\mathcal{Y}/\mathcal{X}}))$. This is done in the following two lemmas.

Lemma 10.36. *Let $\mathcal{Y}$ be an arithmetic variety, $\overline{\mathcal{N}}$ a rank $r$ hermitian vector bundle over $\mathcal{Y}$ and denote $\mathcal{P}=\mathbb{P}^1(\mathcal{N}\oplus\mathcal{O}_{\mathcal{Y}})$, and $\overline{\mathcal{Q}}$ the tautological quotient bundle. Let $\mathcal{Y}_0$ be the cycle defined by the zero section of $\mathcal{P}$. Then*

$$\widehat{c}_r(\overline{\mathcal{Q}})=(\mathcal{Y}_0,(c_r(\overline{\mathcal{Q}}_{\mathbb{C}}),\widetilde{e}(\mathcal{P}_{\mathbb{C}},\overline{\mathcal{Q}}_{\mathbb{C}},s))),\tag{10.37}$$

*where $\widetilde{e}(\mathcal{P}_{\mathbb{C}},\overline{\mathcal{Q}}_{\mathbb{C}},s)$ is the Euler-Green current of lemma 9.4.*

*Proof.* We know that $\widehat{c}_r(\overline{\mathcal{Q}}) = (\mathcal{Y}_0, (c_r(\overline{\mathcal{Q}}_{\mathbb{C}}), \widetilde{e}))$ for certain Green current $\widetilde{e}$. By definition this Green current satisfies

$$\mathrm{d}_{\mathcal{D}}\, \widetilde{e} = c_r(\overline{\mathcal{Q}}_{\mathbb{C}}) - \delta_{\mathcal{Y}_{\mathbb{C}}}.$$

Moreover, since the restriction of $\overline{\mathcal{Q}}_{\mathbb{C}}$ to $D_\infty$ has a global section of constant norm we have that $\widetilde{e}|_{D_\infty} = 0$. Therefore, by lemma 9.4,

$$\widetilde{e} = \widetilde{e}(\mathcal{P}_{\mathbb{C}}, \overline{\mathcal{Q}}_{\mathbb{C}}, s).$$

□

Lemma 10.38. *The following equality hold:*

$$(\pi_{\mathcal{P}})_* \widehat{\mathrm{ch}}(K(\overline{\mathcal{F}}, \overline{\mathcal{N}})_*) = \widehat{\mathrm{ch}}(\overline{\mathcal{F}}) \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}}) + \mathrm{a}(C_T(\overline{\mathcal{F}}, \overline{\mathcal{N}}) - \mathrm{ch}(\mathcal{F}_{\mathbb{C}})\, \mathrm{Td}^{-1}(N_{Y/X}) S_T(N)). \quad (10.39)$$

*Proof.* We just compute, using lemma 10.36,

$$\begin{aligned}
(\pi_{\mathcal{P}})_* \widehat{\mathrm{ch}}(K(\overline{\mathcal{F}}, \overline{\mathcal{N}})_*) &= (\pi_{\mathcal{P}})_* \sum_k (-1)^k \widehat{\mathrm{ch}}(\bigwedge^k \overline{\mathcal{Q}}^\vee) \widehat{\mathrm{ch}}(\pi_{\mathcal{P}}^* \overline{\mathcal{F}}) \\
&= (\pi_{\mathcal{P}})_* (\widehat{c}_r(\overline{\mathcal{Q}}) \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{Q}})) \widehat{\mathrm{ch}}(\overline{\mathcal{F}}) \\
&= \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}}) \widehat{\mathrm{ch}}(\overline{\mathcal{F}}) + \mathrm{a}((\pi_P)_* (\widetilde{e}\, \mathrm{Td}^{-1}(\overline{Q}))\, \mathrm{ch}(\overline{F})) \\
&= \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}}) \widehat{\mathrm{ch}}(\overline{\mathcal{F}}) + \mathrm{a}((\pi_P)_* (T^h(K(\overline{F}, \overline{N})))\, \mathrm{ch}(\overline{F})) \\
&= \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}}) \widehat{\mathrm{ch}}(\overline{\mathcal{F}}) + \mathrm{a}(C_{T^h}(F, N)) \\
&= \widehat{\mathrm{Td}^{-1}}(\overline{\mathcal{N}}) \widehat{\mathrm{ch}}(\overline{\mathcal{F}}) + C_T(F, N) - \mathrm{a}(\mathrm{Td}^{-1}(N)\, \mathrm{ch}(F) S_T(N)).
\end{aligned}$$

□

The equation (10.29) follows by combining equations (10.31), (10.32), (10.33), (10.34), (10.35) and (10.39).
The equation (10.30) follows from equation (10.29) by a straightforward computation. □

Since $T$ is homogeneous if and only if $S_T = 0$, in view of this result, the theory of homogeneous singular Bott-Chern classes is characterized for being the unique theory of singular Bott-Chern classes that provides an exact arithmetic Grothendieck-Riemann-Roch theorem for closed immersions. By contrast, if one uses a theory of singular Bott-Chern classes that is not homogeneous, there is an analogy between the genus $S_T$ and the $R$-genus that appears in the arithmetic Grothendieck-Riemann-Roch theorem for submersions.
Since there is a unique theory of homogeneous singular Bott-Chern classes, the following definition is natural.

Definition 10.40. Let $i\colon (\mathcal{Y}, h_Y) \longrightarrow (\mathcal{X}, h_X)$ be a closed immersion of metrized arithmetic varieties, the *push-forward map*

$$i_*\colon \widehat{K}'(\mathcal{Y}, \mathcal{D}_{\mathrm{cur},Y}) \longrightarrow \widehat{K}'(\mathcal{X}, \mathcal{D}_{\mathrm{cur},Y})$$

is defined as $i_* = i_*^{T_c^h}$.

Corollary 10.41. *The push-forward map makes* $\widehat{K}'(\_, \mathcal{D}_{\mathrm{cur},Y})$ *and* $\widehat{K}(\_, \mathcal{D}_{\mathrm{cur},Y})$ *functors from the category of regular metrized arithmetic varieties and closed immersions to the category of abelian groups.*

Corollary 10.42. *Let* $i\colon (\mathcal{Y}, h_Y) \longrightarrow (\mathcal{X}, h_X)$ *be a closed immersion of regular metrized arithmetic varieties, then*

$$\widehat{\mathrm{ch}}(i_*^T(\alpha))\widehat{\mathrm{Td}}(\mathcal{X}) = i_*(\widehat{\mathrm{ch}}(\alpha)\widehat{\mathrm{Td}}(\mathcal{Y})). \tag{10.43}$$

Remark 10.44. Combining theorem 10.28 with [16] we can obtain an arithmetic Grothendieck-Riemann-Roch theorem for projective morphisms of regular arithmetic varieties.
In a forthcoming paper we will show that the higher torsion forms used to define the direct images for submersions can also be characterized axiomatically.

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# Hicas of Length $\leq 4$

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Abstract. A *hica* is a highest weight, homogeneous, indecomposable, Calabi-Yau category of dimension 0. A hica has length $l$ if its objects have Loewy length $l$ and smaller. We classify hicas of length $\leq 4$, up to equivalence, and study their properties. Over a fixed field $F$, we prove that hicas of length 4 are in one-one correspondence with bipartite graphs. We prove that an algebra $A_\Gamma$ controlling the hica associated to a bipartite graph $\Gamma$ is Koszul, if and only if $\Gamma$ is not a simply laced Dynkin graph, if and only if the quadratic dual of $A_\Gamma$ is Calabi-Yau of dimension 3.

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## 1. Introduction

Once a mathematical definition has been made, the theory surrounding that definition usually begins with a study of small examples. A striking violation of this principle occurred at the birth of category theory, where early theory was concerned with establishing results valid for large and floppy mathematical structures like the category of sets, or the category of groups, or the category of topological spaces. But time has passed, categories have begun to be taken seriously, and they are now objects of detailed study. Since categories are often large and floppy, the 2-category of all categories is very large and very floppy. To prove theorems about categories, it is necessary to make strong restrictions on their structure. To prove classification theorems for categories, it is necessary to make very strong restrictions on their structure.

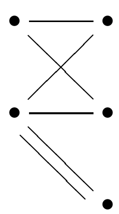
There is by now an extensive collection of categorical classification theorems. A category with one object and invertible morphisms is a group, and there are many examples of classification theorems in group theory. Rings are endowed with various categories, like their module categories. Classification theorems for commutative rings can be thought of as classification theorems in algebraic geometry. There are a number of classification theorems for rings of finite

homological dimension, to which the term noncommutative geometry is applied. For example, hereditary algebras over an algebraically closed field can be parametrised by quivers. Calabi-Yau algebras of dimensions 2 and 3 can be loosely parametrised by quivers with a superpotential [2], [5], [8]. Categorical classification theorems also appear in the representation theory of 2-categories: irreducible integrable representations of 2-Kac-Moody Lie algebras can be parametrised by integral dominant weights [18].
Our paper runs in this vein. A *hica* is a highest weight, homogeneous, indecomposable, Calabi-Yau category of dimension 0. Here, we say a highest weight category is homogeneous if its standard objects all have the same Loewy length, and its costandard objects all have the same Loewy length. We say a hica has length $l$ if its projective objects have Loewy length $l$ and smaller. We classify hicas of length $\leq 4$ up to equivalence.
Hicas show up naturally in group representation theory and in the theory of tilings [20, 3, 14, 15]. A multitude of examples of hicas were constructed by Mazorchuk and Miemietz [13]. Every hica can be realised as the module category of some symmetric quasi-hereditary algebra. If the hica is not semisimple, the corresponding algebra is necessarily infinite dimensional, noncommutative, of infinite homological dimension.
Let us fix a field $F$, and consider hicas over $F$, up to equivalence. The only hica of length 1 is the category of vector spaces over $F$. There are no hicas of length 2. There is a unique hica of length 3, which is the module category of the Brauer tree algebra on a bi-infinite line. Our first main result is

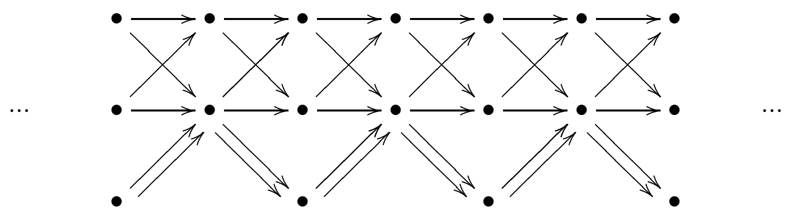
Theorem 1. *There is a natural one-one correspondence*

$$\{\textit{bipartite graphs}\} \leftrightarrow \{\textit{hicas of length } 4\}.$$

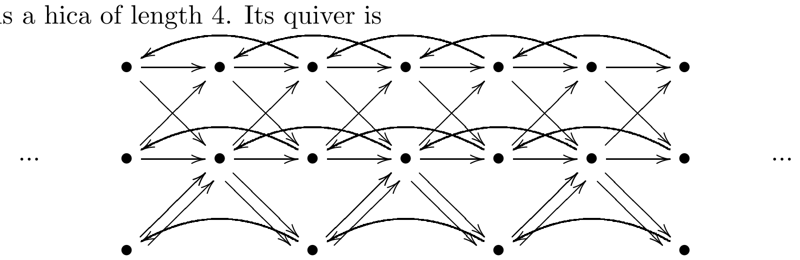
Here, and throughout this paper, a bipartite graph will by definition be connected.
The one-one correspondence of Theorem 1 is obtained from a sequence of three one-one correspondences: a one-one correspondence between bipartite graphs and topsy-turvy quivers; a one-one correspondence between topsy-turvy quivers and basic indecomposable self-injective directed algebras of Loewy length 3; and a one-one correspondence between basic self-injective directed algebras of Loewy length 3 and hicas of length 4.
Let us describe here the construction of a hica $\mathcal{C}_\Gamma$ of length 4 from the following bipartite graph $\Gamma$:



First, we construct a quiver $Q_\Gamma$, by consecutively gluing together opposite orientations of this bipartite graph, one next to the other:



This quiver has an automorphism $\phi$ which shifts a vertex to a vertex which is two steps horizontally to its right. We take the path algebra of this quiver. We now construct a self-injective directed algebra $B_\Gamma$ of Loewy length 3, factoring the path algebra by relations which insist that all squares commute, and that paths $u \to v \to w$ of length 2 are zero, unless $w = \phi(u)$. We define $A_\Gamma$ to be the trivial extension $B_\Gamma \oplus B_\Gamma^*$ of $B_\Gamma$ by its dual. The module category $\mathcal{C}_\Gamma$ of $A_\Gamma$ is a hica of length 4. Its quiver is



The relations for $A_\Gamma$ are those for $B_\Gamma$, along with relations which insist that the product of two leftwards pointing arrows is zero whilst squares involving a pair of parallel leftwards pointing arrows commute. The algebra $A_\Gamma$ has some pleasing properties. It admits a derived self-equivalence $\psi_\gamma$ for every vertex $\gamma$ of $\Gamma$. It also admits a number of $\mathbb{Z}_+^3$-gradings, one for each orientation of the graph $\Gamma$. It is Koszul and its quadratic dual $A_\Gamma^!$ is Calabi-Yau of dimension 3. More generally, we have the following theorem.

Theorem 2. *Suppose* $\Gamma$ *is a connected bipartite graph, and* $\mathcal{C}_\Gamma = A_\Gamma$-mod *the associated hica of length* 4. *The following are equivalent:*

1. $\Gamma$ *is not a simply laced Dynkin graph.*
2. $A_\Gamma$ *is Koszul.*
3. *The quadratic dual of* $A_\Gamma$ *is Calabi-Yau of dimension* 3.

The way this paper evolved was surprising to us. We began with the problem of classifying small hicas, categories whose structural features (Calabi-Yau 0, highest weight) were motivated by exposure to group representation theory. We ended having made contact with mathematics of different kin: bipartite graphs, Calabi-Yau 3s, and Dynkin classifications. The hica restrictions indeed capture some features of Lie theoretic representation theory, but they can also be thought of as noncommutative geometric restrictions: highest weight categories were invented to capture stratification properties appearing in algebraic geometry, whilst 0-Calabi-Yau categories are categories possessing a homological duality with trivial Serre functor.

## 2. Preliminaries

Our main objects of study, *hicas*, are a species of abelian categories. As we study them, we will use freely the languages of abelian categories, algebras, and triangulated categories. Here we give a short phrasebook for these languages.
Let $F$ be a field. The collection of $F$-algebras is a 2-category, whose arrows are bimodules ${}_AM_B$ which are flat on the right, and 2-arrows are bimodule homomorphisms. We have a 2-functor

$$\mathfrak{Algebra} \to \mathfrak{Abelian}$$

from the 2-category $\mathfrak{Algebra}$ of $F$-algebras to the 2-category $\mathfrak{Abelian}$ of abelian categories. This 2-functor takes an algebra $A$ to its module category, a bimodule ${}_AM_B$ to the functor $M \otimes_B -$, and a bimodule homomorphism to a natural transformation. We have a 2-functor

$$\mathfrak{Abelian} \to \mathfrak{Triangulated}$$

taking values in the 2-category of triangulated categories, which takes an abelian category $\mathcal{A}$ to its derived category $D(\mathcal{A})$.
If $X$ is an object of an abelian category of finite composition length, we define the *Loewy length* of $X$ (or *length* of $X$, or $l(X)$) to be the smallest number $l$ for which there exists a filtration of $X$ with $l$ nonzero sections, all of which are semisimple. We define the *head*, or *top* of $X$ to be the maximal semisimple quotient of $X$, and the *socle* of $X$ to be the maximal semisimple submodule. If $\mathcal{A}$ is an abelian category, we define the *length* of $\mathcal{A}$ to be the supremum over all lengths of objects in $\mathcal{A}$. If $A$ is an algebra, we define the *length* of $A$ to be the length of the abelian category $A$-mod of $A$-modules.
Given a finite dimensional $F$-vector space $V$, we denote by $V^*$ the dual $Hom_F(V,F)$ of $V$. We call an object $X$ of a triangulated category *compact* if $\mathrm{Hom}(X,-)$ commutes with infinite direct sums. We say an $F$-linear triangulated category $\mathcal{T}$ is *Calabi-Yau of dimension* $d$ if $\mathrm{Hom}_{\mathcal{T}}(P,X)$ is finite dimensional for objects $X \in \mathcal{T}$, and compact objects $P \in \mathcal{T}$, and

$$\mathrm{Hom}_{\mathcal{T}}(P,X) \cong \mathrm{Hom}_{\mathcal{T}}(X,P[d])^*$$

naturally in objects $X \in \mathcal{T}$, and compact objects $P \in \mathcal{T}$. For background, we recommend a survey article of B. Keller concerning Calabi-Yau triangulated categories [8]. To avoid confusion here, let us emphasise that the definition of a Calabi-Yau triangulated category Keller uses is slightly different from this one since he makes no compactness assumption on $P$.
We say an $F$-linear abelian category $\mathcal{A}$ is Calabi-Yau of dimension $d$ if its derived category $D(\mathcal{A})$ is Calabi-Yau of dimension $d$. We say an $F$-algebra $A$ is Calabi-Yau of dimension $d$ if its module category $A$-mod is Calabi-Yau of dimension $d$.

Suppose $A$ is a basic (not necessarily unital) $F$-algebra satisfying the following assumptions:

(i) $A$ has a countable set $\{e_x \mid x \in \Lambda\}$ of orthogonal primitive idempotents, such that $A = \oplus_{x,y} e_x A e_y$;
(ii) for any $x, y \in \Lambda$ the $F$-vector space $e_x A e_y$ is finite dimensional;
(iii) for any $x \in \Lambda$ there exist only finitely many $y \in \Lambda$ such that $e_x A e_y \neq 0$;
(iv) for any $x \in \Lambda$ there exist only finitely many $y \in \Lambda$ such that $e_y A e_x \neq 0$.

Under these assumptions all indecomposable projective $A$-modules $Ae_x$ and all injective $A$-modules $\mathrm{Hom}_F(e_x A, F)$ are finite-dimensional. $A$-modules $M = {}_A M$ will be left $A$-modules unless they carry a right subscript as in $M_A$ in which case they will be right $A$-modules. We denote by $A$-mod the collection of all finite-dimensional left $A$-modules and by mod-$A$ the collection of all finite-dimensional right $A$-modules. We denote by $A$-perf the subcategory of the derived category of $A$-mod consisting of perfect complexes, that is the smallest thick subcategory of the derived category of $A$-mod containing all projective objects of $A$-mod, or equivalently the subcategory of compact objects in the derived category of $A$. We define $A^*$ to be the $A$-$A$-bimodule $\bigoplus_{x\in\Lambda} \mathrm{Hom}_F(Ae_x, F)$.
We say $A$ is a *symmetric algebra* if $A \cong A^*$ as $A$-$A$-bimodules. Then $A$ is symmetric if and only if $A$-mod is Calabi-Yau of dimension 0 (cf. [17], Theorem 3.1).
Suppose $A$ is an algebra satisfying the above conditions, and $\Lambda$ is ordered. For $\lambda \in \Lambda$, let $J_{\geq\lambda} = \sum_{\mu\geq\lambda} Ae_\mu A$ and $J_{>\lambda} = \sum_{\mu>\lambda} Ae_\mu A$. Let $J_\lambda = J_{\geq\lambda}/J_{>\lambda}$. We say $A$ is *quasi-hereditary* if the product map $J_\lambda e_\lambda \otimes_F e_\lambda J_\lambda \to J_\lambda$ is an isomorphism for every $\lambda \in \Lambda$ [4].

Now suppose $\mathcal{A}$ is an abelian category over $F$, with enough projective objects, enough injective objects, and a countable set $\Lambda$ indexing the isomorphism classes of simple objects of $\mathcal{A}$, such that all objects of $\mathcal{A}$ have a finite composition series with sections in $\Lambda$. Abusing notation, an element $\lambda$ of $\Lambda$ we sometimes take to represent an index, sometimes an isomorphism class of irreducible object, and sometimes a representative of the latter. We denote by $P(\lambda)$ a minimal projective cover of $\lambda$ in $\mathcal{A}$. Such exist, since we have enough projectives, and finite composition series.
We call $\mathcal{A}$ a *highest weight category* [4] if there is an ordering $<$ on $\Lambda$, and a collection of objects $\Delta(\lambda)$, for $\lambda \in \Lambda$, such that

(i) there is an epimorphism $\Delta(\lambda) \twoheadrightarrow \lambda$ whose kernel $X(\lambda)$ has composition factors $\mu < \lambda$;
(ii) $P(\lambda)$ has a filtration with a single section isomorphic to $\Delta(\lambda)$ and every other section isomorphic to $\Delta(\mu)$, for $\mu > \lambda$.

If $A$ is quasi-hereditary, then $A$-mod is a highest weight category, with standard objects ${}_A\Delta(\lambda) = J_\lambda e_\lambda$, and mod-$A$ is a highest weight category with standard modules $\Delta_A(\lambda) = e_\lambda J_\lambda$. Thus $A$ has a filtration by ideals, whose sections are isomorphic to

$$_A\Delta(\lambda) \otimes_F \Delta_A(\lambda).$$

Conversely, if $\mathcal{A}$ is a highest weight category, then $A = \oplus_{\lambda,\mu\in\Lambda} Hom(P(\lambda), P(\mu))$ is a quasi-hereditary algebra.
The left and right costandard modules $_A\nabla(\lambda)$, $\nabla(\lambda)_A$ of $A$ are defined to be the duals of the right and left standard modules $\nabla(\lambda)_A$, $_A\nabla(\lambda)$ of $A$. We write $\Delta = \oplus_\lambda \Delta(\lambda)$ and $\nabla = \oplus_\lambda \nabla(\lambda)$.

LEMMA 3. *Let $A$ be a selfinjective quasi-hereditary algebra. If $A$ is not semisimple, then $A$ is infinite dimensional.*

*Proof.* Nonsemisimple selfinjective algebras have infinite homological dimension, since Heller translation is invertible. Finite dimensional quasi-hereditary algebras have finite homological dimension. □

We say a highest weight category $\mathcal{C}$ is homogeneous if its standard objects all have the same Loewy length, and its costandard objects all have the same Loewy length. Equivalently, $\mathcal{C} = A$-mod, where $A$ is a quasi-hereditary algebra whose left standard modules all have the same Loewy length, and whose right standard modules all have the same Loewy length.

DEFINITION 4. *A* hica *is is a highest weight, homogeneous, indecomposable Calabi-Yau category of dimension* 0.

The collection $\mathfrak{Hica}$ of hicas forms a 2-category (arrows are exact functors, 2-arrows are natural transformations). We denote by $\mathfrak{Hica}_l$ the 2-category of hicas of length $l$.

LEMMA 5. *The* 2*-functor*

$$\{\textit{symmetric, homogeneous, quasihereditary basic algebras}\} \twoheadrightarrow \mathfrak{Hica}$$

*which takes an algebra to its module category is essentially bijective on objects.*

*Proof.* We must define a correspondence between objects of our 2-categories, under which isomorphic algebras correspond to equivalent categories, and vice versa. If $A$ is a symmetric, $\Delta$-homogeneous quasihereditary algebra then $A$-mod is a hica ([4], [17], Theorem 3.1). If $\mathcal{C}$ is a hica, then $A = \oplus_{\lambda\in\Lambda} \mathrm{Hom}(P(\lambda), P(\mu))$ is an algebra such that $A$-mod $= \mathcal{C}$. □

A highest weight category $\mathcal{C}$ has a collection of indecomposable tilting modules $T(\lambda)$ indexed by $\Lambda$, characterised as indecomposable objects with a $\Delta$-filtration and a $\nabla$-filtration. The *Ringel dual* $\mathcal{C}'$ of $\mathcal{C}$ is the module category $A'$-mod of the algebra

$$A' = \oplus_{\lambda,\mu} \mathrm{Hom}_{\mathcal{C}}(T(\lambda), T(\mu)).$$

The Ringel dual $\mathcal{C}'$ of $\mathcal{C}$ is a highest weight category. If $\mathcal{C} = A$-mod, we call $A'$ the Ringel dual of $A$. If $\mathcal{C} \cong \mathcal{C}'$ then we say $\mathcal{C}$ and $A$ are *Ringel self-dual*.

LEMMA 6. *Suppose $\mathcal{C} = A$-mod is a hica. Then*

$$l(A) = l(_A\Delta) + l(\Delta_A) - 1.$$

*Proof.* The length of $A$ is the least number $l$ such that the product of any $l$ elements of the radical of $A$ is zero. This can be otherwise defined as the radical length of the $A \otimes A^{op}$-module $A$. Since $A$ is quasi-hereditary, ${}_AA_A$ has a bimodule filtration with sections ${}_A\Delta(\lambda) \otimes_F \Delta_A(\lambda)$. These sections have radical length $l({}_A\Delta) + l(\Delta_A) - 1$, as $A \otimes A^{op}$-modules. Therefore the Loewy length of $A$ is at least $l({}_A\Delta) + l(\Delta_A) - 1$.
The tops of all of these sections lie in the top of ${}_AA_A$. Since $A$ is symmetric, every irreducible lies in the socle of $A$. Since $A$ is also quasi-hereditary, every irreducible lies in the socle of some standard object $\Delta$. Given $\lambda \in \Lambda$, the socle $Fx_\lambda$ of $Ae_\lambda$ is generated by $\mathrm{soc}({}_A\Delta(\nu)) \otimes_F \mathrm{soc}(\Delta_A(\nu))$, for suitable $\nu$, modulo lower terms in the filtration. The lower terms in the filtration have zero intersection with $Fx_\lambda$, since this space is one dimensional. Therefore, lifting an element of $\mathrm{soc}({}_A\Delta(\nu)) \otimes \mathrm{soc}(\Delta_A(\nu))$ to an element of radical length $l({}_A\Delta) + l(\Delta_A) - 1$ in $A$, we obtain an element of $Fx_\lambda$ of radical length $l({}_A\Delta) + l(\Delta_A) - 1$. It follows that the Loewy length of $A$ is at most $l({}_A\Delta) + l(\Delta_A) - 1$. $\square$

We also wish to consider graded algebras, which may satisfy weaker assumptions than those given above. If $G$ is a group, and $A$ an algebra, then a $G$-grading of $A$ is a decomposition $A = \oplus_{g \in G} A^g$, such that $A^g.A^h \subset A^{gh}$. A graded $A$-module is an $A$ module with a decomposition $M = \oplus_{g \in G} M^g$, such that $A^g.M^h \subset M^{gh}$; a homomorphism $\phi : M \to N$ of graded modules is an $A$-module homomorphism sending $M^g$ to $N^g$, for $g \in G$.
We say $A$ is $\mathbb{Z}_+$-graded if it is $\mathbb{Z}$-graded, with $A^i = 0$ for $i < 0$. Suppose $A$ a $\mathbb{Z}$-graded algebra, whose degree 0 part $A^0$ satisfies the conditions (i)-(iv) above. Then we denote by $A$-mod the abelian subcategory of the category of all $A$-modules generated by $A^0$-mod, and by $A$-gr the abelian subcategory of the category of all graded $A$-modules generated by the category of finite dimensional $A^0$-mod$\langle i \rangle$, for $i \in \mathbb{Z}$. We denote by $A$-grperf the thick subcategory of the the derived category of graded $A$-modules generated by objects of the form $A \otimes_{A^0} X\langle i \rangle$, where $X \in A^0$-mod and $i \in \mathbb{Z}$.

## 3. Elementary constructions

Let us give some elementary constructions of symmetric algebras.
Suppose $B$ is an algebra. Let $A = T(B)$ denote the trivial extension of $B$ by $B^*$. Then $A$ is symmetric, and $A$-mod is Calabi-Yau of dimension 0.
Suppose $B$ is an algebra and $M$ is a $B$-$B$-bimodule such that $e_\lambda M e_\mu$ is finite-dimensional for every $\lambda, \mu \in \Lambda$ and such that for every $\lambda$ only finitely many of $e_\lambda M e_\mu$ and $e_\mu M e_\lambda$ are non-zero. Define $M^* := \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_F(Me_\lambda, F)$ and assume we have a fixed bimodule isomorphism $M \cong M^*$. Then we have a

sequence of bimodule homomorphisms

$$\begin{aligned} B \to \mathrm{Hom}_B(M,M) \cong \mathrm{Hom}_B(M,M^*) &= \mathrm{Hom}_B(M, \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}(Me_\lambda, F)) \\ &\cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_B(M, \mathrm{Hom}(Me_\lambda, F)) \\ &\cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_F(M\otimes_B Me_\lambda, F) \\ &= \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_F((M\otimes_B M)e_\lambda, F) \\ &= (M\otimes_B M)^*, \end{aligned}$$

noting that $M \otimes_B M$ satisfies that $e_\lambda M \otimes_B Me_\mu = \bigoplus_{\nu\in\Lambda} e_\lambda Me_\nu \otimes_B e_\nu Me_\mu$ is finite-dimensional (finitely many finite-dimensional direct summands) for all $\lambda, \mu$, and for every $\lambda$ only finitely many of $e_\lambda M \otimes_B Me_\mu$ and $e_\mu M \otimes_B Me_\lambda$ are nonzero. The obtained bimodule homomorphisms compose to give a bimodule homomorphism $B \to (M \otimes_B M)^*$. Let $\mu : M \otimes_B M \to B^*$ denote the dual map.
Associated to the data $(B, M)$, we have a $\mathbb{Z}$-graded algebra $U = U(B, M)$ concentrated in degrees 0,1, and 2 whose degree 0,1,2 part is $B$,$M$, $B^*$ respectively. The product map $U^0 \otimes U^i \to U^i$ is given by the left action of $B$ on the bimodule $U^i$, for $i = 0, 1, 2$. The product map $U^i \otimes U^0 \to U^i$ is given by the right action of $B$ on the bimodule $U^i$. We define the product $U^1 \otimes_{U^0} U^1 \to U^2$ to be given by $\mu$. The product is associative since the product of three components $U^i \otimes U^j \otimes U^k$ is non-zero if and only if $i+j+k \leq 2$, in which case associativity is clearly visible.

LEMMA 7. $U(B, M)$-mod *is Calabi-Yau of dimension zero.*

*Proof.* We have a bimodule isomorphism $U \cong U^*$ which exchanges $U^0$ and $U^2$, and sends $U^1$ to $U^{1*}$ via the fixed isomorphism $M \cong M^*$. □

## 4. TOPSY-TURVY QUIVERS

Given a vertex $w$ in a quiver $Q$, let $\mathcal{P}(w)$ denote the collection of vertices $v$ of $Q$ for which there is an arrow pointing from $v$ to $w$ (the *past* of $w$), counted with multiplicity. Let $\mathcal{F}(u)$ denote the collection of vertices $v$ of $Q$ for which there is an arrow pointing from $u$ to $v$ (the *future* of $u$), counted with multiplicity.

DEFINITION 8. *A connected quiver is* topsy-turvy *if it contains at least one arrow, and there is an automorphism $\phi$ of the vertices of $Q$ such that $\mathcal{F}(u) = \mathcal{P}(u^\phi)$ for every vertex $u$ of $Q$.*

For any topsy-turvy quiver, the automorphism $\phi$ extends to a quiver automorphism, since arrows from $x$ to $y$ can be placed in bijection with arrows from $y$ to $x^\phi$, which can be placed in bijection with arrows from $x^\phi$ to $y^\phi$.

LEMMA 9. *If $Q$ is a topsy-turvy quiver, then $\mathcal{PF}(w) = \mathcal{FP}(w)$ for all vertices $w$ of $Q$.*

*Proof.* Any $x$ in $\mathcal{FP}(w)$ lies in the future of some $u$ in the past of $w$, and therefore lies in the past of $u^\phi$; since $Q$ is topsy-turvy, $u^\phi$ also lies in the future of $w$ and $x$ lies in $\mathcal{PF}(w)$. By symmetry, if $x$ lies in $\mathcal{PF}(w)$ then $x$ also lies in $\mathcal{FP}(w)$. □

A directed topsy-turvy quiver $Q$ can be $\mathbb{Z}$-graded in the following way: take an arbitrary vertex $u$ of $Q$ and place it in degree 0. We say another vertex $v$ in $Q$ is in degree $k$ if there exist $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ such that $v \in \mathcal{P}^{i_1}\mathcal{F}^{j_1} \ldots \mathcal{P}^{i_r}\mathcal{F}^{j_r}(u)$ and $\sum_{1\leq s\leq r} j_s - \sum_{1\leq s\leq r} i_s = k$. This is well-defined since $\mathcal{PF}(w) = \mathcal{FP}(w)$. It follows that all arrows in $Q$ point from degree $i$ to degree $i+1$ and that $\phi$ has degree 2.
A *bipartite graph* is a countable connected graph $\Gamma$ whose set $V$ of vertices decomposes into two nonempty subsets $V = V_l \cup \Gamma_r$ such that no edges of $\Gamma$ connect $V_l$ to $V_l$, or $V_r$ to $V_r$. Note that we do *not* call the graph with one vertex and no arrows bipartite.
Given a graph $\Gamma$ with a bipartite decomposition of vertices $V = V_l \cup V_r$, we have an associated directed topsy-turvy quiver $Q_\Gamma$, obtained by orienting $\mathbb{Z}$ copies of $\Gamma$, identifying, for $i$ even, the $r$-vertices of $i^{th}$ copy of $\Gamma$ with the $r$-vertices of the $i+1^{th}$ copy of $\Gamma$, the $l$-vertices of $i^{th}$ copy of $\Gamma$ with the $l$-vertices of the $i-1^{th}$ copy of $\Gamma$, and insisting that arrows in the $i^{th}$ copy of $\Gamma$ point from the $i-1^{th}$ copy to the $i+1^{th}$ copy, for $i \in \mathbb{Z}$. Note that if we label our bipartite decomposition with the opposite orientation, we obtain an isomorphic topsy-turvy quiver.

LEMMA 10. *We have a one-one correspondence $\Gamma \leftrightarrow Q_\Gamma$ between bipartite graphs and directed topsy-turvy quivers.*

*Proof.* Given a directed topsy-turvy quiver, we have a $\mathbb{Z}$-grading of the set of vertices $V = \amalg_{i\in\mathbb{Z}} V_i$, see above. Let $A_i$ denote the set of arrows from $V_i$ to $V_{i+1}$. The set of arrows of our quiver is graded $A = \amalg_{i\in\mathbb{Z}} A_i$. The automorphism $\phi$ defines isomorphisms between $V_i$ and $V_j$ and between $A_i$ and $A_j$ when $i$ and $j$ are both even, or when $i$ and $j$ are both odd. We can thus identify the $V_i$ for $i$ even with a single vertex set $V_{even}$, the $V_i$ for $i$ odd with a single vertex set $V_{odd}$, the $A_i$ for $i$ even with a single arrow set $A_{eo}$ from $V_{even}$ to $V_{odd}$, the $A_i$ for $i$ odd with a single arrow set $A_{oe}$ from $V_{odd}$ to $V_{even}$. The topsy-turviness of the quiver means precisely that $A_{eo}$ is the opposite of $A_{oe}$. We thus obtain a graph with vertices $V_{even} \cup V_{odd}$, and with edges between $V_{even}$ and $V_{odd}$, such that directing edges from $V_{even}$ to $V_{odd}$ gives us $A_{eo}$ and directing edges from $V_{odd}$ to $V_{even}$ gives us $A_{oe}$. This is a bipartite graph, by definition.
Reversing the above argument, from any bipartite quiver, we obtain a directed topsy-turvy quiver. □

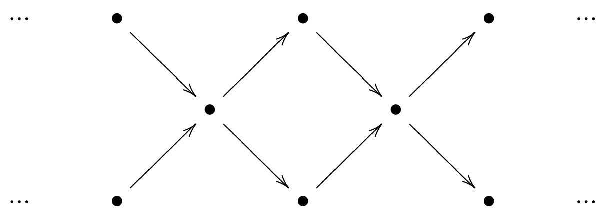
EXAMPLE 11 The bipartite graph $\bullet \!-\!\!-\!\!-\! \bullet$ with two vertices and a single edge results in a topsy-turvy quiver which can be depicted as an oriented line:

$$\cdots \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \cdots$$

The bipartite graph



results in a topsy-turvy quiver which can be depicted as a directed square lattice in $\mathbb{R}^2$:



The bipartite graph whose vertices are elements of the square lattice lattice in $\mathbb{R}^2$ results in a topsy-turvy quiver whose arrows can be thought of as the diagonals of a face-centred cubic lattice in $\mathbb{R}^3$.

## 5. Self-injective directed algebras of length $\leq 3$

Throughout the following, let $B$ be an indecomposable self-injective directed algebra. Here self-injective means that $B \cong \bigoplus_{x\in\Lambda} \operatorname{Hom}_F(Ae_x, F)$ as left $B$-modules or, equivalently, that all projective $B$-modules are also injective, and vice versa. Directed is understood to mean that the $\operatorname{Ext}^1$-quiver of $B$ is a directed quiver.
Note that such an algebra is necessarily infinite-dimensional, since directed implies quasi-hereditary which, in the finite-dimensional case, implies finite global dimension, contradicting self-injectivity.

Lemma 12. *If $B$ is radical-graded, all projective $B$-modules have the same Loewy length.*

*Proof.* For finite-dimensional algebras, this was shown in [12, Theorem 3.3]. We remark that the same proof holds for algebras in our setup, as the comparisons of Loewy length only need to be done using neighbouring projectives in the Ext-quiver. □

Let us now assume that $B$ be an indecomposable self-injective algebra of Loewy length $\leq 3$.

Lemma 13. *$B$ is radical-graded.*

*Proof.* Set $A_0 := \bigoplus_{x\in\Lambda} Fe_x \cong A/\operatorname{Rad} A$ realized by the semisimple algebra generated by the idempotents, this is obviously a subalgebra. It acts naturally on the bimodule $A_1 \cong \operatorname{Rad} A/\operatorname{Rad}^2 A$ given by the arrows in the Ext-quiver and on $A_2 := \operatorname{Rad}^2 A$. Obviously the multiplication maps $A_1 \otimes A_1$ to $A_2$, so $A$ is radical-graded. □

Corollary 14. *All projectives of $B$ have the same Loewy length.*

Lemma 15. *The quiver of $B$ is a topsy-turvy quiver.*

*Proof.* A projective indecomposable $B$-module $P(\lambda)$ can be identified with an injective indecomposable $B$-module $I(\lambda^\phi)$. Here $\phi$ is a quiver isomorphism, corresponding to the Nakayama automorphism of $B$. Since $B$ is selfinjective Loewy length 3, elements of $\mathcal{F}(\lambda)$ correspond to composition factors in the heart of $P(\lambda) =_B Be_\lambda$. Switching from left action to right action, we find elements of $\mathcal{P}(\lambda^\phi)$ correspond to composition factors in the heart of $e_\lambda B_B$. Taking duals, we find elements of $\mathcal{P}(\lambda^\phi)$ correspond to composition factors in the heart of $I(\lambda^\phi)$. Since $P(\lambda) = I(\lambda^\phi)$, we conclude $\mathcal{F}(\lambda) = \mathcal{P}(\lambda^\phi)$. Thus $B$ has a topsy-turvy quiver, as required. □

To any topsy-turvy quiver $Q$, we can associate a self-injective algebra $R(Q)$ of Loewy length 3 by factoring out relations from the path algebra as follows: make products of arrows of $Q$ which do not lie in some $\mathcal{F}(u) \cup \mathcal{P}(u^\phi)$ equal to zero; make squares in $\mathcal{F}(u) \cup \mathcal{P}(u^\phi)$ commute.
Let us now assume $B$ is directed.

Lemma 16. (a) *If $B$ has Loewy length 2, it is isomorphic to the $FQ/I$, where $Q$ is the infinite quiver*

$$\cdots \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \cdots$$

*and $I$ is the quadratic ideal generated by all paths of length two.*

(b) *If $B$ has Loewy length 3, it is given by $R(Q)$, where $Q$ is a directed, topsy-turvy quiver.*

*Proof.* (a) Obvious.
(b) Since projectives are injectives, both have irreducible head and socle. Since $B$ is directed, projectives have structure

$$\begin{array}{c}\lambda\\ \mu_1 \oplus \ldots \oplus \mu_n\\ \nu,\end{array}$$

where $\nu < \mu_i < \lambda$ all $i$. We only have to worry about the nonzero relations. These take the form $ac = \xi bd$, for $\xi \in F^\times$, where $a, b$ are arrows in $F(u)$ and $c, d$ are arrows in $P(u^\phi)$ for some $u$. We want to remove the scalars $\xi$ from this description.
Let us write $B = FQ/I$. Then $Q$ is topsy-turvy with $\phi$ described by the Nakayama automorphism of $B$. Since $Q$ is directed as well, we can give the collection of vertices of our quiver a $\mathbb{Z}$-grading, so that arrows have degree 1, and $\phi$ has degree 2. We now alter scalars inductively. Arrows from vertices of degree 0 to vertices of degree 1 we leave alone. An arrow $a$ from degree 1 to degree 2 lies in $P(t(a))$, and in no other $P(w)$. Therefore, multiplying arrows between vertices of degree 1 and degree 2 by nonzero scalars if necessary, we can force squares in quiver degree $0, 1, 2$ to commute. Similarly, multiplying arrows in degree $2, 3$ by scalars, we can force squares in quiver degree $1, 2, 3$

to commute. And so on. Working backwards, make squares in degree $-1, 0, 1$ commute and so on. $\square$

Suppose $\Gamma$ is a bipartite graph. The double quiver of $\Gamma$ is the quiver which has vertices as $\Gamma$ and a pair of opposing arrows running along each edge of $\Gamma$.

Definition 17. *Let $B_\Gamma$ denote the self-injective directed algebra $R(Q_\Gamma)$. Let $A_\Gamma$ denote the trivial extension $T(B_\Gamma)$ of $B_\Gamma$. Let $\mathcal{C}_\Gamma$ denote the category $A_\Gamma$-mod.*

We define $Z_\Gamma$ to be the zigzag algebra associated to $\Gamma$ [7]. It is the path algebra of the double quiver associated to $\Gamma$ modulo relations insisting that all quadratic paths based at a single vertex are equal, whilst all other quadratic relations are zero. Since the relations are homogeneous, $Z_\Gamma$ is a $\mathbb{Z}_+$-graded algebra with homogeneous elements graded by path length.

Lemma 18. *The category $Z_\Gamma$-mod is Calabi-Yau of dimension 0. We have an equivalence*

$$Z_\Gamma\text{-gr} \simeq B_\Gamma\text{-mod}^{\oplus 2}$$

*between the category $Z_\Gamma$-gr of graded modules of $Z_\Gamma$, taken with respect to the $\mathbb{Z}_+$-grading by path length, and the direct sum of two copies of $B_\Gamma$-mod.*
*Under this equivalence, twisting by the automorphism $\phi$ of $Q_\Gamma$ corresponds to a degree shift by 2 in $Z_\Gamma$-gr.*

*Proof.* The irreducible objects of $Z_\Gamma$-gr are $S\langle i\rangle$, where $S$ is an irreducible $Z_\Gamma$-module concentrated in degree 0. There are homomorphisms in $Z_\Gamma$-gr between $S\langle i\rangle$ and $T\langle j\rangle$ precisely when $S = T$ and $i = j$. There is an extension in $Z_\Gamma$-gr of $S\langle i\rangle$ by $T\langle j\rangle$ precisely when there is an extension between $S$ by $T$ in $Z_\Gamma$-mod and $j = i+1$. In particular when there exists such an extension, $S$ corresponds to a vertex in $V_l$ and $T$ corresponds to a vertex in $V_r$. We thus have two blocks in $Z_\Gamma$-gr: one block is generated by $S\langle i\rangle$ where $S$ lies in $V_l$ and $i$ is even or $S$ lies in $V_r$ and $i$ is odd; the other block is generated by $S\langle i\rangle$ where $S$ lies in $V_r$ and $i$ is even or $S$ lies in $V_l$ and $i$ is odd. It is not difficult to see that each block is isomorphic to $B_\Gamma$-mod so that the automorphism $\phi$ corresponds to a degree shift $\langle 2\rangle$. $\square$

For a quiver $Q$, we define $P_Q$ to be the path algebra of $Q$, modulo the ideal of all paths of length $\geq 2$.

Lemma 19. *For every orientation $\vec{\Gamma}$ of the bipartite graph $\Gamma$, we have an isomorphism*

$$Z_\Gamma \cong T(P_{\vec{\Gamma}})$$

*between $Z_\Gamma$ and the trivial extension algebra $T(P_{\vec{\Gamma}})$ of $P_{\vec{\Gamma}}$ by its dual.*

*Proof.* Projectives for $P_{\vec{\Gamma}}$ take two shapes: they are either of Loewy length two, hence have a simple top with a certain number of extensions, or they are simple. Similarly injectives are simple in the first case or of length two with a simple socle and a certain number of simples in the top in the second case. Projectives for $T(P_{\vec{\Gamma}})$ are extensions of projectives for $P_{\vec{\Gamma}}$ by injectives for the same algebra, hence either of a module of Loewy length two with a certain number of simples in the socle by a simple or of a simple by a module of Loewy length two with a simple socle and some composition factors in the top. In both cases top and socle of the resulting extension have to be simple which forces, in the first case, all of the simples in the socle of the $P_{\vec{\Gamma}}$-projective to extend the simple $P_{\vec{\Gamma}}$-injective, and in the second case, the simple $P_{\vec{\Gamma}}$-projective to extend all the simples in the top of the $P_{\vec{\Gamma}}$-injective. This is the same as saying that for every arrow in $\vec{\Gamma}$ the quiver for $T(P_{\vec{\Gamma}})$ has an arrow in the opposite direction as well, and that all quadratic paths based at a single vertex are the same (we can easily get rid of scalars by rescaling the arrows) while all other quadratic relations are zero. This exactly describes the algebra $Z_\Gamma$. □

In this way, every orientation $\vec{\Gamma}$ of the graph $\Gamma$ defines a $\mathbb{Z}_+^{\{f,a\}}$-grading on $Z_\Gamma$, whose $f$ component corresponds to the $\mathbb{Z}_+$-grading of $P_{\vec{\Gamma}}$ by path length, and whose $a$ component corresponds to the $\mathbb{Z}_+$-grading of $T(P_{\vec{\Gamma}})$ which puts $P_{\vec{\Gamma}}$ in degree 0 and its dual in degree 1.

Correspondingly, the orientation $\vec{\Gamma}$ of $\Gamma$ gives rise to a $\mathbb{Z}_+^{\{f,a\}}$-grading of the associated selfinjective directed algebra $B_\Gamma$ as follows: define a bigrading of the corresponding topsy-turvy quiver by grading arrows with an $f$ if they run with the orientation $\vec{\Gamma}$ of $\Gamma$, and grading them $a$ if they run against the orientation. This grading extends to a $\mathbb{Z}_+^{\{f,a\}}$-grading of $B_\Gamma$.

## 6. Hicas of length ≤ 4

The following is a classical statement which holds for any quasi-hereditary algebra:

Lemma 20. (a) ${}_A\Delta \cong (\nabla_A)^*$
(b) ${}_A\nabla \cong (\Delta_A)^*$

Lemma 21. *Suppose* $\mathcal{C} = A$-mod *is a highest weight category which is Calabi-Yau of dimension* 0*, and Ringel self-dual. Then* $A$ *is quasi-hereditary with respect to two orders, denoted* ▲ *and* ▼*, and we have*

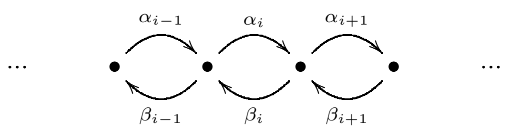
(a) ${}_A\Delta^{\blacktriangle} \cong {}_A\nabla^{\blacktriangledown}$
(b) ${}_A\Delta^{\blacktriangledown} \cong {}_A\nabla^{\blacktriangle}$

*Proof.* Let us suppose the quasi-hereditary structure on $A$ is given by the partial order ▲, and the one induced by Ringel duality is ▼. Since $A$ is Ringel self-dual, we have an isomorphism $A \cong A'$. Say that under this homomorphism the right projective $e_xA$ corresponding to $x \in \Lambda$ goes to the right projective $e'_yA'$ for some $y \in \Lambda$. Then by $\mathrm{Hom}_A(Ae_x, A) \cong e_xA \cong e'_yA' = \mathrm{Hom}_A(T(y), A)$ for $T(y)$ the tilting module for $y$ and the fact that any projective for $A$ is also injective and therefore tilting, it follows that $T(y) = P(x)$. So all tilting modules are projective $A$-modules. So, there is a 1-1-correspondence between tilting modules and projective modules for $A$, say it is, in the above scenario given by $y = \sharp x$. In particular this gives a one-to-one correspondence between standard modules and their socles $x = \operatorname{soc} \Delta^{\blacktriangle}(\sharp x)$. This makes the definition $\Delta^{\blacktriangledown}(x) := \nabla^{\blacktriangle}(\sharp x)$ well-defined. Filtrations of projectives by $\Delta^{\blacktriangledown}$s as well as the respective ordering conditions follow immediately from the dual statments for injectives (=projectives) and $\nabla^{\blacktriangle}$s. □

We wish to classify hicas of length $\leq 4$. To warm up, let us classify hicas of length $\leq 3$.

Lemma 22. *Hicas of length* 1 *are semisimple. There are no hicas of length* 2*. There is a unique hica of length* 3*, which is the module category of the Brauer tree algebra associated to a bi-infinite line.*

*Proof.* Length 1 hicas are trivially semisimple.
Suppose $\mathcal{C} = A$-mod is a hica of length 2. Standard objects in $\mathcal{C}$ must have length 2, since $\mathcal{C}$ is indecomposable, but not semisimple. Since $\mathcal{C}$ itself has length 2, all projective objects in $\mathcal{C}$ also have length 2. Thus standard objects are projective, and the socle of a projective indecomposable object $Ae_x$ has irreducible summands indexed by elements $y$ of $\Lambda$ with $y < x$. Since $A$ is a symmetric algebra, the top and socle of $Ae_x$ are equal, which is a contradiction. Therefore there are no hicas of length 2.
Suppose $\mathcal{C} = A$-mod is a hica of length 3. Then $l({}_A\Delta) + l(\Delta_A) = 4$, by Lemma 6. We have $1 \leq l({}_A\Delta), l(\Delta_A) \leq 3$ since $\mathcal{C}$ has length 3. It is impossible that $l({}_A\Delta) = 3$, since this would imply standard objects are projective, leading to a contradiction as in the case when $\mathcal{C}$ is a hica of length 2. It is dually impossible that $l(\Delta_A) = 3$. Therefore $l({}_A\Delta) = l(\Delta_A) = 2$. The next step is to show our hica $\mathcal{C}$ of length 3 is Ringel self-dual. This follows just in the proof of Ringel duality for hicas of length 4 in Lemma 25 below: it is only necessary to replace the numbers 4 and 3 by the numbers 3 and 2. Since a standard object $\Delta(x)$ is a costandard object for some other ordering, by Lemma 21, $\Delta(x)$ must have an irreducible socle $x_{-1}$, as well as an irreducible top $x = x_0$. Likewise, $x$ is the socle of some standard object $\Delta(x_1)$, for some $x_1 > x$. The projective $Ae_x$ has a filtration whose sections are $\Delta(x_1)$ and $\Delta(x_0)$; it is not possible there are any other standard objects in a $\Delta$-filtration since the existence of such would imply either the socle or top of $Ae_x$ was not irreducible. We conclude $Ae_x$ has top and socle isomorphic to $x_i$, and top modulo socle isomorphic to $x_{-1} \oplus x_1$. Inductively, we find $x_i \in \Lambda$, for $i \in \mathbb{Z}$, such that $Ae_{x_i}$ has a filtration whose top and socle are isomorphic to $x_i$, and top modulo socle isomorphic to $x_{i-1} \oplus x_{i+1}$. It follows $A$ is isomorphic to the path algebra of the quiver




modulo relations $\alpha_{i+1}\alpha_i = \beta_i\beta_{i+1} = 0$, and relations $\alpha_i\beta_i - \lambda_i\beta_{i+1}\alpha_{i+1} = 0$, for some nonzero $\lambda_i \in k$. Rescaling the generators if necessary, we may take all $\lambda_i = 0$. Thus $A$ is isomorphic to the Brauer tree algebra associated to a bi-infinite line. □

Let us now assume $\mathcal{C}$ is a hica of length 4. Thus $\mathcal{C} = A$-mod for a symmetric quasi-hereditary $\Delta$-homogeneous algebra $A$ of Loewy length 4.

Lemma 23. *The endomorphism ring of a projective indecomposable object in $\mathcal{C}$ is isomorphic to $F[d]/d^2$.*

*Proof.* The top and socle of a projective indecomposable are isomorphic, and such a simple cannot appear in either of the middle radical layers as this would imply a self-extension of the simple, contradicting quasi-heredity. □

Lemma 24. *Either ${}_A\Delta$ has length 3 and $\Delta_A$ has length 2, or else ${}_A\Delta$ has length 2 and $\Delta_A$ has length 3.*

*Proof.* Since $A$ is a hica, we have

$$l({}_A\Delta) + l(\Delta_A) = 5.$$

It is impossible that $l({}_A\Delta) = 1$ since this would imply that $A_A = \Delta_A$, which contradicts Lemma 23. Likewise it is impossible that $l(\Delta_A) = 1$. It follows that $\{l({}_A\Delta), l(\Delta_A)\} = \{2, 3\}$, as required. □

We use $<$ to mean "less than, in the order $\blacktriangle$".

Lemma 25. *$\mathcal{C}$ is Ringel self-dual.*

*Proof.* To say that $\mathcal{C}$ is Ringel self-dual is to say that ${}_AA$ is a full tilting module for $A$. This is equivalent to saying that $A_A$ is a full tilting module for $A$ (consider finite dimensional quotients/subalgebras, and pass to a limit). In other words, $A$ is left Ringel self-dual if and only if $A$ is right Ringel self-dual. To establish the Ringel self-duality of $\mathcal{C}$, we may therefore assume that ${}_A\Delta$ has length 3, by Lemma 24.
Suppose $\mathcal{C}$ is not Ringel self-dual. Then we have a nonprojective indecomposable tilting module $T(\lambda)$, which has a filtration with sections

$$\Delta(\lambda), \Delta(\lambda_2), ..., \Delta(\lambda_n).$$

Note that $\Delta(\lambda)$ is the bottom section, and up to scalar we have a unique homomorphism from $P(\lambda)$ to $T$ whose image is $\Delta(\lambda)$ (reference Ringel). Since $T$ is nonprojective, it has length $< 4$. Since the sections all have length 3, the tilting module has length 3, and the tops of the sections all lie in the top of $T$. The module $T$ also has a $\nabla$-filtration since it is tilting. Any simple in the top of $T$ must lie in the top of some $\nabla$ of length 2. In particular, $\lambda$ itself must lie in the top of some $\nabla(\mu)$ of length 2. The resulting homomorphism $P(\lambda) \to \nabla(\mu)$ must lift to a homomorphism $P(\lambda) \to T$. Up to scalar, there is a unique such homomorphism whose image is $\Delta(\lambda)$, implying that $\mu$ is a factor of $\Delta(\lambda)$. Thus, $\lambda$ is a factor of $\nabla(\mu)$ and $\mu$ is a factor of $\Delta(\lambda)$. Thus $\lambda > \mu > \lambda$ which is a contradiction. □

Lemma 26. *Standard modules for $A$ have irreducible head and socle.*

*Proof.* A standard module in one ordering is isomorphic to a costandard module in another ordering, by Lemmas 21 and 25. □

If there is a nonsplit extension of $\lambda$ by $\mu$ then either $\lambda > \mu$ or $\lambda < \mu$. We define *Rel* to be the set of relations $\lambda > \mu$ or $\lambda < \mu$ of this kind. We define $\uparrow$ to be the partial order on $\Lambda$ generated by *Rel*. The order $\blacktriangle$ is a refinement of $\uparrow$.
We define $\downarrow$ to be the ordering on $\Lambda$ which is Ringel dual to $\uparrow$.

Lemma 27. *$\mathcal{C}$ is a highest weight category with respect to the partial order $\uparrow$ on $\Lambda$.*

*Proof.* $\mathcal{C}$ has length 4, which implies that either left or right standard modules have length two and, by Lemma 26, are therefore uniserial. The quasi-hereditary structure induced by $\blacktriangle$ is already determined by these non-split extensions and therefore the order $\uparrow$ already induces the same quasi-hereditary structure as its refinement $\blacktriangle$. □

From now on, whenever we refer to standard or costandard modules, or to orderings, without specifying the order, we mean the order $\uparrow$.
We say an $A$-module $M$ is *directed*, if given a subquotient of $M$ which is a non-split extension of a simple module $\lambda$ by a simple module $\mu$, $\lambda$ is greater than $\mu$.

Lemma 28. *Standard $A$-modules are directed.*

*Proof.* We want that all standard modules are directed, which means for any subquotient of a standard module which is a non-split extension of simple modules $\lambda$ by $\mu$, $\lambda$ is greater than $\mu$. This is trivial for a standard module of Loewy length 2.
Let $\Delta(x)$ be a standard module of Loewy length 3. It must have an irreducible socle $y$ by Lemma 26. Thus $\Delta(x)$ appears in a $\Delta$-filtration of $P(y)$. $\Delta(y)$ appears as the top factor of a $\Delta$-filtration of $P(y)$. Indeed, since $P(y)$ has length 4 with irreducible top and socle, a $\Delta$-filtration of $P(y)$ has precisely two factors, namely $\Delta(x)$ and $\Delta(y)$.
The module $\Delta(y)$ must have irreducible socle $z$, where $y > z$, by Lemma 26. Since $P(y)$ has length 4 and $\Delta(y)$ has length 3, we conclude there is an extension of $z$ by $y$. Since $\nabla(y)$ is dual to a $\Delta$ which has length 2, $\nabla(y)$ itself has length 2, and it must in fact be this extension of $z$ by $y$.
For any other nonsplit extension of an irreducible modules $w$ by $y$, we must have $w > y$ by Lemma 27. These are precisely the extensions of $w$ by $y$ contained in $\Delta(x)$. The extensions of $x$ by $w$ contained in $\Delta(x)$ imply $x > w$ by definition of a standard module. Thus any extension of $\lambda$ by $\mu$ in $\Delta(x)$ implies $\lambda > \mu$ as required. □

Corollary 29. *The orders $\uparrow$ and $\downarrow$ on $\Lambda$ are opposite.*

*Proof.* Just as standard modules are directed in the $\uparrow$ ordering, costandard modules are directed in the $\downarrow$ ordering. But standard modules in the $\uparrow$ ordering are equal to costandard modules in the $\downarrow$ ordering by Lemmas 21 and 25. Therefore $\uparrow$ and $\downarrow$ orderings are opposite, as required. □

REMARK 30 If a finite-dimensional algebra is quasi-hereditary with respect to two opposing orders then it must be directed, in which case the standard modules are projectives in one ordering, and simples in the opposite ordering. This can easily be proved by induction on the size of the indexing set. Symmetric quasi-hereditary algebras are never directed, since their projective indecomposable objects have isomorphic head and socle.

REMARK 31 It is not necessarily the case that a Ringel self-dual hica is a highest weight category with respect to two opposing orderings. Examples of length 5 are found amongst module categories of rhombal algebras [3].

Let $X(\lambda)$ denote the kernel of the surjective homomorphism $\Delta(\lambda) \twoheadrightarrow \lambda$, for $\lambda \in \Lambda$.

DEFINITION 32. *The* $\Delta$-quiver *of $A$ is the quiver with vertices indexed by $\Lambda$, and with arrows $\lambda \to \mu$ corresponding to simple composition factors $\mu$ in the top of $X(\lambda)$.*

LEMMA 33. *Components of the $\Delta$-quiver of length 2 are directed lines. Components of the $\Delta$-quiver of length 3 are directed topsy-turvy quivers.*

*Proof.* The length 2 case is easy.
In length 3, we have a permutation $\phi \circlearrowleft \Lambda$ which takes $\lambda$ to the socle of $\Delta(\lambda)$. We prove that $\mathcal{F}(\lambda) = \mathcal{P}(\lambda^\phi)$ via a sequence of correspondences: arrows emanating from $\lambda$ in the $\Delta$-quiver are in correspondence with simple composition factors $\mu$ in the top of $X(\lambda)$; simple composition factors $\mu$ in the top of $X(\lambda)$ are in correspondence with extensions of $\lambda$ by $\mu$ such that $\lambda > \mu$; since $\Delta^\uparrow(\lambda) = \nabla^\downarrow(\lambda^\phi)$, whilst $\uparrow$ and $\downarrow$ are opposites, extensions of $\lambda$ by $\mu$ such that $\lambda > \mu$ are in one-one correspondence with extensions of $\mu$ by $\lambda^\phi$ such that $\mu > \lambda^\phi$; extensions of $\mu$ by $\lambda^\phi$ such that $\mu > \lambda^\phi$ are in correspondence with simple composition factors $\lambda^\phi$ in the top of $X(\mu)$; simple composition factors $\lambda^\phi$ in the top of $X(\mu)$ are in one-one correspondence with arrows into $\lambda^\phi$ in the $\Delta$-quiver.
Since standard modules are directed, the $\Delta$-quivers are also directed (ie they generate a poset). $\square$

We next find $\Delta$-subalgebra of $A$, in the sense of S. Koenig [10].

LEMMA 34. *$A$ has a $\Delta$-subalgebra $B$.*

*Proof.* We want to find $B$ such that ${}_B\Delta \cong {}_BB$. Let us write $A = FQ/I$ as the path algebra of $Q$ modulo relations, where $Q$ is the $Ext^1$-quiver of $A$. If there is a *positive* arrow $x \to y$ in $Q$, that is to say an arrow $x \to y$ in $Q$ such that $x > y$, then $x$ and $y$ lie in the same component of the $\Delta$-quiver. Since all standard modules are directed, the connected component of the quiver generated by these arrows are the components of the $\Delta$-quiver.
Let $B$ be the subalgebra of $A$ generated by arrows $x \to y$ in $Q$ such that $x > y$. Since all standard modules are directed, composing the natural maps

$Be_\lambda \to Ae_\lambda \to \Delta(\lambda)$ gives us a surjection $Be_\lambda \twoheadrightarrow \Delta(\lambda)$. To establish this composition map is an isomorphism, we have to worry about its kernel, which must lie in $\mathrm{Rad}^2(B)e_\lambda$, which is the socle of $B$ since $A$ has length 4. Assume there is a simple $S$ in the kernel. Then $S$ would have to be a factor of $\Delta(\mu)$ in a $\Delta$-filtration of $Ae_\lambda$; restrictions imply $S$ would lie in the socle of some $\Delta(\mu)$ of length 2, where $\mu > \lambda$ (otherwise if $\Delta(\mu)$ has length 3 then $\lambda$ lies in the socle of $\Delta(\mu)$ so $S > \lambda$, $\lambda > S$ since $S$ appears in $Be_\lambda$, contradiction). Since $S$ lies in $\mathrm{Rad}^2(B)e_\lambda$, we have positive arrows $\lambda \to \nu \to S$, for some $\nu$, so $S$ must lie in $\Delta(\nu)$, and there is an arrow $\lambda \to \nu$ in the $\Delta$-component of $\lambda$. There are now two possibilities. Either $\Delta(\lambda)$ has length 3, implying $S$ lies in a $\Delta$-quiver component of length 2 (for $\mu$), and a $\Delta$-quiver component of length 3 (for $\lambda$)- contradiction. Else $\Delta(\lambda)$ has length 2, which implies we have a $\Delta$-quiver component of length 2 containing the quiver $\mu \to S \leftarrow \nu$ - contradiction (the structure of any length 2 $\Delta$-quiver component is an oriented line by Lemma 33). We conclude that the map $B \twoheadrightarrow \Delta$ must in fact have zero kernel, ie $B$ is a $\Delta$-subalgebra.

□

Let $B$ be the $\Delta$-subalgebra of $A$.

Lemma 35. *Suppose $B$ has length 3. Then the algebra homomorphism $B \to A$ splits.*

*Proof.* Let $I$ denote the ideal of $A$ which is a sum of spaces $AaA$ where $a$ is a *negative* arrow in the quiver $Q$ of $A$. Then the kernel $J$ of the $A$-module homomorphism $A \to {}_A\Delta$ is contained in $I$, since $A$ has length 4 and $\Delta$s have length 3, implying $J$ is generated in the top of the radical of $A$. Also, $J$ contains $I$ since $I$ is generated as a vector space by products of 1, 2, or 3 arrows in the quiver, at least one of which lies in $I$, and these products all lie in $J$ since all $\Delta$s are directed. Thus the kernel of $A \to {}_A\Delta$ is equal to $I$. By symmetry, the homomorphism of right $A$-modules $A \to \Delta_A$ also has kernel $I$. Therefore $B \oplus I \to A$ is an isomorphism of $B$-$B$-bimodules, and the algebra homomorphisms

$$B \to A \to A/I$$

compose to give an algebra isomorphism $B \cong A/I$. Therefore the homomorphism $B \to A$ splits as required. □

Lemma 36. *$B$ is self-injective.*

*Proof.* We write ${}^\uparrow B$ for the ${}_A\Delta$-subalgebra taken with respect to the $\uparrow$ ordering, and $B^\downarrow$ the $\Delta_A$-subalgebra taken with respect to the $\downarrow$ ordering. We know that

$$B = {}^\uparrow B \cong \bigoplus_{x\in\Lambda} {}_A\Delta^\uparrow(x) \cong \bigoplus_{x\in\Lambda} {}_A\nabla^\downarrow(x) \cong \bigoplus_{x\in\Lambda} (\Delta^\downarrow_A(x))^* \cong (B^\downarrow)^*,$$

where $B^\downarrow$ is also a $\Delta$-subalgebra of $A$. Thus ${}^\uparrow B \cong (B^\downarrow)^*$ as $A$-modules, and therefore as ${}^\uparrow B$-modules. To prove ${}^\uparrow B$ is self-injective we must show that ${}^\uparrow B \cong B^\downarrow$. Indeed, ${}^\uparrow B$ is defined to be the subalgebra generated by left positive

$\uparrow$-arrows, whilst $B^{\downarrow}$ is defined to be the subalgebra generated by right positive $\downarrow$-arrows. Passing from the left regular action of an algebra on itself to the right regular action reverses arrow orientation. Therefore left positive $\uparrow$-arrows are equal to right negative $\uparrow$-arrows which are equal to right positive $\downarrow$-arrows. Thus ${}^{\uparrow}B \cong B^{\downarrow}$ as required. □

Lemma 37. *If $B$ has Loewy length 3, then $A$ is isomorphic to $T(B)$, the trivial extension algebra of $B$ by its dual.*
*If $B$ has Loewy length 2, then $A$ is isomorphic to $U(B, M)$ where $M$ is a self-dual $B$-$B$-bimodule.*

*Proof.* We may assume $B = B^{\uparrow}$ has Loewy length 3, in which case $B^{\downarrow}$ has Loewy length 2. We have a surjection of algebras $A \twoheadrightarrow B$ which splits, via an algebra embedding $B \hookrightarrow A$. Dually, we have an embedding of $A$-$A$-bimodules $B^* \hookrightarrow A^*$. Since $A \cong A^*$ as bimodules, we have a homomorphism of $A$-$A$-bimodules $B^* \hookrightarrow A$. Taking the sum of our two embeddings gives us a homomorphism of $B$-$B$-bimodules,

$$B \oplus B^* \to A.$$

This homomorphism is a bimodule isomorphism, because every projective indecomposable $A$-module has a canonical $\Delta$-filtration featuring precisely two $\Delta(\lambda)$s, one of which is a summand of $B$, and the other of which is a summand of $B^*$. We can thus identify the image of $B^*$ in $A$ with the kernel of the algebra homomorphism $A \to B$. The image of $B^*$ in $A$ multiplies to zero, because the map $B^* \to A$ is a homomorphism of $A$-$A$-bimodules, on which the kernel of the surjection $A \twoheadrightarrow B$ acts trivially. The image of $B$ in $A$ multiplies via according to multiplication in $B$. In other words, the map $T(B) = B \oplus B^* \to A$ is an algebra isomorphism, as required.
The algebra $A$ has a $\mathbb{Z}_+^2$-grading whose first component comes from the radical grading on $B^{\uparrow}$, and whose second component comes from the trivial extension grading, with $B^{\uparrow}$ in degree 0 and its dual in degree 1. In other words, the degree $(*, 0)$ part of $A$ is $B^{\uparrow}$. We can then identify the degree $(0, *)$ part of $A$ with $B^{\downarrow}$, which is self-injective of Loewy length 2. The degree $(2, *)$ part of $A$ is then isomorphic to $B^{\downarrow *}$, and we define $M$ to be the degree $(1, *)$ part of $A$. The isomorphism $A \cong A^*$ exchanges the $B^{\downarrow}$-$B^{\downarrow}$-bimodules $B^{\downarrow}$ and $B^{\downarrow *}$, whilst it defines an isomorphism $M \cong M^*$. This way, we obtain the algebra isomorphism $A \cong U(B^{\downarrow}, M)$. □

Let $\mathfrak{Bip}$ denote the 2-category whose objects are bipartite graphs; whose arrows $\Gamma \to \Gamma'$ are given by sequences $(\gamma_1, ..., \gamma_n)$ of distinct vertices of $\Gamma$, such that $\Gamma' = \Gamma \backslash \{\gamma_1, ..., \gamma_n\}$; whose 2-arrows are given by permutations of such sequences.
The following result is a refinement of Theorem 1.

Theorem 38. *The correspondence $\Gamma \mapsto \mathcal{C}_\Gamma$ extends to a 2-functor*

$$\mathfrak{Bip} \to \mathfrak{Hica}_4$$

*which is essentially bijective on objects.*

*Proof.* The correspondence $\Gamma \mapsto A_\Gamma$ -mod is essentially bijective on objects, by Lemmas 16, 34, 36, and 37.
We have to associate functors and natural transformations in $\mathfrak{Hica}_4$ to arrows and 2-arrows in $\mathfrak{Bip}$. Suppose $\gamma \in \Gamma$ is a vertex of a bipartite graph, and $\Gamma' = \Gamma \backslash \gamma$. We have an isomorphism $A_{\Gamma'} \cong e_{\Gamma'} A_\Gamma e_{\Gamma'}$, and therefore an exact functor

$$F_\gamma = e_{\Gamma'} A_\Gamma \otimes_{A_\Gamma} : A_\Gamma\text{-mod} \to A_{\Gamma'}\text{-mod}$$

which sends the irreducible corresponding to a vertex $v$ to the irreducible corresponding to a vertex $v$, if $v \neq \gamma$ and to zero if $v = \gamma$. To a sequence $(\gamma_1, ..., \gamma_n)$ we associate the composition functor $F_{\gamma_n} ... F_{\gamma_1}$. There are natural isomorphisms between various functors corresponding to isomorphisms of bimodules. □

Let $B^\uparrow = FQ^\uparrow / R^\uparrow$, $B^\downarrow = FQ^\downarrow / R^\downarrow$ be minimal presentations of $B^\uparrow$ and $B^\downarrow$ by quiver and relations.
Let $Q$ be the union of $Q^\uparrow$ and $Q^\downarrow$ in which we identify the vertices of these quivers if they represent the same irreducible $A$-module. Let $R$ be the union of $R^\uparrow$, $R^\downarrow$ and $R^\updownarrow$. Let $R^\updownarrow$ denote the set of relations which insist that squares in $Q$ involving two arrows of $Q^\uparrow$ and two arrows of $Q^\downarrow$ commute.

Lemma 39. *$A = FQ/R$ is a minimal presentation of $A$ by quiver and relations.*

*Proof.* We have a surjective map $FQ \twoheadrightarrow A$. It is not difficult to see this must factor through a map $FQ/R \twoheadrightarrow A$. We now want to bound the dimension of a projective of $FQ/R$. Without loss of generality assum that $B = B^\uparrow$ has Loewy length 3 and $B^\downarrow$ therefore has Loewy length 2. So $Q^\uparrow$ is a topsy-turvy quiver and $Q^\downarrow$ is linear. We claim that a spanning set of $(FQ/R)e_x$ is given by $abe_x$ where $b \in B$ and $a$ is either an idempotent or an arrow from $Q^\downarrow$. Without a doubt a spanning set is given by the union of all elements of the form $a_1 b_1 \cdots a_r b_r e_x$ where $a_i$ are either idempotents or arrows in $Q^\downarrow$ and $b_i \in B$. However, if we have an arrow $a$ in $Q^\downarrow$ (say with source $y$ and target $\phi^{-1}(y)$) and and arrow $b \in Q^\uparrow$ starting in $\phi^{-1}(y)$ , the product $bae_y = be_{\phi^{-1}y} ae_y$ equals $a'b'e_y$ where $b' = \phi(b)$ and $a'$ is the unique arrow starting at the end vertex of $b'e_y$. Indeed, $Q^\uparrow$ being topsy turvy implies the existence of $b'$ and in $Q^\downarrow$ there is an arrow from $x$ to $\phi^{-1}x$ for every x. So denoting by $z$ the end vertex of $b$, there is a square

$$\begin{array}{ccc} y & \xrightarrow{a} & \phi^{-1}(y) \\ \downarrow{\scriptstyle \phi(b)} & & \downarrow{\scriptstyle b} \\ z & \xrightarrow{a'} & \phi^{-1}(z) \end{array} .$$

By the required relations this has to commute and we obtain $bae_y = a'b'e_y$. Hence the path $a_1 b_1 \cdots a_r b_r e_x$ is equivalent modulo $R$ to a path $a'_1 \cdots a'_r b'_1 \cdots b'_r = a'_1 \cdots a'_r b'$. However, by the relations in $B^\downarrow$, any product of arrows in $Q^\downarrow$ is zero, so we obtain the claim that $(FQ/R)e_x$ is spanned by $abe_x$ where $b \in B$ and $a$ is either an idempotent or an arrow from $Q^\downarrow$.

This implies that $\dim(FQ/R)e_x \leq 2\dim Be_x = \dim(B+B^*)e_x = \dim Ae_x$, the equality $\dim Be_x = \dim(B+B^*)e_x$ coming from the fact that $B$ is self-injective. Combining the above surjection $FQ/R \twoheadrightarrow A$ and this inequality, we obtain the statment of the lemma. □

## 7. Koszulity

For an algebra $C$ we denote by $C^!$ the quadratic dual of $C$.

Theorem 40. *The following are equivalent:*

1. $\Gamma$ *is not a simply laced Dynkin graph.*
2. $Z_\Gamma$ *is Koszul.*
3. $B_\Gamma$ *is Koszul.*
4. $A_\Gamma$ *is Koszul.*
5. $A_\Gamma^!$-mod *is Calabi-Yau of dimension* 3.

The length of the proof of this result is the length of the section.

1 is equivalent to 2, by a theorem of Martínez-Villa [11].

2 is equivalent to 3, since $B_\Gamma$-mod$^{\oplus 2}$ is equivalent to $Z_\Gamma$-gr by Lemma 18. The implication $3 \Rightarrow 4$ follows from the following lemma, in case $A = A_\Gamma$, and $B = B_\Gamma$.

Lemma 41. *If $B$ is a self-injective Koszul algebra of length $n$, the trivial extension algebra $A = B \oplus B^*\langle n\rangle$ is Koszul.*

*Proof.* Since $B$ is selfinjective, we have an isomorphism $B \cong B^*$ of $B$-modules. The algebra $A$ is a trivial extension $A = B \oplus B^*$, and we thus have a map $A \to A$ of $B$-modules extending to a map of $A$-modules whose kernel is $B^*$ and whose cokernel is $B$. Stringing these together gives us a projective resolution

$$... \to A \to A \to B$$

of $B$ as a left $A$-module. Since $B$ is self-injective and radical graded, every injective $B$-module has length $n$, and consequently this is a linear resolution of $B$ as a left $A$-module. Taking summands, we find that every projective $B$-module has a linear resolution as a left $A$-module.

If $B$ is Koszul, then $B^0$ has a linear resolution by projective $B$-modules. Thus $B^0$ is quasi-isomorphic to a linear complex of projective $B$-modules. Since projective $B$-modules are quasi-isomorphic to a linear complex of projective $A$-modules, we deduce $B^0$ is isomorphic to a linear complex of projective $A$-modules. That is to say, $A^0 = B^0$ has a linear resolution by projective $A$-modules. In other words, $A$ is Koszul. □

The implication $4 \Rightarrow 3$ follows from the following lemma, in case $A = A_\Gamma$, and $B = B_\Gamma$.

Lemma 42. *If $B$ is a radical-graded selfinjective algebra of length $n$, such that $A = B \oplus B^*\langle n\rangle$ is Koszul, then $B$ is Koszul.*

*Proof.* We have a $\mathbb{Z}_+ \times \mathbb{Z}_+$-grading on $A$ in which $B$ lies in degree $(?, 0)$, the dual of $B$ lies in degree $(?, 1)$, and in which the inherent $\mathbb{Z}_+$-grading on $B$ is the radical grading. This corresponds to the action of a two-dimensional torus $\mathbb{T}$ on $A$. Thus $\mathbb{T}$ acts on $A^1$ and we have an exact sequence

$$0 \to R \to A^1 \otimes_{A^0} A^1 \to A^2 \to 0$$

of $\mathbb{T}$-modules, where $R$ denotes the relations for $A$ and $A^j$ refers to the $j$th component in the total grading, whose dual

$$0 \leftarrow A^{!2} \leftarrow A^{!1} \otimes_{A^{!0}} A^{!1} \leftarrow R^! \leftarrow 0$$

is also an exact sequence of $\mathbb{T}$-modules. Since $A^!$ is quadratic by definition, with relations $R^!$, we have an action of $\mathbb{T}$ on $A^!$, which gives a $\mathbb{Z}_+ \times \mathbb{Z}_+$-grading on $A^!$. We have a linear resolution of $W = A^{(0,0)}$, given by the Koszul complex

$$A \otimes_W A^{!*}$$

of $A$ ([1], 2.8). Here $A^{!*}$ denotes the graded dual of $A^!$. The differential on the Koszul complex respects the $\mathbb{Z}_+ \times \mathbb{Z}_+$-grading on $A$ and $A^!$ (see [1], 2.6). In other words, it sends terms involving arrows in $A^{(0,1)}$ or $A^{!(0,1)*}$ to terms involving arrows in $A^{(0,1)}$ or $A^{!(0,1)*}$, and terms not involving arrows in $A^{(0,1)}$ or $A^{!(0,1)*}$ to terms not involving arrows in $A^{(0,1)}$ or $A^{!(0,1)*}$. Consequently the subcomplex $A^{(?,0)} \otimes_R A^{!(-,0)*})$ is a direct summand of the Koszul complex regarded as a complex of $B$-modules. Taking this component gives us a linear resolution of $R = B^0$ as a $B$-module. Therefore $B$ is Koszul. □

If $C$ is a graded algebra and $C$-mod is Calabi-Yau of dimension $n$, then $Ext^*_C(C^0, C^0)$ is a super-symmetric algebra concentrated in degrees $0, 1, ..., n$, by Van den Bergh A.5.2 [2]. We have a converse which applies for Koszul algebras:

Theorem 43. *Suppose $K$ is a Koszul algebra such that $K^!$ is super-symmetric of length $n+1$, then $K$-mod is Calabi-Yau of dimension $n$.*

*Proof.* There is an equivalence between derived categories of graded modules for $K^!$ and $K$ via the Koszul complex. Since $K^!$ is locally finite dimensional, this restricts an equivalence of bounded derived categories, by a theorem of Beilinson, Ginzburg, and Soergel ([1], Theorem 2.12.6). Under this equivalence, simple $K^!$-modules correspond to projective indecomposable $K$-modules. Since $K^!$ is locally finite-dimensional The equivalence therefore restricts to an equivalence between $D^b(K^!\text{-gr})$ and $D^b(K\text{-grperf})$. Also under this equivalence, injective $K^!$-modules correspond to simple $K$-modules, whilst shifts $\langle i \rangle$ in $D^b(K^!\text{-gr})$ correspond to shifts in degree $\langle -i \rangle[-i]$ in $D^b(K\text{-grperf})$. This homological shift in degree means that the Calabi-Yau-$n$ property for $K$-mod is equivalent to the super-Calabi-Yau-0 property for $K^!$-perf, thanks to Van den Bergh's calculation A.5.2 [2]. To prove the super-Calabi-Yau-0 property for $K^!$-perf, it is enough to check that $K^!$ is a super-symmetric algebra (cf [17], Theorem 3.1). □

Assume 4. Then the Koszul dual $A^!$ of $A$ is Calabi-Yau of dimension 3. The Koszul dual of a supersymmetric algebra of length $n+1$ is Calabi-Yau of dimenion $n$ by Theorem 43. The trivial extension algebra $A = B + B^*\langle 3\rangle$ is super-symmetric with $B$ concentrated in degrees $0, 1, 2$, with $B^*\langle 3\rangle$ concentrated in degrees $3, 2, 1$, and with bilinear form pairing $B^i$ and $(B^i)^*\langle 3\rangle$ via

$$<\beta, b>= \beta(b) \qquad <b, \beta>= (-1)^{i(3-i)}\beta(b),$$

for $b \in B^i$, $\beta \in (B^i)^*$. Thus 4 implies 5.

Assume 5. Since $A^!$ is Calabi-Yau of dimension 3, its relations are the derivatives of a superpotential, and its degree 0 part has a 4-term resolution, its Jacobi resolution [2]. The superpotential must be cubic, since $A^!$ is quadratic. This implies further that the Jacobi resolution of $A^{!0}$ is linear, so $A^!$ must be Koszul. Thus 5 implies 4.

We have now shown that $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$, completing the proof of Theorem 40.

Remark 44 If $\Gamma$ is a bipartite graph, then an orientation of $\Gamma$ gives rise to a $\mathbb{Z}_+^3$-grading on $A_\Gamma$. If every vertex of $\Gamma$ is attached to at least two vertices, then this leads to a $\mathbb{Z}_+^3$-grading of the Calabi-Yau algebra $A^!$ of dimension 3, which can otherwise be thought of as the action of a 3-dimensional torus on $A^!$. The algebra $A^!$ has homological dimension 3, and admits the action of a 3-dimensional torus. It thus belongs to the realm of 3-dimensional noncommutative toric geometry.

Example 45 If $\Gamma$ is given by tiling of a bi-infinite line

$$\cdots \quad \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \cdots$$

then the Calabi-Yau algebra of dimension 3 we obtain is familiar from toric geometry. It is the algebra associated to the brane tiling of the plane by hexagons [6]. Its quiver can be thought of as an orientation of the $A_2$-lattice (for a picture, see section 8, assumption 3). If we give $\Gamma$ an alternating orientation,

$$\cdots \quad \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \quad \cdots$$

then in the resulting grading on $A^!$, the three copies of $\mathbb{Z}_+$ correspond to the three directions of arrows in the $A_2$-lattice.

## 8. Relaxing the assumptions

We have given a combinatorial classification of hicas of length $\leq 4$ by bipartite graphs. Here we show that the relaxation of any of the homological assumptions on our categories would necessarily introduce further combinatorial complexity into the classification.

*Assumption 1: highest weight structure.*

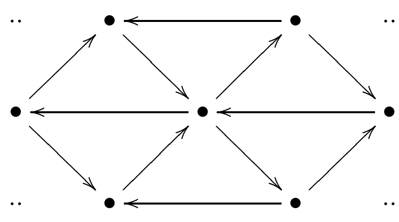
The existence of a highest weight structure on a category is a strong assumption, and the assumptions of Ringel self-duality and homogeneity of standard modules require the existence of a highest weight structure on the category. It is a cinch to give examples of indecomposable Calabi-Yau 0 categories of length 4 which are not highest weight categories, such as the module category of the local symmetric algebra $F[x]/x^4$.

*Assumption 2: Calabi-Yau* 0 *property.*
The Calabi-Yau property is another strong homological restriction on a category. An example of a length 4 highest weight category which is indecomposable and Ringel self-dual, and whose standard modules are homogeneous, is the path algebra of the linear quiver



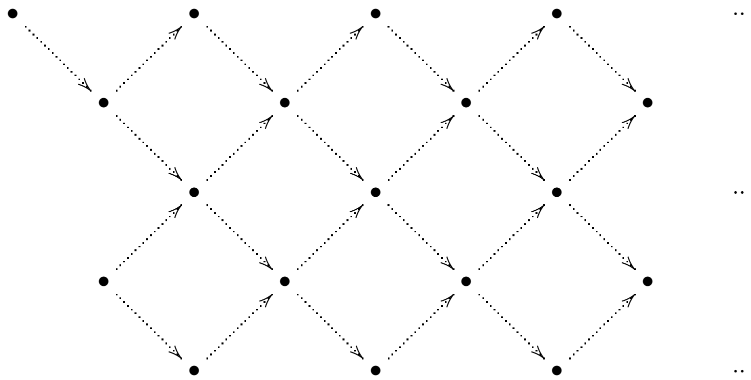
modulo all relations of degree $\geq 4$.

*Assumption 3: homogeneity.*
The homogeneity restriction on a hica is fairly natural, since the known examples of highest weight Calabi-Yau 0 categories arising in group representation theory and the theory of tilings satisfy this restriction. However, some interesting combinatorics arise in length 4 if the condition is dropped.
For example, let $\mathcal{C}_\Gamma$ be the hica associated to a bi-infinite line $\Gamma$, whose quiver is an orientation

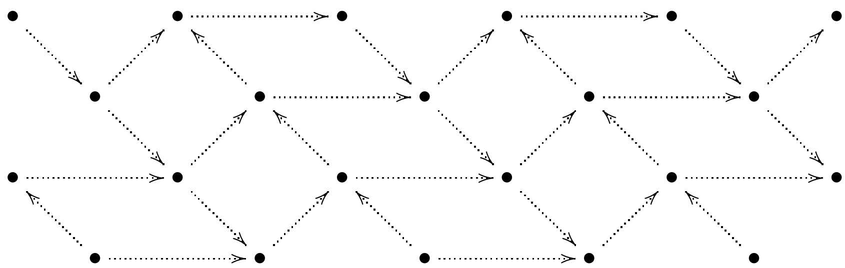


of the $A_2$ lattice, by example 45, and by construction comes with a (horizontal) projection $\pi$ onto $\Gamma$.
There are two natural ways to obtain highest weight indecomposable Calabi-Yau 0 categories which are not homogeneous from $\mathcal{C}_\Gamma$. The first is by choosing a section of $\pi$, that is a path in the $A_2$ lattice which projects homeomorphically onto $\Gamma$ via $\pi$. The elements of $\Lambda$ to the right of the path form a coideal in the poset. Truncating $\mathcal{C}_\Gamma$ at this coideal gives us a highest weight category of length 4 which is CY-0, but not homogeneous (cf. [4], 3.5(b)). Such a truncation is not Ringel self-dual. Beneath is a portion of such a truncated poset, whose left edge defines an orientation of $\Gamma$. We use dotted arrows to represent directions in a partial order on the vertices of the lattice, rather than solid arrows which

represent arrows in a quiver:



The second way to obtain an inhomogeneous highest weight indecomposable Calabi-Yau 0 category from $\mathcal{C}_\Gamma$ is by merely altering the ordering of the vertices. Certain orderings of the vertices of the $A_2$ lattice give $A_\Gamma$-mod is an inhomogeneous highest weight category, which is Ringel self-dual. Here is an example of a portion of such a partial ordering:



*Assumption 4: length* $\leq 4$.
We have studied hicas of length 4, since 4 is the shortest length in which a nontrivial classification is possible. There are two kinds of hicas of length 5: those whose left and right standard modules have length 4 and 2, and those whose left and right standard modules have length 3 and 3.
The category of graded modules over a radical-graded symmetric algebra of length 4 is equivalent to the category of modules over a directed self-injective algebras of length 4. Trivial extensions of directed self-injective algebras of length 4 by their duals give examples of hicas of length 5 whose left and right standard modules have length 4 and 2.
Michael Peach's rhombal algebras give examples of hicas of length 5 associated to rhombic tilings of the plane whose left and right standard modules have length 3 and 3.
We would be interested to learn more about hicas of length 5.

## 9. Tilting

There are natural self-equivalences of the derived categories of $\mathcal{C}_\Gamma$, which are obtained from a standard tilting procedure for symmetric algebras:

Lemma 46. *Suppose $A$ is a symmetric algebra, and suppose that the endomorphism ring $e_\lambda A e_\lambda$ is isomorphic to the dual numbers $F[d]/d^2$. Then we have an exact self-equivalence $\psi_\lambda$ of the derived category of $A$ given by tensoring with the two-term complex*

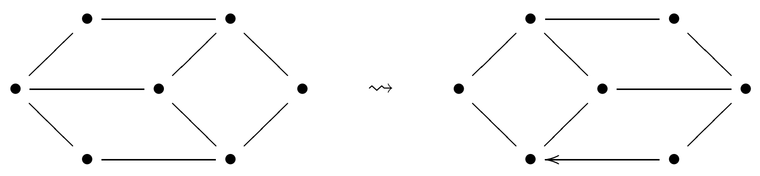
$$Ae_\lambda \otimes_F e_\lambda A \to A,$$

*whose arrow is the multiplication map.*

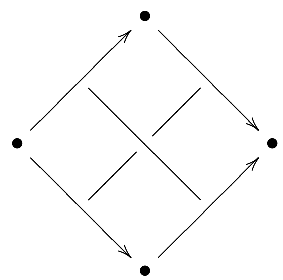
*Proof.* This functor is obviously exact. It fixes all simple modules with the exception of the simple top $\lambda$ of $Ae_\lambda$, which it sends to $\Omega(\lambda)$. The module $\Omega(\lambda)$ has simple socle $\lambda$ since $A$ is symmetric, and other composition factors different from $\lambda$ since $e_\lambda A e_\lambda$ is isomorphic to the dual numbers. Therefore collection of simples $\mu \neq \lambda$, along with $\Omega$ generate $D^b(A\text{-mod})$, and $\psi_\lambda$ is an equivalence. □

The self-equivalence $\psi_\lambda$ is called a spherical twist, because the cohomology ring of the sphere can be identified with the dual numbers (cf. [19]).
One way to obtain self-equivalences of $D^b(\mathcal{C}_\Gamma)$ from spherical twists is by lifting self-equivalences of the derived category of the zigzag algebra $Z_\Gamma$, whose projective indecomposable modules for the algebra $Z_\Gamma$ all have an endomorphism ring isomorphic to the algebra of dual numbers. A second way to obtain self-equivalences of $D^b(\mathcal{C}_\Gamma)$ is to apply spherical twists directly to $\mathcal{C}_\Gamma$, whose projective indecomposable objects also have endomorphism rings isomorphic to the dual numbers.
Let us consider the first case. The projective indecomposable modules for the algebra $Z_\Gamma$ all have an endomorphism ring isomorphic to the algebra of dual numbers. Standard tilts generate an action of a 2-category $\mathcal{T}_\Gamma$ on $D^b(Z_\Gamma\text{-gr})$ which lifts to an action of $\mathcal{T}_\Gamma$ on $D^b(\mathcal{C}_\Gamma)$, by a result of Rickard [16, Thm 3.1]. A second way to obtain self-equivalences of $D^b(\mathcal{C}_\Gamma)$ is to apply Seidel-Thomas twists directly to $A_\Gamma$, whose projective indecomposable modules have endomorphism rings isomorphic to the dual numbers. Standard tilts generate the action of a 2-category $\mathcal{U}_\Gamma$ on $D^b(\mathcal{C}_\Gamma)$, whose combinatorics is rather different from that of $\mathcal{T}_\Gamma$.

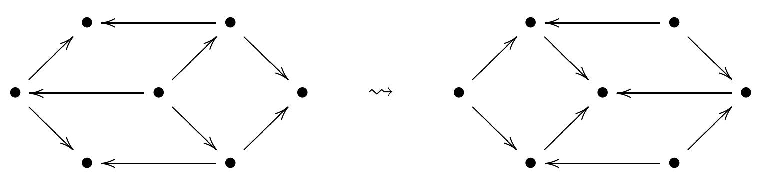
Example 47 When $\Gamma$ is a bi-infinite line, we have an action of the braid 2-category $\mathcal{BC}_\infty$ on a bi-infinite line on the derived category of $Z_\Gamma$, by a theorem of Seidel and Thomas [9]. The action of $\mathcal{BC}_\infty$ on $D^b(Z_\Gamma\text{-mod})$ lifts to an action of $\mathcal{BC}_\infty$ on $\mathcal{C}_\Gamma$. Arrows in $\mathcal{BC}_\infty$ are braids with an infinite number of strands, and 2-arrows are braid cobordisms, such as Reidemeister moves pictured as follows:



In this picture, a rhombus represents a pair of braids running parallel to the two sides and crossing in the middle.



We have a natural 2-functor $\mathcal{BC}_\infty \to \mathcal{T}_\Gamma$. The Reidemeister move depicted above therefore corresponds in a natural way to a 2-*arrow* in $\mathcal{T}_\Gamma$. However, this Reidemeister move also corresponds naturally to an *arrow* in $\mathcal{S}_\Gamma$. Let us explain how. Suppose we remove edges of the $A_2$-lattice to give a rhombic tiling $T$ of the plane, whose edges lie in the quiver $Q$ of $A_\Gamma$. We have a grading of $A_\Gamma$ which places arrows in $Q$ which are edges of $T$ in degree 0 and arrows in $Q$ which are not edges of $T$ in degree 1. Let us denote by $D_T$ the degree 0 part of $A$ taken with respect to this tiling. The algebra $A_\Gamma$ is a trivial extension of $D_T$ by $D_T^*$. If $T'$ is obtained from $T$ by a Reidemeister move centred on the vertex $\lambda$,



then $D_{T'}$ is derived equivalent to $D_T$, because the complex of $D_T$-modules given by the sum of $D_Te_\lambda \otimes e_\lambda D_T \to D_T$ and $D_Te_\lambda \to 0$ is a tilting complex whose derived endomorphism ring is isomorphic to $D_{T'}$. This derived equivalence between $D_T$ and $D_{T'}$ lifts to an equivalence of trivial extensions, that is to say a self-equivalence of $D^b(A_\Gamma\text{-mod}) = D^b(\mathcal{C}_\Gamma)$; this self-equivalence of $D^b(\mathcal{C}_\Gamma)$ is precisely the spherical twist $\psi_\lambda$.

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# On Higher Order Estimates in Quantum Electrodynamics

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Abstract. We propose a new method to derive certain higher order estimates in quantum electrodynamics. Our method is particularly convenient in the application to the non-local semi-relativistic models of quantum electrodynamics as it avoids the use of iterated commutator expansions. We re-derive higher order estimates obtained earlier by Fröhlich, Griesemer, and Schlein and prove new estimates for a non-local molecular no-pair operator.

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## 1. Introduction

The main objective of this paper is to present a new method to derive higher order estimates in quantum electrodynamics (QED) of the form

$$\big\| \, H_{\mathrm{f}}^{n/2} \, (H+C)^{-n/2} \, \big\| \leqslant \text{const} < \infty \,, \tag{1.1}$$

$$\big\| \, [H_{\mathrm{f}}^{n/2} \, , \, H] \, (H+C)^{-n/2} \, \big\| \leqslant \text{const} < \infty \,, \tag{1.2}$$

for all $n \in \mathbb{N}$, where $C > 0$ is sufficiently large. In these bounds $H_{\mathrm{f}}$ denotes the radiation field energy of the quantized photon field and $H$ is the full Hamiltonian generating the time evolution of an interacting electron-photon system. For instance, estimates of this type serve as one of the main technical ingredients in the mathematical analysis of Rayleigh scattering. In this context, (1.1) has been proven by Fröhlich et al. in the case where $H$ is the non- or semi-relativistic Pauli-Fierz Hamiltonian [4]; a slightly weaker version of (1.2) has been obtained in [4] for all even values of $n$. Higher order estimates of the form (1.1) also turn out to be useful in the study of the existence of ground states in a no-pair model of QED [8]. In fact, they imply that every eigenvector of the Hamiltonian $H$ or spectral subspaces of $H$ corresponding to some bounded interval are contained in the domains of higher powers of $H_{\mathrm{f}}$. This information is very helpful in order to overcome numerous technical difficulties which are caused by the non-locality of the no-pair operator. In these applications it is actually necessary to have some control on the norms in (1.1) and (1.2) when the operator $H$ gets modified. To this end we shall give rough bounds on the right hand sides of (1.1) and (1.2) in terms of the ground state energy and integrals involving the form factor and the dispersion relation.

Various types of higher order estimates have actually been employed in the mathematical analysis of quantum field theories since a very long time. Here we only mention the classical works [5, 11] on $P(\phi)_2$ models and the more recent articles [2] again on a $P(\phi)_2$ model and [1] on the Nelson model.

In what follows we briefly describe the organization and the content of the present article. In Section 2 we develop the main idea behind our techniques in a general setting. By the criterion established there the proof of the higher order estimates is essentially boiled down to the verification of certain form bounds on the commutator between $H$ and a regularized version of $H_{\mathrm{f}}^{n/2}$. After that, in Section 3, we introduce some of the most important operators appearing in QED and establish some useful norm bounds on certain commutators involving them. These commutator estimates provide the main ingredients necessary to apply the general criterion of Section 2 to the QED models treated in this article. Their derivation is essentially based on the pull-through formula which is always employed either way to derive higher order estimates in quantum field theories [1, 2, 4, 5, 11]; compare Lemma 3.2 below. In Sections 4, 5, and 6 the general strategy from Section 2 is applied to the non- and semi-relativistic Pauli-Fierz operators and to the no-pair operator, respectively. The latter operators are introduced in detail in these sections. Apart from the fact that our estimate (1.2) is slightly stronger than the corresponding one of [4] the results of Sections 4 and 5 are not new and have been obtained earlier in [4]. However, in order to prove the higher order estimate (1.1) for the no-pair operator we virtually have to re-derive it for the semi-relativistic Pauli-Fierz operator by our own method anyway. Moreover, we think that the arguments employed in Sections 4 and 5 are more convenient and less involved than the procedure carried through in [4]. The main text is followed by an appendix where we show that the semi-relativistic Pauli-Fierz operator for a molecular system with static nuclei is semi-bounded below, provided that all Coulomb coupling constants are less than or equal to $2/\pi$. Moreover, we prove the same result for a molecular no-pair operator assuming that all Coulomb coupling constants are strictly less than the critical coupling constant of the Brown-Ravenhall model [3]. The results of the appendix are based on corresponding estimates for hydrogen-like atoms obtained in [10]. (We remark that the considerably stronger stability of matter of the second kind has been proven for a molecular no-pair operator in [9] under more restrictive assumptions on the involved physical parameters.) No restrictions on the values of the fine-structure constant or on the ultra-violet cut-off are imposed in the present article.

The main new results of this paper are Theorem 2.1 and its corollaries which provide general criteria for the validity of higher order estimates and Theorem 6.1 where higher order estimates for the no-pair operator are established.

*Some frequently used notation.* For $a, b \in \mathbb{R}$, we write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. $\mathcal{D}(T)$ denotes the domain of some operator $T$ acting in some Hilbert space and $\mathcal{Q}(T)$ its form domain, when $T$ is semi-bounded below. $C(a, b, \ldots), C'(a, b, \ldots)$, etc. denote constants that depend only on the quantities $a, b, \ldots$ and whose value might change from one estimate to another.

## 2. Higher order estimates: a general criterion

The following theorem and its succeeding corollaries present the key idea behind of our method. They essentially reduce the derivation of the higher order estimates to the verification of a certain sequence of form bounds. These form bounds can be verified easily without any further induction argument in the QED models treated in this paper.

Theorem 2.1. *Let $H$ and $F_\varepsilon$, $\varepsilon > 0$, be self-adjoint operators in some Hilbert space $\mathscr{K}$ such that $H \geqslant 1$, $F_\varepsilon \geqslant 0$, and each $F_\varepsilon$ is bounded. Let $m \in \mathbb{N} \cup \{\infty\}$, let $\mathscr{D}$ be a form core for $H$, and assume that the following conditions are fulfilled:*

*(a) For every $\varepsilon > 0$, $F_\varepsilon$ maps $\mathscr{D}$ into $\mathcal{Q}(H)$ and there is some $c_\varepsilon \in (0, \infty)$ such that*

$$\big\langle F_\varepsilon \psi \, \big| \, H F_\varepsilon \psi \big\rangle \leqslant c_\varepsilon \langle \psi \, | \, H \psi \rangle , \qquad \psi \in \mathscr{D} .$$

*(b) There is some $c \in [1, \infty)$ such that, for all $\varepsilon > 0$,*

$$\langle \psi \, | \, F_\varepsilon^2 \, \psi \rangle \leqslant c^2 \langle \psi \, | \, H \psi \rangle , \qquad \psi \in \mathscr{D} .$$

*(c) For every $n \in \mathbb{N}$, $n < m$, there is some $c_n \in [1, \infty)$ such that, for all $\varepsilon > 0$,*

$$\begin{aligned} &\big| \langle H \varphi_1 \, | \, F_\varepsilon^n \varphi_2 \rangle - \langle F_\varepsilon^n \varphi_1 \, | \, H \varphi_2 \rangle \big| \\ &\quad \leqslant c_n \big\{ \langle \varphi_1 \, | \, H \varphi_1 \rangle + \langle F_\varepsilon^{n-1} \varphi_2 \, | \, H F_\varepsilon^{n-1} \varphi_2 \rangle \big\} , \qquad \varphi_1, \varphi_2 \in \mathscr{D}. \end{aligned}$$

*Then it follows that, for every $n \in \mathbb{N}$, $n < m + 1$,*

$$\| F_\varepsilon^n H^{-n/2} \| \leqslant C_n := 4^{n-1} c^n \prod_{\ell=1}^{n-1} c_\ell . \tag{2.1}$$

*(An empty product equals 1 by definition.)*

*Proof.* We define

$$T_\varepsilon(n) := H^{1/2} [F_\varepsilon^{n-1}, H^{-1}] H^{-(n-2)/2}, \qquad n \in \{2, 3, 4, \ldots\}.$$

$T_\varepsilon(n)$ is well-defined and bounded because of the closed graph theorem and Condition (a), which implies that $F_\varepsilon \in \mathscr{L}(\mathcal{Q}(H))$, where $\mathcal{Q}(H) = \mathcal{D}(H^{1/2})$

is equipped with the form norm. We shall prove the following sequence of assertions by induction on $n \in \mathbb{N}$, $n < m+1$.

$A(n)$ :⇔ The bound (2.1) holds true and, if $n > 3$, we have

$$\forall\, \varepsilon > 0 \;:\quad \|T_\varepsilon(n)\| \,\leqslant\, C_n/4c^2\,. \tag{2.2}$$

For $n = 1$, the bound (2.1) is fulfilled with $C_1 = c$ on account of Condition (b). Next, assume that $n \in \mathbb{N}$, $n < m$, and that $A(1), \dots, A(n)$ hold true. To find a bound on $\|F_\varepsilon^{n+1}\, H^{-(n+1)/2}\|$ we write

$$F_\varepsilon^{n+1}\, H^{-(n+1)/2} \,=\, Q_1 \,+\, Q_2 \tag{2.3}$$

with

$$Q_1 \,:=\, F_\varepsilon\, H^{-1}\, F_\varepsilon^n\, H^{-(n-1)/2}\,, \qquad Q_2 \,:=\, F_\varepsilon \left[\, F_\varepsilon^n\,,\, H^{-1}\,\right] H^{-(n-1)/2}\,.$$

By the induction hypothesis we have

$$\|Q_1\| \leqslant \|F_\varepsilon\, H^{-1/2}\|\, \|H^{-1/2}\, F_\varepsilon\,\|\, \|F_\varepsilon^{n-1}\, H^{-(n-1)/2}\| \leqslant c^2\, C_{n-1}\,, \tag{2.4}$$

where $C_0 := 1$. Moreover, we observe that

$$\|Q_2\| \,=\, \|F_\varepsilon\, H^{-1/2}\, T_\varepsilon(n+1)\| \,\leqslant\, c\, \|T_\varepsilon(n+1)\|\,. \tag{2.5}$$

To find a bound on $\|T_\varepsilon(n+1)\|$ we recall that $F_\varepsilon$ maps the form domain of $H$ continuously into itself. In particular, since $\mathscr{D}$ is a form core for $H$ the form bound appearing in Condition (c) is available, for all $\varphi_1, \varphi_2 \in \mathcal{Q}(H)$. Let $\phi, \psi \in \mathscr{D}$. Applying Condition (c), extended in this way, with

$$\varphi_1 \,=\, \delta^{1/2}\, H^{-1/2}\, \phi \in \mathcal{Q}(H)\,, \qquad \varphi_2 \,=\, \delta^{-1/2}\, H^{-(n+1)/2}\, \psi \in \mathcal{Q}(H)\,,$$

for some $\delta > 0$, we obtain

$$\begin{aligned}
&|\langle\, \phi \,|\, T_\varepsilon(n+1)\, \psi \,\rangle| \\
&= \big|\big\langle\, H\, H^{-1/2}\, \phi \,\big|\, F_\varepsilon^n\, H^{-(n+1)/2}\, \psi \,\big\rangle - \big\langle\, F_\varepsilon^n\, H^{-1/2}\, \phi \,\big|\, H\, H^{-(n+1)/2}\, \psi \,\big\rangle\big| \\
&\leqslant c_n \inf_{\delta>0} \big\{\, \delta\, \|\phi\|^2 + \delta^{-1}\, \|\{H^{1/2}\, F_\varepsilon^{n-1}\, H^{-n/2}\}\, H^{-1/2}\, \psi\|^2 \,\big\} \\
&\leqslant 2\, c_n\, \|\{H^{1/2}\, F_\varepsilon^{n-1}\, H^{-n/2}\}\|\, \|\phi\|\, \|\psi\|\,.
\end{aligned}$$

The operator $\{\cdots\}$ is just the identity when $n = 1$. For $n > 1$, it can be written as

$$H^{1/2}\, F_\varepsilon^{n-1}\, H^{-n/2} \,=\, \{H^{-1/2}\, F_\varepsilon\}\, F_\varepsilon^{n-2}\, H^{-(n-2)/2} \,+\, T_\varepsilon(n)\,. \tag{2.6}$$

Applying the induction hypothesis and $c, c_\ell \geqslant 1$, we thus get $\|T_\varepsilon(2)\| \leqslant 2\, c_1$, $\|T_\varepsilon(3)\| \leqslant 6\, c\, c_1 c_2$, $\|T_\varepsilon(4)\| \leqslant 14\, c^2 c_1 c_2 c_3 < C_4/4c^2$, and

$$\begin{aligned}
c\, \|T_\varepsilon(n+1)\| &= c \sup \big\{\, |\langle\, \phi \,|\, T_\varepsilon(n+1)\, \psi \,\rangle| \;:\; \phi, \psi \in \mathscr{D}\,,\; \|\phi\| = \|\psi\| = 1 \,\big\} \\
&\leqslant 2\, c_n\, (c^2\, C_{n-2} + C_n/4c) \,<\, c_n\, C_n \,=\, C_{n+1}/4c\,, \qquad n > 3\,,
\end{aligned}$$

since $c^2\, C_{n-2} \leqslant C_n/16$, for $n > 3$. Taking (2.3)–(2.5) into account we arrive at $\|F_\varepsilon^2\, H^{-1}\| \leqslant c^2 + 2c\, c_1 < C_2$, $\|F_\varepsilon^3\, H^{-3/2}\| \leqslant c^3 + 6c^2 c_1 c_2 < C_3$, and

$$\|F_\varepsilon^{n+1}\, H^{-(n+1)/2}\| \,<\, c^2\, C_{n-2} + C_{n+1}/4c \,<\, C_{n+1}\,, \qquad n > 3\,,$$

which concludes the induction step. □

Corollary 2.2. *Assume that $H$ and $F_\varepsilon$, $\varepsilon > 0$, are self-adjoint operators in some Hilbert space $\mathscr{K}$ that fulfill the assumptions of Theorem 2.1 with (c) replaced by the stronger condition*

*(c') For every $n \in \mathbb{N}$, $n < m$, there is some $c_n \in [1, \infty)$ such that, for all $\varepsilon > 0$,*

$$\begin{aligned} &\big|\langle\, H\,\varphi_1 \,|\, F_\varepsilon^n\,\varphi_2\,\rangle - \langle\, F_\varepsilon^n\,\varphi_1\,|\,H\,\varphi_2\,\rangle\big| \\ &\quad \leqslant c_n \left\{\|\varphi_1\|^2 + \langle\, F_\varepsilon^{n-1}\,\varphi_2\,|\,H\,F_\varepsilon^{n-1}\,\varphi_2\,\rangle\right\}, \qquad \varphi_1, \varphi_2 \in \mathscr{D}. \end{aligned}$$

*Then, in addition to* (2.1)*, it follows that, for $n \in \mathbb{N}$, $n < m$, $[F_\varepsilon^n\,,\,H]\,H^{-n/2}$ defines a bounded sesquilinear form with domain $\mathcal{Q}(H) \times \mathcal{Q}(H)$ and*

$$\big\|\,[F_\varepsilon^n\,,\,H]\,H^{-n/2}\,\big\| \leqslant C_n' := 4^n\,c^{n-1}\prod_{\ell=1}^{n} c_\ell\,. \tag{2.7}$$

*Proof.* Again, the form bound in (c') is available, for all $\varphi_1, \varphi_2 \in \mathcal{Q}(H)$, whence

$$\begin{aligned} &\big|\langle\, H\,\phi\,|\,F_\varepsilon^n\,H^{-n/2}\,\psi\,\rangle - \langle\, F_\varepsilon^n\,\phi\,|\,H\,H^{-n/2}\,\psi\,\rangle\big| \\ &\leqslant c_n \inf_{\delta>0}\left\{\delta\,\|\phi\|^2 + \delta^{-1}\,\big\|H^{1/2}\,F_\varepsilon^{n-1}\,H^{-n/2}\,\psi\big\|^2\right\} \leqslant 2\,c_n\,\|H^{1/2}\,F_\varepsilon^{n-1}\,H^{-n/2}\|\,, \end{aligned}$$

for all normalized $\phi, \psi \in \mathcal{Q}(H)$. The assertion now follows from (2.1), (2.6), and the bounds on $\|T_\varepsilon(n)\|$ given in the proof of Theorem 2.1. □

Corollary 2.3. *Let $H \geqslant 1$ and $A \geqslant 0$ be two self-adjoint operators in some Hilbert space $\mathscr{K}$. Let $\kappa > 0$, define*

$$f_\varepsilon(t) := t/(1+\varepsilon\,t)\,, \quad t \geqslant 0\,, \qquad F_\varepsilon := f_\varepsilon^\kappa(A)\,,$$

*for all $\varepsilon > 0$, and assume that $H$ and $F_\varepsilon$, $\varepsilon > 0$, fulfill the hypotheses of Theorem 2.1, for some $m \in \mathbb{N} \cup \{\infty\}$. Then $\mathrm{Ran}(H^{-n/2}) \subset \mathcal{D}(A^{\kappa\,n})$, for every $n \in \mathbb{N}$, $n < m+1$, and*

$$\big\|\,A^{\kappa\,n}\,H^{-n/2}\,\big\| \leqslant 4^{n-1}\,c^n\prod_{\ell=1}^{n-1} c_\ell\,.$$

*If $H$ and $F_\varepsilon$, $\varepsilon > 0$, fulfill the hypotheses of Corollary 2.2, then, for every $n \in \mathbb{N}$, $n < m$, it additionally follows that $A^{\kappa\,n}\,H^{-n/2}$ maps $\mathcal{D}(H)$ into itself so that $[A^{\kappa\,n}\,,\,H]\,H^{-n/2}$ is well-defined on $\mathcal{D}(H)$, and*

$$\big\|\,[A^{\kappa\,n}\,,\,H]\,H^{-n/2}\,\big\| \leqslant 4^n\,c^{n-1}\prod_{\ell=1}^{n} c_\ell\,.$$

*Proof.* Let $U : \mathscr{K} \to L^2(\Omega, \mu)$ be a unitary transformation such that $a = U\,A\,U^*$ is a maximal operator of multiplication with some non-negative measurable function – again called $a$ – on some measure space $(\Omega, \mathfrak{A}, \mu)$. We pick some $\psi \in \mathscr{K}$, set $\phi_n := U\,H^{-n/2}\,\psi$, and apply the monotone convergence theorem to conclude that

$$\int_\Omega a(\omega)^{2\kappa\, n}\,|\phi_n(\omega)|^2\,d\mu(\omega) \;=\; \lim_{\varepsilon\searrow 0}\int_\Omega f_\varepsilon^\kappa(a(\omega))^{2n}\,|\phi_n(\omega)|^2\,d\mu(\omega)$$
$$= \lim_{\varepsilon\searrow 0}\|F_\varepsilon^n\,H^{-n/2}\,\psi\|^2 \;\leqslant\; C_n\,\|\psi\|^2,$$

for every $n\in\mathbb{N}$, $n<m+1$, which implies the first assertion. Now, assume that $H$ and $F_\varepsilon$, $\varepsilon>0$, fulfill Condition (c') of Corollary 2.2. Applying the dominated convergence theorem in the spectral representation introduced above we see that $F_\varepsilon^n\,\psi\to A^{\kappa\,n}\,\psi$, for every $\psi\in\mathcal{D}(A^{\kappa\,n})$. Hence, (2.7) and $\mathrm{Ran}(H^{-n/2})\subset\mathcal{D}(A^{\kappa\,n})$ imply, for $n<m$ and $\phi,\psi\in\mathcal{D}(H)$,

$$\begin{aligned}
&\big|\langle\,\phi\,|\,A^{\kappa\,n}H^{-n/2}\,H\,\psi\,\rangle-\langle\,H\,\phi\,|\,A^{\kappa\,n}\,H^{-n/2}\,\psi\,\rangle\big|\\
&\quad=\lim_{\varepsilon\searrow 0}\big|\langle\,F_\varepsilon^n\,\phi\,|\,H\,H^{-n/2}\,\psi\,\rangle-\langle\,H\,\phi\,|\,F_\varepsilon^n\,H^{-n/2}\,\psi\,\rangle\big|\\
&\quad\leqslant\limsup_{\varepsilon\searrow 0}\big\|\,[F_\varepsilon^n\,,\,H]\,H^{-n/2}\,\big\|\,\|\phi\|\,\|\psi\|\;\leqslant\;C_n'\,\|\phi\|\,\|\psi\|\,.
\end{aligned}$$

Thus, $|\langle\,H\,\phi\,|\,A^{\kappa\,n}\,H^{-n/2}\,\psi\,\rangle|\leqslant\|\phi\|\,\|A^{\kappa\,n}\,H^{-n/2}\|\,\|H\,\psi\|+C_n'\|\phi\|\,\|\psi\|$, for all $\phi,\psi\in\mathcal{D}(H)$. In particular, $A^{\kappa\,n}\,H^{-n/2}\,\psi\in\mathcal{D}(H^*)=\mathcal{D}(H)$, for all $\psi\in\mathcal{D}(H)$, and the second asserted bound holds true. □

## 3. Commutator estimates

In this section we derive operator norm bounds on commutators involving the quantized vector potential, $\mathbf{A}$, the radiation field energy, $H_{\mathrm{f}}$, and the Dirac operator, $D_{\mathbf{A}}$. The underlying Hilbert space is

$$\mathscr{H}\;:=\;L^2(\mathbb{R}^3_{\mathbf{x}}\times\mathbb{Z}_4)\otimes\mathscr{F}_{\mathrm{b}}\;=\;\int_{\mathbb{R}^3}^{\oplus}\mathbb{C}^4\otimes\mathscr{F}_{\mathrm{b}}\,d^3\mathbf{x}\,,$$

where the bosonic Fock space, $\mathscr{F}_{\mathrm{b}}$, is modeled over the one-photon Hilbert space

$$\mathscr{F}_{\mathrm{b}}^{(1)}\;:=\;L^2(\mathcal{A}\times\mathbb{Z}_2,dk)\,,\qquad\int dk\;:=\;\sum_{\lambda\in\mathbb{Z}_2}\int_{\mathcal{A}}d^3\mathbf{k}\,.$$

With regards to the applications in [8] we define $\mathcal{A}:=\{\mathbf{k}\in\mathbb{R}^3:\,|\mathbf{k}|\geqslant m\}$, for some $m\geqslant 0$. We thus have

$$\mathscr{F}_{\mathrm{b}}\;=\;\bigoplus_{n=0}^{\infty}\mathscr{F}_{\mathrm{b}}^{(n)},\qquad\mathscr{F}_{\mathrm{b}}^{(0)}:=\mathbb{C}\,,\quad\mathscr{F}_{\mathrm{b}}^{(n)}:=\mathcal{S}_n\,L^2\big((\mathcal{A}\times\mathbb{Z}_2)^n\big),\;n\in\mathbb{N},$$

where $\mathcal{S}_n=\mathcal{S}_n^2=\mathcal{S}_n^*$ is given by

$$(\mathcal{S}_n\,\psi^{(n)})(k_1,\dots,k_n):=\frac{1}{n!}\sum_{\pi\in\mathfrak{S}_n}\psi^{(n)}(k_{\pi(1)},\dots,k_{\pi(n)}),$$

for every $\psi^{(n)}\in L^2\big((\mathcal{A}\times\mathbb{Z}_2)^n\big)$, $\mathfrak{S}_n$ denoting the group of permutations of $\{1,\dots,n\}$. The vector potential is determined by a certain vector-valued function, $\mathbf{G}$, called the form factor.

Hypothesis 3.1. *The dispersion relation,* $\omega : \mathcal{A} \to [0,\infty)$*, is a measurable function such that* $0 < \omega(k) := \omega(\mathbf{k}) \leqslant |\mathbf{k}|$*, for* $k = (\mathbf{k},\lambda) \in \mathcal{A}\times\mathbb{Z}_2$ *with* $\mathbf{k} \neq 0$*. For every* $k \in (\mathcal{A}\setminus\{0\})\times\mathbb{Z}_2$ *and* $j \in \{1,2,3\}$*,* $G^{(j)}(k)$ *is a bounded continuously differentiable function,* $\mathbb{R}^3_{\mathbf{x}} \ni \mathbf{x} \mapsto G^{(j)}_{\mathbf{x}}(k)$*, such that the map* $(\mathbf{x},k) \mapsto G^{(j)}_{\mathbf{x}}(k)$ *is measurable and* $G^{(j)}_{\mathbf{x}}(-\mathbf{k},\lambda) = \overline{G^{(j)}_{\mathbf{x}}(\mathbf{k},\lambda)}$*, for almost every* $\mathbf{k}$ *and all* $\mathbf{x} \in \mathbb{R}^3$ *and* $\lambda \in \mathbb{Z}_2$*. Finally, there exist* $d_{-1}, d_0, d_1, \ldots \in (0,\infty)$ *such that*

$$2\int \omega(k)^{\ell}\,\|\mathbf{G}(k)\|^2_\infty\,dk \;\leqslant\; d_\ell^2\,, \qquad \ell \in \{-1,0,1,2,\ldots\}\,, \tag{3.1}$$

$$2\int \omega(k)^{-1}\,\|\nabla_{\mathbf{x}}\wedge\mathbf{G}(k)\|^2_\infty\,dk \;\leqslant\; d_1^2\,, \tag{3.2}$$

*where* $\mathbf{G} = (G^{(1)},G^{(2)},G^{(3)})$ *and* $\|\mathbf{G}(k)\|_\infty := \sup_{\mathbf{x}}|\mathbf{G}_{\mathbf{x}}(k)|$*, etc.*

*Example.* In the physical applications the form factor is often given as

$$\mathbf{G}^{e,\Lambda}_{\mathbf{x}}(k) \;:=\; -e\,\frac{\mathbb{1}_{\{|\mathbf{k}|\leqslant\Lambda\}}}{2\pi\sqrt{|\mathbf{k}|}}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,\boldsymbol{\varepsilon}(k), \tag{3.3}$$

for $(\mathbf{x},k) \in \mathbb{R}^3\times(\mathbb{R}^3\times\mathbb{Z}_2)$ with $\mathbf{k}\neq 0$. Here the physical units are chosen such that energies are measured in units of the rest energy of the electron. Length are measured in units of one Compton wave length divided by $2\pi$. The parameter $\Lambda > 0$ is an ultraviolet cut-off and the square of the elementary charge, $e > 0$, equals Sommerfeld's fine-structure constant in these units; we have $e^2 \approx 1/137$ in nature. The polarization vectors, $\boldsymbol{\varepsilon}(\mathbf{k},\lambda)$, $\lambda\in\mathbb{Z}_2$, are homogeneous of degree zero in $\mathbf{k}$ such that $\{\mathring{\mathbf{k}}, \boldsymbol{\varepsilon}(\mathring{\mathbf{k}},0), \boldsymbol{\varepsilon}(\mathring{\mathbf{k}},1)\}$ is an orthonormal basis of $\mathbb{R}^3$, for every $\mathring{\mathbf{k}} \in S^2$. This corresponds to the Coulomb gauge for $\nabla_{\mathbf{x}}\cdot\mathbf{G}^{e,\Lambda} = 0$. We remark that the vector fields $S^2 \ni \mathring{\mathbf{k}} \mapsto \boldsymbol{\varepsilon}(\mathring{\mathbf{k}},\lambda)$ are necessarily discontinuous. ◇

It is useful to work with more general form factors fulfilling Hypothesis 3.1 since in the study of the existence of ground states in QED one usually encounters truncated and discretized versions of the physical choice $\mathbf{G}^{e,\Lambda}$. For the applications in [8] it is necessary to know that the higher order estimates established here hold true uniformly in the involved parameters and Hypothesis 3.1 is convenient way to handle this.
We recall the definition of the creation and the annihilation operators of a photon state $f \in \mathscr{F}_{\mathrm{b}}^{(1)}$,

$$(a^\dagger(f)\,\psi)^{(n)}(k_1,\ldots,k_n) = n^{-1/2}\sum_{j=1}^{n} f(k_j)\,\psi^{(n-1)}(\ldots,k_{j-1},k_{j+1},\ldots),\;\; n\in\mathbb{N},$$

$$(a(f)\,\psi)^{(n)}(k_1,\ldots,k_n) = (n+1)^{1/2}\int \overline{f}(k)\,\psi^{(n+1)}(k,k_1,\ldots,k_n)\,dk,\;\; n\in\mathbb{N}_0,$$

and $(a^\dagger(f)\,\psi)^{(0)} = 0$, $a(f)\,(\psi^{(0)},0,0,\ldots) = 0$, for all $\psi = (\psi^{(n)})_{n=0}^\infty \in \mathscr{F}_{\mathrm{b}}$ such that the right hand sides again define elements of $\mathscr{F}_{\mathrm{b}}$. $a^\dagger(f)$ and $a(f)$ are formal adjoints of each other on the dense domain

$$\mathscr{C}_0 := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{S}_n \, L^\infty_{\mathrm{comp}}\big((\mathcal{A} \times \mathbb{Z}_2)^n\big) . \qquad \text{(Algebraic direct sum.)}$$

For a three-vector of functions $\mathbf{f} = (f^{(1)}, f^{(2)}, f^{(3)}) \in (\mathscr{F}_{\mathrm{b}}^{(1)})^3$, we write $a^\sharp(\mathbf{f}) := (a^\sharp(f^{(1)}), a^\sharp(f^{(2)}), a^\sharp(f^{(3)}))$, where $a^\sharp$ is $a^\dagger$ or $a$. Then the quantized vector potential is the triplet of operators given by

$$\mathbf{A} \equiv \mathbf{A}(\mathbf{G}) := a^\dagger(\mathbf{G}) + a(\mathbf{G}) , \qquad a^\sharp(\mathbf{G}) := \int_{\mathbb{R}^3}^{\oplus} \mathbb{1}_{\mathbb{C}^4} \otimes a^\sharp(\mathbf{G}_{\mathbf{x}}) \, d^3\mathbf{x} .$$

The radiation field energy is the direct sum $H_{\mathrm{f}} = \bigoplus_{n=0}^{\infty} d\Gamma^{(n)}(\omega) : \mathcal{D}(H_{\mathrm{f}}) \subset \mathscr{F}_{\mathrm{b}} \to \mathscr{F}_{\mathrm{b}}$, where $d\Gamma^{(0)}(\omega) := 0$, and $d\Gamma^{(n)}(\omega)$ denotes the maximal multiplication operator in $\mathscr{F}_{\mathrm{b}}^{(n)}$ associated with the symmetric function $(k_1, \ldots, k_n) \mapsto \omega(k_1) + \cdots + \omega(k_n)$. By the permutation symmetry and Fubini's theorem we thus have

$$\big\langle H_{\mathrm{f}}^{1/2} \phi \,\big|\, H_{\mathrm{f}}^{1/2} \psi \big\rangle = \int \omega(k) \, \langle \, a(k) \, \phi \, | \, a(k) \, \psi \, \rangle \, dk , \qquad \phi, \psi \in \mathcal{D}(H_{\mathrm{f}}^{1/2}) , \tag{3.4}$$

where we use the notation

$$(a(k) \, \psi)^{(n)}(k_1, \ldots, k_n) = (n+1)^{1/2} \, \psi^{(n+1)}(k, k_1, \ldots, k_n) , \quad n \in \mathbb{N}_0 ,$$

almost everywhere, and $a(k) \, (\psi^{(0)}, 0, 0, \ldots) = 0$. For a measurable function $f : \mathbb{R} \to \mathbb{R}$ and $\psi \in \mathcal{D}(f(H_{\mathrm{f}}))$, the following identity in $\mathscr{F}_{\mathrm{b}}^{(n)}$,

$$(a(k) \, f(H_{\mathrm{f}}) \, \psi)^{(n)} = f\big(\omega(k) + d\Gamma^{(n)}(\omega)\big) \, (a(k) \, \psi)^{(n)} , \qquad n \in \mathbb{N}_0 ,$$

valid for almost every $k$, is called the pull-through formula. Finally, we let $\alpha_1, \alpha_2, \alpha_3$, and $\beta := \alpha_0$ denote hermitian four times four matrices that fulfill the Clifford algebra relations

$$\alpha_i \, \alpha_j + \alpha_j \, \alpha_i = 2 \, \delta_{ij} \, \mathbb{1} , \qquad i, j \in \{0, 1, 2, 3\} . \tag{3.5}$$

They act on the second tensor factor in $L^2(\mathbb{R}^3_{\mathbf{x}} \times \mathbb{Z}_4) = L^2(\mathbb{R}^3_{\mathbf{x}}) \otimes \mathbb{C}^4$. As a consequence of (3.5) and the $C^*$-equality we have

$$\|\boldsymbol{\alpha} \cdot \mathbf{v}\|_{\mathscr{L}(\mathbb{C}^4)} = |\mathbf{v}| , \quad \mathbf{v} \in \mathbb{R}^3 , \qquad \|\boldsymbol{\alpha} \cdot \mathbf{z}\|_{\mathscr{L}(\mathbb{C}^4)} \leqslant \sqrt{2} \, |\mathbf{z}| , \quad \mathbf{z} \in \mathbb{C}^3 , \tag{3.6}$$

where $\boldsymbol{\alpha} \cdot \mathbf{z} := \alpha_1 \, z^{(1)} + \alpha_2 \, z^{(2)} + \alpha_3 \, z^{(3)}$, for $\mathbf{z} = (z^{(1)}, z^{(2)}, z^{(3)}) \in \mathbb{C}^3$. A standard exercise using the inequality in (3.6), the Cauchy-Schwarz inequality, and the canonical commutation relations,

$$[a^\sharp(f) \, , \, a^\sharp(g)] = 0 , \qquad [a(f) \, , \, a^\dagger(g)] = \langle \, f \, | \, g \, \rangle \, \mathbb{1} , \qquad f, g \in \mathscr{F}_{\mathrm{b}}^{(1)} ,$$

reveals that every $\psi \in \mathcal{D}(H_{\mathrm{f}}^{1/2})$ belongs to the domain of $\boldsymbol{\alpha} \cdot a^\sharp(\mathbf{G})$ and

$$\|\boldsymbol{\alpha} \cdot a(\mathbf{G}) \, \psi\| \leqslant d_{-1} \, \|H_{\mathrm{f}}^{1/2} \, \psi\| , \quad \|\boldsymbol{\alpha} \cdot a^\dagger(\mathbf{G}) \, \psi\|^2 \leqslant d_{-1}^2 \, \|H_{\mathrm{f}}^{1/2} \, \psi\|^2 + d_0^2 \, \|\psi\|^2 . \tag{3.7}$$

(Here and in the following we identify $H_{\mathrm{f}} \equiv \mathbb{1} \otimes H_{\mathrm{f}}$, etc.) These relative bounds imply that $\boldsymbol{\alpha} \cdot \mathbf{A}$ is symmetric on the domain $\mathcal{D}(H_{\mathrm{f}}^{1/2})$.

The operators whose norms are estimated in (3.9) and the following lemmata are always well-defined a priori on the following dense subspace of $\mathscr{H}$,

$$\mathscr{D} := C_0^\infty(\mathbb{R}^3 \times \mathbb{Z}_4) \otimes \mathscr{C}_0 . \qquad \text{(Algebraic tensor product.)}$$

Given some $E \geqslant 1$ we set

$$\check{H}_{\mathrm{f}} := H_{\mathrm{f}} + E \tag{3.8}$$

in the sequel. We already know from [10] that, for every $\nu \geqslant 0$, there is some constant, $C_\nu \in (0,\infty)$, such that

$$\big\| [\boldsymbol{\alpha} \cdot \mathbf{A}\,,\, \check{H}_{\mathrm{f}}^{-\nu}]\, \check{H}_{\mathrm{f}}^{\nu} \big\| \leqslant C_\nu / E^{1/2} , \qquad E \geqslant 1 . \tag{3.9}$$

In our first lemma we derive a generalization of (3.9). Its proof resembles the one of (3.9) given in [10]. Since we shall encounter many similar but slightly different commutators in the applications it makes sense to introduce the numerous parameters that obscure its statement (but simplify its proof).

Lemma 3.2. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1. Let $\varepsilon \geqslant 0$, $E \geqslant 1$, $\kappa, \nu \in \mathbb{R}$, $\gamma, \delta, \sigma, \tau \geqslant 0$, such that $\gamma + \delta + \sigma + \tau \leqslant 1/2$, and define*

$$f_\varepsilon(t) := \frac{t + E}{1 + \varepsilon t + \varepsilon E} , \qquad t \in [0, \infty) . \tag{3.10}$$

*Then the operator $\check{H}_{\mathrm{f}}^{\nu+\gamma}\, f_\varepsilon^{\sigma}(H_{\mathrm{f}})\, [\boldsymbol{\alpha} \cdot \mathbf{A}\,,\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})]\, \check{H}_{\mathrm{f}}^{-\nu+\delta} f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}})$, defined a priori on $\mathscr{D}$, extends to a bounded operator on $\mathscr{H}$ and*

$$\begin{aligned} &\big\| \check{H}_{\mathrm{f}}^{\nu+\gamma}\, f_\varepsilon^{\sigma}(H_{\mathrm{f}})\, [\boldsymbol{\alpha} \cdot \mathbf{A}\,,\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})]\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}}) \big\| \\ &\qquad\qquad \leqslant |\kappa|\, 2^{(\rho+1)/2}\, (d_1 + d_\rho)\, E^{\gamma+\delta+\sigma+\tau-1/2} , \end{aligned} \tag{3.11}$$

*where $\rho$ is the smallest integer greater or equal to $3 + 2|\kappa| + 2|\nu|$.*

*Proof.* We notice that all operators $\check{H}_{\mathrm{f}}^{s}$ and $f_\varepsilon^{s}(H_{\mathrm{f}})$ leave the dense subspace $\mathscr{D}$ invariant and that $\boldsymbol{\alpha} \cdot a^\sharp(\mathbf{G})$ maps $\mathscr{D}$ into $\mathcal{D}(\check{H}_{\mathrm{f}}^{s})$, for every $s \in \mathbb{R}$. Now, let $\varphi, \psi \in \mathscr{D}$. Then

$$\begin{aligned} &\big\langle \varphi \,\big|\, \check{H}_{\mathrm{f}}^{\nu+\gamma}\, f_\varepsilon^{\sigma}(H_{\mathrm{f}})\, [\boldsymbol{\alpha} \cdot \mathbf{A}\,,\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})]\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}})\, \psi \big\rangle \\ &= \big\langle \varphi \,\big|\, \check{H}_{\mathrm{f}}^{\nu+\gamma}\, f_\varepsilon^{\sigma}(H_{\mathrm{f}})\, [\boldsymbol{\alpha} \cdot a(\mathbf{G})\,,\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})]\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}})\, \psi \big\rangle \qquad (3.12) \\ &\quad - \big\langle f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}})\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, [\boldsymbol{\alpha} \cdot a(\mathbf{G})\,,\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})]\, f_\varepsilon^{\sigma}(H_{\mathrm{f}})\, \check{H}_{\mathrm{f}}^{\nu+\gamma}\, \varphi \,\big|\, \psi \big\rangle . \qquad (3.13) \end{aligned}$$

For almost every $k$, the pull-through formula yields the following representation,

$$\check{H}_{\mathrm{f}}^{\nu+\gamma}\, f_\varepsilon^{\sigma}(H_{\mathrm{f}})\, [a(k)\,,\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})]\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}})\, \psi = F(k; H_{\mathrm{f}})\, a(k)\, \check{H}_{\mathrm{f}}^{-1/2}\, \psi ,$$

where

$$\begin{aligned} F(k;t) \,:=\, & (t+E)^{\nu+\gamma}\, f_\varepsilon^\sigma(t)\,\big(f_\varepsilon^\kappa(t+\omega(k)) - f_\varepsilon^\kappa(t)\big) \\ & \cdot (t+E+\omega(k))^{-\nu+\delta+1/2}\, f_\varepsilon^{-\kappa+\tau}(t+\omega(k)) \\ =\, & \Big(\frac{t+E}{t+E+\omega(k)}\Big)^\nu (t+E)^\gamma\,(t+E+\omega(k))^{\delta+1/2} \\ & \cdot \int_0^1 \frac{d}{ds} f_\varepsilon^\kappa(t+s\,\omega(k))\,ds\,\frac{f_\varepsilon^\sigma(t)\,f_\varepsilon^\tau(t+\omega(k))}{f_\varepsilon^\kappa(t+\omega(k))}\,, \end{aligned}$$

for $t \geqslant 0$. We compute

$$\frac{d}{ds} f_\varepsilon^\kappa(t+s\,\omega(k)) \,=\, \frac{\kappa\,\omega(k)\,f_\varepsilon^\kappa(t+s\,\omega(k))}{(t+s\,\omega(k)+E)(1+\varepsilon\,t+\varepsilon\,s\,\omega(k)+\varepsilon\,E)}\,. \tag{3.14}$$

Using that $f_\varepsilon$ is increasing in $t \geqslant 0$ and that

$$(t+\omega(k)+E)/(t+s\,\omega(k)+E) \,\leqslant\, 1+\omega(k)\,, \qquad s\in[0,1]\,,$$

thus

$$f_\varepsilon^\kappa(t+s\,\omega(k))/f_\varepsilon^\kappa(t+\omega(k)) \,\leqslant\, (1+\omega(k))^{-(0\wedge\kappa)}\,, \qquad s\in[0,1]\,,$$

it is elementary to verify that

$$|F_\varepsilon(k;t)| \,\leqslant\, |\kappa|\,\omega(k)\,(1+\omega(k))^{\delta+\tau-(0\wedge\kappa)-(0\wedge\nu)+1/2}\,E^{\gamma+\delta+\sigma+\tau-1/2}\,,$$

for all $t\geqslant 0$ and $k$. We deduce that the term in (3.12) can be estimated as

$$\begin{aligned} & \big|\big\langle\,\varphi\,\big|\,\check{H}_{\mathrm{f}}^{\nu+\gamma}\,f_\varepsilon^\sigma(H_{\mathrm{f}})\,[\boldsymbol{\alpha}\cdot a(\mathbf{G})\,,\,f_\varepsilon^\kappa(H_{\mathrm{f}})]\,\check{H}_{\mathrm{f}}^{-\nu+\delta}\,f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}})\,\psi\,\big\rangle\big| \\ & \leqslant \int \|\varphi\|\,\big\|\boldsymbol{\alpha}\cdot\mathbf{G}(k)\,\check{H}_{\mathrm{f}}^{\nu+\gamma}\,f_\varepsilon^\sigma(H_{\mathrm{f}})\,[a(k)\,,\,f_\varepsilon^\kappa(H_{\mathrm{f}})]\,\check{H}_{\mathrm{f}}^{-\nu+\delta}\,f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}})\,\psi\big\|\,dk \\ & \leqslant \sqrt{2}\int \|\varphi\|\,\|\mathbf{G}(k)\|_\infty\,\|F_\varepsilon(k;H_{\mathrm{f}})\|\,\|a(k)\,\check{H}_{\mathrm{f}}^{-1/2}\,\psi\|\,dk \\ & \leqslant |\kappa|\,\sqrt{2}\Big(\int \omega(k)\,(1+\omega(k))^{2(\delta+\tau)-(0\wedge 2\kappa)-(0\wedge 2\nu)+1}\,\|\mathbf{G}(k)\|_\infty^2\,dk\Big)^{1/2} \\ & \qquad \cdot \Big(\int \omega(k)\,\big\|\,a(k)\,\check{H}_{\mathrm{f}}^{-1/2}\,\psi\,\big\|^2\,dk\Big)^{1/2}\,\|\varphi\|\,E^{\gamma+\delta+\sigma+\tau-1/2} \\ & \leqslant\, |\kappa|\,2^{(\rho-1)/2}\,(d_1+d_\rho)\,\|\varphi\|\,\big\|\,H_{\mathrm{f}}^{1/2}\,\check{H}_{\mathrm{f}}^{-1/2}\,\psi\,\big\|\,E^{\gamma+\delta+\sigma+\tau-1/2}\,. \end{aligned} \tag{3.15}$$

In the last step we used $\delta+\tau\leqslant 1/2$ and applied (3.4). (3.15) immediately gives a bound on the term in (3.13), too. For we have

$$\begin{aligned} & f_\varepsilon^{-\kappa+\tau}(H_{\mathrm{f}})\,\check{H}_{\mathrm{f}}^{-\nu+\delta}\,[\boldsymbol{\alpha}\cdot a(\mathbf{G})\,,\,f_\varepsilon^\kappa(H_{\mathrm{f}})]\,f_\varepsilon^\sigma(H_{\mathrm{f}})\,\check{H}_{\mathrm{f}}^{\nu+\gamma}\,\varphi \\ & \qquad = \check{H}_{\mathrm{f}}^{-\nu+\delta}\,f_\varepsilon^\tau(H_{\mathrm{f}})\,[f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,,\,\boldsymbol{\alpha}\cdot a(\mathbf{G})]\,\check{H}_{\mathrm{f}}^{\nu+\gamma}\,f_\varepsilon^{\kappa+\sigma}(H_{\mathrm{f}})\,\varphi\,, \end{aligned}$$

which after the replacements $(\nu,\kappa,\gamma,\delta,\sigma,\tau)\mapsto(-\nu,-\kappa,\delta,\gamma,\tau,\sigma)$ and $\varphi\mapsto-\psi$ is precisely the term we just have treated. □

Lemma 3.2 provides all the information needed to apply Corollary 2.3 to non-relativistic QED. For the application of Corollary 2.3 to the non-local semi-relativistic models of QED it is necessary to study commutators that involve resolvents and sign functions of the Dirac operator,

$$D_{\mathbf{A}} := \boldsymbol{\alpha}\cdot(-i\nabla_{\mathbf{x}}+\mathbf{A})+\beta\,.$$

An application of Nelson's commutator theorem with test operator $-\Delta+H_f+1$ shows that $D_{\mathbf{A}}$ is essentially self-adjoint on $\mathscr{D}$. The spectrum of its unique closed extension, again denoted by the same symbol, is contained in the union of two half-lines, $\sigma[D_{\mathbf{A}}]\subset(-\infty,-1]\cup[1,\infty)$. In particular, it makes sense to define

$$R_{\mathbf{A}}(iy) := (D_{\mathbf{A}}-iy)^{-1}\,,\qquad y\in\mathbb{R}\,,$$

and the spectral calculus yields

$$\|R_{\mathbf{A}}(iy)\|\leqslant(1+y^2)^{-1/2},\qquad \int_{\mathbb{R}}\big\|\,|D_{\mathbf{A}}|^{1/2}\,R_{\mathbf{A}}(iy)\,\psi\,\big\|^2\,\frac{dy}{\pi} = \|\psi\|^2,\ \ \psi\in\mathscr{H}.$$

The next lemma is a straightforward extension of [10, Corollary 3.1] where it is also shown that $R_{\mathbf{A}}(iy)$ maps $\mathcal{D}(H_{\mathrm{f}}^{\nu})$ into itself, for every $\nu>0$.

Lemma 3.3. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1. Then, for all $\kappa,\nu\in\mathbb{R}$, we find $k_i\equiv k_i(\kappa,\nu,d_1,d_\rho)\in[1,\infty)$, $i=1,2$, such that, for all $y\in\mathbb{R}$, $\varepsilon\geqslant0$, and $E\geqslant k_1$, there exist $\Upsilon_{\kappa,\nu}(iy),\widetilde{\Upsilon}_{\kappa,\nu}(iy)\in\mathscr{L}(\mathscr{H})$ satisfying*

$$R_{\mathbf{A}}(iy)\,\check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}}) = \check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,R_{\mathbf{A}}(iy)\,\Upsilon_{\kappa,\nu}(iy) \tag{3.16}$$

$$= \check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,\widetilde{\Upsilon}_{\kappa,\nu}(iy)\,R_{\mathbf{A}}(iy)\,, \tag{3.17}$$

*on $\mathcal{D}(\check{H}_{\mathrm{f}}^{-\nu})$, and $\|\Upsilon_{\kappa,\nu}(iy)\|,\|\widetilde{\Upsilon}_{\kappa,\nu}(iy)\|\leqslant k_2$, where $\rho$ is defined in Lemma 3.2.*

*Proof.* Without loss of generality we may assume that $\varepsilon>0$ for otherwise we could simply replace $\nu$ by $\nu+\kappa$ and $f_0^\kappa$ by $f_0^0=1$. First, we assume in addition that $\nu\geqslant0$. We observe that

$$T_0 := \big[\,\check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}\,\big]\,\check{H}_{\mathrm{f}}^{\nu}\,f_\varepsilon^{\kappa}(H_{\mathrm{f}}) = T_1+T_2$$

on $\mathscr{D}$, where

$$T_1 := [\check{H}_{\mathrm{f}}^{-\nu}\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}]\,\check{H}_{\mathrm{f}}^{\nu}\,,\qquad T_2 := \check{H}_{\mathrm{f}}^{-\nu}\,[f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}]\,f_\varepsilon^{\kappa}(H_{\mathrm{f}})\,\check{H}_{\mathrm{f}}^{\nu}.$$

Due to (3.9) (or (3.11) with $\varepsilon=0$) the operator $T_1$ extends to a bounded operator on $\mathscr{H}$ and $\|T_1\|\leqslant C_\nu/E^{1/2}$. According to (3.11) we further have $\|T_2\|\leqslant C_{\kappa,\nu}(d_1+d_\rho)/E^{1/2}$. We pick some $\phi\in\mathscr{D}$ and compute

$$\begin{aligned}\big[\,R_{\mathbf{A}}(iy)\,,\,\check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,\big]\,(D_{\mathbf{A}}-iy)\,\phi &= R_{\mathbf{A}}(iy)\,\big[\,\check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,,\,D_{\mathbf{A}}\,\big]\,\phi\\ &= R_{\mathbf{A}}(iy)\,T_0\,\check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,\phi\\ &= R_{\mathbf{A}}(iy)\,\overline{T}_0\,\check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,R_{\mathbf{A}}(iy)\,(D_{\mathbf{A}}-iy)\,\phi\,. \qquad (3.18)\end{aligned}$$

Since $(D_{\mathbf{A}}-iy)\,\mathscr{D}$ is dense in $\mathscr{H}$ and since $\check{H}_{\mathrm{f}}^{-\nu}$ and $f_\varepsilon^{\kappa}(H_{\mathrm{f}})$ are bounded (here we use that $\nu\geqslant0$ and $\varepsilon>0$), this identity implies

$$R_{\mathbf{A}}(iy)\,\check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}}) = \big(\mathbb{1}+R_{\mathbf{A}}(iy)\,\overline{T}_0\big)\,\check{H}_{\mathrm{f}}^{-\nu}\,f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\,R_{\mathbf{A}}(iy)\,.$$

Taking the adjoint of the previous identity and replacing $y$ by $-y$ we obtain

$$\check{H}_{\mathrm{f}}^{-\nu}\, f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\, R_{\mathbf{A}}(iy) \,=\, R_{\mathbf{A}}(iy)\, \check{H}_{\mathrm{f}}^{-\nu}\, f_\varepsilon^{-\kappa}(H_{\mathrm{f}})\, (\mathbb{1} + T_0^*\, R_{\mathbf{A}}(iy))\,.$$

In view of the norm bounds on $T_1$ and $T_2$ we see that (3.16) and (3.17) are valid with $\Upsilon_{\kappa,\nu}(iy) := \sum_{\ell=0}^{\infty}\{-T_0^*\, R_{\mathbf{A}}(iy)\}^\ell$ and $\widetilde{\Upsilon}_{\kappa,\nu}(iy) := \sum_{\ell=0}^{\infty}\{-R_{\mathbf{A}}(iy)\, T_0^*\}^\ell$, provided that $E$ is sufficiently large, depending only on $\kappa, \nu, d_1$, and $d_\rho$, such that the Neumann series converge.
Now, let $\nu < 0$. Then we write $T_0$ on the domain $\mathscr{D}$ as

$$T_0 \,=\, \check{H}_{\mathrm{f}}^{-\nu}\, f_\varepsilon^{-\kappa}(H_{\mathrm{f}}) \left[\boldsymbol{\alpha}\cdot\mathbf{A}\,,\, \check{H}_{\mathrm{f}}^{\nu}\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})\right],$$

and deduce that

$$R_{\mathbf{A}}(iy)\, \check{H}_{\mathrm{f}}^{\nu}\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})\, (\mathbb{1} + \overline{T}_0\, R_{\mathbf{A}}(iy)) \,=\, \check{H}_{\mathrm{f}}^{\nu}\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})\, R_{\mathbf{A}}(iy)$$

by a computation analogous to (3.18). Taking the adjoint of this identity with $y$ replaced by $-y$ we get

$$\big(\mathbb{1} + R_{\mathbf{A}}(iy)\, T_0^*\big)\, \check{H}_{\mathrm{f}}^{\nu}\, f_\varepsilon^{\kappa}\, R_{\mathbf{A}}(iy) \,=\, R_{\mathbf{A}}(iy)\, \check{H}_{\mathrm{f}}^{\nu}\, f_\varepsilon^{\kappa}(H_{\mathrm{f}})\,.$$

Next, we invert $\mathbb{1} + R_{\mathbf{A}}(iy)\, T_0^*$ by means of the same Neumann series as above. As a result we obtain

$$\check{H}_{\mathrm{f}}^{\nu} f_\varepsilon^{\kappa}(H_{\mathrm{f}})\, R_{\mathbf{A}}(iy) = R_{\mathbf{A}}(iy)\, \Upsilon_{\kappa,\nu}(iy)\, \check{H}_{\mathrm{f}}^{\nu} f_\varepsilon^{\kappa}(H_{\mathrm{f}}) = \widetilde{\Upsilon}_{\kappa,\nu}(iy)\, R_{\mathbf{A}}(iy)\, \check{H}_{\mathrm{f}}^{\nu} f_\varepsilon^{\kappa}(H_{\mathrm{f}}),$$

where the definition of $\Upsilon_{\kappa,\nu}$ and $\widetilde{\Upsilon}_{\kappa,\nu}$ has been extended to negative $\nu$. It follows that $R_{\mathbf{A}}(iy)\, \Upsilon_{\kappa,\nu}(iy) = \widetilde{\Upsilon}_{\kappa,\nu}(iy)\, R_{\mathbf{A}}(iy)$ maps $\mathcal{D}(\check{H}_{\mathrm{f}}^{-\nu}) = \mathcal{D}(\check{H}_{\mathrm{f}}^{-\nu}\, f_\varepsilon^{-\kappa}(H_{\mathrm{f}})) = \mathrm{Ran}(\check{H}_{\mathrm{f}}^{\nu}\, f_\varepsilon^{\kappa}(H_{\mathrm{f}}))$ into itself and that (3.16) and (3.17) still hold true when $\nu$ is negative. □

In order to control the Coulomb singularity $1/|\mathbf{x}|$ in terms of $|D_{\mathbf{A}}|$ and $H_{\mathrm{f}}$ in the proof of the following corollary, we shall employ the bound [10, Theorem 2.3]

$$\frac{2}{\pi}\frac{1}{|\mathbf{x}|} \,\leqslant\, |D_{\mathbf{A}}| + H_{\mathrm{f}} + k\, d_1^2\,, \tag{3.19}$$

which holds true in sense of quadratic forms on $\mathcal{Q}(|D_{\mathbf{A}}|) \cap \mathcal{Q}(H_{\mathrm{f}})$. Here $k \in (0,\infty)$ is some universal constant. We abbreviate the sign function of the Dirac operator, which can be represented as a strongly convergent principal value [6, Lemma VI.5.6], by

$$S_{\mathbf{A}}\, \psi \,:=\, D_{\mathbf{A}}\, |D_{\mathbf{A}}|^{-1}\, \psi \,=\, \lim_{\tau\to\infty} \int_{-\tau}^{\tau} R_{\mathbf{A}}(iy)\, \psi\, \frac{dy}{\pi}\,. \tag{3.20}$$

We recall from [10, Lemma 3.3] that $S_{\mathbf{A}}$ maps $\mathcal{D}(H_{\mathrm{f}}^{\nu})$ into itself, for every $\nu > 0$. This can also be read off from the proof of the next corollary.

COROLLARY 3.4. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1. Let $\kappa, \nu \in \mathbb{R}$. Then we find some $C \equiv C(\kappa, \nu, d_1, d_\rho) \in (0,\infty)$ such that, for all $\gamma, \delta, \sigma, \tau \geqslant 0$*

*with* $\gamma+\delta+\sigma+\tau \leqslant 1/2$ *and all* $\varepsilon \geqslant 0$, $E \geqslant k_1$,

$$\big\| \check{H}_{\mathrm{f}}^{\nu}\, f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})\, S_{\mathbf{A}}\, \check{H}_{\mathrm{f}}^{-\nu}\, f_{\varepsilon}^{-\kappa}(H_{\mathrm{f}}) \big\| \leqslant C, \tag{3.21}$$

$$\big\| |D_{\mathbf{A}}|^{1/2}\, \check{H}_{\mathrm{f}}^{\nu+\gamma}\, f_{\varepsilon}^{\sigma}(H_{\mathrm{f}})\, [S_{\mathbf{A}}\,,\, f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})]\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, f_{\varepsilon}^{-\kappa+\tau}(H_{\mathrm{f}}) \big\| \leqslant C, \tag{3.22}$$

$$\big\| |\mathbf{x}|^{-1/2}\, \check{H}_{\mathrm{f}}^{\nu}\, f_{\varepsilon}^{\sigma}(H_{\mathrm{f}})\, [S_{\mathbf{A}}\,,\, f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})]\, \check{H}_{\mathrm{f}}^{-\nu-\sigma-\tau}\, f_{\varepsilon}^{-\kappa+\tau}(H_{\mathrm{f}}) \big\| \leqslant C. \tag{3.23}$$

*($k_1$ is the constant appearing in Lemma 3.3, $\check{H}_{\mathrm{f}}$ is given by* (3.8)*, $f_\varepsilon$ by* (3.10)*.)*

*Proof.* First, we prove (3.22). Using (3.20), writing

$$[R_{\mathbf{A}}(iy)\,,\, f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})] \,=\, R_{\mathbf{A}}(iy)\,[f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})\,,\, \boldsymbol{\alpha}\cdot\mathbf{A}]\,R_{\mathbf{A}}(iy)$$

on $\mathscr{D}$ and employing (3.16), (3.17), and (3.11) we obtain the following estimate, for all $\varphi,\psi\in\mathscr{D}$, and $E\geqslant k_1$,

$$\begin{aligned}
&\big|\big\langle\, |D_{\mathbf{A}}|^{1/2}\,\varphi \,\big|\, \check{H}_{\mathrm{f}}^{\nu+\gamma}\, f_{\varepsilon}^{\sigma}(H_{\mathrm{f}})\, [S_{\mathbf{A}}\,,\, f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})]\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, f_{\varepsilon}(H_{\mathrm{f}})^{-\kappa+\tau}\,\psi\,\big\rangle\big| \\
&\leqslant \int_{\mathbb{R}} \Big|\Big\langle\, \check{H}_{\mathrm{f}}^{\nu+\gamma}\,|D_{\mathbf{A}}|^{1/2}\,\varphi \,\Big|\, f_{\varepsilon}^{\sigma}(H_{\mathrm{f}})\,[f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})\,,\,R_{\mathbf{A}}(iy)]\,\times \\
&\qquad\times\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, f_{\varepsilon}^{-\kappa+\tau}(H_{\mathrm{f}})\,\psi\,\Big\rangle\Big|\,\frac{dy}{\pi} \\
&= \int_{\mathbb{R}} \Big|\Big\langle\, \varphi \,\Big|\, |D_{\mathbf{A}}|^{1/2}\,R_{\mathbf{A}}(iy)\,\Upsilon_{\sigma,\nu+\gamma}(iy)\,\check{H}_{\mathrm{f}}^{\nu+\gamma}\,f_{\varepsilon}^{\sigma}(H_{\mathrm{f}})\,[f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}]\,\times \\
&\qquad\times\, \check{H}_{\mathrm{f}}^{-\nu+\delta}\, f_{\varepsilon}^{-\kappa+\tau}(H_{\mathrm{f}})\,\widetilde{\Upsilon}_{\kappa-\tau,\nu-\delta}(iy)\,R_{\mathbf{A}}(iy)\,\psi\,\Big\rangle\Big|\,\frac{dy}{\pi} \\
&\leqslant C_{\kappa,\nu}\,(d_1+d_\rho)\,E^{\gamma+\delta+\sigma+\tau-1/2}\,\sup_{y\in\mathbb{R}}\{\|\Upsilon_{\sigma,\nu+\gamma}(iy)\|\,\|\widetilde{\Upsilon}_{\kappa-\tau,\nu-\delta}(iy)\|\} \\
&\qquad\cdot\Big(\int_{\mathbb{R}}\big\|\,|D_{\mathbf{A}}|^{1/2}\,R_{\mathbf{A}}(iy)\,\varphi\,\big\|^2\,\frac{dy}{\pi}\Big)^{1/2}\Big(\int_{\mathbb{R}}\big\|\,R_{\mathbf{A}}(iy)\,\psi\,\big\|^2\,\frac{dy}{\pi}\Big)^{1/2} \\
&\leqslant C_{\kappa,\nu,d_1,d_\rho}\,E^{\gamma+\delta+\sigma+\tau-1/2}\,\|\varphi\|\,\|\psi\|\,.
\end{aligned}$$

This estimate shows that the vector in the right entry of the scalar product in the first line belongs to $\mathcal{D}((|D_{\mathbf{A}}|^{1/2})^*) = \mathcal{D}(|D_{\mathbf{A}}|^{1/2})$ and that (3.22) holds true. Next, we observe that (3.23) follows from (3.22) and (3.19). Finally, (3.21) follows from $\|X\| \leqslant \mathrm{const}(\nu,\kappa,d_1,d_\rho)$, where $X := \check{H}_{\mathrm{f}}^{\nu}\, f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})\,[S_{\mathbf{A}}\,,\,\check{H}_{\mathrm{f}}^{-\nu}\, f_{\varepsilon}^{-\kappa}(H_{\mathrm{f}})]$. Such a bound on $\|X\|$ is, however, an immediate consequence of (3.22) (where we can choose $\varepsilon=0$) because

$$X \,=\, [\check{H}_{\mathrm{f}}^{\nu}\,,\,S_{\mathbf{A}}]\,\check{H}_{\mathrm{f}}^{-\nu} \,+\, \check{H}_{\mathrm{f}}^{\nu}\,[f_{\varepsilon}^{\kappa}(H_{\mathrm{f}})\,,\,S_{\mathbf{A}}]\,f_{\varepsilon}^{-\kappa}(H_{\mathrm{f}})\,\check{H}_{\mathrm{f}}^{-\nu}$$

on the domain $\mathscr{D}$. □

## 4. Non-relativistic QED

The Pauli-Fierz operator for a molecular system with static nuclei and $N\in\mathbb{N}$ electrons interacting with the quantized radiation field is acting in the Hilbert space

$$\mathscr{H}_N \,:=\, \mathcal{A}_N L^2\big((\mathbb{R}^3\times\mathbb{Z}_4)^N\big)\otimes\mathscr{F}_{\mathrm{b}}\,, \tag{4.1}$$

where $\mathcal{A}_N = \mathcal{A}_N^2 = \mathcal{A}_N^*$ denotes anti-symmetrization,

$$(\mathcal{A}_N\,\Psi)(X) \,:=\, \frac{1}{N!}\sum_{\pi\in\mathfrak{S}_N}(-1)^\pi\,\Psi\big(\mathbf{x}_{\pi(1)},\varsigma_{\pi(1)},\ldots,\mathbf{x}_{\pi(N)},\varsigma_{\pi(N)}\big)\,,$$

for $\Psi \in L^2((\mathbb{R}^3\times\mathbb{Z}_4)^N)$ and a.e. $X = (\mathbf{x}_i,\varsigma_i)_{i=1}^N \in (\mathbb{R}^3\times\mathbb{Z}_4)^N$. a priori it is defined on the dense domain

$$\mathscr{D}_N \,:=\, \mathcal{A}_N C_0^\infty\big((\mathbb{R}^3\times\mathbb{Z}_4)^N\big)\otimes\mathscr{C}_0\,,$$

the tensor product understood in the algebraic sense, by

$$H^V_{\mathrm{nr}} \,\equiv\, H^V_{\mathrm{nr}}(\mathbf{G}) \,:=\, \sum_{i=1}^N (D^{(i)}_{\mathbf{A}})^2 \,+\, V \,+\, H_{\mathrm{f}}\,. \tag{4.2}$$

A superscript $(i)$ indicates that the operator below is acting on the pair of variables $(\mathbf{x}_i,\varsigma_i)$. In fact, the operator defined in (4.2) is a two-fold copy of the usual Pauli-Fierz operator which acts on two-spinors and the energy has been shifted by $N$ in (4.2). For (3.5) implies

$$D^2_{\mathbf{A}} = \mathcal{T}_{\mathbf{A}}\oplus\mathcal{T}_{\mathbf{A}}\,,\quad \mathcal{T}_{\mathbf{A}} := \big(\boldsymbol{\sigma}\cdot(-i\nabla_{\mathbf{x}}+\mathbf{A})\big)^2+1\,. \tag{4.3}$$

Here $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ is a vector containing the Pauli matrices (when $\alpha_j$, $j\in\{0,1,2,3\}$, are given in Dirac's standard representation). We write $H^V_{\mathrm{nr}}$ in the form (4.2) to maintain a unified notation throughout this paper.
We shall only make use of the following properties of the potential $V$.

Hypothesis 4.1. *$V$ can be written as $V = V_+ - V_-$, where $V_\pm \geqslant 0$ is a symmetric operator acting in $\mathcal{A}_N L^2\big((\mathbb{R}^3\times\mathbb{Z}_4)^4\big)$ such that $\mathscr{D}_N\subset\mathcal{D}(V_\pm)$. There exist $a\in(0,1)$ and $b\in(0,\infty)$ such that $V_-\leqslant a\,H^0_{\mathrm{nr}}+b$ in the sense of quadratic forms on $\mathscr{D}_N$.*

*Example.* The Coulomb potential generated by $K\in\mathbb{N}$ fixed nuclei located at the positions $\{\mathbf{R}_1,\ldots,\mathbf{R}_K\}\subset\mathbb{R}^3$ is given as

$$V_{\mathrm{C}}(X) \,:=\, -\sum_{i=1}^N\sum_{k=1}^K\frac{e^2\,Z_k}{|\mathbf{x}_i-\mathbf{R}_k|} \,+\, \sum_{\substack{i,j=1\\ i<j}}^N\frac{e^2}{|\mathbf{x}_i-\mathbf{x}_j|}\,, \tag{4.4}$$

for some $e, Z_1,\ldots,Z_K > 0$ and a.e. $X = (\mathbf{x}_i,\varsigma_i)_{i=1}^N \in (\mathbb{R}^3\times\mathbb{Z}_4)^N$. It is well-known that $V_{\mathrm{C}}$ is infinitesimally $H^0_{\mathrm{nr}}$-bounded and that $V_{\mathrm{C}}$ fulfills Hypothesis 4.1. ◇

It follows immediately from Hypothesis 4.1 that $H^V_{\mathrm{nr}}$ has a self-adjoint Friedrichs extension – henceforth denoted by the same symbol $H^V_{\mathrm{nr}}$ – and that $\mathscr{D}_N$ is a form core for $H^V_{\mathrm{nr}}$. Moreover, we have

$$(D^{(1)}_{\mathbf{A}})^2,\ldots,(D^{(N)}_{\mathbf{A}})^2,\,V_+,\,H_{\mathrm{f}}\,\leqslant\,H^{V_+}_{\mathrm{nr}}\,\leqslant\,(1-a)^{-1}\,(H^V_{\mathrm{nr}}+b) \tag{4.5}$$

on $\mathscr{D}_N$. In [4] it is shown that $\mathcal{D}((H^V_{\mathrm{nr}})^{n/2})\subset\mathcal{D}(H_{\mathrm{f}}^{n/2})$, for every $n\in\mathbb{N}$. We re-derive this result by means of Corollary 2.3 in the next theorem where

$$E_{\mathrm{nr}} \,:=\, \inf\sigma[H^V_{\mathrm{nr}}]\,,\qquad H'_{\mathrm{nr}} \,:=\, H^V_{\mathrm{nr}} - E_{\mathrm{nr}} + 1\,.$$

Theorem 4.2. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1 and that $V$ fulfills Hypothesis 4.1. Assume in addition that*

$$2\int \omega(k)^{\ell}\,\|\nabla_{\mathbf{x}}\wedge\mathbf{G}(k)\|_{\infty}^{2}\,dk \leqslant d_{\ell+2}^{2}\,, \tag{4.6}$$

$$\int \omega(k)^{\ell}\,\|\nabla_{\mathbf{x}}\cdot\mathbf{G}(k)\|_{\infty}^{2}\,dk \leqslant d_{\ell+2}^{2}\,, \tag{4.7}$$

*for all $\ell\in\{-1,0,1,2,\ldots\}$. Then, for every $n\in\mathbb{N}$, we have $\mathcal{D}((H_{\mathrm{nr}}^{V})^{n/2})\subset\mathcal{D}(H_{\mathrm{f}}^{n/2})$, $H_{\mathrm{f}}^{n/2}\,(H_{\mathrm{nr}}')^{-n/2}$ maps $\mathcal{D}(H_{\mathrm{nr}}^{V})$ into itself, and*

$$\big\|\,H_{\mathrm{f}}^{n/2}\,(H_{\mathrm{nr}}')^{-n/2}\,\big\| \leqslant C(N,n,a,b,d_{-1},d_1,d_{5+n})\,(|E_{\mathrm{nr}}|+1)^{(3n-2)/2}\,,$$
$$\big\|\,[H_{\mathrm{f}}^{n/2}\,,\,H_{\mathrm{nr}}^{V}]\,(H_{\mathrm{nr}}')^{-n/2}\,\big\| \leqslant C'(N,n,a,b,d_{-1},d_1,d_{5+n})\,(|E_{\mathrm{nr}}|+1)^{(3n-1)/2}\,.$$

*Proof.* We pick the function $f_\varepsilon$ defined in (3.10) with $E=1$ and verify that the operators $F_\varepsilon^n := f_\varepsilon^{n/2}(H_{\mathrm{f}})$, $\varepsilon>0$, $n\in\mathbb{N}$, and $H_{\mathrm{nr}}'$ fulfill the conditions (a), (b), and (c') of Theorem 2.1 and Corollary 2.2 with $m=\infty$. Then the assertion follows from Corollary 2.3. We set $\check{H}_{\mathrm{f}} := H_{\mathrm{f}}+E$ in what follows. By means of (4.5) we find

$$\langle\,\Psi\,|\,F_\varepsilon^2\,\Psi\,\rangle \leqslant \langle\,\Psi\,|\,\check{H}_{\mathrm{f}}\,\Psi\,\rangle \leqslant \frac{E_{\mathrm{nr}}+b+E}{1-a}\,\langle\,\Psi\,|\,H_{\mathrm{nr}}'\,\Psi\,\rangle\,, \tag{4.8}$$

for all $\Psi\in\mathscr{D}_N$, which is Condition (b). Next, we observe that $F_\varepsilon$ maps $\mathscr{D}_N$ into itself. Employing (4.5) once more and using $-V_-\leqslant 0$ and the fact that $V_+\geqslant 0$ and $F_\varepsilon$ act on different tensor factors we deduce that

$$\begin{aligned}\big\langle\,F_\varepsilon\,\Psi\,\big|\,(V+H_{\mathrm{f}})\,F_\varepsilon\,\Psi\,\big\rangle &\leqslant \|f_\varepsilon\|_\infty\,\big\langle\,\Psi\,\big|\,(V_+ + H_{\mathrm{f}})\,\Psi\,\big\rangle\\ &\leqslant \|f_\varepsilon\|_\infty\,\frac{E_{\mathrm{nr}}+b+E}{1-a}\,\langle\,\Psi\,|\,H_{\mathrm{nr}}'\,\Psi\,\rangle\,,\end{aligned} \tag{4.9}$$

for every $\Psi\in\mathscr{D}_N$. Thanks to (3.11) with $\kappa=1/2$, $\nu=\gamma=\delta=\sigma=\tau=0$, and (4.5) we further find some $C\in(0,\infty)$ such that

$$\begin{aligned}\big\|\,D_{\mathbf{A}}^{(i)}\,F_\varepsilon\,\Psi\,\big\|^2 &\leqslant 2\,\|f_\varepsilon\|_\infty\,\|D_{\mathbf{A}}^{(i)}\,\Psi\|^2 + 2\,\|f_\varepsilon\|_\infty\,\big\|\,F_\varepsilon^{-1}\,[\boldsymbol{\alpha}\cdot\mathbf{A}\,,\,F_\varepsilon]\,\big\|^2\,\|\Psi\|^2\\ &\leqslant C\,\|f_\varepsilon\|_\infty\,\langle\,\Psi\,|\,H_{\mathrm{nr}}'\,\Psi\,\rangle\,,\end{aligned} \tag{4.10}$$

for all $\Psi\in\mathscr{D}_N$. (4.9) and (4.10) together show that Condition (a) is fulfilled, too. Finally, we verify the bound in (c'). We use

$$[\boldsymbol{\alpha}\cdot(-i\nabla_{\mathbf{x}})\,,\,\boldsymbol{\alpha}\cdot\mathbf{A}] = \boldsymbol{\Sigma}\cdot\mathbf{B} - i\,(\nabla_{\mathbf{x}}\cdot\mathbf{A})\,,$$

where $\mathbf{B} := a^\dagger(\nabla_{\mathbf{x}}\wedge\mathbf{G}) + a(\nabla_{\mathbf{x}}\wedge\mathbf{G})$ is the magnetic field and the $j$-th entry of the formal vector $\boldsymbol{\Sigma}$ is $-i\,\epsilon_{jk\ell}\,\alpha_k\,\alpha_\ell$, $j,k,\ell\in\{1,2,3\}$, to write the square of the Dirac operator on the domain $\mathscr{D}$ as

$$D_{\mathbf{A}}^2 = D_{\mathbf{0}}^2 + \boldsymbol{\Sigma}\cdot\mathbf{B} - i\,(\nabla_{\mathbf{x}}\cdot\mathbf{A}) + (\boldsymbol{\alpha}\cdot\mathbf{A})^2 + 2\,\boldsymbol{\alpha}\cdot\mathbf{A}\,\boldsymbol{\alpha}\cdot(-i\nabla_{\mathbf{x}})\,.$$

This yields

$$[H'_{\mathrm{nr}}, F_\varepsilon^n] = \sum_{i=1}^N \big[(D_{\mathbf{A}}^{(i)})^2, F_\varepsilon^n\big] = \sum_{i=1}^N \big\{ [\boldsymbol{\Sigma}\cdot\mathbf{B}^{(i)}\,,\,F_\varepsilon^n] - i\,[(\nabla_{\mathbf{x}}\cdot\mathbf{A}^{(i)})\,,\,F_\varepsilon^n]$$
$$+\,\boldsymbol{\alpha}\cdot\mathbf{A}^{(i)}\,[\boldsymbol{\alpha}\cdot\mathbf{A}^{(i)}\,,\,F_\varepsilon^n] + [\boldsymbol{\alpha}\cdot\mathbf{A}^{(i)}\,,\,F_\varepsilon^n]\,(2D_{\mathbf{A}}^{(i)} - \boldsymbol{\alpha}\cdot\mathbf{A}^{(i)} - 2\beta)\,\big\}$$

on $\mathscr{D}_N$. For every $i \in \{1,\dots,N\}$, we further write

$$[\boldsymbol{\alpha}\cdot\mathbf{A}^{(i)}\,,\,F_\varepsilon^n]\,D_{\mathbf{A}}^{(i)} \;=\; Q_{\varepsilon,n}^{(i)}\,\big(\,D_{\mathbf{A}}^{(i)}\,F_\varepsilon^{n-1} - Q_{\varepsilon,n-1}^{(i)}\,F_\varepsilon^{n-2}\,\big)$$

on $\mathscr{D}_N$, where

$$Q_{\varepsilon,n}^{(i)} \;:=\; [\boldsymbol{\alpha}\cdot\mathbf{A}^{(i)}\,,\,F_\varepsilon^n]\,F_\varepsilon^{1-n}, \quad n\in\mathbb{N}\,, \qquad Q_{\varepsilon,0}^{(i)} \;:=\; 0\,. \tag{4.11}$$

According to (3.11) we have $\|Q_{\varepsilon,n}^{(i)}\| \leqslant n\,2^{(n+2)/2}\,(d_1 + d_{3+n})$, $\|\check{H}_{\mathrm{f}}^{1/2}\,Q_{\varepsilon,n}^{(i)}\,\check{H}_{\mathrm{f}}^{-1/2}\| \leqslant n\,2^{(n+3)/2}(d_1 + d_{4+n})$. Likewise, we write

$$[\boldsymbol{\alpha}\cdot\mathbf{A}^{(i)}\,,\,F_\varepsilon^n]\,\boldsymbol{\alpha}\cdot\mathbf{A}^{(i)} \;=\; Q_{\varepsilon,n}^{(i)}\,\big(\,\{\boldsymbol{\alpha}\cdot\mathbf{A}^{(i)}\,\check{H}_{\mathrm{f}}^{-1/2}\}\,\check{H}_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1} - Q_{\varepsilon,n-1}^{(i)}\,F_\varepsilon^{n-2}\,\big)$$

on $\mathscr{D}_N$, where $\|\boldsymbol{\alpha}\cdot\mathbf{A}\,\check{H}_{\mathrm{f}}^{-1/2}\|^2 \leqslant 2\,d_0^2 + 4\,d_{-1}^2$ by (3.7). Furthermore, we observe that Lemma 3.2 is applicable to $\boldsymbol{\Sigma}\cdot\mathbf{B}$ as well instead of $\boldsymbol{\alpha}\cdot\mathbf{A}$; we simply have to replace the form factor $\mathbf{G}$ by $\nabla_{\mathbf{x}}\wedge\mathbf{G}$ and to notice that $\|\boldsymbol{\Sigma}\cdot\mathbf{v}\|_{\mathscr{L}(\mathbb{C}^4)} = |\mathbf{v}|$, $\mathbf{v}\in\mathbb{R}^3$, in analogy to (3.6). Note that the indices of $d_\ell$ are shifted by 2 because of (4.6). Finally, we observe that Lemma 3.2 is applicable to $\nabla_{\mathbf{x}}\cdot\mathbf{A}$, too. To this end we have to replace $\mathbf{G}$ by $(\nabla_{\mathbf{x}}\cdot\mathbf{G},0,0)$ and $d_\ell$ by some universal constant times $d_{2+\ell}$ because of (4.7). Taking all these remarks into account we arrive at

$$\big|\big\langle\,\Psi_1\,\big|\,[H'_{\mathrm{nr}}\,,\,F_\varepsilon^n]\,\Psi_2\,\big\rangle\big| \;\leqslant\; \sum_{i=1}^N \Big\{\,\|\Psi_1\|\,\big\|\,[\boldsymbol{\Sigma}\cdot\mathbf{B}^{(i)}\,,\,F_\varepsilon^n]\,F_\varepsilon^{1-n}\,\big\|\,\|F_\varepsilon^{n-1}\,\Psi_2\|$$
$$+\,\|\Psi_1\|\,\big\|\,[\mathrm{div}\,\mathbf{A}^{(i)}\,,\,F_\varepsilon^n]\,F_\varepsilon^{1-n}\,\big\|\,\|F_\varepsilon^{n-1}\,\Psi_2\|$$
$$+\,\|\Psi_1\|\,\|\boldsymbol{\alpha}\cdot\mathbf{A}\,\check{H}_{\mathrm{f}}^{-1/2}\|\,\big\|\,\check{H}_{\mathrm{f}}^{1/2}\,Q_{\varepsilon,n}^{(i)}\,\check{H}_{\mathrm{f}}^{-1/2}\,\big\|\,\|\check{H}_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,\Psi_2\|$$
$$+\,\|\Psi_1\|\,\|Q_{\varepsilon,n}^{(i)}\|\,\big(\,2\,\|D_{\mathbf{A}}^{(i)}\,F_\varepsilon^{n-1}\,\Psi_2\| + \|\boldsymbol{\alpha}\cdot\mathbf{A}\,\check{H}_{\mathrm{f}}^{-1/2}\|\,\|\check{H}_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,\Psi_2\|\,\big)$$
$$+\,3\,\|\Psi_1\|\,\|Q_{\varepsilon,n}^{(i)}\|\,\|Q_{\varepsilon,n-1}^{(i)}\|\,\|F_\varepsilon^{n-2}\,\Psi_2\| + 2\,\|\Psi_1\|\,\|Q_{\varepsilon,n}^{(i)}\|\,\|\beta\|\,\|F_\varepsilon^{n-1}\,\Psi_2\|\,\Big\}\,,$$

for all $\Psi_1,\Psi_2\in\mathscr{D}_N$. From this estimate, Lemma 3.2, and (4.5) we readily infer that Condition (c') is valid with $c_n = (|E_{\mathrm{nr}}|+1)\,C''(N,n,a,b,d_{-1},\dots,d_{5+n})$. □

## 5. The semi-relativistic Pauli-Fierz operator

The semi-relativistic Pauli-Fierz operator is also acting in the Hilbert space $\mathscr{H}_N$ introduced in (4.1). It is obtained by substituting the non-local operator $|D_{\mathbf{A}}|$ for $D_{\mathbf{A}}^2$ in $H_{\mathrm{nr}}^V$. We thus define, a priori on the dense domain $\mathscr{D}_N$,

$$H_{\mathrm{sr}}^V \;\equiv\; H_{\mathrm{sr}}^V(\mathbf{G}) \;:=\; \sum_{i=1}^N |D_{\mathbf{A}}^{(i)}| \;+\; V \;+\; H_{\mathrm{f}}\,,$$

where $V$ is assumed to fulfill Hypothesis 4.1 with $H^0_{\mathrm{nr}}$ replaced by $H^0_{\mathrm{sr}}$. To ensure that in the case of the Coulomb potential $V_{\mathrm{C}}$ defined in (4.4) this yields a well-defined self-adjoint operator we have to impose appropriate restrictions on the nuclear charges.

*Example.* In Proposition A.1 we show that $H^{V_{\mathrm{C}}}_{\mathrm{sr}}$ is semi-bounded below on $\mathscr{D}_N$ provided that $Z_k \in (0, 2/\pi e^2]$, for all $k \in \{1, \dots, K\}$. Its proof is actually a straightforward consequence of (3.19) and a commutator estimate obtained in [10]. If all atomic numbers $Z_k$ are strictly less than $2/\pi e^2$ we thus find $a \in (0,1)$ and $b \in (0,\infty)$ such that

$$\sum_{i=1}^{N} \sum_{k=1}^{K} \frac{e^2\, Z_k}{|\mathbf{x}_i - \mathbf{R}_k|} \leqslant a\, H^0_{\mathrm{sr}} + b \tag{5.1}$$

in the sense of quadratic forms on $\mathscr{D}_N$. In particular, $V_{\mathrm{C}}$ fulfills Hypothesis 4.1 with $H^0_{\mathrm{nr}}$ replaced by $H^0_{\mathrm{sr}}$ as long as $Z_k \in (0, 2/\pi e^2)$, for $k \in \{1, \dots, K\}$. ⋄

For potentials $V$ as above $H^V_{\mathrm{sr}}$ has a self-adjoint Friedrichs extension which we denote again by the same symbol $H^V_{\mathrm{sr}}$. Moreover, $\mathscr{D}_N$ is a form core for $H^V_{\mathrm{sr}}$ and we have the following analogue of (4.5),

$$|D^{(1)}_{\mathbf{A}}|, \dots, |D^{(N)}_{\mathbf{A}}|,\, V_+,\, H_{\mathrm{f}} \leqslant H^{V_+}_{\mathrm{sr}} \leqslant (1-a)^{-1}\,(H^V_{\mathrm{sr}} + b) \tag{5.2}$$

on $\mathscr{D}_N$. In order to apply Corollary 2.3 to the semi-relativistic Pauli-Fierz operator we recall the following special case of [7, Corollary 3.7]:

Lemma 5.1. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1. Let $\tau \in (0,1]$. Then there exist $\delta > 0$ and $C \equiv C(\delta, \tau, d_1) \in (0,\infty)$ such that*

$$C + |D_{\mathbf{A}}| + \tau\, H_{\mathrm{f}} \geqslant \delta\,(|D_{\mathbf{0}}| + H_{\mathrm{f}}) \geqslant \delta\,(|D_{\mathbf{0}}| + \tau\, H_{\mathrm{f}}) \geqslant \delta^2\, |D_{\mathbf{A}}| - \delta\, C \tag{5.3}$$

*in the sense of quadratic forms on $\mathscr{D}$.*

In the next theorem we re-derive the higher order estimates obtained in [4] for the semi-relativistic Pauli-Fierz operator by means of Corollary 2.3. (The second estimate of Theorem 5.2 is actually slightly stronger than the corresponding one stated in [4].) The estimates of the following proof are also employed in Section 6 where we treat the no-pair operator. We set

$$E_{\mathrm{sr}} := \inf \sigma[H_{\mathrm{sr}}]\,, \qquad H'_{\mathrm{sr}} := H^V_{\mathrm{sr}} - E_{\mathrm{sr}} + 1\,.$$

Theorem 5.2. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1 and that $V$ fulfills Hypothesis 4.1 with $H^0_{\mathrm{nr}}$ replaced by $H^0_{\mathrm{sr}}$. Then, for every $m \in \mathbb{N}$, it follows that $\mathcal{D}((H^V_{\mathrm{sr}})^{m/2}) \subset \mathcal{D}(H^{m/2}_{\mathrm{f}})$, $H^{m/2}_{\mathrm{f}}\,(H'_{\mathrm{sr}})^{-m/2}$ maps $\mathcal{D}(H^V_{\mathrm{sr}})$ into itself, and*

$$\big\|\, H^{m/2}_{\mathrm{f}}\,(H'_{\mathrm{sr}})^{-m/2} \,\big\| \leqslant C(N, m, a, b, d_1, d_{3+m})\,(|E_{\mathrm{sr}}| + 1)^{(3m-2)/2}\,,$$

$$\big\|\, [H^{m/2}_{\mathrm{f}}\,,\, H^V_{\mathrm{sr}}]\,(H'_{\mathrm{sr}})^{-m/2} \,\big\| \leqslant C'(N, m, a, b, d_1, d_{3+m})\,(|E_{\mathrm{sr}}| + 1)^{(3m-1)/2}\,.$$

*Proof.* Let $m \in \mathbb{N}$. We pick the function $f_\varepsilon$ defined in (3.10) with $E = k_1 \vee C$. ($k_1$ is the constant appearing in Lemma 3.3 with $\kappa = m/2$, $\nu = 0$, and depends on $m$, $d_1$, and $d_{3+m}$; $C$ is the one in (5.3).) We fix some $n \in \mathbb{N}$, $n \leqslant m$, and verify Conditions (a), (b), and (c') of Theorem 2.1 and Corollary 2.2 with $F_\varepsilon = f_\varepsilon^{1/2}(H_\mathrm{f})$, $\varepsilon > 0$. The estimates (4.8) and (4.9) are still valid without any further change when the subscript nr is replaced by sr. Employing (5.3) twice and using (5.2) we obtain the following substitute of (4.10),

$$\begin{aligned}\big\langle\, F_\varepsilon\,\Psi \,\big|\, |D_\mathbf{A}|\,F_\varepsilon\,\Psi\,\big\rangle &\leqslant \delta^{-1}\|\,|D_\mathbf{0}|^{1/2}\,F_\varepsilon\,\Psi\|^2+\delta^{-1}\,\|\check{H}_\mathrm{f}^{1/2}\,F_\varepsilon\,\Psi\|^2\\ &\leqslant \delta^{-1}\,\|f_\varepsilon\|_\infty\,\big(\|\,|D_\mathbf{0}|^{1/2}\,\Psi\|^2+\|\check{H}_\mathrm{f}^{1/2}\,\Psi\|^2\big)\leqslant C'\,\|f_\varepsilon\|_\infty\,\big\langle\,\Psi\,\big|\,H'_\mathrm{sr}\,\Psi\,\big\rangle\,,\end{aligned}$$

for all $\Psi \in \mathscr{D}_N$. Altogether we see that Conditions (a) and (b) are satisfied. In order to verify (c') we set

$$U^{(i)}_{\varepsilon,n} := [S^{(i)}_\mathbf{A}\,,\,F^n_\varepsilon]\,F^{1-n}_\varepsilon = F^n_\varepsilon\,[F^{-n}_\varepsilon\,,\,S^{(i)}_\mathbf{A}]\,F_\varepsilon\,,\qquad i\in\{1,\ldots,N\}\,. \tag{5.4}$$

By virtue of (3.22) we know that the norms of $U^{(i)}_{\varepsilon,n}$ and $U^{(i)}_{\varepsilon,n}\,|D^{(i)}_\mathbf{A}|^{1/2}$ are bounded uniformly in $\varepsilon > 0$ by some constant, $C \in (0,\infty)$, that depends only on $n$, $d_1$, and $d_{3+n}$. We employ the notation (4.11) and (5.4) to write

$$\begin{aligned}[H'_\mathrm{sr}\,,\,F^n_\varepsilon] &= \sum_{i=1}^N \big[\,|D^{(i)}_\mathbf{A}|\,,\,F^n_\varepsilon\,\big] = \sum_{i=1}^N\big[\,S^{(i)}_\mathbf{A}\,D^{(i)}_\mathbf{A}\,,\,F^n_\varepsilon\,\big]\\ &= \sum_{i=1}^N\Big\{\{U^{(i)}_{\varepsilon,n}\,|D^{(i)}_\mathbf{A}|^{1/2}\}\,S^{(i)}_\mathbf{A}\,|D^{(i)}_\mathbf{A}|^{1/2}\,F^{n-1}_\varepsilon\\ &\qquad\qquad -U^{(i)}_{\varepsilon,n}\,Q^{(i)}_{\varepsilon,n-1}\,F^{n-2}_\varepsilon+S^{(i)}_\mathbf{A}\,Q^{(i)}_{\varepsilon,n}\,F^{n-1}_\varepsilon\Big\}.\end{aligned}$$

The previous identity, (5.2), and $|D_\mathbf{A}| \geqslant 1$ permit to get

$$\begin{aligned}\big|\big\langle\,\Psi_1\,\big|\,[H'_\mathrm{sr}\,,\,F^n_\varepsilon]\,\Psi_2\,\big\rangle\big| &\leqslant \sum_{i=1}^N\|\Psi_1\|\,\big\{C\,\big\|\,|D^{(i)}_\mathbf{A}|^{1/2}\,F^{n-1}_\varepsilon\,\Psi_2\big\|\\ &\qquad + C\,\|Q^{(i)}_{\varepsilon,n}\|\,\|F^{n-2}_\varepsilon\,\Psi_2\|+\|Q^{(i)}_{\varepsilon,n}\|\,\|F^{n-1}_\varepsilon\,\Psi_2\|\big\}\\ &\leqslant c_n\,\big\{\,\|\Psi_1\|^2+\big\langle\,F^{n-1}_\varepsilon\,\Psi_2\,\big|\,H'_\mathrm{sr}\,F^{n-1}_\varepsilon\,\Psi_2\,\big\rangle\,\big\}\,,\end{aligned}$$

for all $\Psi_1, \Psi_2 \in \mathscr{D}_N$ and some constant $c_n = C''(n,a,b,d_1,d_{3+n})\,(|E_\mathrm{sr}|+1)$. So (c') is fulfilled also and the assertion follows from Corollary 2.3. □

## 6. The no-pair operator

We introduce the spectral projections

$$P^+_\mathbf{A} := E_{[0,\infty)}(D_\mathbf{A}) = \frac{1}{2}\,\mathbb{1}+\frac{1}{2}\,S_\mathbf{A}\,,\qquad P^-_\mathbf{A} := \mathbb{1}-P^+_\mathbf{A}\,. \tag{6.1}$$

The no-pair operator acts in the projected Hilbert space

$$\mathscr{H}^+_N \equiv \mathscr{H}^+_N(\mathbf{G}) := P^+_{\mathbf{A},N}\,\mathscr{H}_N\,,\qquad P^+_{\mathbf{A},N} := \prod_{i=1}^N P^{+,(i)}_\mathbf{A}\,,$$

and is a priori defined on the dense domain $P^+_{\mathbf{A},N}\,\mathscr{D}_N$ by

$$H^V_{\mathrm{np}} \;\equiv\; H^V_{\mathrm{np}}(\mathbf{G}) \;:=\; P^+_{\mathbf{A},N}\,\Big\{\,\sum_{i=1}^N D^{(i)}_{\mathbf{A}} + V + H_{\mathrm{f}}\,\Big\}\,P^+_{\mathbf{A},N}\,.$$

Notice that all operators $D^{(1)}_{\mathbf{A}},\dots,D^{(N)}_{\mathbf{A}}$ and $P^{+,(1)}_{\mathbf{A}},\dots,P^{+,(N)}_{\mathbf{A}}$ commute in pairs owing to the fact that the components of the vector potential $A^{(i)}(\mathbf{x})$, $A^{(j)}(\mathbf{y})$, $\mathbf{x},\mathbf{y}\in\mathbb{R}^3$, $i,j\in\{1,2,3\}$, commute in the sense that all their spectral projections commute; see the appendix to [9] for more details. (Here we use the assumption that $\mathbf{G}_{\mathbf{x}}(-\mathbf{k},\lambda)=\overline{\mathbf{G}_{\mathbf{x}}(\mathbf{k},\lambda)}$.) So the order of the application of the projections $P^{+,(i)}_{\mathbf{A}}$ is immaterial. In this section we restrict the discussion to the case where $V$ is given by the Coulomb potential $V_{\mathrm{C}}$ defined in (4.4). To have a handy notation we set

$$v_i \;:=\; -\sum_{k=1}^K \frac{e^2\,Z_k}{|\mathbf{x}_i-\mathbf{R}_k|}\,,\qquad w_{ij}\;:=\;\frac{e^2}{|\mathbf{x}_i-\mathbf{x}_j|}\,,$$

for all $i\in\{1,\dots,N\}$ and $1\leqslant i<j\leqslant N$, respectively. Thanks to [10, Proof of Lemma 3.4(ii)], which implies that $P^+_{\mathbf{A}}$ maps $\mathscr{D}$ into $\mathcal{D}(|D_{\mathbf{0}}|)\cap\mathcal{D}(H_{\mathrm{f}}^\nu)$, for every $\nu>0$, and Hardy's inequality, we actually know that $H^{V_{\mathrm{C}}}_{\mathrm{np}}$ is well-defined on $\mathscr{D}_N$. In order to apply Corollary 2.3 to $H^{V_{\mathrm{C}}}_{\mathrm{np}}$ we extend $H^{V_{\mathrm{C}}}_{\mathrm{np}}$ to a continuously invertible operator on the whole space $\mathscr{H}_N$: We pick the complementary projection,

$$P^\perp_{\mathbf{A},N}\;:=\;\mathbb{1}-P^+_{\mathbf{A},N}\,,$$

abbreviate

$$P^{+,(i,j)}_{\mathbf{A}}\;:=\;P^{+,(i)}_{\mathbf{A}}\,P^{+,(j)}_{\mathbf{A}}\;=\;P^{+,(j)}_{\mathbf{A}}\,P^{+,(i)}_{\mathbf{A}}\,,\qquad 1\leqslant i<j\leqslant N\,,$$

and define the operator $\widetilde{H}_{\mathrm{np}}$ a priori on the domain $\mathscr{D}_N$ by

$$\widetilde{H}_{\mathrm{np}}\;:=\;\sum_{i=1}^N\big\{\,|D^{(i)}_{\mathbf{A}}|+P^{+,(i)}_{\mathbf{A}}\,v_i\,P^{+,(i)}_{\mathbf{A}}\,\big\}+\sum_{\substack{i,j=1\\ i<j}}^N P^{+,(i,j)}_{\mathbf{A}}\,w_{ij}\,P^{+,(i,j)}_{\mathbf{A}}$$

$$+\,P^+_{\mathbf{A},N}\,H_{\mathrm{f}}\,P^+_{\mathbf{A},N}+P^\perp_{\mathbf{A},N}\,H_{\mathrm{f}}\,P^\perp_{\mathbf{A},N}\,. \tag{6.2}$$

Evidently, we have $[\widetilde{H}_{\mathrm{np}}\,,\,P^+_{\mathbf{A},N}]=0$ and $\widetilde{H}_{\mathrm{np}}\,P^+_{\mathbf{A},N}=H^{V_{\mathrm{C}}}_{\mathrm{np}}\,P^+_{\mathbf{A},N}$ on $\mathscr{D}_N$. In Proposition A.2 we show that the quadratic forms of the no-pair operator $H^{V_{\mathrm{C}}}_{\mathrm{np}}$ and of $\widetilde{H}_{\mathrm{np}}$ are semi-bounded below on $\mathscr{D}_N$ provided that the atomic numbers $Z_1,\dots,Z_K\geqslant 0$ are less than the critical one of the Brown-Ravenhall model determined in [3],

$$Z_{\mathrm{np}}\;:=\;(2/e^2)/(2/\pi+\pi/2)\,. \tag{6.3}$$

Therefore, both $H^{V_{\mathrm{C}}}_{\mathrm{np}}$ and $\widetilde{H}_{\mathrm{np}}$ possess self-adjoint Friedrichs extensions which are again denoted by the same symbols in the sequel. $\mathscr{D}_N$ is a form core for

$\widetilde{H}_{\mathrm{np}}$ and we have the bound

$$(6.4)\quad \widetilde{H}_{\mathrm{np}} - \sum_{i=1}^{N} P_{\mathbf{A}}^{+,(i)}\, v_i\, P_{\mathbf{A}}^{+,(i)} \leqslant \frac{Z_{\mathrm{np}} + |\mathscr{Z}|}{Z_{\mathrm{np}} - |\mathscr{Z}|}\big(\widetilde{H}_{\mathrm{np}} + C(N, \mathscr{Z}, \mathscr{R}, d_{-1}, d_1, d_5)\big)$$

on $\mathscr{D}_N$, where $|\mathscr{Z}| := \max\{Z_1, \dots, Z_K\} < Z_{\mathrm{np}}$. Moreover, it makes sense to define

$$E_{\mathrm{np}} := \inf \sigma[H_{\mathrm{np}}^{V_{\mathrm{C}}}],$$

so that

$$H'_{\mathrm{np}} := \widetilde{H}_{\mathrm{np}} - E_{\mathrm{np}}\, P_{\mathbf{A},N}^{+} + \mathbb{1} \geqslant \mathbb{1}\,.$$

Theorem 6.1. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1 and let $N, K \in \mathbb{N}$, $e > 0$, $\mathscr{Z} = (Z_1, \dots, Z_K) \in [0, Z_{\mathrm{np}})^K$, and $\mathscr{R} = \{\mathbf{R}_1, \dots, \mathbf{R}_K\} \subset \mathbb{R}^3$, where $Z_{\mathrm{np}}$ is defined in* (6.3). *Then $\mathcal{D}((H'_{\mathrm{np}})^{m/2}) \subset \mathcal{D}(H_{\mathrm{f}}^{m/2})$, for every $m \in \mathbb{N}$, and*

$$\begin{aligned}\big\| H_{\mathrm{f}}^{m/2}\restriction_{\mathscr{H}_N^+} (H_{\mathrm{np}} - (E_{\mathrm{np}} - 1)\,\mathbb{1}_{\mathscr{H}_N^+})^{-m/2} \big\|_{\mathscr{L}(\mathscr{H}_N^+, \mathscr{H}_N)} &\leqslant \big\| H_{\mathrm{f}}^{m/2}\, (H'_{\mathrm{np}})^{-m/2} \big\| \\ &\leqslant C(N, m, \mathscr{Z}, \mathscr{R}, e, d_{-1}, d_1, d_{5+m})\,(1 + |E_{\mathrm{np}}|)^{(3m-2)/2} < \infty\,.\end{aligned}$$

*Proof.* Let $m \in \mathbb{N}$. Again we pick the function $f_\varepsilon$ defined in (3.10) and set $F_\varepsilon := f_\varepsilon^{1/2}(H_{\mathrm{f}})$, $\varepsilon > 0$. This time we choose $E = \max\{k\, d_1^2, k_1, C\}$ where $k$ is the constant appearing in (3.19), $C \equiv C(d_1)$ is the one in (5.3), and $k_1$ the one appearing in Lemma 3.3 with $|\kappa| = (m+1)/2$, $|\nu| = 1/2$. Thus $k_1$ depends only on $m$, $d_1$, and $d_{5+m}$. On account of Corollary 2.3 it suffices to show that the conditions (a)–(c) of Theorem 2.1 are fulfilled. To this end we observe that on $\mathscr{D}_N$ the extended no-pair operator can be written as $H'_{\mathrm{np}} = H_{\mathrm{sr}}^0 + \mathbb{1} + W$, where

$$\begin{aligned} W := \sum_{i=1}^{N} P_{\mathbf{A}}^{+,(i)}\, v_i\, P_{\mathbf{A}}^{+,(i)} &+ \sum_{\substack{i,j=1\\ i<j}}^{N} P_{\mathbf{A}}^{+,(i,j)}\, w_{ij}\, P_{\mathbf{A}}^{+,(i,j)} \\ &- E_{\mathrm{np}}\, P_{\mathbf{A},N}^{+} - 2\mathrm{Re}\big[P_{\mathbf{A},N}^{+}\, H_{\mathrm{f}}\, P_{\mathbf{A},N}^{\perp}\big]\,.\end{aligned}$$

The semi-relativistic Pauli-Fierz operator $H_{\mathrm{sr}}^0$ has already been treated in the previous section and the bound

$$(6.5)\qquad H_{\mathrm{f}} \leqslant 2P_{\mathbf{A},N}^{+}\, H_{\mathrm{f}}\, P_{\mathbf{A},N}^{+} + 2P_{\mathbf{A},N}^{\perp}\, H_{\mathrm{f}}\, P_{\mathbf{A},N}^{\perp}$$

together with (6.4) implies

$$(6.6)\qquad H_{\mathrm{sr}}^0 \leqslant 2\,\widetilde{H}_{\mathrm{np}} - 2\sum_{i=1}^{N} P_{\mathbf{A}}^{+,(i)}\, v_i\, P_{\mathbf{A}}^{+,(i)} \leqslant C'\,(1 + |E_{\mathrm{np}}|)\, H'_{\mathrm{np}}$$

on $\mathscr{D}_N$, for some $C' \equiv C'(N, \mathscr{Z}, \mathscr{R}, d_{-1}, d_1, d_5) \in (0, \infty)$. Hence, it only remains to consider the operator $W$.

We fix some $n \in \mathbb{N}$, $n \leqslant m$. When we verify (a) we can ignore the potentials $v_i$ since they are negative. Using $[F_\varepsilon^n, P^\perp_{\mathbf{A},N}] = [P^+_{\mathbf{A},N}, F_\varepsilon^n]$ we obtain

$$\begin{aligned}
&\big|2\mathrm{Re}\,\big\langle\, P^+_{\mathbf{A},N}\, F_\varepsilon\, \Psi \,\big|\, H_\mathrm{f}\, P^\perp_{\mathbf{A},N}\, F_\varepsilon\, \Psi \,\big\rangle\big| \\
&\qquad \leqslant \big\|\, H_\mathrm{f}^{1/2}\, P^+_{\mathbf{A},N}\, F_\varepsilon\, \Psi \,\big\|^2 + \big\|\, H_\mathrm{f}^{1/2}\, P^\perp_{\mathbf{A},N}\, F_\varepsilon\, \Psi \,\big\|^2 \\
&\qquad \leqslant 2\,\|f_\varepsilon\|_\infty\, \big\|H_\mathrm{f}^{1/2}\, P^+_{\mathbf{A},N}\, \Psi\big\|^2 + 2\,\|f_\varepsilon\|_\infty\, \big\|H_\mathrm{f}^{1/2}\, P^\perp_{\mathbf{A},N}\, \Psi\big\|^2 \\
&\qquad\quad + 4\,\big\|\, H_\mathrm{f}^{1/2}\, [P^+_{\mathbf{A},N}\,,\, F_\varepsilon]\, \check{H}_\mathrm{f}^{-1/2}\,\big\|\, \|\check{H}_\mathrm{f}^{1/2}\, \Psi\|^2,
\end{aligned}$$

for every $\Psi \in \mathscr{D}_N$, where, for all $n \in \mathbb{N}$ and $\nu \in \mathbb{R}$,

$$\begin{aligned}
\check{H}_\mathrm{f}^{\nu}\, [P^+_{\mathbf{A},N}\,,\, F_\varepsilon^n]\, \check{H}_\mathrm{f}^{-\nu}\, F_\varepsilon^{1-n} \;=\; \sum_{i=1}^{N} \Big\{ \prod_{j=1}^{i-1} \check{H}_\mathrm{f}^{\nu}\, P^{+,(j)}_{\mathbf{A}}\, \check{H}_\mathrm{f}^{-\nu} \Big\} \times \\
\times\; \big\{ \check{H}_\mathrm{f}^{\nu}\, [P^{+,(i)}_{\mathbf{A}}, F_\varepsilon^n]\, \check{H}_\mathrm{f}^{-\nu}\, F_\varepsilon^{1-n} \big\} \Big\{ \prod_{k=i+1}^{N} \check{H}_\mathrm{f}^{\nu}\, F_\varepsilon^{n-1}\, P^{+,(k)}_{\mathbf{A}}\, \check{H}_\mathrm{f}^{-\nu}\, F_\varepsilon^{1-n} \Big\}
\end{aligned}$$

on $\mathscr{D}_N$. On account of Corollary 3.4 we thus have, for $|\nu| \leqslant 1/2$,

$$\sup_{\varepsilon>0} \big\|\, H_\mathrm{f}^{\nu}\, [P^+_{\mathbf{A},N}\,,\, F_\varepsilon^n]\, \check{H}_\mathrm{f}^{-\nu}\, F_\varepsilon^{1-n}\,\big\| \;\leqslant\; C(N, n, d_1, d_{4+n})\,. \tag{6.7}$$

Likewise we have

$$\begin{aligned}
\big|\big\langle\, F_\varepsilon\, \Psi \,\big|\, P^{+,(i,j)}_{\mathbf{A}}\, w_{ij}\, P^{+,(i,j)}_{\mathbf{A}}\, F_\varepsilon\, \Psi \,\big\rangle\big| \;&\leqslant\; 2\,\|f_\varepsilon\|\, \big\|\, w_{ij}^{1/2}\, P^{+,(i,j)}_{\mathbf{A}}\, \Psi \,\big\|^2 \\
&\quad + 4\,\big\|\, w_{ij}^{1/2}\, [P^{+,(i,j)}_{\mathbf{A}}\,,\, F_\varepsilon]\, \check{H}_\mathrm{f}^{-1/2}\,\big\|^2\, \|\check{H}_\mathrm{f}^{1/2}\, \Psi\|^2,
\end{aligned} \tag{6.8}$$

where the first norm in the second line of (6.8) is bounded (uniformly in $\varepsilon > 0$) due to Lemma 6.2. Taking these remarks, $v_i \leqslant 0$, (6.4), and (6.5) into account we infer that

$$\big\langle\, F_\varepsilon\, \Psi \,\big|\, H'_\mathrm{np}\, F_\varepsilon\, \Psi \,\big\rangle \;\leqslant\; c_\varepsilon \big\langle\, \Psi \,\big|\, H'_\mathrm{np}\, \Psi \,\big\rangle\,, \qquad \Psi \in \mathscr{D}_N\,,$$

showing that (a) is fulfilled. The condition (b) with some constant $c^2 = C(N, \mathscr{Z}, \mathscr{R}, d_{-1}, d_1, d_5)(1+|E_\mathrm{np}|)$ follows immediately from $F_\varepsilon^2 \leqslant \check{H}_\mathrm{f} \leqslant H^0_\mathrm{sr} + E$ on $\mathscr{D}_N$ and (6.6). Finally, we turn to Condition (c). To this end let $P^\sharp_{\mathbf{A},N}$ and $P^\flat_{\mathbf{A},N}$ be $P^+_{\mathbf{A},N}$ or $P^\perp_{\mathbf{A},N}$. On $\mathscr{D}_N$ we clearly have

$$\big[\, P^\sharp_{\mathbf{A},N}\, H_\mathrm{f}\, P^\flat_{\mathbf{A},N}\,,\, F_\varepsilon^n \,\big] \;=\; \pm\,[P^+_{\mathbf{A},N}\,,\, F_\varepsilon^n]\, H_\mathrm{f}\, P^\flat_{\mathbf{A},N} \;\pm\; P^\sharp_{\mathbf{A},N}\, H_\mathrm{f}\, [P^+_{\mathbf{A},N}\,,\, F_\varepsilon^n]\,. \tag{6.9}$$

For $\Psi_1, \Psi_2 \in \mathscr{D}_N$, we thus obtain

$$\begin{aligned}
&\big|\big\langle\, \Psi_1 \,\big|\, \big[\, P^\sharp_{\mathbf{A},N}\, H_\mathrm{f}\, P^\flat_{\mathbf{A},N}\,,\, F_\varepsilon^n \,\big]\, \Psi_2 \,\big\rangle\big| \\
&\qquad \leqslant \|\check{H}_\mathrm{f}^{1/2}\, \Psi_1\|\, \big\|\, \check{H}_\mathrm{f}^{-1/2}\, [P^+_{\mathbf{A},N}\,,\, F_\varepsilon^n]\, H_\mathrm{f}^{1/2} F^{1-n}\,\big\|\, \big\|H_\mathrm{f}^{1/2}\, F_\varepsilon^{n-1}\, P^\flat_{\mathbf{A},N}\, \Psi_2\big\| \\
&\qquad + \|H_\mathrm{f}^{1/2}\, P^\sharp_{\mathbf{A},N}\, \Psi_1\|\, \big\|H_\mathrm{f}^{1/2}\, [P^+_{\mathbf{A},N}\,,\, F_\varepsilon^n]\, \check{H}_\mathrm{f}^{-1/2}\, F_\varepsilon^{1-n}\big\|\, \|\check{H}_\mathrm{f}^{1/2}\, F_\varepsilon^{n-1}\, \Psi_2\|\,,
\end{aligned} \tag{6.10}$$

where we can further estimate

$$\begin{aligned}
&\big\|H_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,P^\flat_{\mathbf{A},N}\,\Psi_2\big\| \\
&\qquad\leqslant\;\big\{1+\big\|H_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,P^+_{\mathbf{A},N}\,\check{H}_{\mathrm{f}}^{-1/2}\,F_\varepsilon^{1-n}\big\|\big\}\,\|\check{H}_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,\Psi_2\| \\
&\qquad\leqslant\;\big\{1+\|H_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,P^+_{\mathbf{A}}\,\check{H}_{\mathrm{f}}^{-1/2}\,F_\varepsilon^{1-n}\|^N\big\}\,\|\check{H}_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,\Psi_2\|\,, \qquad (6.11)
\end{aligned}$$

and, of course,

$$\|\check{H}_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,\Psi_2\|\;\leqslant\;\|\check{H}_{\mathrm{f}}^{1/2}\,P^+_{\mathbf{A},N}\,F_\varepsilon^{n-1}\,\Psi_2\|+\|\check{H}_{\mathrm{f}}^{1/2}\,P^\perp_{\mathbf{A},N}\,F_\varepsilon^{n-1}\,\Psi_2\|\,. \qquad (6.12)$$

The operator norms in (6.10) can be estimated by means of (6.7) with $\nu=\pm1/2$, the one in the last line of (6.11) is bounded by some $C(n,d_1,d_{3+n})\in(0,\infty)$ due to (3.21). In a similar fashion we obtain, for all $i,j\in\{1,\dots,N\}$, $i<j$, and $\Psi_1,\Psi_2\in\mathscr{D}_N$,

$$\begin{aligned}
&\big|\big\langle\,\Psi_1\,\big|\,[P_{\mathbf{A}}^{+,(i,j)}\,w_{ij}P_{\mathbf{A}}^{+,(i,j)}\,,\,F_\varepsilon^n]\,\Psi_2\,\big\rangle\big| \\
&\leqslant\big\|\,F_\varepsilon^{1-n}\,w_{ij}^{1/2}\,[F_\varepsilon^n\,,\,P_{\mathbf{A}}^{+,(i,j)}]\,\check{H}_{\mathrm{f}}^{-1/2}\,\big\|\,\|\check{H}_{\mathrm{f}}^{1/2}\,\Psi_1\|\,\big\|F_\varepsilon^{n-1}\,w_{ij}^{1/2}\,P_{\mathbf{A}}^{+,(i,j)}\,\Psi_2\big\| \\
&\quad+\big\|\,w_{ij}^{1/2}\,P_{\mathbf{A}}^{+,(i,j)}\,\Psi_1\big\|\,\big\|\,w_{ij}^{1/2}\,[P_{\mathbf{A}}^{+,(i,j)},\,F_\varepsilon^n]\,F_\varepsilon^{1-n}\,\check{H}_{\mathrm{f}}^{-1/2}\big\|\,\|\check{H}_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,\Psi_2\|. \qquad (6.13)
\end{aligned}$$

Here we can further estimate

$$\begin{aligned}
\big\|\,w_{ij}^{1/2}\,F_\varepsilon^{n-1}\,P_{\mathbf{A}}^{+,(i,j)}\,\Psi_2\big\|\;&\leqslant\;\big\|\,w_{ij}^{1/2}\,P_{\mathbf{A}}^{+,(i,j)}\,F_\varepsilon^{n-1}\,\Psi_2\big\| \\
&\quad+\;\big\|\,w_{ij}^{1/2}\,[F_\varepsilon^{n-1}\,,\,P_{\mathbf{A}}^{+,(i,j)}]\,\check{H}_{\mathrm{f}}^{-1/2}F_\varepsilon^{1-n}\,\big\|\,\|\check{H}_{\mathrm{f}}^{1/2}\,F_\varepsilon^{n-1}\,\Psi_2\|\,. \qquad (6.14)
\end{aligned}$$

Lemma 6.2 below ensures that all operator norms in (6.13) and (6.14) that involve $w_{ij}^{1/2}$ are bounded uniformly in $\varepsilon>0$ by constants depending only on $e,n,d_1$, and $d_{5+n}$. Furthermore, it is now clear how to treat the terms involving $v_i$ or $E_{\mathrm{np}}$. (In order to treat $v_i$ just replace $P_{\mathbf{A}}^{+,(i,j)}$ by $P_{\mathbf{A}}^{+,(i)}$, $w_{ij}$ by $v_i$, and $w_{ij}^{1/2}$ by $|v_i|^{1/2}$ in (6.13) and (6.14).) Combining (6.9)–(6.14) and their analogues for the remaining operators in $W$ we arrive at

$$\begin{aligned}
&\big|\big\langle\,\Psi_1\,\big|\,[W\,,\,F_\varepsilon^n]\,\Psi_2\,\big\rangle\big| \\
&\leqslant C\sum_{\sharp\in\{+,\perp\}}\big\{\big\langle\,\Psi_1\,\big|\,P^\sharp_{\mathbf{A},N}\,H_{\mathrm{f}}\,P^\sharp_{\mathbf{A},N}\,\Psi_1\,\big\rangle+\big\langle\,F_\varepsilon^{n-1}\,\Psi_2\,\big|\,P^\sharp_{\mathbf{A},N}\,H_{\mathrm{f}}\,P^\sharp_{\mathbf{A},N}\,F_\varepsilon^{n-1}\,\Psi_2\,\big\rangle\big\} \\
&+C\sum_{\substack{i,j=1\\ i<j}}^N\big\{\big\langle\,\Psi_1\,\big|\,P_{\mathbf{A}}^{+,(i,j)}w_{ij}P_{\mathbf{A}}^{+,(i,j)}\Psi_1\,\big\rangle \\
&\qquad\qquad+\big\langle\,F_\varepsilon^{n-1}\Psi_2\,\big|\,P_{\mathbf{A}}^{+,(i,j)}w_{ij}P_{\mathbf{A}}^{+,(i,j)}F_\varepsilon^{n-1}\Psi_2\,\big\rangle\big\} \\
&+C\sum_{i=1}^N\big\{\big\langle\,\Psi_1\,\big|\,P_{\mathbf{A}}^{+,(i)}\,|v_i|\,P_{\mathbf{A}}^{+,(i)}\,\Psi_1\,\big\rangle+\big\langle\,F_\varepsilon^{n-1}\,\Psi_2\,\big|\,P_{\mathbf{A}}^{+,(i)}\,|v_i|\,P_{\mathbf{A}}^{+,(i)}\,F_\varepsilon^{n-1}\,\Psi_2\,\big\rangle\big\} \\
&+C\,(1+|E_{\mathrm{np}}|)\,\big\{\|\Psi_1\|^2+\|F_\varepsilon^{n-1}\,\Psi_2\|^2\big\}\,,
\end{aligned}$$

for all $\Psi_1,\Psi_2\in\mathscr{D}_N$ and some $\varepsilon$-independent $C\equiv C(N,n,e,d_1,d_{5+n})\in(0,\infty)$. Employing successively (3.19), which implies $|v_i|\leqslant(\pi e^2|\mathscr{Z}|/2)(|D_{\mathbf{A}}^{(i)}|+\check{H}_{\mathrm{f}})$, after that (3.21), which yields $\|\check{H}_{\rm f}^{1/2}\,P^{+,(i)}_{\mathbf{A}}\,\Psi\|^2 \leqslant C(d_1,d_4)(\|\check{H}_{\rm f}^{1/2}\,P^{+}_{\mathbf{A},N}\,\Psi\|^2 + \|\check{H}_{\rm f}^{1/2}\,P^{\perp}_{\mathbf{A},N}\,\Psi\|^2)$, and finally (6.4) we conclude that Condition (c) is fulfilled with $c_n = C(N,n,\mathscr{Z},\mathscr{R},e,d_{-1},d_1,d_{5+n})(1+|E_{\rm np}|)$. □

Lemma 6.2. *For all $i,j\in\{1,\dots,N\}$, $i<j$, $n\in\mathbb{Z}$, and $\sigma,\tau\geqslant 0$ with $\sigma+\tau\leqslant 1$,*

$$\sup_{\varepsilon>0}\big\|F_\varepsilon^{\sigma-n}\,w_{ij}^{1/2}\,[F_\varepsilon^n\,,\,P^{+,(i,j)}_{\mathbf{A}}]\,\check{H}_{\rm f}^{-1/2}\,F_\varepsilon^{\tau}\,\big\|$$

$$=\sup_{\varepsilon>0}\big\|\,w_{ij}^{1/2}\,F_\varepsilon^{\sigma}\,[P^{+,(i,j)}_{\mathbf{A}}\,,\,F_\varepsilon^{-n}]\,\check{H}_{\rm f}^{-1/2}\,F_\varepsilon^{n+\tau}\,\big\|\;\leqslant\; e\,C(n,d_1,d_{5+n})\;<\;\infty\,.$$

*Proof.* We write

$$w_{ij}^{1/2}\,F_\varepsilon^{\sigma}\,[P^{+,(i)}_{\mathbf{A}}\,P^{+,(j)}_{\mathbf{A}}\,,\,F_\varepsilon^{-n}]\,\check{H}_{\rm f}^{-1/2}F_\varepsilon^{n+\tau}\;=\;Y_1+w_{ij}^{1/2}\,Y_2+Y_3\,,$$

where

$$Y_1 := \{w_{ij}^{1/2}\,F_\varepsilon^{\sigma}\,[P^{+,(i)}_{\mathbf{A}}\,,\,F_\varepsilon^{-n}]\,\check{H}_{\rm f}^{-1/2}\,F_\varepsilon^{n+\tau}\}\{\check{H}_{\rm f}^{1/2}\,F_\varepsilon^{-n-\tau}\,P^{+,(j)}_{\mathbf{A}}\,\check{H}_{\rm f}^{-1/2}\,F_\varepsilon^{n+\tau}\},$$

$$Y_2 := P^{+,(i)}_{\mathbf{A}}\,F_\varepsilon^{\sigma}\,[P^{+,(j)}_{\mathbf{A}}\,,\,F_\varepsilon^{-n}]\,\check{H}_{\rm f}^{-1/2}\,F_\varepsilon^{n+\tau},$$

$$Y_3 := w_{ij}^{1/2}\,[F_\varepsilon^{\sigma}\,,\,P^{+,(i)}_{\mathbf{A}}]\,[P^{+,(j)}_{\mathbf{A}}\,,\,F_\varepsilon^{-n}]\,\check{H}_{\rm f}^{-1/2}\,F_\varepsilon^{n+\tau}.$$

Applying Corollary 3.4 we immediately see that $\|Y_1\|\leqslant e\,C(n,d_1,d_{5+n})$ and that

$$\|Y_3\|\leqslant\big\|\,w_{ij}^{1/2}\,[F_\varepsilon^{\sigma}\,,\,P^{+,(i)}_{\mathbf{A}}]\,F_\varepsilon^{-\sigma}\big\|\,\big\|\,F_\varepsilon^{\sigma}\,[P^{+,(j)}_{\mathbf{A}}\,,\,F_\varepsilon^{-n}]\,F_\varepsilon^{n+\tau}\,\big\|\leqslant e\,C(n,d_1,d_{3+n})$$

uniformly in $\varepsilon>0$. Employing (3.19) (with respect to the variable $\mathbf{x}_j$ for each fixed $\mathbf{x}_i$) and using $[|D^{(j)}_{\mathbf{A}}|^{1/2},P^{+,(i)}_{\mathbf{A}}]=0$, we further get

$$\begin{aligned}&\big\|\,w_{ij}^{1/2}\,Y_2\,\Psi\big\|^2\\&\leqslant\;(\pi e^2/2)\,\|P^{+,(i)}_{\mathbf{A}}\|^2\,\big\|\,|D^{(j)}_{\mathbf{A}}|^{1/2}\,F_\varepsilon^{\sigma}\,[P^{+,(j)}_{\mathbf{A}}\,,\,F_\varepsilon^{-n}]\,F_\varepsilon^{n+\tau}\big\|^2\,\|\check{H}_{\rm f}^{-1/2}\|^2\\&\quad+(\pi e^2/2)\,\big\|\check{H}_{\rm f}^{1/2}\,P^{+,(i)}_{\mathbf{A}}\,\check{H}_{\rm f}^{-1/2}\big\|^2\,\big\|\check{H}_{\rm f}^{1/2}\,F_\varepsilon^{\sigma}\,[P^{+,(i)}_{\mathbf{A}}\,,\,F_\varepsilon^{-n}]\,\check{H}_{\rm f}^{-1/2}\,F_\varepsilon^{n+\tau}\big\|^2.\end{aligned}$$

By Corollary 3.4 all norms on the right hand side are bounded uniformly in $\varepsilon>0$ by constants depending only on $n,d_1$, and $d_{4+n}$. □

## Appendix A. Semi-boundedness of $H^{V_{\rm C}}_{\rm sr}$ and $H^{V_{\rm C}}_{\rm np}$

In this appendix we verify that the semi-relativistic Pauli-Fierz and no-pair operators with Coulomb potential are semi-bounded below for all nuclear charges less than the critical charges without radiation fields. We do not attempt to give good lower bounds on their spectra since this is not the topic addressed in this paper. Our aim here is essentially only to ensure that these operators possess self-adjoint Friedrichs extensions. We recall that the stability of matter of the second kind has been proven for the no-pair operator in [9] under certain restrictions on the fine-structure constant, the ultra-violet cut-off, and the nuclear charges. The stability of matter of the second kind is a much stronger property than mere semi-boundedness. It says that the operator is bounded below by some constant which is proportional to the total number of nuclei and electrons and uniform in the nuclear positions. The restrictions imposed

on the physical parameters in [9] do, however, not allow for all atomic numbers less than $Z_{\mathrm{np}}$.
First, we consider the semi-relativistic Pauli-Fierz operator. The following proposition is a simple generalization of the bound (3.19) proven in [10] to the case of $N \in \mathbb{N}$ electrons and $K \in \mathbb{N}$ nuclei.

Proposition A.1. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1 and let $N, K \in \mathbb{N}$, $e > 0$, $\mathscr{Z} = (Z_1, \ldots, Z_K) \in (0, 2/\pi e^2]^K$, and $\mathscr{R} = \{\mathbf{R}_1, \ldots, \mathbf{R}_K\} \subset \mathbb{R}^3$. Then*

$$\sum_{i=1}^{N} |D_{\mathbf{A}}^{(i)}| + V_{\mathrm{C}} + \delta\, H_{\mathrm{f}} \geqslant -C(\delta, N, \mathscr{Z}, \mathscr{R}, d_1) > -\infty\,, \tag{A.1}$$

*for every $\delta > 0$ in the sense of quadratic forms on $\mathscr{D}_N$.*

*Proof.* In view of (3.19) we only have to explain how to localize the non-local kinetic energy terms. To begin with we recall the following bounds proven in [10, Lemmata 3.5 and 3.6]: For every $\chi \in C^\infty(\mathbb{R}^3_{\mathbf{x}}, [0,1])$,

$$\|\,[\chi, S_{\mathbf{A}}]\,\| \leqslant \|\nabla\chi\|_\infty\,, \qquad \big\|D_{\mathbf{A}}\,\big[\chi\,,\,[\chi, S_{\mathbf{A}}]\,\big]\big\| \leqslant 2\,\|\nabla\chi\|_\infty^2\,. \tag{A.2}$$

Now, let $\mathcal{B}_r(\mathbf{z})$ denote the open ball of radius $r > 0$ centered at $\mathbf{z} \in \mathbb{R}^3$ in $\mathbb{R}^3$. We set $\varrho := \min\{|\mathbf{R}_k - \mathbf{R}_\ell| : k \neq \ell\}/2$ and pick a smooth partition of unity on $\mathbb{R}^3$, $\{\chi_k\}_{k=0}^K$, such that $\chi_k \equiv 1$ on $\mathcal{B}_{\varrho/2}(\mathbf{R}_k)$ and $\mathrm{supp}(\chi_k) \subset \mathcal{B}_\varrho(\mathbf{R}_k)$, for $k = 1, \ldots, K$, and such that $\sum_{k=0}^K \chi_k^2 = 1$. Then we have the following IMS type localization formula,

$$|D_{\mathbf{A}}| = \sum_{k=0}^{K} \Big\{ \chi_k\,|D_{\mathbf{A}}|\,\chi_k + \frac{1}{2}\,\big[\chi_k\,,\,[\chi_k, |D_{\mathbf{A}}|\,]\,\big] \Big\} \tag{A.3}$$

on $\mathscr{D}$, for every $i \in \{1, \ldots, N\}$. A direct calculation shows that

$$\big[\chi_k\,,\,[\chi_k, |D_{\mathbf{A}}|\,]\,\big] = 2\,i\boldsymbol{\alpha}\cdot(\nabla\chi_k)\,[\chi_k, S_{\mathbf{A}}] + D_{\mathbf{A}}\,\big[\chi_k\,,\,[\chi_k, S_{\mathbf{A}}]\,\big] \tag{A.4}$$

on $\mathscr{D}$. By virtue of (3.6) and (A.2) we thus get

$$\big\|\,\big[\chi_k\,,\,[\chi_k, |D_{\mathbf{A}}|\,]\,\big]\,\big\| \leqslant 4\,\|\nabla\chi\|_\infty^2\,, \tag{A.5}$$

for all $k \in \{0, \ldots, K\}$. Since we are able to localize the kinetic energy terms and since, by the choice of the partition of unity, the functions $\mathbb{R}^3 \ni \mathbf{x} \mapsto |\mathbf{x} - \mathbf{R}_k|^{-1}\,\chi_\ell^2(\mathbf{x})$ are bounded, for $k \in \{1, \ldots, K\}$, $\ell \in \{0, \ldots, K\}$, $k \neq \ell$, the bound (A.1) is now an immediate consequence of (3.19) (with $\delta$ replaced by $\delta/N$). (Here we also make use of the fact that the hypotheses on $\mathbf{G}$ are translation invariant.) □

Next, we turn to the no-pair operator discussed in Section 6. The semi-boundedness of the molecular $N$-electron no-pair operator is essentially a consequence of the following inequality [10, Equation (2.14)], valid for all $\omega$ and $\mathbf{G}$ fulfilling Hypothesis 3.1, $\gamma \in (0, 2/(2/\pi + \pi/2))$, and $\delta > 0$,

$$P_{\mathbf{A}}^+\,(D_{\mathbf{A}}^{(i)} - \gamma/|\mathbf{x}| + \delta\,H_{\mathrm{f}})\,P_{\mathbf{A}}^+ \geqslant P_{\mathbf{A}}^+\,(c(\gamma)\,|D_{\mathbf{0}}| - C)\,P_{\mathbf{A}}^+\,, \tag{A.6}$$

in the sense of quadratic forms on $P^+_{\mathbf{A}}\,\mathscr{D}$. Here $C \equiv C(\delta,\gamma,d_{-1},d_0,d_1) \in (0,\infty)$ and $c(\gamma) \in (0,\infty)$ depends only on $\gamma$.

Proposition A.2. *Assume that $\omega$ and $\mathbf{G}$ fulfill Hypothesis 3.1 and let $N, K \in \mathbb{N}$, $e > 0$, $\mathscr{Z} = (Z_1,\dots,Z_K) \in (0, Z_{\mathrm{np}})^K$, and $\mathscr{R} = \{\mathbf{R}_1,\dots,\mathbf{R}_K\} \subset \mathbb{R}^3$, where $Z_{\mathrm{np}}$ is defined in* (6.3). *Then the quadratic form associated with the operator $\widetilde{H}_{\mathrm{np}}$ defined in* (6.2) *is semi-bounded below,*

$$\widetilde{H}_{\mathrm{np}} \,\geqslant\, -C(N,K,\mathscr{Z},\mathscr{R},d_{-1},d_1,d_5) \,>\, -\infty\,,$$

*in the sense of quadratic forms on $\mathscr{D}_N$.*

*Proof.* We again employ the parameter $\varrho > 0$ and the partition of unity introduced in the paragraph succeeding (A.2). Thanks to [10, Proof of Lemma 3.4(ii)] we know that $P^+_{\mathbf{A}}$ maps $\mathcal{D}(D_{\mathbf{0}} \otimes H_{\mathrm{f}}^{\nu})$ into $\mathcal{D}(D_{\mathbf{0}} \otimes H_{\mathrm{f}}^{\nu-1})$, for every $\nu \geqslant 1$. The IMS localization formula thus yields

$$\begin{aligned}
&P^{+,(i)}_{\mathbf{A}}\, v_i\, P^{+,(i)}_{\mathbf{A}} \\
&\quad= \sum_{k=0}^{K} \Big\{ \chi_k^{(i)}\, P^{+,(i)}_{\mathbf{A}}\, v_i\, P^{+,(i)}_{\mathbf{A}}\, \chi_k^{(i)} \,+\, \frac{1}{2}\,\big[\chi_k^{(i)}\,,\,[\chi_k^{(i)}\,,\,P^{+,(i)}_{\mathbf{A}}\, v_i\, P^{+,(i)}_{\mathbf{A}}]\,\big]\Big\}
\end{aligned}$$

on $\mathcal{D}(D_{\mathbf{0}} \otimes H_{\mathrm{f}})$, where a superscript $(i)$ indicates that $\chi_k = \chi_k^{(i)}$ depends on the variable $\mathbf{x}_i$. Using $v_i \leqslant 0$, we observe that

$$\begin{aligned}
&\big[\chi_k^{(i)}\,,\,[\chi_k^{(i)}\,,\,P^{+,(i)}_{\mathbf{A}}\, v_i\, P^{+,(i)}_{\mathbf{A}}]\,\big] \\
&\qquad= -2\,[\chi_k^{(i)}\,,\,P^{+,(i)}_{\mathbf{A}}]\, v_i\,[P^{+,(i)}_{\mathbf{A}}\,,\,\chi_k^{(i)}] \\
&\qquad\quad + 2\,\mathrm{Re}\,\big\{\, P^{+,(i)}_{\mathbf{A}}\, v_i\, \big[\chi_k^{(i)}\,,\,[\chi_k^{(i)}\,,\,P^{+,(i)}_{\mathbf{A}}]\,\big]\,\big\} \\
&\qquad\geqslant\; 2\,\mathrm{Re}\,\big\{\, P^{+,(i)}_{\mathbf{A}}\, v_i\, \big[\chi_k^{(i)}\,,\,[\chi_k^{(i)}\,,\,P^{+,(i)}_{\mathbf{A}}]\,\big]\,\big\}\,.
\end{aligned} \tag{A.7}$$

We recall the following estimate proven in [10, Lemma 3.6], for every $\chi \in C^\infty(\mathbb{R}^3_{\mathbf{x}},[0,1])$,

$$\big\|\tfrac{1}{|\mathbf{x}|}\,\big[\chi\,,\,[\chi\,,\,P^+_{\mathbf{A}}]\,\big]\,\check{H}_{\mathrm{f}}^{-1/2}\,\big\| \;\leqslant\; 8^{3/2}\,\|\nabla\chi\|_\infty^2\,,$$

where $\check{H}_{\mathrm{f}} = H_{\mathrm{f}} + E$ with $E \geqslant 1 \vee (4d_1)^2$. Together with (A.7) it implies

$$\begin{aligned}
&\big\langle\, \Psi \,\big|\, \big[\chi_k^{(i)}\,,[\chi_k^{(i)}\,,\,P^{+,(i)}_{\mathbf{A}}\, v_i\, P^{+,(i)}_{\mathbf{A}}]\,\big]\,\Psi\,\big\rangle \\
&\qquad\qquad \geqslant\; -\delta\,\langle\, \Psi \,|\, \check{H}_{\mathrm{f}}\,\Psi\,\rangle - (8^3\,\|\nabla\chi_k\|_\infty^4/\delta)\,\|\Psi\|^2,
\end{aligned}$$

for all $k \in \{0,\dots,K\}$, $i \in \{1,\dots,N\}$, $\delta > 0$, and $\Psi \in \mathcal{D}(D_{\mathbf{0}} \otimes H_{\mathrm{f}})$. Next, we pick cut-off functions, $\zeta_1,\dots,\zeta_K \in C_0^\infty(\mathbb{R}^3_{\mathbf{x}},[0,1])$, such that $\zeta_k = 1$ in a neighborhood of $\mathbf{R}_k$ and $\mathrm{supp}(\zeta_k) \subset \mathcal{B}_{\varrho/4}(\mathbf{R}_k)$, for $k \in \{1,\dots,K\}$. By construction, $\mathrm{supp}(\zeta_k) \cap \mathrm{supp}(\chi_\ell) = \varnothing$, for all $k \in \{1,\dots,K\}$ and $\ell \in \{0,\dots,K\}$ with $k \neq \ell$. Denoting $\overline{\zeta}_k := 1 - \zeta_k$ and using the superscript $(i)$ to indicate that $\zeta_k = \zeta_k^{(i)}$

is a function of the variable $\mathbf{x}_i$, we obtain

$$\big\langle\, \Psi \,\big|\, \chi_k^{(i)}\, P_{\mathbf{A}}^{+,(i)}\, v_i\, P_{\mathbf{A}}^{+,(i)}\, \chi_k^{(i)}\, \Psi \,\big\rangle$$

$$= -\Big\langle\, \Psi \,\Big|\, \chi_k^{(i)}\, P_{\mathbf{A}}^{+,(i)}\, \frac{e^2\, Z_k}{|\mathbf{x}_i - \mathbf{R}_k|}\, P_{\mathbf{A}}^{+,(i)}\, \chi_k^{(i)}\, \Psi \,\Big\rangle$$

$$- \sum_{\substack{\ell=1\\ \ell\neq k}}^{K} \Big\langle\, \Psi \,\Big|\, \chi_k^{(i)}\, P_{\mathbf{A}}^{+,(i)}\, \frac{e^2\, Z_\ell\, \zeta_\ell^{(i)}}{|\mathbf{x}_i - \mathbf{R}_\ell|}\, P_{\mathbf{A}}^{+,(i)}\, \chi_k^{(i)}\, \Psi \,\Big\rangle \tag{A.8}$$

$$- \sum_{\substack{\ell=1\\ \ell\neq k}}^{K} \Big\langle\, \Psi \,\Big|\, \chi_k^{(i)}\, P_{\mathbf{A}}^{+,(i)}\, \frac{e^2\, Z_\ell\, \overline{\zeta}_\ell^{(i)}}{|\mathbf{x}_i - \mathbf{R}_\ell|}\, P_{\mathbf{A}}^{+,(i)}\, \chi_k^{(i)}\, \Psi \,\Big\rangle, \tag{A.9}$$

for all $\Psi \in \mathcal{D}(D_{\mathbf{0}} \otimes H_{\mathrm{f}})$. The operators appearing in the scalar products in (A.9) are bounded by definition of $\overline{\zeta}_\ell$. Their norms depend only on $\mathscr{R}$ since $e^2\, Z_\ell < 1$. Furthermore, by virtue of Lemma A.3 below the term in (A.8) is bounded from below by $-\delta\, \langle\, \Psi \,|\, H_{\mathrm{f}}\, \Psi \,\rangle - C_\delta\, \|\Psi\|^2$, for all $\delta > 0$ and some $C_\delta \equiv C_\delta(\mathscr{R}, d_1, d_4) \in (0, \infty)$; see (A.11).
Taking all the previous remarks into account, using (A.3)–(A.5), $w_{ij} \geqslant 0$, $|D_{\mathbf{A}}^{(i)}| \geqslant P_{\mathbf{A}}^{+,(i)}\, D_{\mathbf{A}}^{(i)}\, P_{\mathbf{A}}^{+,(i)}$, and writing

$$H_{\mathrm{f}} \,=\, \frac{1}{N} \sum_{i=1}^{N} \sum_{k=0}^{K} \chi_k^{(i)}\, (P_{\mathbf{A}}^{+,(i)} + P_{\mathbf{A}}^{-,(i)})\, H_{\mathrm{f}}\, \chi_k^{(i)}\,,$$

we deduce that

$$\begin{aligned}
&\widetilde{H}_{\mathrm{np}}\\
&\geqslant (1-3\delta)\, P_{\mathbf{A},N}^{+}\, H_{\mathrm{f}}\, P_{\mathbf{A},N}^{+} \,+\, (1-3\delta)\, P_{\mathbf{A},N}^{\perp}\, H_{\mathrm{f}}\, P_{\mathbf{A},N}^{\perp}\\
&+ \sum_{\sharp\in\{+,\perp\}} \sum_{k=0}^{K} P_{\mathbf{A},N}^{\sharp} \Big\{ \sum_{i=1}^{N} \chi_k^{(i)}\, P_{\mathbf{A}}^{+,(i)} \Big( D_{\mathbf{A}}^{(i)} - \frac{e^2\, Z_k}{|\mathbf{x}_i - \mathbf{R}_k|} + \frac{\delta}{N}\, H_{\mathrm{f}} \Big) P_{\mathbf{A}}^{+,(i)}\, \chi_k^{(i)}\\
&+ \frac{\delta}{N} \sum_{i=1}^{N} \Big( \chi_k^{(i)}\, P_{\mathbf{A}}^{-,(i)}\, H_{\mathrm{f}}\, P_{\mathbf{A}}^{-,(i)}\, \chi_k^{(i)} + \sum_{\flat=\pm} \chi_k^{(i)} P_{\mathbf{A}}^{\flat,(i)}\, [P_{\mathbf{A}}^{\flat,(i)}, H_{\mathrm{f}}]\, \chi_k^{(i)} \Big) \Big\}\, P_{\mathbf{A},N}^{\sharp}\\
&-\, \mathrm{const}(N, \mathscr{R}, d_1, d_4)
\end{aligned}$$

on $\mathscr{D}_N$, for every $\delta > 0$. Thanks to Corollary 3.4 (with $\varepsilon = 0$) we know that $[P_{\mathbf{A}}^{\flat,(i)}, H_{\mathrm{f}}]\, \check{H}_{\mathrm{f}}^{-1/2}$ extends to an element of $\mathscr{L}(\mathscr{H}_N)$ whose norm is bounded by some constant depending only on $d_1$ and $d_5$, whence

$$\begin{aligned}
&\frac{\delta}{N} \sum_{i=1}^{N} \sum_{k=0}^{K} \big\langle\, \chi_k^{(i)}\, P_{\mathbf{A},N}^{\sharp}\, \Psi \,\big|\, P_{\mathbf{A}}^{\flat,(i)}\, [P_{\mathbf{A}}^{\flat,(i)}, H_{\mathrm{f}}]\, \chi_k^{(i)}\, P_{\mathbf{A},N}^{\sharp}\, \Psi \,\big\rangle\\
&\qquad\qquad \geqslant\, -(\delta/2)\, \|\check{H}_{\mathrm{f}}^{1/2}\, P_{\mathbf{A},N}^{\sharp}\, \Psi\|^2 - (\delta/2)\, \big\|[P_{\mathbf{A}}^{\flat,(i)}, H_{\mathrm{f}}]\check{H}_{\mathrm{f}}^{-1/2}\big\|^2\, \|\Psi\|^2,
\end{aligned}$$

for every $\Psi \in \mathscr{D}_N$, $\sharp \in \{+,\perp\}$, and $\flat = \pm$. For a sufficiently small choice of $\delta > 0$, the assertion of the proposition now follows from the semi-boundedness of $P^{+,(i)}_{\mathbf{A}}\,(D^{(i)}_{\mathbf{A}} - e^2\,Z_k/|\mathbf{x}_i - \mathbf{R}_k| + (\delta/N)\,H_{\mathrm{f}})\,P^{+,(i)}_{\mathbf{A}}$ ensured by (A.6) and the condition $Z_k < Z_{\mathrm{np}}$. □

Lemma A.3. *Let* $\zeta \in C_0^\infty(\mathbb{R}^3,[0,1])$, $\chi \in C^\infty(\mathbb{R}^3,[0,1])$, *such that* $0 \in \mathrm{supp}(\zeta)$ *and* $\mathrm{supp}(\zeta)\cap\mathrm{supp}(\chi) = \varnothing$. *Set* $\check{H}_{\mathrm{f}} := H_{\mathrm{f}} + E$, *where* $E \geqslant k_1 \vee d_1^2$. *Then*

$$\big\|\, D_{\mathbf{A}}\, H_{\mathrm{f}}^{1/2}\,\zeta\, P^+_{\mathbf{A}}\,\chi\,\check{H}_{\mathrm{f}}^{-1/2}\,\big\| \,\leqslant\, C(\zeta,\chi,d_1,d_4)\,, \tag{A.10}$$

$$\big\|\, \tfrac{\zeta}{|\mathbf{x}|}\, P^+_{\mathbf{A}}\,\chi\,\check{H}_{\mathrm{f}}^{-1/2}\,\big\| \,\leqslant\, C'(\zeta,\chi,d_1,d_4)\,. \tag{A.11}$$

*Proof.* We pick some $\widetilde{\chi} \in C^\infty(\mathbb{R}^3,[0,1])$ such that $\mathrm{supp}(\widetilde{\chi})\cap\mathrm{supp}(\zeta) = \varnothing$ and $\widetilde{\chi} \equiv 1$ on $\mathrm{supp}(\nabla\chi)$. Using $\zeta\,\chi = 0 = \zeta\,\widetilde{\chi}$ we infer that, for all $\varphi,\psi \in \mathscr{D}$,

$$\begin{aligned}
&\big|\big\langle\, D_{\mathbf{A}}\,\varphi\,\big|\, H_{\mathrm{f}}^{1/2}\,\zeta\,P^+_{\mathbf{A}}\,\chi\,\check{H}_{\mathrm{f}}^{-1/2}\,\psi\,\big\rangle\big| \,=\, \big|\big\langle\, D_{\mathbf{A}}\,\varphi\,\big|\, H_{\mathrm{f}}^{1/2}\,\zeta\,[P^+_{\mathbf{A}}\,,\,\chi]\,\check{H}_{\mathrm{f}}^{-1/2}\,\psi\,\big\rangle\big|\\
&\quad\leqslant \int_{\mathbb{R}} \Big|\Big\langle\, D_{\mathbf{A}}\,\varphi\,\Big|\, H_{\mathrm{f}}^{1/2}\,\zeta\,[R_{\mathbf{A}}(iy)\,,\,\widetilde{\chi}]\,i\boldsymbol{\alpha}\cdot\nabla\chi\,R_{\mathbf{A}}(iy)\,\check{H}_{\mathrm{f}}^{-1/2}\,\psi\,\Big\rangle\Big|\frac{dy}{2\pi}\\
&\quad= \int_{\mathbb{R}} \Big|\Big\langle\, D_{\mathbf{A}}\,\varphi\,\Big|\, H_{\mathrm{f}}^{1/2}\,\zeta\,R_{\mathbf{A}}(iy)\,i\boldsymbol{\alpha}\cdot\nabla\widetilde{\chi}\,R_{\mathbf{A}}(iy)\,i\boldsymbol{\alpha}\cdot\nabla\chi\,R_{\mathbf{A}}(iy)\,\check{H}_{\mathrm{f}}^{-1/2}\,\psi\,\Big\rangle\Big|\frac{dy}{2\pi}\\
&\quad= \int_{\mathbb{R}} \Big|\Big\langle\, \zeta\,D_{\mathbf{A}}\,\varphi\,\Big|\, R_{\mathbf{A}}(iy)\,\Upsilon_{0,1/2}(iy)\,i\boldsymbol{\alpha}\cdot\nabla\widetilde{\chi}\,R_{\mathbf{A}}(iy)\,\Upsilon_{0,1/2}(iy)\,\times\\
&\qquad\qquad\qquad\times\, i\boldsymbol{\alpha}\cdot\nabla\chi\,R_{\mathbf{A}}(iy)\,\Upsilon_{0,1/2}(iy)\,\psi\,\Big\rangle\Big|\frac{dy}{2\pi}\,.
\end{aligned}$$

In the last step we repeatedly applied (3.16). Commuting $\zeta$ and $D_{\mathbf{A}}$ and using $\|D_{\mathbf{A}}\,R_{\mathbf{A}}(iy)\| \leqslant 1$, $\|R_{\mathbf{A}}(iy)\|^2 \leqslant (1+y^2)^{-1}$, and the fact that $\|\Upsilon_{0,1/2}(iy)\|$ is uniformly bounded in $y \in \mathbb{R}$, we readily deduce that

$$\big|\big\langle\, D_{\mathbf{A}}\,\varphi\,\big|\, H_{\mathrm{f}}^{1/2}\,\zeta\,P^+_{\mathbf{A}}\,\chi\,\check{H}_{\mathrm{f}}^{-1/2}\,\psi\,\big\rangle\big| \,\leqslant\, C(\zeta,\chi,\widetilde{\chi},d_1,d_4)\,\|\varphi\|\,\|\psi\|\,,$$

which implies (A.10). The bound (A.11) follows from (A.10) and the inequality

$$\|\,|\mathbf{x}|^{-1}\,\varphi\,\|^2 \,\leqslant\, 4\,\|\,D_{\mathbf{A}}\,\varphi\,\|^2 + 4\,\|\check{H}_{\mathrm{f}}^{1/2}\,\varphi\|\,,\qquad \varphi \in \mathcal{D}(D_{\mathbf{0}}\otimes H_{\mathrm{f}}^{1/2})\,,$$

which is a simple consequence of standard arguments (see, e.g., [10, Equation (4.7)]). □

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