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Chow–Künneth Decomposition for Some Moduli Spaces[1]

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Abstract. In this paper we investigate Murre's conjecture on the Chow–Künneth decomposition for universal families of smooth curves over spaces which dominate the moduli space $\mathcal{M}_g$, in genus at most 8 and show existence of a Chow–Künneth decomposition. This is done in the setting of equivariant cohomology and equivariant Chow groups to get equivariant Chow–Künneth decompositions.

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Contents

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1. Introduction

Suppose X is a nonsingular projective variety defined over the complex numbers. We consider the rational Chow group $CH^i(X)_{\mathbb{Q}} = CH^i(X) \otimes \mathbb{Q}$ of algebraic cycles of codimension i on X. The conjectures of S. Bloch and A. Beilinson predict a finite descending filtration $\{F^jCH^i(X)_{\mathbb{Q}}\}$ on $CH^i(X)_{\mathbb{Q}}$ and satisfying certain compatibility conditions. A candidate for such a filtration has been proposed by J. Murre and he has made the following conjecture [Mu2],
Murre's conjecture: The motive (X, Δ) of X has a Chow-Künneth decomposition:

$$\Delta = \sum_{i=0}^{2d} \pi_i \in CH^d(X \times X) \otimes \mathbb{Q}$$

such that π_i are orthogonal projectors, lifting the Künneth projectors in $H^{2d-i}(X) \otimes H^i(X)$. Furthermore, these algebraic projectors act trivially on the rational Chow groups in a certain range.
These projectors give a candidate for a filtration of the rational Chow groups, see §2.1.
This conjecture is known to be true for curves, surfaces and a product of a curve and surface [Mu1], [Mu3]. A variety X is known to have a Chow–Künneth decomposition if X is an abelian variety/scheme [Sh],[De-Mu], a uniruled threefold [dA-Mü1], universal families over modular varieties [Go-Mu], [GHM2] and the universal family over one Picard modular surface [MMWYK], where a partial set of projectors are found. Finite group quotients (maybe singular) of an abelian variety also satisfy the above conjecture [Ak-Jo]. Furthermore, for some varieties with a nef tangent bundle, Murre's conjecture is proved in [Iy]. A criterion for existence of such a decomposition is also given in [Sa]. Some other examples are also listed in [Gu-Pe].
Gordon-Murre-Hanamura [GHM2], [Go-Mu] obtained Chow–Künneth projectors for universal families over modular varieties. Hence it is natural to ask if the universal families over the moduli space of curves of higher genus also admit a Chow–Künneth decomposition. In this paper, we investigate the existence of Chow–Künneth decomposition for families of smooth curves over spaces which closely approximate the moduli spaces of curves $\mathcal{M}_g$ of genus at most 8, see §5.
In this example, we take into account the non-trivial action of a linear algebraic group G acting on the spaces. This gives rise to the equivariant cohomology and equivariant Chow groups, which were introduced and studied by Borel, Totaro, Edidin-Graham [Bo], [To], [Ed-Gr]. Hence it seems natural to formulate Murre's conjecture with respect to the cycle class maps between the rational equivariant Chow groups and the rational equivariant cohomology, see §4.5. Since in concrete examples, good quotients of non-compact varieties exist, it became necessary to extend Murre's conjecture for non-compact smooth varieties, by taking only the bottom weight cohomology $W_iH^i(X, \mathbb{Q})$ (see [D]), into consideration. This is weaker than the formulation done in [BE]. For our purpose though, it suffices to look at this weaker formulation. We then

construct a category of equivariant Chow motives, fixing an algebraic group G (see [dB-Az], [Ak-Jo], for a category of motives of quotient varieties, under a finite group action).
With this formalism, we show (see §5.2);

Theorem 1.1. *The equivariant Chow motive of a universal family of smooth curves $\mathcal{X} \to U$ over spaces U which dominate the moduli space of curves $\mathcal{M}_g$, for $g \leq 8$, admits an equivariant Chow–Künneth decomposition, for a suitable linear algebraic group G acting non-trivially on $\mathcal{X}$.*

Whenever smooth good quotients exist under the action of G, then the equivariant Chow-Künneth projectors actually correspond to the absolute Chow–Künneth projectors for the quotient varieties. In this way, we get orthogonal projectors for universal families over spaces which closely approximate the moduli spaces $\mathcal{M}_g$, when g is at most 8.
One would like to try to prove a Chow–Künneth decomposition for $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$ (which parametrizes curves with marked points) and we consider our work a step forward. However since we only work on an open set U one has to refine projectors after taking closures a bit in a way we don't yet know.
Other examples that admit a Chow–Künneth decomposition are Fano varieties of r-dimensional planes contained in a general complete intersection in a projective space, see Corollary 5.3.
The proofs involve classification of curves in genus at most 8 by Mukai [Muk],[Muk2] with respect to embeddings as complete intersections in homogeneous spaces. This allows us to use Lefschetz theorem and construct orthogonal projectors.
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2. Preliminaries

The category of nonsingular projective varieties over $\mathbb{C}$ will be denoted by $\mathcal{V}$. Let $CH^i(X)_{\mathbb{Q}} = CH^i(X) \otimes \mathbb{Q}$ denote the rational Chow group of codimension i algebraic cycles modulo rational equivalence.
Suppose $X, Y \in Ob(\mathcal{V})$ and $X = \cup X_i$ be a decomposition into connected components X_i and $d_i = \dim X_i$. Then $\mathrm{Corr}^r(X,Y) = \oplus_i CH^{d_i+r}(X_i \times Y)_{\mathbb{Q}}$ is called a space of correspondences of degree r from X to Y.
A category $\mathcal{M}$ of Chow motives is constructed in [Mu2]. Suppose X is a nonsingular projective variety over $\mathbb{C}$ of dimension d. Let $\Delta \subset X \times X$ be the diagonal. Consider the Künneth decomposition of the class of Δ in the Betti Cohomology:

$$[\Delta] = \oplus_{i=0}^{2d} \pi_i^{hom}$$

where $\pi_i^{hom} \in H^{2d-i}(X,\mathbb{Q}) \otimes H^i(X,\mathbb{Q})$.

Definition 2.1. *The motive of X is said to have Künneth decomposition if each of the classes π_i^{hom} is algebraic, i.e., π_i^{hom} is the image of an algebraic* *cycle π_i under the cycle class map from the rational Chow groups to the Betti cohomology.*

Definition 2.2. *The motive of X is said to have a Chow–Künneth decomposition if each of the classes π_i^{hom} is algebraic and they are orthogonal projectors, i.e., $\pi_i \circ \pi_j = \delta_{i,j}\pi_i$.*

Lemma 2.3. *If X and Y have a Chow–Künneth decomposition then $X \times Y$ also has a Chow–Künneth decomposition.*

Proof. If π_i^X and π_j^Y are the Chow–Künneth components for $h(X)$ and $h(Y)$ respectively then

$$\pi_i^{X\times Y} = \sum_{p+q=i} \pi_p^X \times \pi_q^Y \in CH^*(X \times Y \times X \times Y)_{\mathbb{Q}}$$

are the Chow–Künneth components for $X \times Y$. Here the product $\pi_p^X \times \pi_q^Y$ is taken after identifying $X \times Y \times X \times Y \simeq X \times X \times Y \times Y$. $\square$

2.1. Murre's conjectures. J. Murre [Mu2], [Mu3] has made the following conjectures for any smooth projective variety X.
(A) The motive $h(X) := (X, \Delta_X)$ of X has a Chow-Künneth decomposition:

$$\Delta_X = \sum_{i=0}^{2n} \pi_i \in CH^n(X \times X) \otimes \mathbb{Q}$$

such that π_i are orthogonal projectors.
(B) The correspondences $\pi_0, \pi_1, ..., \pi_{j-1}, \pi_{2j+1}, ..., \pi_{2n}$ act as zero on $CH^j(X) \otimes \mathbb{Q}$.
(C) Suppose

$$F^r CH^j(X) \otimes \mathbb{Q} = \mathrm{Ker}\pi_{2j} \cap \mathrm{Ker}\pi_{\mathrm{2j-1}} \cap ... \cap \mathrm{Ker}\pi_{\mathrm{2j-r+1}}.$$

Then the filtration $F^\bullet$ of $CH^j(X) \otimes \mathbb{Q}$ is independent of the choice of the projectors π_i.
(D) Further, $F^1 CH^i(X) \otimes \mathbb{Q} = (CH^i(X) \otimes \mathbb{Q})_{hom}$, the cycles which are homologous to zero.

In §4, we will extend (A) in the setting of equivariant Chow groups.

3. Equivariant Chow groups and equivariant Chow motives

In this section, we recall some preliminary facts on the equivariant groups to formulate Murre's conjectures for a smooth variety X of dimension d, which is equipped with an action by a linear reductive algebraic group G. The equivariant groups and their properties that we recall below were defined by Borel, Totaro, Edidin-Graham, Fulton [Bo],[To],[Ed-Gr], [Fu2].

3.1. Equivariant cohomology $H^i_G(X,\mathbb{Z})$ of X. Suppose X is a variety with an action on the left by an algebraic group G. Borel defined the equivariant cohomology $H^*_G(X)$ as follows. There is a contractible space EG on which G acts freely (on the right) with quotient $BG := EG/G$. Then form the space

$$EG \times_G X := EG \times X/(e.g, x) \sim (e, g.x).$$

In other words, $EG \times_G X$ represents the (topological) quotient stack $[X/G]$.

Definition 3.1. *The equivariant cohomology of X with respect to G is the ordinary singular cohomology of $EG \times_G X$:*

$$H^i_G(X) = H^i(EG \times_G X).$$

For the special case when X is a point, we have

$$H^i_G(point) = H^i(BG)$$

For any X, the map $X \to point$ induces a pullback map $H^i(BG) \to H^i_G(X)$. Hence the equivariant cohomology of X has the structure of a $H^i(BG)$-algebra, at least when $H^i(BG) = 0$ for odd i.

3.2. Equivariant Chow groups $CH^i_G(X)$ of X. [Ed-Gr]
As in the previous subsection, let X be a smooth variety of dimension n, equipped with a left G-action. Here G is an affine algebraic group of dimension g. Choose an l-dimensional representation V of G such that V has an open subset U on which G acts freely and whose complement has codimension more than $n-i$. The diagonal action on $X \times U$ is also free, so there is a quotient in the category of algebraic spaces. Denote this quotient by $X_G := (X \times U)/G$.

Definition 3.2. *The i-th equivariant Chow group $CH^G_i(X)$ is the usual Chow group $CH_{i+l-g}(X_G)$. The codimension i equivariant Chow group $CH^i_G(X)$ is the usual codimension i Chow group $CH^i(X_G)$.*

Note that if X has pure dimension n then

$$\begin{aligned} CH^i_G(X) &= CH^i(X_G) \\ &= CH_{n+l-g-i}(X_G) \\ &= CH^G_{n-i}(X). \end{aligned}$$

Proposition 3.3. *The equivariant Chow group $CH^G_i(X)$ is independent of the representation V, as long as $V-U$ has codimension more than $n-i$.*

Proof. See [Ed-Gr, Definition-Proposition 1]. □

If $Y \subset X$ is an m-dimensional subvariety which is invariant under the G-action, and compatible with the G-action on X, then it has a G-equivariant fundamental class $[Y]_G \in CH^G_m(X)$. Indeed, we can consider the product $(Y \times U) \subset X \times U$, where U is as above and the corresponding quotient $(Y \times U)/G$ canonically embeds into X_G. The fundamental class of $(Y \times U)/G$ defines the class $[Y]_G \in CH^G_m(X)$. More generally, if V is an l-dimensional representation of G and $S \subset X \times V$ is an $m+l$-dimensional subvariety which is invariant under the G-action, then the quotient $(S \cap (X \times U))/G \subset (X \times U)/G$ defines the G-equivariant fundamental class $[S]_G \in CH^G_m(X)$ of S.

Proposition 3.4. *If $\alpha \in CH^G_m(X)$ then there exists a representation V such that $\alpha = \sum a_i[S_i]_G$, for some G-invariant subvarieties S_i of $X \times V$.*

Proof. See [Ed-Gr, Proposition 1]. □

3.3. Functoriality properties. Suppose $f : X \to Y$ is a G-equivariant morphism. Let $\mathcal{S}$ be one of the following properties of schemes or algebraic spaces: proper, flat, smooth, regular embedding or l.c.i.

Proposition 3.5. *If $f : X \to Y$ has property $\mathcal{S}$, then the induced map $f_G : X_G \to Y_G$ also has property $\mathcal{S}$.*

Proof. See [Ed-Gr, Proposition 2]. □

Proposition 3.6. *Equivariant Chow groups have the same functoriality as ordinary Chow groups for equivariant morphisms with property $\mathcal{S}$.*

Proof. See [Ed-Gr, Proposition 3]. □

If X and Y have G-actions then there are exterior products

$$CH^G_i(X) \otimes CH^G_j(Y) \to CH^G_{i+j}(X \times Y).$$

In particular, if X is smooth then there is an intersection product on the equivariant Chow groups which makes $\oplus_j CH^G_j(X)$ into a graded ring.

3.4. Cycle class maps. [Ed-Gr, §2.8]
Suppose X is a complex algebraic variety and G is a complex algebraic group. The equivariant Borel-Moore homology $H^G_{BM,i}(X)$ is the Borel-Moore homology $H_{BM,i}(X_G)$, for $X_G = X \times_G U$. This is independent of the representation as long as $V - U$ has sufficiently large codimension. This gives a cycle class map,

$$cl_i : CH^G_i(X) \to H^G_{BM,2i}(X, \mathbb{Z})$$

compatible with usual operations on equivariant Chow groups. Suppose X is smooth of dimension d then X_G is also smooth. In this case the Borel-Moore cohomology $H^G_{BM,2i}(X, \mathbb{Z})$ is dual to $H^{2d-i}(X_G) = H^{2d-i}(X \times_G U)$.
This gives the cycle class maps

$$cl^i : CH^i_G(X) \to H^{2i}_G(X, \mathbb{Z}). \tag{1}$$

There are also maps from the equivariant groups to the usual groups:

$$H^i_G(X, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \tag{2}$$

and

$$CH^i_G(X) \to CH^i(X). \tag{3}$$

3.5. Weight filtration $W_{\cdot}$ on $H^i_G(X,\mathbb{Z})$. In this paper, we assign only the bottom weight W_i of the equivariant cohomology in the simplest situation. Consider a smooth variety X equipped with a left G action as above.
We can define
$$W_iH^i_G(X,\mathbb{Q}) := W_iH^i((X\times U)/G,\mathbb{Q}),$$
for $U\subset V$ an open subset with a free G-action, where codim $V-U$ is at least $n-i$.

Lemma 3.7. *The group $W_iH^i_G(X,\mathbb{Q})$ is independent of the choice of the G-representation V as long as codim $V-U$ is at least $n-i$.*

Proof. The proof of independence of V in the case of equivariant Chow groups [Ed-Gr, Definition-Proposition 1] applies directly in the case of the bottom weight equivariant cohomology. □

3.6. Equivariant Chow motives and the category of equivariant Chow motives. When G is a finite group then a category of Chow motives for (maybe singular) quotients of varieties under the G-action was constructed in [dB-Az], [Ak-Jo]. More generally, we consider the following situation, taking into account the equivariant cohomology and the equivariant rational Chow groups, which does not seem to have been considered before.
Fix an affine complex algebraic group G. Let $\mathcal{V}_G$ be the category whose objects are complex smooth projective varieties with a G-action and the morphisms are G-equivariant morphisms.
For any $X,Y,Z\in Ob(\mathcal{V}_G)$, consider the projections
$$X\times Y\times Z\xrightarrow{p_{XY}} X\times Y,$$
$$X\times Y\times Z\xrightarrow{p_{YZ}} Y\times Z,$$
$$X\times Y\times Z\xrightarrow{p_{XZ}} X\times Z.$$
which are G-equivariant.
Let d be the dimension of X. The group of correspondences from X to Y of degree r is defined as
$$\mathrm{Corr}^r_G(X\times Y) := CH^{r+d}_G(X\times Y).$$
Every G-equivariant morphism $X\to Y$ defines an element in $\mathrm{Corr}^0_G(X\times Y)$, by taking the graph cycle.
For any $f\in \mathrm{Corr}^r_G(X,Y)$ and $g\in\mathrm{Corr}^e_G(Y,Z)$ define their composition
$$g\circ f\in \mathrm{Corr}^{r+e}_G(X,Z)$$
by the prescription
$$g\circ f = p_{XZ*}(p^*_{XY}(f).p^*_{YZ}(g)).$$
This gives a linear action of correspondences on the equivariant Chow groups
$$\mathrm{Corr}^r_G(X,Y)\times CH^s_G(X)_{\mathbb{Q}}\longrightarrow CH^{r+s}_G(Y)_Q$$
$$(\gamma,\alpha)\mapsto p_{Y*}(p^*_X\alpha.\gamma)$$
for the projections $p_X : X\times Y\longrightarrow X$, $p_Y: X\times Y\longrightarrow Y$.

The category of pure equivariant G-motives with rational coefficients is denoted by $\mathcal{M}_G^+$. The objects of $\mathcal{M}_G^+$ are triples $(X,p,m)_G$, for $X \in Ob(\mathcal{V}_G)$, $p \in \mathrm{Corr}_G^0(X,X)$ is a projector, i.e., $p\circ p = p$ and $m \in \mathbb{Z}$. The morphisms between the objects $(X,p,m)_G, (Y,q,n)_G$ in $\mathcal{M}_G^+$ are given by the correspondences $f \in \mathrm{Corr}_G^{n-m}(X,Y)$ such that $f\circ p = q\circ f = f$. The composition of the morphisms is the composition of correspondences. This category is pseudoabelian and $\mathbb{Q}$-linear [Mu2]. Furthermore, it is a tensor category defined by

$$(X,p,m)_G \otimes (Y,q,n)_G = (X\times Y, p\otimes q, m+n)_G.$$

The object $(\mathrm{Spec}\,\mathbb{C}, id, 0)_G$ is the unit object and the Lefschetz motive $\mathbb{L}$ is the object $(\mathrm{Spec}\,\mathbb{C}, id, -1)_G$. Here $\mathrm{Spec}\,\mathbb{C}$ is taken with a trivial G-action. The Tate twist of a G-motive M is $M(r) := M\otimes \mathbb{L}^{\otimes -r} = (X,p,m+r)_G$.

Definition 3.8. *The theory of equivariant Chow motives (*[Sc]*) provides a functor*

$$h : \mathcal{V}_G \longrightarrow \mathcal{M}_G^+.$$

For each $X \in Ob(\mathcal{V}_G)$ the object $h(X) = (X,\Delta,0)_G$ is called the equivariant Chow motive of X. Here Δ is the class of the diagonal in $CH^*(X\times X)_{\mathbb{Q}}$, which is G-invariant for the diagonal action on $X\times X$ and hence lies in $Corr_G^0(X,X) = CH_G^*(X\times X)_{\mathbb{Q}}$.

4. Murre's conjectures for the equivariant Chow motives

Suppose X is a complex smooth variety of dimension d, equipped with a G-action. Consider the product variety $X\times X$ together with the diagonal action of the group G.
The cycle class map

$$cl^d : CH^d(X\times X)_{\mathbb{Q}} \to H^{2d}(X\times X,\mathbb{Q}). \tag{4}$$

actually maps to the weight $2d$ piece $W_{2d}H^{2d}(X\times X,\mathbb{Q})$ of the ordinary cohomology group.
Applying this to the spaces $X\times U$, for open subset $U\subset V$ as in §3.2, (4) holds for the equivariant groups as well and there are cycle class maps:

$$cl^d : CH_G^d(X\times X)_{\mathbb{Q}} \to W_{2d}H_G^{2d}(X\times X,\mathbb{Q}). \tag{5}$$

Lemma 4.1. *The image of the diagonal cycle $[\Delta_X]$ under the cycle class map cl^d lies in the subspace*

$$\bigoplus_i W_{2d-i}H_G^{2d-i}(X)\otimes W_iH_G^i(X)$$

of $W_{2d}H_G^{2d}(X\times X,\mathbb{Q})$.

Proof. First we prove the assertion for the ordinary cohomology of non-compact smooth varieties and next apply it to the product spaces $X\times U$, which is equipped with a free G-action and the quotient space X_G.

If X is a compact smooth variety then we notice that the weight $2d$ piece coincides with the cohomology group $H^{2d}(X \times X, \mathbb{Q})$ and by the Künneth formula for products the statement follows in the usual cohomology. Suppose X is not compact. Using (4), notice that the image of the diagonal cycle $[\Delta_X]$ lies in $W_{2d}H^{2d}(X \times X, \mathbb{Q})$. Choose a smooth compactification $\overline{X}$ of X and consider the commutative diagram:

$$\begin{array}{ccc} \bigoplus_i H^{2d-i}(\overline{X}) \otimes H^i(\overline{X}) & \stackrel{\simeq}{\to} & H^{2d}(\overline{X} \times \overline{X}, \mathbb{Q}) \\ \downarrow & & \downarrow \\ \bigoplus_i W_{2d-i}H^{2d-i}(X) \otimes W_iH^i(X) & \stackrel{k}{\to} & W_{2d}H^{2d}(X \times X, \mathbb{Q}). \end{array}$$

The vertical arrows are surjective maps, defined by the localization. Hence the map k is surjective. The injectivity follows because this is the Künneth product map, restricted to the bottom weight cohomology. This shows that k is an isomorphism.
In particular, the isomorphism k can be applied to the bottom weights of the ordinary cohomology groups of the smooth variety $X \times U$, for any open subset $U \subset V$ of large complementary codimension and V is a G-representation. But this is essentially the bottom weight of the equivariant cohomology group of X. To conclude, we need to observe that the diagonal cycle $[\Delta_X]$ is G-invariant. $\square$

Denote the decomposition of the G-invariant diagonal cycle

$$\Delta_X = \oplus_{i=0}^{2d} \pi_i^G \in W_{2d}H_G^{2d}(X \times X, \mathbb{Q}) \tag{6}$$

such that π_i^G lies in the space $W_{2d-i}H_G^{2d-i}(X) \otimes W_iH_G^i(X)$.
We defined the equivariant Chow motive of a smooth projective variety with a G-action in §3.6. We extend the notion of orthogonal projectors on a smooth variety equipped with a G-action, as follows.

Definition 4.2. *Suppose X is a smooth variety equipped with a G-action. The equivariant Chow motive $(X, \Delta_X)_G$ of X is said to have an* equivariant Künneth decomposition *if the classes π_i^G are algebraic, i.e., they have a lift in the equivariant Chow group $CH_G^d(X \times X)_{\mathbb{Q}}$. Furthermore, if X admits a smooth compactification $X \subset \overline{X}$ such that the action of G extends on $\overline{X}$ and the Künneth projectors extend to orthogonal projectors on $\overline{X}$ then we say that X has an* equivariant Chow–Künneth decomposition.

Remark 4.3. *When G is a linear algebraic group, using the results of Sumihiro* [Su]*, Bierstone-Milman* [Bi-Mi, Theorem 13.2]*, Reichstein-Youssin* [Re-Yo]*, one can always choose a smooth compactification $\overline{X} \supset X$ such that action of G extends to $\overline{X}$. Since any affine algebraic group is linear, we can always find smooth G-equivariant compactifications in our set-up.*

Suppose X is a smooth variety with a free G-action so that we can form the quotient variety $Y := X/G$. Using [Ed-Gr], we have the identification of the

rational Chow groups
$$CH^*(Y)_{\mathbb{Q}} = CH^*_G(X)_{\mathbb{Q}}$$
and
$$CH^*(Y \times Y)_{\mathbb{Q}} = CH^*_G(X \times X)_{\mathbb{Q}}.$$
Furthermore, these identifications respect the ring structure on the above rational Chow groups. A similar identification also holds for the rational cohomology groups. In view of this, we make the following definition.

Definition 4.4. *Suppose X is a smooth variety with a G-action and G acts freely on X. Denote the quotient space $Y := X/G$. The absolute Chow–Künneth decomposition of Y is defined to be the equivariant Chow–Künneth decomposition of X.*

We can now extend Murre's conjecture to smooth varieties with a G-action, as follows.

Conjecture 4.5. *Suppose X is a smooth variety with a G-action. Then X has an equivariant Chow–Künneth decomposition.*

In particular, if the action of G is trivial then we can extend Murre's conjecture to a (not necessarily compact) smooth variety, by taking only the bottom weight cohomology $W_iH^i(X)$ of the ordinary cohomology. This is weaker than obtaining projectors for the ordinary cohomology. We remark a projector π_1 in the case of quasi–projective varieties has been constructed by Bloch and Esnault [BE].

5. Families of curves

Our goal in this paper is to find an (explicit) absolute Chow–Künneth decomposition for the universal families of curves over close approximations of the moduli space of smooth curves of small genus. We begin with the following situation which motivates the statements on universal curves.

Lemma 5.1. *Any smooth hypersurface $X \subset \mathbb{P}^n$ of degree d has an absolute Chow–Künneth decomposition. If $L \subset X$ is any line, then the blow-up $X' \to X$ also has a Chow–Künneth decomposition.*

Proof. Notice that the cohomology of X is algebraic except in the middle dimension $H^{n-1}(X, \mathbb{Q})$. By the Lefschetz Hyperplane section theorem, the algebraic cohomology $H^{2j}(X, Q)$, $j \neq n-1$, is generated by the hyperplane section H^j. So the projectors are simply
$$\pi_r := \frac{1}{d}.H^{n-1-r} \times H^r \in CH^{n-1}(X \times X)_{\mathbb{Q}}$$
for $r \neq n-1$. We can now take $\pi_{n-1} := \Delta_X - \sum_{r, r \neq n-1} \pi_r$. This gives a complete set of orthogonal projectors and a Chow–Künneth decomposition for X. Since $X' \to X$ is a blow-up along a line, the new cohomology is again algebraic, by the blow-up formula. Similarly we get a Chow–Künneth decomposition for X' (see also [dA-Mü2, Lemma 2] for blow-ups). $\square$

The above lemma can be generalized to the following situation.

Lemma 5.2. *Suppose Y is a smooth projective variety of dimension r over $\mathbb{C}$ which has only algebraic cohomology groups $H^i(Y)$ for all $0 \leq i \leq m$ for some $m < r$. Then we can construct orthogonal projectors*

$$\pi_0, \pi_1, ..., \pi_m, \pi_{2r-m}, \pi_{2r-m+1}, ..., \pi_{2r}$$

in the usual Chow group $CH^r(Y \times Y)_{\mathbb{Q}}$, and where π_{2i} acts as $\delta_{i,p}$ on $H^{2p}(Y)$ and $\pi_{2i-1} = 0$. Moreover, if there is an affine complex algebraic group G acting on Y, then we can lift the above projectors in the equivariant Chow group $CH^r_G(Y \times Y)_{\mathbb{Q}}$ as orthogonal projectors.

Proof. See also [dA-Mü1, dA-Mü2]. Let $H^{2p}(Y)$ be generated by cohomology classes of cycles $C_1, \ldots, C_s$ and $H^{2r-2p}(Y)$ be generated by cohomology classes of cycles $D_1, \ldots, D_s$. We denote by M the intersection matrix with entries

$$M_{ij} = C_i \cdot D_j \in \mathbb{Z}.$$

After base change and passing to $\mathbb{Q}$–coefficients we may assume that M is diagonal, since the cup–product $H^{2p}(Y, \mathbb{Q}) \otimes H^{2r-2p}(Y, \mathbb{Q}) \to \mathbb{Q}$ is non–degenerate. We define the projector π_{2p} as

$$\pi_{2p} = \sum_{k=1}^{s} \frac{1}{M_{kk}} D_k \times C_k.$$

It is easy to check that $\pi_{2p*}(C_k) = D_k$. Define π_{2r-2p} as the adjoint, i.e., transpose of π_{2p}. Via the Gram–Schmidt process from linear algebra we can successively make all projectors orthogonal. □

Suppose $X \subset \mathbb{P}^n$ is a smooth complete intersection of multidegree $d_1 \leq d_2 \leq ... \leq d_s$. Let $F_r(X)$ be the variety of r-dimensional planes contained in X. Let $\delta := \min\{(r+1)(n-r) - \binom{d+r}{r}, n - 2r - s\}$.

Corollary 5.3. *If X is general then $F_r(X)$ is a smooth projective variety of dimension δ and it has an absolute Chow–Künneth decomposition.*

Proof. The first assertion on the smoothness of the variety $F_r(X)$ is well–known, see [Al-Kl], [ELV], [De-Ma]. For the second assertion, notice that $F_r(X)$ is a subvariety of the Grassmanian $G(r, \mathbb{P}^n)$ and is the zero set of a section of a vector bundle. Indeed, let S be the tautological bundle on $G(r, \mathbb{P}^n)$. Then a section of $\oplus_{i=1}^s Sym^{d_i} H^0(\mathbb{P}^n, \mathcal{O}(1))$ induces a section of the vector bundle $\oplus_{i=1}^s Sym^{d_i} S^*$ on $G(r, \mathbb{P}^n)$. Thus, $F_r(X)$ is the zero locus of the section of the $\bigoplus_{i=1}^s Sym^{d_i} S^*$ induced by the equations defining the complete intersection X. A Lefschetz theorem is proved in [De-Ma, Theorem 3.4]:

$$H^i(G(r, \mathbb{P}^n), \mathbb{Q}) \to H^i(F_r(X), \mathbb{Q})$$

is bijective, for $i \leq \delta - 1$. We can apply Lemma 5.2 to get the orthogonal projectors in all degrees except in the middle dimension. The projector corresponding to the middle dimension can be gotten by subtracting the sum of these projectors from the diagonal class.

□

Corollary 5.4. *Suppose $X \subset \mathbb{P}^n$ is a smooth projective variety of dimension d. Let $r = 2d - n$. Then we can construct orthogonal projectors*

$$\pi_0, \pi_1, ..., \pi_r, \pi_{2d-r}, \pi_{2d-r+1}, ..., \pi_{2d}.$$

Proof. Barth [Ba] has proved a Lefschetz theorem for higher codimensional subvarieties in projective spaces:

$$H^i(\mathbb{P}^n, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$

is bijective if $i \leq 2d - n$ and is injective if $i = 2d - n + 1$. The claim now follows from Lemma 5.2. □

Remark 5.5. *The above corollary says that if we can embed a variety X in a low dimensional projective space then we get at least a partial set of orthogonal projectors. A conjecture of Hartshorne's says that any codimension two subvariety of $\mathbb{P}^n$ for $n \geq 6$ is a complete intersection. This gives more examples for subvarieties with several algebraic cohomology groups.*

5.1. Chow–Künneth decomposition for the universal plane curve. We want to find explicit equivariant Chow–Künneth projectors for the universal plane curve of degree d. Let $d \geq 1$ and consider the linear system $\mathbb{P} = |\mathcal{O}_{\mathbb{P}^2}(d)|$ and the universal plane curve

$$\begin{array}{ccc} \mathcal{C} & \subset & \mathbb{P}^2 \times \mathbb{P} \\ \downarrow & & \\ \mathbb{P}. & & \end{array}$$

Furthermore, we notice that the general linear group $G := GL_3(\mathbb{C})$ acts on $\mathbb{P}^2$ and hence acts on the projective space $\mathbb{P} = |\mathcal{O}_{\mathbb{P}^2}(d)|$. This gives an action on the product space $\mathbb{P}^2 \times \mathbb{P}$ and leaves the universal smooth plane curve $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}$ invariant under the G-action.

Lemma 5.6. *The variety $\mathcal{C}$ has an absolute Chow–Künneth decomposition and an absolute equivariant Chow–Künneth decomposition.*

Proof. We observe that $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}$ is a smooth hypersurface of bi-degree $(d, 1)$ with variables in $\mathbb{P}^2$ whose coefficients are polynomial functions on $\mathbb{P}$. Notice that $\mathbb{P}^2 \times \mathbb{P}$ has a Chow–Künneth decomposition and Lefschetz theorems hold for the embedding $\mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}$, since $\mathcal{O}(d, 1)$ is very ample. Now we can repeat the arguments from Lemma 5.2 to get an absolute Chow–Künneth decomposition and absolute equivariant Chow–Künneth decomposition, for the variety $\mathcal{C}$. □

5.2. Families of curves contained in homogeneous spaces. We notice that when $d = 3$ in the previous subsection, the family of plane cubics restricted to the loci of stable curves is a complete family of genus one stable curves. If $d \geq 4$, then the above family of plane curves is no longer a complete family of genus g curves. Hence to find families which closely approximate over the moduli spaces of stable curves, we need to look for curves embedded as complete intersections in other simpler looking varieties. For this purpose, we look at the curves embedded in special Fano varieties of small genus $g \leq 8$, which was studied by S. Mukai [Muk], [Muk2], [Muk3], [Muk5] and Ide-Mukai [IdMuk]. We recall the main result that we need.

Theorem 5.7. *Suppose C is a generic curve of genus $g \leq 8$. Then C is a complete intersection in a smooth projective variety which has only algebraic cohomology.*

Proof. This is proved in [Muk], [Muk2], [Muk3], [IdMuk] and [Muk5]. The below classification is for the generic curve.
When $g \leq 5$ then it is well-known that the generic curve is a linear section of a Grassmanian.
When $g = 6$ then a curve has finitely many g^1_4 if and only if it is a complete intersection of a Grassmanian and a smooth quadric , see [Muk3, Theorem 5.2].
When $g = 7$ then a curve is a linear section of a 10-dimensional spinor variety $X \subset \mathbb{P}^{15}$ if and only if it is non-tetragonal, see [Muk3, Main theorem].
When $g = 8$ then it is classically known that the generic curve is a linear section of the grassmanian $G(2, 6)$ in its Plücker embedding.

□

Suppose $\mathbb{P}(g)$ is the parameter space of linear sections of a Grassmanian or of a spinor variety, which depends on the genus, as in the proof of above Theorem 5.7. $\mathbb{P}(g)$ is a product of projective spaces on which an algebraic group G (copies of PGL_N) acts. Generic curves are isomorphic, if they are in the same orbit of G.

Proposition 5.8. *Suppose $\mathbb{P}(g)$ is as above, for $g \leq 8$. Then there is a universal curve*

$$\mathcal{C}_g \to \mathbb{P}(g)$$

such that the classifying (rational) map $\mathbb{P}(g) \to \mathcal{M}_g$ is dominant. The smooth projective variety $\mathcal{C}_g$ has an absolute Chow–Künneth decomposition and an absolute equivariant Chow-Künneth decomposition for the natural G–action mentioned above.

Proof. The first assertion follows from Theorem 5.7. For the second assertion notice that the universal curve, when $g \leq 8$, is a complete intersection in $\mathbb{P}(g) \times V$ where V is either a Grassmanian or a spinor variety, which are homogeneous varieties. In other words, $\mathcal{C}_g$ is a complete intersection in a space which has only algebraic cohomology. Hence, by Lemma 5.2, $\mathcal{C}_g$ has orthogonal projectors $\pi_0, \pi_1, ..., \pi_m, \pi_{2r-m}, \pi_{2r-m+1}, ..., \pi_{2r}$, where $r := \dim\mathcal{C}_g$ and $m = \dim\mathcal{C}_g - 1$, using Lefschetz hyplerplane section theorem. Taking $\pi_{m+1} = \Delta_{\mathcal{C}_g} - \sum_{i\neq m+1}\pi_i$, gives an absolute Chow–Künneth decomposition for $\mathcal{C}_g$. Now a homogeneous variety looks like $V = G/P$ where G is an (linear) algebraic group and P is a parabolic subgroup. Hence the group G acts on the variety V. This induces an action on the linear system $\mathbb{P}(g)$ and hence G acts on the ambient variety $\mathbb{P}(g)\times V$ and leaves the universal curve $\mathcal{C}_g$ invariant. Hence we can again apply Lemma 5.2 to obtain absolute equivariant Chow–Künneth decomposition for $\mathcal{C}_g$. □

Consider the universal family of curves $\mathcal{C}_g \to \mathbb{P}(g)$ as obtained above, which are equipped with an action of a linear algebraic group G.
Suppose there is an open subset $U_g \subset \mathbb{P}(g)$, with the universal family $\mathcal{C}_{U_g} \to U_g$, on which G acts freely to form a good quotient family

$$Y_g := \mathcal{C}_{U_g}/G \to S_g := U_g/G.$$

Notice that the classifying map $S_g \to \mathcal{M}_g$ is dominant.

Corollary 5.9. *The smooth variety Y_g has an absolute Chow–Künneth decomposition.*

Proof. Consider the localization sequence, for the embedding $j : \mathcal{C}_{U_g}\times\mathcal{C}_{U_g} \hookrightarrow \mathcal{C}_g\times\mathcal{C}_g$,

$$CH^d_G(\mathcal{C}_g\times\mathcal{C}_g)_{\mathbb{Q}} \xrightarrow{j^*} CH^d_G(\mathcal{C}_{U_g}\times\mathcal{C}_{U_g})_{\mathbb{Q}} \to 0.$$

Here d is the dimension of $\mathcal{C}_g$. Then the map j^* is an equivariant ring homomorphism and transforms orthogonal projectors to orthogonal projectors. Similarly there is a commuting diagram between the equivariant cohomologies:

$$\begin{array}{ccc} \bigoplus_i H^{2d-i}_G(\mathcal{C}_g)\otimes H^i_G(\mathcal{C}_g) & \stackrel{\simeq}{\to} & H^{2d}_G(\mathcal{C}_g,\mathbb{Q}) \\ \downarrow & & \downarrow \\ \bigoplus_i W_{2d-i}H^{2d-i}_G(\mathcal{C}_{U_g})\otimes W_iH^i_G(\mathcal{C}_{U_g}) & \stackrel{\simeq}{\to} & W_{2d}H^{2d}_G(\mathcal{C}_{U_g},\mathbb{Q}) \end{array}$$

The vertical arrows are surjective maps mapping onto the bottom weights of the equivariant cohomology groups. By Proposition 5.8, the variety $\mathcal{C}_g$ has an absolute equivariant Chow–Künneth decomposition. Hence the images of the equivariant Chow–Künneth projectors for the complete smooth variety $\mathcal{C}_g$, under the morphism j^* give equivariant Chow–Künneth projectors for the smooth variety $\mathcal{C}_{U_g}$.
Using [Ed-Gr], we have the identification of the rational Chow groups

$$CH^*(Y_g)_{\mathbb{Q}} = CH^*_G(\mathcal{C}_{U_g})_{\mathbb{Q}}$$

and

$$CH^*(Y_g\times Y_g)_{\mathbb{Q}} = CH^*_G(\mathcal{C}_{U_g}\times\mathcal{C}_{U_g})_{\mathbb{Q}}.$$

Furthermore, this respects the ring structure on the above rational Chow groups. A similar identification also holds for the rational cohomology groups. This means that the equivariant Chow–Künneth projectors for the variety $\mathcal{C}_{U_g}$

correspond to a complete set of absolute Chow–Künneth projectors for the quotient variety Y_g. □

Remark 5.10. *Since Mukai has a similar classification for the non-generic curves in genus* ≤ 8*, one can obtain absolute equivariant Chow–Künneth decomposition for these special families of smooth curves, by applying the proof of Proposition 5.8. There is also a classification for* $K3$*-surfaces and in many cases the generic* $K3$*-surface is obtained as a linear section of a Grassmanian* [Muk]. *Hence we can apply the above results to families of* $K3$*-surfaces over spaces which dominate the moduli space of* $K3$*-surfaces.*

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Hecke Operators on Quasimaps into Horospherical Varieties

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Abstract. Let G be a connected reductive complex algebraic group. This paper and its companion [GN] are devoted to the space Z of meromorphic quasimaps from a curve into an affine spherical G-variety X. The space Z may be thought of as an algebraic model for the loop space of X. The theory we develop associates to X a connected reductive complex algebraic subgroup $\check{H}$ of the dual group $\check{G}$. The construction of $\check{H}$ is via Tannakian formalism: we identify a certain tensor category $\mathrm{Q}(Z)$ of perverse sheaves on Z with the category of finite-dimensional representations of $\check{H}$.

In this paper, we focus on horospherical varieties, a class of varieties closely related to flag varieties. For an affine horospherical G-variety X_{horo}, the category $\mathrm{Q}(Z_{\mathrm{horo}})$ is equivalent to a category of vector spaces graded by a lattice. Thus the associated subgroup $\check{H}_{\mathrm{horo}}$ is a torus. The case of horospherical varieties may be thought of as a simple example, but it also plays a central role in the general theory. To an arbitrary affine spherical G-variety X, one may associate a horospherical variety X_{horo}. Its associated subgroup $\check{H}_{\mathrm{horo}}$ turns out to be a maximal torus in the subgroup $\check{H}$ associated to X.

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1. Introduction

Let G be a connected reductive complex algebraic group. In this paper and its companion [GN], we study the space Z of meromorphic quasimaps from a curve into an affine spherical G-variety X. A G-variety X is said to be spherical if a Borel subgroup of G acts on X with a dense orbit. Examples include flag varieties, symmetric spaces, and toric varieties. A meromorphic quasimap consists of a point of the curve, a G-bundle on the curve, and a meromorphic section of the associated X-bundle with a pole only at the distinguished point. The space Z may be thought of as an algebraic model for the loop space of X. The theory we develop identifies a certain tensor category $\mathrm{Q}(Z)$ of perverse sheaves on Z with the category of finite-dimensional representations of a connected reductive complex algebraic subgroup $\check{H}$ of the dual group $\check{G}$. Our method is to use Tannakian formalism: we endow $\mathrm{Q}(Z)$ with a tensor product, a fiber functor to vector spaces, and the necessary compatibility constraints so that it must be equivalent to the category of representations of such a group. Under this equivalence, the fiber functor corresponds to the forgetful functor which assigns to a representation of $\check{H}$ its underlying vector space. In the paper [GN], we define the category $\mathrm{Q}(Z)$, and endow it with a tensor product and fiber functor. This paper provides a key technical result needed for the construction of the fiber functor.

Horospherical G-varieties form a special class of G-varieties closely related to flag varieties. A subgroup $S \subset G$ is said to be horospherical if it contains the unipotent radical of a Borel subgroup of G. A G-variety X is said to be horospherical if for each point $x \in X$, its stabilizer $S_x \subset G$ is horospherical. When X is an affine horospherical G-variety, the subgroup $\check{H}$ we associate to it turns out to be a torus. To see this, we explicitly calculate the functor which corresponds to the restriction of representations from $\check{G}$. Representations of $\check{G}$ naturally act on the category $\mathrm{Q}(Z)$ via the geometric Satake correspondence. The restriction of representations is given by applying this action to the object of $\mathrm{Q}(Z)$ corresponding to the trivial representation of $\check{H}$. The main result of this paper describes this action in the horospherical case. The statement does not mention $\mathrm{Q}(Z)$, but rather what is needed in [GN] where we define and study $\mathrm{Q}(Z)$.

In the remainder of the introduction, we first describe a piece of the theory of geometric Eisenstein series which the main result of this paper generalizes. This may give the reader some context from which to approach the space Z and our main result. We then define Z and state our main result. Finally, we collect notation and preliminary results needed in what follows. Throughout the introduction, we use the term space for objects which are strictly speaking stacks and ind-stacks.

1.1. Background.

One way to approach the results of this paper is to interpret them as a generalization of a theorem of Braverman-Gaitsgory [BG, Theorem 3.1.4] from the theory of geometric Eisenstein series. Let C be a

smooth complete complex algebraic curve. The primary aim of the geometric Langlands program is to construct sheaves on the moduli space Bun_G of G-bundles on C which are eigensheaves for Hecke operators. These are the operators which result from modifying G-bundles at prescribed points of the curve C. Roughly speaking, the theory of geometric Eisenstein series constructs sheaves on Bun_G starting with local systems on the moduli space Bun_T, where T is the universal Cartan of G. When the original local system is sufficiently generic, the resulting sheaf is an eigensheaf for the Hecke operators.
At first glance, the link between Bun_T and Bun_G should be the moduli stack Bun_B of B-bundles on C, where $B \subset G$ is a Borel subgroup with unipotent radical $U \subset B$ and reductive quotient $T = B/U$. Unfortunately, naively working with the natural diagram

$$\begin{array}{ccc} \mathrm{Bun}_B & \to & \mathrm{Bun}_G \\ \downarrow & & \\ \mathrm{Bun}_T & & \end{array}$$

leads to difficulties: the fibers of the horizontal map are not compact. The eventual successful construction depends on V. Drinfeld's relative compactification of Bun_B along the fibers of the map to Bun_G. The starting point for the compactification is the observation that Bun_B also classifies data

$$(\mathcal{P}_G \in \mathrm{Bun}_G, \mathcal{P}_T \in \mathrm{Bun}_T, \sigma : \mathcal{P}_T \to \mathcal{P}_G \overset{G}{\times} G/U)$$

where σ is a T-equivariant bundle map to the $\mathcal{P}_G$-twist of G/U. From this perspective, it is natural to be less restrictive and allow maps into the $\mathcal{P}_G$-twist of the fundamental affine space

$$\overline{G/U} = \mathrm{Spec}(\mathbb{C}[G]^U).$$

Here $\mathbb{C}[G]$ denotes the ring of regular functions on G, and $\mathbb{C}[G]^U \subset \mathbb{C}[G]$ the (right) U-invariants. Following V. Drinfeld, we define the compactification $\overline{\mathrm{Bun}}_B$ to be that classifying quasimaps

$$(\mathcal{P}_G \in \mathrm{Bun}_G, \mathcal{P}_T \in \mathrm{Bun}_T, \sigma : \mathcal{P}_T \to \mathcal{P}_G \overset{G}{\times} \overline{G/U})$$

where σ is a T-equivariant bundle map which factors

$$\sigma|_{C'} : \mathcal{P}_T|_{C'} \to \mathcal{P}_G \overset{G}{\times} G/U|_{C'} \to \mathcal{P}_G \overset{G}{\times} \overline{G/U}|_{C'},$$

for some open curve $C' \subset C$. Of course, the quasimaps that satisfy

$$\sigma : \mathcal{P}_T \to \mathcal{P}_G \overset{G}{\times} G/U$$

form a subspace canonically isomorphic to Bun_B.
Since the Hecke operators on Bun_G do not lift to $\overline{\mathrm{Bun}}_B$, it is useful to introduce a version of $\overline{\mathrm{Bun}}_B$ on which they do. Following [BG, Section 4], we define the space ${}_\infty\overline{\mathrm{Bun}}_B$ to be that classifying meromorphic quasimaps

$$(c \in C, \mathcal{P}_G \in \mathrm{Bun}_G, \mathcal{P}_T \in \mathrm{Bun}_T, \sigma : \mathcal{P}_T|_{C \setminus c} \to \mathcal{P}_G \overset{G}{\times} \overline{G/U}|_{C \setminus c})$$

where σ is a T-equivariant bundle map which factors

$$\sigma|_{C'} : \mathcal{P}_T|_{C'} \to \mathcal{P}_G \overset{G}{\times} G/U|_{C'} \to \mathcal{P}_G \overset{G}{\times} \overline{G/U}|_{C'},$$

for some open curve $C' \subset C \setminus c$. We call $c \in C$ the pole point of the quasimap. Given a meromorphic quasimap with G-bundle $\mathcal{P}_G$ and pole point $c \in C$, we may modify $\mathcal{P}_G$ at c and obtain a new meromorphic quasimap. In this way, the Hecke operators on Bun_G lift to ${}_\infty\overline{\mathrm{Bun}}_B$.
Now the result we seek to generalize [BG, Theorem 3.1.4] describes how the Hecke operators act on a distinguished object of the category $\mathrm{P}({}_\infty\overline{\mathrm{Bun}}_B)$ of perverse sheaves with $\mathbb{C}$-coefficients on ${}_\infty\overline{\mathrm{Bun}}_B$. Let $\Lambda_G = \mathrm{Hom}(\mathbb{C}^\times, T)$ be the coweight lattice, and let $\Lambda_G^+ \subset \Lambda$ be the semigroup of dominant coweights of G. For $\lambda \in \Lambda_G^+$, we have the Hecke operator

$$H_G^\lambda : \mathrm{P}({}_\infty\overline{\mathrm{Bun}}_B) \to \mathrm{P}({}_\infty\overline{\mathrm{Bun}}_B)$$

given by convolving with the simple spherical modification of coweight λ. (See [BG, Section 4] or Section 5 below for more details.) For $\mu \in \Lambda_G$, we have the locally closed subspace ${}_\infty\overline{\mathrm{Bun}}_B^\mu \subset {}_\infty\overline{\mathrm{Bun}}_B$ that classifies data for which the map

$$\mathcal{P}_T(\mu \cdot c)|_{C\setminus c} \overset{\sigma}{\to} \mathcal{P}_G \overset{G}{\times} \overline{G/U}|_{C\setminus c}$$

extends to a holomorphic map

$$\mathcal{P}_T(\mu \cdot c) \overset{\sigma}{\to} \mathcal{P}_G \overset{G}{\times} \overline{G/U}$$

which factors

$$\mathcal{P}_T(\mu \cdot c) \overset{\sigma}{\to} \mathcal{P}_G \overset{G}{\times} G/U \to \mathcal{P}_G \overset{G}{\times} \overline{G/U}.$$

We write ${}_\infty\overline{\mathrm{Bun}}_B^{\leq\mu} \subset {}_\infty\overline{\mathrm{Bun}}_B$ for the closure of ${}_\infty\overline{\mathrm{Bun}}_B^\mu \subset {}_\infty\overline{\mathrm{Bun}}_B$, and

$$\mathrm{IC}^{\leq\mu}_{{}_\infty\overline{\mathrm{Bun}}_B} \in \mathrm{P}({}_\infty\overline{\mathrm{Bun}}_B)$$

for the intersection cohomology sheaf of ${}_\infty\overline{\mathrm{Bun}}_B^{\leq\mu} \subset {}_\infty\overline{\mathrm{Bun}}_B$.

THEOREM 1.1.1. [BG, Theorem 3.1.4] *For $\lambda \in \Lambda_G^+$, there is a canonical isomorphism*

$$H_G^\lambda(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_B}) \simeq \sum_{\mu\in\Lambda_T} \mathrm{IC}^{\leq\mu}_{{}_\infty\overline{\mathrm{Bun}}_B} \otimes \mathrm{Hom}_{\check{T}}(V_{\check{T}}^\mu, V_{\check{G}}^\lambda)$$

Here we write $V_{\check{G}}^\lambda$ for the irreducible representation of the dual group $\check{G}$ with highest weight $\lambda \in \Lambda_G^+$, and $V_{\check{T}}^\mu$ for the irreducible representation of the dual torus $\check{T}$ of weight $\mu \in \Lambda_G$.
In the same paper of Braverman-Gaitsgory [BG, Section 4], there is a generalization [BG, Theorem 4.1.5] of this theorem from the Borel subgroup $B \subset G$ to other parabolic subgroups $P \subset G$. We recall and use this generalization in Section 5 below. It is the starting point for the results of this paper.

1.2. Main result. The main result of this paper is a version of [BG, Theorem 3.1.4] for X an arbitrary affine horospherical G-variety with a dense G-orbit $\mathring{X} \subset X$. For any point in the dense G-orbit $\mathring{X} \subset X$, we refer to its stabilizer $S \subset G$ as the generic stabilizer of X. All such subgroups are conjugate to each other. By choosing such a point, we obtain an identification $\mathring{X} \simeq G/S$.

To state our main theorem, we first introduce some more notation. Satz 2.1 of [Kn] states that the normalizer of a horospherical subgroup $S \subset G$ is a parabolic subgroup $P \subset G$ with the same derived group $[P,P] = [S,S]$. We write A for the quotient torus P/S, and $\Lambda_A = \mathrm{Hom}(\mathbb{C}^\times, A)$ for its coweight lattice. Similarly, for the identity component $S^0 \subset S$, we write A_0 for the quotient torus P/S_0, and $\Lambda_{A_0} = \mathrm{Hom}(\mathbb{C}^\times, A_0)$ for its coweight lattice. The natural maps $T \to A_0 \to A$ induce maps of coweight lattices

$$\Lambda_T \xrightarrow{q} \Lambda_{A_0} \xrightarrow{i} \Lambda_A,$$

where q is a surjection, and i is an injection. For a conjugate of S, the associated tori are canonically isomorphic to those associated to S. Thus when S is the generic stabilizer of a horospherical G-variety X, the above tori, lattices and maps are canonically associated to X.

For an affine horospherical G-variety X with dense G-orbit $\mathring{X} \subset X$, we define the space Z to be that classifying mermorphic quasimaps into X. Such a quasimap consists of data

$$(c \in C, \mathcal{P}_G \in \mathrm{Bun}_G, \sigma : C \setminus c \to \mathcal{P}_G \overset{G}{\times} X|_{C\setminus c})$$

where σ is a section which factors

$$\sigma|_{C'} : C' \to \mathcal{P}_G \overset{G}{\times} \mathring{X}|_{C'} \to \mathcal{P}_G \overset{G}{\times} X|_{C'},$$

for some open curve $C' \subset C \setminus c$.

Given a meromorphic quasimap into X with G-bundle $\mathcal{P}_G$ and pole point $c \in C$, we may modify $\mathcal{P}_G$ at c and obtain a new meromorphic quasimap. But in this context the resulting Hecke operators on Z do not in general preserve the category of perverse sheaves. Instead, we must consider the bounded derived category $\mathrm{Sh}(Z)$ of sheaves of $\mathbb{C}$-modules on Z. For $\lambda \in \Lambda_G^+$, we have the Hecke operator

$$H_G^\lambda : \mathrm{Sh}(Z) \to \mathrm{Sh}(Z)$$

given by convolving with the simple spherical modification of coweight λ. (See Section 5 below for more details.) For $\kappa \in \Lambda_{A_0}$, we have a locally closed subspace $Z^\kappa \subset Z$ consisting of meromorphic quasimaps that factor

$$\sigma : C \setminus c \to \mathcal{P}_G \overset{G}{\times} \mathring{X}|_{C\setminus c} \to \mathcal{P}_G \overset{G}{\times} X|_{C\setminus c}$$

and have a singularity of type κ at $c \in C$. (See Section 3.5 below for more details.) We write $Z^{\leq\kappa} \subset Z$ for the closure of $Z^\kappa \subset Z$, and

$$\mathrm{IC}_Z^{\leq\kappa} \in \mathrm{Sh}(Z)$$

for its intersection cohomology sheaf.

Our main result is the following.

Theorem 1.2.1. *For $\lambda \in \Lambda_G^+$, there is an isomorphism*

$$H_G^\lambda(\mathrm{IC}_{\overline{Z}}^{\leq 0}) \simeq \sum_{\kappa \in \Lambda_{A_0}} \sum_{\mu \in \Lambda_T, q(\mu)=\kappa} \mathrm{IC}_{\overline{Z}}^{\leq \kappa} \otimes \mathrm{Hom}_{\check{T}}(V_{\check{T}}^\mu, V_{\check{G}}^\lambda)[\langle 2\check{\rho}_M, \mu\rangle].$$

Here the torus A_0 and its coweight lattice Λ_{A_0} are those associated to the generic stabilizer $S \subset G$. We write M for the Levi quotient of the normalizer $P \subset G$ of the generic stabilizer $S \subset G$, and $2\check{\rho}_M$ for the sum of the positive roots of M.

In the context of the companion paper [GN], the theorem translates into the following fundamental statement. The tensor category $\mathrm{Q}(Z)$ associated to X is the category of semisimple perverse sheaves with simple summands $\mathrm{IC}_{\overline{Z}}^{\leq \kappa}$, for $\kappa \in \Lambda_{A_0}$, and the dual subgroup $\check{H}$ associated to X is the subtorus $\mathrm{Spec}\, \mathbb{C}[\Lambda_{A_0}] \subset \check{T}$.

1.3. Notation. Throughout this paper, let G be a connected reductive complex algebraic group, let $B \subset G$ be a Borel subgroup with unipotent radical $U(B)$, and let $T = B/U(B)$ be the abstract Cartan.

Let $\check{\Lambda}_G$ denote the weight lattice $\mathrm{Hom}(T, \mathbb{C}^\times)$, and $\check{\Lambda}_G^+ \subset \check{\Lambda}_G$ the semigroup of dominant weights. For $\lambda \in \check{\Lambda}_G^+$, we write V_G^λ for the irreducible representation of G of highest weight λ.

Let Λ_G denote the coweight lattice $\mathrm{Hom}(\mathbb{C}^\times, T)$, and $\Lambda_G^+ \subset \Lambda_G$ the semigroup of dominant coweights. For $\lambda \in \Lambda_G^+$, let $V_{\check{G}}^\lambda$ denote the irreducible representation of the dual group $\check{G}$ of highest weight λ.

Let $\Lambda_G^{\mathrm{pos}} \subset \Lambda_G$ denote the semigroup of coweights in Λ_G which are non-negative on $\check{\Lambda}_G^+$, and let $R_G^{\mathrm{pos}} \subset \Lambda_G^{\mathrm{pos}}$ denote the semigroup of positive coroots.

Let $P \subset G$ be a parabolic subgroup with unipotent radical $U(P)$, and let M be the Levi factor $P/U(P)$.

We have the natural map

$$\check{r} : \check{\Lambda}_{M/[M,M]} \to \check{\Lambda}_G$$

of weights, and the dual map

$$r : \Lambda_G \to \Lambda_{M/[M,M]}$$

of coweights.

Let $\check{\Lambda}_{G,P}^+ \subset \check{\Lambda}_{M/[M,M]}$ denote the inverse image $\check{r}^{-1}(\check{\Lambda}_G^+)$. Let $\Lambda_{G,P}^{\mathrm{pos}} \subset \Lambda_{M/[M,M]}$ denote the semigroup of coweights in $\Lambda_{M/[M,M]}$ which are non-negative on $\check{\Lambda}_{G,P}^+$. Let $R_{G,P}^{\mathrm{pos}} \subset \Lambda_{G,P}^{\mathrm{pos}}$ denote the image $r(R_G^{\mathrm{pos}})$.

Let $\mathcal{W}_M$ denote the Weyl group of M, and let $\mathcal{W}_M \check{\Lambda}_G^+ \subset \check{\Lambda}_G$ denote the union of the translates of $\check{\Lambda}_G^+$ by $\mathcal{W}_M$. Let $\tilde{\Lambda}_{G,P}^{\mathrm{pos}} \subset \Lambda_M^+$ denote the semigroup of dominant coweights of M which are nonnegative on $\mathcal{W}_M \check{\Lambda}_G^+$.

Finally, let $\langle \cdot, \cdot \rangle : \check{\Lambda}_G \times \Lambda_G \to \mathbb{Z}$ denote the natural pairing, and let $\check{\rho}_M \in \check{\Lambda}_G$ denote half the sum of the positive roots of M.

1.4. Bundles and Hecke correspondences. Let C be a smooth complete complex algebraic curve.
For a connected complex algebraic group H, let Bun_H be the moduli stack of H-bundles on C. Objects of Bun_H will be denoted by $\mathcal{P}_H$.
Let $\mathcal{H}_H$ be the Hecke ind-stack that classifies data

$$(c \in C, \mathcal{P}^1_H, \mathcal{P}^2_H \in \mathrm{Bun}_H, \alpha : \mathcal{P}^1_H|_{C\setminus c} \xrightarrow{\sim} \mathcal{P}^2_H|_{C\setminus c})$$

where α is an isomorphism of H-bundles. We have the maps

$$\mathrm{Bun}_H \overset{h^{\leftarrow}_H}{\leftarrow} \mathcal{H}_H \overset{h^{\rightarrow}_H}{\rightarrow} \mathrm{Bun}_H$$

defined by

$$h^{\leftarrow}_H(c, \mathcal{P}^1_H, \mathcal{P}^2_H, \alpha) = \mathcal{P}^1_H \qquad h^{\rightarrow}_H(c, \mathcal{P}^1_H, \mathcal{P}^2_H, \alpha) = \mathcal{P}^2_H,$$

and the map

$$\pi : \mathcal{H}_H \to C$$

defined by

$$\pi(c, \mathcal{P}^1_H, \mathcal{P}^2_H, \alpha) = c.$$

It is useful to have another description of the Hecke ind-stack $\mathcal{H}_H$ for which we introduce some more notation. Let $\mathcal{O}$ be the ring of formal power series $\mathbb{C}[[t]]$, let $\mathcal{K}$ be the field of formal Laurent series $\mathbb{C}((t))$, and let D be the formal disk $\mathrm{Spec}(\mathcal{O})$. For a point $c \in C$, let $\mathcal{O}_c$ be the completed local ring of C at c, and let D_c be the formal disk $\mathrm{Spec}(\mathcal{O}_c)$. Let $\mathrm{Aut}(\mathcal{O})$ be the group-scheme of automorphisms of the ring $\mathcal{O}$. Let $H(\mathcal{O})$ be the group of $\mathcal{O}$-valued points of H, and let $H(\mathcal{K})$ be the group of $\mathcal{K}$-valued points of H. Let Gr_H be the affine Grassmannian of H. It is an ind-scheme whose set of $\mathbb{C}$-points is the quotient $H(\mathcal{K})/H(\mathcal{O})$.
Now consider the $(H(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))$-torsor

$$\widehat{\mathrm{Bun}_H \times C} \to \mathrm{Bun}_H \times C$$

that classifies data

$$(c \in C, \mathcal{P}_H \in \mathrm{Bun}_H, \beta : D \times H \xrightarrow{\sim} \mathcal{P}_H|_{D_c}, \gamma : D \xrightarrow{\sim} D_c)$$

where β is an isomorphism of H-bundles, and γ is an identification of formal disks. We have an identification

$$\mathcal{H}_H \simeq \widehat{\mathrm{Bun}_H \times C} \overset{(H(\mathcal{O})\rtimes\mathrm{Aut}(\mathcal{O}))}{\times} \mathrm{Gr}_H$$

such that the projection $h^{\rightarrow}_H$ corresponds to the obvious projection from the twisted product to Bun_H.
For H reductive, the $(H(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))$-orbits $\mathrm{Gr}^\lambda_H \subset \mathrm{Gr}_H$ are indexed by $\lambda \in \Lambda^+_H$. For $\lambda \in \Lambda^+_H$, we write $\mathcal{H}^\lambda_H \subset \mathcal{H}_H$ for the substack

$$\mathcal{H}^\lambda_H \simeq \widehat{\mathrm{Bun}_H \times C} \overset{(H(\mathcal{O})\rtimes\mathrm{Aut}(\mathcal{O}))}{\times} \mathrm{Gr}^\lambda_H .$$

For a parabolic subgroup $P \subset H$, the connected components $S_{P,\theta} \subset \mathrm{Gr}_P$ are indexed by $\theta \in \Lambda_P/\Lambda_{[P,P]^{sc}}$, where $[P,P]^{sc}$ denotes the simply connected cover of $[P,P]$. For $\theta \in \Lambda_P/\Lambda_{[P,P]^{sc}}$, we write $\mathcal{S}_{P,\theta} \subset \mathcal{H}_P$ for the ind-substack

$$\mathcal{S}_{P,\theta} \simeq \widehat{\mathrm{Bun}_P \times C} \overset{(P(\mathcal{O})\rtimes \mathrm{Aut}(\mathcal{O}))}{\times} S_{P,\theta}.$$

For $\theta \in \Lambda_P/\Lambda_{[P,P]^{sc}}$, and $\lambda \in \Lambda^+_H$, we write $\mathcal{S}^\lambda_{P,\theta} \subset \mathcal{H}_P$ for the ind-substack

$$\mathcal{S}^\lambda_{P,\theta} \simeq \widehat{\mathrm{Bun}_P \times C} \overset{(P(\mathcal{O})\rtimes \mathrm{Aut}(\mathcal{O}))}{\times} S^\lambda_{P,\theta}$$

where $S^\lambda_{P,\theta}$ denotes the intersection $S_{P,\theta} \cap \mathrm{Gr}^\lambda_H$.

For any ind-stack $\mathcal{Z}$ over $\mathrm{Bun}_H \times C$, we have the $(H(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))$-torsor

$$\widehat{\mathcal{Z}} \to \mathcal{Z}$$

obtained by pulling back the $(H(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))$-torsor

$$\widehat{\mathrm{Bun}_H \times C} \to \mathrm{Bun}_H \times C.$$

We also have the Cartesian diagram

$$\begin{array}{ccc} \mathcal{H}_H \underset{\mathrm{Bun}_H \times C}{\times} \mathcal{Z} & \overset{h^{\rightarrow}_H}{\to} & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{H}_H & \overset{h^{\rightarrow}_H}{\to} & \mathrm{Bun}_H \end{array}$$

and an identification

$$\mathcal{H}_H \underset{\mathrm{Bun}_H \times C}{\times} \mathcal{Z} \simeq \widehat{\mathcal{Z}} \overset{(H(\mathcal{O})\rtimes \mathrm{Aut}(\mathcal{O}))}{\times} \mathrm{Gr}_H$$

such that the projection $h^{\rightarrow}_H$ corresponds to the obvious projection from the twisted product to $\mathcal{Z}$. For $\mathcal{F} \in \mathrm{Sh}(\mathcal{Z})$, and $\mathcal{P} \in \mathrm{P}_{(H(\mathcal{O})\rtimes \mathrm{Aut}(\mathcal{O}))}(\mathrm{Gr}_H)$, we may form the twisted product

$$(\mathcal{F}\widetilde{\boxtimes}\mathcal{P})^r \in \mathrm{Sh}(\mathcal{H}_H \underset{\mathrm{Bun}_H \times C}{\times} \mathcal{Z}).$$

with respect to the map $h^{\rightarrow}_H$. In particular, for $\lambda \in \Lambda^+_H$, we may take $\mathcal{P}$ to be the intersection cohomology sheaf $\mathcal{A}^\lambda_G$ of the closure $\overline{\mathrm{Gr}}^\lambda_H \subset \mathrm{Gr}_H$ of the $(H(\mathcal{O}) \rtimes \mathrm{Aut}(\mathcal{O}))$-orbit $\mathrm{Gr}^\lambda_H \subset \mathrm{Gr}_H$.

2. Affine horospherical G-varieties

A subgroup $S \subset G$ is said to be horospherical if it contains the unipotent radical of a Borel subgroup of G. A G-variety X is said to be horospherical if for each point $x \in X$, its stabilizer $S_x \subset G$ is horospherical. A G-variety X is said to be spherical if a Borel subgroup of G acts on X with a dense orbit. Note that a horospherical G-variety contains a dense G-orbit if and only if it is spherical.

Let X be an affine G-variety. As a representation of G, the ring of regular functions $\mathbb{C}[X]$ decomposes into isotypic components

$$\mathbb{C}[X] \simeq \sum_{\lambda \in \check{\Lambda}_G^+} \mathbb{C}[X]_\lambda.$$

We say that $\mathbb{C}[X]$ is graded if

$$\mathbb{C}[X]_\lambda \mathbb{C}[X]_\mu \subset \mathbb{C}[X]_{\lambda+\mu},$$

for all $\lambda, \mu \in \check{\Lambda}_G^+$. We say that $\mathbb{C}[X]$ is simple if the irreducible representation V^λ of highest weight λ occurs in $\mathbb{C}[X]_\lambda$ with multiplicity 0 or 1, for all $\lambda \in \check{\Lambda}_G^+$.

Proposition 2.0.1. *Let X be an affine G-variety.*
(1) [Pop, Proposition 8, (3)] *X is horospherical if and only if $\mathbb{C}[X]$ is graded.*
(2) [Pop, Theorem 1] *X is spherical if and only if $\mathbb{C}[X]$ is simple.*

We see by the proposition that affine horospherical G-varieties containing a dense G-orbit are classified by finitely-generated subsemigroups of $\check{\Lambda}_G^+$. To such a variety X, one associates the subsemigroup

$$\check{\Lambda}_X^+ \subset \check{\Lambda}_G^+$$

of dominant weights λ with $\dim \mathbb{C}[X]_\lambda > 0$.

2.1. Structure of generic stabilizer.

Theorem 2.1.1. [Kn, Satz 2.2] *If X is an irreducible horospherical G-variety, then there is an open G-invariant subset $W \subset X$, and a G-equivariant isomorphism $W \simeq G/S \times Y$, where $S \subset G$ is a horospherical subgroup, and Y is a variety on which G acts trivially.*

Note that for any two such open subsets $W \subset X$ and isomorphisms $W \simeq G/S \times Y$, the subgroups $S \subset G$ are conjugate. We refer to such a subgroup $S \subset G$ as the generic stabilizer of X.

Lemma 2.1.2. [Kn, Satz 2.1] *If $S \subset G$ is a horospherical subgroup, then its normalizer is a parabolic subgroup $P \subset G$ with the same derived group $[P, P] = [S, S]$ and unipotent radical $U(P) = U(S)$.*

Note that the identity component $S^0 \subset S$ is also horospherical with the same derived group $[S^0, S^0] = [S, S]$ and unipotent radical $U(S^0) = U(S)$.
Let $S \subset G$ be a horospherical subgroup with identity component $S^0 \subset S$, and normalizer $P \subset G$. We write A for the quotient torus P/S, and Λ_A for its coweight lattice $\mathrm{Hom}(\mathbb{C}^\times, A)$. Similarly, we write A_0 for the quotient torus P/S^0, and Λ_{A_0} for its coweight lattice $\mathrm{Hom}(\mathbb{C}^\times, A_0)$. The natural maps

$$T \to A_0 \to A$$

induce maps of coweight lattices

$$\Lambda_T \xrightarrow{q} \Lambda_{A_0} \xrightarrow{i} \Lambda_A,$$

where q is a surjection, and i is an injection. For a conjugate of S, the associated tori, lattices, and maps are canonically isomorphic to those associated to S.

Thus when S is the generic stabilizer of a horospherical G-variety X, the tori, lattices and maps are canonically associated to X.
We shall need the following finer description of which subgroups $S \subset G$ may appear as the generic stabilizer of an affine horospherical G-variety. To state it, we introduce some more notation used throughout the paper. For a horospherical subgroup $S \subset G$ with identity component $S^0 \subset S$, and normalizer $P \subset G$, let M be the Levi quotient $P/U(P)$, let M_S be the Levi quotient $S/U(S)$, and let M_S^0 be the identity component of M_S. The natural maps

$$S^0 \to S \to P$$

induce isomorphisms of derived groups

$$[M_S^0, M_S^0] \xrightarrow{\sim} [M_S, M_S] \xrightarrow{\sim} [M, M].$$

We write $\Lambda_{M/[M,M]}$ for the coweight lattice of the torus $M/[M,M]$, and $\Lambda_{M_S^0/[M_S,M_S]}$ for the coweight lattice of the torus $M_S^0/[M_S, M_S]$. The natural maps

$$M_S^0/[M_S, M_S] \to M/[M,M] \to A_0$$

induce a short exact sequence of coweight lattices

$$0 \to \Lambda_{M_S^0/[M_S,M_S]} \to \Lambda_{M/[M,M]} \to \Lambda_{A_0} \to 0.$$

Proposition 2.1.3. *Let $S \subset G$ be a horospherical subgroup. Then S is the generic stabilizer of an affine horospherical G-variety containing a dense G-orbit if and only if*

$$\Lambda_{M_S^0/[M_S,M_S]} \cap \Lambda^{\rm pos}_{G,P} = \langle 0 \rangle.$$

Proof. The proof of the proposition relies on the following lemma. Let $\check{V}$ be a finite-dimensional real vector space, and let $\check{V}^+$ be an open set in $\check{V}$ which is preserved by the action of $\mathbb{R}^{>0}$. Let V be the dual of $\check{V}$, and let $V^{\rm pos}$ be the closed cone of covectors in V that are nonnegative on all vectors in $\check{V}^+$. For a linear subspace $\check{W} \subset \check{V}$, we write $\check{W}^\perp \subset V$ for its orthogonal.

Lemma 2.1.4. *The map $\check{W} \mapsto \check{W}^\perp$ provides a bijection from the set of all linear subspaces $\check{W} \subset \check{V}$ such that $\check{W} \cap \check{V}^+ \neq \emptyset$ to the set of all linear subspaces $W \subset V$ such that $W \cap V^{\rm pos} = \langle 0 \rangle$.*

Proof. If $\check{W} \cap \check{V}^+ \neq \emptyset$, then clearly $\check{W}^\perp \cap V^{\rm pos} = \langle 0 \rangle$. Conversely, if $W \cap V^{\rm pos} = \langle 0 \rangle$, then since $\check{V}^+$ is open, there is a hyperplane $H \subset V$ such that $W \subset H$, and $H \cap V^{\rm pos} = \langle 0 \rangle$. Thus $H^\perp \subset W^\perp$, and $H^\perp \cap \check{V}^+ \neq \emptyset$, and so $W^\perp \cap \check{V}^+ \neq \emptyset$. □

Now suppose X is an affine horospherical G-variety with an open G-orbit and generic stabilizer $S \subset G$ with normalizer $P \subset G$. Then we have $\check{\Lambda}^+_X \subset \check{\Lambda}^+_{G,P}$, since otherwise $[S,S]$ would be smaller. We also have that $\check{\Lambda}^+_X$ intersects the interior of $\check{\Lambda}^+_{G,P}$, since otherwise $[S,S]$ would be larger. Applying Lemma 2.1.4, we conclude

$$\Lambda_{M_S^0/[M_S,M_S]} \cap \Lambda^{\rm pos}_{G,P} = \langle 0 \rangle.$$

Conversely, suppose $S \subset G$ is a horospherical subgroup with normalizer $P \subset G$. We define X to be the spectrum of the ring $\mathbb{C}[X]$ of (right) S-invariants in the ring of regular functions $\mathbb{C}[G]$. Then $\mathbb{C}[X]$ is finitely-generated, since S contains the unipotent radical of a Borel subgroup of G. We have $\check{\Lambda}_X^+ \subset \check{\Lambda}_{G,P}^+$, since otherwise $[S,S]$ would be smaller. Suppose

$$\Lambda_{M_S^0/[M_S,M_S]} \cap \Lambda_{G,P}^{\mathrm{pos}} = \langle 0 \rangle.$$

Applying Lemma 2.1.4, we conclude that $\check{\Lambda}_X^+$ intersects the interior of $\check{\Lambda}_{G,P}^+$. Therefore $S/[S,S]$ consists of exactly those elements of $P/[P,P]$ annhilated by $\check{\Lambda}_X^+$, and so S is the generic stabilizer of X. □

2.2. Canonical affine closure. Let $S \subset G$ be the generic stabilizer of an affine horospherical G-variety X containing a dense G-orbit. Let $\mathbb{C}[G]$ be the ring of regular functions on G, and let $\mathbb{C}[G]^S \subset \mathbb{C}[G]$ be the (right) S-invariants. We call the affine variety

$$\overline{G/S} = \mathrm{Spec}(\mathbb{C}[G]^S)$$

the canonical affine closure of G/U. We have the natural map

$$\overline{G/S} \to X$$

corresponding to the restriction map

$$\mathbb{C}[X] \to \mathbb{C}[G/S] \simeq \mathbb{C}[G]^S.$$

Since S is horospherical, the ring $\mathbb{C}[G]^S$ is simple and graded, and so the affine variety $\overline{G/S}$ is spherical and horospherical.
Although we do not use the following, it clarifies the relation between X and the canonical affine closure $\overline{G/S}$.

Proposition 2.2.1. *Let X be an affine horospherical G-variety containing a dense G-orbit and generic stabilizer $S \subset G$. The semigroup $\check{\Lambda}_{\overline{G/S}}^+ \subset \check{\Lambda}_G$ is the intersection of the dominant weights $\check{\Lambda}_G^+ \subset \check{\Lambda}_G$ with the group generated by the semigroup $\check{\Lambda}_X^+ \subset \check{\Lambda}_G$.*

Proof. Let $P \subset G$ be the normalizer of $S \subset G$. The intersection of $\check{\Lambda}_G^+$ and the group generated by $\check{\Lambda}_X^+$ consists of exactly those weights in $\check{\Lambda}_{G,P}^+$ that annhilate $S/[S,S]$. □

3. Ind-stacks

As usual, let C be a smooth complete complex algebraic curve.

3.1. Labellings. Fix a pair $(\Lambda, \Lambda^{\mathrm{pos}})$ of a lattice Λ and a semigroup $\Lambda^{\mathrm{pos}} \subset \Lambda$. We shall apply the following to the pair $(\Lambda_{M/[M,M]}, \Lambda_{G,P}^{\mathrm{pos}})$.
For $\theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}$, we write $\mathfrak{U}(\theta^{\mathrm{pos}})$ for a decomposition

$$\theta^{\mathrm{pos}} = \sum_m n_m \theta_m^{\mathrm{pos}}$$

where $\theta_m^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}} \setminus \{0\}$ are pairwise distinct and n_m are positive integers.

For $\theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}$, and a decomposition $\mathfrak{U}(\theta^{\mathrm{pos}})$, we write $C^{\mathfrak{U}(\theta^{\mathrm{pos}})}$ for the partially symmetrized power $\prod_m C^{(n_m)}$ of the curve C. We write $C_0^{\mathfrak{U}(\theta^{\mathrm{pos}})} \subset C^{\mathfrak{U}(\theta^{\mathrm{pos}})}$ for the complement of the diagonal divisor.

For Θ a pair $(\theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$ consisting of $\theta \in \Lambda$, and $\mathfrak{U}(\theta^{\mathrm{pos}})$ a decomposition of $\theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}$, we write C^{Θ} for the product $C \times C^{\mathfrak{U}(\theta^{\mathrm{pos}})}$. We write $C_0^{\Theta} \subset C^{\Theta}$ for the complement of the diagonal divisor. Although C^{Θ} is independent of θ, it is notationally convenient to denote it as we do.

3.2. $\overline{\textbf{Ind-stack}}$ ASSOCIATED TO PARABOLIC SUBGROUP. Fix a parabolic subgroup $P \subset G$, and let M be its Levi quotient $P/U(P)$. For our application, P will be the normalizer of the generic stabilizer $S \subset G$ of an irreducible affine horospherical G-variety.

Let ${}_\infty\overline{\mathrm{Bun}}_P$ be the ind-stack that classifies data

$$(c \in C, \mathcal{P}_G \in \mathrm{Bun}_G, \mathcal{P}_{M/[M,M]} \in \mathrm{Bun}_{M/[M,M]},$$
$$\sigma : \mathcal{P}_{M/[M,M]}|_{C\setminus c} \to \mathcal{P}_G \overset{G}{\times} \overline{G/[P,P]}|_{C\setminus c})$$

where σ is an $M/[M,M]$-equivariant section which factors

$$\sigma|_{C'} : \mathcal{P}_{M/[M,M]}|_{C'} \to \mathcal{P}_G \overset{G}{\times} G/[P,P]|_{C'} \to \mathcal{P}_G \overset{G}{\times} \overline{G/[P,P]}|_{C'}$$

for some open curve $C' \subset C \setminus c$.

3.2.1. *Stratification.* Let Θ be a pair $(\theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$, with $\theta \in \Lambda_{M/[M,M]}$, and $\theta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}}$. We recall that we have a locally closed embedding

$$j_\Theta : \mathrm{Bun}_P \times C_0^{\Theta} \to {}_\infty\overline{\mathrm{Bun}}_P$$

defined by

$$j_\Theta(\mathcal{P}_P, (c, \sum_{m,n} \theta_m^{\mathrm{pos}} \cdot c_{m,n})) = (c, \mathcal{P}_P \overset{P}{\times} G, \mathcal{P}_P \overset{P}{\times} [P,P](-\theta \cdot c - \sum_{m,n} \theta_m^{\mathrm{pos}} \cdot c_{m,n}), \sigma)$$

where σ is the natural map

$$\mathcal{P}_P \overset{P}{\times} [P,P](-\theta \cdot c - \sum_{m,n} \theta_m^{\mathrm{pos}} \cdot c_{m,n})|_{C\setminus c} \to \mathcal{P}_P \overset{P}{\times} G \overset{G}{\times} \overline{G/[P,P]}|_{C\setminus c}$$

induced by the inclusion

$$\mathcal{P}_P \overset{P}{\times} \overline{P/[P,P]} \subset \mathcal{P}_P \overset{P}{\times} \overline{G/[P,P]} \simeq \mathcal{P}_P \overset{P}{\times} G \overset{G}{\times} \overline{G/[P,P]}.$$

The following is an ind-version of [BG, Propositions 6.1.2 & 6.1.3], or [BFGM, Proposition 1.5], and we leave the proof to the reader.

PROPOSITION 3.2.2. *Let Θ be a pair $(\theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$, with $\theta \in \Lambda_{M/[M,M]}$, and $\theta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}}$.*

Every closed point of ${}_\infty\overline{\mathrm{Bun}}_P$ belongs to the image of a unique j_Θ.

For Θ a pair $(\theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$, with $\theta \in \Lambda_{M/[M,M]}$, and $\theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}_{G,P}$, we write ${}_\infty\overline{\mathrm{Bun}}^{\Theta}_P \subset {}_\infty\overline{\mathrm{Bun}}_P$ for the image of j_Θ, and ${}_\infty\overline{\mathrm{Bun}}^{\leq\Theta}_P \subset {}_\infty\overline{\mathrm{Bun}}_P$ for the closure of ${}_\infty\overline{\mathrm{Bun}}^{\Theta}_P \subset {}_\infty\overline{\mathrm{Bun}}_P$.
For Θ a pair $(\theta, \mathfrak{U}(0))$, with $\theta \in \Lambda_{M/[M,M]}$, the substack ${}_\infty\overline{\mathrm{Bun}}^{\Theta}_P \subset {}_\infty\overline{\mathrm{Bun}}_P$ classifies data $(c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}, \sigma)$ for which the map

$$\mathcal{P}_{M/[M,M]}(\theta\cdot c)|_{C\setminus c} \xrightarrow{\sigma} \mathcal{P}_G \overset{G}{\times} \overline{G/[P,P]}|_{C\setminus c}$$

extends to a holomorphic map

$$\mathcal{P}_{M/[M,M]}(\theta\cdot c) \xrightarrow{\sigma} \mathcal{P}_G \overset{G}{\times} \overline{G/[P,P]}$$

which factors

$$\mathcal{P}_{M/[M,M]}(\theta\cdot c) \xrightarrow{\sigma} \mathcal{P}_G \overset{G}{\times} G/[P,P] \to \mathcal{P}_G \overset{G}{\times} \overline{G/[P,P]}.$$

In this case, we write j_θ in place of j_Θ, ${}_\infty\overline{\mathrm{Bun}}^{\theta}_P$ in place of ${}_\infty\overline{\mathrm{Bun}}^{\Theta}_P$, and ${}_\infty\overline{\mathrm{Bun}}^{\leq\theta}_P$ in place of ${}_\infty\overline{\mathrm{Bun}}^{\leq\Theta}_P$. For example, ${}_\infty\overline{\mathrm{Bun}}^{\leq 0}_P \subset {}_\infty\overline{\mathrm{Bun}}_P$ is the closure of the canonical embedding

$$j_0 : \mathrm{Bun}_P \times C \to {}_\infty\overline{\mathrm{Bun}}_P.$$

3.3. $\widetilde{\textbf{Ind-stack}}$ associated to parabolic subgroup. Fix a parabolic subgroup $P \subset G$, and let M be its Levi quotient $P/U(P)$. As usual, for our application, P will be the normalizer of the generic stabilizer $S \subset G$ of an irreducible affine horospherical G-variety.
Let ${}_\infty\widetilde{\mathrm{Bun}}_P$ be the ind-stack that classifies data

$$(c \in C, \mathcal{P}_G \in \mathrm{Bun}_G, \mathcal{P}_M \in \mathrm{Bun}_M, \sigma : \mathcal{P}_M|_{C\setminus c} \to \mathcal{P}_G \overset{G}{\times} \overline{G/U(P)}|_{C\setminus c})$$

where σ is an M-equivariant section which factors

$$\sigma|_{C'} : \mathcal{P}_M|_{C'} \to \mathcal{P}_G \overset{G}{\times} G/U(P)|_{C'} \to \mathcal{P}_G \overset{G}{\times} \overline{G/U(P)}|_{C'}$$

for some open curve $C' \subset C \setminus c$.

3.3.1. *Stratification.* For $\theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}_{G,P}$, we write $\tilde{\mathfrak{U}}(\theta^{\mathrm{pos}})$ for a collection of (not necessarily distinct) elements $\tilde{\theta}^{\mathrm{pos}}_m \in \tilde{\Lambda}^{\mathrm{pos}}_{G,P} \setminus \{0\}$ such that

$$\theta^{\mathrm{pos}} = \sum_m r(\tilde{\theta}^{\mathrm{pos}}_m).$$

We write $r(\tilde{\mathfrak{U}}(\theta^{\mathrm{pos}}))$ for the decomposition such a collection defines.
Let $\tilde{\Theta}$ be a pair $(\tilde{\theta}, \tilde{\mathfrak{U}}(\theta^{\mathrm{pos}}))$ with $\tilde{\theta} \in \Lambda^+_M$, and $\theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}_{G,P}$, and let Θ be the associated pair $(r(\tilde{\theta}), r(\tilde{\mathfrak{U}}(\theta^{\mathrm{pos}})))$. We define the Hecke ind-stack

$$\mathcal{H}^{\tilde{\Theta}}_{M,0} \to C^{\Theta}_0$$

to be that with fiber over $(c, c_{\mathfrak{U}(\theta^{\mathrm{pos}})}) \in C_0^{\Theta}$, where $c_{\mathfrak{U}(\theta^{\mathrm{pos}})} = \sum_m r(\tilde{\theta}_m^{\mathrm{pos}}) \cdot c_m$, the fiber product

$$\mathcal{H}_M^{\tilde{\theta}}|_c \underset{\mathrm{Bun}_M}{\times} \prod_{\mathrm{Bun}_M} \mathcal{H}_M^{\tilde{\theta}_m^{\mathrm{pos}}}|_{c_m}.$$

The following is an ind-version of [BG, Proposition 6.2.5], or [BFGM, Proposition 1.9], and we leave the proof to the reader.

Proposition 3.3.2. *Let $\tilde{\Theta}$ be a pair $(\tilde{\theta}, \tilde{\mathfrak{U}}(\theta^{\mathrm{pos}}))$ with $\tilde{\theta} \in \Lambda_M^+$, and $\theta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}}$. On the level of reduced ind-stacks, there is a locally closed embedding*

$$j_{\tilde{\Theta}} : \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_{M,0}^{\tilde{\Theta}} \to {}_\infty\widetilde{\mathrm{Bun}}_P.$$

Every closed point of ${}_\infty\widetilde{\mathrm{Bun}}_P$ belongs to the image of a unique $j_{\tilde{\Theta}}$.

For $\tilde{\Theta}$ a pair $(\tilde{\theta}, \tilde{\mathfrak{U}}(\theta^{\mathrm{pos}}))$, with $\tilde{\theta} \in \Lambda_M^+$, and $\theta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}}$, we write ${}_\infty\widetilde{\mathrm{Bun}}_P^{\tilde{\Theta}} \subset {}_\infty\widetilde{\mathrm{Bun}}_P$ for the image of $j_{\tilde{\Theta}}$, and ${}_\infty\widetilde{\mathrm{Bun}}_P^{\leq\tilde{\Theta}} \subset {}_\infty\widetilde{\mathrm{Bun}}_P$ for the closure of ${}_\infty\widetilde{\mathrm{Bun}}_P^{\tilde{\Theta}} \subset {}_\infty\widetilde{\mathrm{Bun}}_P$.

For $\tilde{\Theta}$ a pair $(\tilde{\theta}, \tilde{\mathfrak{U}}(0))$, with $\tilde{\theta} \in \Lambda_M^+$, we write $j_{\tilde{\theta}}$ in place of $j_{\tilde{\Theta}}$, ${}_\infty\widetilde{\mathrm{Bun}}_P^{\tilde{\theta}}$ in place of ${}_\infty\widetilde{\mathrm{Bun}}_P^{\tilde{\Theta}}$, and ${}_\infty\widetilde{\mathrm{Bun}}_P^{\leq\tilde{\theta}}$ in place of ${}_\infty\widetilde{\mathrm{Bun}}_P^{\leq\tilde{\Theta}}$ For example, ${}_\infty\widetilde{\mathrm{Bun}}_P^{\leq 0}$ is the closure of the canonical embedding

$$j_{\tilde{0}} : \mathrm{Bun}_P \times C \to {}_\infty\widetilde{\mathrm{Bun}}_P.$$

3.4. $\overline{\textbf{Ind-stack}}$ associated to generic stabilizer. Let X be an irreducible affine horospherical G-variety with generic stabilizer $S \subset G$. Recall that the normalizer of S is a parabolic subgroup $P \subset G$ with the same derived group $[P, P] = [S, S]$ and unipotent radical $U(P) = U(S)$. Let M be the Levi quotient $P/U(P)$, and let M_S be the Levi quotient $S/U(S)$.

Let $\overline{Z}_{\mathrm{can}}$ be the ind-stack that classifies data

$$(c \in C, \mathcal{P}_G \in \mathrm{Bun}_G, \mathcal{P}_{M_S/[M_S,M_S]} \in \mathrm{Bun}_{M_S/[M_S,M_S]},$$
$$\sigma : \mathcal{P}_{M_S/[M_S,M_S]}|_{C\setminus c} \to \mathcal{P}_G \overset{G}{\times} \overline{G/[S,S]}|_{C\setminus c})$$

where σ is an $M_S/[M_S, M_S]$-equivariant section which factors

$$\sigma|_{C'} : \mathcal{P}_{M_S/[M_S,M_S]}|_{C'} \to \mathcal{P}_G \overset{G}{\times} G/[S,S]|_{C'} \to \mathcal{P}_G \overset{G}{\times} \overline{G/[S,S]}|_{C'}$$

for some open curve $C' \subset C \setminus c$.

The following is immediate from the definitions.

Proposition 3.4.1. *The diagram*

$$\begin{array}{ccc} \overline{Z}_{\mathrm{can}} & \to & {}_\infty\overline{\mathrm{Bun}}_P \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{M_S/[M_S,M_S]} & \to & \mathrm{Bun}_{M/[M,M]} \end{array}$$

is Cartesian.

3.4.2. *Stratification.* Let Θ be a pair $(\theta, \mathfrak{U}(\theta^{\text{pos}}))$, with $\theta \in \Lambda_{M/[M,M]}$, and $\theta^{\text{pos}} \in \Lambda^{\text{pos}}_{G,P}$.
We write $\overline{Z}^{\Theta}_{\text{can}} \subset \overline{Z}_{\text{can}}$ for the substack which completes the Cartesian diagram

$$\begin{array}{ccc} \overline{Z}^{\Theta}_{\text{can}} & \to & {}_{\infty}\overline{\text{Bun}}^{\Theta}_{P} \\ \downarrow & & \downarrow \\ \text{Bun}_{M_S/[M_S,M_S]} & \to & \text{Bun}_{M/[M,M]}, \end{array}$$

and $\overline{Z}^{\leq\Theta}_{\text{can}} \subset \overline{Z}_{\text{can}}$ for the closure of $\overline{Z}^{\Theta}_{\text{can}} \subset \overline{Z}_{\text{can}}$.
For Θ a pair $(\theta, \mathfrak{U}(0))$, with $\theta \in \Lambda_{M/[M,M]}$, we write $\overline{Z}^{\theta}_{\text{can}}$ in place of $\overline{Z}^{\Theta}_{\text{can}}$, and $\overline{Z}^{\leq\theta}_{\text{can}}$ in place of $\overline{Z}^{\leq\Theta}_{\text{can}}$. For example, $\overline{Z}^{\leq 0}_{\text{can}}$ is the closure of the canonical embedding

$$\text{Bun}_S \times C \subset \overline{Z}_{\text{can}}.$$

3.5. Naive ind-stack associated to X. Let X be an affine horospherical G-variety with dense G-orbit $\overset{\circ}{X} \subset X$ and generic stabilizer $S \subset G$.
Let Z be the ind-stack that classifies data

$$(c \in C, \mathcal{P}_G \in \text{Bun}_G, \sigma : C \setminus c \to \mathcal{P}_G \overset{G}{\times} X|_{C\setminus c})$$

where σ is a section which factors

$$\sigma|_{C'} : C' \to \mathcal{P}_G \overset{G}{\times} \overset{\circ}{X}|_{C'} \to \mathcal{P}_G \overset{G}{\times} X|_{C'}$$

for some open curve $C' \subset C \setminus c$.
For the canonical affine closure $\overline{G/S}$, we write Z_{can} for the corresponding ind-stack.
We call the ind-stack Z naive, since there is no auxilliary bundle in its definition: it classifies honest sections. Let ${}^{\star}Z$ be the ind-stack that classifies data

$$(c \in C, \mathcal{P}_G \in \text{Bun}_G, \mathcal{P}_{M/M_S} \in \text{Bun}_{M/M_S}, \sigma : \mathcal{P}_{M/M_S}|_{C\setminus c} \to \mathcal{P}_G \overset{G}{\times} X|_{C\setminus c})$$

where σ is an M/M_S-equivariant section which factors

$$\sigma|_{C'} : \mathcal{P}_{M/M_S}|_{C'} \to \mathcal{P}_G \overset{G}{\times} \overset{\circ}{X}|_{C'} \to \mathcal{P}_G \overset{G}{\times} X|_{C'}$$

for some open curve $C' \subset C \setminus c$. Here as usual, we write M for the Levi quotient $P/U(P)$ of the normalizer $P \subset G$ of the generic stabilizer $S \subset G$, and M_S for the Levi quotient $S/U(S)$.
For the canonical affine closure $\overline{G/S}$, we write ${}^{\star}Z_{\text{can}}$ for the corresponding ind-stack.
The following analogue of Proposition 3.4.1 is immediate from the definitions.

Proposition 3.5.1. *The diagram*

$$\begin{array}{ccc} Z & \to & {}^{\star}Z \\ \downarrow & & \downarrow \\ \text{Bun}_{\langle 1\rangle} & \to & \text{Bun}_{M/M_S} \end{array}$$

is Cartesian.

3.5.2. *Stratification.* We shall content ourselves here with defining the substacks of the naive ind-stack Z which appear in our main theorem. (See [GN] for a different perspective involving a completely local definition.) Recall that we write A for the quotient torus P/S, and Λ_A for its coweight lattice. Similarly, for the identity component $S^0 \subset S$, we write A_0 for the quotient torus P/S^0, and Λ_{A_0} for its coweight lattice. The natural map $A_0 \to A$ provides an inclusion of coweight lattices $\Lambda_{A_0} \to \Lambda_A$. For $\kappa \in \Lambda_A$, we shall define a closed substack $Z^{\leq\kappa} \subset Z$. When $\kappa \in \Lambda_{A_0}$, the closed substack $Z^{\leq\kappa} \subset Z$ appears in our main theorem.

For $\kappa \in \Lambda_A$, let ${}^\star Z^\kappa \subset {}^\star Z$ be the locally closed substack that classifies data $(c, \mathcal{P}_G, \mathcal{P}_{M/M_S}, \sigma)$ for which the natural map

$$\mathcal{P}_{M/M_S}(\kappa \cdot c)|_{C\setminus c} \xrightarrow{\sigma} \mathcal{P}_G \overset{G}{\times} X|_{C\setminus c}$$

extends to a holomorphic map

$$\mathcal{P}_{M/M_S}(\kappa \cdot c) \xrightarrow{\sigma} \mathcal{P}_G \overset{G}{\times} X$$

which factors

$$\mathcal{P}_{M/M_S}(\kappa \cdot c) \xrightarrow{\sigma} \mathcal{P}_G \overset{G}{\times} \overset{\circ}{X} \to \mathcal{P}_G \overset{G}{\times} X.$$

We write ${}^\star Z^{\leq\kappa} \subset {}^\star Z$ for the closure of ${}^\star Z^\kappa \subset {}^\star Z$.

For $\kappa \in \Lambda_A$, let $Z^\kappa \subset Z$ be the locally closed substack completing the Cartesian diagram

$$\begin{array}{ccc} Z^\kappa & \to & {}^\star Z^\kappa \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{\langle 1 \rangle} & \to & \mathrm{Bun}_{M/M_S}. \end{array}$$

We write $Z^{\leq\kappa} \subset Z$ for the closure of $Z^\kappa \subset Z$.

4. Maps

4.1. The map $\mathfrak{r} : {}_\infty\widetilde{\mathrm{Bun}}_P \to {}_\infty\overline{\mathrm{Bun}}_P$. Let Θ be a pair $(\theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$, with $\theta \in \Lambda_{M/[M,M]}$, and $\theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}_{G,P}$. and $\mathfrak{U}(\theta^{\mathrm{pos}})$ a decomposition $\theta^{\mathrm{pos}} = \sum_m n_m \theta^{\mathrm{pos}}_m$. Let ${}_\infty\widetilde{\mathrm{Bun}}^{\Theta}_P \subset {}_\infty\widetilde{\mathrm{Bun}}_P$ be the inverse image of ${}_\infty\overline{\mathrm{Bun}}^{\Theta}_P \subset {}_\infty\overline{\mathrm{Bun}}_P$ under the natural map

$$\mathfrak{r} : {}_\infty\widetilde{\mathrm{Bun}}_P \to {}_\infty\overline{\mathrm{Bun}}_P.$$

We would like to describe the fibers of the restriction of $\mathfrak{r}$ to the substack ${}_\infty\widetilde{\mathrm{Bun}}^{\Theta}_P \subset {}_\infty\widetilde{\mathrm{Bun}}_P$.

First, we define the Hecke ind-substack

$$\mathcal{H}^{\flat(\theta)}_M \subset \mathcal{H}_M$$

to be the union of the spherical Hecke substacks

$$\mathcal{H}^{\mu}_M \subset \mathcal{H}_M,$$

for $\mu \in \Lambda^+_M$ such that $r(\mu) = \theta$.

Second, if there exists $\tilde{\mu}^{\rm pos} \in \tilde{\Lambda}^{\rm pos}_{G,P}$ such that $r(\tilde{\mu}^{\rm pos}) = \theta^{\rm pos}$, we define the Hecke substack

$$\mathcal{H}_M^{\flat(\theta^{\rm pos})} \subset \mathcal{H}_M$$

to be the union of the spherical Hecke substacks

$$\mathcal{H}_M^{\tilde{\mu}^{\rm pos}} \subset \mathcal{H}_M,$$

for $\tilde{\mu}^{\rm pos} \in \tilde{\Lambda}^{\rm pos}_{G,P}$ such that $r(\tilde{\mu}^{\rm pos}) = \theta^{\rm pos}_m$.
Finally, we define the Hecke ind-stack

$$\mathcal{H}_{M,0}^{\flat(\Theta)} \to C_0^\Theta$$

to be that with fiber over $(c, c_{\mathfrak{U}(\theta^{\rm pos})}) \in C_0^\Theta$, where $c_{\mathfrak{U}(\theta^{\rm pos})} = \sum_{m,n} \theta^{\rm pos}_m \cdot c_{m,n}$, the fiber product

$$\mathcal{H}_M^{\flat(\theta)}|_c \underset{\mathrm{Bun}_M}{\times} \prod_{\mathrm{Bun}_M} \mathcal{H}_M^{\flat(\theta^{\rm pos}_m)}|_{c_{m,n}}.$$

The following is an ind-version of [BG, Proposition 6.2.5], or [BFGM, Proposition 1.9], and we leave the proof to the reader. It is also immediately implied by Proposition 3.3.2.

Proposition 4.1.1. *Let Θ be a pair $(\theta, \mathfrak{U}(\theta^{\rm pos}))$, with $\theta \in \Lambda_{M/[M,M]}$, $\theta^{\rm pos} \in \Lambda^{\rm pos}_{G,P}$, and $\mathfrak{U}(\theta^{\rm pos})$ a decomposition $\theta^{\rm pos} = \sum_m n_m \theta^{\rm pos}_m$.*
If for all m there exists $\tilde{\mu}^{\rm pos}_m \in \tilde{\Lambda}^{\rm pos}_{G,P}$ such that $r(\tilde{\mu}^{\rm pos}_m) = \theta^{\rm pos}_m$, then on the level of reduced stacks there is a canonical isomorphism

$${}_\infty\widetilde{\mathrm{Bun}}_P^\Theta \simeq \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_{M,0}^{\flat(\Theta)}$$

such that the following diagram commutes

$$\begin{array}{ccc} {}_\infty\widetilde{\mathrm{Bun}}_P^\Theta & \simeq & \mathrm{Bun}_P \underset{\mathrm{Bun}_M}{\times} \mathcal{H}_{M,0}^{\flat(\Theta)} \\ \downarrow & & \downarrow \\ {}_\infty\overline{\mathrm{Bun}}_P^\Theta & \simeq & \mathrm{Bun}_P \times C_0^\Theta \end{array}$$

where the right hand side is the obvious projection.
If there is an m such that $\theta^{\rm pos}_m$ is not equal to $r(\tilde{\mu}^{\rm pos})$, for any $\tilde{\mu}^{\rm pos} \in \tilde{\Lambda}^{\rm pos}_{G,P}$, then ${}_\infty\widetilde{\mathrm{Bun}}_P^\Theta$ is empty.

4.2. The map $\mathfrak{p} : \overline{Z}_{\rm can} \to Z_{\rm can}$. Let X be an irreducible affine horospherical G-variety with generic stabilizer $S \subset G$. Recall that the normalizer of a horospherical subgroup $S \subset G$ is a parabolic subgroup $P \subset G$ with the same derived group $[P,P] = [S,S]$ and unipotent radical $U(P) = U(S)$. We write M for the Levi quotient $P/U(P)$, M_S for the Levi quotient $S/U(S)$, and M_S^0 for the identity component of M_S. We write A for the quotient torus P/S, and Λ_A for its coweight lattice. Similarly, for the identity component $S^0 \subset S$, we write A_0 for the quotient torus P/S^0, and Λ_{A_0} for its coweight lattice. The natural map $M/[M,M] \to A_0$ induces a surjection of coweight lattices $\Lambda_{M/[M,M]} \to \Lambda_{A_0}$ which we denote by p. The kernel of p is the coweight lattice $\Lambda_{M_S^0/[M_S,M_S]}$. (Note that the component group of M_S is abelian.)

Associated to the canonical affine closure $\overline{G/S}$, we have a Cartesian diagram of ind-stacks

$$\begin{array}{ccc} \overline{Z}_{\mathrm{can}} & \to & {}_{\infty}\overline{\mathrm{Bun}}_P \\ \mathfrak{p}\downarrow & & \downarrow\mathfrak{p} \\ Z_{\mathrm{can}} & \to & {}^{\star}Z_{\mathrm{can}} \end{array}$$

We would like to describe some properties of the vertical maps.

PROPOSITION 4.2.1. *The map* $\mathfrak{p} : {}_{\infty}\overline{\mathrm{Bun}}_P \to {}^{\star}Z_{\mathrm{can}}$ *is ind-finite.*
For $\theta \in \Lambda_{M/[M,M]}$, *its restriction to* ${}_{\infty}\overline{\mathrm{Bun}}_P^{\theta}$ *is an embedding with image* ${}^{\star}Z_{\mathrm{can}}^{p(\theta)}$, *and its restriction to* ${}_{\infty}\overline{\mathrm{Bun}}_P^{\leq\theta}$ *is finite with image* ${}^{\star}Z_{\mathrm{can}}^{\leq p(\theta)}$.

Proof. For a point $(c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}, \overline{\sigma}) \in {}_{\infty}\overline{\mathrm{Bun}}_P$, we write $(c, \mathcal{P}_G, \mathcal{P}_{M/M_S}, \sigma) \in {}^{\star}Z_{\mathrm{can}}$ for its image under $\mathfrak{p}$. Observe that for $\theta \in \Lambda_{M/[M,M]}$, the point $(c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}(\theta\cdot c), \overline{\sigma}) \in {}_{\infty}\overline{\mathrm{Bun}}_P$ maps to $(c, \mathcal{P}_G, \mathcal{P}_{M/M_S}(p(\theta)\cdot c), \sigma) \in {}^{\star}Z_{\mathrm{can}}$ under $\mathfrak{p}$. Therefore to prove the proposition, it suffices to show that the restriction of $\mathfrak{p}$ to the canonical embedding $\mathrm{Bun}_P \subset {}_{\infty}\overline{\mathrm{Bun}}_P$ is an embedding with image the canonical embedding $\mathrm{Bun}_P \subset {}^{\star}Z_{\mathrm{can}}$, and its restriction to ${}_{\infty}\overline{\mathrm{Bun}}_P^{\leq 0}$ is a finite map with image ${}^{\star}Z_{\mathrm{can}}^{\leq 0}$. The first assertion is immediate from the definitions. To prove the second, recall that by [BG, Proposition 1.3.6], ${}_{\infty}\overline{\mathrm{Bun}}_P$ is proper over Bun_G, and so the map $\mathfrak{p}$ is proper since it respects the projection to Bun_G. Therefore it suffices to check that the fibers over closed points of the restriction of $\mathfrak{p}$ to ${}_{\infty}\overline{\mathrm{Bun}}_P^{\leq 0}$ are finite.
Let Θ be a pair $(0, \mathfrak{U}(\theta^{\mathrm{pos}}))$, with $\theta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}}$. The stack ${}_{\infty}\overline{\mathrm{Bun}}_P^{\Theta}$ classifies data

$$(c, \mathcal{P}_P, c_{\Theta}, \mathcal{P}_{M/[M,M]})$$

together with an isomorphism

$$\alpha : \mathcal{P}_P \overset{P}{\times} P/[P,P] \simeq \mathcal{P}_{M/[M,M]}(c_{\Theta}).$$

The fiber of $\mathfrak{p}$ through such a point classifies data

$$(\mathcal{P}_P, c'_{\Theta'}, \mathcal{P}'_{M/[M,M]})$$

together with an isomorphism

$$\alpha' : \mathcal{P}_P \overset{P}{\times} P/[P,P] \simeq \mathcal{P}'_{M/[M,M]}(c'_{\Theta'})$$

such that the labelling $c_{\Phi} = c_{\Theta} - c'_{\Theta'}$ takes values in $\Lambda_{M_S^0/[M_S,M_S]}$. Therefore we need only check that for $\theta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}}$, there are only a finite number of $\phi \in \Lambda_{M_S^0/[M_S,M_S]}$ such that $\theta^{\mathrm{pos}} + \phi \in \Lambda_{G,P}^{\mathrm{pos}}$. By Proposition 2.1.3, the lattice $\Lambda_{M_S^0/[M_S,M_S]}$ intersects the semigroup $\Lambda_{G,P}^{\mathrm{pos}}$ only at 0. Since $\Lambda_{G,P}^{\mathrm{pos}}$ is finitely-generated, this implies that for $\theta^{\mathrm{pos}} \in \Lambda_{M/[M,M]}$, the coset $\theta^{\mathrm{pos}} + \Lambda_{M_S^0/[M_S,M_S]}$ intersects $\Lambda_{G,P}^{\mathrm{pos}}$ in a finite set. □

Corollary 4.2.2. *The map* $\mathfrak{p} : \overline{Z}_{\mathrm{can}} \to Z_{\mathrm{can}}$ *is ind-finite.*
For $\theta \in \Lambda_{M/[M,M]}$, *its restriction to* $\overline{Z}^{\theta}_{\mathrm{can}}$ *is an embedding with image* $Z^{p(\theta)}_{\mathrm{can}}$, *and its restriction to* $\overline{Z}^{\leq\theta}_{\mathrm{can}}$ *is finite with image* $Z^{\leq p(\theta)}_{\mathrm{can}}$.

4.3. The map $\mathfrak{s} : Z_{\mathrm{can}} \to Z$. Let X be an affine horospherical variety with dense G-orbit $\mathring{X} \subset X$ and generic stabilizer $S \subset G$.
Associated to the natural map $\overline{G/S} \to X$, we have a Cartesian diagram of ind-stacks

$$\begin{array}{ccc} Z_{\mathrm{can}} & \to & {}^{\star}Z_{\mathrm{can}} \\ \mathfrak{s}\downarrow & & \downarrow\mathfrak{s} \\ Z & \to & {}^{\star}Z. \end{array}$$

We would like to describe some properties of the vertical maps.

Proposition 4.3.1. *The map* $\mathfrak{s} : {}^{\star}Z_{\mathrm{can}} \to {}^{\star}Z$ *is a closed embedding.*
For $\kappa \in \Lambda_A$, *its restriction to* ${}^{\star}Z^{\kappa}_{\mathrm{can}}$ *is an embedding with image* ${}^{\star}Z^{\kappa}$, *and its restriction to* ${}^{\star}Z^{\leq\kappa}_{\mathrm{can}}$ *is a closed embedding with image* ${}^{\star}Z^{\leq\kappa}$.

Proof. First note that $\mathfrak{s}$ is injective on scheme-valued points since for $(c, \mathcal{P}_G, \mathcal{P}_{M/M_S}\sigma) \in {}^{\star}Z_{\mathrm{can}}$, the map

$$\sigma : \mathcal{P}_{M/M_S}|_{C\setminus c} \to \mathcal{P}_G \overset{G}{\times} \overline{G/S}|_{C\setminus c}$$

factors

$$\sigma|_{C'} : \mathcal{P}_{M/M_S}|_{C'} \to \mathcal{P}_G \overset{G}{\times} G/S|_{C'} \to \mathcal{P}_G \overset{G}{\times} \overline{G/S}|_{C'},$$

for some open curve $C' \subset C \setminus c$, and the map $\overline{G/S} \to X$ restricted to G/S is an embedding.
Now to see $\mathfrak{s}$ is a closed embedding, it suffices to check that $\mathfrak{s}$ satisfies the valuative criterion of properness. Let $D = \operatorname{Spec}\mathbb{C}[[t]]$ be the disk, and $D^\times = \operatorname{Spec}\mathbb{C}((t))$ the punctured disk. Let $f : D \to Z$ be a map with a partial lift $F^\times : D^\times \to Z_{\mathrm{can}}$. Let $\mathcal{P}^f_G$ be the D-family of G-bundles defined by f, and let $\mathcal{P}^f_{M/M_S}$ be the D-family of M/M_S-bundles defined by f. We must check that any partial lift

$$\Sigma^\times : \mathcal{P}^f_{M/M_S}|_{(C\setminus c)\times D^\times} \to \mathcal{P}^f_G \overset{G}{\times} \overline{G/S}|_{(C\setminus c)\times D^\times}$$

of a map

$$\sigma : \mathcal{P}^f_{M/M_S}|_{(C\setminus c)\times D} \to \mathcal{P}^f_G \overset{G}{\times} X|_{(C\setminus c)\times D}$$

which factors

$$\sigma|_{C'\times D} : \mathcal{P}^f_{M/M_S}|_{C'\times D} \to \mathcal{P}^f_G \overset{G}{\times} G/S|_{C'\times D} \to \mathcal{P}^f_G \overset{G}{\times} X|_{C'\times D},$$

for some open curve $C' \subset C \setminus c$, extends to $(C \setminus c) \times D$. Since $\overline{G/S} \to X$ restricted to G/S is an embedding with image G/S, we may lift $\sigma|_{C'\times D}$ to extend $\Sigma^\times$ to $C' \times D$. But then $\Sigma^\times$ extends completely since $\mathcal{P}^f_{M/M_S}|_{(C\setminus c)\times D}$ is normal and the complement of $\mathcal{P}^f_{M/M_S}|_{C'\times D}$ is of codimension 2.

Finally, for a point $(c, \mathcal{P}_G, \mathcal{P}_{M/M_S}, \sigma_{\mathrm{can}}) \in {}^\star Z_{\mathrm{can}}$, we write $(c, \mathcal{P}_G, \mathcal{P}_{M/M_S}, \sigma) \in {}^\star Z$ for its image under $\mathfrak{s}$. Observe that for $\kappa \in \Lambda_A$, the point $(c, \mathcal{P}_G, \mathcal{P}_{M/M_S}(\kappa \cdot c), \sigma_{\mathrm{can}}) \in {}^\star Z_{\mathrm{can}}$ maps to $(c, \mathcal{P}_G, \mathcal{P}_{M/M_S}(\kappa \cdot c), \sigma) \in {}^\star Z$ under $\mathfrak{s}$. Therefore to complete the proof of the proposition, it suffices to show that the restriction of $\mathfrak{s}$ to the canonical embedding $\mathrm{Bun}_S \times C \subset {}^\star Z_{\mathrm{can}}$ has image the canonical embedding $\mathrm{Bun}_S \times C \subset {}^\star Z$. This is immediate from the definitions. □

COROLLARY 4.3.2. *The map $\mathfrak{s} : Z_{\mathrm{can}} \to Z$ is a closed embedding.*
For $\kappa \in \Lambda_A$, its restriction to Z^κ_{can} is an embedding with image Z^κ, and its restriction to $Z^{\leq\kappa}_{\mathrm{can}}$ is a closed embedding with image $Z^{\leq\kappa}$.

5. CONVOLUTION

Let X be an affine horospherical G-variety with dense G-orbit $\overset{\circ}{X} \subset X$ and generic stabilizer $S \subset G$.

The following diagram summarizes the ind-stacks and maps under consideration

$$\begin{array}{ccccccc}
{}_\infty\widetilde{\mathrm{Bun}}_P & \xrightarrow{\mathfrak{r}} & {}_\infty\overline{\mathrm{Bun}}_P & \xrightarrow{\mathfrak{p}} & {}^\star Z_{\mathrm{can}} & & \\
 & & \uparrow \mathfrak{k} & & \uparrow \mathfrak{k} & & \\
 & & \overline{Z}_{\mathrm{can}} & \xrightarrow{\mathfrak{p}} & Z_{\mathrm{can}} & \xrightarrow{\mathfrak{s}} & Z.
\end{array}$$

Each of the ind-stacks of the diagram projects to $C \times \mathrm{Bun}_G$, and the maps of the diagram commute with the projections.

Let $\mathcal{Z}$ be any one of the ind-stacks from the diagram, and form the diagram

$$\begin{array}{ccccc}
\mathcal{Z} & \xleftarrow{h^{\leftarrow}_G} & \mathcal{H}_G \underset{Bun_G \times C}{\times} \mathcal{Z} & \xrightarrow{h^{\rightarrow}_G} & \mathcal{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_G & \xleftarrow{h^{\leftarrow}_G} & \mathcal{H}_G & \xrightarrow{h^{\rightarrow}_G} & \mathrm{Bun}_G
\end{array}$$

in which each square is Cartesian.

For $\lambda \in \Lambda^+_G$, we define the convolution functor

$$H^\lambda_G : \mathrm{Sh}(\mathcal{Z}) \to \mathrm{Sh}(\mathcal{Z})$$

on an object $\mathcal{F} \in \mathrm{Sh}(\mathcal{Z})$ to be

$$H^\lambda_G(\mathcal{F}) = h^{\leftarrow}_{G\,!}(\mathcal{A}^\lambda_G \widetilde{\boxtimes} \mathcal{F})^r$$

where $(\mathcal{A}^\lambda_G \widetilde{\boxtimes} \mathcal{F})^r$ is the twisted product defined with respect to $h^{\rightarrow}_G$, and $\mathcal{A}^\lambda_G$ is the simple spherical sheaf on the fibers of $h^{\rightarrow}_G$ corresponding to λ. (See Section 1.4 for more on the twisted product and spherical sheaf.)

5.1. CONVOLUTION ON ${}_\infty\widetilde{\mathrm{Bun}}_P$.

Recall that for a reductive group H, and $\lambda \in \Lambda^+_H$, we write $V^\lambda_{\check{H}}$ for the irreducible representation of the dual group $\check{H}$ of highest weight λ.

We shall deduce our results from the following.

THEOREM 5.1.1. [BG, Theorem 4.1.5]. *For $\lambda \in \Lambda_G^+$, there is a canonical isomorphism*

$$H_G^\lambda(\mathrm{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P}) \simeq \sum_{\mu\in\Lambda_M^+} \mathrm{IC}^{\leq\mu}_{{}_\infty\widetilde{\mathrm{Bun}}_P} \otimes \mathrm{Hom}_{\check{M}}(V_{\check{M}}^\mu, V_{\check{G}}^\lambda).$$

5.2. CONVOLUTION ON ${}_\infty\overline{\mathrm{Bun}}_P$. Recall that $r : \Lambda_M \to \Lambda_{M/[M,M]}$ denotes the natural projection, $2\check{\rho}_M$ the sum of the positive roots of M, and $\langle 2\check{\rho}_M, \mu\rangle$ the natural pairing, for $\mu \in \Lambda_M$.

THEOREM 5.2.1. *For $\lambda \in \Lambda_G^+$, there is an isomorphism*

$$H_G^\lambda(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P}) \simeq \sum_{\theta\in\Lambda_{M/[M,M]}} \sum_{\mu\in\Lambda_M, r(\mu)=\theta} \mathrm{IC}^{\leq\theta}_{{}_\infty\overline{\mathrm{Bun}}_P} \otimes \mathrm{Hom}_{\check{T}}(V_{\check{T}}^\mu, V_{\check{G}}^\lambda)[\langle 2\check{\rho}_M, \mu\rangle].$$

Proof. Step 1. For the projection

$$\mathfrak{r} : {}_\infty\widetilde{\mathrm{Bun}}_P \to {}_\infty\overline{\mathrm{Bun}}_P,$$

we clearly have

$$H_G^\lambda(\mathfrak{r}_! \, \mathrm{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P}) \simeq \mathfrak{r}_! H_G^\lambda(\mathrm{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P}). \tag{1}$$

Let us first analyze the left hand side of equation 1. We may write the push-forward $\mathfrak{r}_! \, \mathrm{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P}$ in the form

$$\mathfrak{r}_! \, \mathrm{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P} \simeq \mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P} \oplus \mathcal{J}^{\leq 0}$$

where $\mathcal{J}^{\leq 0} \in \mathrm{Sh}({}_\infty\overline{\mathrm{Bun}}_P)$ is isomorphic to a direct sum of shifts of sheaves of the form

$$\mathrm{IC}^{\leq\Theta}_{{}_\infty\overline{\mathrm{Bun}}_P}, \text{ for pairs } \Theta = (0, \mathfrak{U}(\theta^{\mathrm{pos}})), \text{ with } \theta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}} \setminus \{0\}.$$

The asserted form of $\mathcal{J}^{\leq 0}$ follows from the Decomposition Theorem, the fact that the restrictions of $\mathrm{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P}$ to the strata of ${}_\infty\widetilde{\mathrm{Bun}}_P$ are constant [BFGM, Theorem 1.12], and the structure of the map $\mathfrak{r}$ described in Proposition 4.1.1. For any $\eta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}} \setminus \{0\}$, and decomposition $\mathfrak{U}(\eta^{\mathrm{pos}})$, we have the finite map

$$\tau_{\mathfrak{U}(\eta^{\mathrm{pos}})} : C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \times {}_\infty\overline{\mathrm{Bun}}_P \to {}_\infty\overline{\mathrm{Bun}}_P$$

defined by

$$\tau_{\mathfrak{U}(\eta^{\mathrm{pos}})}(\sum_{m,n} \eta_m^{\mathrm{pos}} \cdot c_{m,n}, (c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}, \sigma))$$
$$= (c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}(-\sum_{m,n} \eta_m^{\mathrm{pos}} \cdot c_{m,n}), \sigma).$$

Note that for $\eta \in \Lambda_{M/[M,M]}$, and Θ the pair $(\eta, \mathfrak{U}(\eta^{\mathrm{pos}}))$, the restriction of $\tau_{\mathfrak{U}(\eta^{\mathrm{pos}})}$ provides an isomorphism

$$\tau_{\mathfrak{U}(\eta^{\mathrm{pos}})} : (C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \times {}_\infty\overline{\mathrm{Bun}}_P^\eta)_0 \xrightarrow{\sim} {}_\infty\overline{\mathrm{Bun}}_P^\Theta$$

where the domain completes the Cartesian square

$$\begin{array}{ccc} (C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \times {}_\infty\overline{\mathrm{Bun}}_P^\eta)_0 & \to & C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \times {}_\infty\overline{\mathrm{Bun}}_P^\eta \\ \downarrow & & \downarrow \\ (C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \times C)_0 & \to & C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \times C \end{array}$$

where as usual

$$(C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \times C)_0 \subset C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \times C$$

denotes the complement to the diagonal divisor.

We define the strict full triangulated subcategory of irrelevant sheaves

$$\mathrm{IrrelSh}({}_\infty\overline{\mathrm{Bun}}_P) \subset \mathrm{Sh}({}_\infty\overline{\mathrm{Bun}}_P)$$

to be that generated by sheaves of the form

$$\tau_{\mathfrak{U}(\eta^{\mathrm{pos}})!}(\mathrm{IC}_C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \boxtimes \mathcal{F})$$

where η^{pos} runs through $\Lambda_{G,P}^{\mathrm{pos}} \setminus \{0\}$, $\mathfrak{U}(\eta^{\mathrm{pos}})$ runs through decompositions of η^{pos}, $\mathrm{IC}_C^{\mathfrak{U}(\eta^{\mathrm{pos}})}$ denotes the intersection cohomology sheaf of $C^{\mathfrak{U}(\eta^{\mathrm{pos}})}$, and $\mathcal{F}$ runs through objects of $\mathrm{Sh}({}_\infty\overline{\mathrm{Bun}}_P)$.

Lemma 5.2.2. *The sheaf $\mathcal{I}^{\leq 0}$ is irrelevant.*

Proof. Let Θ be a pair $(\theta, \mathfrak{U}(\theta^{\mathrm{pos}}))$, with $\theta \in \Lambda_{M/[M,M]}$, and $\theta^{\mathrm{pos}} \in \Lambda_{G,P}^{\mathrm{pos}} \setminus \{0\}$. Then we may realize the sheaf $\mathrm{IC}^{\leq\Theta}_{{}_\infty\overline{\mathrm{Bun}}_P}$ as the pushforward

$$\mathrm{IC}^{\leq\Theta}_{{}_\infty\overline{\mathrm{Bun}}_P} \simeq \tau_{\mathfrak{U}(\theta^{\mathrm{pos}})!}(\mathrm{IC}_C^{\mathfrak{U}(\theta^{\mathrm{pos}})} \boxtimes \mathrm{IC}^\theta_{{}_\infty\overline{\mathrm{Bun}}_P})$$

To see this, we use the isomorphism

$$\tau_{\mathfrak{U}(\theta^{\mathrm{pos}})} : (C^{\mathfrak{U}(\theta^{\mathrm{pos}})} \times {}_\infty\overline{\mathrm{Bun}}_P^\theta)_0 \xrightarrow{\sim} {}_\infty\overline{\mathrm{Bun}}_P^\Theta,$$

and the fact that $\tau_{\mathfrak{U}(\theta^{\mathrm{pos}})}$ is finite. □

Lemma 5.2.3. *If $\mathcal{E}$ is an irrelevant sheaf, then $H_G^\lambda(\mathcal{E})$ is an irrelevant sheaf.*

Proof. Clearly we have a canonical isomorphism

$$H_G^\lambda(\tau_{\mathfrak{U}(\eta^{\mathrm{pos}})!}(\mathrm{IC}_C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \boxtimes \mathcal{F})) \simeq \tau_{\mathfrak{U}(\eta^{\mathrm{pos}})!}(\mathrm{IC}_C^{\mathfrak{U}(\eta^{\mathrm{pos}})} \boxtimes H_G^\lambda(\mathcal{F})).$$

□

By the preceding lemmas, we may write the left hand side of equation 1 in the form

$$H_G^\lambda(\mathfrak{r}_! \,\mathrm{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P}) \simeq H_G^\lambda(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P}) \oplus H_G^\lambda(\mathcal{I}^{\leq 0}) \tag{2}$$

where $H_G^\lambda(\mathcal{I}^{\leq 0})$ is an irrelevant sheaf.

Let us next analyze the right hand side of equation 1. By Theorem 5.1.1, we have

$$\mathfrak{r}_! H_G^\lambda(\mathrm{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P}) \simeq \sum_{\mu \in \Lambda_M^+} \mathfrak{r}_! \,\mathrm{IC}^{\leq\mu}_{{}_\infty\widetilde{\mathrm{Bun}}_P} \otimes \mathrm{Hom}_{\check{M}}(V_{\check{M}}^\mu, V_{\check{G}}^\lambda).$$

Lemma 5.2.4. *For $\mu \in \Lambda_M^+$, we have*

$$\mathfrak{r}_! \operatorname{IC}^{\leq \mu}_{{}_\infty\widetilde{\mathrm{Bun}}_P} \simeq \sum_{\nu \in \Lambda_M} (\operatorname{IC}^{\leq r(\mu)}_{{}_\infty\overline{\mathrm{Bun}}_P} \oplus \mathcal{I}^{\leq \mu}) \otimes \operatorname{Hom}_{\check{T}}(V^{\nu}_{\check{T}}, V^{\mu}_{\check{M}})[\langle 2\check{\rho}_M, \nu\rangle].$$

where $\mathcal{I}^{\leq \mu}$ is isomorphic to a direct sum of shifts of sheaves of the form

$$\operatorname{IC}^{\leq \Theta}_{{}_\infty\overline{\mathrm{Bun}}_P}, \textit{ for pairs } \Theta = (\theta, \mathfrak{U}(\theta^{\mathrm{pos}})).$$

Proof. We may form the diagram

$$\begin{tikzcd} {}_\infty\widetilde{\mathrm{Bun}}_P \arrow[d] & \mathcal{H}_M \underset{\mathrm{Bun}_M \times C}{\times} {}_\infty\widetilde{\mathrm{Bun}}_P \arrow[l, "h^{\leftarrow}_M"'] \arrow[r, "h^{\rightarrow}_M"] \arrow[d] & {}_\infty\widetilde{\mathrm{Bun}}_P \arrow[d] \\ \mathrm{Bun}_G & \mathcal{H}_G \arrow[l, "h^{\leftarrow}_M"'] \arrow[r, "h^{\rightarrow}_M"] & \mathrm{Bun}_G \end{tikzcd}$$

in which each square is Cartesian. We define the convolution functor

$$H^{\mu}_M : \mathrm{Sh}({}_\infty\widetilde{\mathrm{Bun}}_P) \to \mathrm{Sh}({}_\infty\widetilde{\mathrm{Bun}}_P)$$

on an object $\mathcal{F} \in \mathrm{Sh}({}_\infty\widetilde{\mathrm{Bun}}_P)$ to be

$$H^{\mu}_M(\mathcal{F}) = h^{\leftarrow}_{M!}(\mathcal{A}^{\mu}_M \widetilde{\boxtimes} \mathcal{F})^r$$

where $(\mathcal{A}^{\mu}_M \widetilde{\boxtimes} \mathcal{F})^r$ is the twisted product defined with respect to $h^{\rightarrow}_M$, and $\mathcal{A}^{\mu}_M$ is the simple spherical sheaf on the fibers of $h^{\rightarrow}_M$ corresponding to μ. Theorem 4.1.3 of [BG] provides a canonical isomorphism

$$H^{\mu}_M(\operatorname{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P}) \simeq \operatorname{IC}^{\leq \mu}_{{}_\infty\widetilde{\mathrm{Bun}}_P}.$$

We also have a commutative diagram

$$\begin{tikzcd} {}_\infty\widetilde{\mathrm{Bun}}_P \arrow[d, "\mathfrak{r}"'] & \mathcal{H}_M \underset{\mathrm{Bun}_M \times C}{\times} {}_\infty\widetilde{\mathrm{Bun}}_P \arrow[l, "h^{\leftarrow}_M"'] \arrow[d, "\mathfrak{r}'"] \\ {}_\infty\overline{\mathrm{Bun}}_P & \mathcal{H}_{M/[M,M]} \underset{\mathrm{Bun}_{M/[M,M]} \times C}{\times} {}_\infty\overline{\mathrm{Bun}}_P \arrow[l, "h^{\leftarrow}_{M/[M,M]}"'] \end{tikzcd}$$

where the modification map $h^{\leftarrow}_{M/[M,M]}$ is given by

$$h^{\leftarrow}_{M/[M,M]}(\theta, (c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}, \sigma)) = (c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}(-\theta \cdot c), \sigma).$$

We conclude that there is an isomorphism

$$\mathfrak{r}_! \operatorname{IC}^{\leq \mu}_{{}_\infty\widetilde{\mathrm{Bun}}_P} \simeq h^{\leftarrow}_{M/[M,M]!} \mathfrak{r}'_!(\mathcal{A}^{\mu}_M \widetilde{\boxtimes} \operatorname{IC}^{\leq 0}_{{}_\infty\widetilde{\mathrm{Bun}}_P})^r.$$

Now the map $\mathfrak{r}'$ factors into the projection of the left hand factor

$$\mathcal{H}_M \underset{\mathrm{Bun}_M \times C}{\times} {}_\infty\widetilde{\mathrm{Bun}}_P \to \mathcal{H}_{M/[M,M]} \underset{\mathrm{Bun}_{M/[M,M]} \times C}{\times} {}_\infty\widetilde{\mathrm{Bun}}_P$$

followed by the projection of the right hand factor

$$\mathcal{H}_{M/[M,M]} \underset{\mathrm{Bun}_{M/[M,M]} \times C}{\times} {}_\infty\widetilde{\mathrm{Bun}}_P \xrightarrow{\mathfrak{r}} \mathcal{H}_{M/[M,M]} \underset{\mathrm{Bun}_{M/[M,M]} \times C}{\times} {}_\infty\overline{\mathrm{Bun}}_P.$$

Thus we have an isomorphism

$$\mathfrak{r}'_!(\mathcal{A}^\mu_M \widetilde{\boxtimes} \mathrm{IC}^{\leq 0}_{\infty \widetilde{\mathrm{Bun}}_P})^r \simeq \sum_{\nu \in \Lambda_M} (\mathrm{IC}^{\leq 0}_{\infty \overline{\mathrm{Bun}}_P} \oplus \mathcal{I}^{\leq 0}) \otimes \mathrm{Hom}_{\check{T}}(V^\nu_{\check{T}}, V^\mu_{\check{M}})[\langle 2\check{\rho}_M, \nu\rangle]$$

where as before

$$\mathfrak{r}_! \, \mathrm{IC}^{\leq 0}_{\infty \widetilde{\mathrm{Bun}}_P} \simeq \mathrm{IC}^{\leq 0}_{\infty \overline{\mathrm{Bun}}_P} \oplus \mathcal{I}^{\leq 0}$$

where $\mathcal{I}^{\leq 0}$ is isomorphic to a direct sum of shifts of sheaves of the form

$$\mathrm{IC}^{\leq \Theta}_{\infty \overline{\mathrm{Bun}}_P}, \text{ for pairs } \Theta = (0, \mathfrak{U}(\theta^{\mathrm{pos}})), \text{ with } \theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}_{G,P} \setminus \{0\}.$$

Finally, applying the modification $h^{\leftarrow}_{M/[M,M]!}$ with twist $r(\mu)$ to the above isomorphism, we obtain an isomorphism

$$\mathfrak{r}_! \, \mathrm{IC}^{\leq \mu}_{\infty \widetilde{\mathrm{Bun}}_P} \simeq \sum_{\nu \in \Lambda_M} (\mathrm{IC}^{\leq r(\mu)}_{\infty \overline{\mathrm{Bun}}_P} \oplus \mathcal{I}^{\leq \mu}) \otimes \mathrm{Hom}_{\check{T}}(V^\nu_{\check{T}}, V^\mu_{\check{M}})[\langle 2\check{\rho}_M, \nu\rangle].$$

Here we write $\mathcal{I}^{\leq \mu}$ for the result of applying the modification $h^{\leftarrow}_{M/[M,M]!}$ with twist $r(\mu)$ to $\mathcal{I}^{\leq 0}$. Clearly the modification $h^{\leftarrow}_{M/[M,M]!}$ takes strata to strata so we conclude that $\mathcal{I}^{\leq \mu}$ is isomorphic to a direct sum of shifts of sheaves of the form

$$\mathrm{IC}^{\leq \Theta}_{\infty \overline{\mathrm{Bun}}_P}, \text{ for pairs } \Theta = (\theta, \mathfrak{U}(\theta^{\mathrm{pos}})).$$

□

Note that the proof actually shows that $\mathcal{I}^{\leq \mu}$ is isomorphic to a direct sum of shifts of sheaves of the form

$$\mathrm{IC}^{\leq \Theta}_{\infty \overline{\mathrm{Bun}}_P}, \text{ for pairs } \Theta = (0, \mathfrak{U}(\theta^{\mathrm{pos}})), \text{ with } \theta^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}_{G,P} \setminus \{0\},$$

and so in particular is irrelevant, but we shall have no need for this.
Combining the formulas given by Theorem 5.1.1 and the preceding lemma, we may write the right hand side of equation 1 in the form
(3)

$$\mathfrak{r}_! H^\lambda_G(\mathrm{IC}^{\leq 0}_{\infty \widetilde{\mathrm{Bun}}_P}) \simeq \sum_{\theta \in \Lambda_{M/[M,M]}} \sum_{\mu \in \Lambda_M, r(\mu) = \theta} \mathrm{IC}^{\leq \theta}_{\infty \overline{\mathrm{Bun}}_P} \otimes \mathrm{Hom}_{\check{T}}(V^\mu_{\check{T}}, V^\lambda_{\check{G}})[\langle 2\check{\rho}_M, \mu\rangle] \oplus \mathcal{J}$$

where $\mathcal{J}$ is isomorphic to a direct sum of shifts of sheaves of the form

$$\mathrm{IC}^{\leq \Theta}_{\infty \overline{\mathrm{Bun}}_P}, \text{ for pairs } \Theta = (\theta, \mathfrak{U}(\theta^{\mathrm{pos}})).$$

Finally, comparing the left hand side (equation 2) and the right hand side (equation 3), and noting that $\mathrm{IC}^{\leq \theta}_{\infty \overline{\mathrm{Bun}}_P}$ is not irrelevant, we conclude that

$$H^\lambda_G(\mathrm{IC}^{\leq 0}_{\infty \overline{\mathrm{Bun}}_P}) \simeq \sum_{\theta \in \Lambda_{M/[M,M]}} \sum_{\mu \in \Lambda_M, r(\mu) = \theta} \mathrm{IC}^{\leq \theta}_{\infty \overline{\mathrm{Bun}}_P} \otimes \mathrm{Hom}_{\check{T}}(V^\mu_{\check{T}}, V^\lambda_{\check{G}})[\langle 2\check{\rho}_M, \mu\rangle] \oplus \mathcal{M}$$

where $\mathcal{M}$ is is isomorphic to a direct sum of shifts of sheaves of the form

$$\mathrm{IC}^{\leq \Theta}_{\infty \overline{\mathrm{Bun}}_P}, \text{ for pairs } \Theta = (\theta, \mathfrak{U}(\theta^{\mathrm{pos}})).$$

Step 2. Now we shall show that $\mathcal{M}$ is in fact zero. To do this, we shall show that its restriction to each stratum of ${}_\infty\overline{\mathrm{Bun}}_P$ is zero.

Let Φ be a pair $(\phi, \mathfrak{U}(\phi^{\mathrm{pos}}))$, with $\phi \in \Lambda_{M/[M,M]}$, and $\phi^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}_{G,P}$. Let $H^\lambda_G(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P})_\Phi$ be the restriction of $H^\lambda_G(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P})$ to the stratum ${}_\infty\overline{\mathrm{Bun}}^\Phi_P$. For $\theta \in \Lambda_{M/[M,M]}$, let $\mathcal{A}^\theta_\Phi$ be the restriction of $\mathrm{IC}^{\leq\theta}_{{}_\infty\overline{\mathrm{Bun}}_P}$ to the stratum ${}_\infty\overline{\mathrm{Bun}}^\Phi_P$, and let $\mathcal{M}_\Phi$ be the restriction of $\mathcal{M}$. Note that by step 1, [BFGM, Theorem 7.3] and Lemma 5.2.5 below, all of the restrictions are locally constant.

We shall calculate $H^\lambda_G(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P})_\Phi$ in two different ways and compare the results.

On the one hand, by Step 1, we have

(4)

$$H^\lambda_G(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P})_\Phi \simeq \sum_{\theta\in\Lambda_{M/[M,M]}} \sum_{\mu\in\Lambda_M, r(\mu)=\theta} \mathcal{A}^\theta_\Phi \otimes \mathrm{Hom}_{\check{T}}(V^\mu_{\check{T}}, V^\lambda_{\check{G}})[\langle 2\check{\rho}_M, \mu\rangle] \oplus \mathcal{M}_\Phi$$

On the other hand, let us return to the definition of the convolution, and consider the diagram

$$\begin{array}{ccccc}
{}_\infty\overline{\mathrm{Bun}}_P & \overset{h^{\leftarrow}_G}{\leftarrow} & \mathcal{H}_G \underset{\mathrm{Bun}_G\times C}{\times} {}_\infty\overline{\mathrm{Bun}}^{\leq 0}_P & \overset{h^{\rightarrow}_G}{\rightarrow} & {}_\infty\overline{\mathrm{Bun}}^{\leq 0}_P \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Bun}_G & \overset{h^{\leftarrow}_G}{\leftarrow} & \mathcal{H}_G & \overset{h^{\rightarrow}_G}{\rightarrow} & \mathrm{Bun}_G
\end{array}$$

Recall that by definition

$$H^\lambda_G(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P}) = h^{\leftarrow}_{G\,!}(\mathcal{A}^\lambda_G \widetilde{\boxtimes}\, \mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P})^r$$

where $(\mathcal{A}^\lambda_G \widetilde{\boxtimes}\, \mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P})^r$ is the twisted product defined with respect to $h^{\rightarrow}_G$, and $\mathcal{A}^\lambda_G$ is the simple spherical sheaf on the fibers of $h^{\rightarrow}_G$ corresponding to λ.

To calculate $H^\lambda_G(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P})_\Phi$, consider the inverse image $h^{\leftarrow\,-1}_G({}_\infty\overline{\mathrm{Bun}}^\Phi_P)$. Projecting along $h^{\rightarrow}_G$, we may decompose the inverse image into a union of locally closed substacks

$$h^{\leftarrow\,-1}_G({}_\infty\overline{\mathrm{Bun}}^\Phi_P) \simeq \bigsqcup_{\xi\in R^{\mathrm{pos}}_{G,P}} \mathcal{S}^\lambda_{P,\phi-\xi} \underset{\mathrm{Bun}_P}{\times} {}_\infty\overline{\mathrm{Bun}}^{(\xi,\mathfrak{U}(\phi^{\mathrm{pos}}))}_P.$$

Projecting each piece back along $h^{\leftarrow}_G$, we arrive at a spectral sequence for $H^\lambda_G(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P})_\Phi$ with E_2 term

$$\sum_{\xi\in R^{\mathrm{pos}}_{G,P}} \sum_{\mu\in\Lambda_M, r(\mu)=\phi-\xi} \mathcal{A}^0_{(\xi,\mathfrak{U}(\phi^{\mathrm{pos}}))} \otimes \mathrm{Hom}_{\check{T}}(V^\mu_{\check{T}}, V^\lambda_{\check{G}})[\langle 2\check{\rho}_M, \mu\rangle]$$

In fact, the spectral sequence degenerates here for reasons of parity, but we shall not need this. What we do need is the following cyclicity.

Lemma 5.2.5. *Let Ψ be a pair $(\psi, \mathfrak{U}(\psi^{\mathrm{pos}}))$, with $\psi \in \Lambda_{M/[M,M]}$, and $\psi^{\mathrm{pos}} \in \Lambda^{\mathrm{pos}}_{G,P}$. Let $\theta \in \Lambda_{M/[M,M]}$. Then $\mathcal{A}^0_{(\psi,\mathfrak{U}(\psi^{\mathrm{pos}}))} \simeq \mathcal{A}^\theta_{(\psi+\theta,\mathfrak{U}(\psi^{\mathrm{pos}}))}$.*

Proof. The modification

$$(c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}, \sigma) \mapsto (c, \mathcal{P}_G, \mathcal{P}_{M/[M,M]}(\theta \cdot c), \sigma).$$

defines an isomorphism ${}_\infty\overline{\mathrm{Bun}}_P \xrightarrow{\sim} {}_\infty\overline{\mathrm{Bun}}_P$ which restricts to an isomorphism

$${}_\infty\overline{\mathrm{Bun}}_P^{(\psi, \mathfrak{U}(\psi^{\mathrm{pos}}))} \xrightarrow{\sim} {}_\infty\overline{\mathrm{Bun}}_P^{(\psi+\theta, \mathfrak{U}(\psi^{\mathrm{pos}}))}.$$

$\square$

We apply the lemma with $\psi = \xi$, $\psi^{\mathrm{pos}} = \phi^{\mathrm{pos}}$, and make the substitution $\theta = \phi - \xi$, to write the E_2 term

$$\sum_{\phi-\theta \in R^{\mathrm{pos}}_{G,P}} \sum_{\mu \in \Lambda_M, r(\mu)=\theta} \mathcal{A}^{\theta}_{(\phi, \mathfrak{U}(\phi^{\mathrm{pos}}))} \otimes \mathrm{Hom}_{\check{T}}(V^{\mu}_{\check{T}}, V^{\lambda}_{\check{G}})[\langle 2\check{\rho}_M, \mu\rangle] \tag{5}$$

Comparing our two calculations (equations 4 and 5), we conclude by a dimension count that $\mathcal{M}_\Phi$ must be zero. $\square$

5.3. Convolution on $\overline{Z}_{\mathrm{can}}$.

Theorem 5.3.1. *For $\lambda \in \Lambda^+_G$, there is an isomorphism*

$$H^{\lambda}_G(\mathrm{IC}^{\leqq 0}_{\overline{Z}_{\mathrm{can}}}) \simeq \sum_{\theta \in \Lambda_{M/[M,M]}} \sum_{\mu \in \Lambda_M, r(\mu)=\theta} \mathrm{IC}^{\leqq \theta}_{\overline{Z}_{\mathrm{can}}} \otimes \mathrm{Hom}_{\check{T}}(V^{\mu}_{\check{T}}, V^{\lambda}_{\check{G}})[\langle 2\check{\rho}_M, \mu\rangle].$$

Proof. By Proposition 3.4.1, for $\theta \in \Lambda_{M/[M,M]}$, we have

$$\mathfrak{k}^* \,\mathrm{IC}^{\leq \theta}_{{}_\infty\overline{\mathrm{Bun}}_P} \simeq \mathrm{IC}^{\leqq \theta}_{\overline{Z}_{\mathrm{can}}},$$

Clearly the pullback $\mathfrak{k}^*$ commutes with convolution

$$H^{\lambda}_G(\mathfrak{k}^* \,\mathrm{IC}^{\leq \theta}_{{}_\infty\overline{\mathrm{Bun}}_P}) \simeq \mathfrak{k}^* H^{\lambda}_G(\mathrm{IC}^{\leq \theta}_{{}_\infty\overline{\mathrm{Bun}}_P}).$$

Thus by Theorem 5.2.1, we conclude

$$\begin{aligned}
&H^{\lambda}_G(\mathrm{IC}^{\leqq 0}_{\overline{Z}_{\mathrm{can}}}) \\
&\simeq H^{\lambda}_G(\mathfrak{k}^* \,\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P}) \\
&\simeq \mathfrak{k}^* H^{\lambda}_G(\mathrm{IC}^{\leq 0}_{{}_\infty\overline{\mathrm{Bun}}_P}) \\
&\simeq \sum_{\theta \in \Lambda_{M/[M,M]}} \sum_{\mu \in \Lambda_M, r(\mu)=\theta} \mathfrak{k}^* \,\mathrm{IC}^{\leq \theta}_{{}_\infty\overline{\mathrm{Bun}}_P} \otimes \mathrm{Hom}_{\check{T}}(V^{\mu}_{\check{T}}, V^{\lambda}_{\check{G}})[\langle 2\check{\rho}_M, \mu\rangle] \\
&\simeq \sum_{\theta \in \Lambda_{M/[M,M]}} \sum_{\mu \in \Lambda_M, r(\mu)=\theta} \mathrm{IC}^{\leqq \theta}_{\overline{Z}_{\mathrm{can}}} \otimes \mathrm{Hom}_{\check{T}}(V^{\mu}_{\check{T}}, V^{\lambda}_{\check{T}})[\langle 2\check{\rho}_M, \mu\rangle].
\end{aligned}$$

$\square$

5.4. Convolution on Z. Recall the map of coweight lattices

$$q : \Lambda_M \xrightarrow{r} \Lambda_{M/[M,M]} \xrightarrow{p} \Lambda_{A_0}.$$

Theorem 5.4.1. *For $\lambda \in \Lambda_G^+$, there is an isomorphism*

$$H_G^\lambda(\mathrm{IC}_{\overline{Z}}^{\leq 0}) \simeq \sum_{\kappa \in \Lambda_{A_0}} \sum_{\mu \in \Lambda_T, q(\mu)=\kappa} \mathrm{IC}_{\overline{Z}}^{\leq \kappa} \otimes \mathrm{Hom}_{\check{T}}(V_{\check{T}}^\mu, V_{\check{G}}^\lambda)[\langle 2\check{\rho}_M, \mu\rangle].$$

Proof. By Corollary 4.2.2, for $\theta \in \Lambda_{M/[M,M]}$, we have

$$\mathfrak{p}_! \mathrm{IC}_{\overline{Z}_{\mathrm{can}}}^{\leq \theta} \simeq \mathrm{IC}_{\overline{Z}_{\mathrm{can}}}^{\leq p(\theta)},$$

By Corollary 4.3.2, for $\kappa \in \Lambda_{A_0}$, we have

$$\mathfrak{s}_! \mathrm{IC}_{\overline{Z}_{\mathrm{can}}}^{\leq \kappa} \simeq \mathrm{IC}_{\overline{Z}}^{\leq \kappa}.$$

Clearly the pushforwards $\mathfrak{p}_!$ and $\mathfrak{s}_!$ commute with convolution

$$H_G^\lambda(\mathfrak{s}_!\mathfrak{p}_! \mathrm{IC}_{\overline{Z}_{\mathrm{can}}^{\leq 0}}) \simeq \mathfrak{s}_!\mathfrak{p}_! H_G^\lambda(\mathrm{IC}_{\overline{Z}_{\mathrm{can}}^{\leq 0}}).$$

Thus by Theorem 5.3.1, we conclude

$$\begin{aligned}
&H_G^\lambda(\mathrm{IC}_{\overline{Z}}^{\leq 0}) \\
&\simeq H_G^\lambda(\mathfrak{s}_!\mathfrak{p}_! \mathrm{IC}_{\overline{Z}_{\mathrm{can}}}^{\leq 0}) \\
&\simeq \mathfrak{s}_!\mathfrak{p}_! H_G^\lambda(\mathrm{IC}_{\overline{Z}_{\mathrm{can}}}^{\leq 0}) \\
&\simeq \sum_{\theta \in \Lambda_{M/[M,M]}} \sum_{\mu \in \Lambda_M, r(\mu)=\theta} \mathfrak{s}_!\mathfrak{p}_! \mathrm{IC}_{\overline{Z}_{\mathrm{can}}}^{\leq \theta} \otimes \mathrm{Hom}_{\check{T}}(V_{\check{T}}^\mu, V_{\check{G}}^\lambda)[\langle 2\check{\rho}_M, \mu\rangle] \\
&\simeq \sum_{\kappa \in \Lambda_{A_0}} \sum_{\mu \in \Lambda_T, q(\mu)=\kappa} \mathrm{IC}_{\overline{Z}}^{\leq \kappa} \otimes \mathrm{Hom}_{\check{T}}(V_{\check{T}}^\mu, V_{\check{G}}^\lambda)[\langle 2\check{\rho}_M, \mu\rangle].
\end{aligned}$$

□

6. Complements

For our application [GN], we need a slight modification of our main result. As usual, let X be an affine horospherical G-variety with dense G-orbit $\overset{\circ}{X} \subset X$ and generic stabilizer $S \subset G$. Let S^0 be the identity component of S, and let $\pi_0(S)$ be the component group S/S^0.

For a scheme $\mathcal{S}$, we write $C_{\mathcal{S}}$ for the product $\mathcal{S} \times C$. For an $\mathcal{S}$-point $(c, \mathcal{P}_G, \sigma)$ of the ind-stack Z, the section σ defines a reduction of the G-bundle $\mathcal{P}_G$ to an S-bundle $\mathcal{P}'_S$ over an open subscheme $C'_{\mathcal{S}} \subset C_{\mathcal{S}}$ which is the complement $C_{\mathcal{S}} \setminus \mathcal{D}$ of a subscheme $\mathcal{D} \subset C_{\mathcal{S}}$ which is finite and flat over $\mathcal{S}$. By induction, the S-bundle $\mathcal{P}'_S$ defines a $\pi_0(S)$-bundle over $C'_{\mathcal{S}}$. We call this the generic $\pi_0(S)$-bundle associated to the point $(c, \mathcal{P}_G, \sigma)$.

We define $'Z \subset Z$ to be the ind-substack whose $\mathcal{S}$-points $(c, \mathcal{P}_G, \sigma)$ have the property that for every geometric point $s \in \mathcal{S}$, the restriction of the associated generic $\pi_0(S)$-bundle to $\{s\} \times C \subset C_{\mathcal{S}}$ is trivial. It is not difficult (see [GN]) to show that $'Z$ is closed in Z. Observe that we have a short exact sequence

$$0 \to \Lambda_{A_0} \to \Lambda_A \to S/S^0 \to 0.$$

Thus for $\kappa \in \Lambda_{A_0}$, it makes sense to consider the locally closed substack $'Z^\kappa \subset {}'Z$ and its closure $'Z^{\leq \kappa} \subset {}'Z$. Observe as well that from the fibration $S \to G \to G/S$, we have an exact sequence

$$\pi_1(G) \to \pi_1(\overset{\circ}{X}) \to \pi_0(S).$$

Thus for $\lambda \in \Lambda_G^+$, we have the convolution functor

$$H_G^\lambda : \mathrm{Sh}('Z) \to \mathrm{Sh}('Z).$$

The same arguments show that our main result holds equally well in this context.

Theorem 6.0.2. *For $\lambda \in \Lambda_G^+$, there is an isomorphism*

$$H_G^\lambda(\mathrm{IC}_{'Z}^{\leq 0}) \simeq \sum_{\kappa \in \Lambda_{A_0}} \sum_{\mu \in \Lambda_T, q(\mu)=\kappa} \mathrm{IC}_{'Z}^{\leq \kappa} \otimes \mathrm{Hom}_{\check{T}}(V_{\check{T}}^\mu, V_{\check{G}}^\lambda)[\langle 2\check{\rho}_M, \mu\rangle].$$

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Projective Homogeneous Varieties Birational to Quadrics

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Abstract. We will consider an explicit birational map between a quadric and the projective variety $X(J)$ of traceless rank one elements in a simple reduced Jordan algebra J. $X(J)$ is a homogeneous G-variety for the automorphism group $G = \mathrm{Aut}(J)$. We will show that the birational map is a blow up followed by a blow down. This will allow us to use the blow up formula for motives together with Vishik's work on the motives of quadrics to give a motivic decomposition of $X(J)$.

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Recently Totaro has solved the birational classification problem for a large class of quadrics [To08]. In particular, let ϕ be an r-Pfister form over a field k of characteristic not 2, and $b = \langle b_1, \cdots b_n\rangle$ be a non-degenerate quadratic form with $n \geq 2$.

Proposition 0.1. *[To08, Thm. 6.3] The birational class of the quadric defined by*

$$q = \phi \otimes \langle b_1, \cdots , b_{n-1}\rangle \perp \langle b_n\rangle$$

only depends on the isometry classes of ϕ and $\phi \otimes b$, and not on the choice of diagonalization of b.

The Sarkisov program [Co94] predicts that any birational map between quadrics (in fact between any two Mori fibre spaces) factors as a chain of composites of "elementary links". In 2.16 we will explicitly factor many of Totaro's birational maps into chains of elementary links, and also prove the following theorem.

Theorem 0.2. *For $r = 0, 1, 2$ and $n \geq 3$, or $r = 3$ and $n = 3$, for each of the birational equivalences from Prop. 0.1, there is a birational map which factors into two elementary links, each of which is the blow up of a reduced subscheme followed by a blow down. Furthermore, if $r \neq 1$ or ϕ is not hyperbolic, then the intermediate Mori fibre space of this factorization will be the projective homogeneous variety $X(J)$ of traceless rank one elements in a Jordan algebra J.*

The birational map from a quadric to $X(J)$ will be the codimension 1 restriction of a birational map between projective space and the projective variety V_J of rank one elements of J, first written down by Jacobson [Ja85, 4.26].

0.3 Motivic decompositions. Let G a semisimple linear algebraic group of inner type, and X a projective homogeneous G-variety such that G splits over the function field of X, which is to say, X is generically split (see [PSZ08, 3.6] for a convenient table). Then [PSZ08] gives a direct sum decomposition of the Chow motive $\mathcal{M}(X; \mathbb{Z}/p\mathbb{Z})$ of X. They show that it is the direct sum of some Tate twists of a single indecomposable motive $\mathcal{R}_p(G)$, which generalizes the Rost motive. This work unified much of what was previously known about motivic decompositions of anisotropic projective homogeneous varieties.
In the non-generically split cases less is known. Quadrics are in general not generically split, but much is known by the work of Vishik and others, especially in low dimensions [Vi04].

Theorem 0.4. *(See Thm. 3.6) The motive of the projective quadric defined by the quadratic forms in Prop. 0.1 may be decomposed into the sum, up to Tate twists, of Rost motives and higher forms of Rost motives.*

In the present paper we will use this knowledge of motives of quadrics to produce motivic decompositions for the non-generically split projective homogeneous G-varieties $X(J)$ which appear in Thm. 0.2. The algebraic groups G are of Lie type ${}^2A_{n-1}$, C_n and F_4, and are automorphism groups of simple reduced Jordan algebras of degree ≥ 3. These varieties $X(J)$ come in four different types which we label $r = 0, 1, 2$ or 3, corresponding to the 2^r dimensional composition algebra of the simple Jordan algebra J (see Thm. 2.4 for a description of $X(J)$ as G/P for a parabolic subgroup P).

Theorem 0.5. *(See Thm. 3.12) The motive of $X(J)$ is the direct sum of a higher form of a Rost motive, F_n^r, together with several Tate twisted copies of the Rost motive R^r.*

The $r = 1$ case of this theorem provides an alternate proof of Krashen's motivic equivalence [Kr07, Thm. 3.3]. On the other hand, the $r = 1$ case of this theorem is shown in [SZ08, Thm. (C)] by using Krashen's result (See Remark 3.14).

0.6 Notational conventions. We will fix a base field k of characteristic 0 (unless stated otherwise), and an algebraically closed (equivalently, a separably closed) field extension $\bar{k}$ of k. We only use the characteristic 0 assumption to

show the varieties $X(J)$ and Z_1 are homogeneous. We will assume a *scheme* over k is a separated scheme of finite type over k, and a *variety* will be an irreducible reduced scheme.
For a scheme X over k, $\overline{X} = X \times_k \bar{k}$.
G denotes an algebraic group over k.
a_i are coefficients of the r-Pfister form ϕ over k.
b_i are coefficients of the n-dimensional quadratic form b over k.
q denotes a quadratic form over k, and Q is the associated projective quadric.
$i_W(q)$ is the Witt index of the quadratic form q.
C is a composition algebra (not to be confused with the Lie type C_n), and c_i are elements of C.
J is a Jordan algebra, x is an element of J, and u is an idempotent in J.
$X(J)$, $Q(J,u)$, Z_1 and Z_2 are complete schemes over k defined in Section 2.
F_n^r and R^r are motives defined in Section 3.1 (not to be confused with the Lie type F_4).
$\mathcal{M}(X)$ denotes the motive of a smooth complete scheme X, and $M\{i\}$ denotes the i^{th} Tate twist of the motive M.
The paper is organized as follows. In Section 1 we will recall the terminology and classification of reduced simple Jordan algebras. In Section 2 we describe the variety $X(J)$ and show it is homogeneous. Also we will define the birational map v_2 from a quadric to $X(J)$ and show that it is a Sarkisov link by analyzing its scheme of base points. In Section 3 we deduce motivic decompositions for a class of quadrics, as well as for the indeterminacy locus of v_2 introduced in Section 2. Finally we put these decompositions together to give a motivic decomposition of $X(J)$.

1 Jordan algebras

A *Jordan algebra* over k is a commutative, unital (not necessarily associative) k-algebra J whose elements obey the identity

$$x^2(xy) = x(x^2y) \text{ for all } x, y \in J.$$

A *simple* Jordan algebra is one with no proper ideals. An *idempotent* in J is an element $u^2 = u \neq 0 \in J$. Two idempotents are *orthogonal* if they multiply to zero, and an idempotent is *primitive* if it is not the sum of two orthogonal idempotents in J. For any field extension l/k, we can *extend scalars* to l by taking $J_l = J \otimes_k l$, for example $\bar{J} = J \otimes \bar{k}$. A Jordan algebra has *degree* n if the identity in $\bar{J}$ decomposes into n pairwise orthogonal primitive idempotents over $\bar{k}$. A degree n Jordan algebra is *reduced* if the identity decomposes into n orthogonal primitive idempotents over k.
The classification of reduced simple Jordan algebras of degree ≥ 3 is closely related to the classification of composition algebras. A *composition algebra* over k is a unital k-algebra C together with a non-degenerate quadratic form ϕ on C (called the *norm form*) such that for any $c_1, c_2 \in C$ we have that

$\phi(c_1c_2) = \phi(c_1)\phi(c_2)$. Two composition algebras are isomorphic as k-algebra iff their norm forms are isometric. Every norm form is an r-fold *Pfister form*, which is to say

$$\phi = \langle\langle a_1, \cdots, a_r\rangle\rangle := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_r\rangle.$$

Furthermore, r must be $0, 1, 2$ or 3, and for any such r-fold Pfister form ϕ, there is a composition algebra with ϕ as its norm form and a canonical conjugation map $^- : C \to C$.

Let C be a composition algebra with norm form $\phi = \langle\langle a_1, \cdots, a_r\rangle\rangle$, and let $b = \langle b_1, \cdots, b_n\rangle$ be a non-degenerate quadratic form. Then we can define a reduced Jordan algebra in the following way. Let $\Gamma = \mathrm{diag}(b_1, \cdots, b_n)$, and let $\sigma_b(x) := \Gamma^{-1}\bar{x}^t\Gamma$ define a map from $M_n(C)$ to $M_n(C)$. Then σ_b is an involution (i.e. an anti-homomorphism such that $\sigma_b^2 = \sigma_b$), so we can define $\mathrm{Sym}(M_n(C), \sigma_b)$ to be the commutative algebra of symmetric elements (i.e. elements x such that $\sigma_b(x) = x$). The product structure is defined by $x \circ y = \frac{1}{2}(xy + yx)$, using the multiplication in C. When C is associative (i.e. $r = 0, 1$ or 2) we know $\mathrm{Sym}(M_n(C), \sigma_b)$ is Jordan. For $r = 3$, it is only Jordan when $n \leq 3$, so in what follows we will always impose this condition in the $r = 3$ case.

The Jordan algebra isomorphism class of $\mathrm{Sym}(M_n(C), \sigma_b)$ only depends on the isomorphism classes of b and C, and not on the diagonalization we have chosen for b. The following theorem states that in degrees ≥ 3 these make up all of the reduced Jordan algebras up to isomorphism.

Theorem 1.1. *(Coordinatization [Mc04, 17],[Ja68, p.137]) Let J be a reduced simple Jordan algebra of degree $n \geq 3$. Then there exists a composition algebra C and an n-dimensional quadratic form b such that $J \cong \mathrm{Sym}(M_n(C), \sigma_b)$.*

2 The Sarkisov link

We will define a birational map from a projective quadric to a projective homogeneous variety, $X(J)$, and show it is an elementary link in terms of Sarkisov (see 2.17).

Let $r = 0, 1, 2, 3$ and $n \geq 3$, and if $r = 3$ then $n = 3$. Throughout we will fix a composition algebra C of dimension 2^r over k, and elements $b_i \in k^*$ such that $b = \langle b_1, \cdots, b_n\rangle$ is a non-degenerate quadratic form. Let $J = \mathrm{Sym}(M_n(C), \sigma_b)$ (see Section 1). Then J is a central simple reduced Jordan algebra. Jacobson defined the closed subset $V_J \subset \mathbb{P}J$ of *rank 1* elements of J (he used the terminology *reduced elements*) and showed it is a variety defined over k [Ja85, §4].

2.1 The Veronese map. The following rational map is a generalization of the $r = 0$ case where it is the degree 2 Veronese morphism [Ch06, 3] [Za93, Last page].

$$v_2 : \mathbb{P}(C^n) \dashrightarrow \mathbb{P}J$$
$$[c_1, \cdots, c_n] \mapsto [b_i c_i \bar{c_j}].$$

If the composition algebra is associative (so $r \neq 3$), then the set-theoretic image of v_2 (where it is defined) is precisely V_J. If $r = 3$, then the set-theoretic image of v_2 isn't closed, but its closure is V_J [Ch06, Prop. 4.2]. Note that this map specifies a choice of n orthogonal primitive idempotents, $v_2([0, \cdots, 1, \cdots, 0])$, so it depends on more than just the isomorphism class of J.
Let us restrict the map v_2 to the projective space defined by $c_n \in k1$, and abuse notation by sometimes considering v_2 as a rational map from $\mathbb{P}(C^{n-1} \times k) \dashrightarrow V_J$. This map is an isomorphism on the open subset $U = (c_n \neq 0) \subset \mathbb{P}(C^{n-1} \times k)$ [Ja85, Thm. 4.26], and hence birational. The projective homogeneous variety we will be interested in is $X(J) \subset V_J$ the hyperplane of traceless matrices, which has dimension $2^r(n-1) - 1$.

2.2 The quadric $Q(J, u)$. Define the quadric $Q(J, u) \subset \mathbb{P}(C^{n-1} \times k)$ by

$$\phi \otimes \langle b_1, \cdots b_{n-1} \rangle \perp \langle b_n \rangle = (\sum_{i=1}^{n-1} b_i c_i \bar{c_i}) + b_n c_n^2 = 0.$$

Here ϕ is the norm form of C. The right hand side is simply the trace in V_J, so the restriction of the birational map v_2 to $Q(J, u)$ has image in $X(J)$. We will often further abuse notation and consider v_2 to be the birational map from $Q(J, u)$ to $X(J)$.
Although the definition of $Q(J, u)$ depends on the diagonalization of b, the isomorphism class of $Q(J, u)$ depends only on the isomorphism class of J together with a choice of primitive idempotent u, which we will usually take to be $u = \mathrm{diag}(0, \cdots, 0, 1) \in J$, as we have done above.

Remark 2.3. Since the birational class of $Q(J, u)$ is independent of $u \in J$, we have another proof of Prop. 0.1 when $r \leq 3$, and if $r = 3$ then $n = 3$. For more on this, see 2.16.

For connected algebraic groups G over $\bar{k}$, projective homogeneous G-varieties G/P are classified by conjugacy classes of parabolic subgroups P in G. Furthermore, the conjugacy classes of parabolics are classified by specifying subsets θ of the set Δ of nodes of the Dynkin diagram of G, as in [Ti65, 1.6]. In fact we will use the complement to his notation, so that $\theta = \Delta$ corresponds to a Borel subgroup $P_\Delta = B$, and $\theta = \emptyset$ corresponds to $P_\emptyset = G$. We use the Bourbaki root numberings. G^0 denotes the connected component of the identity in G.

Theorem 2.4. *V_J is the union of two* $\mathrm{Aut}(J)$*-orbits: $X(J)$ and $V_J - X(J)$. Furthermore, we have:*
(r=0): $\overline{X(J)} \cong G/P_\theta$, for $G = \mathrm{Aut}(\bar{J}) \cong \mathrm{SO}(n)$, if $n \neq 4$ then $\theta = \{1\}$, and if $n = 4$ then the Dynkin diagram is two disjoint nodes, where θ is both nodes.

In all cases, these varieties are quadrics.
(r=1): $\overline{X(J)} \cong G^0/P_\theta$, *for* $G = \mathrm{Aut}(\bar{J}) \cong \mathbb{Z}/2 \ltimes \mathrm{PGL}(n)$ *and* $\theta = \{1, n-1\}$, *this is the variety of flags of dimension 1 and codimension 1 linear subspaces in a vector space.*
(r=2): $\overline{X(J)} \cong G/P_\theta$, *for* $G = \mathrm{Aut}(\bar{J}) \cong \mathrm{PSp}(2n)$ *and* $\theta = \{2\}$, *this is the second symplectic Grassmannian.*
(r=3): $\overline{X(J)} \cong G/P_\theta$, *for* $G = \mathrm{Aut}(\bar{J}) \cong F_4$ *and* $\theta = \{4\}$, *this may be viewed as a hyperplane section of the Cayley plane.*

Proof. $\mathrm{Aut}(J)$ acts on V_J, since the rank is preserved by automorphisms. So it is sufficient to prove this theorem for $k = \bar{k}$. Every element of $V_J - X(J)$ is $[u]$ for some rank one idempotent u [Ch06, Prop. 3.8], and $\mathrm{Aut}(J)$ is transitive on rank one idempotents by Jacobson's coordinatization theorem, since the field is algebraically closed [Mc04, 17].
Clearly $X(J)$ is preserved by $\mathrm{Aut}(J)$, since the trace is preserved by automorphisms. All that remains is to show that $\mathrm{Aut}(J)$ is transitive on $X(J)$, which we will do in cases. Consider the $2^{r-1}n(n-1)+n$ dimensional $\mathrm{Aut}(J)$ representation $J = k \oplus J_0$, where J_0 is the subrepresentation of traceless elements in J. In all cases we will show that J_0 is an irreducible $\mathrm{Aut}(J)$ representation, find the highest weight, and show that there is a closed orbit in $\mathbb{P}(J_0)$ which is contained in $X(J)$ and is of the same dimension. Therefore, by uniqueness of the closed orbit, which follows from the irreducibility of J_0, $X(J)$ is the closed orbit.

Case $r = 0$: For simplicity, we will modify the definition of J. Instead of taking $n \times n$ matrices such that $x^t = x$, we will take matrices such that $M^{-1}x^tM = x$ where

$$M = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} \text{ for } n = 2m, \text{ and } M = \begin{bmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for } n = 2m+1.$$

This change is justified by recalling that any two orthogonal involutions in the same matrix algebra over an algebraically closed field are isomorphic. Now the Lie algebra of derivations $\mathrm{Der}(J) \cong \mathfrak{so}(n)$ is in the more standard form, and we can choose elements of the Cartan subalgebra $\mathfrak{h}$ as diagonal matrices $H_i = E_{i,i} - E_{m+i,m+i}$ as in [FH91, 18]. Following the conventions of [FH91], we have a dual basis $L_i(H_j) = \delta_{ij}$ of $\mathfrak{h}^*$, and we wish to find the highest weight of the representation J_0.
For $n = 2m$, the roots of $\mathfrak{so}(2m)$ are $\pm L_i \pm L_j$ for $1 \le i \ne j \le m$. One can check that the non-zero weights of J_0 are $\pm L_i \pm L_j$ for all i, j. In particular, the element $E_{1,m+1}$ is a weight vector in J_0 for the weight $2L_1$, and the irreducible representation with highest weight $2L_1$ is of the same dimension as J_0. Therefore J_0 is the irreducible representation with highest weight $2L_1$, and since $\mathrm{Aut}(J)$ is simple, there is a unique closed orbit in $\mathbb{P}(J_0)$, and it is the orbit of $E_{1,m+1}$. To determine the dimension of the orbit, we ask which root spaces $\mathfrak{g}_{-\alpha_i}$ in the Lie algebra for the negative simple roots $-\alpha_i$, kill the

weight space of $2L_1$. For $n = 4$, neither root space, for $-\alpha_1 = -L_1 - L_2$ nor $\alpha_2 = -L_1 + L_2$, kills this weight space. For any $n \geq 6$ even, all of the negative simple root spaces kill the weight space $2L_1$ except for the one for $-L_1 + L_2$. In either case the dimension of the parabolic fixing $E_{1,m+1}$ is $2m^2 - 3m + 2$, so the dimension of the orbit is $n-2$. This is the dimension of the closed invariant subset $X(J)$, which must contain a closed orbit. Since there is only one closed orbit, $X(J)$ must be the entire orbit.
A similar analysis may be carried out in the $n = 2m + 1$ case, where again $E_{1,m+1}$ is a weight vector for the highest weight $2L_1$.

Case $r = 1$*:* We have the action of the connected component $\mathrm{Aut}(J)^0 = PGL(n)$ on $J \cong M_n(k)$, acting by conjugation. The induced action of the Lie algebra of derivations $\mathrm{Der}(J) \cong \mathfrak{sl}(n)$ on J_0 is just the adjoint action on $\mathfrak{sl}(n)$. With the standard diagonal Cartan subalgebra, and choice of positive roots dual to $H_i = E_{i,i} - E_{i+1,i+1}$, the highest weight is in the representation J_0 is $2L_1 + L_2 + \cdots + L_{n-1}$ with multiplicity 1. A dimension count shows this representation is irreducible, and the dimension of the parabolic fixing a highest weight vector is $n^2 - 2n + 2$. So the dimension of the unique closed orbit is $2n - 3$, which is the dimension of $X(J)$. Therefore $X(J)$ is the closed orbit.

Case $r = 2$*:* As in the $r = 0$ case, we will change our symplectic involution $\sigma(x) = \bar{x}^t$ to $\sigma_M(x) = M^{-1}x^tM$ for

$$M = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Then the Lie algebra of derivations $\mathrm{Der}(J) \cong \mathfrak{sp}(2n)$ is in the standard form, by choosing a Cartan subalgebra of diagonal matrices, with $H_i = E_{i,i} - E_{n+i,n+i}$ and dual basis $L_i \in \mathfrak{h}^*$. The roots of $\mathfrak{sp}(2n)$ are $\pm L_i \pm L_j$ for all i, j, and the non-zero weights of J_0 are $\pm L_i \pm L_j$ for $i \neq j$. In particular, the highest weight is $L_1 + L_2$ in the standard weight ordering of [FH91, p.257]. Comparing dimensions shows that J_0 is irreducible, and the parabolic fixing a highest weight vector is of dimension $2n^2 - 3n + 5$. So the unique closed orbit in $\mathbb{P}(J_0)$ is of dimension $4n - 5$, which is the same as the dimension of $X(J)$. Therefore $X(J)$ is the unique closed orbit.

Case $r = 3$*:* First notice that J_0 is a 26-dimensional non-trivial representation of $F_4 = \mathrm{Aut}(J)$. It is well-known that such a representation is unique, and has a 15-dimensional unique closed orbit in $\mathbb{P}(J_0)$. Since $X(J)$ is a 15-dimensional closed invariant subset, it must be equal to the closed orbit. □

Remark 2.5. Over the complex numbers the varieties with exactly two G-orbits for some semisimple algebraic group G, one of which is of codimension one, have been classified by [Ah86]. The varieties V_J account for most of these.

2.6 Blowing up the base loci

Any birational map of projective varieties over a field can be expressed as a blow up followed a blow down of closed subschemes (Prop. 2.7). In this section

we will show that these closed subschemes, for our birational map from $Q(J,u)$ to $X(J)$, are (usually) smooth varieties, and hence see that the map is an elementary link in terms of Sarkisov.
Given a rational map between projective varieties $f : Y \dashrightarrow X$, we can define the scheme of base points of f as a closed subscheme of Y [Ha77, II. Example 7.17.3].

Proposition 2.7. *Let $f : Y \dashrightarrow X$ be a birational map of projective varieties over a field k with $g : X \dashrightarrow Y$ the inverse birational map. Let Z_Y and Z_X be the schemes of base points of f and g respectively. Then the blow up $\tilde{Y}$ of Y along Z_Y is isomorphic to the blow up $\tilde{X}$ of X along Z_X.*

Proof. Let $U \subset Y$ be the open subset on which f is an isomorphism. Then the graph Γ_f of $f|_U$ is a subset of $U \times f(U) \subset Y \times X$. The closure of Γ_f in $Y \times X$, given the structure of a closed reduced subscheme, is the blow up $\tilde{Y}$ [EH00, Prop. IV.22][1].
Similarly, $\tilde{X}$ is the closure of $\Gamma_g \subset U \times f(U)$. Since the inverse of f on U is g, we have that $\tilde{X}$ and $\tilde{Y}$ are both closures in $Y \times X$ of the same subset of $U \times f(U)$. So they have the same structure as reduced schemes, and hence $\tilde{X} \cong \tilde{Y}$. □

2.8 Indeterminacy locus of v_2. Let Z_1 be the closed *reduced* subscheme associated to the scheme of base points in $Q(J,u)$ of the birational map v_2. We will show that Z_1 is isomorphic to the scheme of base points. We denote by $\mathrm{Aut}(J,u)$ the subgroup of automorphisms of J that fix the primitive idempotent u.

Theorem 2.9. *Z_1 is homogeneous under an action of* $\mathrm{Aut}(J,u)$.

Proof. To describe the action we will use the vector space isomorphism $C^{n-1} \cong J_{\frac{1}{2}}(u) = \{x \in J | x \cdot u = \frac{1}{2}x\}$. Here, as above, we take $u = \mathrm{diag}(0,\cdots,0,1) = E_{n,n}$. This isomorphism is given by sending an element $c \in C^{n-1}$ to the matrix element in $J_{\frac{1}{2}}(u) \subset M_n(C)$ with n^{th} row equal to $[c, 0]$.
So we have an $\mathrm{Aut}(J,u)$ action on $\mathbb{P}(C^{n-1})$. By considering the defining equations, one see that Z_1 is isomorphic to the reduced subscheme of $\mathbb{P}(J_{\frac{1}{2}}(u))$ defined by the matrix equation $x^2 = 0$. So it is clear that the underlying closed subset is stable under $\mathrm{Aut}(J,u)$.
Finally, to show the action is transitive, it is enough to show it after extending scalars to an algebraically closed field $\bar{k}$. We will use similar arguments as in the proof of Thm. 2.4.

Case $r = 2$: Using the notation from the proof of Thm. 2.4, the roots of the Lie algebra of $\mathrm{Aut}(J,u)$ are $\pm L_i \pm L_j$ for $i, j \leq n-1$ together with $\pm 2L_n$. One can check that the non-zero weights of the representation $J_{\frac{1}{2}}(u)$ are $\pm L_i \pm L_n$ for $i \leq n-1$. A dimension count reveals that $J_{\frac{1}{2}}(u)$ is therefore

[1] They assume Y is affine, but we can drop this assumption since the blow up is determined locally.

an irreducible representation with highest weight $L_1 + L_n$. The only negative simple roots that don't kill a highest weight vector are $L_2 - L_1$ and $-2L_n$, so the dimension of the parabolic subgroup that fixes a point in the unique closed orbit in $\mathbb{P}(J_{\frac{1}{2}}(u))$ is $2n^2 - 5n + 6$. So the dimension of this orbit is $2n - 2$.

To see this is the same as the dimension of Z_1, consider the affine cone $\tilde{Z}_1$ over Z_1 inside $J_{\frac{1}{2}}(u)$. Then consider the Jacobian matrix of the equations given by $\{x_i\bar{x}_j = 0\}$ with respect to the $4(n-1)$ variables: 4 variables for each coordinate $x_i \in C$. The rank of this matrix at any point in the affine cone over Z_1 is $\leq \dim(J_{\frac{1}{2}}(u)) - \dim(\tilde{Z}_1)$, where equality holds if the ideal spanned by the polynomials $\{x_i\bar{x}_j\}$ is radical. By choosing a convenient point, we see that the dimension of Z_1 is at most $2n-2$, which is the dimension of the closed orbit. So if Z_1 contained another $\mathrm{Aut}(J,u)$-orbit, then it would contain another closed orbit. But the closed orbit is unique, and therefore Z_1 is the closed orbit.

Case $r = 3$: It is well known that the $\mathrm{Aut}(J,u) \cong \mathrm{Spin}(9)$ representation given by $J_{\frac{1}{2}}(u)$ for $u = E_{3,3}$ is the 16-dimensional spin representation. The unique closed orbit in $\mathbb{P}(J_{\frac{1}{2}}(u))$ is therefore the 10-dimensional spinor variety. Using a similar argument to the $r = 2$ case, we can show the dimension of Z_1 is at most 10, so by the uniqueness of the closed orbit we can conclude that Z_1 is the closed orbit.

Case $r = 1$: This case is slightly different from the other two because $\mathrm{Aut}(J,u) \cong \mathbb{Z}/2 \ltimes GL(n-1)$ is a disconnected group, and the connected component has *two* closed orbits in $\mathbb{P}(J_{\frac{1}{2}}(u))$. The argument is similar to the $r = 2$ case, except we find that the $\mathfrak{sl}(n-1)$-representation $J_{\frac{1}{2}}(u)$ is the direct sum of the standard representation V with its dual V^*. So the two closed orbits in $\mathbb{P}(J_{\frac{1}{2}}(u))$ are the orbits of weight vectors for the weights $L_1 - L_n$ and $L_n - L_1$, which are the respective closed orbits in $\mathbb{P}V$ and $\mathbb{P}V^*$. Each $\mathfrak{sl}(n-1)$-orbit has dimension $n-2$. Furthermore, the $\mathbb{Z}/2$ part of $\mathrm{Aut}(J,u)$ swaps these two representations, since it acts on matrices as the transpose. So there is a unique closed $\mathrm{Aut}(J,u)$-orbit, and it is of dimension $n-2$.

As in the $r = 2$ case, by considering the rank of the Jacobian at a closed point in $\tilde{Z}_1$, we see that the dimension of Z_1 is at most $n-2$. Since Z_1 is $\mathrm{Aut}(J,u)$-stable, we can conclude that it is the closed orbit. $\square$

Corollary 2.10. *The reduced scheme Z_1 is isomorphic to the scheme of base points of v_2 in $Q(J,u)$.*

Proof. The $r = 0$ case is trivial, since v_2 is a morphism and hence Z_1 is empty. It is sufficient to assume k is algebraically closed.

The other cases follow from the proof of Thm. 2.9, as follows. We can choose a convenient closed point in the scheme of base points, and show that the rank of the Jacobian of the defining polynomials given by $\{v_2(x) = 0\}$ is equal to the codimension. This implies the scheme is smooth at that point (and therefore at all points), so in particular, it is reduced. $\square$

Corollary 2.11. *Over $\bar{k}$, the smooth subscheme Z_1 is isomorphic to the following.*
$(r = 0) : \emptyset$
$(r = 1) : \mathbb{P}^{n-2} \sqcup \mathbb{P}^{n-2}$
$(r = 2) : \mathbb{P}^1 \times \mathbb{P}^{2n-3}$
$(r = 3) :$ *The 10-dimensional spinor variety*

Proof. This follows from our representation theoretic understanding of Z_1 from the proof of Thm. 2.9.
There are much more explicit ways of understanding the $r \neq 3$ cases. For example, in the $r = 2$ case, if $c = [c_1, \cdots, c_{n-1}] \in \mathbb{P}(M_2(\bar{k})^{n-1})$ is in $\overline{Z_1}$, then the c_i's are rank 1 matrices that have a common non-zero vector in their kernels. This can be used to get an explicit isomorphism with $\mathbb{P}^1 \times \mathbb{P}^{2n-3}$. $\square$

Remark 2.12. These varieties are written in [Za93, Final pages], where it is implicitly suggested that they are the base locus of the rational map v_2.

Remark 2.13. It is shown in [Kr07] that $Z_1 \cong \mathrm{Spec}(k(\sqrt{a_1})) \times_k \mathbb{P}^{n-2}$, where $\langle\langle a_1 \rangle\rangle$ is the norm form associated to C. So the above corollary shows that Z_1 is irreducible over k except for the single case when $r = 1$ and C is split.

2.14 Indeterminacy locus of v_2^{-1}. Let Z_2 be the scheme of base points of the inverse birational map $v_2^{-1} : X(J_n) \dashrightarrow Q(J, u)$. We have that $v_2^{-1}([x_{ij}]) = [x_{n,1}, \cdots, x_{n,n}]$, where this is defined.
We will use the notation $J_{n-1} = \mathrm{Sym}(M_{n-1}(C), \sigma_{\langle b_1, \cdots, b_{n-1}\rangle})$, and sometimes $J_n = J$ for emphasis. The isomorphism class of J_{n-1} depends on the choice of primitive idempotent $u = E_{n,n} \in J$, but is otherwise independent of the diagonalization of $\langle b_1, \cdots, b_{n-1}\rangle$.

Lemma 2.15. *The scheme of base points Z_2 is isomorphic to the smooth subvariety $X(J_{n-1})$.*

Proof. The indeterminacy locus of v_2^{-1} is simply the closed subset of matrices in $X(J_n)$ whose bottom row (and therefore right-most column) is zero. In other words, Z_2 is defined by linear polynomials. The ideal of these polynomials is radical, and therefore the scheme Z_2 is reduced. For $n \geq 4$, one sees that Z_2 is isomorphic to $X(J_{n-1})$. For $n = 3$, by considering the matrix equation $x^2 = 0$, we see that the base locus of Z_2 is the quadric defined by $\phi \otimes \langle b_1 \rangle \perp \langle b_2 \rangle = 0$. We will define $X(J_2)$ to be this quadric. $\square$

2.16 The chain between two quadrics

The Sarkisov program [Co94] predicts that any birational map between two Mori fibre spaces X and Y factors into a chain of elementary links between intermediate Mori fibre spaces. An example of such a link (of type II [Co94, 3.4.2]) would be $X \leftarrow W \rightarrow V$ where both morphisms are blow ups of smooth subvarieties, and X and V are projective homogeneous varieties with Picard number 1 (and hence Mori fibre spaces).

Theorem 2.17. *For $r \neq 1$ or C non-split, the birational map v_2 from $Q(J,u)$ to $X(J)$ is an elementary link of type II.*

Proof. We have that Z_1 is irreducible (see Remark 2.13). The blow up of an irreducible smooth subscheme increases the Picard number by 1, and a blow down decreases it by 1. So in this situation, by Lemma 2.15 and Lemma 2.10 we see that $X(J)$ has Picard number 1. So by Prop. 2.7 we have that v_2 is a blow up of a smooth subvariety followed by a blow down to a smooth subvariety, and therefore it is an elementary link of type II. $\square$

Let $b' = \langle b'_1, \cdots, b'_n\rangle$, and $q' = \phi \otimes \langle b'_1, \cdots, b'_{n-1}\rangle \perp \langle b'_n\rangle$. Then Totaro's Prop. 0.1 states that if $\phi \otimes b \cong \phi \otimes b'$, then the quadrics defined by q and q' are birational. By defining the Jordan algebra J' using ϕ and b', we have a birational map v'_2 from $Q(J', u')$ to $X(J')$.

Proof of Thm. 0.2. If $\phi \otimes b \cong \phi \otimes b'$, then the Jordan algebras $J \cong J'$ are isomorphic as algebras ([KMRT98, Prop. 4.2, p. 43], [Ja68, Ch. V.7, p. 210]), and therefore the varieties $X(J) \cong X(J')$ are also isomorphic. So, as noted in Remark 2.3, $Q(J,u)$ is birational to $Q(J', u')$, and moreover by Thm. 2.17 this map is the composition of two elementary links, with intermediate variety $X(J)$. Notice that if C is a split composition algebra (equivalently, ϕ is hyperbolic) then $Q(J,u)$ and $Q(J',u')$ are already isomorphic. $\square$

2.18 Transposition maps. Now we will explicitly factor the birational maps of Roussey ([Ro05]) and Totaro ([To08]), which in general have more than two elementary links. The most basic case they consider, though, is that of *transposition*. This corresponds to finding a birational map between quadrics q and q', where $b'_i = b_i$ for $1 \leq i \leq n-2$, and $b'_{n-1} = b_n$, $b'_n = b_{n-1}$. So b and b' differ by transposing the last two entries. Totaro proves Prop. 0.1 by finding a suitable chain of such transposition maps.

Proposition 2.19. *For $r = 0, 1, 2$ and $n \geq 3$, and if $r = 3$ then $n = 3$, Totaro's transposition map factors as the composite of two elementary links.*

Proof. Let q and q' be as above, and let $J = \mathrm{Sym}(M_n(C_\phi), \sigma_b)$. Then the quadric $(q = 0) = Q(J,u)$ is defined using the idempotent $u = \mathrm{diag}(0, \cdots 0, 1) \in J$ (see 2.2). General rational points on this quadric are elements in $\mathbb{P}(C^{n-1} \times k)$ such that $v_2([c_1, \cdots, c_n]) \in \mathbb{P}J$ has trace zero. Here $c_i \in C$ for $i \neq n$, and $c_n \in k$. The inverse birational map v_2^{-1} simply takes the n^{th} row of the matrix in J.
Then the quadric for $(q' = 0) = Q(J, u')$ can be defined using the idempotent $u' = \mathrm{diag}(0, \cdots, 1, 0) \in J$. General rational points on this quadric are elements in $\mathbb{P}(C^{n-2} \times k \times C)$ such that $v'_2([c'_1, \cdots, c'_n]) \in \mathbb{P}J$ has trace zero, where we use the same Jordan algebra J. Here $c'_i \in C$ for $i \neq n-1$, and $c'_{n-1} \in k$. The inverse birational map $(v'_2)^{-1}$ takes the $n-1^{th}$ row of the matrix in J.
So the composition $(v'_2)^{-1} \circ v_2$ defines a birational map from $Q(J,u)$ to $Q(J,u')$. From Thm. 2.17 this is the composite of two elementary links. So it remains to show this composite is the same as Totaro's transposition map.

To see this, consider the map $(v_2')^{-1} \circ v_2$ over $\bar{k}$, and observe where it sends a general point from $Q(J,u)$. Recall that v_2 sends $[c_1, \cdots, c_n]$ to the matrix $[b_i c_i \overline{c_j}] \in X(J)$, and then taking the $n-1^{th}$ row of this matrix gives us

$$[b_{n-1}c_{n-1}\overline{c_1}, \cdots, b_{n-1}c_{n-1}\overline{c_{n-1}}, b_{n-1}c_{n-1}\overline{c_n}] \in Q(J,u') \subset \mathbb{P}(C^{n-2} \times \bar{k} \times C).$$

After using the isomorphism $\mathbb{P}(C^{n-2} \times k \times C) \cong \mathbb{P}(C^{n-1} \times k)$ to swap the last two coordinates, we can now recognize that this is exactly a map from [To08, Lemma 5.1], where the "multiplication" of elements in C, is $x * y := x\bar{y}$. $\square$

Remark 2.20. We may also view this chain of birational maps as a "weak factorization" in the sense of [AKMW02]. They prove that any birational map between smooth projective varieties can be factored into a sequence of blow ups and blow downs of smooth subvarieties. But a chain of Sarkisov links (of type II) is stronger, because then each blow up is immediately followed by a blow down, and the intermediate varieties are Mori fibre spaces.

3 Motives

For a smooth complete scheme X defined over k, we will denote the Chow motive of X with coefficients in a ring Λ by $\mathcal{M}(X;\Lambda)$, following [EKM08] (see also [Vi04], [Ma68]). We will briefly recall the definition of the category of graded Chow motives with coefficients in Λ.

Let us define the category $\mathcal{C}(k,\Lambda)$. The objects will be pairs (X,i) for X a smooth complete scheme over k, and $i \in \mathbb{Z}$, and the morphisms will be *correspondences*:

$$\mathrm{Hom}_{\mathcal{C}(k,\Lambda)}((X,i),(Y,j)) = \bigsqcup_m \mathrm{CH}_{\dim(X_m)+i-j}(X_m \times_k Y, \Lambda).$$

Here $\{X_m\}$ is the set of irreducible components of X. If $f : X \to Y$ is a morphism of k-schemes, then the graph of f is an element of $\mathrm{Hom}_{\mathcal{C}(k,\Lambda)}((X,0),(Y,0))$. There is a natural composition on correspondences that generalizes the composition of morphisms of schemes.

We denote the *additive completion* of this pre-additive category by $CR(k,\Lambda)$. Its objects are finite direct sums of objects in $\mathcal{C}(k,\Lambda)$, and the morphisms are matrices of morphisms in $\mathcal{C}(k,\Lambda)$. Then $CR(k,\Lambda)$ is the category of *graded correspondences* over k with coefficients in Λ.

Finally, we let $CM(k,\Lambda)$ be the *idempotent completion* of $CR(k,\Lambda)$. Here the objects are pairs (A,e), where A is an object in $CR(k,\Lambda)$ and $e \in \mathrm{Hom}_{CR(k,\Lambda)}(A,A)$ such that $e \circ e = e$. Then the morphisms are

$$\mathrm{Hom}_{CM(k,\Lambda)}((A,e),(B,f)) = f \circ \mathrm{Hom}_{CR(k,\Lambda)}(A,B) \circ e.$$

This is the category of graded Chow motives over k with coefficients in Λ. For any smooth complete scheme X over k, we denote $\mathcal{M}(X) = ((X,0), id_X)$ its

Chow motive, and $\mathcal{M}(X)\{i\} = ((X,i), id_X)$ its i^{th} Tate twist. Any object in $CM(k,\Lambda)$ is the direct summand of a finite sum of motives $\mathcal{M}(X)\{i\}$.
In this section we will describe direct sum motivic decompositions of $Q(J,u)$, Z_1 and finally $X(J)$. A non-degenerate quadratic form q of dimension ≥ 2 defines a smooth projective quadric Q, and we will sometimes write $\mathcal{M}(q) = \mathcal{M}(Q)$.

3.1 Motives of neighbours of multiples of Pfister quadrics

In this section until 3.8 we can assume our base field k is of any characteristic other than 2, and $r \geq 1$ may be arbitrarily large. Given an r-fold Pfister form ϕ and an n-dimensional non-degenerate quadratic form $b = \langle b_1, \cdots, b_n\rangle$ over k we will describe the motivic decomposition of the projective quadric Q defined by

$$q = \phi \otimes \langle b_1, \cdots, b_{n-1}\rangle \perp \langle b_n\rangle.$$

This quadric is dependent on the choice of diagonalization of b. The following is Vishik's motivic decomposition of the quadric defined by $\phi \otimes b$.

Theorem 3.2. *([Vi04, 6.1])*
For $n \geq 1$, there exists a motive F_n^r such that

$$\mathcal{M}(\phi\otimes b) = \bigoplus_{i=0}^{2^r-1} F_n^r\{i\} \oplus \begin{cases} \emptyset & \text{if } n \text{ is even} \\ \mathcal{M}(\phi)\{2^{r-1}(n-1)\} & \text{if } n \text{ is odd.} \end{cases}$$

Vishik uses the notation $F_\phi(\mathcal{M}(b))$ for F_n^r, and calls it a *higher form of* $\mathcal{M}(b)$. It only depends on the isometry classes of ϕ and b.
If ϕ is anisotropic, Rost defined an indecomposable motive R^r such that $\mathcal{M}(\phi)$ is the direct sum of Tate twists of R^r. This is called the Rost motive of ϕ. If ϕ is split, then this motive is no longer indecomposable, but we will still call $R^r = \mathbb{Z} \oplus \mathbb{Z}\{2^{r-1}-1\}$ the Rost motive. In fact, F_2^r is just the Rost motive of $\phi \otimes b$ (which is similar to a Pfister form). Also note that $F_1^r = 0$.
In particular, for $n \geq 1$, by counting Tate motives one sees that

$$F_n^r|_{\bar{k}} = \bigoplus_{i=0}^{\lfloor \frac{n}{2}\rfloor -1} (\mathbb{Z}\{2^r i\} \oplus \mathbb{Z}\{2^r(n-1) - 2^r i - 1\}).$$

So the summand has $2\lfloor\frac{n}{2}\rfloor$ Tate motives, which is the same number that $\mathcal{M}(b)|_{\bar{k}}$ has.
A summand M is said to *start at d* if $d = \min\{i|\mathbb{Z}\{i\}$ is a summand of $M_{\bar{k}}\}$.
Similarly, a summand M *ends at d* if $d = \max\{i|\mathbb{Z}\{i\}$ is a summand of $M_{\bar{k}}\}$.
We will use the following theorem of Vishik. Here $i_W(q)$ denotes the Witt index of the quadratic form q. This is the number of hyperbolic plane summands in q.

Theorem 3.3. *([Vi04, 4.15]) Let P, Q be smooth projective quadrics over k, and $d \geq 0$. Assume that for every field extension E/k, we have that*

$$i_W(p|_E) > d \Leftrightarrow i_W(q|_E) > m.$$

Then there is an indecomposable summand in $\mathcal{M}(P)$ starting at d, and it is isomorphic to a (Tate twisted) indecomposable summand in $\mathcal{M}(Q)$ starting at m.

With this theorem, it becomes straight forward to prove the following motivic decomposition (Thm. 3.6), by translating it into some elementary facts about multiples of Pfister forms. First we will state two lemmas for convenience.

Lemma 3.4. *Let ϕ be an r-fold Pfister form ($r \geq 1$) and let b be an n-dimensional non-degenerate quadratic form ($n \geq 2$). For any $0 \leq d \leq \lfloor \frac{n}{2} \rfloor - 1$, we have $i_W(\phi \otimes b) > 2^r d$ implies $i_W(\phi \otimes b) > 2^r(d+1) - 1$.*

Proof. This follows from the fact that if ϕ is anisotropic then 2^r divides $i_W(\phi \otimes b)$ [Vi04, Lemma 6.2] or [WS77, Thm. 2(c)]. ◻

Lemma 3.5. *If Q is a smooth projective quadric of dimension N, then for any $0 \leq d \leq N$, an indecomposable summand of $\mathcal{M}(Q)$ starting at d is isomorphic (up to Tate twist) to an indecomposable summand of $\mathcal{M}(Q)$ ending at $N - d$. The same is true for indecomposable summands of F_n^r for any $r \geq 1$ and $n \geq 1$.*

Proof. This is proved in [Vi04, Thm. 4.19] for anisotropic Q, but it is also true for isotropic Q by using [Vi04, Prop. 2.1] to reduce to the anisotropic case. The statement for the motive F_n^r follows easily from its construction. ◻

Theorem 3.6. *Let ϕ be an r-fold Pfister form ($r \geq 1$), and for non-zero b_i and $n \geq 2$ we let $q = \phi \otimes \langle b_1, \cdots, b_{n-1} \rangle \perp \langle b_n \rangle$ over k of characteristic not 2. Then we have the following motivic decomposition.*

$$\mathcal{M}(q) = F_n^r \oplus \bigoplus_{i=1}^{2^r-1} F_{n-1}^r\{i\} \oplus \begin{cases} \emptyset & \text{if } n \text{ is odd} \\ \bigoplus_{j=1}^{2^{r-1}-1} R^r\{2^{r-1}(n-1) - j\} & \text{if } n \text{ is even.} \end{cases}$$

Proof. We will split the proof into steps, including one step for each of the three summands. We will use the notation $b' = \langle b_1, \cdots, b_{n-1} \rangle$ and $b = b' \perp \langle b_n \rangle$. Note that we can assume that ϕ is anisotropic, because when it is isotropic both sides split into Tate motives, and we get the isomorphism by checking that on the right hand side there is exactly one copy of $\mathbb{Z}\{i\}$ for each $0 \leq i < 2^r(n-1)$.

Step 1: The first summand. To show that F_n^r is isomorphic to a summand of $\mathcal{M}(q)$, we need to show that given an indecomposable summand in F_n^r starting at d, then there is an isomorphic indecomposable summand in $\mathcal{M}(q)$ starting at d. In fact, by Lemma 3.5 it is enough to only consider indecomposable summands starting in the 'first half', which is to say starting at $i < 2^{r-1}(n-1)$. Since the only Tate motives in the first half of $F_n^r|_{\bar{k}}$ are $\mathbb{Z}\{2^r d\}$ for some $0 \leq d \leq \lfloor \frac{n}{2} \rfloor - 1$, by Thm. 3.3 it is enough to show that for each such d and E/k field extension we have $i_W(\phi \otimes b|_E) > 2^r d$ iff $i_W(\phi \otimes b' \perp \langle b_n \rangle|_E) > 2^r d$.
The "if" part is clear. So assume $i_W(\phi \otimes b|_E) > 2^r d$. Then by Lemma 3.4 we know $i_W(\phi \otimes b|_E) \geq 2^r(d+1)$. So the $2^r(d+1)$-dimensional totally isotropic

subspace must intersect the $2^r - 1$-codimensional subform $\phi \otimes b' \perp \langle b_n \rangle \subset \phi \otimes b$ in dimension at least $2^r d + 1$. In other words, $i_W(\phi \otimes b' \perp \langle b_n \rangle|_E) > 2^r d$.

Step 2: The second summand. Fix a $1 \leq i \leq 2^r - 1$. As argued in Step 1, we want to show that if $0 \leq d \leq \lfloor \frac{n}{2} \rfloor - 1$, and if there is an indecomposable summand of F^r_{n-1} starting at $2^r d$, then there is an isomorphic indecomposable summand of $\mathcal{M}(q)$ starting at $2^r d + i$. By Thm. 3.3 it is enough to show that for any E/k we have $i_W(\phi \otimes b'|_E) > 2^r d$ iff $i_W(\phi \otimes b' \perp \langle b_n \rangle|_E) > 2^r d + i$.

$$\begin{aligned} i_W(\phi \otimes b'|_E) > 2^r d &\Rightarrow i_W(\phi \otimes b' \perp \langle b_n \rangle|_E) > 2^r(d+1) - 1 && \text{Lemma 3.4} \\ &\Rightarrow i_W(\phi \otimes b' \perp \langle b_n \rangle|_E) > 2^r d + i && \\ &\Rightarrow i_W(\phi \otimes b') > 2^r d && \text{See below} \end{aligned}$$

The last implication follows since the $\geq 2^r d + 2$ dimensional totally isotropic subspace must intersect the codimension 1 subform in dimension at least $2^r d+1$. So, by Lemma 3.5, we have shown that $F^r_{n-1}\{i\}$ is isomorphic to a summand of $\mathcal{M}(q)$ for $1 \leq i \leq 2^r - 1$.

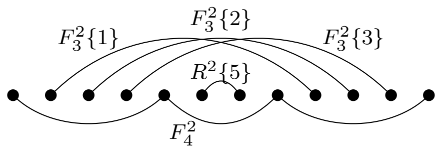
Step 3: The third summand. Assume n is even. Since the summand is empty for $r = 1$, we can assume $r \geq 2$. Fix an $2^{r-1}(n-2) < i < 2^{r-1}(n-1)$.

$$\begin{aligned} i_W(\phi) > 0 &\Rightarrow i_W(\phi) = 2^{r-1} && \text{Property of Pfister forms} \\ &\Rightarrow i_W(\phi \otimes b' \perp \langle b_n \rangle) > i && \\ &\Rightarrow i_W(\phi) > 0 && \text{See below} \end{aligned}$$

For the last implication, we have that the hyperbolic part of $\phi \otimes b' \perp \langle b_n \rangle$ is of dimension $\geq 2^r(n-2)+4$. So the anisotropic part is of dimension $\leq 2^r-2$. So by the Arason-Pfister hauptsatz, $\phi \otimes b'$ is hyperbolic. Now if ϕ were anisotropic, then $2\dim(\phi)$ would divide $\dim(\phi \otimes b')$ [WS77, Thm. 2(c)]. But this says $2^{r+1} | 2^r(n-1)$, which is impossible for n even. Therefore ϕ is isotropic.
To finish Step 3, we use Thm. 3.3 to get the isomorphism of motivic summands.

Step 4: Counting Tate motives. To finish the proof, one needs to show that the summands we have described in these three steps are all possible summands. This can easily be checked by counting the Tate motives over $\bar{k}$. For a visualization of this, see Example 3.7 below.
We have implicitly used [Vi04, Cor. 4.4] here. Note also that for the $n = 2$ case the second summand is zero. □

Example 3.7. As an illustration of the counting argument in Step 4 above, consider $r = 2$ and $n = 4$. Then Thm. 3.6 says that $\mathcal{M}(\langle\langle a_1, a_2 \rangle\rangle \otimes \langle b_1, b_2, b_3 \rangle \perp \langle b_4 \rangle)$ has 5 motivic (possibly decomposable) summands in this decomposition. We can visualize this decomposition, as in [Vi04], with a node for each of the 12 Tate motives over $\bar{k}$, and a line between the nodes if they are in the same summand. Then the motive of the 11-dimensional quadric, $\mathcal{M}(q)$, is as follows, with each summand labelled:



Notice that these summands might be decomposable, for example if the Pfister form $\langle\langle a_1, a_2\rangle\rangle$ is split. So this differs slightly from Vishik's diagrams, since he used solid lines to denote indecomposable summands, and dotted lines for possibly decomposable ones.

3.8 The motive of the base locus Z_1

Now we will use our understanding of Z_1 from Thm. 2.9 and its proof, to decompose its motive into the direct sum of Tate twisted Rost motives.

Proposition 3.9. *(1) For $r = 1$, we have that $\mathcal{M}(Z_1, \mathbb{Z}/2) \cong \oplus_{i=0}^{n-1} R^1\{i\}$*
(2) For $r = 2$, we have that $\mathcal{M}(Z_1, \mathbb{Z}/2) \cong \oplus_{i=0}^{2n-3} R^2\{i\}$.
(3) For $r = 3$, we have that $\mathcal{M}(Z_1, \mathbb{Z}/2) \cong \oplus_{i=0}^{7} R^3\{i\}$.

Proof. For $r = 1$, it is shown in [Kr07] that $Z_1 \cong \mathbb{P}^{n-2} \times_k \operatorname{Spec}(k\sqrt{a_1})$. We know that $\mathcal{M}(\operatorname{Spec}(k[\sqrt{a_1}])) \cong R^1$, so the result follows because the motive of projective space splits into Tate motives.
We have seen that in all cases Z_1 is a smooth scheme that is homogeneous for $\operatorname{Aut}(J, u)$. Moreover, for $r = 2$ or 3, we know that Z_1 is a *generically split* variety in the sense of [PSZ08]. So by their theorem [PSZ08, 5.17] we have that $\mathcal{M}(Z_1, \mathbb{Z}/2)$ is isomorphic to a direct sum of Tate twisted copies of an indecomposable motive $\mathcal{R}_2(\operatorname{Aut}(J, u))$.
Now let V be the projective quadric defined by the r-Pfister form ϕ, the norm form of the composition algebra C. It is a homogeneous $\mathrm{SO}(\phi)$ variety. Since C splits over the function field $k(V)$, by Jacobson's coordinatization theorem J must also split over $k(V)$, and therefore so does the group $\operatorname{Aut}(J, u)$. Furthermore, over $k(Z_1)$, we have a rational point in Z_1. Then for any non-zero coordinate $c_i \in C$ of such a point, there exists $0 \neq y \in C$ such that $c_i y = 0$ in C. But then $\phi(c_i)y = (\bar{c}_i c_i)y = \bar{c}_i(c_i y) = 0$, and so C has an isotropic vector, and is therefore split. Therefore $\mathrm{SO}(\phi)$ splits over $k(Z_1)$.
Now we may apply [PSZ08, Prop. 5.18(iii)] to conclude that $\mathcal{R}_2(\operatorname{Aut}(J, u)) \cong \mathcal{R}_2(\mathrm{SO}(\phi))$. Finally, observe that $\mathcal{R}_2(\mathrm{SO}(\phi))$ is isomorphic to the Rost motive of ϕ ([PSZ08, Last example in 7]), which is the motive R^r. The proposition can be deduced now by counting the Betti numbers of Z_1 (see [Kö91]). $\square$

3.10 Motivic decomposition of $X(J)$

We are ready to decompose the motive $\mathcal{M}(X(J))$ for any reduced simple Jordan algebra J. Recall that $X(J)$ is a homogeneous space for $\operatorname{Aut}(J)$ (Lemma 2.4).

PROPOSITION 3.11. *Let $r = 0, 1, 2$ or 3 and $n \geq 3$, and if $r = 3$ then $n = 3$. We have the following isomorphism of motives with coefficients in $\mathbb{Z}$.*

$$\mathcal{M}(Q(J_n, u)) \oplus \bigoplus_{i=1}^{d_1-1} \mathcal{M}(Z_1)\{i\} \cong \mathcal{M}(X(J_n)) \oplus \bigoplus_{i=1}^{d_2-1} \mathcal{M}(X(J_{n-1}))\{i\}.$$

Here d_i are the respective codimensions of the subschemes Z_i. In particular, for $r \neq 0$, $d_1 = 2^{r-1}n - 2$ and $d_2 = 2^r$.

Proof. If n is the degree of J_n, we have by Section 2.6 that the blow up of $X(J_n)$ along the smooth subvariety $X(J_{n-1})$ is isomorphic to the blow up of $Q(J_n, u)$ along the smooth subscheme Z_1. So by applying the blow up formula for motives [Ma68, p.463], we get the above isomorphism. □

THEOREM 3.12. *Let $r = 0, 1, 2$ or 3, and $n \geq 3$ (and if $r = 3$ then $n = 3$). And let $J = \mathrm{Sym}(M_n(C), \sigma_b)$ where C is a 2^r-dimensional composition algebra over k, and $b = \langle b_1, \cdots, b_n \rangle$ is a non-degenerate quadratic form over k. Then*
$(r = 0)$:

$$\mathcal{M}(X(J)) \cong F_n^0 = \mathcal{M}(b),$$

$(r = 1)$:

$$\mathcal{M}(X(J), \mathbb{Z}/2) \cong F_n^1 \oplus \bigoplus_{j=0}^{\lfloor \frac{n-3}{2} \rfloor} \left(\bigoplus_{i=1}^{2\lfloor \frac{n}{2} \rfloor} R^1\{i + 2j\} \right),$$

$(r = 2)$:

$$\mathcal{M}(X(J), \mathbb{Z}/2) \cong F_n^2 \oplus \bigoplus_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} \left(\bigoplus_{i=1}^{4\lfloor \frac{n-1}{2} \rfloor + 1} R^2\{i + 4j\} \right),$$

$(r = 3)$:

$$\mathcal{M}(X(J), \mathbb{Z}/2) \cong F_3^3 \oplus \bigoplus_{i=1}^{11} R^3\{i\}.$$

Proof. The motive of $Q(J, u)$ may be decomposed in terms of the motives F_n^r, F_{n-1}^r and R^r (Thm. 3.6). The motive of Z_1 with $\mathbb{Z}/2$ coefficients may be decomposed in terms of R^r (Prop. 3.9). The subvariety $X(J_2)$ is isomorphic to the quadric defined by $\phi \otimes \langle b_1 \rangle \perp \langle b_2 \rangle$ (see proof of Lemma 2.15), so we have already decomposed its motive in terms of F_2^r and R^r (Thm. 3.6).
So the last ingredient we need is the cancellation theorem. It gives conditions for when it is true that an isomorphism of motives $A \oplus B \cong A \oplus C$ implies an isomorphism of motives $B \cong C$. This does not hold in general; there are counter-examples when $\Lambda = \mathbb{Z}$ [CPSZ06, Remark 2.8]. But if we take Λ to be

any field, then the stronger Krull-Schmidt theorem holds, which says that any motivic decomposition into indecomposables is unique [CM06, Thm. 34][2].
When we put these pieces into the isomorphism from Prop. 3.11, we may proceed by induction on n. One sees that we can cancel the F^r_{n-1} terms in the decomposition, leaving us with the motive $\mathcal{M}(X(J))$ on the right hand side, F^r_n on the left hand side, and several Tate twisted copies of R^r on both sides. To finish the proof one just needs to count the number of copies of R^r remaining after the cancellation theorem, and verify that the given expressions are correct. We leave this induction argument to the reader. □

Remark 3.13. When ϕ is isotropic, the above motives split. When ϕ is anisotropic, R^r is indecomposable, but the motive F^r_n could still be decomposable, depending on the quadratic form b.

Remark 3.14. The $r = 1$ case of the above theorem may be used to prove Krashen's motivic equivalence [Kr07, Thm. 3.3]. To see this, notice that a 1−Pfister form ϕ defines a quadratic étale extension l/k, and any hermitian form h over l/k is defined by a quadratic form b over k. So in Krashen's notation, $V(h) = X(J)$. Furthermore, his $V(q_h)$ is the projective quadric defined by $\phi \otimes b$, and his $\mathbb{P}_L(N)$ is isomorphic to the base locus Z_1. So in the notation of this paper, his motivic equivalence is

$$\mathcal{M}(\phi \otimes b) \oplus \bigoplus_{i=1}^{n-2} \mathcal{M}(Z_1)\{i\} \cong \mathcal{M}(X(J)) \oplus \mathcal{M}(X(J))\{1\}.$$

Since we have motivic decompositions of all of these summands in terms of F^1_n and R^1 (see Thm. 3.2, Prop. 3.9 and Thm. 3.12), it is easy to verify his motivic equivalence, at least for $\mathbb{Z}/2$ coefficients.
On the other hand, the $r = 1$ case of Thm. 3.12 follows from Krashen's motivic equivalence, together with the $r = 1$ cases of Thm. 3.2 and Prop. 3.9; this is pointed out in [SZ08, Thm. (C)].

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[2] Although this theorem is only stated for Λ a discrete valuation ring, the same proof works for any field.

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