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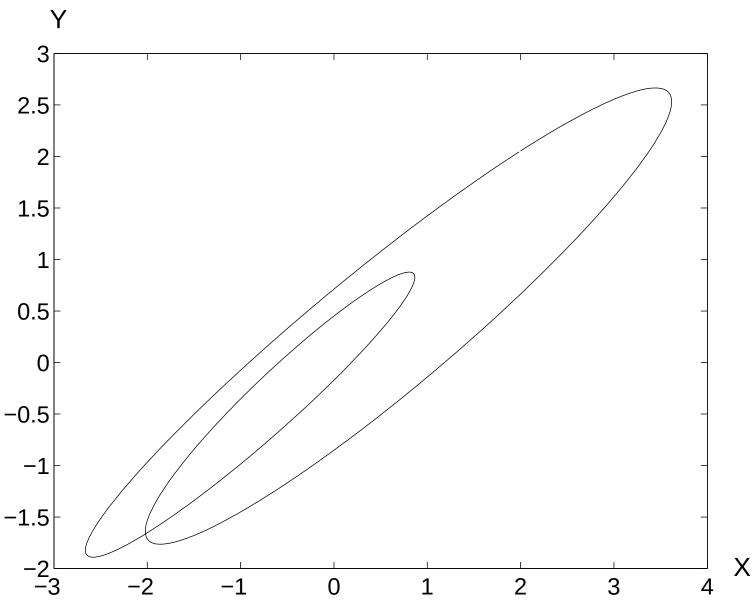


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# Second Order Freeness and Fluctuations of Random Matrices III. Higher order freeness and free cumulants

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Abstract. We extend the relation between random matrices and free probability theory from the level of expectations to the level of all correlation functions (which are classical cumulants of traces of products of the matrices). We introduce the notion of "higher order freeness" and develop a theory of corresponding free cumulants. We show that two independent random matrix ensembles are free of arbitrary order if one of them is unitarily invariant. We prove R-transform formulas for second order freeness. Much of the presented theory relies on a detailed study of the properties of "partitioned permutations".

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## 1. Introduction

Random matrix models and their large dimension behavior have been an important subject of study in Mathematical Physics and Statistics since Wishart and Wigner. Global fluctuations of the eigenvalues (that is, linear functionals of the eigenvalues) of random matrices have been widely investigated in the last decade; see, e.g., [Joh98, Dia03, Rad06, AZ06, MN04, MŚS07]. Roughly speaking, the trend of these investigations is that for a wide class of converging random matrix models, the non-normalized trace asymptotically behaves like a Gaussian variable whose variance only depends on macroscopic parameters such as moments. The philosophy of these results, together with the freeness results of Voiculescu served as a motivation for our series of papers on second order freeness.

One of the main achievements of the free probability theory of Voiculescu [Voi91, VDN92] was an abstract description via the notion of "freeness" of the expectation of these Gaussian variables for a large class of non-commuting tuples of random matrices.

In the previous articles of this series [MS06, MŚS07] we showed that for many interesting ensembles of random matrices an analogue of the results of Voiculescu for expectations holds also true on the level of variances as well; thus pointing in the direction that the structure of random matrices and the fine structure of their eigenvalues can be studied in much more detail by using the new concept of "second order freeness". One of the main obstacles for such a detailed study was the absence of an effective machinery for doing concrete calculations in this framework. Within free probability theory of first order, such a machinery was provided by Voiculescu with the concept of the $R$-transform, and by Speicher with the concept of free cumulants; see, e.g., [VDN92, NSp06].

One of the main achievements of the present article is to develop a theory of second order cumulants (and show that the original definition of second order freeness from Part I of this series [MS06] is equivalent to the vanishing of mixed second order cumulants) and provide the corresponding $R$-transform machinery.

In Section 2 we will give a more detailed (but still quite condensed) survey of the connection between Voiculescu's free probability theory and random matrix theory. We will there also provide the main motivation, notions and concepts for our extension of this theory to the level of fluctuations (second order), as well as the statement of our main results concerning second order cumulants and $R$-transforms.

Having first and second order freeness it is, of course, a natural question whether this theory can be generalized to higher orders. It turns out that this is the case, most of the general theory is the same for all orders. So we will in this paper consider freeness of all orders from the very beginning and develop a general theory of higher order freeness and higher order cumulants. Let us, however, emphasize that first and second order freeness seem to be more important than the higher order ones. Actually, we can prove some of the most important results (e.g. the $R$-transform machinery) only for first and second order, mainly because of the complexity of the underlying combinatorial objects.

The basic combinatorial notion behind the (usual) free cumulants are non-crossing partitions. Basically, passage to higher order free cumulants corresponds to a change to multi-annular non-crossing permutations [MN04], or more general objects which we call "partitioned permutations". For much of the conceptual framework there is no difference between different levels of freeness, however for many concrete questions it seems that increasing the order makes some calculations much harder. This relates to the fact that $n$-th order freeness is described in terms of planar permutations which connect points on $n$ different circles. Whereas enumeration of all non-crossing permutations in the case of one circle is quite easy, the case of two circles gets more complicated, but is still feasible; for the case of three or more circles, however, the answer does not seem to be of a nice compact form.

In the present paper we develop the notion and combinatorial machinery for freeness of all orders by a careful analysis of the main example: unitarily invariant random matrices. We start with the calculation of mixed correlation functions for random matrices and use the structure which we observe there as a motivation for our combinatorial setup. In this way the concept of partitioned permutations and the moment–cumulant relations appear quite canonically.

We want to point out that even though our notion of second and higher order freeness is modeled on the situation found for correlation functions of random matrices, this notion and theory also have some far-reaching applications. Let us mention in this respect two points.

Firstly, recently one of us [Śni06] developed a quite general theory for fluctuations of characters and shapes of random Young diagrams contributing to many natural representations of symmetric groups. The results presented there are closely (though, not explicitly) related to combinatorics of higher order cumulants. This connection will be studied in detail in the part IV of this series where we prove that under some mild technical conditions Jucys-Murphy elements, which arise naturally in the study of symmetric groups, are examples of free random variables of higher order.

In another direction, the description of subfactors in von Neumann algebras via planar algebras [Jon99] relies very much on the notions of annular non-crossing partitions and thus resembles the combinatorial objects lying at the basis of our theory of second order freeness. This indicates that our results could have some relevance for subfactors.

OVERVIEW OF THE ARTICLE. In Section 2 we will give a compact survey of the connection between Voiculescu's free probability theory and random matrix theory, provide the main motivation, notions and concepts for our extension of this theory to the level of fluctuations (second order), as well as the statement of our main results concerning second order cumulants and $R$-transforms. We will also make a few general remarks about higher order freeness.

In Section 3 we will introduce the basic notions and relevant results on permutations, partitions, classical cumulants, Haar unitary random matrices, and the Weingarten function.
In Section 4 we study the correlation functions (classical cumulants of traces) of random matrix models. We will see how those are related to cumulants of entries of the matrices for unitarily invariant random matrices and we will in particular look on the correlation functions for products of two independent ensembles of random matrices, one of which is unitarily invariant. The limit of those formulas if the size $N$ of the matrices goes to infinity will be the essence of what we are going to call "higher order freeness". Also our main combinatorial objects, "partitioned permutations", will arise very naturally in these calculations.
In Section 5 we will forget for a while random variables and just look on the combinatorial essence of our formulas, thus dealing with multiplicative functions on partitioned permutations and their convolution. The Zeta and Möbius functions on partitioned permutations will play an important role in these considerations.
In Section 6 we will derive, for the case of second order, the analogue of the R-transform formulas.
In Section 7 we will finally come back to a (non-commutative) probabilistic context, give the definition and work out the basic properties of "higher order freeness".
In Section 8 we introduce the notion of "asymptotic higher order freeness" and show the relevance of our work for Itzykson-Zuber integrals.
In an appendix, Section 9, we provide a graphical interpretation of partitioned permutations as a special case of "surfaced permutations".

## 2. Motivation and Statement of our Main Results Concerning Second Order Freeness and Cumulants

In this section we will first recall in a quite compact form the main connection between Voiculescu's free probability theory and questions about random matrices. Then we want to motivate our notion of second order freeness by extending these questions from the level of expectations to the level of fluctuations. We will recall the relevant results from the papers [MS06, MŚS07] and state the main new results of the present paper. Even though in the later parts of the paper our treatment will include freeness of arbitrarily high order, we restrict ourselves in this section mainly to the second order. The reason for this is that (apart from first order) second order freeness seems to be the most important order for applications, so that it seems worthwhile to spell out our general results for this case more explicitly. Furthermore, it is only there that we have an analogue of $R$-transform formulas. We will make a few general remarks about higher order freeness at the end of this section.

### 2.1. Moments of random matrices and asymptotic freeness.

Assume we know the eigenvalue distribution of two matrices $A$ and $B$. What can we say about the eigenvalue distribution of the sum $A+B$ of the matrices? Of course, the latter is not just determined by the eigenvalues of $A$ and the eigenvalues of $B$, but also by the relation between the eigenspaces of $A$ and of $B$. Actually, it is quite a hard problem (Horn's conjecture) — which was only solved recently — to characterize all possible eigenvalue distributions of $A+B$. However, if one is asking this question in the context of $N \times N$-random matrices, then in many situations the answer becomes deterministic in the limit $N \to \infty$.

DEFINITION 2.1. Let $A = (A_N)_{N\in\mathbb{N}}$ be a sequence of $N \times N$-random matrices. We say that $A$ has a *first order limit distribution* if the limit of all moments

$$\alpha_n := \lim_{N\to\infty} E[\mathrm{tr}(A_N^n)] \qquad (n \in \mathbb{N})$$

exists and for all $r > 1$ and all $n_1, \dots, n_r \in \mathbb{N}$

$$\lim_{N\to\infty} k_r(\mathrm{tr}(A_N^{n_1}), \mathrm{tr}(A_N^{n_2}), \dots, \mathrm{tr}(A_N^{n_r})) = 0,$$

where $E$ denotes the expectation, tr the normalized trace, and $k_r$ the $r^{th}$ classical cumulant.

In this language, our question becomes: Given two random matrix ensembles of $N \times N$-random matrices, $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$, with first order limit distribution, does also their sum $C = (C_N)_{N\in\mathbb{N}}$, with $C_N = A_N + B_N$, have a first order limit distribution, and furthermore, can we calculate the limit moments $\alpha_n^C$ of $C$ out of the limit moments $(\alpha_k^A)_{k\geq 1}$ of $A$ and the limit moments $(\alpha_k^B)_{k\geq 1}$ of $B$ in a deterministic way. It turns out that this is the case if the two ensembles are in generic position, and then the rule for calculating the limit moments of $C$ are given by Voiculescu's concept of "freeness". Let us recall this fundamental result of Voiculescu.

THEOREM 2.2 (Voiculescu [Voi91]). *Let $A$ and $B$ be two random matrix ensembles of $N \times N$-random matrices, $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$, each of them with a first order limit distribution. Assume that $A$ and $B$ are independent (i.e., for each $N \in \mathbb{N}$, all entries of $A_N$ are independent from all entries of $B_N$), and that at least one of them is unitarily invariant (i.e., for each $N$, the joint distribution of the entries does not change if we conjugate the random matrix with an arbitrary unitary $N \times N$ matrix). Then $A$ and $B$ are asymptotically free in the sense of the following definition.*

DEFINITION 2.3 (Voiculescu [Voi85]). Two random matrix ensembles $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$ with limit eigenvalue distributions are *asymptotically free* if we have for all $p \geq 1$ and all $n(1), m(1), \dots, n(p), m(p) \geq 1$ that

$$\lim_{N\to\infty} E\Big[\mathrm{tr}\big\{(A_N^{n(1)} - \alpha_{n(1)}^A 1) \cdot (B_N^{m(1)} - \alpha_{m(1)}^B 1) \cdots \\ \cdots (A^{n(p)} - \alpha_{n(p)}^A 1) \cdot (B^{m(p)} - \alpha_{m(p)}^B 1)\big\}\Big] = 0$$

One should realize that asymptotic freeness is actually a rule which allows to calculate all mixed moments in $A$ and $B$, i.e. all expressions

$$\lim_{N\to\infty} E[\mathrm{tr}(A^{n(1)}B^{m(1)}A^{n(2)}B^{m(2)}\cdots A^{n(p)}B^{m(p)})]$$

out of the limit moments of $A$ and the limit moments of $B$. In particular, this means that all limit moments of $A+B$ (which are sums of mixed moments) exist and are actually determined in terms of the limit moments of $A$ and the limit moments of $B$. The actual calculation rule is not directly clear from the above definition but a basic result of Voiculescu shows how this can be achieved by going over from the moments $\alpha_n$ to new quantities $\kappa_n$. In [Spe94], the combinatorial structure behind these $\kappa_n$ was revealed and the name "free cumulants" was coined for them. Whereas in the later parts of this paper we will have to rely crucially on the combinatorial description and their extensions to higher orders, as well as on the definition of more general "mixed" cumulants, we will here state the results in the simplest possible form in terms of generating power series, which avoids the use of combinatorial objects.

DEFINITION 2.4 (Voiculescu [Voi86], Speicher [Spe94]). Given the moments $(\alpha_n)_{n\geq 1}$ of some distribution (or limit moments of some random matrix ensemble), we define the corresponding *free cumulants* $(\kappa_n)_{n\geq 1}$ by the following relation between their generating power series: If we put

$$M(x) := 1 + \sum_{n\geq 1} \alpha_n x^n \qquad \text{and} \qquad C(x) := 1 + \sum_{n\geq 1} \kappa_n x^n,$$

then we require as a relation between these formal power series that

$$C(xM(x)) = M(x).$$

Voiculescu actually formulated the relation above in a slightly different way using the so-called $R$-transform $\mathcal{R}(x)$, which is related to $C(x)$ by the relation

$$C(x) = 1 + x\mathcal{R}(x)$$

and in terms of the Cauchy transform $G(x)$ corresponding to a measure with moments $\alpha_n$, which is related to $M(x)$ by

$$G(x) = \frac{M(\frac{1}{x})}{x}.$$

In these terms the equation $C(xM(x)) = M(x)$ says that

$$\frac{1}{G(x)} + \mathcal{R}(G(x)) = x, \tag{1}$$

i.e., that $G(x)$ and $K(x) := \frac{1}{x} + \mathcal{R}(x)$ are inverses of each other under composition.

One should also note that the relation $C(xM(x)) = M(x)$ determines the moments uniquely in terms of the cumulants and the other way around. The relevance of the $\kappa_n$ and the $R$-transform for our problem comes from the following result of Voiculescu, which provides, together with (1), a very efficient way for calculating eigenvalue distributions of the sum of asymptotically free random matrices.

Theorem 2.5 (Voiculescu [Voi86]). *Let $A$ and $B$ be two random matrix ensembles which are asymptotically free. Denote by $\kappa_n^A$, $\kappa_n^B$, $\kappa_n^{A+B}$ the free cumulants of $A$, $B$, $A+B$, respectively. Then one has for all $n \geq 1$ that*

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B.$$

*Alternatively,*

$$\mathcal{R}^{A+B}(x) = \mathcal{R}^A(x) + \mathcal{R}^B(x).$$

This theorem is one reason for calling the $\kappa_n$ cumulants, but there is also another justification for this, namely they are also the limit of classical cumulants of the entries of our random matrix, in the case that this is unitarily invariant. This description will follow from our formulas (28) and (30). We denote the classical cumulants by $k_n$, considered as multi-linear functionals in $n$ arguments.

Theorem 2.6. *Let $A = (A_N)_{N\in\mathbb{N}}$ be a unitarily invariant random matrix ensemble of $N \times N$ random matrices $A_N$ whose first order limit distribution exists. Then the free cumulants of this matrix ensemble can also be expressed as the limit of special classical cumulants of the entries of the random matrices: If $A_N = (a_{ij}^{(N)})_{i,j=1}^N$, then*

$$\kappa_n^A = \lim_{N\to\infty} N^{n-1} k_n(a_{i(1)i(2)}^{(N)}, a_{i(2)i(3)}^{(N)}, \ldots, a_{i(n),i(1)}^{(N)})$$

*for any choice of distinct $i(1), \ldots, i(n)$.*

2.2. Fluctuations of random matrices and asymptotic second order freeness. There are many more refined questions about the limiting eigenvalue distribution of random matrices. In particular, questions around fluctuations have received a lot of interest in the last decade or so. The main motivation for introducing the concept of "second order freeness" was to understand the global fluctuations of the eigenvalues, which means that we look at the probabilistic behavior of traces of powers of our matrices. The limiting eigenvalue distribution, as considered in the last section, gives us the limit of the average of this traces. However, one can make more refined statements about their distributions. Consider a random matrix $A = (A_N)_{N\in\mathbb{N}}$ and look on the normalized traces $\mathrm{tr}(A_N^k)$. Our assumption of a limit eigenvalue distribution means that the limits $\alpha_k := \lim_{N\to\infty} E[\mathrm{tr}(A_N^k)]$ exist. It turned out that in many cases the fluctuation around this limit,

$$\mathrm{tr}(A_N^k) - \alpha_k$$

is asymptotically Gaussian of order $1/N$; i.e., the random variable

$$N \cdot (\mathrm{tr}(A_N^k) - \alpha_k) = \mathrm{Tr}(A_N^k) - N\alpha_k = \mathrm{Tr}(A_N^k - \alpha_k 1)$$

(where Tr denotes the unnormalized trace) converges for $N \to \infty$ to a normal variable. Actually, the whole family of centered unnormalized traces $(\mathrm{Tr}(A_N^k) -$

$N\alpha_k)_{k\geq 1}$ converges to a centered Gaussian family. (One should note that we restrict all our considerations to complex random matrices; in the case of real random matrices there are additional complications, which will be addressed in some future investigations.) Thus the main information about fluctuations of our considered ensemble is contained in the covariance matrix of the limiting Gaussian family, i.e., in the quantities

$$\alpha_{m,n} := \lim_{N\to\infty} \operatorname{cov}(\operatorname{Tr}(A_N^m), \operatorname{Tr}(A_N^n)).$$

Let us emphasize that the $\alpha_n$ and the $\alpha_{m,n}$ are actually limits of classical cumulants of traces; for the first and second order, with expectation as first and variance as second cumulant, this might not be so visible, but it will become evident when we go over to higher orders. Nevertheless, the $\alpha$'s will behave and will also be treated like moments; accordingly we will call the $\alpha_{m,n}$ 'fluctuation moments'. We will later define some other quantities $\kappa_{m,n}$, which take the role of cumulants in this context.

This kind of convergence to a Gaussian family was formalized in [MS06] as follows. Note that convergence to Gaussian means that all higher order classical cumulants converge to zero. As before, we denote the classical cumulants by $k_n$; so $k_1$ is just the expectation, and $k_2$ the covariance.

DEFINITION 2.7. Let $A = (A_N)_{N\in\mathbb{N}}$ be an ensemble of $N\times N$ random matrices $A_N$. We say that it has a *second order limit distribution* if for all $m, n \geq 1$ the limits

$$\alpha_n := \lim_{N\to\infty} k_1(\operatorname{tr}(A_N^n))$$

and

$$\alpha_{m,n} := \lim_{N\to\infty} k_2(\operatorname{Tr}(A_N^m), \operatorname{Tr}(A_N^n))$$

exist and if

$$\lim_{N\to\infty} k_r\big(\operatorname{Tr}(A_N^{n(1)}), \dots, \operatorname{Tr}(A_N^{n(r)})\big) = 0$$

for all $r \geq 3$ and all $n(1), \dots, n(r) \geq 1$.

We can now ask the same kind of question for the limit fluctuations as for the limit moments; namely, if we have two random matrix ensembles $A$ and $B$ and we know the second order limit distribution of $A$ and the second order limit distribution of $B$, does this imply that we have a second order limit distribution for $A+B$, and, if so, is there an effective way for calculating it. Again, we can only hope for a positive solution to this if $A$ and $B$ are in a kind of generic position. As it turned out, the same requirements as before are sufficient for this. The rule for calculating mixed fluctuations constitutes the essence of the definition of the concept of second order freeness.

THEOREM 2.8 (Mingo, Śniady, Speicher [MŚS07]). *Let $A$ and $B$ be two random matrix ensembles of $N\times N$-random matrices, $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$, each of them having a second order limit distribution. Assume that $A$ and $B$ are independent and that at least one of them is unitarily* *invariant. Then $A$ and $B$ are asymptotically free of second order in the sense of the following definition.*

Definition 2.9 (Mingo, Speicher [MS06]). Consider two random matrix ensembles $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$, each of them with a second order limit distribution. Denote by

$$Y_N\big(n(1), m(1), \dots, n(p), m(p)\big)$$

the random variable

$$\mathrm{Tr}\big((A_N^{n(1)} - \alpha^A_{n(1)}1)(B_N^{m(1)} - \alpha^B_{m(1)}1)\cdots(A_N^{n(p)} - \alpha^A_{n(p)}1)(B_N^{m(p)} - \alpha^B_{m(p)}1)\big).$$

The random matrices $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$ are *asymptotically free of second order* if for all $n, m \geq 1$

$$\lim_{N\to\infty} k_2\big(\mathrm{Tr}(A_N^n - \alpha^A_n 1), \mathrm{Tr}(B_N^m - \alpha^B_m 1)\big) = 0$$

and for all $p, q \geq 1$ and $n(1), \dots, n(p)$,$m(1), \dots, m(p)$,$\tilde{n}(1), \dots, \tilde{n}(q)$, $\tilde{m}(1), \dots, \tilde{m}(q) \geq 1$ we have

$$\lim_{N\to\infty} k_2\Big(Y_N\big(n(1), m(1), \dots, n(p), m(p)\big), Y_N\big(\tilde{n}(1), \tilde{m}(2), \dots, \tilde{n}(q), \tilde{m}(q)\big)\Big) = 0$$

if $p \neq q$, and otherwise (where we count modulo $p$ for the arguments of the indices, i.e., $n(i+p) = n(i)$)

$$\begin{aligned}&\lim_{N\to\infty} k_2\Big(Y_N\big(n(1), m(1), \dots, n(p), m(p)\big), Y_N\big(\tilde{n}(p), \tilde{m}(p), \dots, \tilde{n}(1), \tilde{m}(1)\big)\Big)\\ &= \sum_{k=1}^{p}\prod_{i=1}^{p}\big(\alpha^A_{n(i+k)+\tilde{n}(i)} - \alpha^A_{n(i+k)}\alpha^A_{\tilde{n}(i)}\big)\big(\alpha^B_{m(i+k)+\tilde{m}(i+1)} - \alpha^B_{m(i+k)}\alpha^B_{\tilde{m}(i+1)}\big).\end{aligned}$$

Again, it is crucial to realize that this definition allows one (albeit in a complicated way) to express every second order mixed moment, i.e., a limit of the form

$$\lim_{N\to\infty} k_2\big(\mathrm{Tr}(A_N^{n(1)}B_N^{m(1)}\cdots A_N^{n(p)}B_N^{m(p)}), \mathrm{Tr}(A_N^{\tilde{n}(1)}B_N^{\tilde{m}(1)}\cdots A_N^{\tilde{n}(q)}B_N^{\tilde{m}(q)})\big)$$

in terms of the second order limits of $A$ and the second order limits of $B$. In particular, asymptotic freeness of second order also implies that the sum $A+B$ of our random matrix ensembles has a second order limit distribution and allows one to express them in principle in terms of the second order limit distribution of $A$ and the second order limit distribution of $B$. As in the case of first order freeness, it is not clear at all how this calculation of the fluctuations of $A+B$ out of the fluctuations of $A$ and the fluctuations of $B$ can be performed effectively. It is one of the main results of the present paper to achieve such an effective description. We are able to solve this problem by providing a second order cumulant machinery, similar to the first order case. Again, the idea is to go over to quantities which behave like cumulants in this setting. The actual description of those relies on combinatorial objects (annular non-crossing permutations), but as before this can be reformulated in terms of formal power series. Let us spell out the definition here in this form. (That this is equivalent to our actual definition of the cumulants will follow from Theorem 6.3.)

DEFINITION 2.10. Let $(\alpha_n)_{n\geq 1}$ and $(\alpha_{m,n})_{m,n\geq 1}$ describe the first and second order limit moments of a random matrix ensemble. We define the corresponding *first and second order free cumulants* $(\kappa_n)_{n\geq 1}$ and $(\kappa_{m,n})_{m,n\geq 1}$ by the following requirement in terms of the corresponding generating power series. Put

$$C(x) := 1 + \sum_{n\geq 1} \kappa_n x^n, \qquad C(x,y) := \sum_{m,n\geq 1} \kappa_{m,n} x^m y^n$$

and

$$M(x) := 1 + \sum_{n\geq 1} \alpha_n x^n, \qquad M(x,y) := \sum_{m,n\geq 1} \alpha_{m,n} x^m y^n.$$

Then we require as relations between these formal power series that

$$C(xM(x)) = M(x) \tag{2}$$

and for the second order

$$M(x,y) = H\big(xM(x), yM(y)\big) \cdot \frac{\frac{d}{dx}(xM(x))}{M(x)} \cdot \frac{\frac{d}{dy}(yM(y))}{M(y)}, \tag{3}$$

where

$$H(x,y) := C(x,y) - xy \frac{\partial^2}{\partial x \partial y} \log\Big(\frac{xC(y) - yC(x)}{x-y}\Big), \tag{4}$$

or equivalently,

$$\begin{aligned} M(x,y) = C\big(xM(x), yM(y)\big) \cdot \frac{\frac{d}{dx}(xM(x))}{M(x)} \cdot \frac{\frac{d}{dy}(yM(y))}{M(y)} \\ + xy\Big(\frac{\frac{d}{dx}(xM(x)) \cdot \frac{d}{dy}(yM(y))}{(xM(x) - yM(y))^2} - \frac{1}{(x-y)^2}\Big). \end{aligned} \tag{5}$$

From equation (5) one can calculate the second order version of *moment-cumulant* relations.

$$\begin{gathered}
\alpha_{1,1} = \kappa_{1,1} + \kappa_2 \\
\alpha_{2,1} = \kappa_{1,2} + 2\kappa_1\kappa_{1,1} + 2\kappa_3 + 2\kappa_1\kappa_2 \\
\alpha_{2,2} = \kappa_{2,2} + 4\kappa_1\kappa_{2,1} + 4\kappa_1^2\kappa_{1,1} + 4\kappa_4 + 8\kappa_1\kappa_3 + 2\kappa_2^2 + 4\kappa_1^2\kappa_2 \\
\alpha_{1,3} = \kappa_{1,3} + 3\kappa_1\kappa_{2,1} + 3\kappa_2\kappa_{1,1} + 3\kappa_1^2\kappa_{1,1} + 3\kappa_4 + 6\kappa_1\kappa_3 + 3\kappa_2^2 + 3\kappa_1^2\kappa_2 \\
\alpha_{2,3} = \kappa_{2,3} + 2\kappa_1\kappa_{1,3} + 3\kappa_1\kappa_{2,2} + 3\kappa_2\kappa_{1,2} + 9\kappa_1^2\kappa_{1,2} + 6\kappa_1\kappa_2\kappa_{1,1} + 6\kappa_1^3\kappa_{1,1} \\
+ 6\kappa_5 + 18\kappa_1\kappa_4 + 12\kappa_2\kappa_3 + 18\kappa_1^2\kappa_3 + 12\kappa_1\kappa_2^2 + 6\kappa_1^3\kappa_2 \\
\alpha_{3,3} = \kappa_{3,3} + 6\kappa_1\kappa_{2,3} + 6\kappa_2\kappa_{1,3} + 6\kappa_1^2\kappa_{1,3} + 9\kappa_1^2\kappa_{2,2} + 18\kappa_1\kappa_2\kappa_{1,2} + 18\kappa_1^3\kappa_{1,2} \\
+ 9\kappa_2^2\kappa_{1,1} + 18\kappa_1^2\kappa_2\kappa_{1,1} + 9\kappa_1^4\kappa_{1,1} + 9\kappa_6 + 36\kappa_1\kappa_5 + 27\kappa_2\kappa_4 + 54\kappa_1^2\kappa_4 \\
+ 9\kappa_3^2 + 72\kappa_1\kappa_2\kappa_3 + 36\kappa_1^3\kappa_3 + 12\kappa_2^3 + 36\kappa_1^2\kappa_2^2 + 9\kappa_1^4\kappa_2
\end{gathered}$$

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$$\begin{gathered}
\kappa_{1,1} = \alpha_1^2 - \alpha_2 + \alpha_{1,1} \\
\kappa_{1,2} = -4\alpha_1^3 + 6\alpha_1\alpha_2 - 2\alpha_3 - 2\alpha_1\alpha_{1,1} + \alpha_{1,2}
\end{gathered}$$

$$\kappa_{2,2} = 18\alpha_1^4 - 36\alpha_1^2\alpha_2 + 6\alpha_2^2 + 16\alpha_1\alpha_3 - 4\alpha_4 + 4\alpha_1^2\alpha_{1,1} - 4\alpha_1\alpha_{1,2} + \alpha_{2,2}$$
$$\kappa_{1,3} = 15\alpha_1^4 - 30\alpha_1^2\alpha_2 + 6\alpha_2^2 + 12\alpha_1\alpha_3 - 3\alpha_4 + 6\alpha_1^2\alpha_{1,1} - 3\alpha_2\alpha_{1,1} - 3\alpha_1\alpha_{1,2} + \alpha_{1,3}$$
$$\begin{aligned}\kappa_{2,3} = {} & -72\alpha_1^5 + 180\,\alpha_1^3\,\alpha_2 - 72\alpha_1\alpha_2^2 - 84\,\alpha_1^2\alpha_3 + 24\alpha_2\alpha_3 + 30\alpha_1\alpha_4 - 6\alpha_5 \\ & - 12\alpha_1^3\alpha_{1,1} + 6\alpha_1\alpha_2\alpha_{1,1} + 12\alpha_1^2\alpha_{1,2} - 3\alpha_2\alpha_{1,2} - 2\alpha_1\alpha_{1,3} - 3\alpha_1\alpha_{2,2} + \alpha_{2,3}\end{aligned}$$
$$\begin{aligned}\kappa_{3,3} = {} & 300\alpha_1^6 - 900\alpha_1^4\alpha_2 + 576\alpha_1^2\alpha_2^2 - 48\alpha_2^3 + 432\alpha_1^3\alpha_3 - 288\alpha_1\alpha_2\alpha_3 + 18\alpha_3^2 \\ & - 180\alpha_1^2\alpha_4 + 45\alpha_2\alpha_4 + 54\alpha_1\alpha_5 - 9\alpha_6 + 36\alpha_1^4\alpha_{1,1} - 36\alpha_1^2\alpha_2\alpha_{1,1} + 9\alpha_2^2\alpha_{1,1} \\ & - 36\alpha_1^3\alpha_{1,2} + 18\alpha_1\alpha_2\alpha_{1,2} + 12\alpha_1^2\alpha_{1,3} - 6\alpha_2\alpha_{1,3} + 9\alpha_1^2\alpha_{2,2} - 6\alpha_1\alpha_{2,3} + \alpha_{3,3}\end{aligned}$$

As in the first order case, instead of the moment power series $M(x,y)$ one can consider a kind of second order Cauchy transform, defined by

$$G(x,y) := \frac{M(\frac{1}{x},\frac{1}{y})}{xy}.$$

If we also define a kind of second order $R$ transform $\mathcal{R}(x,y)$ by

$$\mathcal{R}(x,y) := \frac{1}{xy}C(x,y),$$

then the formula (5) takes on a particularly nice form:

$$G(x,y) = G'(x)G'(y)\Big\{\mathcal{R}(G(x),G(y)) + \frac{1}{(G(x)-G(y))^2}\Big\} - \frac{1}{(x-y)^2}. \tag{6}$$

$G(x)$ is here, as before, the first order Cauchy transform, $G(x) = \frac{1}{x}M(1/x)$. The $\kappa_{m,n}$ defined above deserve the name "cumulants" as they linearize the problem of adding random matrices which are asymptotically free of second order. Namely, as will follow from our Theorem 7.15, we have the following theorem, which provides, together with (6), an effective machinery for calculating the fluctuations of the sum of asymptotically free random matrices.

Theorem 2.11. *Let $A$ and $B$ be two random matrix ensembles which are asymptotically free. Then one has for all $m,n \geq 1$ that*

$$\kappa_n^{A+B} = \kappa_n^A + \kappa_n^B \qquad \textit{and} \qquad \kappa_{m,n}^{A+B} = \kappa_{m,n}^A + \kappa_{m,n}^B.$$

*Alternatively,*

$$\mathcal{R}^{A+B}(x) = \mathcal{R}^A(x) + \mathcal{R}^B(x)$$

*and*

$$\mathcal{R}^{A+B}(x,y) = \mathcal{R}^A(x,y) + \mathcal{R}^B(x,y).$$

Again, one can express the second order cumulants as limits of classical cumulants of entries of a unitarily invariant matrix. In contrast to the first order case, we have now to run over two disjoint cycles in the indices of the matrix entries. This theorem will follow from our formulas (28) and (30).

Theorem 2.12. *Let $A = (A_N)_{N\in\mathbb{N}}$ be a unitarily invariant random matrix ensemble which has a second order limit distribution. Then the second order free cumulants of this matrix ensemble can also be expressed as the limit of classical cumulants of the entries of the random matrices: If $A_N = (a_{ij}^{(N)})_{i,j=1}^N$, then*

$$\kappa^A_{m,n} = \lim_{N\to\infty} N^{m+n} k_{m+n}\big(a^{(N)}_{i(1)i(2)}, a^{(N)}_{i(2)i(3)}, \dots, a^{(N)}_{i(m),i(1)}, \\ a^{(N)}_{j(1)j(2)}, a^{(N)}_{j(2)j(3)}, \dots, a^{(N)}_{j(n),j(1)}\big)$$

*for any choice of distinct $i(1), \dots, i(m), j(1), \dots, j(n)$.*

This latter theorem makes it quite obvious that the second order cumulants for Gaussian as well as for Wishart matrices vanish identically, i.e., $\mathcal{R}(x,y) = 0$ and thus we obtain in these cases that the second order Cauchy transform is totally determined in terms of the first order Cauchy transform (i.e., in terms of the limiting eigenvalue distribution) via

$$G(x,y) = \frac{G'(x)G'(y)}{(G(x)-G(y))^2} - \frac{1}{(x-y)^2}. \tag{7}$$

This formula for fluctuations of Wishart matrices was also derived by Bai and Silverstein in [BS04].

2.3. Higher order freeness. The idea for higher order freeness is the same as for second order one. For a random matrix ensemble $A = (A_N)_{N\in\mathbb{N}}$ we define $r$-th order limit moments as the scaled limit of classical cumulants of $r$ traces of powers of our matrices,

$$\alpha_{n_1,\dots,n_r} := \lim_{N\to\infty} N^{r-2} k_r\big(\mathrm{Tr}(A_N^{n(1)}), \dots, \mathrm{Tr}(A_N^{n(r)})\big).$$

(The choice of $N^{r-2}$ is motivated by the fact that this is the leading order for many interesting random matrix ensembles, e.g. Gaussian or Wishart. Thus our theory of higher order freeness captures the features of random matrix ensembles whose cumulants of traces decay in the same way as Gaussian random matrices.) Then we look at two random matrix ensembles $A$ and $B$ which are independent, and one of them unitarily invariant. The mixed moments in $A$ and $B$ of order $r$ are, in leading order in the limit $N\to\infty$, determined by the limit moments of $A$ up to order $r$ and the limit moments of $B$ up to order $r$. The structure of these formulas motivates directly the definition of cumulants of the considered order. The definition of those is in terms of a moment-cumulant formula, which gives a moment in terms of cumulants by summing over special combinatorial objects, which we call "partitioned permutations". Most of the theory we develop relies on an in depth analysis of properties of these partitioned permutations and the corresponding convolution of multiplicative functions on partitioned permutations. Our definition of "higher order freeness" is then in terms of the vanishing of mixed cumulants. It follows quite easily that in the first and second order case this gives the same as the relations in Definitions 2.3 and 2.9, respectively. For higher orders, however, we are not able to find an explicit relation of that type.

This reflects somehow the observation that our general formulas in terms of sums over partitioned permutations are the same for all orders, but that evaluating or simplifying these sums (by doing partial summations) is beyond our abilities for orders greater than 2. Reformulating the combinatorial relation between moments and cumulants in terms of generating power series is one prominent example for this. Whereas this is quite easy for first order, the complexity of the arguments and the solution (given in Definition 2.10) is much higher for second order, and out of reach for higher order.

One should note that an effective (analytic or symbolic) calculation of higher order moments of a sum $A+B$ for $A$ and $B$ free of higher order relies usually on the presence of such generating power series formulas. In this sense, we have succeeded in providing an effective machinery for dealing with fluctuations (second order), but we were not able to do so for higher order.

Our results for higher orders are more of a theoretical nature. One of the main problems we have to address there is the associativity of the notion of higher order freeness. Namely, in order to be an interesting concept, our definition that $A$ and $B$ are free of higher order should of course imply that any function of $A$ is also free of higher order from any function of $B$. Whereas for first and second order this follows quite easily from the equivalent characterization of freeness in terms of moments as in Definitions 2.3 and 2.9, the absence of such a characterization for higher orders makes this a more complicated matter. Namely, what we have to see is that the vanishing of mixed cumulants in random variables implies also the vanishing of mixed cumulants in elements from the generated algebras. This is quite a non-trivial fact and requires a careful analysis, see section 7.

## 3. Preliminaries

3.1. Some general notation. For natural numbers $m, n \in \mathbb{N}$ with $m < n$, we denote by $[m, n]$ the interval of natural numbers between $m$ and $n$, i.e.,

$$[m, n] := \{m, m+1, m+2, \dots, n-1, n\}.$$

For a matrix $A = (a_{ij})_{i,j=1}^{N}$, we denote by Tr the unnormalized and by tr the normalized trace,

$$\mathrm{Tr}(A) := \sum_{i=1}^{N} a_{ii}, \qquad \mathrm{tr}(A) := \frac{1}{N}\mathrm{Tr}(A).$$

3.2. Permutations. We will denote the set of permutations on $n$ elements by $S_n$. We will quite often use the cycle notation for such permutations, i.e., $\pi = (i_1, i_2, \dots, i_r)$ is a cycle which sends $i_k$ to $i_{k+1}$ ($k = 1, \dots, r$), where $i_{r+1} = i_1$.

3.2.1. *Length function.* For a permutation $\pi \in S_n$ we denote by $\#\pi$ the number of cycles of $\pi$ and by $|\pi|$ the minimal number of transpositions needed to write $\pi$ as a product of transpositions. Note that one has

$$|\pi| + \#\pi = n \qquad \text{for all } \pi \in S_n.$$

3.2.2. *Non-crossing permutations.* Let us denote by $\gamma_n \in S_n$ the cycle

$$\gamma_n = (1, 2, \dots, n).$$

For all $\pi \in S_n$ one has that

$$|\pi| + |\gamma_n \pi^{-1}| \leq n - 1.$$

If we have equality then we call $\pi$ *non-crossing*. Note that this is equivalent to

$$\#\pi + \#(\gamma_n \pi^{-1}) = n + 1.$$

If $\pi$ is non-crossing, then so are $\gamma_n \pi^{-1}$ and $\pi^{-1}\gamma_n$; the latter is called the *(Kreweras) complement* of $\pi$.
We will denote the set of non-crossing permutations in $S_n$ by $NC(n)$. Note that such a non-crossing permutation can be identified with a non-crossing partition, by forgetting the order on the cycles. There is exactly one cyclic order on the blocks of a non-crossing partition which makes it into a non-crossing permutation.

3.2.3. *Annular non-crossing permutations.* Fix $m, n \in \mathbb{N}$ and denote by $\gamma_{m,n}$ the product of the two cycles

$$\gamma_{m,n} = (1, 2, \dots, m)(m + 1, m + 2, \dots, m + n).$$

More generally, we shall denote by $\gamma_{m_1,\dots,m_k}$ the product of the corresponding $k$ cycles.
We call a $\pi \in S_{m+n}$ *connected* if the pair $\pi$ and $\gamma_{m,n}$ generates a transitive subgroup in $S_{m+n}$. A connected permutation $\pi \in S_{m+n}$ always satisfies

$$|\pi| + |\gamma_{m,n}\pi^{-1}| \leq m + n. \tag{8}$$

If $\pi$ is connected and if we have equality in that equation then we call $\pi$ *annular non-crossing*. Note that if $\pi$ is annular non-crossing then $\gamma_{m,n}\pi^{-1}$ is also annular non-crossing. Again, we call the latter the *complement* of $\pi$. Of course, all the above notations depend on the pair $(m, n)$; if we want to emphasize this dependency we will also speak about $(m, n)$-connected permutations and $(m, n)$-annular non-crossing permutations.
We will denote the set of $(m, n)$-annular non-crossing permutations by $S_{NC}(m, n)$. A cycle of a $\pi \in S_{NC}(m, n)$ is called a *through-cycle* if it contains points on both cycles. Each $\pi \in S_{NC}(m, n)$ is connected and must thus have at least one through-cycle. The subset of $S_{NC}(m, n)$ where all cycles are through-cycles will be denoted by $S_{NC}^{all}(m, n)$.
Again one can go over from annular non-crossing permutations to annular non-crossing partitions by forgetting the cyclic orders on cycles; however, in the annular case, the relation between non-crossing permutation and non-crossing

partition is not one-to-one. Since we will not use the language of annular partitions in the present paper, this is of no relevance here.
Annular non-crossing permutations and partitions were introduced in [MN04]; there, many different characterizations—in particular, the one (8) above in terms of the length function—were given.

3.3. Partitions. We say that $\mathcal{V} = \{V_1, \dots, V_k\}$ is a partition of a set $[1, n]$ if the sets $V_i$ are disjoint and non–empty and their union is equal to $[1, n]$. We call $V_1, \dots, V_k$ the blocks of partition $\mathcal{V}$.
If $\mathcal{V} = \{V_1, \dots, V_k\}$ and $\mathcal{W} = \{W_1, \dots, W_l\}$ are partitions of the same set, we say that $\mathcal{V} \leq \mathcal{W}$ if for every block $V_i$ there exists some block $W_j$ such that $V_i \subseteq W_j$. For a pair of partitions $\mathcal{V}, \mathcal{W}$ we denote by $\mathcal{V} \vee \mathcal{W}$ the smallest partition $\mathcal{U}$ such that $\mathcal{V} \leq \mathcal{U}$ and $\mathcal{W} \leq \mathcal{U}$. We denote by $1_n = \big\{[1, n]\big\}$ the biggest partition of the set $[1, n]$.
If $\pi \in S_n$ is a permutation, then we can associate to $\pi$ in a natural way a partition whose blocks consist exactly of the cycles of $\pi$; we will denote this partition either by $0_\pi \in \mathcal{P}(n)$ or, if the context makes the meaning clear, just by $\pi \in \mathcal{P}(n)$.
For a permutation $\pi \in S_n$ we say that a partition $\mathcal{V}$ is $\pi$-invariant if $\pi$ preserves each block of $\mathcal{V}$. This means that $0_\pi \leq \mathcal{V}$ (which we will usually write just as $\pi \leq \mathcal{V}$).
If $\mathcal{V} = \{V_1, \dots, V_k\}$ is a partition of the set $[1, n]$ and if, for $1 \leq i \leq k$, $\pi_i$ is a permutation of the set $V_i$ we denote by $\pi_1 \times \cdots \times \pi_k \in S_n$ the concatenation of these permutations. We say that $\pi = \pi_1 \times \cdots \times \pi_k$ is a cycle decomposition if additionally every factor $\pi_i$ is a cycle.

3.4. Classical cumulants. Given some classical probability space $(\Omega, P)$ we denote by E the expectation with respect to the corresponding probability measure,

$$\mathrm{E}(a) := \int_\Omega a(\omega) dP(\omega)$$

and by $L^{\infty-}(\Omega, P)$ the algebra of random variables for which all moments exist. Let us for the following put $\mathcal{A} := L^{\infty-}(\Omega, P)$.
We extend the linear functional $\mathrm{E} : \mathcal{A} \to \mathbb{C}$ to a corresponding multiplicative functional on all partitions by ($\mathcal{V} \in \mathcal{P}(n)$, $a_1, \dots, a_n \in \mathcal{A}$)

$$\mathrm{E}_\mathcal{V}[a_1, \dots, a_n] := \prod_{V \in \mathcal{V}} \mathrm{E}[a_1, \dots, a_n|_V], \tag{9}$$

where we use the notation

$$\mathrm{E}[a_1, \dots, a_n|_V] := \mathrm{E}(a_{i_1} \cdots a_{i_s}) \qquad \text{for} \qquad V = (i_1 < \cdots < i_s) \in \mathcal{V}.$$

Then, for $\mathcal{V} \in \mathcal{P}(n)$, we define the *classical cumulants* $k_\mathcal{V}$ as multilinear functionals on $\mathcal{A}$ by

$$k_\mathcal{V}[a_1, \dots, a_n] = \sum_{\substack{\mathcal{W} \in \mathcal{P}(n) \\ \mathcal{W} \leq \mathcal{V}}} \mathrm{E}_\mathcal{W}[a_1, \dots, a_n] \cdot \mathrm{Möb}_{\mathcal{P}(n)}(\mathcal{W}, \mathcal{V}), \tag{10}$$

where $\mathrm{M\ddot{o}b}_{\mathcal{P}(n)}$ denotes the Möbius function on $\mathcal{P}(n)$ (see [Rot64]).
The above definition is, by Möbius inversion on $\mathcal{P}(n)$, equivalent to

$$\mathrm{E}(a_1 \cdots a_n) = \sum_{\pi \in \mathcal{P}(n)} k_\pi[a_1, \dots, a_n].$$

The $k_\pi$ are also multiplicative with respect to the blocks of $\mathcal{V}$ and thus determined by the values of

$$k_n(a_1, \dots, a_n) := k_{1_n}[a_1, \dots, a_n].$$

Note that we have in particular

$$k_1(a) = \mathrm{E}(a) \qquad \text{and} \qquad k_2(a_1, a_2) = \mathrm{E}(a_1 a_2) - \mathrm{E}(a_1)\mathrm{E}(a_2).$$

An important property of classical cumulants is the following formula of Leonov and Shiryaev [LS59] for cumulants with products as arguments.
Let $m, n \in \mathbb{N}$ and $1 \leq i(1) < i(2) < \cdots < i(m) = n$. Define $\mathcal{U} \in \mathcal{P}(n)$ by

$$\mathcal{U} = \big\{ \big(1, \dots, i(1)\big), \big(i(1)+1, \dots, i(2)\big), \dots, \big(i(m-1)+1, \dots, i(m)\big) \big\}.$$

Consider now random variables $a_1, \dots, a_n \in \mathcal{A}$ and define

$$\begin{aligned} A_1 &:= a_1 \cdots a_{i(1)} \\ A_2 &:= a_{i(1)+1} \cdots a_{i(2)} \\ &\vdots \\ A_m &:= a_{i(m-1)+1} \cdots a_{i(m)}. \end{aligned}$$

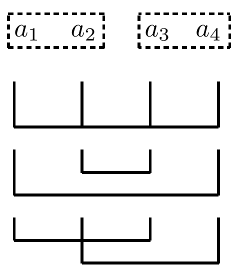
Then we have

$$k_m(A_1, A_2, \dots, A_m) = \sum_{\substack{\mathcal{V} \in \mathcal{P}(n) \\ \mathcal{V} \vee \mathcal{U} = 1_n}} k_{\mathcal{V}}[a_1, \dots, a_n]. \tag{11}$$

The sum on the right-hand side is running over those partitions of $n$ elements which satisfy $\mathcal{V} \vee \mathcal{U} = 1_n$, which are, informally speaking, those partitions which connect all the arguments of the cumulant $k_m$, when written in terms of the $a_i$.
Here is an example for this formula; for $k_2(a_1 a_2, a_3 a_4)$. In order to reduce the number of involved terms we will restrict to the special case where $\mathrm{E}(a_i) = 0$ (and thus also $k_1(a_i) = 0$) for all $i = 1, 2, 3, 4$. There are three partitions $\pi \in \mathcal{P}(4)$ without singletons which satisfy

$$\pi \vee \{(1,2),(3,4)\} = 1_4,$$

namely



and thus formula (11) gives in this case

$$k_2(a_1a_2, a_3a_4) = k_4(a_1, a_2, a_3, a_4) \\ + k_2(a_1, a_4)k_2(a_2, a_3) + k_2(a_1, a_3)k_2(a_2, a_4).$$

As a consequence of (11) one has the following important corollary: If $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are independent then

$$k_{\mathcal{W}}[a_1b_1, \dots, a_nb_n] = \sum_{\substack{\mathcal{V},\mathcal{V}' \in \mathcal{P}(n) \\ \mathcal{V} \vee \mathcal{V}' = \mathcal{W}}} k_{\mathcal{V}}[a_1, \dots, a_n] \cdot k_{\mathcal{V}'}[b_1, \dots, b_n]. \tag{12}$$

3.5. HAAR DISTRIBUTED UNITARY RANDOM MATRICES AND THE WEINGARTEN FUNCTION. In the following we will be interested in the asymptotics of special matrix integrals over the group $\mathcal{U}(N)$ of unitary $N \times N$-matrices. We always equip the compact group $\mathcal{U}(N)$ with its Haar probability measure. A random matrix whose distribution is this measure will be called a *Haar distributed unitary random matrix*. Thus the expectation E over this ensemble is given by integrating with respect to the Haar measure.

The expectation of products of entries of Haar distributed unitary random matrices can be described in terms of a special function on the permutation group. Since such considerations go back to Weingarten [Wei78], Collins [Col03] calls this function the *Weingarten function* and denotes it by Wg. We will follow his notation. In the following we just recall the relevant information about this Weingarten function, for more details we refer to [Col03, CŚ06].

We use the following definition of the Weingarten function. For $\pi \in S_n$ and $N \geq n$ we put

$$\mathrm{Wg}(N, \pi) = \mathrm{E}[u_{11} \cdots u_{nn} \overline{u_{1\pi(1)}} \cdots \overline{u_{n\pi(n)}}],$$

where $U = (u_{ij})_{i,j=1}^N$ is an $N \times N$ Haar distributed unitary random matrix. Sometimes we will suppress the dependency on $N$ and just write $\mathrm{Wg}(\pi)$. This $\mathrm{Wg}(N, \pi)$ depends only on the conjugacy class of $\pi$. General matrix integrals over the unitary group can be calculated as follows:

$$\mathrm{E}[u_{i'_1 j'_1} \cdots u_{i'_n j'_n} \overline{u_{i_1 j_1}} \cdots \overline{u_{i_n j_n}}] \\ = \sum_{\alpha, \beta \in S_n} \delta_{i_1 i'_{\alpha(1)}} \cdots \delta_{i_n i'_{\alpha(n)}} \delta_{j_1 j'_{\beta(1)}} \cdots \delta_{j_n j'_{\beta(n)}} \mathrm{Wg}(\beta\alpha^{-1}). \tag{13}$$

This formula for the calculation of moments of the entries of a Haar unitary random matrix bears some resemblance to the Wick formula for the joint moments of the entries of Gaussian random matrices; thus we will call (13) the *Wick formula for Haar unitary matrices*.

The Weingarten function is quite a complicated object, and its full understanding is at the basis of questions around Itzykson-Zuber integrals. One knows (see, e.g., [Col03, CŚ06]) that the leading order in $1/N$ is given by $|\pi| + n$ and increases in steps of 2.

3.6. Cumulants of the Weingarten function. We will also need some (classical) relative cumulants of the Weingarten function, which were introduced in [Col03, §2.3]. As before, let $\text{Möb}_{\mathcal{P}(n)}$ be the Möbius function on the partially ordered set of partitions of $[1,n]$ ordered by inclusion.
Let us first extend the Weingarten function by multiplicative extension, for $\mathcal{V} \geq \pi$, by

$$\text{Wg}(\mathcal{V},\pi) := \prod_{V\in\mathcal{V}} \text{Wg}(\pi|_V),$$

where $\pi|_V$ denotes the restriction of $\pi$ to the block $V \in \mathcal{V}$ (which is invariant under $\pi$ since $\pi \leq \mathcal{V}$).
The *relative cumulant of the Weingarten function* is now, for $\sigma \leq \mathcal{V} \leq \mathcal{W}$, defined by

$$C_{\mathcal{V},\mathcal{W}}(\sigma) = \sum_{\substack{\mathcal{U}\in\mathcal{P}(n)\\ \mathcal{V}\leq\mathcal{U}\leq\mathcal{W}}} \text{Möb}(\mathcal{U},\mathcal{W}) \cdot \text{Wg}(\mathcal{U},\sigma). \tag{14}$$

Note that, by Möbius inversion, this is, for any $\sigma \leq \mathcal{V} \leq \mathcal{W}$, equivalent to

$$\text{Wg}(\mathcal{W},\sigma) = \sum_{\substack{\mathcal{U}\in\mathcal{P}(n)\\ \mathcal{V}\leq\mathcal{U}\leq\mathcal{W}}} C_{\mathcal{V},\mathcal{U}}(\sigma). \tag{15}$$

In [Col03, Cor. 2.9] it was shown that the order of $C_{\mathcal{V},\mathcal{W}}(\sigma)$ is at most

$$N^{-2n+\#\sigma+2\#\mathcal{W}-2\#\mathcal{V}}. \tag{16}$$

## 4. Correlation functions for random matrices

4.1. Correlation functions and partitioned permutations. Let us consider $N\times N$-random matrices $B_1,\dots,B_n : \Omega \to M_N(\mathbb{C})$. The main information we are interested in are the "correlation functions" $\varphi_n$ of these matrices, given by classical cumulants of their traces, i.e.,

$$\varphi_n(B_1,\dots,B_n) := k_n(\text{Tr}(B_1),\dots,\text{Tr}(B_n)).$$

Even though these correlation functions are cumulants, it is more adequate to consider them as a kind of moments for our random matrices. Thus, we will also call them sometimes *correlation moments*.
We will also need to consider traces of products which are best encoded via permutations. Thus, for $\pi \in S_n$, $\varphi(\pi)[B_1,\dots,B_n]$ shall mean that we take cumulants of traces of products along the cycles of $\pi$. For an $n$-tuple $B = (B_1,\dots,B_n)$ of random matrices and a cycle $c = (i_1,i_2,\dots,i_k)$ with $k \leq n$ we denote

$$B|_c := B_{i_1}B_{i_2}\cdots B_{i_k}.$$

(We do not distinguish between products which differ by a cyclic rotation of the factors; however, in order to make this definition well-defined we could normalize our cycle $c = (i_1,i_2,\dots,i_k)$ by the requirement that $i_1$ is the smallest among the appearing numbers.) For any $\pi \in S(n)$ and any $n$-tuple $B = (B_1,\dots,B_n)$ of random matrices we put

$$\varphi(\pi)[B_1,\dots,B_n] := \varphi_r(B|_{c_1},\dots,B|_{c_r}),$$

where $\pi$ consists of the cycles $c_1, \dots, c_r$.
Example:

$$\begin{aligned}\varphi((1,3)(2,5,4))[B_1, B_2, B_3, B_4, B_5] &= \varphi_2(B_1B_3, B_2B_5B_4)\\ &= k_2(\mathrm{Tr}(B_1B_3), \mathrm{Tr}(B_2B_5B_4))\end{aligned}$$

Furthermore, we also need to consider more general products of such $\varphi(\pi)$'s. In order to index such products we will use pairs $(\mathcal{V}, \pi)$ where $\pi$ is, as above, an element in $S_n$, and $\mathcal{V} \in \mathcal{P}(n)$ is a partition which is compatible with the cycle structure of $\pi$, i.e., each block of $\mathcal{V}$ is fixed under $\pi$, or to put it another way, $\mathcal{V} \geq \pi$. In the latter inequality we use the convention that we identify a permutation with the partition corresponding to its cycles if this identification is obvious from the structure of the formula; we will write this partition $0_\pi$ or just 0 if no confusion will result.

Notation 4.1. A *partitioned permutation* is a pair $(\mathcal{V}, \pi)$ consisting of $\pi \in S_n$ and $\mathcal{V} \in \mathcal{P}(n)$ with $\mathcal{V} \geq \pi$. We will denote the set of partitioned permutations of $n$ elements by $\mathcal{PS}(n)$. We will also put

$$\mathcal{PS} := \bigcup_{n \in \mathbb{N}} \mathcal{PS}(n).$$

For such a $(\mathcal{V}, \pi) \in \mathcal{PS}$ we denote finally

$$\varphi(\mathcal{V}, \pi)[B_1, \dots, B_n] := \prod_{V \in \mathcal{V}} \varphi(\pi|_V)[B_1, \dots, B_n|_V].$$

Example:

$$\begin{aligned}&\varphi\big(\{1,3,4\}\{2\}, (1,3)(2)(4)\big)[B_1, B_2, B_3, B_4]\\ &\qquad = \varphi_2(B_1B_3, B_4) \cdot \varphi_1(B_2)\\ &\qquad = k_2(\mathrm{Tr}(B_1B_3), \mathrm{Tr}(B_4)) \cdot k_1(\mathrm{Tr}(B_2))\end{aligned}$$

Let us denote by $\mathrm{Tr}_\sigma$ as usual a product of traces along the cycles of $\sigma$. Then we have the relation

$$\mathrm{E}\{\mathrm{Tr}_\sigma[A_1, \dots, A_n]\} = \sum_{\substack{\mathcal{W} \in \mathcal{P}(n)\\ \mathcal{W} \geq \sigma}} \varphi(\mathcal{W}, \sigma)[A_1, \dots, A_n].$$

By using the formula (11) of Leonov and Shiryaev one sees that in terms of the entries of our matrices $B_k = (b_{ij}^{(k)})_{i,j=1}^N$ our $\varphi(\mathcal{U}, \gamma)$ can also be written as

$$\varphi(\mathcal{U}, \gamma)[B_1, \dots, B_n] = \sum_{\substack{\mathcal{V} \leq \mathcal{U}\\ \mathcal{V} \vee \gamma = \mathcal{U}}} \sum_{i(1), \dots, i(n) = 1}^{N} k_{\mathcal{V}}[b^{(1)}_{i(1)i(\gamma(1))}, \dots, b^{(n)}_{i(n)i(\gamma(n))}]. \tag{17}$$

4.2. Moments of unitarily invariant random matrices. For unitarily invariant random matrices there exists a definite relation between cumulants of traces and cumulants of entries. We want to work out this connection in this section. Related considerations were presented by Capitaine and Casalis in [CC06].

Definition 4.2. Random matrices $A_1, \dots, A_n$ are called *unitarily invariant* if the joint distribution of all their entries does not change by global conjugation with any unitary matrix, i.e., if, for any unitary matrix $U$, the matrix-valued random variables $A_1, \dots, A_n : \Omega \to M_N(\mathbb{C})$ have the same joint distribution as the matrix-valued random variables $UA_1U^*, \dots, UA_nU^* : \Omega \to M_N(\mathbb{C})$.

Let $A_1, \dots, A_n$ be unitarily invariant random matrices. We will now try expressing the microscopic quantities "cumulants of entries of the $A_i$" in terms of the macroscopic quantities "cumulants of traces of products of the $A_i$".
In order to make this connection we have to use the unitary invariance of our ensemble. By definition, this means that $A_1, \dots, A_n$ has the same distribution as $\tilde{A}_1, \dots, \tilde{A}_n$ where $\tilde{A}_i := UA_iU^*$. Since this holds for any unitary $U$, the same is true after averaging over such $U$, i.e., we can take in the definition of the $\tilde{A}_i$ the $U$ as Haar distributed unitary random matrices, independent from $A_1, \dots, A_n$. This reduces calculations for unitarily invariant ensembles essentially to properties of Haar unitary random matrices; in particular, the Wick formula for the $U$'s implies that we have an analogous Wick formula for joint moments in the entries of the $A_i$. Let us write $A_k = (a_{ij}^{(k)})_{i,j=1}^N$ and $\tilde{A}_k = (\tilde{a}_{ij}^{(k)})_{i,j=1}^N$. Then we can calculate:

$$\begin{aligned}
\mathrm{E}\{a_{p_1r_1}^{(1)} \cdots\ a_{p_nr_n}^{(n)}\} &= \mathrm{E}\{\tilde{a}_{p_1r_1}^{(1)} \cdots\ \tilde{a}_{p_nr_n}^{(n)}\} \\
&= \sum_{i,j} \mathrm{E}\{u_{p_1i_1} a_{i_1j_1}^{(1)} \overline{u_{r_1j_1}} \cdots u_{p_ni_n} a_{i_nj_n}^{(n)} \overline{u_{r_nj_n}}\} \\
&= \sum_{i,j} \mathrm{E}\{u_{p_1i_1} \overline{u_{r_1j_1}} \cdots u_{p_ni_n} \overline{u_{r_nj_n}}\} \cdot \mathrm{E}\{a_{i_1j_1}^{(1)} \cdots a_{i_nj_n}^{(n)}\} \\
&= \sum_{i,j} \sum_{\pi,\sigma \in S_n} \delta_{r,p\circ\pi} \delta_{j,i\circ\sigma} \mathrm{Wg}(\sigma\pi^{-1}) \cdot \mathrm{E}\{a_{i_1j_1}^{(1)} \cdots a_{i_nj_n}^{(n)}\} \\
&= \sum_{\pi \in S_n} \delta_{r,p\circ\pi} \cdot \mathcal{G}(\pi)[A_1, \dots, A_n],
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}(\pi)[A_1, \dots, A_n] :&= \sum_{\sigma \in S_n} \mathrm{Wg}(\sigma\pi^{-1}) \cdot \sum_i \mathrm{E}\{a_{i_1i_{\sigma(1)}}^{(1)} \cdots a_{i_ni_{\sigma(n)}}^{(n)}\} \\
&= \sum_{\sigma \in S_n} \mathrm{Wg}(\sigma\pi^{-1}) \cdot \mathrm{E}\{\mathrm{Tr}_\sigma[A_1, \dots, A_n]\}. \\
&= \sum_{\sigma \in S_n} \mathrm{Wg}(\sigma\pi^{-1}) \cdot \sum_{\substack{\mathcal{W} \in \mathcal{P}(n) \\ \mathcal{W} \geq \sigma}} \varphi(\mathcal{W}, \sigma)[A_1, \dots, A_n] \\
&= \sum_{(\mathcal{W},\sigma) \in \mathcal{PS}(n)} \mathrm{Wg}(\sigma\pi^{-1}) \cdot \varphi(\mathcal{W}, \sigma)[A_1, \dots, A_n].
\end{aligned} \tag{18}$$

The important point here is that $\mathcal{G}(\pi)[A_1, \dots, A_n]$ depends only on the macroscopic correlation moments of $A$.

We can extend the above to products of expectations by

$$\mathrm{E}_{\mathcal{V}}[a_{p_1r_1},\dots a_{p_nr_n}] = \sum_{\substack{\pi\in S_n\\ \pi\le\mathcal{V}}} \delta_{r,p\circ\pi}\cdot \mathcal{G}(\mathcal{V},\pi)[A_1,\dots,A_n],$$

where $\mathcal{G}(\mathcal{V},\pi)$ is given by multiplicative extension:

$$\begin{aligned}\mathcal{G}(\mathcal{V},\pi)[A_1,\dots,A_n] &:= \prod_{V\in\mathcal{V}} \mathcal{G}(\pi|_V)[A_1,\dots,A_n|_V]\\ &= \sum_{\substack{(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ \mathcal{W}\le\mathcal{V}}} \mathrm{Wg}(\mathcal{V},\sigma\pi^{-1})\cdot\varphi(\mathcal{W},\sigma)[A_1,\dots,A_n]. \end{aligned}\tag{19}$$

Now we can look on the cumulants of the entries of our unitarily invariant matrices $A_i$; they are given by

$$\begin{aligned} k_{\mathcal{V}}\{a^{(1)}_{p_1r_1},\dots,a^{(n)}_{p_nr_n}\} &= \sum_{\substack{\mathcal{U}\in\mathcal{P}(n)\\ \mathcal{U}\le\mathcal{V}}} \mathrm{Möb}_{\mathcal{P}(n)}(\mathcal{U},\mathcal{V})\cdot \mathrm{E}_{\mathcal{U}}[a^{(1)}_{p_1r_1},\dots a^{(n)}_{p_nr_n}]\\ &= \sum_{\mathcal{U}\le\mathcal{V}}\sum_{\substack{\pi\in S_n\\ \pi\le\mathcal{U}}} \delta_{r,p\circ\pi}\cdot\mathrm{Möb}_{\mathcal{P}(n)}(\mathcal{U},\mathcal{V})\cdot\mathcal{G}(\mathcal{U},\pi)[A_1,\dots,A_n]\\ &= \sum_{\substack{\pi\in S_n\\ \pi\le\mathcal{V}}}\delta_{r,p\circ\pi}\sum_{\substack{\mathcal{U}\in\mathcal{P}(n)\\ \mathcal{V}\ge\mathcal{U}\ge\pi}} \mathrm{Möb}_{\mathcal{P}(n)}(\mathcal{U},\mathcal{V})\cdot\mathcal{G}(\mathcal{U},\pi)[A_1,\dots,A_n]. \end{aligned}$$

With the definition

$$\kappa(\mathcal{V},\pi)[A_1,\dots,A_n] := \sum_{\substack{\mathcal{U}\in\mathcal{P}(n)\\ \mathcal{V}\ge\mathcal{U}\ge\pi}} \mathrm{Möb}_{\mathcal{P}(n)}(\mathcal{U},\mathcal{V})\cdot\mathcal{G}(\mathcal{U},\pi)[A_1,\dots,A_n].\tag{20}$$

we thereby get

$$k_{\mathcal{V}}\{a^{(1)}_{p_1r_1},\dots,a^{(n)}_{p_nr_n}\} = \sum_{\substack{\pi\in S_n\\ \pi\le\mathcal{V}}} \delta_{r,p\circ\pi}\cdot\kappa(\mathcal{V},\pi)[A_1,\dots,A_n].\tag{21}$$

It follows that

$$\begin{aligned}\varphi(\mathcal{U},\gamma)[A_1,\dots,A_n] &= \sum_{\substack{\mathcal{V}\le\mathcal{U}\\ \mathcal{V}\vee\gamma=\mathcal{U}}}\sum_{i(1),\dots,i(n)=1}^{N} k_{\mathcal{V}}[a^{(1)}_{i(1)i(\gamma(1))},\dots,a^{(n)}_{i(n)i(\gamma(n))}]\\ &= \sum_{\substack{\mathcal{V}\le\mathcal{U}\\ \mathcal{V}\vee\gamma=\mathcal{U}}}\sum_{i(1),\dots,i(n)=1}^{N}\sum_{\substack{\pi\in S_n\\ \pi\le\mathcal{V}}} \delta_{i\circ\gamma,i\circ\pi}\cdot\kappa(\mathcal{V},\pi)[A_1,\dots,A_n]\\ &= \sum_{\substack{\mathcal{V}\le\mathcal{U}\\ \mathcal{V}\vee\gamma=\mathcal{U}}}\sum_{\substack{\pi\in S_n\\ \pi\le\mathcal{V}}} \kappa(\mathcal{V},\pi)[A_1,\dots,A_n]\cdot N^{\#(\gamma\pi^{-1})}. \end{aligned}$$

Since $\mathcal{V}\vee\gamma=\mathcal{U}$ is, under the assumption $\pi\le\mathcal{V}$, equivalent to $\mathcal{V}\vee\gamma\pi^{-1}=\mathcal{U}$ we can write this also as

$$\varphi(\mathcal{U},\gamma)[A_1,\dots,A_n] = \sum_{\substack{(\mathcal{V},\pi)\in\mathcal{PS}(n)\\ \mathcal{V}\vee\gamma\pi^{-1}=\mathcal{U}}} \kappa(\mathcal{V},\pi)[A_1,\dots,A_n]\cdot N^{\#(\gamma\pi^{-1})}.\tag{22}$$

*Remark* 4.3. 1) Note that although the quantity $\kappa$ is defined by (20) in terms of the macroscopic moments of the $A_i$, they have also a very concrete meaning in terms of cumulants of entries of the $A_i$. Namely, if we choose $\pi \in S_n$ and distinct $1 \le i(1), \dots, i(n) \le N$ then equation (21) becomes, when we set $\mathcal{V} = 1_n$,

$$\kappa(1_n, \pi)[A_1, \dots, A_n] = k_n\big(a^{(1)}_{i(1)i(\pi(1))}, \dots, a^{(n)}_{i(n)i(\pi(n))}\big) \tag{23}$$

as the the only term in the sum that survives is the one for $\pi$.
2) Equation (22) should be considered as a kind of moment-cumulant formula in our context, thus it should contain all information for defining the "cumulants" $\kappa$ in terms of the moments $\varphi$. Actually, we can solve this linear system of equations for $\kappa$ in terms of $\varphi$, by using equation (20) to define $\kappa$ and equation (19) for $\mathcal{G}$.

$$\begin{aligned}
&\kappa(\mathcal{V}, \pi)[A_1, \dots, A_n] \\
&= \sum_{\substack{\mathcal{U} \in \mathcal{P}(n) \\ \mathcal{V} \ge \mathcal{U} \ge \pi}} \mathrm{M\ddot{o}b}_{\mathcal{P}(n)}(\mathcal{U}, \mathcal{V}) \cdot \sum_{\substack{(\mathcal{W}, \sigma) \in \mathcal{PS}(n) \\ \mathcal{W} \le \mathcal{U}}} \mathrm{Wg}(\mathcal{U}, \sigma\pi^{-1}) \cdot \varphi(\mathcal{W}, \sigma)[A_1, \dots, A_n] \\
&= \sum_{(\mathcal{W}, \sigma) \in \mathcal{PS}(n)} \varphi(\mathcal{W}, \sigma)[A_1, \dots, A_n] \cdot \sum_{\substack{\mathcal{U} \in \mathcal{P}(n) \\ \mathcal{V} \ge \mathcal{U} \ge \pi \vee \mathcal{W}}} \mathrm{M\ddot{o}b}_{\mathcal{P}(n)}(\mathcal{U}, \mathcal{V}) \cdot \mathrm{Wg}(\mathcal{U}, \sigma\pi^{-1}).
\end{aligned}$$

Thus, by using the relative cumulants of the Weingarten function from (14), we get finally

$$\kappa(\mathcal{V}, \pi)[A_1, \dots, A_n] = \sum_{\substack{(\mathcal{W}, \sigma) \in \mathcal{PS}(n) \\ \mathcal{W} \le \mathcal{V}}} \varphi(\mathcal{W}, \sigma)[A_1, \dots, A_n] \cdot C_{\pi \vee \mathcal{W}, \mathcal{V}}(\sigma\pi^{-1}). \tag{24}$$

3) One should also note that we have defined the Weingarten function only for $N \ge n$; thus in the above formulas we should always consider sufficiently large $N$. This is consistent with the observation that the system of equations (22) might not be invertible for $N$ too small; the matrix $\big(N^{\#(\sigma\pi^{-1})}\big)_{\sigma, \pi \in S_n}$ is invertible for $N \ge n$, however, in general not for all $N < n$ (e.g, clearly not for $N = 1$). One can make sense of some formulas involving the Weingarten function also for $N < n$ (see [CŚ06]). However, since we are mainly interested in the asymptotic behavior of our formulas for $N \to \infty$, we will not elaborate on this.

4.3. PRODUCT OF TWO INDEPENDENT ENSEMBLES. Let us now calculate the correlation functions for a product of two independent ensembles $A_1, \dots, A_n$ and $B_1, \dots, B_n$ of random matrices, where we assume that one of them, let's say the $B_i$'s, is unitarily invariant. We have, by using (17) and the special version (12) of the formula of Leonov and Shiryaev, the following:

$$\begin{aligned}
&\varphi(\mathcal{U},\gamma)[A_1B_1,\dots,A_nB_n]\\
&=\sum_{\substack{i(1),\cdots,i(n)\\ j(1),\cdots,j(n)}}\sum_{\substack{\mathcal{V},\mathcal{V}'\leq\mathcal{U}\\ \mathcal{V}\vee\mathcal{V}'\vee\gamma=\mathcal{U}}} k_{\mathcal{V}}[a^{(1)}_{j(1)i(1)},\dots,a^{(n)}_{j(n)i(n)}]\cdot k_{\mathcal{V}'}[b^{(1)}_{i(1)j(\gamma(1))},\dots,b^{(n)}_{i(n)j(\gamma(n))}]\\
&\overset{(20)}{=}\sum_{i,j}\sum_{\substack{\mathcal{V},\mathcal{V}'\leq\mathcal{U}\\ \mathcal{V}\vee\mathcal{V}'\vee\gamma=\mathcal{U}}}\sum_{\substack{\pi\in S_n\\ \pi\leq\mathcal{V}}}\delta_{i,j\circ\pi}\cdot\kappa(\mathcal{V},\pi)[A_1,\dots,A_n]\cdot k_{\mathcal{V}'}[b^{(1)}_{i(1)j(\gamma(1))},\dots,b^{(n)}_{i(n)j(\gamma(n))}]\\
&=\sum_{\pi\in S_n}\sum_{\substack{\mathcal{V}\in\mathcal{P}(n)\\ \mathcal{U}\geq\mathcal{V}\geq\pi}}\kappa(\mathcal{V},\pi)[A_1,\dots,A_n]\cdot\\
&\qquad\qquad\cdot\Big(\sum_{\substack{\mathcal{V}'\leq\mathcal{U}\\ \mathcal{V}'\vee\mathcal{V}\vee\gamma=\mathcal{U}}}\sum_i k_{\mathcal{V}'}[b^{(1)}_{i(1)i(\pi^{-1}\gamma(1))},\dots,b^{(n)}_{i(n)i(\pi^{-1}\gamma(n))}]\Big).
\end{aligned}$$

In order to evaluate the second factor we note first that, under the assumption $\pi\leq\mathcal{V}$, the condition $\mathcal{V}'\vee\mathcal{V}\vee\gamma=\mathcal{U}$ is equivalent to $\mathcal{V}'\vee\mathcal{V}\vee\pi^{-1}\gamma=\mathcal{U}$. Next, we rewrite the sum over all $\mathcal{V}'\in\mathcal{P}(n)$ with $\mathcal{V}'\leq\mathcal{U}$ and $\mathcal{V}'\vee\mathcal{V}\vee\pi^{-1}\gamma=\mathcal{U}$ as a double sum over all $\mathcal{W}\in\mathcal{P}(n)$ with $\mathcal{V}\vee\mathcal{W}=\mathcal{U}$ and all $\mathcal{V}'\in\mathcal{P}(n)$ with $\mathcal{V}'\leq\mathcal{W}$ and $\mathcal{V}'\vee\pi^{-1}\gamma=\mathcal{W}$.

$$\begin{aligned}
&\sum_{\substack{\mathcal{V}'\in\mathcal{P}(n)\\ \mathcal{V}'\leq\mathcal{U},\mathcal{V}'\vee\mathcal{V}\vee\gamma=\mathcal{U}}}\sum_i k_{\mathcal{V}'}[b^{(1)}_{i(1)i(\pi^{-1}\gamma(1))},\dots,b^{(n)}_{i(n)i(\pi^{-1}\gamma(n))}]\\
&\qquad=\sum_{\substack{\mathcal{W}\in\mathcal{P}(n)\\ \mathcal{V}\vee\mathcal{W}=\mathcal{U}}}\sum_{\substack{\mathcal{V}'\leq\mathcal{W}\\ \mathcal{V}'\vee\pi^{-1}\gamma=\mathcal{W}}}\sum_i k_{\mathcal{V}'}[b^{(1)}_{i(1)i(\pi^{-1}\gamma(1))},\dots,b^{(n)}_{i(n)i(\pi^{-1}\gamma(n))}]\\
&\qquad=\sum_{\substack{\mathcal{W}\in\mathcal{P}(n)\\ \mathcal{W}\geq\pi^{-1}\gamma,\mathcal{V}\vee\mathcal{W}=\mathcal{U}}}\varphi(\mathcal{W},\pi^{-1}\gamma)[B_1,\dots,B_n].
\end{aligned}$$

Thus we finally get

$$\begin{aligned}
&\varphi(\mathcal{U},\gamma)[A_1B_1,\dots,A_nB_n]\\
&=\sum_{\pi\in S_n}\sum_{\substack{\mathcal{V}\in\mathcal{P}(n)\\ \mathcal{U}\geq\mathcal{V}\geq\pi}}\sum_{\substack{\mathcal{W}\in\mathcal{P}(n)\\ \mathcal{W}\geq\pi^{-1}\gamma,\mathcal{V}\vee\mathcal{W}=\mathcal{U}}}\kappa(\mathcal{V},\pi)[A_1,\dots,A_n]\cdot\varphi(\mathcal{W},\pi^{-1}\gamma)[B_1,\dots,B_n]\\
&=\sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ \mathcal{V}\vee\mathcal{W}=\mathcal{U},\pi\sigma=\gamma}}\kappa(\mathcal{V},\pi)[A_1,\dots,A_n]\cdot\varphi(\mathcal{W},\pi^{-1}\gamma)[B_1,\dots,B_n].
\end{aligned}$$

Let us summarize the result of our calculations in the following theorem. In order to indicate that our main formulas are valid for any fixed $N$, we will decorate the relevant quantities with a superscript $^{(N)}$. Note that up to now we have not made any asymptotic consideration.

Theorem 4.4. *Let $\mathcal{M}_N:=M_N\otimes L^\infty(\Omega)$ be an ensemble of $N\times N$-random matrices. Define correlation functions $\varphi_n^{(N)}$ on $\mathcal{M}_N$ by ($n\in\mathbb{N}$, $D_1,\dots,D_n\in$*

*$\mathcal{M}_N$)*

$$\varphi_n^{(N)}(D_1,\dots,D_n) := k_n(\mathrm{Tr}(D_1),\dots,\mathrm{Tr}(D_n)) \tag{25}$$

*and corresponding "cumulant functions" $\kappa^{(N)}$ (for $n \le N$) by*

$$\kappa^{(N)}(\mathcal{V},\pi)[D_1,\dots,D_n] = \sum_{\substack{\mathcal{W}\in\mathcal{P}(n),\,\sigma\in S_n\\ W\le\mathcal{V}}} \varphi^{(N)}(\mathcal{W},\sigma)[D_1,\dots,D_n]\cdot C^{(N)}_{\pi\vee\mathcal{W},\mathcal{V}}(\sigma\pi^{-1}), \tag{26}$$

*or equivalently by the implicit system of equations*

$$\varphi^{(N)}(\mathcal{U},\gamma)[D_1,\dots,D_n] = \sum_{\mathcal{V},\pi} \kappa^{(N)}(\mathcal{V},\pi)[D_1,\dots,D_n]\cdot N^{\#(\gamma\pi^{-1})}, \tag{27}$$

*where the sum is over all $\mathcal{V}\in\mathcal{P}(n)$ all $\pi\in S_n$ such that $\pi\le\mathcal{V}$ and $\mathcal{V}\vee\gamma\pi^{-1} = \mathcal{U}$.*
*1) Let $\mathcal{A}_N$ be an algebra of unitarily invariant random matrices in $\mathcal{M}_N$. Then we have for all $n\le N$, all distinct $i(1),\dots,i(n)$, all $A_k = \big(a_{ij}^{(k)}\big)_{i,j=1}^N \in \mathcal{A}_N$, and all $\pi\in S_n$ that*

$$\kappa^{(N)}(1_n,\pi)[A_1,\dots,A_n] = k_n\big(a^{(1)}_{i(1)i(\pi(1))},\dots,a^{(n)}_{i(n)i(\pi(n))}\big). \tag{28}$$

*2) Assume that we have two subalgebras $\mathcal{A}_N$ and $\mathcal{B}_N$ of $\mathcal{M}_N$ such that*
- *$\diamond$ $\mathcal{A}_N$ is a unitarily invariant ensemble,*
- *$\diamond$ $\mathcal{A}_N$ and $\mathcal{B}_N$ are independent.*

*Then we have for all $n\in\mathbb{N}$ with $n\le N$ and all $A_1,\dots,A_n\in\mathcal{A}_N$ and $B_1,\dots,B_n\in\mathcal{B}_N$:*

$$\begin{aligned}\varphi^{(N)}(\mathcal{U},\gamma)&[A_1B_1,\dots,A_nB_n]\\ &= \sum_{\mathcal{V},\pi,\mathcal{W},\sigma} \kappa^{(N)}(\mathcal{V},\pi)[A_1,\dots,A_n]\cdot\varphi^{(N)}(\mathcal{W},\sigma)[B_1,\dots,B_n],\end{aligned} \tag{29}$$

*where the sum is over all $\mathcal{V},\mathcal{W}\in\mathcal{P}(n)$ and all $\pi,\sigma\in S_n$ such that $\pi\le\mathcal{V}$, $\sigma\le\mathcal{W}$, $\mathcal{V}\vee\mathcal{W}=\mathcal{U}$, and $\gamma=\pi\sigma$.*

4.4. LARGE $N$ ASYMPTOTICS FOR MOMENTS AND CUMULANTS. Our main interest in this paper will be the large $N$ limit of formula (29). This structure in leading order between independent ensembles of random matrices which are randomly rotated against each other will be captured in our abstract notion of higher order freeness.

Of course, now we must make an assumption about the asymptotic behavior in $N$ of our correlation functions. We will require that the cumulants of traces of our random matrices decays in $N$ with the same order as in the case of Gaussian or Wishart random matrices. In these cases one has very detailed "genus expansions" for those cumulants; see, e.g. [Oko00, MN04] and one knows that the $n$-th cumulant of unnormalized traces in polynomials of those random matrices decays like $N^{2-n}$ (see e.g. [MS06, Thm. 3.1 and Thm. 3.5]).

Definition 4.5. Let, for each $N \in \mathbb{N}$, $B_1^{(N)}, \dots, B_r^{(N)} \subset M_N \otimes L^{\infty-}(\Omega)$ be $N \times N$-random matrices. Suppose that the leading term of the correlation moments of $B_1^{(N)}, \dots, B_r^{(N)}$ are of order $2-n$, i.e., that for all $n \in \mathbb{N}$ and all polynomials $p_1, \dots, p_t$ in $r$ non-commuting variables the limits

$$\lim_{N\to\infty} \varphi_n^{(N)}(p_1(B_1^{(N)}, \dots, B_r^{(N)}), \dots, p_t(B_1^{(N)}, \dots, B_r^{(N)})) \cdot N^{n-2}$$

exist. Then we will say that $\{B_1^{(N)}, \dots, B_r^{(N)}\}$ has *limit distributions of all orders*. Let $\mathcal{B}$ be the free algebra generated by generators $b_1, \dots, b_r$. Then we define the *limit correlation functions* of $\mathcal{B}$ by

$$\begin{aligned}&\varphi_n(p_1(b_1, \dots, b_r), \dots, p_t(b_1, \dots, b_r))\\ &\quad = \lim_{N\to\infty} \varphi_n^{(N)}(p_1(B_1^{(N)}, \dots, B_r^{(N)}), \dots, p_t(B_1^{(N)}, \dots, B_r^{(N)})) \cdot N^{n-2}\end{aligned}$$

Note that this assumption implies that the leading term for the quantities $\varphi^{(N)}(\mathcal{V}, \pi)$ is of order $2\#(\mathcal{V}) - \#(\pi)$. Indeed, if $\mathcal{V}$ has $k$ blocks and the $i^{th}$ block of $\mathcal{V}$ contains $r_i$ cycles of $\pi$ then $\varphi^{(N)}(\mathcal{V}, \pi) = \varphi_{r_1} \cdots \varphi_{r_k}$ and each $\varphi_{r_i}$ has order $2 - r_i$. Then the order of $\varphi^{(N)}(\mathcal{V}, \pi)$ is $(2 - r_1) + \cdots + (2 - r_k) = 2k - (r_1 + \cdots + r_k) = 2\,\#(\mathcal{V}) - \#(\pi)$. Thus

$$\begin{aligned}&\varphi(\mathcal{V}, \pi)(p_1(b_1, \dots, b_r), \dots, p_t(b_1, \dots, b_r))\\ &\quad = \lim_{N\to\infty} \varphi^{(N)}(\mathcal{V}, \pi)(p_1(B_1^{(N)}, \dots, B_r^{(N)}), \dots, p_t(B_1^{(N)}, \dots, B_r^{(N)}))\\ &\qquad\quad \cdot N^{-2\#(\mathcal{V})+\#(\pi)}\end{aligned}$$

From formula (27) one can deduce that the leading order of $\kappa^{(N)}(\mathcal{V}, \pi)$ is given by the term $(\mathcal{U}, \gamma) = (\mathcal{V}, \pi)$ and thus must be of order

$$N^{-n+2\#\mathcal{V}-\#\pi}.$$

(Indeed, this also follows from equation (24) and the leading order of the relative cumulant of the Weingarten function given in equation (16).)
Thus we can define the *limiting cumulant functions* to be the limit of the leading order of the cumulants by the equation

$$\kappa(\mathcal{V}, \pi)[b_1, \dots, b_n] := \lim_{N\to\infty} N^{n-2\#\mathcal{V}+\#\pi} \cdot \kappa^{(N)}(\mathcal{V}, \pi)[B_1^{(N)}, \dots, B_n^{(N)}] \tag{30}$$

When $(\mathcal{V}, \pi) = (1_n, \gamma_n)$ and $B_1 = B_2 = \cdots = B_n = B$ equation (28) becomes

$$\kappa^{(N)}(1_n, \gamma_n)[B, \dots, B] = k_n\big(b_{i(1)i(2)}^{(N)}, \dots, b_{i(n)i(1)}^{(N)}\big)$$

Thus to prove Theorem 2.6 we must show that $\kappa^{(N)}(1_n, \gamma_n)[B, \dots, B] \cdot N^{n-1}$ converges to $\kappa_n^b$ the $n^{th}$ free cumulant of the limiting eigenvalue distribution of $B^{(N)}$.
When $(\mathcal{V}, \pi) = (1_{m+n}, \gamma_{m,n})$ equation (28) becomes

$$\kappa^{(N)}(1_{m+n}, \gamma_{m,n})[B, \dots, B] = k_{m+n}\big(b_{i(1)i(2)}^{(N)}, \dots, b_{i(m)i(1)}^{(N)}, b_{j(1)j(2)}^{(N)}, \dots, b_{j(n),j(1)}^{(N)}\big)$$

Thus to prove Theorem 2.12 we must show that $\kappa^{(N)}(1_{m+n}, \gamma_{m,n})[B, \ldots, B] \cdot N^{m+n}$ converges to $\kappa^b_{m,n}$ the $(m,n)^{th}$ free cumulant of second order of the limiting second order distribution of $B^{(N)}$.

4.5. LENGTH FUNCTIONS. We want to understand the asymptotic behavior of formula (29). The leading order in $N$ of the right hand side is given by

$$-n + 2\#\mathcal{V} - \#\pi + 2\#\mathcal{W} - \#\sigma = n + (|\pi| - 2|\mathcal{V}|) + (|\sigma| - 2|\mathcal{W}|),$$

whereas the leading order of the left hand side is given by

$$2\#\mathcal{U} - \#\gamma = 2\#(\mathcal{V} \vee \mathcal{W}) - \#(\sigma\pi) = n + (|\pi\sigma| - 2|\mathcal{V} \vee \mathcal{W}|).$$

This suggests the introducing of the following "length functions" for permutations, partitions, and partitioned permutations.

NOTATION 4.6.

(1) For $\mathcal{V} \in \mathcal{P}(n)$ and $\pi \in S_n$ we put

$$|\mathcal{V}| := n - \#\mathcal{V}$$
$$|\pi| := n - \#\pi.$$

(2) For any $(\mathcal{V}, \pi) \in \mathcal{PS}(n)$ we put

$$|(\mathcal{V}, \pi)| := 2|\mathcal{V}| - |\pi| = n - (2\#\mathcal{V} - \#\pi).$$

Let us first observe that these quantities behave actually like a length. It is clear from the definition that they are always non-negative; that they also obey a triangle inequality is the content of the next lemma.

LEMMA 4.7.

(1) *For all $\pi, \sigma \in S_n$ we have*

$$|\pi\sigma| \leq |\pi| + |\sigma|.$$

(2) *For all $\mathcal{V}, \mathcal{W} \in \mathcal{P}(n)$ we have*

$$|\mathcal{V} \vee \mathcal{W}| \leq |\mathcal{V}| + |\mathcal{W}|.$$

(3) *For all partitioned permutations $(\mathcal{V}, \pi), (\mathcal{W}, \sigma) \in \mathcal{PS}(n)$ we have*

$$|(\mathcal{V} \vee \mathcal{W}, \pi\sigma)| \leq |(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)|.$$

*Proof.* (1) This is well-known, since $|\pi|$ is the minimal number of factors needed to write $\pi$ as a product of transpositions.
(2) Each block $B$ of $\mathcal{W}$ can glue at most $\#B - 1$ many blocks of $\mathcal{V}$ together, i.e., $\mathcal{W}$ can glue at most $n - \#\mathcal{W}$ many blocks of $\mathcal{V}$ together, thus the difference between $|\mathcal{V}|$ and $|\mathcal{V} \vee \mathcal{W}|$ cannot exceed $n - \#\mathcal{W}$ and hence

$$\#\mathcal{V} - \#(\mathcal{V} \vee \mathcal{W}) \leq n - \#\mathcal{W}.$$

This is equivalent to our assertion.
(3) We prove this, for fixed $\pi$ and $\sigma$ by induction over $|\mathcal{V}| + |\mathcal{W}|$. The smallest possible value of the latter appears for $|\mathcal{V}| = |\pi|$ and $|\mathcal{W}| = |\sigma|$ (i.e., $\mathcal{V} = 0_\pi$ and $\mathcal{W} = 0_\sigma$). But then we have (since $\mathcal{V} \vee \mathcal{W} \geq \pi\sigma$)

$$2|\mathcal{V} \vee \mathcal{W}| - |\pi\sigma| \leq |\mathcal{V} \vee \mathcal{W}| \leq |\mathcal{V}| + |\mathcal{W}|,$$

which is exactly our assertion for this case. For the induction step, on the other side, one only has to observe that if one increases $|\mathcal{V}|$ (or $|\mathcal{W}|$) by one then $|\mathcal{V} \vee \mathcal{W}|$ can also increase by at most 1. □

*Remark* 4.8. 1) Note that the triangle inequality for partitioned permutations together with (29) implies the following. Given random matrices $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$ which have limit distributions of all orders. If $A$ and $B$ are independent and at least one of them is unitarily invariant, then $C = (C_N)_{N\in\mathbb{N}}$ with $C_N := A_N B_N$ also has limit distributions of all orders.
2) Since we know that Gaussian and Wishart random matrices have limit distributions of all orders (see e.g. [MS06, Thm. 3.1 and Thm. 3.5]), and since they are unitarily invariant, it follows by induction from the previous part that any polynomial in independent Gaussian and Wishart matrices has limit distributions of all orders.

4.6. MULTIPLICATION OF PARTITIONED PERMUTATIONS. Suppose $\{B_1^{(N)}, \dots, B_n^{(N)}\}$ has limit distributions of all orders. Then the left hand side of equation (27) has order $N^{2\#(\mathcal{U})-\#(\gamma)}$ and the right hand side of equation (27) has order $N^{-n+2\#(\mathcal{V})-\#(\pi)+|\gamma\pi^{-1}|}$. Thus the only terms of the right hand side that have order $N^{2\#(\mathcal{U})-\#(\gamma)}$ are those for which

$$2\#(\mathcal{U}) - \#(\gamma) = -n + 2\#(\mathcal{V}) - \#(\pi) + |\gamma\pi^{-1}|$$

i.e. for which $|(\mathcal{U}, \gamma)| = |(\mathcal{V}, \pi)| + |\gamma\pi^{-1}|$. Hence

$$\varphi^{(N)}(\mathcal{U},\gamma)[B_1^{(N)},\dots,B_n^{(N)}] = \sum_{\substack{(\mathcal{V},\pi)\in\mathcal{PS}(n)\\ \mathcal{V}\vee\gamma\pi^{-1}=\mathcal{U}\\ |(\mathcal{U},\gamma)|=|(\mathcal{V},\pi)|+|\gamma\pi^{-1}|}} \kappa^{(N)}(\mathcal{V},\pi)[B_1^{(N)},\dots,B_n^{(N)}]\cdot N^{|\gamma\pi^{-1}|} + O(N^{2\#(\mathcal{U})-\#(\gamma)-2})$$

Thus after taking limits we have

$$\varphi(\mathcal{U},\gamma)[b_1,\dots,b_n] = \sum_{(\mathcal{V},\pi)\in\mathcal{PS}(n)} \kappa(\mathcal{V},\pi)[b_1,\dots,b_n] \tag{31}$$

where the sum is over all $(\mathcal{V},\pi)$ in $\mathcal{PS}(n)$ such that $\mathcal{V}\vee\gamma\pi^{-1} = \mathcal{U}$ and $|(\mathcal{U},\gamma)| = |(\mathcal{V},\pi)| + |\gamma\pi^{-1}|$.
A similar analysis of equation (29) gives that for independent $\{A_1^{(N)},\dots,A_n^{(N)}\}$ and $\{B_1^{(N)},\dots,B_n^{(N)}\}$ with the $A_i^{(N)}$'s unitarily invariant and both having limit distributions of all orders we have

$$\varphi^{(N)}(\mathcal{U},\gamma)[A_1^{(N)}B_1^{(N)},\dots,A_n^{(N)}B_n^{(N)}]$$
$$= \sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ \mathcal{V}\vee\mathcal{W}=\mathcal{U},\ \pi\sigma=\gamma\\ |(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|=|(\mathcal{V}\vee\mathcal{W},\pi\sigma)|}} \kappa^{(N)}(\mathcal{V},\pi)[A_1^{(N)},\dots,A_n^{(N)}]\cdot\varphi^{(N)}(\mathcal{W},\sigma)[B_1^{(N)},\dots,B_n^{(N)}]$$
$$+ O(N^{2\#(\mathcal{U})-\#(\gamma)-2})$$

and again after taking limits

$$\varphi(\mathcal{U},\gamma)[a_1b_1,\dots,a_nb_n] = \sum_{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)} \kappa(\mathcal{V},\pi)[a_1,\dots,a_n]\cdot\varphi(\mathcal{W},\sigma)[b_1,\dots,b_n] \tag{32}$$

where the sum is over all $(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)$ such that

- ⋄ $\mathcal{V}\vee\mathcal{W}=\mathcal{U}$
- ⋄ $\pi\sigma=\gamma$
- ⋄ $|(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|=|(\mathcal{U},\gamma)|$

In order to write this in a more compact form it is convenient to define a multiplication for partitioned permutations (in $\mathbb{C}\mathcal{PS}(n)$) as follows.

DEFINITION 4.9. For $(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)$ we define their product as follows.

$$(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma) := = \begin{cases}(\mathcal{V}\vee\mathcal{W},\pi\sigma) & \text{if } |(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|=|(\mathcal{V}\vee\mathcal{W},\pi\sigma)|,\\ 0 & \text{otherwise.}\end{cases} \tag{33}$$

PROPOSITION 4.10. *The multiplication defined in Definition 4.9 is associative.*

*Proof.* We have to check that

$$\big((\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)\big)\cdot(\mathcal{U},\tau) = (\mathcal{V},\pi)\cdot\big((\mathcal{W},\sigma)\cdot(\mathcal{U},\tau)\big). \tag{34}$$

Since both sides are equal to $(\mathcal{V}\vee\mathcal{W}\vee\mathcal{U},\pi\sigma\tau)$ in case they do not vanish, we have to see that the conditions for non-vanishing are for both sides the same. The conditions for the left hand side are

$$|(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|=|(\mathcal{V}\vee\mathcal{W},\pi\sigma)|$$

and

$$|(\mathcal{V}\vee\mathcal{W},\pi\sigma)|+|(\mathcal{U},\tau)|=|(\mathcal{V}\vee\mathcal{W}\vee\mathcal{U},\pi\sigma\tau)|.$$

These imply

$$\begin{aligned}|(\mathcal{V},\pi)|+|(\mathcal{W},\sigma)|+|(\mathcal{U},\tau)| &= |(\mathcal{U}\vee\mathcal{W}\vee\mathcal{U},\pi\sigma\tau)|\\ &\le |(\mathcal{V},\pi)|+|(\mathcal{W}\vee\mathcal{U},\sigma\tau)|,\end{aligned}$$

However, the triangle inequality

$$|(\mathcal{W}\vee\mathcal{U},\sigma\tau)|\le|(\mathcal{W},\sigma)|+|(\mathcal{U},\tau)|$$

yields that we have actually equality in the above inequality, thus leading to

$$|(\mathcal{W},\sigma)| + |(\mathcal{U},\tau)| = |(\mathcal{W}\vee\mathcal{U},\sigma\tau)|$$

and

$$|(\mathcal{V},\pi)| + |(\mathcal{W}\vee\mathcal{U},\sigma\tau)| = |(\mathcal{V}\vee\mathcal{W}\vee\mathcal{U},\pi\sigma\tau)|.$$

These are exactly the two conditions for the vanishing of the right hand side of (34). The other direction goes analogously. □

Now we can write formulas (31) and (32) in convolution form

$$\varphi(\mathcal{U},\gamma)[b_1,\dots,b_n] = \sum_{\substack{(\mathcal{V},\pi)\in\mathcal{PS}(n)\\ (\mathcal{V},\pi)\cdot(0,\gamma\pi^{-1})=(\mathcal{U},\gamma)}} \kappa(\mathcal{V},\pi)[b_1,\dots,b_n] \tag{35}$$

and

$$\varphi(\mathcal{U},\gamma)[a_1b_1,\dots,a_nb_n] = \sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ (\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}} \kappa(\mathcal{V},\pi)[a_1,\dots,a_n]\cdot\varphi(\mathcal{W},\sigma)[b_1,\dots,b_n]. \tag{36}$$

Note that both $\varphi(\mathcal{V},\pi)$ and $\kappa(\mathcal{V},\pi)$ are multiplicative in the sense that they factor according to the decomposition of $\mathcal{V}$ into blocks.
The philosophy for our definition of higher order freeness will be that equation (35) is the analogue of the moment-cumulant formula and shall be used to define the quantities $\kappa$, which will thus take on the role of cumulants in our theory – whereas the $\varphi$ are the moments (see Definition 7.4). We shall define higher order freeness by requiring the vanishing of mixed cumulants, see Definition 7.6. On the other hand, equation (36) would be another way of expressing the fact that the $a$'s are free from the $b$'s. Of course, we will have to prove that those two possibilities are actually equivalent (see Theorem 7.9).

## 5. Multiplicative functions on partitioned permutations and their convolution

5.1. Convolution of multiplicative functions. Formulas (35) and (36) above are a generalization of the formulas describing first order freeness in terms of cumulants and convolution of multiplicative functions on non-crossing partitions. Since the dependence on the random matrices is irrelevant for this structure we will free ourselves in this section from the random matrices and look on the combinatorial heart of the observed formulas. In Section 7, we will return to the more general situation involving multiplicative functions which depend also on random matrices or more generally elements from an algebra.

Definition 5.1.

(1) We denote by $\mathcal{PS}$ the set of partitioned permutations on an arbitrary number of elements, i.e.,

$$\mathcal{PS} = \bigcup_{n\in\mathbb{N}} \mathcal{PS}(n).$$

(2) For two functions

$$f, g : \mathcal{PS} \to \mathbb{C}$$

we define their convolution

$$f * g : \mathcal{PS} \to \mathbb{C}$$

by

$$(f * g)(\mathcal{U}, \gamma) := \sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ (\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}} f(\mathcal{V}, \pi)\; g(\mathcal{W}, \sigma)$$

for any $(\mathcal{U}, \gamma) \in \mathcal{PS}(n)$.

DEFINITION 5.2. A function $f : \mathcal{PS} \to \mathbb{C}$ is called *multiplicative* if $f(1_n, \pi)$ depends only on the conjugacy class of $\pi$ and we have

$$f(\mathcal{V}, \pi) = \prod_{V \in \mathcal{V}} f(1_V, \pi|_V).$$

Our main interest will be in multiplicative functions. It is easy to see that the convolution of two multiplicative functions is again multiplicative. It is clear that a multiplicative function is determined by the values of $f(1_n, \pi)$ for all $n \in \mathbb{N}$ and all $\pi \in S_n$.
An important example of a multiplicative function is the $\delta$-function presented below.

NOTATION 5.3. The $\delta$-function on $\mathcal{PS}$ is the multiplicative function determined by

$$\delta(1_n, \pi) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $(\mathcal{U}, \pi) \in \mathcal{PS}(n)$

$$\delta(\mathcal{U}, \pi) = \begin{cases} 1 & \text{if } (\mathcal{U}, \pi) = \big(0_n, (1)(2)\dots(n)\big) \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 5.4. *The convolution of multiplicative functions on $\mathcal{PS}$ is commutative and $\delta$ is the unit element.*

*Proof.* It is clear that $\delta$ is the unit element. For commutativity, we note that for multiplicative functions we have

$$f(\mathcal{V}, \pi) = f(\mathcal{V}, \pi^{-1}),$$

and thus

$$(g * f)(\mathcal{U}, \gamma) = (g * f)(\mathcal{U}, \gamma^{-1}) = \sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ (\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma^{-1})}} g(\mathcal{V}, \pi) f(\mathcal{W}, \sigma).$$

Since the condition $(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (\mathcal{U}, \gamma^{-1})$ is equivalent to the condition $(\mathcal{W}, \sigma^{-1}) \cdot (\mathcal{V}, \pi^{-1}) = (\mathcal{U}, \gamma)$ we can continue with

$$(g * f)(\mathcal{U}, \gamma) = \sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ (\mathcal{W},\sigma^{-1})\cdot(\mathcal{V},\pi^{-1})=(\mathcal{U},\gamma)}} f(\mathcal{W}, \sigma^{-1}) g(\mathcal{V}, \pi^{-1}) = (f * g)(\mathcal{U}, \gamma).$$

□

5.2. FACTORIZATIONS. Let us now try to characterize the non-trivial factorizations $(\mathcal{U},\gamma)=(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)$ appearing in the definition of our convolution. Let us first observe some simple general inequalities.

LEMMA 5.5.

(1) *For permutations* $\pi,\sigma\in S(n)$ *we have*

$$|\pi|+|\sigma|+|\pi\sigma|\geq 2|\pi\vee\sigma|.$$

(2) *For partitions* $\mathcal{V}_2\leq\mathcal{V}_1$ *and* $\mathcal{W}_2\leq\mathcal{W}_1$ *we have*

$$|\mathcal{W}_1|+|\mathcal{V}_1|+|\mathcal{V}_2\vee\mathcal{W}_2|\geq|\mathcal{V}_1\vee\mathcal{W}_1|+|\mathcal{W}_2|+|\mathcal{V}_2|$$

*and*

$$|\mathcal{V}_1\vee\mathcal{W}_2|+|\mathcal{V}_2\vee\mathcal{W}_1|\geq|\mathcal{V}_1\vee\mathcal{W}_1|+|\mathcal{V}_2\vee\mathcal{W}_2|.$$

*Proof.* (1) By the triangle inequality for partitioned permutations we have

$$|(0_\pi\vee 0_\sigma,\pi\sigma)|\leq|(0_\pi,\pi)|+|(0_\sigma,\sigma)|,$$

i.e.,

$$2|\pi\vee\sigma|-|\pi\sigma|\leq|\pi|+|\sigma|. \tag{37}$$

(2) Consider first the special case $\mathcal{W}_1=\mathcal{W}_2=\mathcal{W}$. Then we clearly have

$$\#(\mathcal{V}_2\vee\mathcal{W})-\#(\mathcal{V}_1\vee\mathcal{W})\leq\#\mathcal{V}_2-\#\mathcal{V}_1,$$

which leads to

$$|\mathcal{V}_1\vee\mathcal{W}|-|\mathcal{V}_2\vee\mathcal{W}|\leq|\mathcal{V}_1|-|\mathcal{V}_2|.$$

From this the general case follows by

$$\begin{aligned}|\mathcal{V}_1\vee\mathcal{W}_1|-|\mathcal{V}_2\vee\mathcal{W}_2|&=|\mathcal{V}_1\vee\mathcal{W}_1|-|\mathcal{V}_1\vee\mathcal{W}_2|+|\mathcal{V}_1\vee\mathcal{W}_2|-|\mathcal{V}_2\vee\mathcal{W}_2|\\&\leq|\mathcal{W}_1|-|\mathcal{W}_2|+|\mathcal{V}_1|-|\mathcal{V}_2|.\end{aligned}$$

The second inequality follows from this as follows:

$$\begin{aligned}|\mathcal{V}_1\vee\mathcal{W}_1|-|\mathcal{V}_1\vee\mathcal{W}_2|&=|\mathcal{V}_1\vee(\mathcal{V}_2\vee\mathcal{W}_1)|-|\mathcal{V}_1\vee(\mathcal{V}_2\vee\mathcal{W}_2)|\\&\leq|\mathcal{V}_2\vee\mathcal{W}_1|-|\mathcal{V}_2\vee\mathcal{W}_2|.\end{aligned}$$

□

THEOREM 5.6. *For* $(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)$ *the equation*

$$(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{V}\vee\mathcal{W},\pi\sigma)$$

*is equivalent to the conjunction of the following four conditions:*

$$\begin{aligned}|\pi|+|\sigma|+|\pi\sigma|&=2|\pi\vee\sigma|,\\|\mathcal{V}|+|\pi\vee\sigma|&=|\pi|+|\mathcal{V}\vee\sigma|,\\|\mathcal{W}|+|\pi\vee\sigma|&=|\sigma|+|\pi\vee\mathcal{W}|,\\|\mathcal{V}\vee\sigma|+|\pi\vee\mathcal{W}|&=|\mathcal{V}\vee\mathcal{W}|+|\pi\vee\sigma|.\end{aligned}$$

*Proof.* Adding the four inequalities given by Lemma 5.5

$$|\pi| + |\sigma| + |\pi\sigma| \geq 2|\pi \vee \sigma|,$$
$$2|\mathcal{V}| + 2|\pi \vee \sigma| \geq 2|\pi| + 2|\mathcal{V} \vee \sigma|,$$
$$2|\mathcal{W}| + 2|\pi \vee \sigma| \geq 2|\sigma| + 2|\pi \vee \mathcal{W}|,$$
$$2|\mathcal{V} \vee \sigma| + 2|\pi \vee \mathcal{W}| \geq 2|\mathcal{V} \vee \mathcal{W}| + 2|\pi \vee \sigma|$$

gives

$$2|\mathcal{V}| - |\pi| + 2|\mathcal{W}| - |\sigma| \geq 2|\mathcal{V} \vee \mathcal{W}| - |\pi\sigma|,$$

i.e.,

$$|(\mathcal{V}, \pi)| + |(\mathcal{W}, \sigma)| \geq |(\mathcal{V} \vee \mathcal{W}, \pi\sigma).$$

Since $(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (\mathcal{V} \vee \mathcal{W}, \pi\sigma)$ means that we require equality in the last inequality, this is equivalent to having equality in all the four inequalities. □

The conditions describing our factorizations have a quite geometrical meaning. Let us elaborate on this in the following.

Definition 5.7. Let $\gamma \in S(n)$ be a fixed permutation.

(1) A permutation $\pi \in S(n)$ is called *$\gamma$-planar* if

$$|\pi| + |\pi^{-1}\gamma| + |\gamma| = 2|\pi \vee \gamma|.$$

(2) A partitioned permutation $(\mathcal{V}, \pi) \in \mathcal{PS}(n)$ is called *$\gamma$-minimal* if

$$|\mathcal{V} \vee \gamma| - |\pi \vee \gamma| = |\mathcal{V}| - |\pi|.$$

*Remark* 5.8. *i*) It is easy to check (for example, by calculating the Euler characteristic) that $\gamma$-planarity of $\pi$ corresponds indeed to a planar diagram, i.e. one can draw a planar graph representing permutations $\gamma$ and $\pi$ without any crossings. The most important cases are when $\gamma$ consists of a single cycle [Bia97] and when $\gamma$ consists of two cycles [MN04].
*ii*) The notion of $\gamma$-minimality of $(\mathcal{V}, \pi)$ means that $\mathcal{V}$ connects only blocks of $\pi$ which are not already connected by $\gamma$.
*iii*) If $(\mathcal{V}, \pi)$ satisfies both (1) and (2) of Definition 5.7 then $(\mathcal{V}, \pi)(0, \pi^{-1}\gamma) = (1, \gamma)$, by Theorem 5.6.

Corollary 5.9. *Assume that we have the equation*

$$(\mathcal{U}, \gamma) = (\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma).$$

*Then $\pi$ and $\sigma$ must be $\gamma$-planar and $(\mathcal{V}, \pi)$ and $(\mathcal{W}, \sigma)$ must be $\gamma$-minimal.*

5.3. Factorizations of disc and tunnel permutations.

Notation 5.10. *i* ) We call $(\mathcal{V}, \pi) \in \mathcal{PS}_n$ a *disc permutation* if $\mathcal{V} = 0_\pi$; the latter is equivalent to the condition $|\mathcal{V}| = |\sigma|$. For $\pi \in S_n$, by $(0, \pi)$ we will always mean the disc permutation

$$(0, \pi) := (0_\pi, \pi) \in \mathcal{PS}(n).$$

*ii*) We call $(\mathcal{V}, \pi) \in \mathcal{PS}_n$ a *tunnel permutation* if $|\mathcal{V}| = |\pi| + 1$. This means that $\mathcal{V}$ is obtained from $\pi$ by joining a pair of cycles; i.e. one block of $\mathcal{V}$ contains exactly two cycles of $\pi$ and all other blocks contain only one cycle of $\pi$.

A motivation for those names comes from the identification between partitioned permutations and so-called surfaced permutations; see the Appendix for more information on this.

Our goal is now to understand more explicitly the factorizations of disc and tunnel permutations. (It will turn out that those are the relevant ones for first and second order freeness). For this, note that we can rewrite the crucial condition for our product of partitioned permutations,

$$2|\mathcal{V}| - |\pi| + 2|\mathcal{W}| - |\sigma| = 2|\mathcal{V} \vee \mathcal{W}| - |\pi\sigma|,$$

in the form

$$\big(|\mathcal{V}| - |\pi|\big) + \big(|\mathcal{W}| - |\sigma|\big) + \big(|\mathcal{V}| + |\mathcal{W}| - |\mathcal{V} \vee \mathcal{W}|\big) = \big(|\mathcal{V} \vee \mathcal{W}| - |\pi\sigma|\big).$$

Since all terms in brackets are non-negative integers this formula can be used to obtain explicit solutions to our factorization problem for small values of the right hand side. Essentially, this tells us that factorizations of a disc permutation can only be of the form disc $\times$ disc; and factorizations of a tunnel permutation can only be of the form disc $\times$ disc, disc $\times$ tunnel, and tunnel $\times$ disc. Of course, one can generalize the following arguments to higher order type permutations, however, the number of possibilities grows quite quickly.

Proposition 5.11.

(1) *The solutions to the equation*

$$(1_n, \gamma_n) = (0, \gamma_n) = (\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma)$$

*are exactly of the form*

$$(1_n, \gamma_n) = (0, \pi) \cdot (0, \pi^{-1}\gamma_n),$$

*for some* $\pi \in NC(n)$.

(2) *The solutions to the equation*

$$(1_{m+n}, \gamma_{m,n}) = (\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma)$$

*are exactly of the following three forms:*

(a)

$$(1_{m+n}, \gamma_{m,n}) = (0, \pi) \cdot (0, \pi^{-1}\gamma_{m,n}),$$

*where* $\pi \in S_{NC}(m, n)$*;*

(b)

$$(1_{m+n}, \gamma_{m,n}) = (0, \pi) \cdot (\mathcal{W}, \pi^{-1}\gamma_{m,n}),$$

*where* $\pi \in NC(m) \times NC(n)$, $|\mathcal{W}| = |\pi^{-1}\gamma_{m,n}| + 1$, *and* $\mathcal{W}$ *connects a cycle of* $\pi^{-1}\gamma_{m,n}$ *in* $NC(m)$ *with a cycle in* $NC(n)$*;*

(c)

$$(1_{m+n}, \gamma_{m,n}) = (\mathcal{V}, \pi) \cdot (0, \pi^{-1}\gamma_{m,n}),$$

*where* $\pi \in NC(m) \times NC(n)$, $|\mathcal{V}| = |\pi| + 1$, *and* $|\mathcal{W}| = |\pi^{-1}\gamma_{m,n}| + 1$, *and* $\mathcal{V}$ *connects a cycle of* $\pi$ *in* $NC(m)$ *with a cycle in* $NC(n)$.

*Proof.* (1) The correspondence between non-crossing partitions and permutations was studied in detail by Biane [Bia97]. In this case we have

$$\big(|\mathcal{V}| - |\pi|\big) + \big(|\mathcal{W}| - |\sigma|\big) + \big(|\mathcal{V}| + |\mathcal{W}| - |\mathcal{V} \vee \mathcal{W}|\big) = |1_n| - |\gamma_n| = 0.$$

Since all three terms in brackets are greater or equal to zero, all of them must vanish, i.e.,

$$|\mathcal{V}| = |\pi|, \qquad \text{thus} \qquad \mathcal{V} = 0_\pi$$

$$|\mathcal{W}| = |\sigma|, \qquad \text{thus} \qquad \mathcal{W} = 0_\sigma$$

and

$$|\pi| + |\sigma| = |\mathcal{V}| + |\mathcal{W}| = |\mathcal{V} \vee \mathcal{W}| = |\gamma| = n - 1.$$

(2) Now we have

$$\big(|\mathcal{V}| - |\pi|\big) + \big(|\mathcal{W}| - |\sigma|\big) + \big(|\mathcal{V}| + |\mathcal{W}| - |\mathcal{V} \vee \mathcal{W}|\big) = \big(|\mathcal{V} \vee \mathcal{W}| - |\pi\sigma|\big) = 1,$$

which means that two of the terms on the left-hand side must be equal to 0, and the other term must be equal to 1. Thus we have the following three possibilities.

(a)

$$\begin{aligned} |\mathcal{V}| &= |\pi|, \qquad \text{thus} \qquad & \mathcal{V} = 0_\pi, \\ |\mathcal{W}| &= |\sigma|, \qquad \text{thus} \qquad & \mathcal{W} = 0_\sigma \end{aligned}$$

and

$$|\pi| + |\sigma| = |\mathcal{V}| + |\mathcal{W}| = |\mathcal{V} \vee \mathcal{W}| + 1 = m + n.$$

Note that

$$\pi \vee \sigma = \mathcal{V} \vee \mathcal{W} = 1_{m+n},$$

and thus $\pi$ connects the two cycles of $\gamma_{m,n}$. This means that $\pi$ is a non-crossing $(m, n)$-permutation.

(b)

$$|\mathcal{V}| = |\pi|, \qquad \text{thus} \qquad \mathcal{V} = 0_\pi,$$

$$|\mathcal{W}| = |\sigma| + 1,$$

and

$$|\mathcal{V}| + |\mathcal{W}| = |\mathcal{V} \vee \mathcal{W}| = m + n - 1.$$

This implies

$$|\pi| + |\gamma_{m,n}\pi^{-1}| = m + n - 2,$$

which means that $\pi$ must be a disconnected non-crossing $(m, n)$-annular permutation, i.e.,

$$\pi = \pi_1 \times \pi_2 \qquad \text{with} \qquad \pi_1 \in NC(m), \pi_2 \in NC(n).$$

(c)
$$|\mathcal{V}| = |\pi| + 1,$$
$$|\mathcal{W}| = |\sigma| + 1, , \qquad \text{thus} \qquad \mathcal{W} = 0_\sigma$$

and
$$|\mathcal{V}| + |\mathcal{W}| = |\mathcal{V} \vee \mathcal{W}| = m + n - 1.$$

This implies
$$|\pi| + |\gamma_{m,n}\pi^{-1}| = m + n - 2,$$

which means that $\pi$ must be a disconnected non-crossing $(m, n)$-annular permutation, i.e.,
$$\pi = \pi_1 \times \pi_2 \qquad \text{with} \qquad \pi_1 \in NC(m), \pi_2 \in NC(n).$$

□

EXAMPLE 5.12. We can now use the previous description of factorizations of disc and tunnel permutations to write down explicit first and second order formulas for our convolution of multiplicative functions.
1) In the first order case we have

$$(f * g)(1_n, \gamma_n) = (f * g)(0, \gamma_n) = \sum_{\pi \in NC(n)} f(0, \pi) g(0, \pi^{-1}\gamma_n). \tag{38}$$

This equation is exactly the formula for the convolution of multiplicative functions on non-crossing partitions, which is the cornerstone of the combinatorial description of first order freeness [NSp97]. (Note that $\pi^{-1}\gamma_n$ is in this case the Kreweras complement of $\pi$.)
2) In the second order case we have

$$(f * g)(1_{m+n}, \gamma_{m,n}) = \sum_{\pi \in S_{NC}(m,n)} f(0, \pi) g(0, \pi^{-1}\gamma_{m,n})$$
$$+ \sum_{\substack{\pi \in NC(m) \times NC(n) \\ |\mathcal{V}| = |\pi| + 1}} \big( f(0, \gamma_{m,n}\pi^{-1}) g(\mathcal{V}, \pi) + f(\mathcal{V}, \pi) g(0, \pi^{-1}\gamma_{m,n}) \big).$$

We should expect that this formula is the combinatorial key for the understanding of second order freeness. However, in this form it does not match exactly the formulas appearing in [MŚS07]. Let us, however, for a multiplicative function $f$ put, for $\pi \in NC(n)$,

$$\tilde{f}_1(\pi) := f(1_n, \pi) \qquad (\pi \in NC(n)) \tag{39}$$

and, for $\pi_1 \in NC(m)$ and $\pi_2 \in NC(n)$,

$$\tilde{f}_2(\pi_1, \pi_2) = \sum_{\substack{\mathcal{V} \geq \pi_1 \times \pi_2, \\ |\mathcal{V}| = |\pi| + 1, \\ \mathcal{V} \vee (\pi_1 \times \pi_2) = 1_{m+n}}} f(\mathcal{V}, \pi_1 \times \pi_2). \tag{40}$$

Note that in the definition of $\tilde{f}_2$ the sum is running over all $\mathcal{V}$ which connect exactly one cycle of $\pi_1$ with one cycle of $\pi_2$.

Then, with $h = f * g$, we have

$$\tilde{h}_2(1_m, 1_n) = \sum_{\pi \in S_{NC}(m,n)} \tilde{f}_1(\pi)\tilde{g}_1(\pi^{-1}\gamma_{m,n})$$
$$+ \sum_{\pi_1,\pi_2 \in NC(m)\times NC(n)} \big(\tilde{f}_2(\pi_1,\pi_2)\tilde{g}_1(\pi_1^{-1} \times \pi_2^{-1}\gamma_{m.n})$$
$$+ \tilde{f}_1(\pi_1 \times \pi_2)\tilde{g}_2(\pi_1^{-1}\gamma_m, \pi_2^{-1}\gamma_n)\big).$$

In this form we recover exactly the structure of the formula (10) from [MŚS07], which describes second order freeness. The descriptions in terms of $f$ and in terms of $\tilde{f}_2$ are equivalent. Whereas $f$ is multiplicative, $\tilde{f}_2$ satisfies a kind of cocycle property. From our present perspective the description of second (and higher) order freeness in terms of multiplicative functions seems more natural. In any case, we see that our convolution of multiplicative functions on partitioned permutations is a generalization of the structure underlying first and second order freeness.

5.4. ZETA AND MÖBIUS FUNCTION. In the definition of our convolution we are running over factorizations of $(\mathcal{U}, \gamma)$ into products $(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma)$. In the first order case the second factor is determined if the first factor is given. In the general case, however, we do not have such a uniqueness of the decomposition; if we fix $(\mathcal{V}, \pi)$ there might be different choices for $(\mathcal{W}, \sigma)$. For example, this situation was considered in Proposition 5.11 in the case (2b). However, in the case when $(\mathcal{W}, \sigma)$ is a disc permutation, it must be of the form $(0_{\pi^{-1}\gamma}, \pi^{-1}\gamma)$ and is thus uniquely determined. Note that factorizations of such a special form appear in our formula (35) and thus deserve special attention.

NOTATION 5.13. Let $(\mathcal{U}, \gamma) \in \mathcal{PS}$ be a fixed partitioned permutation. We say that $(\mathcal{V}, \pi) \in \mathcal{PS}$ is $(\mathcal{U}, \gamma)$*–non-crossing* if

$$(\mathcal{V}, \pi) \cdot (0_{\pi^{-1}\gamma}, \pi^{-1}\gamma) = (\mathcal{U}, \gamma).$$

The set of $(\mathcal{U}, \gamma)$–non-crossing partitioned permutations will be denoted by $\mathcal{PS}_{NC}(\mathcal{U}, \gamma)$, see Remark 5.8.

To justify this notation we point out that $(1_n, \gamma_n)$–non-crossing partitioned permutations can be identified with non-crossing permutations; to be precise

$$\mathcal{PS}_{NC}(1_n, \gamma_n) = \{(0_\pi, \pi) \mid \pi \in NC(n)\}.$$

Furthermore,

$$\mathcal{PS}_{NC}(1_{m+n}, \gamma_{m,n}) = \{(0_\pi, \pi) \mid \pi \in S_{NC}(m,n)\} \cup$$
$$\cup \{(\mathcal{V}, \pi_1 \times \pi_2) \mid \pi_1 \in NC(m), \pi_2 \in NC(n), \mathcal{V} \geq \pi, |\mathcal{V}| = |\pi| + 1$$
$$\text{and } \mathcal{V} \text{ connects one cycle of } \pi_1 \text{ to a cycle of } \pi_2\}.$$

We can now also use a special multiplicative function, which we will call Zeta-function $\zeta$, to single out such factorizations. It will be useful to be able to invert formula (35), which means we need also the inverse of $\zeta$ under our convolution.

This inverse, called the Möbius-function $\mu$, is a key object in the theory and contains a lot of important information.

NOTATION 5.14.

(1) The *Zeta-function* $\zeta$ is the multiplicative function on $\mathcal{PS}$ which is determined by

$$\zeta(1_n, \pi) = \begin{cases} 1 & \text{if } (1_n, \pi) \text{ is a disc permutation, i.e., if } 1_n = 0_\pi, \\ 0 & \text{otherwise.} \end{cases}$$

(2) The *Möbius function* $\mu$ is the inverse of $\zeta$ under convolution, i.e., it is determined by

$$\zeta * \mu = \delta = \mu * \zeta.$$

Note that in general

$$\zeta(\mathcal{V}, \pi) = \begin{cases} 1, & \text{if } \mathcal{V} = 0_\pi \\ 0, & \text{if } \mathcal{V} > 0_\pi. \end{cases}$$

It is also quite easy to see that the Möbius function exists and is uniquely determined as the inverse of the Zeta-function — the determining equations can be solved recursively. Indeed letting $\mu_n = \mu(1_n, \gamma_n)$ and $\mu_{m,n} = \mu(1_{m+n}, \gamma_{m,n})$ we have

$$\begin{gathered} 0 = \mu_{1,1} + \mu_2 \\ 0 = \mu_{1,2} + 2\mu_1\mu_{1,1} + 2\mu_3 + 2\mu_1\mu_2 \\ 0 = \mu_{2,2} + 4\mu_1\mu_{2,1} + 4\mu_1^2\mu_{1,1} + 4\mu_4 + 8\mu_1\mu_3 + 2\mu_2^2 + 4\mu_1^2\mu_2 \\ 0 = \mu_{1,3} + 3\mu_1\mu_{2,1} + 3\mu_2\mu_{1,1} + 3\mu_4 + 6\mu_1\mu_3 + 3\mu_2^2 + 3\mu_1^2\mu_2 \\ 0 = \mu_{2,3} + 2\mu_1\mu_{1,3} + 3\mu_1\mu_{2,2} + 3\mu_2\mu_{1,2} + 9\mu_1^2\mu_{1,2} + 6\mu_1\mu_2\mu_{1,1} + 6\mu_1^3\mu_{1,1} \\ + 6\mu_5 + 18\mu_1\mu_4 + 12\mu_2\mu_3 + 18\mu_1^2\mu_3 + 12\mu_1\mu_2^2 + 6\mu_1^3\mu_2 \\ 0 = \mu_{3,3} + 6\mu_1\mu_{2,3} + 6\mu_2\mu_{1,3} + 6\mu_1^2\mu_{1,3} + 9\mu_1^2\mu_{2,2} + 18\mu_1\mu_2\mu_{1,2} + 18\mu_1^3\mu_{1,2} \\ + 9\mu_2^2\mu_{1,1} + 18\mu_1^2\mu_2\mu_{1,1} + 9\mu_1^4\mu_{1,1} + 9\mu_6 + 36\mu_1\mu_5 + 27\mu_2\mu_4 + 54\mu_1^2\mu_4 \\ + 9\mu_3^2 + 72\mu_1\mu_2\mu_3 + 36\mu_1^3\mu_3 + 12\mu_2^3 + 36\mu_1^2\mu_2^2 + 9\mu_1^4\mu_2 \end{gathered}$$

This shows how, knowing the first order Möbius function $\mu_n$, the second order Möbius function $\mu_{m,n}$ can be calculated recursively.
One should observe that with these notations we have

$$(f * \zeta)(\mathcal{U}, \gamma) = \sum_{(\mathcal{V},\pi) \in \mathcal{PS}_{NC}(\mathcal{U},\gamma)} f(\mathcal{V}, \pi).$$

In the following we will use the notation

$$\zeta^{*p} = \underbrace{\zeta * \cdots * \zeta}_{p\text{-times}}.$$

It is clear, by definition, that $\zeta^{*p}$ counts factorizations into the product of $p$ disc permutations, thus we have the following result.

PROPOSITION 5.15. *For $(\mathcal{U}, \gamma) \in \mathcal{PS}$ and $p \geq 1$ we have*

$$\zeta^{*p}(\mathcal{U}, \gamma) := \#\{(\pi_1, \ldots, \pi_p) \mid (\mathcal{U}, \gamma) = (0, \pi_1) \cdots (0, \pi_p)\}.$$

Of special interest for us is the case $p = 2$.

PROPOSITION 5.16. *We have for all* $r \geq 1$ *and* $n(1), \ldots, n(r) \in \mathbb{N}$, $n := n(1) + \cdots + n(r)$ *that*

$$(\zeta * \zeta)(1_n, \gamma_{n(1),\ldots,n(r)}) = \# S_{NC}(n(1), \ldots, n(r)).$$

*Proof.* As noted above, $(\zeta * \zeta)(1_n, \gamma_{n(1),\cdots,n(r)})$ counts the number of factorizations of $(1_n, \gamma_{n(1),\ldots,n(r)})$ into a product of two disc permutations, i.e., the number of factorizations of the form

$$(1_n, \gamma_{n(1),\ldots,n(r)}) = (0, \pi) \cdot (0, \pi^{-1}\gamma_{n(1),\ldots,n(r)}),$$

with

$$|\pi| + |\pi^{-1}\gamma| = |\gamma| = n - r$$

and $\pi \vee \gamma = 1_n$. But this describes exactly connected $(n(1), \ldots, n(r))$-annular permutations $\pi \in S_{NC}(n(1), \ldots, n(r))$. $\square$

NOTATION 5.17. We put

$$c_{n(1),\ldots,n(r)} := \# S_{NC}(n(1), \ldots, n(r)).$$

Note in particular that $c_n$ counts the number of non-crossing partitions of $n$ elements and thus is the Catalan number

$$c_n = \frac{1}{n+1}\binom{2n}{n},$$

and that $c_{m,n}$ counts the number of non-crossing $(m, n)$-annular permutations, and thus [MN04]

$$c_{m,n} = \frac{2mn}{m+n}\binom{2m-1}{m}\binom{2n-1}{n}.$$

More generally, an explicit formula for the number of factorizations into $p$ factors was derived by Bousquet-Mélou and Schaeffer [BMS00], namely one has (with $n := n(1) + \cdots + n(r)$)

$$\zeta^{*p}(1_n, \gamma_{n(1),\cdots,n(r)}) = p\frac{[(p-1)n-1]!}{[(p-1)n-r+2]!}\prod_{i=1}^{r}\left[n(i)\binom{pn(i)-1}{n(i)}\right],$$

and thus in particular

$$c_{n(1),\ldots,n(r)} = 2\frac{(n-1)!}{(n-r+2)!}\prod_{i=1}^{r}\left[n(i)\binom{2n(i)-1}{n(i)}\right].$$

For our purposes, however, the following recursive formula for the number of factorizations is more interesting.
In the next theorem we will show how to reduce the problem of counting the number of disc factorizations on $[n]$ to counting the factorizations on $[n-1]$. This will enable of to obtain a recursive formula for $c_{n_1,\ldots,n_r}$.

Notation 5.18. Let $(\mathcal{U}, \gamma)$ be a partitioned permutation of $[n]$ with $\gamma(1) \neq 1$. Let $\hat{\gamma}_k$ be the restriction of $(1,k)\gamma(1,\gamma^{-1}(k))$ to the invariant subset $[2,n] := \{2,3,4,\dots,n\}$. Then

$$|\hat{\gamma}_k| = \begin{cases} |\gamma| & \text{if } 1 \text{ and } k \text{ are in different cycles of } \gamma, \\ |\gamma|-1 & \text{if } k = 1 \text{ or } \gamma(1) \\ |\gamma|-2 & \text{if } 1 \text{ and } k \text{ are in the same cycle of } \gamma, \\ & \text{but } k \neq 1 \text{ and } k \neq \gamma(1), \end{cases}$$

Let $\overline{\mathcal{U}} = \mathcal{U}|_{[2,n]}$ be the restriction of $\mathcal{U}$ to $[2,n]$, *i.e.* if the blocks of $\mathcal{U}$ are $U_1, \dots, U_r$ and $1 \in U_1$, then the blocks of $\overline{\mathcal{U}}$ are $\overline{U}_1, U_2, \dots, U_r$ where $\overline{U}_1 = U_1 \cap [2,n]$. In the theorem below we sum over a set of partitions $\mathcal{P}_k$ of $[2,n]$ described as follows.

For $k = 1$, $\gamma(1)$ or $k$ not in the $\gamma$-orbit of 1, $\mathcal{P}_k = \{\overline{\mathcal{U}}\}$ *i.e.* $\mathcal{P}_k$ consists of the single partition $\overline{\mathcal{U}}$.

For $k$ in the $\gamma$-orbit of 1 but $k \neq 1$, $\gamma(1)$, $\mathcal{P}_k = \{\widehat{\mathcal{U}} \mid \hat{\gamma}_k \leq \widehat{\mathcal{U}}, |\widehat{\mathcal{U}}| = |\mathcal{U}| - 2$, and $\overline{\mathcal{U}} = \widehat{\mathcal{U}} \vee (k, \gamma^{-1}(k))\}$. In words this means $\overline{U}_1$ is split into two blocks:

- the first containing the cycle of $\hat{\gamma}_k$ containing $\gamma^{-1}(k)$ and some (possibly none) of the other cycles of $\gamma$ contained in $U_1$
- the second containing the cycle of $\hat{\gamma}_k$ containing $k$ and the remaining (possibly none) cycles of $\gamma$ contained in $U_1$.

More explicitly, in the case $k$ is in the $\gamma$-orbit of 1 but $k \neq 1$, $\gamma(1)$, let us write $\gamma$ as as a product of cycles $d_1 \cdots d_s$ where $d_1 = (1, \gamma(1), \dots, \gamma^t(1))$ is the cycle that contains 1. Let $d_1' = (\gamma(1), \gamma^2(1), \dots, \gamma^{-1}(k))$ and $d_1'' = (k, \dots, \gamma^t(1))$. Then $\hat{\gamma}_k = d_1' d_1'' d_2 \cdots d_s$. $\mathcal{P}_k$ consists of all partitions $\widehat{\mathcal{U}}$ of $[2,n]$ such that $\widehat{\mathcal{U}} = \{U_1', U_1'', U_2, \dots, U_r\}$ where $U_1' \cup U_1'' = \overline{U}_1$, $U_1' \cap U_1'' = \emptyset$, $U_1'$ contains $d_1'$, $U_1''$ contains $d_1''$, and each cycle of $\gamma$ that was in $U_1$ is now in either $U_1'$ or $U_1''$, *i.e.* $\hat{\gamma}_k \leq \widehat{\mathcal{U}}$ and $|\widehat{\mathcal{U}}| = |\mathcal{U}| - 2$.

Theorem 5.19.

$$\zeta^{*^2}(U, \gamma) = \sum_{k=1}^{n} \sum_{\widehat{\mathcal{U}} \in \mathcal{P}_k} \zeta^{*^2}(\widehat{\mathcal{U}}, \widehat{\gamma_k}) \tag{41}$$

*Proof.* We must show that for each factorization $(0,\pi)\cdot(0,\sigma)$ of $(\mathcal{U},\gamma)$ there are $k := \pi(1)$, $\widehat{\mathcal{U}} \in \mathcal{P}_k$, and permutations of $[2,n]$, $\hat{\pi}$ and $\hat{\sigma}$ such that $(0,\hat{\pi})\cdot(0,\hat{\sigma}) = (\widehat{\mathcal{U}}, \hat{\gamma}_k)$. Conversely we must show that given $k$, $\widehat{\mathcal{U}} \in \mathcal{P}_k$ and a factorization $(0,\hat{\pi}) \cdot (0,\hat{\sigma})$ of $(\widehat{\mathcal{U}}, \hat{\gamma}_k)$ there are $\pi$ and $\sigma$ such that $(0,\pi) \cdot (0,\sigma) = (\mathcal{U},\gamma)$ and $\pi(1) = k$. Moreover we must show that these two maps are inverses of each other. The relation between $\pi$, $\sigma$ and $\hat{\pi}$, $\hat{\sigma}$ is given by $\hat{\pi} = (1,k)\pi|_{[2,n]}$, $\hat{\sigma} = \sigma(1,\gamma^{-1}(k))|_{[2,n]}$. So on the level of permutations we have a bijection. The main work of the proof is to show that starting with $\pi$ and $\sigma$ we have $\widehat{\mathcal{U}} := \hat{\pi} \vee \hat{\sigma} \in \mathcal{P}_k$ and $2|\widehat{\mathcal{U}}| - |\hat{\gamma}_k| = |\hat{\pi}| + |\hat{\sigma}|$; and then conversely starting with $\widehat{\mathcal{U}} \in \mathcal{P}_k$ and a factorization $(0,\hat{\pi}) \cdot (0,\hat{\sigma})$ of $(\widehat{\mathcal{U}}, \hat{\gamma}_k)$ then $2|\mathcal{U}| - |\gamma| = |\pi| + |\sigma|$ and $\pi \vee \sigma = \mathcal{U}$.

Note that we have for all $k$

$$|\hat{\pi}| = \begin{cases} |\pi| - 1 & k \neq 1 \\ |\pi| & k = 1 \end{cases}$$

$$|\hat{\sigma}| = \begin{cases} |\sigma| - 1 & k \neq \gamma(1) \\ |\sigma| & k = \gamma(1) \end{cases}$$

It is necessary to break the proof into four cases: $k$ is not in the $\gamma$-orbit of 1; $k$ is in the $\gamma$-orbit of 1 but $k \neq 1, \gamma(1)$; $k = 1$; and $k = \gamma(1)$.
Suppose we have a factorization

$$(\mathcal{U}, \gamma) = (0, \pi) \cdot (0, \sigma),$$

i.e., $\gamma = \pi\sigma$, $\mathcal{U} = \pi \vee \sigma$, and

$$2|\mathcal{U}| - |\gamma| = |\pi| + |\sigma|$$

with $k := \pi(1)$ not in the $\gamma$-orbit of 1. Then$|\hat{\gamma}_k| = |\gamma|$ and $\mathcal{P}_k$ contains only the partition of $[2, n]$ which results from $\mathcal{U}$ by removing 1, *i.e.* $\widehat{\mathcal{U}} = \overline{\mathcal{U}}$. Then we have $|\hat{\mathcal{U}}| = |\mathcal{U}| - 1$. Hence $|\hat{\pi}| + |\hat{\sigma}| = |\pi| + |\sigma| - 2 = 2|\mathcal{U}| - |\gamma| - 2 = |\widehat{\mathcal{U}}| - |\gamma| = |\widehat{\mathcal{U}}| - |\hat{\gamma}_k|$.
Also $0_\pi|_{[2,n]} = 0_{\hat{\pi}}$ and $0_\gamma|_{[2,n]} \leq 0_{\hat{\gamma}_k}$. Thus $\widehat{\mathcal{U}} = (\pi \vee \gamma)|_{[2,n]} \leq \hat{\pi} \vee \hat{\gamma}_k$. On the other hand the difference between $0_\gamma|_{[2,n]}$ and $0_{\hat{\gamma}_k}$ is that the blocks containing 1 and $k$ have been joined. However these points were already connected by $\pi$. Thus $\hat{\pi} \vee \hat{\gamma}_k \leq \widehat{\mathcal{U}}$, and so $\hat{\mathcal{U}} = \hat{\pi} \vee \hat{\sigma}$, and thus

$$(\hat{\mathcal{U}}, \hat{\gamma}) = (0, \hat{\pi}) \cdot (0, \hat{\sigma}).$$

Conversely, given a factorization $(0, \hat{\pi}) \cdot (0, \hat{\sigma})$ of $(\widehat{\mathcal{U}}, \hat{\gamma}_k)$, let $\pi = (1, k)\hat{\pi}$ and $\sigma = \hat{\sigma}(1, \gamma^{-1}(k))$. Then $\pi \vee \sigma = \mathcal{U}$ because 1 has been connected to the block of $\widehat{\mathcal{U}}$ containing $k$. Also $\#(\pi) = \#(\hat{\pi})$ and $\#(\sigma) = \#(\hat{\sigma})$; thus $|\pi| = |\hat{\pi}| - 1$ and $|\sigma| = |\hat{\sigma}| - 1$, and so $|\pi| + |\sigma| = 2|\mathcal{U}| - |\gamma|$. This establishes the bijection when $k$ is not in the $\gamma$-orbit of 1.
Let us now consider the case that 1 and $k$ are in the same cycle of $\gamma$, but $k \neq 1, \gamma(1)$. Again suppose that $(0, \pi) \cdot (0, \sigma)$ is a factorization of $(\mathcal{U}, \gamma)$ with $\pi(1) = k$. In this case we have that $|\hat{\gamma}_k| = |\gamma| - 2$ and so by the triangle inequality, Lemma 4.7

$$\begin{aligned} 2|\hat{\pi} \vee \hat{\sigma}| - |\gamma| + 2 &= 2|\hat{\pi} \vee \hat{\sigma}| - |\hat{\gamma}| \\ &= |(\hat{\pi} \vee \hat{\sigma}, \hat{\pi}\hat{\sigma})| \\ &\leq |(0, \hat{\pi})| + |(0, \hat{\sigma})| \\ &= |\hat{\pi}| + |\hat{\sigma}| \\ &= |\pi| + |\sigma| - 2 \\ &= 2|\mathcal{U}| - |\gamma| - 2, \end{aligned}$$

and thus

$$|\hat{\pi} \vee \hat{\sigma}| \leq |\mathcal{U}| - 2.$$

On the other hand, let us compare

$$\hat{\pi} \vee \hat{\sigma} = \hat{\pi} \vee \hat{\gamma} \qquad \text{with} \qquad \mathcal{U} = \pi \vee \gamma.$$

Note that all our changes of the permutations affected only what happens on the first cycle of $\gamma$. Since the transition from $\gamma$ to $\hat{\gamma}$ consists in removing the point 1 and splitting the first cycle of $\gamma$ into two cycles, we can lose at most one block by going over from $\hat{\pi} \vee \hat{\gamma}$ to $\pi \vee \gamma$. Thus

$$|\hat{\pi} \vee \hat{\sigma}| = (n-1) - \#(\hat{\pi} \vee \hat{\sigma}) \geq (n-1) - (\#\mathcal{U} + 1) = |\mathcal{U}| - 2,$$

so that we necessarily have the equality

$$|\hat{\pi} \vee \hat{\sigma}| = |\mathcal{U}| - 2.$$

Thus $\widehat{\mathcal{U}} := \hat{\pi} \vee \hat{\sigma} \in \mathcal{P}_k$ and $2|\hat{\pi} \vee \hat{\sigma}| - |\hat{\gamma}_k| = |\hat{\pi}| + |\hat{\sigma}|$. Hence $(0, \hat{\pi}) \cdot (0, \hat{\sigma})$ is a factorization of $(\widehat{\mathcal{U}}, \hat{\gamma}_k)$.

Conversely let us suppose that $k$ is in the $\gamma$-orbit of 1 but $k \neq 1$ or $\gamma(1)$ and $\widehat{\mathcal{U}} \in \mathcal{P}_k$ and $(0, \hat{\pi}) \cdot (0, \hat{\sigma})$ is a factorization of $(\widehat{\mathcal{U}}, \hat{\gamma}_k)$. We must show that $\pi \vee \sigma = \mathcal{U}$ and that $|\pi| + |\sigma| = 2|\mathcal{U}| - |\gamma|$. 1 and $k$ are in the same orbit of $\pi$ and 1 and $\gamma^{-1}(k)$ are in the same orbit of $\sigma$. So the blocks of $\widehat{\mathcal{U}}$ containing $d_1'$ and $d_1''$ are joined in $\pi \vee \sigma$. Thus $\pi \vee \sigma = \mathcal{U}$. Also $|\widehat{\mathcal{U}}| = |\mathcal{U}| - 2$, so $|\pi| + |\sigma| = |\hat{\pi}| + |\hat{\sigma}| + 2 = 2|\widehat{\mathcal{U}}| - |\hat{\gamma}_k| + 2 = |\mathcal{U}| - |\hat{\gamma}_k| - 2 = 2|\mathcal{U}| - |\gamma|$. Thus $(0, \pi) \cdot (0, \sigma)$ is a factorization of $(\mathcal{U}, \gamma)$. This establishes the bijection in the case $k$ is in the $\gamma$-orbit of 1 but $k \neq 1$ or $\gamma(1)$.

Next suppose that $k = 1$ and $(0, \pi) \cdot (0, \sigma)$ is a factorization of $(\mathcal{U}, \gamma)$ with $\pi(1) = 1$. Then $|\hat{\pi}| + |\hat{\sigma}| = |\pi| + |\sigma| - 1 = 2|\mathcal{U}| - |\gamma| - 1 = 2|\widehat{\mathcal{U}}| - |\gamma| + 1 = 2|\widehat{\mathcal{U}}| - |\hat{\gamma}_k|$. Let $U_1$ be the block of $\mathcal{U}$ containing 1 and $\overline{U}_1 = U_1 \cap [2, n]$. We must show that $\overline{U}_1$ is a block of $\hat{\pi} \vee \hat{\gamma}_k$. Since $\pi \vee \gamma = \mathcal{U}$ we know that if $d_i$ and $d_j$ are cycles of $\gamma$ contained in $U_1$ then $\pi$ must connect them. Since $\pi|_{\overline{U}_1} = \hat{\pi}|_{\overline{U}_1}$ we see that $\hat{\pi}$ connects the corresponding cycles of $\hat{\gamma}_k$ (which are unchanged except for the cycle containing 1). Similarly if $f_1$ and $f_2$ are cycles of $\pi$ contained in $U_1$ and neither is a singleton then they are connected by $\gamma$ and thus by $\hat{\gamma}_k$. Thus $(0, \hat{\pi}) \cdot (0, \hat{\sigma})$ is a factorization of $(\widehat{\mathcal{U}}, \hat{\gamma}_k)$.

Conversely suppose that $k = 1$, $\widehat{\mathcal{U}} \in \mathcal{P}_k$, and $(0, \hat{\pi}) \cdot (0, \hat{\sigma})$ is a factorization of $(\widehat{\mathcal{U}}, \hat{\gamma}_k)$. We must show that $\pi(1) = 1$ and $(0, \pi) \cdot (0, \sigma)$ is a factorization of $(\mathcal{U}, \gamma)$. Since $\hat{\pi} \vee \hat{\gamma}_k = \widehat{\mathcal{U}}$ and $\gamma$ connects 1 to $\gamma(1) \in \overline{U}_1$, we have that $\pi \vee \gamma = \mathcal{U}$. Also $|\pi| + |\sigma| = |\hat{\pi}| + |\hat{\sigma}| + 1 = 2|\widehat{\mathcal{U}}| - |\hat{\gamma}_k| + 1 = 2|\mathcal{U}| - |\hat{\gamma}_1| - 1 = 2|\mathcal{U}| - |\gamma|$. Thus $(0, \pi) \cdot (0, \sigma)$ is a factorization of $(\mathcal{U}, \gamma)$. This completes the case when $k = 1$. The proof in the case $k = \gamma(1)$ is exactly the same except that the roles of $\pi$ and $\sigma$ are reversed. □

Let us take a closer look at the meaning of Theorem 5.19 for the case $(\mathcal{U}, \gamma) = (1_n, \gamma_{n(1), \dots, n(r)})$. To reduce the depth of subscripts we shall write $c(n_1, \dots, n_r)$ for $c_{n(1), \dots, n(r)}$.

PROPOSITION 5.20. *We have for all* $r, n_1, \ldots, n_r \in \mathbb{N}$ *the recursion*

$$c(n_1, \ldots, n_r) = \sum_{l=2}^{r} n_l \cdot c(n_1 + n_l - 1, n_2, \ldots, n_{l-1}, n_{l+1}, \ldots, n_r) + \sum_{k=1}^{n_1} \sum_{\substack{A=\{i_1,\ldots,i_s\} \\ B=\{j_1,\ldots,j_t\}}} c(k-1, n_{i_1}, \ldots, n_{i_s}) c(n_1 - k, n_{j_1}, \ldots, n_{j_t}) \tag{42}$$

*where the sum is over all pairs of subsets* $A, B \subset [2, r]$ *such that* $A \cap B = \emptyset$ *and* $A \cup B = [2, r]$ *including the possibility that either* $A$ *or* $B$ *could be empty. We have for all* $m, n \geq 1$

$$c_n = \sum_{1 \leq k \leq n} c_{k-1} c_{n-k},$$

*and*

$$c_{m,n} = \sum_{1 \leq k \leq n} \big(c_{k-1} c_{m,n-k} + c_{m,k-1} c_{n-k}\big) + m c_{m+n-1}, \tag{43}$$

*where we use the convention that* $c_0 = 1$ *but* $c(n_1, \ldots, n_r) = 0$ *if* $r > 1$ *and for some* $i$, $n_i = 0$.

*Proof.* Let $n = n_1 + \cdots + n_r$. By Proposition 5.16 $c(n_1, \ldots, n_r) = \zeta^{*^2}(1_n, \gamma_{n_1,\ldots,n_r})$. So we must give the correspondence between the terms on the right hand side of (41) and the right hand side of (42). In this case $\mathcal{U} = 1_n$ and $\overline{\mathcal{U}} = 1_{n-1}$ (in the notation of 5.18). Thus $\mathcal{P}_k = \{1_{n-1}\}$. Also for $n_1 + \cdots + n_{l-1} < k \leq n_1 + \cdots + n_l$, $\zeta^{*^2}(1_{n-1}, \hat{\gamma}_k) = c(n_1 + n_l - 1, n_2, \ldots, n_{l-1}, n_{l+1}, \ldots, n_r)$. Thus

$$\begin{aligned} \sum_{k=n_1+1}^{n} \sum_{\widehat{\mathcal{U}} \in \mathcal{P}_k} \zeta^{*^2}(\widehat{\mathcal{U}}, \hat{\gamma}_k) &= \sum_{k=n_1+1}^{n} \zeta^{*^2}(1_{n-1}, \hat{\gamma}_k) \\ &= \sum_{l=2}^{r} \sum_{k=n_1+\cdots+n_{l-1}+1}^{n_1+\cdots+n_l} c(n_1 + n_l - 1, n_2, \ldots, n_{l-1}, n_{l+1}, \ldots, n_r) \\ &= \sum_{l=2}^{r} n_l \cdot c(n_1 + n_l - 1, n_2, \ldots, n_{l-1}, n_{l+1}, \ldots, n_r) \end{aligned} \tag{44}$$

For $k \leq n_1$, $\hat{\gamma}_k = d_1' d_1'' d_2 \cdots d_r$, with $d_1'$ a cycle of length $k-1$ and $d_1''$ a cycle of length $n_1 - k$. $\mathcal{P}_k$ is the set of all partitions of the cycles of $\hat{\gamma}_k$ into two blocks such that $d_1'$ and $d_1''$ are in different blocks. Hence

$$\sum_{\widehat{\mathcal{U}} \in \mathcal{P}_k} \zeta^{*^2}(\widehat{\mathcal{U}}, \hat{\gamma}_k) = \sum_{\substack{A=\{i_1,\ldots,i_s\} \\ B=\{j_1,\ldots,j_t\}}} c(n_1 - k, n_{i_1}, \ldots, n_{i_s}) c(k-1, n_{j_1}, \ldots, n_{j_t})$$

where the sum is over all pairs of subsets $A, B \subset [2, r]$ such that $A \cap B = \emptyset$ and $A \cup B = [2, r]$ including the possibility that either $A$ or $B$ could be empty.

Thus

$$\sum_{k=1}^{n_1} \sum_{\widehat{\mathcal{U}} \in \mathcal{P}_k} \zeta^{*^2}(\widehat{\mathcal{U}}, \hat{\gamma}_k) \tag{45}$$

$$= \sum_{k=1}^{n_1} \sum_{\substack{A=\{i_1,\dots,i_s\} \\ B=\{j_1,\dots,j_t\}}} c(n_1 - k, n_{i_1}, \dots, n_{i_s}) c(k-1, n_{j_1}, \dots, n_{j_t})$$

Assembling equations (44) and (45) gives the result. □

In [OZ84], O'Brien and Zuber used a similar formula of this kind in order to compute the asymptotics of, so called, external field matrix integral. See also [BMS00] and Theorem 5.22.

Clearly, our notions around the convolution of functions on $\mathcal{PS}$ are analogous to (and motivated by) the convolution of functions on posets. Even though we are not able to put the above theory into the framework of posets, it seems that this analogy goes quite far. The following description of the Möbius functions is an instance of this—its poset analogue is due to Hall (see [Rot64]). It is essentially the simple observation that one can expand the Möbius function in terms of a geometric series as

$$\mu = \zeta^{*-1} = \big(\delta + (\zeta - \delta)\big)^{*-1} = \sum_{k=0}^{\infty} (-1)^k (\zeta - \delta)^{*k}.$$

Proposition 5.21. *We have for any $(\mathcal{U}, \gamma) \in \mathcal{PS}$ that*

$$\mu(\mathcal{U}, \gamma) = \delta(\mathcal{U}, \gamma) + \sum_{k=1}^{\infty} \sum_{\substack{(\mathcal{U},\gamma)=(0,\pi_1)\cdots(0,\pi_k) \\ \pi_i \neq e \ \forall i}} (-1)^k.$$

*Proof.* As noted above this is just the geometric series for

$$(\delta + (\zeta - \delta))^{*-1}.$$

(Note that we are working for this in the algebra of functions on $\mathcal{PS}$ with the pointwise sum and the convolution as sum and product—we are not bothering about multiplicativity.) The only thing to check is that the sum is finite, and this is the case because the number of factors $k$ is bounded by $|(\mathcal{U}, \gamma)|$, since $|(0, \pi)| \geq 1$ for any $\pi \neq e$. □

This description of the Möbius function allows us now to derive a recursive formula for $\mu$.

Theorem 5.22. *Consider $(\mathcal{U}, \gamma) \in \mathcal{PS}$ such that $\gamma(1) \neq 1$. Then we have*

$$\mu(\mathcal{U}, \gamma) = (-1) \sum_{\substack{(0,(1,k))\cdot(\mathcal{V},\pi)=(\mathcal{U},\gamma) \\ k \neq 1}} \mu(\mathcal{V}, \pi), \tag{46}$$

*where the sum runs over all decompositions of $(\mathcal{U}, \gamma)$ into a product of a disc transposition $(0,(1,k))$ (with $k \geq 2$) and a $(\mathcal{V}, \pi) \in \mathcal{PS}$.*

The proof of this theorem will rely on the following lemma.

Lemma 5.23. *Let $(\mathcal{U}, \gamma) \in \mathcal{PS}$ such that $\gamma(1) \neq 1$. For $p \in \mathbb{N}$, we denote by $\mathcal{S}_p$ the set consisting of all tuples $(\pi_1, \dots, \pi_p)$ of permutations such that $\pi_i \neq e$ for all $i = 1, \dots, p$ and*

$$(0, \pi_1) \cdots (0, \pi_p) = (\mathcal{U}, \gamma).$$

*We consider now the two sums*

$$S_1 := \sum_{p=1}^{\infty} \sum_{(\pi_1, \dots, \pi_p) \in \mathcal{S}_p} (-1)^p \tag{47}$$

*and*

$$S_2 := \sum_{p=1}^{\infty} \sum_{\substack{(\pi_1, \dots, \pi_p) \in \mathcal{S}_p \\ \pi_1 = (1,k) \text{ for } k \neq 1}} (-1)^p \tag{48}$$

*where the second sum $S_2$ is over all tuples $(\pi_1, \dots, \pi_p)$ as for the first sum $S_1$, but now with the additional property that $\pi_1$ is a transposition interchanging the element 1 with some other element.*
*Then the two sums* (47) *and* (48) *are equal,*

$$S_1 = S_2.$$

*Proof.* Let $\pi = (\pi_1, \dots, \pi_p) \in \mathcal{S}_p$. Let $1 \leq q \leq p$ denote the smallest index for which 1 is not a fixed point of $\pi_q$; note that such a $q$ necessarily exists since $\gamma(1) \neq 1$. We shall group all factorizations into three classes: 1a), 1b) and 2). Class 1) consists of factorizations for which $\pi_q$ is a transposition interchanging 1 with some other element. The subclass 1a) consists of factorizations for which $q = 1$ and subclass 1b) of those for which $q \geq 2$. Class 2) consists of all other factorizations.
Let $\Pi = (\pi_1, \dots, \pi_p)$ be a factorization from the class 1b). We define

$$\Pi' = (\pi'_1, \dots, \pi'_{p-1}) = (\pi_1, \dots, \pi_{q-2}, \pi_{q-1}\pi_q, \pi_{q+1}, \dots, \pi_p).$$

In the following we shall prove that $f : \Pi \mapsto \Pi'$ is a bijection between factorizations of class 1*b*) and factorizations of class 2).
Firstly, we prove that $\Pi' \in \mathcal{S}_p$ and is of class 2). Clearly, $\pi'_{q-1} = \pi_{q-1}\pi_q$ is a permutation which does not fix 1, it is not a transposition interchanging 1 with some other element, and we have

$$(0, \pi_{q-1}) \cdot (0, \pi_q) = (0, \pi'_{q-1}).$$

In order to show that $f$ is a bijection we shall describe its inverse. If $\Pi' = (\pi'_1, \dots, \pi'_{p-1}) \in \mathcal{S}_p$ and is of class 2), we define $1 \leq q \leq p-1$ to be the smallest number for which $\pi'_{q-1}$ does not fix 1. There is a unique decomposition $\pi'_{q-1} = \pi_{q-1}\pi_q$ such that 1 is a fixed point of $\pi_{q-1}$ and $\pi_q$ is a transposition interchanging 1 with some other element. Thus $|\pi_{q-1}| + |\pi_q| = |\pi'_{q-1}|$. The assumption that the factorization $\Pi'$ is of class 2) implies that $\pi_{q-1} \neq e$. For $1 \leq i \leq q-2$ we set $\pi_i = \pi'_i$ and for $q+1 \leq i \leq p$ we set $\pi_i = \pi'_{i-1}$. In this

way we defined $\Pi = (\pi_1, \dots, \pi_p)$. Now it is easy to check that $g : \Pi' \mapsto \Pi$ is a left and right inverse of $f$.
Since the factorization $\Pi$ and the corresponding $\Pi'$ contribute to (47) with the opposite signs, the contribution of all factorizations of class 1b) cancels with the contribution of factorizations of class 2). $\square$

*Proof of 5.22.* In the proof we will consider all factorizations $(0, \pi_1) \cdot (0, \pi_2) \cdots (0, \pi_p) = (\mathcal{U}, \gamma)$ with the requirement that $\pi_i \neq e$ for all $i$, *i.e.* $(\pi_1, \dots, \pi_p) \in \mathcal{S}_p$, as in the proof of Lemma 5.23. Sometimes we will require in addition that $\pi_1 = (1, k)$ with $k \neq 1$. To simplify the notation we will not explicitly state every time that $\pi_i \neq e$. Since $\gamma(1) \neq 1$ we have $\delta(\mathcal{U}, \gamma) = 0$. When $\gamma$ is a transposition the right hand side of equation (46) is $-1$; so we can assume that $\gamma$ is not a transposition. So by Proposition 5.21 we have

$$\begin{aligned}
\mu(\mathcal{U}, \gamma) &= \sum_{p=1}^{\infty} \sum_{\substack{(0,\pi_1)\cdots(0,\pi_p) \\ =(\mathcal{U},\gamma)}} (-1)^p \overset{(5.23)}{=} \sum_{p=1}^{\infty} \sum_{\substack{(0,(1,k))\cdots(0,\pi_p) \\ =(\mathcal{U},\gamma)}} (-1)^p \\
&= \sum_{p=2}^{\infty} \sum_{\substack{(0,(1,k)),(\mathcal{V},\pi) \\ (0,(1,k))\cdot(\mathcal{V},\pi)=(\mathcal{U},\gamma)}} \sum_{\substack{(0,\pi_2)\cdots(0,\pi_p) \\ =(\mathcal{V},\pi)}} (-1)^p \\
&= - \sum_{\substack{(0,(1,k)),(\mathcal{V},\pi) \\ (0,(1,k))\cdot(\mathcal{V},\pi)=(\mathcal{U},\gamma)}} \sum_{p=2}^{\infty} \sum_{\substack{(0,\pi_2)\cdots(0,\pi_p) \\ =(\mathcal{V},\pi)}} (-1)^{p-1} \\
&= - \sum_{\substack{(0,(1,k)),(\mathcal{V},\pi) \\ (0,(1,k))\cdot(\mathcal{V},\pi)=(\mathcal{U},\gamma)}} \sum_{p=2}^{\infty} \sum_{\substack{(0,\pi_2)\cdots(0,\pi_p) \\ =(\mathcal{V},\pi)}} (-1)^{p-1} \\
&= - \sum_{\substack{(0,(1,k)),(\mathcal{V},\pi) \\ (0,(1,k))\cdot(\mathcal{V},\pi)=(\mathcal{U},\gamma)}} \mu(\mathcal{V}, \pi)
\end{aligned}$$

$\square$

One observes that the recursion formulas for the Möbius function and for $\zeta^{*2}$ look very similar. However, there are some significant differences. The recursion for $\zeta^{*2}$ effectively expresses $\zeta^{*2}$ for $n$ points in terms of $\zeta^{*2}$ for $n-1$ points. The recursion for the Möbius function does not reduce the number of points. Nevertheless, at least for first and second order one can match the two recursions and connect the values of the Möbius function with the values of the function $\zeta^{*2}$ (i.e., with the number of non-crossing partitions and non-crossing annular permutations). In order to see this let us first specify the meaning of Theorem 5.22 for first and second order. In first order we get

$$\mu(1_n, \gamma_n) = - \sum_{1 \leq k \leq n-1} \mu(1_k, \gamma_k) \mu(1_{n-k}, \gamma_{n-k}),$$

which shows that $(-1)^n\mu(1_{n+1},\gamma_{n+1})$ and $\zeta^{*2}(1_n,\gamma_n)$ satisfy the same recursion (namely the one for the Catalan numbers). This is, of course, just the well-known fact [Kre72, Spe94] that the Möbius function on non-crossing partitions is given by the signed and shifted Catalan numbers. In second order our recursion reads

$$(-1)\mu(1_{m+n},\gamma_{m,n}) = m\cdot\mu(1_{m+n},\gamma_{m+n}) \\ + \sum_{1\le k\le n-1}\big(\mu(1_{m+k},\gamma_{m,k})\mu(1_{n-k},\gamma_{n-k}) + \mu(1_{m+n-k},\gamma_{m,n-k})\mu(1_k,\gamma_k)\big),$$

which we recognize — by taking into account the shifted relation between $\mu$ and $\zeta^{*2}$ on the first level — as the recursion for $(-1)^{m+n}\zeta^{*2}(1_{m+n},\gamma_{m,n})$. Let us collect these explicit results about the Möbius function in the following theorem.

Theorem 5.24. *We have for $m,n\in\mathbb{N}$ that*

$$\mu(1_n,\gamma_n) = (-1)^{n-1}\cdot\#NC(n-1) = (-1)^{n-1}\cdot c_{n-1}$$

*and*

$$\mu(1_{m+n},\gamma_{m,n}) = (-1)^{m+n}\cdot\#S_{NC}(m,n) = (-1)^{m+n}\cdot c_{m,n}.$$

For higher orders we were not able to match the values of $\mu$ with those of $\zeta^{*2}$.

## 6. R-transform formulas

Let us consider the situation that two multiplicative functions $f$ and $h$ on $\mathcal{PS}$ are related by $h = f * \zeta$. We want to understand what this means for the relations between the numbers $\kappa_n := f(1_n,\gamma_n)$ and $\kappa_{m,n} := f(1_{m+n},\gamma_{m,n})$ on one side and the numbers $\alpha_n := h(1_n,\gamma_n)$ and $\alpha_{m,n} := h(1_{m+n},\gamma_{m,n})$ on the other side. In particular, we want to express this in terms of the generating power series of these numbers,

$$C(x) := 1 + \sum_{n\ge 1}\kappa_n x^n, \qquad C(x,y) := \sum_{m,n\ge 1}\kappa_{m,n}x^m y^n$$

and

$$M(x) := 1 + \sum_{n\ge 1}\alpha_n x^n, \qquad M(x,y) := \sum_{m,n\ge 1}\alpha_{m,n}x^m y^n.$$

(Note that the above summation corresponds to putting formally

$$f(1_0,\gamma_0) := 1 \qquad \text{and} \qquad f(1_0,\gamma_{0,0}) := 0$$

for a multiplicative $f$. Our notation is motivated by the fact that the most important realization of the relation $h = f*\zeta$ will be the situation where the $\alpha$'s are the correlation moments and the $\kappa$'s the corresponding cumulants, thus $M$ is a moment series and $C$ is a cumulant series.) On the first order level we have

$$\alpha_n = \sum_{\pi\in NC(n)} f(0_\pi,\pi),$$

which is the usual moment-cumulant formula of free probability theory, and it is well-known [Spe94] that this is equivalent to

$$C\big(xM(x)\big) = M(x).$$

Our main goal now is to derive the analogue of this for the second order level. There we have

$$\alpha_{m,n} = \sum_{\pi \in S_{NC}(m,n)} f(0_\pi, \pi) + \sum_{\substack{\pi_1 \times \pi_2 \in NC(m) \times NC(n) \\ |\mathcal{V}| = |\pi_1 \times \pi_2| + 1}} f(\mathcal{V}, \pi_1 \times \pi_2).$$

It turns out that the second term, the sum over disconnected partitions, is quite easy to deal with. The first term, the sum over connected annular permutations, looks much more involved, however, one can handle this also if one realizes that one can reduce this first term to the second one. Namely, one can sum over all connected annular permutations by first bundling all through-cycles into one through-cycle and secondly decomposing this through-cycle into sub-cycles all of which are through-cycles. In this way one can reduce the problem of dealing with all annular non-crossing permutations to the problem of considering permutations with exactly one through-cycle and the problem of considering permutations where all cycles are through-cycles. The first problem corresponds exactly to the above sum over disconnected partitions. So we can write

$$\sum_{\pi \in S_{NC}(m,n)} f(0_\pi, \pi) = \sum_{\substack{\pi_1 \times \pi_2 \in NC(m) \times NC(n) \\ |\mathcal{V}| = |\pi_1 \times \pi_2| + 1}} \tilde{f}(\mathcal{V}, \pi_1 \times \pi_2),$$

where $\tilde{f}$ is now the multiplicative function corresponding to

$$\tilde{f}(1_n, \gamma_n) = \tilde{\kappa}_n, \qquad \tilde{f}(1_{m+n}, \gamma_{m,n}) = \tilde{\kappa}_{m,n}$$

with

$$\tilde{\kappa}_n := \kappa_n$$

and

$$\tilde{\kappa}_{m,n} := \sum_{\pi \in S_{NC}^{all}(m,n)} f(0_\pi, \pi).$$

Thus we can combine this to get finally

$$\begin{aligned}
\alpha_{m,n} &= \sum_{\substack{\pi_1 \times \pi_2 \in NC(m) \times NC(n) \\ |\mathcal{V}| = |\pi_1 \times \pi_2| + 1}} \big(f(\mathcal{V}, \pi_1 \times \pi_2) + \tilde{f}(\mathcal{V}, \pi_1 \times \pi_2)\big) \\
&= \sum_{\substack{\pi_1 \times \pi_2 \in NC(m) \times NC(n) \\ |\mathcal{V}| = |\pi_1 \times \pi_2| + 1}} g(\mathcal{V}, \pi_1 \times \pi_2),
\end{aligned}$$

where $g$ is the multiplicative function corresponding to

$$g(1_n, \gamma_n) = \tilde{\alpha}_n, \qquad g(1_{m+n}, \gamma_{m,n}) = \tilde{\alpha}_{m,n}$$

with

$$\tilde{\alpha}_n = \tilde{\kappa}_n = \kappa_n$$

and

$$\tilde{\alpha}_{m,n} = \kappa_{m,n} + \tilde{\kappa}_{m,n}.$$

So we have to translate the relation between $\tilde{\kappa}_{m,n}$ and $f$ and the relation between $\alpha_{m,n}$ and $g$ into relations between the corresponding formal power series.

PROPOSITION 6.1. *Let $f$ be a multiplicative function on $\mathcal{PS}$ with*

$$f(1_n, \gamma_n) =: \kappa_n \qquad \text{and} \qquad C(x) := 1 + \sum_{n \geq 1} \kappa_n x^n.$$

*Put*

$$\tilde{\kappa}_{m,n} := \sum_{\pi \in S_{NC}^{all}(m,n)} f(0_\pi, \pi),$$

*where $S_{NC}^{all}(m,n)$ denotes the permutations in $S_{NC}(m,n)$ for which all cycles are through-cycles. Consider the corresponding generating power series*

$$\tilde{C}(x,y) := \sum_{m,n \geq 1} \tilde{\kappa}_{m,n} x^m y^n.$$

*Then we have*

$$\tilde{C}(x,y) = -xy \frac{\partial^2}{\partial x \partial y} \log\Big(\frac{xC(y) - yC(x)}{x - y}\Big),$$

*or equivalently*

$$\tilde{C}(x,y) = -xy\Big(\frac{\big(C(x) - xC'(x)\big)\big(C(y) - yC'(y)\big)}{\big(xC(y) - yC(x)\big)^2} - \frac{1}{(x-y)^2}\Big).$$

*Proof.* Note that we can parameterize an element $\pi \in S_{NC}^{all}(m,n)$ in a bijective way by specifying the number of cycles, the number of elements on each circle for all cycles, the position of a fixed element (let's say 1) in its cycle and the first element on the other circle of this cycle. Let us denote the number of cycles by $r$, the number of elements of the cycles on the first circle by $i_1, \dots, i_r$ and the number of elements of those cycles on the other circle by $j_1, \dots, j_r$. Thus the $l$-th cycle contains $i_l + j_l$ elements and makes the contribution $\kappa_{i_l + j_l}$ in the calculation of $\tilde{\kappa}_{m,n}$. We normalize things so that the first cycle contains the element 1. Fixing $i_1, \dots, i_r$ and $j_1, \dots, j_r$ we thus have $i_1$ possibilities for where 1 sits in the first cycle and $n$ possibilities for the first element of this cycle on the other circle. This means we have

$$\tilde{\kappa}_{m,n} = \sum_{r \geq 1} \sum_{\substack{i_1, \dots, i_r \geq 1 \\ i_1 + \cdots + i_r = m}} \sum_{\substack{j_1, \dots, j_r \geq 1 \\ j_1 + \cdots + j_r = n}} i_1 n \kappa_{i_1 + j_1} \cdots \kappa_{i_r + j_r}$$

and thus

$$\begin{aligned}\tilde{C}(x,y) &= \sum_{r\geq 1}\sum_{i_1,\dots,i_r\geq 1}\sum_{j_1,\dots,j_r\geq 1} i_1(j_1+\cdots j_r)\kappa_{i_1+j_1}\cdots\kappa_{i_r+j_r}x^{i_1}\cdots x^{i_r}y^{j_1}\cdots y^{j_r}\\
&= \sum_{r\geq 1}\sum_{i_1,\dots,i_r\geq 1}\sum_{j_1,\dots,j_r\geq 1} i_1 y\frac{\partial}{\partial y}\big(\kappa_{i_1 gb+j_1}\cdots\kappa_{i_r+j_r}x^{i_1}\cdots x^{i_r}y^{j_1}\cdots y^{j_r}\big)\\
&= \sum_{r\geq 1} y\frac{\partial}{\partial y}\Big(\big(\sum_{i_1,j_1\geq 1} i_1\kappa_{i_1+j_1}x^{i_1}y^{j_1}\big)\cdot\big(\sum_{i_2,j_2\geq 1}\kappa_{i_2+j_2}x^{i_2}y^{j_2}\big)\cdots\big(\sum_{i_r,j_r\geq 1}\kappa_{i_r+j_r}x^{i_r}y^{j_r}\big)\Big)\end{aligned}$$

Let us now use the notation

$$\hat{C}(x,y) := \sum_{i,j\geq 1}\kappa_{i+j}x^i y^j.$$

Then we can continue with

$$\begin{aligned}\tilde{C}(x,y) &= \sum_{r\geq 1} y\frac{\partial}{\partial y}\Big(\big(x\frac{\partial}{\partial x}\hat{C}(x,y)\big)\cdot\hat{C}(x,y)^{r-1}\Big)\\
&= \sum_{r\geq 1} xy\frac{\partial}{\partial y}\Big(\frac{1}{r}\frac{\partial}{\partial x}\big(\hat{C}(x,y)^r\big)\Big)\\
&= xy\frac{\partial}{\partial y}\frac{\partial}{\partial x}\Big(\sum_{r\geq 1}\frac{1}{r}\hat{C}(x,y)^r\Big)\\
&= -xy\frac{\partial}{\partial y}\frac{\partial}{\partial x}\log\big(1-\hat{C}(x,y)\big)\end{aligned}$$

The assertions follow now by noting that

$$\hat{C}(x,y) = 1 - \frac{xC(y)-yC(x)}{x-y}$$

and by working out the partial derivatives. □

Proposition 6.2. *Let $g$ be a multiplicative function on $\mathcal{PS}$. Put*

$$\tilde{\alpha}_{m,n} := g(1_{m+n},\gamma_{m,n})$$

*and denote its generating power series of second order by*

$$H(x,y) := \sum_{m,n\geq 1}\tilde{\alpha}_{m,n}x^m y^n.$$

*Put*

$$\alpha_n := (g*\zeta)(1_n,\gamma_n)$$

*and*

$$\alpha_{m,n} := \sum_{\substack{(\mathcal{V},\pi_1\times\pi_2)\\ |\mathcal{V}|=|\pi_1\times\pi_2|+1}} g(\mathcal{V},\pi)$$

*and denote the corresponding generating functions by*

$$M(x) := 1+\sum_{n\geq 1}\alpha_n x^n \qquad \textit{and} \qquad M(x,y) := \sum_{m,n\geq 1}\alpha_{m,n}x^m y^n.$$

*Then we have the relation*

$$M(x,y) = H(xM(x), yM(y)) \cdot \Big(1 + x\frac{M'(x)}{M(x)}\Big) \cdot \Big(1 + y\frac{M'(y)}{M(y)}\Big).$$

*Proof.* Let us do the summation in the definition of $\alpha_{m,n}$ in the way that we first fix the two cycles $V_1 \in \pi_1$ and $V_2 \in \pi_2$ which are connected by $\mathcal{V}$ and sum over all possibilities for fixed $V_1, V_2$. If $V_1$ has $k$ elements and $V_2$ has $l$ elements then this contributes the factor $\tilde{\alpha}_{k,l}$. Furthermore, $\pi_1 \backslash V_1$ decomposes into $k$ independent non-crossing partitions and the summations over them (for fixed $V_1$) gives the $\alpha_i$ for the intervals between consecutive elements from $V_1$. (Of course, we are counting here modulo $m$.) For the final summation over $V_1$ we have to notice that there are two different possibilities: either a fixed number (let's say 1) is an element of $V_1$ - in which case we can specify the situation by prescribing the number $k$ of elements of $V_1$ and the differences $i_1, \dots, i_k$ between consecutive elements in $V_1$ - or 1 is not an element of $V_1$, — in which case we need an extra factor $i_1$, because we have now $i_1$ different possibilities how 1 can lie between two consecutive elements of $V_1$. Since we have the same situation for $V_2$ we can thus write $\alpha_{m,n}$ in the form

$$\alpha_{m,n} = \sum_{k,l\geq 1} \sum_{\substack{i_1,\dots,i_k\geq 0\\ k+i_1+\cdots+i_k=m}} \sum_{\substack{j_1,\dots,j_l\geq 0\\ l+j_1+\cdots+j_l=n}} \tilde{\alpha}_{k,l}\alpha_{i_1}\cdots\alpha_{i_k}\alpha_{j_1}\cdots\alpha_{j_l}\Big(1 + i_1 + j_1 + i_1 j_1\Big).$$

Translating this into generating power series gives the assertion. □

The combination of the previous two propositions, with

$$H(x,y) = C(x,y) + \tilde{C}(x,y),$$

gives now our main result.

Theorem 6.3. *Let $f$ and $h$ be multiplicative functions on $\mathcal{PS}$ which are related by*

$$h = f * \zeta.$$

*Denote*

$$\kappa_n := f(1_n, \gamma_n), \qquad \kappa_{m,n} := f(1_{m+n}, \gamma_{m,n})$$

*and*

$$\alpha_n := h(1_n, \gamma_n), \qquad \alpha_{m,n} := h(1_{m+n}, \gamma_{m,n})$$

*and define the corresponding generating power series*

$$C(x) := 1 + \sum_{n\geq 1} \kappa_n x^n, \qquad C(x,y) := \sum_{m,n\geq 1} \kappa_{m,n} x^m y^n$$

*and*

$$M(x) := 1 + \sum_{n\geq 1} \alpha_n x^n, \qquad M(x,y) := \sum_{m,n\geq 1} \alpha_{m,n} x^m y^n.$$

*Then we have as formal power series the first order relation*

$$C(xM(x)) = M(x) \tag{49}$$

*and for the second order*

$$M(x,y) = H\big(xM(x), yM(y)\big) \cdot \frac{\frac{d}{dx}(xM(x))}{M(x)} \cdot \frac{\frac{d}{dy}(yM(y))}{M(y)}, \tag{50}$$

*where*

$$H(x,y) := C(x,y) - xy\frac{\partial^2}{\partial x \partial y}\log\Big(\frac{xC(y) - yC(x)}{x-y}\Big), \tag{51}$$

*or equivalently,*

$$\begin{aligned} M(x,y) = C\big(xM(x), yM(y)\big) \cdot \frac{\frac{d}{dx}(xM(x))}{M(x)} \cdot \frac{\frac{d}{dy}(yM(y))}{M(y)} \\ + xy\Big(\frac{\frac{d}{dx}(xM(x)) \cdot \frac{d}{dy}(yM(y))}{(xM(x) - yM(y))^2} - \frac{1}{(x-y)^2}\Big). \end{aligned} \tag{52}$$

*Proof.* The formulation (50) and (51) follows directly from a combination of Propositions 6.1 and 6.2. In order to reformulate this to (52) one uses the equivalence of the two formulas in Proposition 6.1 and the fact that $C(xM(x)) = M(x)$ yields

$$1 - xC'(xM(x)) = \frac{M(x)}{\frac{d}{dx}(xM(x))}.$$

□

If we go over from the moment generating series $M$ to a kind of Cauchy transform like quantity $G$, then these formulas take on a particularly nice form.

Corollary 6.4. *Consider the same situation and notations as in Theorem 6.3. In terms of*

$$G(x) := \frac{1}{x}M(1/x), \quad G(x,y) := \frac{1}{xy}M(1/x, 1/y), \quad \mathcal{R}(x,y) := \frac{1}{xy}C(x,y)$$

*the Equation (52) can be written as*

$$G(x,y) = G'(x)G'(y)\Big\{\mathcal{R}(G(x), G(y)) + \frac{1}{(G(x) - G(y))^2}\Big\} - \frac{1}{(x-y)^2}. \tag{53}$$

$\mathcal{R}(x,y)$ is the second order $R$-transform. Note that Voiculescu's first order $R$-transform $\mathcal{R}$ is defined by the relation $C(x) = 1 + x\mathcal{R}(x)$, and equation (49) says for this

$$\frac{1}{G(x)} + \mathcal{R}(G(x)) = x,$$

i.e., that $G(x)$ and $K(x) := \frac{1}{x} + \mathcal{R}(x)$ are inverses of each other under composition.

Example 6.5. Let us apply our formulas to some examples.
1) If we put $f$ to be the multiplicative function with $\kappa_2 = 1$ and all other $\kappa_n$ and all $\kappa_{m,n}$ vanishing, then $h = f * \zeta$ counts the non-crossing pairings, i.e., in this case $M(x)$ is the generating function of the number of non-crossing

pairings (on one circle) and $M(x,y)$ is the generating function of the number of non-crossing annular pairings (on two circles). Let us calculate it by using the above theorem.
We have

$$C(x) = 1 + x^2, \qquad C(x,y) = 0$$

and we know that $M$ is the generating function of number of non-crossing pairings on a circle. In this case

$$\hat{C}(x,y) = xy,$$

and thus

$$H(x,y) = -xy \frac{\partial^2}{\partial x \partial y} \log(1 - xy) = \frac{xy}{(1-xy)^2},$$

which yields the result

$$M(x,y) = xy \cdot \frac{\frac{d}{dx}(xM(x)) \cdot \frac{d}{dy}(yM(y))}{\big(1 - xyM(x)M(y)\big)^2}.$$

Related formulas are known in the physical literature, see, e.g, [FMP78], [AJM90], [ACKM93] and also [BZ93], [KKP95].
2) If we put $f = \zeta$ then $h = \zeta * \zeta$ counts the non-crossing permutations, i.e., in this case $M(x)$ is the generating function of the number of non-crossing permutations (which is the same as non-crossing partition) on one circle and $M(x,y)$ is the generating function of the number of annular non-crossing permutations (on two circles).
We have

$$C(x) = \frac{1}{1-x}, \qquad C(x,y) = 0.$$

In this case

$$\hat{C}(x,y) = \frac{1-x-y}{(1-x)(1-y)},$$

and thus

$$H(x,y) = -xy \frac{\partial^2}{\partial x \partial y} \log(1 - xy) = \frac{xy}{(1-x-y)^2},$$

which yields

$$M(x,y) = xy \cdot \frac{\frac{d}{dx}(xM(x)) \cdot \frac{d}{dy}(yM(y))}{\big(1 - xM(x) - yM(y)\big)^2}.$$

3) Let us finally see whether we can extract the value of the Möbius function from our formula. Since we have $\delta = \mu * \zeta$, our formula with

$$M(x) = 1 + x, \qquad M(x,y) = 0$$

should allow to solve for $C(x,y)$ which is then the generating function for the annular Möbius function. Note that we already know $M(x)$ in this case to be the generating function of the disc Möbius function.

If $M(x,y)$ vanishes identically this implies that $H(x,y)$ vanishes identically, leading to the identity

$$\begin{aligned} C(x,y) &= xy\frac{\partial^2}{\partial x \partial y}\log(\frac{xC(y)-yC(x)}{x-y}) \\ &= xy\Big(\frac{(C(x)-xC'(x))\cdot(C(y)-yC'(y))}{(xC(y)-yC(x))^2} - \frac{1}{(x-y)^2}\Big) \end{aligned}$$

## 7. Higher order freeness and corresponding cumulants

### 7.1. Abstract framework.

Definition 7.1. A *higher-order (non-commutative) probability space*, or briefly HOPS, $(\mathcal{A},\varphi)$ consists of a unital algebra $\mathcal{A}$ and a collection $\varphi=(\varphi_n)_{n\in\mathbb{N}}$ of maps ($n\in\mathbb{N}$)

$$\varphi_n : \underbrace{\mathcal{A}\times\cdots\times\mathcal{A}}_{n \text{ times}} \to \mathbb{C},$$

which are linear and tracial in each of its $n$ arguments and which are symmetric under exchange of its $n$ arguments and which satisfy

$$\varphi_1(1)=1$$

and

$$\varphi_n(1,a_2,\dots,a_n)=0$$

for all $n\geq 2$ and all $a_2,\dots,a_n\in\mathcal{A}$.

Of course, we can include the usual (first order) non-commutative probability space $(\mathcal{A},\varphi_1)$ into this framework by putting all higher $\varphi_n$ equal to zero. In the same way we recover a second order non-commutative probability space $(\mathcal{A},\varphi_1,\varphi_2)$ by putting $\varphi_n=0$ for all $n\geq 3$.

Definition 7.2. 1) We denote by $\mathcal{PS}(\mathcal{A})$ the set of partitioned permutations decorated with elements from $\mathcal{A}$, i.e.,

$$\mathcal{PS}=\bigcup_{n\in\mathbb{N}}\big(\mathcal{PS}(n)\times\mathcal{A}^n\big).$$

2) For a function

$$\begin{gathered} f:\mathcal{PS}(\mathcal{A})\to\mathbb{C} \\ (\mathcal{V},\pi)\times(a_1,\dots,a_n)\mapsto f(\mathcal{V},\pi)[a_1,\dots,a_n] \end{gathered}$$

and a function

$$g:\mathcal{PS}\to\mathbb{C}$$

we define their convolution

$$f*g:\mathcal{PS}(\mathcal{A})\to\mathbb{C}$$

by

$$(f*g)(\mathcal{U},\gamma)[a_1,\dots,a_n] := \sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ (\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}} f(\mathcal{V},\pi)[a_1,\dots,a_n]\cdot g(\mathcal{W},\sigma)$$

for all $(\mathcal{U},\gamma)\in\mathcal{PS}(n)$ and all $a_1,\dots,a_n\in\mathcal{A}$.

Definition 7.3. A function $f:\mathcal{PS}(\mathcal{A})\to\mathbb{C}$ is called *multiplicative* if we have

$$f(\mathcal{V},\pi)[a_1,\dots,a_n]=\prod_{B\in\mathcal{V}} f(1_B,\pi|_B)[(a_1,\dots,a_n)_B]$$

and

$$f(1_n,\sigma^{-1}\pi\sigma)[a_{\sigma(1)},\dots,a_{\sigma(n)}]=f(1_n,\pi)[a_1,\dots,a_n]$$

for all $a_1,\dots,a_n\in\mathcal{A}$ and all $\pi,\sigma\in S(n)$.

Note that this extension of our formalism on multiplicative functions on $\mathcal{PS}$ and their convolution from the last section is not changing the results from the last section. The structure of all formulas remains the same; one just has to insert the $a_1,\dots,a_n$ as dummy variables at the right positions. Thus, in particular, $\delta$ is still the unit for this extended convolution and $f=g*\zeta$ is equivalent to $g=f*\mu$ for multiplicative $f,g$ on $\mathcal{PS}(\mathcal{A})$. And again, the convolution of a multiplicative function on $\mathcal{PS}(\mathcal{A})$ with a multiplicative function on $\mathcal{PS}$ gives a multiplicative function on $\mathcal{PS}(\mathcal{A})$.
It is clear that a multiplicative function $f$ on $\mathcal{PS}(\mathcal{A})$ is uniquely determined by the values of $f(1_n,\gamma_{n(1),\dots,n(r)})[a_1,\dots,a_n]$ (where we put $n:=n(1)+\dots+n(r)$) for all $r\in\mathbb{N}$, all $n(1),\dots,n(r)\in\mathbb{N}$ and all $a_1,\dots,a_n\in\mathcal{A}$.

7.2. Moment and cumulant functions. Let us now apply this formalism to get moment and cumulant functions for higher order probability spaces. So let a HOPS $(\mathcal{A},\varphi)$ be given. We will use the $\varphi_n$ to produce a multiplicative "moment" function on $\mathcal{PS}(\mathcal{A})$, which we will also denote by $\varphi$. Namely, we put

$$\begin{aligned}\varphi(1_n,\gamma_{n(1),\dots,n(r)})&[a_1,\dots,a_n]\\ &:=\varphi_r(a_1\cdots a_{n(1)};\dots;a_{n(1)+\cdots n(r-1)+1}\cdots a_n)\end{aligned}$$

and extend this by multiplicativity. (Note that we need the $\varphi_n$ to be tracial in their arguments for this extension.)
Here is an example for our function $\varphi$.

$$\varphi\big(\{1,3,4\}\{2\},(1,3)(2)(4)\big)[a_1,a_2,a_3,a_4]=\varphi_2(a_1a_3,a_4)\cdot\varphi_1(a_2)$$

Definition 7.4. For a given HOPS $(\mathcal{A},\varphi)$ we define the corresponding *(higher order) free cumulants* as a function on $\mathcal{PS}(\mathcal{A})$ by

$$\kappa=\varphi*\mu,$$

or more explicitly

$$\kappa(\mathcal{U},\gamma)[a_1,\dots,a_n]:=\sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ (\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}}\varphi(\mathcal{V},\pi)[a_1,\dots,a_n]\cdot\mu(\mathcal{W},\sigma),$$

for all $n\in\mathbb{N}$, $(\mathcal{U},\gamma)\in\mathcal{PS}(n)$, $a_1,\dots,a_n\in\mathcal{A}$.

As we noted before the definition above is equivalent to the statement $\varphi = \kappa * \zeta$, i.e.,

$$\varphi(\mathcal{U},\gamma)[a_1,\dots,a_n] = \sum_{(\mathcal{V},\pi)\in\mathcal{PS}_{NC}(\mathcal{U},\gamma)} \kappa(\mathcal{V},\pi)[a_1,\dots,a_n]$$

for all $(\mathcal{U},\gamma)[a_1,\dots,a_n] \in \mathcal{PS}(\mathcal{A})$.
Furthermore, as with $\varphi$, $\kappa$ is also a multiplicative function on $\mathcal{PS}(\mathcal{A})$. Thus in the same way as all $\varphi(\mathcal{U},\gamma)$ are determined by the knowledge of all

$$\begin{aligned}\varphi(1_n,\gamma_{n(1),\dots,n(r)})&[a_1,\dots,a_n]\\ &= \varphi_r(a_1\cdots a_{n(1)};\dots;a_{n(1)+\cdots+n(r-1)+1}\cdots a_{n(1)+\cdots+n(r)})\end{aligned}$$

the free cumulants $\kappa(\mathcal{U},\gamma)$ are determined by the values of

$$\begin{aligned}\kappa(1_n,\gamma_{n(1),\dots,n(r)})&[a_1,\dots,a_n]\\ &=: \kappa_{n(1),\dots,n(r)}(a_1,\dots,a_{n(1)};\dots;a_{n(1)+\cdots+n(r-1)+1},\dots,a_{n(1)+\cdots+n(r)}).\end{aligned}$$

*Remark* 7.5. Note that whereas on the level of $\varphi$ we also know (by definition) that we can multiply elements along the cycles of $\pi$ (and thus we do not need a comma as separator for those elements along a cycle), this is not true for $\kappa$. Thus we have, e.g.,

$$\varphi(1_3,(1,2)(3))[a_1,a_2,a_3] = \varphi_2(a_1a_2;a_3) = \varphi(1_2,(1),(2))[a_1a_2;a_3],$$

but no clear relation exists among

$$\kappa(1_3,(1,2)(3))[a_1,a_2,a_3] = \kappa_{2,1}(a_1,a_2;a_3)$$

and

$$\kappa(1_2,(1),(2))[a_1a_2;a_3] = \kappa_{1,1}[a_1a_2;a_3].$$

Note also that since our convolution on $\mathcal{PS}$ coincides on the first level with the usual convolution of multiplicative functions on non-crossing partitions, the above definition of cumulants reduces on the first level to the usual free cumulants.

7.3. HIGHER ORDER FREENESS. Equipped with the notion of cumulants we can now define "freeness" by the requirement of vanishing of mixed cumulants.

DEFINITION 7.6. We say that a family $(\mathcal{X}_i)_{i\in I}$ of subsets of $\mathcal{A}$ is *free (of all orders)* if we have the following vanishing of mixed cumulants: For all $n \geq 2$ and all $a_k \in \mathcal{X}_{i(k)}$ $(1 \leq k \leq n)$ such that $i(p) \neq i(q)$ for some $1 \leq p,q \leq n$ we have

$$\kappa(1_n,\pi)[a_1,\dots,a_n] = 0$$

for all $\pi \in S(n)$.

EXAMPLE 7.7. Let us see that this definition includes the definition of Voiculescu [VDN92] for (first order) freeness and the definition of Mingo and Speicher [MS06] for second order freeness.
1) On the first level this follows from the fact that our cumulants reduce then to the usual free cumulants and it is well-known that freeness is equivalent to

the vanishing of mixed cumulants. One can see it directly as follows: Let us consider $a_k \in \mathcal{X}_{i(k)}$ with $i(k) \neq i(k+1)$ and $\varphi_1(a_k) = 0$ for all $k = 1, \dots, n$. Then we have

$$\varphi_1(a_1 \cdots a_n) = \varphi(1_n, \gamma_n)[a_1, \dots, a_n] = \sum_{\pi \in NC(n)} \kappa(0_\pi, \pi)[a_1, \dots, a_n].$$

However the vanishing of mixed moments means now that the only $\pi$ which contribute are those which do not connect elements from different sets. Furthermore, the fact that all our variables are centered excludes singletons. But then it is easy to see that there are no such $\pi$ at all, so the sum is zero.
2) Now we have to consider two cyclically alternating and centered tuples $a_1, \dots, a_m$ and $b_1, \dots, b_n$. Then we have

$$\begin{aligned}\varphi_2(a_1 \cdots a_m; b_1 \cdots b_n) &= \varphi(1_{m+n}, \gamma_{m,n})[a_1, \dots, a_m, b_1, \dots, b_n] \\ &= \sum_{(\mathcal{V},\pi) \in \mathcal{PS}_{NC}(m,n)} \kappa(\mathcal{V}, \pi)[a_1, \dots, a_m, b_1, \dots, b_n].\end{aligned}$$

Again, the vanishing of mixed moments requires that $(\mathcal{V}, \pi)$ connects only elements from the same set and the centeredness of the elements excludes singletons. It is then easy to see that, for $n \geq 2$, the only possibilities for such $(\mathcal{V}, \pi)$ arise for $m = n$ and they have to be disc permutations $(0_\pi, \pi)$ which are pairings $(a_1, b_{1+s})(a_2, b_{2+s}) \cdots (a_n, b_{n+s})$ for some $s$. The factors $k(1_2, (...))[a_k, b_{k+s}]$ are just $\varphi_1(a_2 b_{2+s})$, so that one finally gets, for $n \geq 2$, the formula

$$\varphi_2(a_1 \cdots a_m; b_1 \cdots b_n) = \delta_{mn} \sum_{k=1}^{n} \varphi_1(a_1 b_{1+s}) \cdots \varphi_1(a_n b_{n+s}).$$

For $n = m = 1$ one gets with

$$\varphi_2(a_1; b_1) = k_2(a_1, a_2) + k_{1,1}(a_1; b_1)$$

the conclusion that $\varphi_2(a_1; b_1)$ has to vanish if $a_1$ and $b_1$ are from different sets. Nothing is required if both are from the same set. We see that we get exactly the defining properties for second order freeness from [MS06].
3) It would be nice to be able to reformulate in a similar way the definition of higher order freeness in terms of the $\varphi$ instead of the cumulants. However, the situation with more than two circles is getting much more involved and we are not aware of such a reformulation for third and higher order freeness.

As in the case of the first order freeness, one sees immediately that constants are free from everything.

Proposition 7.8. *Let $(\mathcal{A}, \varphi)$ be a HOPS. Then $\{1\}$ is free of all orders from every subset $\mathcal{X} \subset \mathcal{A}$.*

*Proof.* We have to prove that

$$\kappa(1_n, \gamma_{n(1),\dots,n(r)})[1, a_2, \dots, a_n] = 0,$$

unless $n = 1$. We will do this by induction on $n$. The case $n = 2$ is clear because

$$\kappa(1_2, (12))[1, a_2] = \varphi_1(1 \cdot a_2) - \varphi_1(1) \cdot \varphi_1(a_2) = 0$$

and

$$\kappa(1_2, (1)(2))[1, a_2] = \varphi_2(1; a_2) = 0.$$

In general, one has

$$\begin{aligned}\varphi(1_n, \gamma_{n(1),\dots,n(r)})&[1, a_2, \dots, a_n] \\ &= \sum_{(\mathcal{V},\pi)\in \mathcal{PS}_{NC}(n(1),\dots,n(r))} \kappa(\mathcal{V}, \pi)[1, a_2, \dots, a_n] \\ &= \kappa(1_n, \gamma_{n(1),\dots,n(r)})[1, a_2, \dots, a_n] \\ &\qquad + \sum_{\substack{(\mathcal{V},\pi)\in \mathcal{PS}_{NC}(n(1),\dots,n(r)) \\ |(\mathcal{V},\pi)|<|(1_n,\gamma_{n(1),\dots,n(r)})|}} \kappa(\mathcal{V}, \pi)[1, a_2, \dots, a_n]\end{aligned}$$

By induction hypothesis, in the later sum exactly terms of the form $(\{1\} \cup \tilde{\mathcal{V}}, (1) \cup \tilde{\pi})$ with

$$(\tilde{\mathcal{V}}, \tilde{\pi}) \in \mathcal{PS}_{NC}(n(1) - 1, n(2), \dots, n(r))$$

contribute. In the case $n(1) > 1$ the sum over those yields

$$\varphi(1_{n-1}, \gamma_{n(1)-1,n(2),\dots,n(r)}[a_2, \dots, a_n].$$

In this case, also

$$\varphi(1_n, \gamma_{n(1),\dots,n(r)})[1, a_2, \dots, a_n] = \varphi(1_{n-1}, \gamma_{n(1)-1,n(2),\dots,n(r)}[a_2, \dots, a_n],$$

and thus $\kappa(1_n, \gamma_{n(1),\dots,n(r)})[1, a_2, \dots, a_n] = 0$. If, on the other side, $n(1) = 1$ (i.e., 1 is the only element on its circle), then we have to set

$$\mathcal{PS}_{NC}(0, n(2), \dots, n(r)) = \emptyset,$$

because then the first circle cannot be connected to the others if we ask 1 to be a cycle of its own. But this means that in this case

$$\kappa(1_n, \gamma_{n(1),\dots,n(r)})[1, a_2, \dots, a_n] = \varphi(1_n, \gamma_{n(1),\dots,n(r)})[1, a_2, \dots, a_n]$$

However, for $n(1) = 1$ and $n > 1$ we have

$$\varphi(1_n, \gamma_{1,\dots,n(r)})[1, a_2, \dots, a_n] = 0.$$

□

Note that our definition of freeness behaves clearly very nicely with respect to decompositions of our sets. For example, we have that $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ are free if and only if $\mathcal{X}_1$ and $\mathcal{X}_2 \cup \mathcal{X}_3$ are free and $\mathcal{X}_2$ and $\mathcal{X}_3$ are free. Thus we can reduce the investigation of freeness to the understanding of freeness for the case of two sets. A characterization for this is given in the next theorem.

Theorem 7.9. *Let $(\mathcal{A}, \varphi)$ be a higher order probability space and consider two subsets of $\mathcal{X}_1, \mathcal{X}_2 \subset \mathcal{A}$. Then the following are equivalent.*

(1) *The sets $\mathcal{X}_1, \mathcal{X}_2$ are free of all orders.*

(2) *The sets* $\mathcal{X}_1 \cup \{1\}$, $\mathcal{X}_2 \cup \{1\}$ *are free of all orders.*
(3) *We have*

$$\varphi(\mathcal{U},\gamma)[a_1b_1,\dots,a_nb_n] = \sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} \kappa(\mathcal{V},\pi)[a_1,\dots,a_n]\cdot\varphi(\mathcal{W},\sigma)[b_1,\dots,b_n]$$

*for all* $n \in \mathbb{N}$, *all* $(\mathcal{U},\gamma) \in \mathcal{PS}(n)$ *and all* $a_1,\dots,a_n \in \mathcal{X}_1 \cup \{1\}$, $b_1,\dots,b_n \in \mathcal{X}_2 \cup \{1\}$.
(4) *We have*

$$\varphi(\mathcal{U},\gamma)[a_1b_1,\dots,a_nb_n] = \sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} \varphi(\mathcal{V},\pi)[a_1,\dots,a_n]\cdot\kappa(\mathcal{W},\sigma)[b_1,\dots,b_n]$$

*for all* $n \in \mathbb{N}$, *all* $(\mathcal{U},\gamma) \in \mathcal{PS}(n)$ *and all* $a_1,\dots,a_n \in \mathcal{X}_1 \cup \{1\}$, $b_1,\dots,b_n \in \mathcal{X}_2 \cup \{1\}$.
(5) *We have*

$$\kappa(\mathcal{U},\gamma)[a_1b_1,\dots,a_nb_n] = \sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} \kappa(\mathcal{V},\pi)[a_1,\dots,a_n]\cdot\kappa(\mathcal{W},\sigma)[b_1,\dots,b_n]$$

*for all* $n \in \mathbb{N}$, *all* $(\mathcal{U},\gamma) \in \mathcal{PS}(n)$ *and all* $a_1,\dots,a_n \in \mathcal{X}_1 \cup \{1\}$, $b_1,\dots,b_n \in \mathcal{X}_2 \cup \{1\}$.

In order to prove this we would like to write $\varphi(\mathcal{U},\gamma)[a_1b_1,\dots,a_nb_n]$ in the form $\varphi(\hat{\mathcal{U}},\hat{\gamma})[a_1,b_1,\dots,a_n,b_n]$. Let us introduce the following formalism for this. Let $(\mathcal{U},\gamma) \in \mathcal{PS}(n)$ be a partitioned permutation of the numbers $1,2,3,\dots,n$. Double now this set of numbers by introducing a copy $\bar{1},\bar{2},\bar{3},\dots,\bar{n}$ and interleave the new and old numbers as follows:

$$1,\bar{1},2,\bar{2},3,\bar{3},\dots,n,\bar{n}.$$

If we induce now $(\mathcal{U},\gamma)$ on $1,2,\dots,n$ to $(\hat{\mathcal{U}},\hat{\gamma})$ on $1,\bar{1},\dots,n,\bar{n}$ by putting

$$\hat{\gamma}(k) = \bar{k} \qquad \text{and} \qquad \hat{\gamma}(\bar{k}) = \gamma(k),$$

then this has exactly the desired property. The vanishing of mixed cumulants means that in the factorizations of $(\hat{\mathcal{U}},\hat{\gamma})$ in $(\mathcal{V},\pi)$ times a disc permutation we are only interested in $(\mathcal{V},\pi)$ which have the property that each block of $\mathcal{V}$ contains either only unbarred numbers or only barred numbers, i.e., $(\mathcal{V},\pi)$ must be of the form $(\mathcal{V}_a \cup \mathcal{V}_b, \pi_a \cup \pi_b)$ with

$$(\mathcal{V}_a,\pi_a) \in \mathcal{PS}(1,\dots,n) \qquad \text{and} \qquad (\mathcal{V}_b,\pi_b) \in \mathcal{PS}(\bar{1},\dots,\bar{n}).$$

Let us first observe some simple relations between the quantities on $1,\dots,n$ and their relatives on $1,\bar{1},\dots,n,\bar{n}$.

Lemma 7.10. *1) We have*

$$|\hat{\gamma}| = n + |\gamma|, \qquad |\hat{\mathcal{U}}| = n + |\mathcal{U}|,$$

*and thus*

$$|(\hat{\mathcal{U}}, \hat{\gamma})| = n + |(\mathcal{U}, \gamma)|.$$

*2) We have*

$$|\pi_a \cup \pi_b| = |\pi_a| + |\pi_b|, \qquad |\mathcal{V}_a \cup \mathcal{V}_b| = |\mathcal{V}_a| + |\mathcal{V}_b|,$$

*and thus*

$$|(\mathcal{V}_a \cup \mathcal{V}_b, \pi_a \cup \pi_b)| = |(\mathcal{V}_a, \pi_a)| + |(\mathcal{V}_b, \pi_b)|.$$

*3) We have that $(\pi_a \cup \pi_b)\hat{\gamma}$ maps unbarred to barred and barred to unbarred elements and, for all $k = 1, \dots, n$,*

$$[(\pi_a \cup \pi_b)\hat{\gamma}]^2(\bar{k}) = \pi_b \pi_a \gamma(k),$$

*thus*

$$|(\pi_a \cup \pi_b)\hat{\gamma}| = n + |\pi_b \pi_a \gamma|$$

*Proof.* Only the third part is non-trivial. To see this observe

$$(\pi_a \cup \pi_b)\hat{\gamma}(\bar{k}) = \pi_a(\gamma(k))$$

and thus

$$[(\pi_a \cup \pi_b)\hat{\gamma}]^2(\bar{k}) = \pi_b \overline{\pi_a(\gamma(k))},$$

which is our first equation, with the identification of $\pi_b \in S(\bar{1}, \dots, \bar{1})$ with the corresponding permutation in $S(1, \dots, n)$. Since the mapping between barred and unbarred elements is clear, this yields that $(\pi_a \cup \pi_b)\hat{\gamma}$ and $\pi_b \pi_a \gamma$ have the same number of orbits which gives the last equation. $\square$

This lemma allows us to characterize the contributing factorizations in $(\hat{\mathcal{U}}, \hat{\gamma})$ in terms of special factorizations of $(\mathcal{U}, \gamma)$.

Proposition 7.11. *The statement*

$$(\mathcal{V}_a \cup \mathcal{V}_b, \pi_a \cup \pi_b) \in \mathcal{PS}_{NC}(\hat{\mathcal{U}}, \hat{\gamma})$$

*is equivalent to the statement*

$$(\mathcal{V}_a, \pi_a) \cdot (\mathcal{V}_b, \pi_b) \in \mathcal{PS}_{NC}(\mathcal{U}, \gamma),$$

*where in the last product we identify $(\mathcal{V}_b, \pi) \in \mathcal{PS}(\bar{1}, \dots, \bar{n})$ with the corresponding element in $\mathcal{PS}(1, \dots, n)$.*

*Proof.* Note that $(\mathcal{V}_a \cup \mathcal{V}_b, \pi_a \cup \pi_b) \in \mathcal{PS}_{NC}(\hat{\mathcal{U}}, \hat{\gamma})$ is equivalent to

$$|(\mathcal{V}_a \cup \mathcal{V}_b, \pi_a \cup \pi_b)| + |(\pi_a \cup \pi_b)^{-1}\hat{\gamma}| = |(\hat{\mathcal{U}}, \hat{\gamma})| \tag{54}$$

and

$$\hat{\mathcal{U}} = (\mathcal{V}_a \cup \mathcal{V}_b) \vee \hat{\gamma}. \tag{55}$$

On the other hand, $(\mathcal{V}_a, \pi_a) \cdot (\mathcal{V}_b, \pi_b) \in \mathcal{PS}_{NC}(\mathcal{U}, \gamma)$, means

$$(\mathcal{V}_a, \pi_a) \cdot (\mathcal{V}_b, \pi_b) \cdot (0_{\pi_b^{-1}\pi_a^{-1}\gamma}, \pi_b^{-1}\pi_a^{-1}\gamma) = (\mathcal{U}, \gamma),$$

which is equivalent to

$$|(\mathcal{V}_a,\pi_a)|+|(\mathcal{V}_b,\pi_b)|+|\pi_b^{-1}\pi_a^{-1}\gamma|=|(\mathcal{U},\gamma)| \tag{56}$$

and

$$\mathcal{U}=\mathcal{V}_a\vee\mathcal{V}_b\vee\gamma. \tag{57}$$

Equations (54) and (56) are, by Lemma 7.10, equivalent.
The equivalence between (55) and (57) is also easily checked. □

Equipped with these tools we can now prove our main Theorem 7.9.

*Proof.* The equivalences between (3), (4), and (5) follow by convolving with the $\zeta$ or the $\mu$ function. That (2) is actually the same as (1) follows from Prop. 7.8.
(1) $\Longrightarrow$ (3): We have

$$\begin{aligned}\varphi(\mathcal{U},\gamma)[a_1b_1,\dots,a_nb_n]&=\varphi(\hat{\mathcal{U}},\hat{\gamma})[a_1,b_1,\dots,a_n,b_n]\\&=\sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\hat{\mathcal{U}},\hat{\gamma})}\kappa(\mathcal{V},\pi)[a_1,b_1,\dots,a_n,b_n]\cdot\zeta(\mathcal{W},\sigma)\\&=\sum_{(\mathcal{V},\pi)\in\mathcal{PS}_{NC}(\hat{\mathcal{U}},\hat{\gamma})}\kappa(\mathcal{V},\pi)[a_1,b_1,\dots,a_n,b_n]\end{aligned}$$

By our assumption on the vanishing of mixed cumulants, only $(\mathcal{V},\pi)$ of the form $(\mathcal{V}_1\cup\mathcal{V}_2,\pi_a\cup\pi_b)$ with

$$(\mathcal{V}_a,\pi_a)\in\mathcal{PS}(1,\dots,n)\qquad\text{and}\qquad(\mathcal{V}_b,\pi_b)\in\mathcal{PS}(\bar{1},\dots,\bar{n})$$

contribute and, by the above Proposition 7.11,

$$(\mathcal{V}_a\cup\mathcal{V}_b,\pi_a\cup\pi_b)\in\mathcal{PS}_{NC}(\hat{\mathcal{U}},\hat{\gamma})$$

is equivalent to

$$(\mathcal{V}_a,\pi_a)\cdot(\mathcal{V}_b,\pi_b)\in\mathcal{PS}_{NC}(\mathcal{U},\gamma).$$

Thus we can continue with

$$\begin{aligned}&\varphi(\mathcal{U},\gamma)[a_1b_1,\dots,a_nb_n]\\&=\sum_{(\mathcal{V}_a\cup\mathcal{V}_b,\pi_a\cup\pi_b)\in\mathcal{PS}_{NC}(\hat{\mathcal{U}},\hat{\gamma})}\kappa(\mathcal{V}_a,\pi_a)[a_1,a_2,\dots,a_n]\cdot\kappa(\mathcal{V}_b,\pi_b)[b_1,b_2,\dots,b_n]\\&=\sum_{(\mathcal{V}_a,\pi_a)\cdot(\mathcal{V}_b,\pi_b)\in\mathcal{PS}_{NC}(\mathcal{U},\gamma)}\kappa(\mathcal{V}_a,\pi_a)[a_1,a_2,\dots,a_n]\cdot\kappa(\mathcal{V}_b,\pi_b)[b_1,b_2,\dots,b_n]\\&=\sum_{(\mathcal{V}_a,\pi_a)\cdot(\mathcal{V}_b,\pi_b)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)}\kappa(\mathcal{V}_a,\pi_a)[a_1,a_2,\dots,a_n]\cdot\\&\qquad\qquad\cdot\kappa(\mathcal{V}_b,\pi_b)[b_1,b_2,\dots,b_n]\cdot\zeta(\mathcal{W},\sigma)\\&=\sum_{(\mathcal{V}_a,\pi_a)\cdot(\mathcal{V},\pi)=(\mathcal{U},\gamma)}\kappa(\mathcal{V}_a,\pi_a)[a_1,\dots,a_n]\cdot\varphi(\mathcal{V},\pi)[b_1,\dots,b_n].\end{aligned}$$

(3) $\Longrightarrow$ (1): Note that (3) allows us to calculate all moments of elements from $\mathcal{X}_1\cup\mathcal{X}_2$ out of the moments of elements from $\mathcal{X}_1$ and the moments of elements

from $\mathcal{X}_2$. (In order to do so, we also have to allow some of the $a$'s or $b$'s to be equal to the unit 1.) Since this calculation rule is the same as for free sets, this shows that the sets $\mathcal{X}_1$ and $\mathcal{X}_2$ must be free. $\square$

This theorem is now the key ingredient to transfer freeness from sets to their generated algebras.

THEOREM 7.12. *Let $(\mathcal{A}, \varphi)$ be a HOPS and consider subsets $(\mathcal{X}_i)_{i\in I}$. For each $i \in I$, let $\mathcal{A}_i$ be the unital algebra generated by elements from $\mathcal{X}_i$. Then the following are equivalent.*

1. *The subsets $(\mathcal{X}_i)_{i\in I}$ are free of all orders.*
2. *The subalgebras $(\mathcal{A}_i)_{i\in I}$ are free of all orders.*

*Proof.* Since the cumulant $\kappa(\mathcal{V}, \pi)[a_1, \dots, a_n]$ is a multi-linear functional in the $n$ variables $a_1, \dots, a_n$, it is clear that taking sums of elements within the sets $\mathcal{X}_i$ preserves freeness. What we have to see is that also taking products preserves freeness. Since we can iterate our arguments, it suffices to see the following: if $\mathcal{X}_1$ and $\mathcal{X}_2$ are free, then also $\mathcal{X}_1 \cup \{a_0 a_1 \mid a_0, a_1 \in \mathcal{X}_1\}$ and $\mathcal{X}_2$ are free. Adding one product after the other to $\mathcal{X}_1$ and by Theorem 7.9 it is enough to show that

$$\varphi(\mathcal{U}, \gamma)[a_0 a_1 b_1, a_2 b_2 \dots, a_n b_n] = \sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\mathcal{U},\gamma)} \varphi(\mathcal{V}, \pi)[a_0 a_1, a_2 \dots, a_n] \cdot \kappa(\mathcal{W}, \sigma)[b_1, \dots, b_n]$$

for all $n \in \mathbb{N}$, all $(\mathcal{U}, \gamma) \in \mathcal{PS}(n)$ and all $a_0, a_1, \dots, a_n \in \mathcal{X}_1 \cup \{1\}$, $b_1, \dots, b_n \in \mathcal{X}_2 \cup \{1\}$. Let us induce $(\mathcal{U}, \pi) \in \mathcal{PS}(1, \dots, n)$ to $(\hat{\mathcal{U}}, \hat{\pi}) \in \mathcal{PS}(0, 1, \dots, n)$ by requiring that $\hat{\mathcal{W}}$ and $\hat{\pi}$ restricted to $1, \dots, n$ agree with $\mathcal{W}$ and $\pi$, respectively, and that 0 and 1 are in the same block of $\hat{\mathcal{W}}$ and $\hat{\pi}(0) = 1$. Then we can calculate

$$\begin{aligned}\varphi(\mathcal{U}, \gamma)[a_0 a_1 b_1, a_2 b_2 \dots, a_n b_n] &= \varphi(\hat{\mathcal{U}}, \hat{\pi})[a_0 1, a_1 b_1, a_2 b_2, \dots, a_n b_n] \\ = \sum_{(\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(\hat{\mathcal{U}},\hat{\gamma})} &\varphi(\mathcal{V}, \pi)[a_0, a_1, a_2 \dots, a_n] \cdot \kappa(\mathcal{W}, \sigma)[1, b_1, \dots, b_n].\end{aligned}$$

By Proposition 7.8 we know that $\kappa(\mathcal{W}, \sigma)[1, b_1, \dots, b_n]$ is only different from zero if $\mathcal{W}$ has 0 as a singleton, i.e., $(\mathcal{W}, \sigma)$ has to be of the form

$$\mathcal{W} = \{0\} \cup \tilde{\mathcal{W}}, \qquad \sigma = (0)\tilde{\sigma},$$

with

$$(\tilde{\mathcal{W}}, \tilde{\sigma}) \in \mathcal{PS}(1, \dots, n).$$

But then we must have that $\pi(0) = 1$ and 0 and 1 must be in the same block of $\mathcal{V}$. Thus there is a unique $(\mathcal{V}', \pi')$ so that $(\mathcal{V}, \pi) = (\hat{\mathcal{V}}', \hat{\pi}')$ and

$$(\mathcal{V}, \pi) \cdot (\mathcal{W}, \sigma) = (\hat{\mathcal{U}}, \hat{\gamma})$$

is equivalent to

$$(\mathcal{V}', \pi') \cdot (\tilde{\mathcal{W}}, \tilde{\sigma}) = (\mathcal{U}, \gamma).$$

Note also that in this situation

$$\kappa(\mathcal{W},\sigma)[a_0,a_1,,\dots,a_n]=\kappa(\tilde{\mathcal{W}},\tilde{\sigma})[b_1,b_2,\dots,b_n].$$

and

$$\varphi(\hat{\mathcal{V}}',\hat{\pi}')[a_0,a_1,,\dots,a_n]=\varphi(\mathcal{V}',\pi')[a_0a_1,a_2,\dots,a_n].$$

So we can continue the above calculation as follows

$$\begin{aligned}&\varphi(\mathcal{U},\gamma)[a_0a_1b_1,a_2b_2\dots,a_nb_n]\\ &=\sum_{(\mathcal{V}',\pi')\cdot(\tilde{\mathcal{W}},\tilde{\sigma})=(\mathcal{U},\gamma)}\varphi(\mathcal{V}',\pi')[a_0a_1,a_2\dots,a_n]\cdot\kappa(\tilde{\mathcal{W}},\tilde{\sigma})[b_1,\dots,b_n],\end{aligned}$$

which is exactly what we had to show. □

7.4. DISTRIBUTION OF ONE RANDOM VARIABLE. For the case where we restrict our attention to just one random variable $a\in\mathcal{A}$ we introduce the following notation.

NOTATION 7.13. Let $(\mathcal{A},\varphi)$ be a HOPS and let $a\in\mathcal{A}$.
1) For, $(\mathcal{V},\pi)\in\mathcal{PS}(n)$, we will write

$$\varphi^a(\mathcal{V},\pi):=\varphi(\mathcal{V},\pi)\underbrace{[a,\dots,a]}_{n\text{-times}}$$

and

$$\kappa^a(\mathcal{V},\pi):=\kappa(\mathcal{V},\pi)\underbrace{[a,\dots,a]}_{n\text{-times}}.$$

2) A *Young diagram* is a $\lambda=(\lambda_1,\dots,\lambda_l)$ for some $l\in\mathbb{N}$ and $\lambda_1,\dots,\lambda_l\in\mathbb{N}$ with $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_l$. We put $|\lambda|:=\lambda_1+\cdots+\lambda_l$ (the total number of boxes of the Young diagram $\lambda$). The set of all Young diagrams will be denoted by $\mathbf{Y}$.
3) The information about the higher order moments of $a$ can also be parameterized by Young diagrams as follows: for $\lambda=(\lambda_1,\dots,\lambda_l)$ we put

$$\varphi^a(\lambda):=\varphi(1_{|\lambda|},\pi)\underbrace{[a,\dots,a]}_{n\text{-times}}=\varphi_l(a^{\lambda_1},\dots,a^{\lambda_l})$$

where $\pi$ is any permutation whose conjugacy class corresponds to $\lambda$ (i.e., $\pi\in S_{|\lambda|}$ has cycles of length $\lambda_1,\dots,\lambda_l$. The collection of all higher order moments $(\varphi^a(\lambda))_{\lambda\in\mathbf{Y}}$ is called the *(higher order) distribution of* $a$.
4) Similarly as for moments, we put

$$\kappa^a(\lambda):=\kappa(1_{|\lambda|},\pi)\underbrace{[a,\dots,a]}_{n\text{-times}},$$

where $\pi$ is any permutation whose conjugacy class corresponds to $\lambda$.

*Remark* 7.14. For first and second order moments and cumulants, we used in Section 2 also the following notations:

$$\alpha_n:=\varphi^a(1_n,\gamma_n)\qquad\alpha^a_{m,n}:=\varphi^a(1_{m+n},\gamma_{m,n}),$$

and

$$\kappa_n^a := \kappa^a(1_n, \gamma_n) \qquad \kappa_{m,n} := \kappa^a(1_{m+n}, \gamma_{m,n}),$$

where $\gamma_n$ and $\gamma_{m,n}$ are permutations with one cycle and two cycles, respectively.

The vanishing of mixed cumulants translates in this framework into the additivity of the cumulants for sums of free variables.

Theorem 7.15. *Let $(\mathcal{A}, \varphi)$ be a HOPS and $a, b \in \mathcal{A}$ free of all orders. Then we have*

$$\kappa^{a+b}(\lambda) = \kappa^a(\lambda) + \kappa^b(\lambda)$$

*for all $\lambda \in \mathbf{Y}$.*

*Proof.* By the multilinearity of the cumulants and the vanishing of mixed cumulants for free variable, we have for any $n \in \mathbb{N}$ and $\pi \in S_n$:

$$\begin{aligned}\kappa^{a+b}(1_n, \pi) &= \kappa(1_n, \pi)[a+b, \dots, a+b] \\ &= \kappa(1_n, \pi)[a, \dots, a] + \kappa(1_n, \pi)[b, \dots, b] \\ &= \kappa^a(1_n, \pi) + \kappa^b(1_n, \pi).\end{aligned}$$

□

## 8. Random matrices, Itzykson-Zuber integrals and higher order freeness

8.1. Asymptotic higher order freeness of random matrices. Let us now come back to our original motivation for our theory – the asymptotic behavior of random matrices. In order to reformulate our calculations from Section 4 in our language of higher order freeness we still need to define the notion of "asymptotic freeness".

Definition 8.1. 1) Let $(\mathcal{A}, \varphi)$ and, for each $N \in \mathbb{N}$, $(\mathcal{A}_N, \varphi^{(N)})$ be HOPSs. Let $I$ be an index set and for each $i \in I$, $a_i \in \mathcal{A}$ and $a_i^{(N)} \in \mathcal{A}_N$ $(N \in \mathbb{N})$. We say that the family $(a_i^{(N)} \mid i \in I)$ converges, for $N \to \infty$, to $(a_i \mid i \in I)$, denoted by

$$(a_i^{(N)})_{i\in I} \to (a_i)_{i\in I},$$

if we have for all $n \in \mathbb{N}$ and all polynomials $p_1, \dots, p_n$ in $|I|$-many non-commuting indeterminates that

$$\lim_{N\to\infty} \varphi_n^{(N)} \Big(p_1\big((a_i^{(N)})_{i\in I}\big), \dots, p_n\big((a_i^{(N)})_{i\in I}\big)\Big) = \varphi_n\left(p_1\big((a_i)_{i\in I}\big), \dots, p_n\big((a_i)_{i\in I}\big)\right). \tag{58}$$

2) Let, for each $N \in \mathbb{N}$, $(\mathcal{A}_N, \varphi^{(N)})$ be HOPSs. Let $I$ be an index set and, for each $i \in I$ and $N \in \mathbb{N}$, $a_i^{(N)} \in \mathcal{A}_N$. We say that the sequence of families $(a_i^{(N)})_{i\in I}$ has a *limit distribution of all orders* if there exists a HOPS $(\mathcal{A}, \varphi)$ such that

$$(a_i^{(N)})_{i\in I} \to (a_i)_{i\in I},$$

for some $a_i \in \mathcal{A}$ $(i \in I)$

3) Let, for each $N \in \mathbb{N}$, $(\mathcal{A}_N, \varphi^{(N)})$ be HOPSs. Let $I$ be an index set and, for each $i \in I$ and $N \in \mathbb{N}$, $a_i^{(N)} \in \mathcal{A}_N$. Let $I = I_1 \cup \cdots \cup I_k$ be a decomposition of $I$ into $k$ disjoint subsets. We say that the sets $\{a_i^{(N)} \mid i \in I_1\}, \dots, \{a_i^{(N)} \mid i \in I_k\}$ are *asymptotically free of all orders* if there exists a HOPS $(\mathcal{A}, \varphi)$ such that

$$(a_i^{(N)})_{i\in I} \to (a_i)_{i\in I},$$

for some $a_i \in \mathcal{A}$ $(i \in I)$ and such that the sets $\{a_i \mid i \in I_1\}, \dots, \{a_i \mid i \in I_k\}$ are free of all orders in $(\mathcal{A}, \varphi)$.

With this notation and by invoking Theorem 7.9 we can reformulate our main result on random matrices, Theorem 4.4, in the following form.

THEOREM 8.2. *Let $\mathcal{M}_N := M_N \otimes L^{\infty-}(\Omega)$ be an ensemble of $N \times N$-random matrices. Define rescaled correlation functions $\tilde{\varphi}^{(N)} = (\tilde{\varphi}_n^{(N)})_{n\in\mathbb{N}}$ on $\mathcal{M}_N$ by ($n \in \mathbb{N}$, $D_1, \dots, D_n \in \mathcal{M}_N$)*

$$\tilde{\varphi}_n^{(N)}(D_1, \dots, D_n) := k_n(\mathrm{Tr}(D_1), \dots, \mathrm{Tr}(D_n)) \cdot N^{2-n}. \tag{59}$$

*Assume that we have, for each $N \in \mathbb{N}$, subalgebras $\mathcal{A}_N, \mathcal{B}_N \in \mathcal{M}_N$ such that*

(1) *$\mathcal{A}_N$ is a unitarily invariant ensemble,*
(2) *$\mathcal{A}_N$ and $\mathcal{B}_N$ are independent.*

*Let $(A_i^{(N)})_{i\in I}$ be a family of elements in $(\mathcal{A}_N, \tilde{\varphi}^{(N)})$ which has a higher order limit distribution and let $(B_j^{(N)})_{j\in J}$ ($N \in \mathbb{N}$) be a family of elements in $(\mathcal{B}_N, \tilde{\varphi}^{(N)})$ which has a higher order limit distribution. Then the families $\{A_i^{(N)} \mid i \in I\}$ and $\{B_j^{(N)} \mid j \in J\}$ are asymptotically free of all orders.*

## 8.2. ITZYKSON-ZUBER INTEGRALS.

DEFINITION 8.3. For $N \times N$ matrices $A_N, B_N$ their *Itzykson-Zuber integral* is defined as the following function in $z \in \mathbb{C}$:

$$\mathrm{IZ}(z, A_N, B_N) := N^{-2} \log E(e^{zN\mathrm{Tr}(A_N U B_N U^*)}),$$

where $U$ denotes a Haar unitary $N \times N$-random matrix.

Consider now a sequence of such matrices $A_N$ and $B_N$. Note that $A_N$ and $B_N$ are non-random, thus all distributions of order higher than 1 vanish identically. If we assume that $A_N$ and $B_N$ have a first order (eigenvalue) limit distribution for $N \to \infty$, then it is known (see [Col03]) that each Taylor coefficient about zero of $z \to IZ(z, A_N, B_N)$ admits a limit as $N \to \infty$. Note that the effect of the Haar unitary random matrix in the above Itzykson-Zuber integral was to make $A_N$ and $UB_NU^*$ asymptotically free of all orders. We show now that this kind of result extends also to the case of random matrices $A_N$ and $B_N$, and that our theory allows to identify the limit of the Taylor coefficients very precisely.

THEOREM 8.4. *Let $A = (A_N)_{N\in\mathbb{N}}$ and $B = (B_N)_{N\in\mathbb{N}}$ be two ensembles of $N \times N$-random matrices which are asymptotically free of all orders with respect* *to the rescaled correlation functions* $\tilde{\varphi}^{(N)}$. *Denote the corresponding limiting distribution of* $(A_N)_{N\in\mathbb{N}}$ *by* $\varphi^a$ *and the corresponding limit distribution of* $(B_N)_{N\in\mathbb{N}}$ *by* $\varphi^b$. *Then, as formal power series in* $z$, *we have*

(60)
$$\lim_{N\to\infty} N^{-2}\log \mathrm{E}[e^{zN\mathrm{Tr}(A_NB_N)}] = \sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{\substack{(\mathcal{V},\pi),(\mathcal{W},\sigma)\in\mathcal{PS}(n)\\ (\mathcal{V},\pi)\cdot(\mathcal{W},\sigma)=(1_n,e)}}\kappa^a(\mathcal{V},\pi)\cdot\varphi^b(\mathcal{W},\sigma).$$

*Proof.* Recall that the logarithm of the exponential generating series of the moments of a random variable is the exponential generating series of the classical cumulants of that variable. Thus we have

$$\begin{aligned} &N^{-2}\cdot \log\mathrm{E}[e^{zNA_NB_N}]\\ &\qquad = N^{-2}\sum_{n=1}^{\infty} k_n(N\mathrm{Tr}(A_NB_N),\dots,N\mathrm{Tr}(A_NB_N))\cdot\frac{z^n}{n!}\\ &\qquad = \sum_{n=1}^{\infty} N^{n-2}\cdot\varphi^{(N)}(1_n,e)[A_NB_N,\dots,A_NB_N]\cdot\frac{z^n}{n!}. \end{aligned}$$

By our assumption that $A_N$ and $B_N$ are asymptotically free with respect to $\tilde{\varphi}_n^{(N)} = N^{n-2}\varphi_n^N$, this converges to

$$\sum_{n=1}^{\infty}\varphi(1_n,e)[ab,\dots,ab]\cdot\frac{z^n}{n!},$$

where $a$ and $b$ are free of all orders with respect to $\varphi$. Theorem 7.9 yields then the assertion. □

In a forthcoming work we will investigate the relevance of higher order freeness for Itzykson-Zuber integrals more detailed, in particular, in comparison with and extension of results of Zinn-Justin [ZJ99], Collins [Col03], and Guionnet and Maida [GM05].

## 9. Appendix: Surfaced permutations

In this appendix we will present a more geometrical view on partitioned permutations. As we shall see in the following, partitioned permutations are just special cases of "surfaced permutations"; in particular the results of this article can be equivalently formulated in the language of surfaced permutations. On the other hand, for the purpose of this article we do not need anything more than just partitioned permutations and the Reader not interested in surfaced permutations may skip this Section without much harm.

9.1. Motivations. Our goal is to study factorizations of permutations, i.e. solutions $(\pi_1,\dots,\pi_k)$ of the equation

$$\gamma = \pi_1\cdots\pi_k,$$

where $\gamma\in S_n$ is some fixed permutation and $\pi_1,\dots,\pi_k\in S_n$ are subject to some additional constraints, depending on a particular context. Typically, one

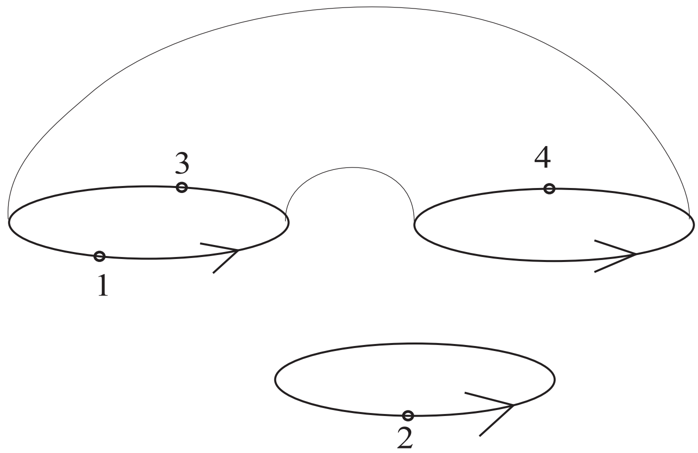



FIGURE 1. Example of a surfaced permutation. Its support is equal to $(1,3)(2)(4) \in S_4$. This surfaced permutation corresponds to a partitioned permutation $\big(\{1,3,4\}\{2\},(1,3)(2)(4)\big)$
.

of these constrains concerns $|\pi_1| + \cdots + |\pi_k|$, the other one concerns the orbits of the action of $\pi_1, \dots, \pi_k$.
It would be very useful to equip permutations $\pi_1, \dots, \pi_k$ with some additional structure in such a way that the product $\tilde{\pi}_1 \cdots \tilde{\pi}_k$ of the resulting enriched permutations $\tilde{\pi}_1, \dots, \tilde{\pi}_k$ would carry both the information about the product $\pi_1 \cdots \pi_k$ of permutations and the information about $|\pi_1| + \cdots + |\pi_k|$. As we shall see in the following, surfaced permutations provide an appropriate tool.

9.2. DEFINITION. We say that $\sigma = (S, j)$ is a *surfaced permutation* of some finite set $A$ if $S$ is a two–dimensional surface with a fixed orientation and with a boundary $\partial S$ and if $j : A \to \partial S$ is a injection. We can think about the information carried by $j$ as follows: some of the points on the boundary $\partial S$ are distinguished and carry different labels from the set $A$. We also require that every connected component of $\partial S$ carries at least one distinguished point. An example of a surfaced permutation is presented on Figure 1.
We identify surfaced permutations $(S_1, j_1)$, $(S_2, j_2)$ of the same set $A$ if there exists a orientation preserving homeomorphism $f : S_1 \to S_2$ such that $f \circ j_1 = j_2$. The set of surfaced permutations of set $\{1, \dots, n\}$ will be denoted by $\mathcal{SS}_n$.

9.3. SURFACED PERMUTATIONS AND THE USUAL PERMUTATIONS. Let $(S, j) \in \mathcal{SS}_n$; the boundary $\partial S$ with the inherited orientation from $S$ is just a collection of oriented circles with some distinguished points labeled $1, \dots, n$ marked on them. In this way we can define a permutation $\sigma \in S_n$, called the support of $(S, j)$, the cycles of which correspond to connected components of $\partial S$, as it can

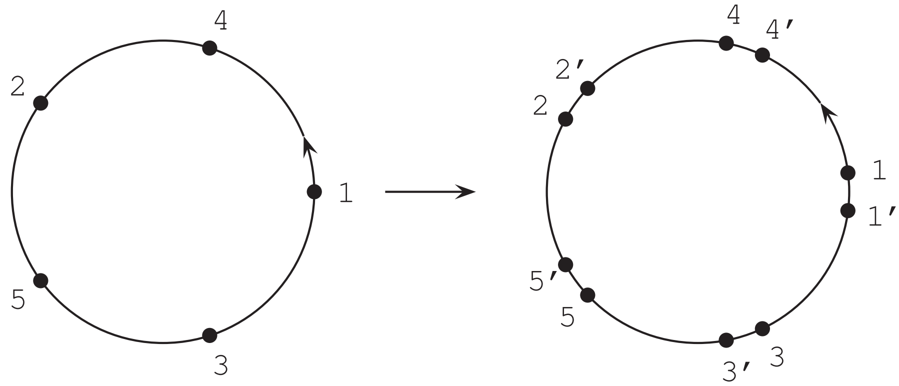



Figure 2. Convention for splitting labels.

be seen on Figure 1. It is therefore a good idea to think that a surfaced permutation is just a (usual) permutation $\sigma \in S_n$ equipped with some additional information carried by the surface $S$.
A surfaced permutation $(S, j) \in \mathcal{SS}_n$ can be uniquely specified (up to the equivalence relation) by its support $\sigma \in S_n$ and by specifying the shape of the connected components of $S$. The latter information is given by an equivalence relation on cycles of $\sigma$ (each class corresponds to a connected component of $S$) and furthermore for each class of this relation we should specify the genus of the corresponding connected component of $S$. Above it should be understood that the genus of a surface $S$ with a boundary is by definition equal to the genus of a surface $S'$ without boundary obtained from $S$ by gluing a disc to every connected component of $\partial S$; for example both a disc and the lateral surface of a cylinder have genus zero.

9.4. Surfaced permutations and partitioned permutations.
Among surfaced permutations a special class will be very important for our purposes, namely surfaced permutations $(S, j)$ such that each connected component of $S$ has genus zero. It is easy to see that there is a bijection between such surfaced permutations $(S, j)$ and partitioned permutations $(\mathcal{V}, \sigma)$ given as follows: $\sigma$ is the support of $(S, j)$ and $\mathcal{V}$ is the partition given by connected components of $S$.

9.5. Products of surfaced permutations. Let surfaced permutations $(S_1, j_1), (S_2, j_2) \in \mathcal{SS}_n$ be given. On the boundary of $S_2$ there are marked points labeled by numbers $1, \dots, n$; let us split every marked point $k$ into a consecutive pair of points $k$ and $k'$, as it is presented on the example from Figure 2. In the second step, for each $k \in \{1, \dots, n\}$ we glue a small neighborhood of the vertex $k \in \partial S_1$ to a small neighborhood of the vertex $k' \in \partial S_2$ in such a way that the orientations of $S_1$ and $S_2$ coincide. In this way we obtain a new surface $S$ which has marked points on its boundary $\partial S$ and these are exactly the vertices from $\partial S_2$ labeled $1, \dots, n$; we denote the resulting surfaced permutation by $(S, j)$ and we call it a product $(S_1, j_1)(S_2, j_2)$ of the original surfaced permutations. This choice of gluing surfaces $S_1$ and $S_2$ implies that the support of $(S_1, j_1)(S_2, j_2)$ is equal to the product of the support of $(S_1, j_1)$ and the support of $(S_2, j_2)$.

It is not difficult to explain now the definition of the product of partitioned permutations (Definition 4.9): we treat partitioned permutations as surfaced permutations and compute their product; if the genus of the resulting surface is zero we can identify it with another partitioned permutation, otherwise we set the product to be zero.

It is not difficult to show that for surfaced permutations the product is associative and the associativity of the product of partitioned permutations is a simple corollary.

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# Motivic Tubular Neighborhoods

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Abstract. We construct motivic versions of the classical tubular neighborhood and the punctured tubular neighborhood, and give applications to the construction of tangential base-points for mixed Tate motives, algebraic gluing of curves with boundary components, and limit motives.

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## 1. Introduction

Let $i : A \to B$ be a closed embedding of finite CW complexes. One useful fact is that $A$ admits a cofinal system of neighborhoods $T$ in $B$ with $A \to T$ a deformation retract. This is often used in the case where $B$ is a differentiable manifold, showing for example that $A$ has the homotopy type of the differentiable manifold $T$. This situation occurs in algebraic geometry, for instance in the case of the inclusion of the special fiber in a degeneration of smooth varieties $\mathcal{X} \to C$ over the complex numbers.

To some extent, one has been able to mimic this construction in purely algebraic terms. The rigidity theorems of Gillet-Thomason [14], extended by Gabber (details appearing a paper of Fujiwara [13]) indicated that, at least through the eyes of torsion étale sheaves, the topological tubular neighborhood can be replaced by the Hensel neighborhood. However, basic examples of non-torsion phenomena, even in the étale topology, show that the Hensel neighborhood cannot always be thought of as a tubular neighborhood, perhaps the simplest example being the sheaf $\mathbb{G}_m$.[1]

Our object in this paper is to construct an algebraic version of the tubular neighborhood which has the basic properties of the topological construction, at least for a reasonably large class of cohomology theories. It turns out that

[1]If $\mathcal{O}$ is a local ring with residue field $k$ and maximal ideal $\mathfrak{m}$, the surjection $\mathbb{G}_m(\mathcal{O}) \to \mathbb{G}_m(k)$ has kernel $(1+\mathfrak{m})^\times$, which is in general non-zero, even for $\mathcal{O}$ Hensel

a "homotopy invariant" version of the Hensel neighborhood does the job, at least for theories which are themselves homotopy invariant. If one requires in addition that the given cohomology theory has a Mayer-Vietoris property for the Nisnevich topology, then one also has an algebraic version of the punctured tubular neighborhood. We extend these constructions to the case of a (reduced) strict normal crossing subscheme by a Mayer-Vietoris procedure, giving us the tubular neighborhood and punctured tubular neighborhood of a normal crossing subscheme of a smooth $k$-scheme.

Morel and Voevodsky have constructed an algebro-geometric version of homotopy theory, in the setting of presheaves of spaces or spectra on the category of smooth varieties over a reasonable base scheme $B$; we concentrate on the $\mathbb{A}^1$-homotopy category of spectra, $\mathcal{SH}_{\mathbb{A}^1}(B)$. For a map $f : X \to Y$, they construct a pair of adjoint functors

$$Rf_* : \mathcal{SH}_{\mathbb{A}^1}(X) \to \mathcal{SH}_{\mathbb{A}^1}(Y)$$
$$Lf^* : \mathcal{SH}_{\mathbb{A}^1}(Y) \to \mathcal{SH}_{\mathbb{A}^1}(X).$$

If we have a closed immersion $i : W \to X$ with open complement $j : U \to X$, then one has the functor

$$Li^*Rj_* : \mathcal{SH}_{\mathbb{A}^1}(U) \to \mathcal{SH}_{\mathbb{A}^1}(W)$$

One of our main results is that, in case $W$ is a strict normal crossing subscheme of a smooth $X$, the restriction of $Li^*Rj_*E$ to a Zariski presheaf on $W$ can be viewed as the evaluation of $E$ on the punctured tubular neighborhood of $W$ in $X$.

Consider a morphism $p : X \to \mathbb{A}^1$ and take $i : W \to X$ to be the inclusion of $p^{-1}(0)$. Following earlier constructions of Spitzweck [43], Ayoub has constructed a "unipotent specialization functor" in the motivic setting, essentially (in the case of a semi-stable degeneration) by evaluating $Li^*Rj_*E$ on a cosimplicial version of the appropriate path space on $\mathbb{G}_m$ with base-point 1. Applying the same idea to our tubular neighborhood construction gives a model for this specialization functor, again only as a Zariski presheaf on $p^{-1}(0)$.

Ayoub has also defined a motivic monodromy operator and monodromy sequence involving the unipotent specialization functor and the functor $Li^*Rj_*$, for theories with $\mathbb{Q}$-coefficients that satisfy a certain additional condition (see definition 9.2.2). We give a model for this construction by combining our punctured tubular neighborhood with a $\mathbb{Q}$-linear version of the $\mathbb{G}_m$-path space mentioned above. We conclude with an application of our constructions to the moduli spaces of smooth curves and a construction of a specialization functor for category of mixed Tate motives, which in some cases yields a purely algebraic construction of tangential base-points. Of course, the construction of Ayoub, when restricted to the triangulated category of Tate motives, also gives such a specialization functor, but we hope that the explicit nature of our construction will be useful for applications.

We have left to another paper the task of checking the compatibilities of our constructions with others via the appropriate realization functor. As we have already mentioned, our punctured tubular neighborhood construction is comparable with the motivic version of the functor $Li^*Rj_*$ for the situation of a normal crossing scheme $i : D \to X$ with open complement $j : X \setminus D \to X$; this should imply that it is a model for the analogous functor after realization. Similarly, our limit cohomology construction should transform after realization to the appropriate version of the sheaf of vanishing cycles, at least in the case of a semi-stable degeneration, and should be comparable with the constructions of Rappaport-Zink [37] as well as the limit mixed Hodge structure of Katz [22] and Steenbrink [44]. Our specialization functor for Tate motives should be compatible with the Betti, étale and Hodge realizations; similarly, realization functors applied to our limit motive should yield for example the limit mixed Hodge structure. We hope that our rather explicit construction of the limiting motive will be useful in giving a geometric view to the limit mixed Hodge structure of a semi-stable degeneration but we have not attempted an investigation of these issues in this paper.

My interest in this topic began as a result of several discussions on limit motives with Spencer Bloch and Hélène Esnault, whom I would like to thank for their encouragement and advice. I would also like to thank Hélène Esnault for clarifying the role of the weight filtration leading to the exactness of Clemens-Schmidt monodromy sequence (see Remark 9.3.6). An earlier version of our constructions used an analytic (i.e. formal power series) neighborhood instead of the Hensel version now employed; I am grateful to Fabien Morel for suggesting this improvement. Finally, I want to thank Joseph Ayoub for explaining his construction of the nearby cycles functor; his comments suggested to us the use of the cosimplicial path space in our construction of limit cohomology. In addition, Ayoub noticed a serious error in our first attempt at constructing the monodromy sequence; the method used in this version is following his suggestions and comments. Finally, we would like to thank the referee for giving unusually thorough and detailed comments and suggestions, which have substantially improved this paper. In particular, the material in sections 7 comparing our construction with the categorical ones of Morel-Voevodsky, as well as the comparison with Ayoub's specialization functor and monodromy sequence in section 8.3 and section 9 was added following the suggestion of the referee, who also supplied the main ideas for the proofs.

## 2. Model structures and other preliminaries

2.1. Presheaves of simplicial sets. We recall some facts on the model structures in categories of simplicial sets, spectra, associated presheaf categories and certain localizations. For details, we refer the reader to [17] and[19].

For a small category $I$ and category $\mathcal{C}$, we will denote the category of functors from $I$ to $\mathcal{C}$ by $\mathcal{C}^I$.

We let **Ord** denote the category with objects the finite ordered sets $[n] := \{0, \ldots, n\}$ (with the standard ordering) and morphisms the order-preserving maps of sets. For a category $\mathcal{C}$, the functor categories $\mathcal{C}^{\mathbf{Ord}}$, $\mathcal{C}^{\mathbf{Ord}^{\mathrm{op}}}$ are the categories of *cosimplicial objects of* $\mathcal{C}$, resp. *simplicial objects of* $\mathcal{C}$. For $\mathcal{C} = \mathbf{Sets}$, we have the category of simplicial sets, $\mathbf{Spc} := \mathbf{Sets}^{\mathbf{Ord}^{\mathrm{op}}}$, and similarly for $\mathcal{C}$ the category of pointed sets, $\mathbf{Sets}_*$, the category of pointed simplicial sets $\mathbf{Spc}_* := \mathbf{Sets}_*^{\mathbf{Ord}^{\mathrm{op}}}$.

We give $\mathbf{Spc}$ and $\mathbf{Spc}_*$ the standard model structures: cofibrations are (pointed) monomorphisms, weak equivalences are weak homotopy equivalences on the geometric realization, and fibrations are detemined by the right lifting property (RLP) with respect to trivial cofibrations; the fibrations are then exactly the Kan fibrations. We let $|A|$ denote the geometric realization, and $[A, B]$ the homotopy classes of (pointed) maps $|A| \to |B|$.

For an essentially small category $\mathcal{C}$, we let $\mathbf{Spc}(\mathcal{C})$ be the category of presheaves of simplicial sets on $\mathcal{C}$. We give $\mathbf{Spc}(\mathcal{C})$ the so-called injective model structure, that is, the cofibrations and weak equivalences are the pointwise ones, and the fibrations are determined by the RLP with respect to trivial cofibrations. We let $\mathcal{H}\mathbf{Spc}(\mathcal{C})$ denote the associated homotopy category (see [17] for details on these model structures for $\mathbf{Spc}$ and $\mathbf{Spc}(\mathcal{C})$).

2.2. Presheaves of spectra. Let $\mathbf{Spt}$ denote the category of spectra. To fix ideas, a spectrum will be a sequence of pointed simplicial sets $E_0, E_1, \ldots$ together with maps of pointed simplicial sets $\epsilon_n : S^1 \wedge E_n \to E_{n+1}$. Maps of spectra are maps of the underlying simplicial sets which are compatible with the attaching maps $\epsilon_n$. The stable homotopy groups $\pi_n^s(E)$ are defined by

$$\pi_n^s(E) := \lim_{m\to\infty} [S^{m+n}, E_m].$$

The category $\mathbf{Spt}$ has the following model structure: Cofibrations are maps $f : E \to F$ such that $E_0 \to F_0$ is a cofibration, and for each $n \geq 0$, the map

$$E_{n+1} \coprod_{S^1\wedge E_n} S^1 \wedge F_n \to F_{n+1}$$

is a cofibration. Weak equivalences are the stable weak equivalences, i.e., maps $f : E \to F$ which induce an isomorphism on $\pi_n^s$ for all $n$. Fibrations are characterized by having the RLP with respect to trivial cofibrations. We write $\mathcal{SH}$ for the homotopy category of $\mathbf{Spt}$.

For $X \in \mathbf{Spc}_*$, we have the suspension spectrum $\Sigma^\infty X := (X, \Sigma X, \Sigma^2 X, \ldots)$ with the identity bonding maps. Dually, for a spectrum $E := (E_0, E_1, \ldots)$ we have the 0-space $\Omega^\infty E := \lim_n \Omega^n E_n$. These operations form a Quillen pair of adjoint functors $(\Sigma^\infty, \Omega^\infty)$ between $\mathbf{Spc}_*$ and $\mathbf{Spt}$, and thus induce adjoint functors on the homotopy categories.

Let $\mathcal{C}$ be a category. A functor $E : \mathcal{C}^{\mathrm{op}} \to \mathbf{Spt}$ is called a *presheaf of spectra* on $\mathcal{C}$.

We use the following model structure on the category of presheaves of spectra (see [19]): Cofibrations and weak equivalences are given pointwise, and fibrations are characterized by having the RLP with respect to trivial cofibrations. We denote this model category by $\mathbf{Spt}(\mathcal{C})$, and the associated homotopy category by $\mathcal{H}\mathbf{Spt}(\mathcal{C})$.

As a particular example, we have the model category of simplicial spectra $\mathbf{Spt}^{\mathbf{Ord}^{\mathrm{op}}} = \mathbf{Spt}(\mathbf{Ord})$. We have the *total spectrum functor*

$$\mathrm{Tot} : \mathbf{Spt}(\mathbf{Ord}) \to \mathbf{Spt}$$

which preserves weak equivalences. The adjoint pair $(\Sigma^\infty, \Omega^\infty)$ extend pointwise to define a Quillen pair on the presheaf categories and an adjoint pair on the homotopy categories.

Let $B$ be a noetherian separated scheme of finite Krull dimension. We let $\mathbf{Sm}/B$ denote the category of smooth $B$-schemes of finite type over $B$. We often write $\mathbf{Spc}(B)$ and $\mathcal{H}\mathbf{Spc}(B)$ for $\mathbf{Spc}(\mathbf{Sm}/B)$ and $\mathcal{H}\mathbf{Spc}(\mathbf{Sm}/B)$, and write $\mathbf{Spt}(B)$ and $\mathcal{H}\mathbf{Spt}(B)$ for $\mathbf{Spt}(\mathbf{Sm}/B)$ and $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/B)$.

For $Y \in \mathbf{Sm}/B$, a subscheme $U \subset Y$ of the form $Y \setminus \cup_\alpha F_\alpha$, with $\{F_\alpha\}$ a possibly infinite set of closed subsets of $Y$, is called *essentially smooth over* $B$; the category of essentially smooth $B$-schemes is denoted $\mathbf{Sm}^{\mathrm{ess}}$.

2.3. Local model structure. If the category $\mathcal{C}$ has a topology, there is often another model structure on $\mathbf{Spc}(\mathcal{C})$ or $\mathbf{Spt}(\mathcal{C})$ which takes this into account. We consider the case of the small Nisnevich site $X_{\mathrm{Nis}}$ on a scheme $X$ (assumed to be noetherian, separated and of finite Krull dimension), and the big Nisnevich sites $\mathbf{Sm}/B_{\mathrm{Nis}}$ or $\mathbf{Sch}/B_{\mathrm{Nis}}$, as well as the Zariski versions $X_{\mathrm{Zar}}$, $\mathbf{Sm}/B_{\mathrm{Zar}}$, etc. We describe the Nisnevich version for spectra below; the definitions and results for the Zariski topology and for spaces are exactly parallel.

Definition 2.3.1. A map $f : E \to F$ of presheaves of spectra on $X_{\mathrm{Nis}}$ is a *local weak equivalence* if the induced map on the Nisnevich sheaf of stable homotopy groups $f_* : \pi^s_m(E)_{\mathrm{Nis}} \to \pi^s_m(F)_{\mathrm{Nis}}$ is an isomorphism of sheaves for all $m$. A map $f : E \to F$ of presheaves of spectra on $\mathbf{Sm}/B_{\mathrm{Nis}}$ or $\mathbf{Sch}/B_{\mathrm{Nis}}$ is a local weak equivalence if the restriction of $f$ to $X_{\mathrm{Nis}}$ is a local weak equivalence for all $X \in \mathbf{Sm}/B$ or $X \in \mathbf{Sch}/B$. □

The Nisnevich local model structure on the category of presheaves of spectra on $X_{\mathrm{Nis}}$ has cofibrations given pointwise, weak equivalences the local weak equivalences and fibrations are characterized by having the RLP with respect to trivial cofibrations. We write $\mathbf{Spt}(X_{\mathrm{Nis}})$ for this model category, and $\mathcal{H}\mathbf{Spt}(X_{\mathrm{Nis}})$ for the associated homotopy category. The Nisnevich local model categories $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$ and $\mathbf{Spt}(\mathbf{Sch}/B_{\mathrm{Nis}})$, with homotopy categories $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$ and $\mathcal{H}\mathbf{Spt}(\mathbf{Sch}/B_{\mathrm{Nis}})$, are defined similarly. A similar localization gives model categories of presheaves of spaces $\mathbf{Spc}(X_{\mathrm{Nis}})$, $\mathbf{Spc}(X_{\mathrm{Zar}})$, $\mathbf{Spc}(\mathbf{Sm}/B_{\mathrm{Nis}})$, etc., and homotopy categories $\mathcal{H}\mathbf{Spc}(X_{\mathrm{Nis}})$, $\mathcal{H}\mathbf{Spc}(X_{\mathrm{Zar}})$, $\mathcal{H}\mathbf{Spc}(\mathbf{Sm}/B_{\mathrm{Nis}})$, etc. We also have the adjoint pair $(\Sigma^\infty, \Omega^\infty)$ in this setting. For details, we refer the reader to [19].

*Remark* 2.3.2. Let $E$ be in $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$, and let

$$\begin{tikzcd} W' \arrow[r, "i'"] \arrow[d] & Y' \arrow[d, "f"] \\ W \arrow[r, "i"'] & X \end{tikzcd}$$

be an *elementary Nisnevich square*, i.e., $f$ is étale, $i : W \to X$ is a closed immersion, the square is cartesian, and $W' \to W$ is an isomorphism, with $X$ and $X'$ in $\mathbf{Sm}/B$ (see [34, Definition 1.3, pg. 96]) .
If $E$ is fibrant in $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$ then $E$ transforms each elementary Nisnevich square to a homotopy cartesian square in $\mathbf{Spt}$. Conversely, suppose that $E$ transforms each elementary Nisnevich square to a homotopy cartesian square in $\mathbf{Spt}$. Then $E$ is *quasi-fibrant*, i.e., for all $Y \in \mathbf{Sm}/B$, the canonical map $E(Y) \to E_{fib}(Y)$, where $E_{fib}$ is the fibrant model of $E$, is a weak equivalence. See [19] for details.
If we define an elementary Zariski square as above, with $X' \to X$ an open immersion, the same holds in the model category $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Zar}})$. More precisely, one can show (see e.g. [45]) that, if $E$ transforms each elementary Zarisk square to a homotopy cartesian square in $\mathbf{Spt}$, then $E$ *satisfies Mayer-Vietoris* for the Zariski topology: if $X \in \mathbf{Sm}/B$ is a union of Zariski open subschemes $U$ and $V$, then the evident sequence

$$E(X) \to E(U) \oplus E(V) \to E(U \cap V)$$

is a homotopy fiber sequence in $\mathcal{SH}$. □

*Remark* 2.3.3. Let $\mathcal{C}$ be a small category with an initial object $\emptyset$ and admiting finite coproducts over $\emptyset$, denoted $X \amalg Y$. A functor $E : \mathcal{C}^{\mathrm{op}} \to \mathbf{Spt}$ is called *additive* if for each $X, Y$ in $\mathcal{C}$, the canonical map

$$E(X \amalg Y) \to E(X) \oplus E(Y)$$

in $\mathcal{SH}$ is an isomorphism. It is easy to show that if $E \in \mathbf{Spt}(\mathbf{Sm}/B)$ satisfies Mayer-Vietoris for the Zariski topology, and $E(\emptyset) \cong 0$ in $\mathcal{SH}$, then $E$ is additive. From now on, we will assume that *all* our presheaves of spectra $E$ satisfy $E(\emptyset) \cong 0$ in $\mathcal{SH}$. □

2.4. $\mathbb{A}^1$-LOCAL STRUCTURE. One can perform a Bousfield localization on $\mathbf{Spc}(\mathbf{Sm}/B_{\mathrm{Nis}})$ or $\mathbf{Spt}(\mathbf{Sm}/B_{\mathrm{Nis}})$ so that the maps $\Sigma^\infty X \times \mathbb{A}^1_+ \to \Sigma^\infty X_+$ induced by the projections $X \times \mathbb{A}^1 \to X$ become weak equivalences. We call the resulting model structure the *Nisnevich-local* $\mathbb{A}^1$*-model structure*, denoted $\mathbf{Spc}_{\mathbb{A}^1}(\mathbf{Sm}/B_{\mathrm{Nis}})$ or $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/B_{\mathrm{Nis}})$. One has the Zariski-local versions as well. We denote the homotopy categories for the Nisnevich version by $\mathcal{H}_{\mathbb{A}^1}(B)$ (for spaces) and $\mathcal{SH}_{\mathbb{A}^1}(B)$ (for spectra). For the Zariski versions, we indicate the topology in the notation. We also have the adjoint pair $(\Sigma^\infty, \Omega^\infty)$ in this setting. For details, see [30, 31, 34].

2.5. Additional notation. Given $W \in \mathbf{Sm}/S$, we have restriction functors

$$\mathbf{Spc}(S) \to \mathbf{Spc}(W_{\mathrm{Zar}})$$
$$\mathbf{Spt}(S) \to \mathbf{Spt}(W_{\mathrm{Zar}});$$

we write the restriction of some $E \in \mathbf{Spc}(S)$ to $\mathbf{Spc}(W_{\mathrm{Zar}})$ as $E(W_{\mathrm{Zar}})$. We use a similar notation for the restriction of $E$ to $\mathbf{Spt}(W_{\mathrm{Zar}})$, or for restrictions to $W_{\mathrm{Nis}}$. More generally, if $p : Y \to W$ is a morphism in $\mathbf{Sm}/S$, we write $E(Y/W_{\mathrm{Zar}})$ for the presheaf $U \mapsto E(Y \times_W U)$ on $W_{\mathrm{Zar}}$.
For $Z \subset Y$ a closed subset, $Y \in \mathbf{Sm}/S$ and for $E \in \mathbf{Spc}(S)$ or $E \in \mathbf{Spt}(S)$, we write $E^Z(Y)$ for the homotopy fiber of the restriction map

$$E(Y) \to E(Y \setminus Z).$$

We define the presheaf $E^{Z_{\mathrm{Zar}}}(Y)$ by setting, for $U \subset Z$ a Zariski open subscheme with closed complement $F$,

$$E^{Z_{\mathrm{Zar}}}(Y)(U) := E^U(Y \setminus F).$$

A *co-presheaf* on a category $\mathcal{C}$ with values in $\mathcal{A}$ is just an $\mathcal{A}$-valued preheaf on $\mathcal{C}^{\mathrm{op}}$.
As usual, we let $\Delta^n$ denote the algebraic $n$-simplex

$$\Delta^n := \operatorname{Spec} \mathbb{Z}[t_0, \ldots, t_n]/\sum_i t_i - 1,$$

and $\Delta^*$ the cosimplicial scheme $n \mapsto \Delta^n$. For a scheme $X$, we have $\Delta^n_X := X \times \Delta^n$ and the cosimplicial scheme $\Delta^*_X$.
Let $B$ be a scheme as above. For $E \in \mathbf{Spc}(B)$ or in $\mathbf{Spt}(B)$, we say that $E$ is *homotopy invariant* if for all $X \in \mathbf{Sm}/B$, the pull-back map $E(X) \to E(X \times \mathbb{A}^1)$ is a weak equivalence (resp., stable weak equivalence). We say that $E$ *satisfies Nisnevich excision* if $E$ transforms elementary Nisnevich squares to homotopy cartesian squares.

## 3. Tubular neighborhoods for smooth pairs

Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$. In this section, we construct the tubular neighborhood $\tau_\epsilon^{\hat{X}}(W)$ of $W$ in $X$ as a functor from $W_{\mathrm{Zar}}$ to cosimplicial pro-$k$-schemes. Given $E \in \mathbf{Spc}(k)$, we can evaluate $E$ on $\tau_\epsilon^{\hat{X}}(W)$, yielding the presheaf of spaces $E(\tau_\epsilon^{\hat{X}}(W))$ on $W_{\mathrm{Zar}}$, which is our main object of study.

3.1. The cosimplicial pro-scheme $\tau_\epsilon^{\hat{X}}(W)$. For a closed immersion $W \to T$ in $\mathbf{Sm}/k$, let $T^W_{\mathrm{Nis}}$ be the category of Nisnevich neighborhoods of $W$ in $T$, i.e., objects are étale maps $p : T' \to T$ of finite type, together with a section $s : W \to T'$ to $p$ over $W$. Morphisms are morphisms over $T$ which respect the sections. Note that $T^W_{\mathrm{Nis}}$ is a left-filtering essentially small category.
Sending $(p : T' \to T, s : W \to T')$ to $T' \in \mathbf{Sm}/k$ defines the pro-object $\hat{T}^h_W$ of $\mathbf{Sm}/k$; the sections $s : W \to T'$ give rise to a map of the constant pro-scheme $W$ to $\hat{T}^h_W$, denoted

$$\hat{i}_W : W \to \hat{T}^h_W.$$

Given a $k$-morphism $f : S \to T$, and closed immersions $i_V : V \to S$, $i_W : W \to T$ such that $f \circ i_V$ factors through $i_W$ (by $\bar{f} : V \to W$), we have the pull-back functor

$$\begin{aligned} &f^* : T^W_{\mathrm{Nis}} \to S^V_{\mathrm{Nis}}, \\ &f^*(T' \to T, s : W \to T') := (T' \times_T S, (s \circ \bar{f}, i_V)). \end{aligned}$$

This gives us the map of pro-objects $f^h : \hat{S}^h_V \to \hat{T}^h_W$, so that sending $W \to T$ to $\hat{T}^h_W$ and $f$ to $f^h$ becomes a pseudo-functor.
We let $f^h : \hat{S}^h_V \to \hat{T}^h_W$ denote the induced map on pro-schemes. If $f$ happens to be a Nisnevich neighborhood of $W \to X$ (so $\bar{f} : V \to W$ is an isomorphism) then $f^h : \hat{S}^h_V \to \hat{T}^h_W$ is clearly an isomorphism.

*Remark* 3.1.1. The pseudo-functor $(W \to T) \mapsto \hat{T}^h_W$ can be rectified to an honest functor by first replacing $T^W_{\mathrm{Nis}}$ with the cofinal subcategory $T^W_{\mathrm{Nis},0}$ of neighborhoods $T' \to T$, $s : W \to T'$ such that each connected component of $T'$ has non-empty intersection with $s(W)$. One notes that $T^W_{\mathrm{Nis},0}$ has only identity automorphisms, so we replace $T^W_{\mathrm{Nis},0}$ with a choice of a full subcategory $T^W_{\mathrm{Nis},00}$ giving a set of representatives of the isomorphism classes in $T^W_{\mathrm{Nis},0}$. Given a map of pairs of closed immersions $f : (V \xrightarrow{i_V} S) \to (W \xrightarrow{i_W} T)$ as above, we modify the pull-back functor $f^*$ defined above by passing to the connected component of $(s \circ \bar{f}, i_V)(V)$ in $T' \times_T S$. We thus have the honest functor $(W \to T) \mapsto T^W_{\mathrm{Nis},00}$ which yields an equivalent pro-object $\hat{T}^h_W$.
As pointed out by the referee, one can also achieve strict functoriality by rectifying the fiber product; in any case, we will use a strictly functorial version from now on without comment. □

For a closed immersion $i : W \to X$ in $\mathbf{Sm}/k$, set $\hat{\Delta}^n_{X,W} := (\widehat{\Delta^n_X})^h_{\Delta^n_W}$. The cosimplicial scheme

$$\begin{aligned} \Delta^*_X &: \mathbf{Ord} \to \mathbf{Sm}/k \\ [n] &\mapsto \Delta^n_X \end{aligned}$$

thus gives rise to the cosimplicial pro-scheme

$$\begin{aligned} \hat{\Delta}^*_{X,W} &: \mathbf{Ord} \to \mathbf{Pro\text{-}Sm}/k \\ [n] &\mapsto \hat{\Delta}^n_{X,W} \end{aligned}$$

The maps $\hat{i}_{\Delta^n_W} : \Delta^n_W \to (\widehat{\Delta^n_X})^h_{\Delta^n_W}$ give the closed immersion of cosimplicial pro-schemes

$$\hat{i}_W : \Delta^*_W \to \hat{\Delta}^*_{X,W}.$$

Also, the canonical maps $\pi_n : \hat{\Delta}^n_{X,W} \to \Delta^n_X$ define the map

$$\pi_{X,W} : \hat{\Delta}^*_{X,W} \to \Delta^*_X.$$

Let $(p : X' \to X, s : W \to X')$ be a Nisnevich neighborhood of $(W, X)$. The map

$$p : \hat{\Delta}^n_{X',W} \to \hat{\Delta}^n_{X,W}$$

is an isomorphism respecting the closed immersions $\hat{i}_W$. Thus, sending a Zariski open subscheme $U \subset W$ with complement $F \subset W \subset X$ to $\hat{\Delta}^n_{X\setminus F,U}$ defines a co-presheaf $\hat{\Delta}^n_{\hat{X},W_{\mathrm{Zar}}}$ on $W_{\mathrm{Zar}}$ with values in pro-objects of $\mathbf{Sm}/k$; we write $\tau_\epsilon^{\hat{X}}(W)$ for the cosimplicial object

$$n \mapsto \hat{\Delta}^n_{\hat{X},W_{\mathrm{Zar}}}.$$

We use the $\hat{X}$ in the notation because the co-presheaf $\hat{\Delta}^n_{\hat{X},W_{\mathrm{Zar}}}$ is depends only on the Nisnevich neighborhood of $W$ in $X$.
Let $\Delta^*_{W_{\mathrm{Zar}}}$ denote the co-presheaf on $W_{\mathrm{Zar}}$ defined by $U \mapsto \Delta^*_U$. The closed immersions $\hat{i}_U$ define the natural transformation

$$\hat{i}_W : \Delta^*_{W_{\mathrm{Zar}}} \to \tau_\epsilon^{\hat{X}}(W).$$

The maps $\pi_{X\setminus F,W\setminus F}$ for $F \subset W$ a Zariski closed subset define the map

$$\pi_{X,W} : \tau_\epsilon^{\hat{X}}(W) \to \Delta^*_{X|W_{\mathrm{Zar}}}$$

where $X|W_{\mathrm{Zar}}$ is the co-presheaf $W \setminus F \mapsto X \setminus F$ on $W_{\mathrm{Zar}}$. We let

$$\bar{\pi}_{X,W} : \tau_\epsilon^{\hat{X}}(W) \to X|W_{\mathrm{Zar}} \tag{3.1.1}$$

denote the composition of $\pi_{X.W}$ with the projection $\Delta^*_{X|W_{\mathrm{Zar}}} \to X|W_{\mathrm{Zar}}$.

3.2. Evaluation on spaces. Let $i : W \to T$ be a closed immersion in $\mathbf{Sm}/k$. For $E \in \mathbf{Spc}(T)$, we have the space $E(\hat{T}^W_h)$, defined by

$$E(\hat{T}^h_W) := \operatorname*{colim}_{(p:T'\to T,s:W\to T')\in T^W_{\mathrm{Nis}}} E(T').$$

Given a Nisnevich neighborhood $(p : T' \to T, s : W \to T')$, we have the isomorphism

$$p^* : E(\hat{T}^h_W) \to E(\hat{T'}^h_{s(W)}).$$

Thus, for each open subscheme $j : U \to W$, we may evaluate $E$ on the cosimplicial pro-scheme $\tau_\epsilon^{\hat{X}}(W)(U)$, giving us the presheaf of simplicial spectra $E(\tau_\epsilon^{\hat{X}}(W))$ on $W_{\mathrm{Zar}}$:

$$E(\tau_\epsilon^{\hat{X}}(W))(U) := E(\tau_\epsilon^{\hat{X}}(W)(U)).$$

Now suppose that $E$ is in $\mathbf{Spc}(k)$. The map $\hat{i}_W : \Delta^*_{W_{\mathrm{Zar}}} \to \tau_\epsilon^{\hat{X}}(W))$ gives us the map of presheaves on $W_{\mathrm{Zar}}$

$$i^*_W : E(\tau_\epsilon^{\hat{X}}(W)) \to E(\Delta^*_{W_{\mathrm{Zar}}}).$$

Similarly, the map $\pi_{X,W}$ gives the map of presheaves on $W_{\mathrm{Zar}}$

$$\pi^*_{X,W} : E(\Delta^*_{X|W_{\mathrm{Zar}}}) \to E(\tau_\epsilon^{\hat{X}}(W)).$$

The main result of this section is

Theorem 3.2.1. *Let $E$ be in $\mathbf{Spc}(k)$. Then the map $i_W^*: E(\tau_\epsilon^{\hat{X}}(W)) \to E(\Delta^*_{W_{\mathrm{Zar}}})$ is a weak equivalence for the Zariski-local model structure, i.e., for each point $w \in W$, the map $i_{W,w}^*$ on the stalks at $w$ is a weak equivalence of the associated total space.*

3.3. Proof of Theorem 3.2.1. The proof relies on two lemmas.

Lemma 3.3.1. *Let $i: W \to X$ be a closed immersion in $\mathbf{Sm}/k$, giving the closed immersion $\mathbb{A}^1_W \to \mathbb{A}^1_X$. For $t \in \mathbb{A}^1(k)$, we have the section $i_t: W \to \mathbb{A}^1_W$, $i_t(w) := (w,t)$. Then for each $E \in \mathbf{Spc}(k)$, the maps*

$$i_0^*, i_1^*: E(\hat{\Delta}^*_{\mathbb{A}^1_X, \mathbb{A}^1_W}) \to E(\hat{\Delta}^*_{X,W})$$

*are homotopic.*

*Proof.* This is just an adaptation of the standard triangulation argument. For each order-preserving map $g = (g_1, g_2): [m] \to [1] \times [n]$, let

$$T_g: \Delta^m \to \Delta^1 \times \Delta^n,$$

be the affine-linear extension of the map on the vertices

$$v_i \mapsto (v_{g_1(i)}, v_{g_2(i)}).$$

$\mathrm{id}_X \times T_g$ induces the map

$$\hat{T}_g: \hat{\Delta}^m_{X,W} \to (\widehat{\Delta^1 \times \Delta^n_X})^h_{\Delta^1 \times \Delta^n_W}$$

We note that the isomorphism $(t_0, t_1) \mapsto t_0$ of $(\Delta^1, v_1, v_0)$ with $(\mathbb{A}^1, 0, 1)$ induces an isomorphism of cosimplicial schemes

$$\hat{\Delta}^*_{\mathbb{A}^1_X, \mathbb{A}^1_W} \cong (\widehat{\Delta^1 \times \Delta^*_X})^h_{\Delta^1 \times \Delta^*_W}.$$

The maps

$$\hat{T}_g^*: E(\hat{\Delta}^n_{\mathbb{A}^1_X, \mathbb{A}^1_W}) \to E(\hat{\Delta}^m_{X,W})$$

induce a simplicial homotopy $T$ between $i_0^*$ and $i_1^*$. Indeed, we have the simplicial sets $\Delta[n]: \mathrm{Hom}_{\mathbf{Ord}}(-, [n])$. Let $(\Delta^1 \times \Delta^*)^{\Delta[1]}$ be the cosimplicial scheme

$$n \mapsto (\Delta^1 \times \Delta^n)^{\Delta[1]([n])} := \prod_{s \in \Delta[1]([n])} \Delta^1 \times \Delta^n$$

where the product is $\times_{\mathbb{Z}}$. The inclusions $\delta_0, \delta_1: [0] \to [1]$ thus induce the maps of cosimplicial schemes

$$\delta_0^*, \delta_1^*: (\Delta^1 \times_k \Delta^*)^{\Delta[1]} \to \Delta^1 \times_k \Delta^*.$$

The maps $T_g$ satisfy the identities necessary to define a map of cosimplicial schemes

$$T: \Delta^* \to (\Delta^1 \times \Delta^*)^{\Delta[1]}.$$

with $\delta_0^* \circ T = i_0$, $\delta_1^* \circ T = i_1$. Applying the functor ${}^h$, we see that the maps $\hat{T}_g$ define the map of cosimplicial schemes

$$\hat{T}: \hat{\Delta}^*_{X,W} \to (\hat{\Delta}^*_{\mathbb{A}^1_X, \mathbb{A}^1_W})^{\Delta[1]},$$

with $\delta_0^* \circ \hat{T} = \hat{i}_0$, $\delta_1^* \circ \hat{T} = \hat{i}_1$; we then apply $E$. □

Lemma 3.3.2. *Take $W \in \mathbf{Sm}_k$. Let $X = \mathbb{A}^n_W$ and let $i : W \to X$ be the 0-section. Then $i^*_W : E(\hat{\Delta}^*_{X,W}) \to E(\Delta^*_W)$ is a homotopy equivalence.*

*Proof.* Let $p : X \to W$ be the projection, giving the map

$$\hat{p} : \hat{\Delta}^*_{X,W} \to \hat{\Delta}^*_{W,W} = \Delta^*_W$$

and $\hat{p}^* : E(\Delta^*_W) \to E(\hat{\Delta}^*_{X,W})$. Clearly $\hat{i}^*_W \circ \hat{p}^* = \mathrm{id}$, so it suffices to show that $\hat{p}^* \circ \hat{i}^*_W$ is homotopic to the identity.
For this, we use the multiplication map $\mu : \mathbb{A}^1 \times \mathbb{A}^n \to \mathbb{A}^n$,

$$\mu(t; x_1, \ldots, x_n) := (tx_1, \ldots, tx_n).$$

The map $\mu \times \mathrm{id}_{\Delta^*}$ induces the map

$$\hat{\mu} : (\mathbb{A}^1 \times \widehat{\mathbb{A}^n_W \times} \Delta^*)^h_{\mathbb{A}^1 \times 0_W \times \Delta^*} \to (\mathbb{A}^n_W \widehat{\times \Delta}^*)^h_{0_W \times \Delta^*}$$

with $\hat{\mu} \circ \hat{i}_0 = \hat{i}_W \circ \hat{p}$ and $\hat{\mu} \circ \hat{i}_1 = \mathrm{id}$. Since $\hat{i}^*_0$ and $\hat{i}^*_1$ are homotopic by Lemma 3.3.1, the proof is complete. $\square$

To complete the proof of Theorem 3.2.1, take a point $w \in W$. Then replacing $X$ with a Zariski open neighborhood of $w$, we may assume there is a Nisnevich neigborhood $X' \to X$, $s : W \to X'$ of $W$ in $X$ such that $W \to X'$ is in turn a Nisnevich neighborhood of the zero-section $W \to \mathbb{A}^n_W$, $n = \mathrm{codim}_X W$. Since $E(\hat{\Delta}^n_{X,W})$ is thus weakly equivalent to $E(\hat{\Delta}^n_{\mathbb{A}^n_W, 0_W})$, the result follows from Lemma 3.3.2.

Corollary 3.3.3. *Suppose that $E \in \mathbf{Spc}(\mathbf{Sm}/k)$, resp. $E \in \mathbf{Spt}(\mathbf{Sm}/k)$ is homotopy invariant. Then for $i : W \to X$ a closed immersion, there is a natural isomorphism in $\mathcal{H}\mathbf{Spc}(W_{\mathrm{Zar}})$, resp. $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$*

$$E(\tau^{\hat{X}}_\epsilon(W)) \cong E(\tau^{\hat{N}_i}_\epsilon(0_W))$$

*Here $N_i$ is the normal bundle of the immersion $i$, and $0_W \subset N_i$ is the 0-section.*

*Proof.* This follows directly from Theorem 3.2.1: Since $E$ is homotopy invariant, the canonical map

$$E(T) \to E(\Delta^*_T)$$

is a weak equivalence for each $T \in \mathbf{Sm}/k$. The desired isomorphism in the respective homotopy category is constructed by composing the isomorphisms

$$\begin{aligned} E(\tau^{\hat{X}}_\epsilon(W)) &\xrightarrow{i^*_W} E(\Delta^*_{W_{\mathrm{Zar}}}) \leftarrow E(W_{\mathrm{Zar}}) \\ &\qquad = E(0_{W\mathrm{Zar}}) \to E(\Delta^*_{0_{W\mathrm{Zar}}}) \xleftarrow{i^*_{0_W}} E(\tau^{\hat{N}_i}_\epsilon(0_W)). \end{aligned}$$

$\square$

## 4. Punctured tubular neighborhoods

Our real interest is not in the tubular neighborhood $\tau^{\hat{X}}_\epsilon(W)$, but in the punctured tubular neighborhood $\tau^{\hat{X}}_\epsilon(W)^0$. In this section, we define this object and discuss its basic properties.

4.1. Definition of the punctured neighborhood. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$. We have the closed immersion of cosimplicial pro-schemes

$$\hat{i} : \Delta^*_W \to \hat{\Delta}^*_{X,W}$$

giving for each $n$ the open complement $\hat{\Delta}^n_{X\setminus W} := \hat{\Delta}^n_{X,W} \setminus \Delta^n_W$. We may pass to the cofinal subcategory of Nisnevich neighborhoods of $\Delta^n_W$ in $\Delta^n_X$,

$$(p : T \to \Delta^n_X, s : \Delta^n_W \to T)$$

for which the diagram

$$\begin{tikzcd} T \setminus s(\Delta^n_W) \arrow[r] \arrow[d] & T \arrow[d] \\ \Delta^n_X \setminus \Delta^n_W \arrow[r] & \Delta^n_X \end{tikzcd}$$

is cartesian, giving us the cosimplicial proscheme $n \mapsto \hat{\Delta}^n_{X\setminus W}$ and the map

$$\hat{j} : \hat{\Delta}^*_{X\setminus W} \to \hat{\Delta}^*_{X,W},$$

which defines the "open complement" $\hat{\Delta}^*_{X\setminus W}$ of $\Delta^*_W$ in $\hat{\Delta}^*_{X,W}$. Extending this construction to all open subschemes of $X$, we have the co-presheaf on $W_{\mathrm{Zar}}$,

$$U = W \setminus F \mapsto \hat{\Delta}^*_{(X\setminus F)\setminus U},$$

which we denote by $\tau^{\hat{X}}_\epsilon(W)^0$.

Let $\Delta^n_{(X\setminus W)_{\mathrm{Zar}}}$ be the constant co-presheaf on $W_{\mathrm{Zar}}$ with value $\Delta^n_{X\setminus W}$, giving the cosimplicial co-presheaf $\Delta^*_{(X\setminus W)_{\mathrm{Zar}}}$. The maps

$$\hat{j}_U : \hat{\Delta}^*_{(X\setminus F)\setminus U} \to \hat{\Delta}^*_{(X\setminus F),U}$$

define the map $\hat{j} : \tau^{\hat{X}}_\epsilon(W)^0 \to \tau^{\hat{X}}_\epsilon(W)$. The maps $\hat{\Delta}^*_{U\setminus W\cap U} \to \Delta^*_{X\setminus W}$ give us the map

$$\pi : \tau^{\hat{X}}_\epsilon(W)^0 \to \Delta^*_{X\setminus W}$$

where we view $\Delta^*_{X\setminus W}$ as the constant co-sheaf on $W_{\mathrm{Zar}}$.

To give a really useful result on the presheaf $E(\tau^{\hat{X}}_\epsilon(W)^0)$, we will need to impose additional conditions on $E$. These are

(1) $E$ is homotopy invariant
(2) $E$ satisfies Nisnevich. excision

One important consequence of these properties is the purity theorem of Morel-Voevodsky:

Theorem 4.1.1 (Purity [34, theorem 2.23]). *Suppose $E \in \mathbf{Spt}(k)$ is homotopy invariant and satisfies Nisnevich excision. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ and $s : W \to N_i$ the 0-section of the normal bundle. Then there is an isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$*

$$E^{W_{\mathrm{Zar}}}(X) \to E^{W_{\mathrm{Zar}}}(N_i)$$

□

Let $E(X|W_{\mathrm{Zar}})$ be the presheaf on $W_{\mathrm{Zar}}$

$$W \setminus F \mapsto E(X \setminus F)$$

and $E(X \setminus W)$ the constant presheaf.
Let

$$\mathrm{res} : E(X|W_{\mathrm{Zar}}) \to E(\tau_\epsilon^{\hat{X}}(W))$$
$$\mathrm{res}^0 : E(X \setminus W) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

be the pull-back by the natural maps $\tau_\epsilon^{\hat{X}}(W)(W \setminus F) \to X \setminus F$, $\tau_\epsilon^{\hat{X}}(W)^0 \to X \setminus W$. Let $E^{\Delta^*_W}(\tau_\epsilon^{\hat{X}}(W)) \in \mathbf{Spt}(W_{\mathrm{Zar}})$ be the homotopy fiber of the restriction map

$$\hat{j} : E(\tau_\epsilon^{\hat{X}}(W)) \to E(\tau_\epsilon^{\hat{X}}(W)^0).$$

The commutative diagram in $\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\begin{tikzcd} E(X|W_{\mathrm{Zar}}) \arrow[r, "j^*"] \arrow[d, "\mathrm{res}"'] & E(X \setminus W) \arrow[d, "\mathrm{res}^0"] \\ E(\tau_\epsilon^{\hat{X}}(W)) \arrow[r, "\hat{j}^*"'] & E(\tau_\epsilon^{\hat{X}}(W)^0) \end{tikzcd}$$

induces the map of homotopy fiber sequences

$$\begin{tikzcd} E^{W_{\mathrm{Zar}}}(X) \arrow[r] \arrow[d, "\psi"'] & E(X|W_{\mathrm{Zar}}) \arrow[r, "j^*"] \arrow[d, "\mathrm{res}"'] & E(X \setminus W) \arrow[d, "\mathrm{res}^0"] \\ E^{\Delta^*_W}(\tau_\epsilon^{\hat{X}}(W)) \arrow[r] & E(\tau_\epsilon^{\hat{X}}(W)) \arrow[r, "j^*"'] & E(\tau_\epsilon^{\hat{X}}(W)^0) \end{tikzcd}$$

We can now state the main result for $E(\tau_\epsilon^{\hat{X}}(W)^0)$.

Theorem 4.1.2. *Suppose that $E \in \mathbf{Spt}(k)$ is homotopy invariant and satisfies Nisnevich excision. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$. Then the map $\psi$ is a Zariski local weak equivalence.*

*Proof.* Let $i_{\Delta^*} : \Delta^*_W \to \Delta^*_X$ be the immersion $\mathrm{id} \times i$. For $U = W \setminus F \subset W$, $\tau_\epsilon^{\hat{X}}(W)^0(U)$ is the cosimplicial pro-scheme with $n$-cosimplices

$$\tau_\epsilon^{\hat{X}}(W)^0(U)^n = \hat{\Delta}^n_{X\setminus F, U} \setminus \Delta^n_U$$

so by Nisnevich excision we have the natural isomorphism

$$\alpha : E^{\Delta^*_{W_{\mathrm{Zar}}}}(\Delta^*_{X|W_{\mathrm{Zar}}}) \to E^{\Delta^*_W}(\tau_\epsilon^{\hat{X}}(W)),$$

where $E^{\Delta^*_{W_{\mathrm{Zar}}}}(\Delta^*_{X|W_{\mathrm{Zar}}})(W\setminus F)$ is the total spectrum of the simplicial spectrum

$$n \mapsto E^{\Delta^n_{W\setminus F}}(\Delta^n_{X\setminus F}).$$

The homotopy invariance of $E$ implies that the pull-back

$$E^{W\setminus F}(X \setminus F) \to E^{\Delta^n_{W\setminus F}}(\Delta^n_{X\setminus F})$$

is a weak equivalence for all $n$, so we have the weak equivalence

$$\beta : E^{W_{\mathrm{Zar}}}(X) \to E^{\Delta^*_{W_{\mathrm{Zar}}}}(\Delta^*_{X|W_{\mathrm{Zar}}}).$$

It follows from the construction that $\psi = \alpha\beta$, completing the proof. $\square$

Corollary 4.1.3. *There is a distinguished triangle in* $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$E^{W_{\mathrm{Zar}}}(X) \to E(W_{\mathrm{Zar}}) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

*Proof.* By Theorem 3.2.1, the map $\hat{i}^* : E(\tau_\epsilon^{\hat{X}}(W)) \to E(\Delta^*_{W_{\mathrm{Zar}}})$ is a weak equivalence; using homotopy invariance again, the map

$$E(W_{\mathrm{Zar}}) \to E(\Delta^*_{W_{\mathrm{Zar}}})$$

is a weak equivalence. Combining this with Theorem 4.1.2 yields the result. $\square$

For homotopy invariant $E \in \mathbf{Spt}(k)$, we let

$$\phi_E : E(\tau_\epsilon^{\hat{N}_i}(0_W)) \to E(\tau_\epsilon^{\hat{X}}(W)).$$

be the isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$ given by corollary 3.3.3.

Corollary 4.1.4. *Suppose that $E \in \mathbf{Spt}(k)$ is homotopy invariant and satisfies Nisnevich excision. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ and let $N_i^0 = N_i \setminus 0_W$.*

*(1) The restriction maps*

$$\mathrm{res} : E(N_i/W_{\mathrm{Zar}}) \to E(\tau_\epsilon^{\hat{N}_i}(0_W))$$

$$\mathrm{res}^0 : E(N_i^0/W_{\mathrm{Zar}}) \to E(\tau_\epsilon^{\hat{N}_i}(0_W)^0)$$

*are weak equivalences in* $\mathbf{Spt}(W_{\mathrm{Zar}})$.

*(2) There is a canonical isomorphism in* $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\phi_E^0 : E(\tau_\epsilon^{\hat{N}_i}(0_W)^0) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

*(3) Consider the diagram (in* $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$*):*

$$\begin{tikzcd}
E^{0_{W_{\mathrm{Zar}}}}(N_i) \arrow[r] \arrow[d, equal] & E(N_i/W_{\mathrm{Zar}}) \arrow[r] \arrow[d, "\mathrm{res}_E"] & E(N_i^0/W_{\mathrm{Zar}}) \arrow[d, "\mathrm{res}_E^0"] \\
E^{0_{W_{\mathrm{Zar}}}}(N_i) \arrow[r] \arrow[d, "\pi"'] & E(\tau_\epsilon^{\hat{N}_i}(0_W)) \arrow[r, "\hat{j_N}^*"] \arrow[d, "\phi_E"] & E(\tau_\epsilon^{\hat{N}_i}(0_W)^0) \arrow[d, "\phi_E^0"] \\
E^{W_{\mathrm{Zar}}}(X) \arrow[r] & E(\tau_\epsilon^{\hat{X}}(W)) \arrow[r, "\hat{j}^*"'] & E(\tau_\epsilon^{\hat{X}}(W)^0) \\
E^{W_{\mathrm{Zar}}}(X) \arrow[r] \arrow[u, equal] & E(X|W_{\mathrm{Zar}}) \arrow[r, "j^*"'] \arrow[u, "\mathrm{res}_E"'] & E(X \setminus W) \arrow[u, "\mathrm{res}_E^0"']
\end{tikzcd}$$

*The first and last rows are the homotopy fiber sequences defining the presheaves $E^{0_{W_{\mathrm{Zar}}}}(N_i)$ and $E^{W_{\mathrm{Zar}}}(X)$, respectively, the second row and third rows are the* *distinguished triangles of Theorem 4.1.2, and* $\pi$ *is the Morel-Voevodsky purity isomorphism. Then this diagram commutes and each triple of vertical maps defines a map of distinguished triangles.*

*Proof.* It follows directly from the weak equivalence (in Theorem 4.1.2) of the homotopy fiber of

$$\hat{j}^* : E(\tau_\epsilon^{\hat{X}}(W)) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

with $E^{W_{\mathrm{Zar}}}(X)$ that the triple $(\mathrm{id}, \mathrm{res}_E, \mathrm{res}_E^0)$ defines a map of distinguished triangles. The same holds for the map of the first row to the second row; we now verify that this latter map is an isomorphism of distinguished triangles. For this, let $s : W \to N_i$ be the zero-section. We have the isomorphism $i_W^* : E(\tau_\epsilon^{\hat{N}_i}(0_W)) \to E(W_{\mathrm{Zar}})$ defined as the diagram of weak equivalences

$$E(\tau_\epsilon^{\hat{N}_i}(0_W)) \xrightarrow{i^*_{\Delta^*_{W_{\mathrm{Zar}}}}} E(\Delta^*_{W_{\mathrm{Zar}}}) \xleftarrow{\iota_{0*}} E(W_{\mathrm{Zar}}).$$

From this, it is easy to check that the diagram

$$\begin{tikzcd} E(N_i/W_{\mathrm{Zar}}) \arrow[r, "\mathrm{res}_E"] \arrow[dr, "s^*"'] & E(\tau_\epsilon^{\hat{N}_i}(0_W)) \arrow[d, "i_W^*"] \\ & E(W_{\mathrm{Zar}}) \end{tikzcd}$$

commutes in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$. As $E$ is homotopy invariant, $s^*$ is an isomorphism, hence $\mathrm{res}_E$ is an isomorphism as well. This completes the proof of (1).

The proof of (2) and (3) uses the standard deformation diagram. Let $\bar{\mu} : \bar{Y} \to X \times \mathbb{A}^1$ be the blow-up of $X \times \mathbb{A}^1$ along $W$, let $\bar{\mu}^{-1}[X \times 0]$ denote the proper transform, and let $\mu : Y \to X \times \mathbb{A}^1$ be the open subscheme $\bar{Y} \setminus \bar{\mu}^{-1}[X \times 0]$. Let $p : Y \to \mathbb{A}^1$ be $p_2 \circ \mu$. Then $p^{-1}(0) = N_i$, $p^{-1}(1) = X \times 1 = X$, and $Y$ contains the proper transform $\bar{\mu}^{-1}[W \times \mathbb{A}^1]$, which is mapped isomorphically by $\mu$ to $W \times \mathbb{A}^1 \subset X \times \mathbb{A}^1$. We let $\tilde{i} : W \times \mathbb{A}^1 \to Y$ be the resulting closed immersion. The restriction of $\tilde{i}$ to $W \times 0$ is the zero-section $s : W \to N_i$ and the restriction of $\tilde{i}$ to $W \times 1$ is $i : W \to X$. The resulting diagram is

(4.1.1)
$$\begin{tikzcd} W \arrow[r, "i_0"] \arrow[d, "s"'] & W \times \mathbb{A}^1 \arrow[d, "\tilde{i}"'] & W \arrow[l, "i_1"'] \arrow[d, "i"] \\ N_i \arrow[r, "i_0"] \arrow[d, "p_0"'] & Y \arrow[d, "p"'] & X \arrow[l, "i_1"'] \arrow[d, "p_1"] \\ 0 \arrow[r, "i_0"'] & \mathbb{A}^1 & 1 \arrow[l, "i_1"] \end{tikzcd}$$

Together with Theorem 4.1.2, diagram (4.1.1) gives us two maps of distinguished triangles:

$$\Big[E^{0_{W\times\mathbb{A}^1}\mathrm{Zar}}(Y) \to E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \to E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)^0)\Big]$$
$$\xrightarrow{i_1^*}$$
$$\Big[E^{W_{\mathrm{Zar}}}(X) \to E(\tau_\epsilon^{\hat{X}}(W)) \to E(\tau_\epsilon^{\hat{X}}(W)^0)\Big]$$

and

$$\Big[E^{0_{W\times\mathbb{A}^1}\mathrm{Zar}}(Y) \to E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \to E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)^0)\Big]$$
$$\xrightarrow{i_0^*}$$
$$\Big[E^{0_{W}\mathrm{Zar}}(N_i) \to E(\tau_\epsilon^{\hat{N}_i}(0_W)) \to E(\tau_\epsilon^{\hat{N}_i}(0_W)^0)\Big]$$

As above, we have the commutative diagram

$$\begin{tikzcd}
E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \arrow[r, "i_0^*"] \arrow[d, "i_{W\times\mathbb{A}^1}^*"'] & E(\tau_\epsilon^{\hat{N}_i}(0_W)) \arrow[d, "i_W^*"'] \\
E(W\times\mathbb{A}^1) \arrow[r, "i_0^*"'] & E(W).
\end{tikzcd}$$

As $E$ is homotopy invariant, the maps $i_W^*$, $i_{W\times\mathbb{A}^1}^*$ and $i_0^* : E(W\times\mathbb{A}^1) \to E(W)$ are isomorphisms, hence

$$i_0^* : E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \to E(\tau_\epsilon^{\hat{N}_i}(0_W))$$

is an isomorphism. Similarly,

$$i_1^* : E(\tau_\epsilon^{\hat{Y}}(W\times\mathbb{A}^1)) \to E(\tau_\epsilon^{\hat{X}}(W))$$

is an isomorphism. The proof of the Morel-Voevodsky purity theorem [34, Theorem 2.23] shows that

$$i_0^* : E^{0_{W\times\mathbb{A}^1}\mathrm{Zar}}(Y) \to E^{0_W\mathrm{Zar}}(N_i)$$
$$i_1^* : E^{0_{W\times\mathbb{A}^1}\mathrm{Zar}}(Y) \to E^{W_{\mathrm{Zar}}}(X)$$

are weak equivalences; the purity isomorphism $\pi$ is by definition $i_1^*\circ(i_0^*)^{-1}$. Thus, both $i_0^*$ and $i_1^*$ define isomorphisms of distinguished triangles, and

$$i_1^*\circ(i_0^*)^{-1} : E(\tau_\epsilon^{\hat{N}_i}(0_W)) \to E(\tau_\epsilon^{\hat{X}}(W))$$

is the map $\phi_E$. Defining $\phi_E^0$ to be the isomorphism

$$i_1^*\circ(i_0^*)^{-1} : E(\tau_\epsilon^{\hat{N}_i}(0_W)^0) \to E(\tau_\epsilon^{\hat{X}}(W)^0)$$

proves both (2) and (3). □

*Remarks* 4.1.5. (1) It follows from the construction of $\phi_E$ and $\phi^0_E$ that both of these maps are natural in $E$.

(2) The maps $\phi^0_E$ are natural in the closed immersion $i : W \to X$ in the following sense: Suppose we have closed immersions $i_j : W_j \to X_j$, $j = 1, 2$ and a morphism $f : (W_1, X_1) \to (W_2, X_2)$ of pairs of immersions such that $f$ restricts to a morphism $X_1 \setminus W_1 \to X_2 \setminus W_2$. Fix $E$ and let $\phi^0_{jE}$ be the map corresponding to the immersions $i_j$ We have the evident maps

$$\iota : \tau^{\hat{X}_1}_\epsilon(W_1)^0 \to \tau^{\hat{X}_2}_\epsilon(W_2)^0 \qquad \eta : \tau^{\hat{N}_{i_1}}_\epsilon(W_1)^0 \to \tau^{\hat{N}_{i_2}}_\epsilon(W_2)^0$$

Then the diagram

$$\begin{tikzcd} E(\tau^{\hat{N}_{i_2}}_\epsilon(W)^0) \arrow[r, "\phi_{2E}"] \arrow[d, "\eta^*"'] & E(\tau^{\hat{Y}}_\epsilon(W)^0) \arrow[d, "\iota^*"] \\ E(\tau^{\hat{N}_{i_1}}_\epsilon(W)^0) \arrow[r, "\phi_{1E}"'] & E(\tau^{\hat{X}}_\epsilon(W)^0) \end{tikzcd}$$

commutes. Indeed, the map $f$ induces a map of deformation diagrams.
□

## 5. The exponential map

If $i : M' \to M$ is a submanifold of a differentiable manifold $M$, there is a diffeomorphism exp of the normal bundle $N_{M'/M}$ of $M'$ in $M$ with the tubular neighborhood $\tau^M_\epsilon(M')$. In addition, exp restricts to a diffeomorphism $\exp^0$ of the punctured normal bundle $N_{M'/M} \setminus 0_{M'}$ with the punctured tubular neighborhood $\tau^M_\epsilon(M') \setminus M'$. Classically, this has been used to define the boundary map in the Gysin sequence for $M' \to M$, by using the restriction map $\exp^{0*} : H^*(M \setminus M') \to H^*(N_{M'/M} \setminus 0_{M'})$ followed by the Thom isomorphism $H^*(N_{M'/M} \setminus 0_{M'}) \cong H^{*-d}(M')$.
In this section, we use our punctured tubular neighborhood to construct a purely algebraic version of the classical exponential map, at least for the associated suspension spectra. We will use this in section 11 to define a purely algebraic version of the gluing of Riemann surfaces along boundary components.

5.1. Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ with normal bundle $p : N_i \to W$. We have the map

$$\exp : N_i \to X$$

in $\mathbf{Spc}(k)$, defined as the composition $N_i \to W \to X$. We also have the Morel-Voevodsky purity isomorphism

$$\pi : \mathrm{Th}(N_i) \to X/(X \setminus W)$$

in $\mathcal{H}(k)$. In fact:

Lemma 5.1.1. *The diagram*

(5.1.1)
$$\begin{tikzcd} N_i \arrow[r, "q'"] \arrow[d, "\exp"'] & \mathrm{Th}(N_i) \arrow[d, "\pi"] \\ X \arrow[r, "q"'] & X/(X\setminus W) \end{tikzcd}$$

*commutes in* $\mathcal{H}(k)$.

*Proof.* As we have already seen, the construction of the purity isomorphism $\pi$ relies on the deformation to the normal bundle; we retain the notation from the proof of corollary 4.1.4. We have the total space $Y \to \mathbb{A}^1$ of the deformation. The fiber $Y_0$ over $0 \in \mathbb{A}^1$ is canonically isomorphic to $N_i$ and the fiber $Y_1$ over 1 is canonically isomorphic to $X$; the inclusions $W \times 0 \to Y_0$, $W \times 1 \to Y_1$ are isomorphic to the zero-section $s : W \to N_i$ and the original inclusion $i : W \to X$, respectively. The proper transform $\mu^{-1}[W \times \mathbb{A}^1]$ is isomorphic to $W \times \mathbb{A}^1$, giving the closed immersion $\iota : W \times \mathbb{A}^1 \to Y$. The diagram thus induces maps in $\mathbf{Spc}(k)$:

$$\bar{i}_0 : \mathrm{Th}(N_i) \to Y/(Y \setminus W \times \mathbb{A}^1)$$
$$\bar{i}_1 : X/X \setminus W \to Y/(Y \setminus W \times \mathbb{A}^1)$$

which are isomorphisms in $\mathcal{H}(k)$ (see [34, Thm. 2.23]); the purity isomorphism is by definition $\pi := \bar{i}_1^{-1} \circ \bar{i}_0$.

We have the commutative diagram in $\mathbf{Spc}(k)$:

$$\begin{tikzcd} W \arrow[r, "i_0"] \arrow[d, shift left, "s"] \arrow[rr, bend left, "\mathrm{id}"] & W \times \mathbb{A}^1 \arrow[r, shift left, "p_1"] \arrow[d, "\iota"] & W \arrow[l, shift left, "i_1"] \arrow[d, "i"] \\ N_i \arrow[u, shift left, "p"] \arrow[r] \arrow[d, "q'"'] & Y \arrow[d] & X \arrow[l] \arrow[d, "q"] \\ \mathrm{Th}(N_i) \arrow[r, "\bar{i}_0"'] & Y/(Y \setminus W \times \mathbb{A}^1) & X/X \setminus W \arrow[l, "\bar{i}_1"] \end{tikzcd}$$

from which the result follows directly. □

*Remark* 5.1.2. Since we have the homotopy cofiber sequences:

$$N_i \setminus 0_W \to N_i \to \mathrm{Th}(N_i) \to \Sigma(N_i \setminus 0_W)_+$$
$$X \setminus W \to X \to X/(X \setminus W) \to \Sigma(X \setminus W)_+$$

the diagram (5.1.1) induces a map

$$\Sigma(N_i \setminus 0_W)_+ \to \Sigma(X \setminus W)_+$$

in $\mathcal{H}(k)$, however, this map is not uniquely determined, hence is not canonical. □

5.2. The construction. In this section we define a canonical map

$$\exp^0 : \Sigma^\infty(N_i \setminus 0_W)_+ \to \Sigma^\infty(X \setminus W)_+$$

in $\mathcal{SH}_{\mathbb{A}^1}(k)$ which yields the map of distinguished triangles in $\mathcal{SH}_{\mathbb{A}^1}(k)$:

$$\begin{tikzcd}
\Sigma^\infty(N_i \setminus 0_W)_+ \arrow[r] \arrow[d, "\exp^0"'] & \Sigma^\infty N_{i+} \arrow[r] \arrow[d, "\exp"'] & \Sigma^\infty \mathrm{Th}(N_i) \arrow[d, "\pi"] \\
\Sigma^\infty(X \setminus W)_+ \arrow[r] & \Sigma^\infty X_+ \arrow[r] & \Sigma^\infty X/(X \setminus W)
\end{tikzcd}$$

To define $\exp^0$, we apply Corollary 4.1.4 with $E$ a fibrant model of $\Sigma^\infty(X\setminus W)_+$. Denote the composition

$$E(X \setminus W) \xrightarrow{\mathrm{res}^0_E} E(\tau_\epsilon^{\hat{X}}(W)^0)(W) \xrightarrow{(\phi^0_E)^{-1}} E(\tau_\epsilon^{\hat{N}_i}(0_W)^0)(W) \xrightarrow{(\mathrm{res}^0_E)^{-1}} E(N_i^0)$$

by $\exp_E^{0*}$. Since $E$ is fibrant, we have canonical isomorphisms

$$\begin{aligned}
\pi_0 E(N_i^0) &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty N_{i+}^0, E) \\
&\cong \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty N_{i+}^0, \Sigma^\infty(X \setminus W)_+)
\end{aligned}$$

$$\begin{aligned}
\pi_0 E(X \setminus W) &\cong \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty(X \setminus W)_+, E) \\
&\cong \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty(X \setminus W)_+, \Sigma^\infty(X \setminus W)_+)
\end{aligned}$$

so $\exp_E^{0*}$ induces the map

$$\mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty(X \setminus W)_+, \Sigma^\infty(X \setminus W)_+) \xrightarrow{\exp_E^{0*}} \mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(\Sigma^\infty N_{i+}^0, \Sigma^\infty(X \setminus W)_+).$$

We set

$$\exp^0 := \exp_E^{0*}(\mathrm{id}).$$

To finish the construction, we show

Proposition 5.2.1. *The diagram, with rows the evident homotopy cofiber sequences,*

$$\begin{tikzcd}
\Sigma^\infty(N_i \setminus 0_W)_+ \arrow[r] \arrow[d, "\exp^0"'] & \Sigma^\infty N_{i+} \arrow[r] \arrow[d, "\exp"'] & \Sigma^\infty \mathrm{Th}(N_i) \arrow[r, "\partial"] \arrow[d, "\pi"] & \Sigma\Sigma^\infty(N_i \setminus 0_W)_+ \arrow[d, "\Sigma \exp^0"'] \\
\Sigma^\infty(X \setminus W)_+ \arrow[r] & \Sigma^\infty X_+ \arrow[r] & \Sigma^\infty X/(X \setminus W) \arrow[r, "\partial"'] & \Sigma\Sigma^\infty(X \setminus W)_+
\end{tikzcd}$$

*commutes in* $\mathcal{SH}_{\mathbb{A}^1}(k)$.

*Proof.* It suffices to show that, for all fibrant $E \in \mathbf{Spt}(k)$, the diagram formed by applying $\mathrm{Hom}_{\mathcal{SH}_{\mathbb{A}^1}(k)}(-, E)$ to our diagram commutes. This latter diagram is the same as applying $\pi_0$ to the diagram

(5.2.1)
$$\begin{tikzcd} E(N_i \setminus 0_W) & E(N_i) \arrow[l] & E^{0_W}(N_i) \arrow[l] & \Omega E(N_i \setminus 0_W)) \arrow[l, "\partial"'] \\ E(X \setminus W) \arrow[u, "\exp^{0*}"] & E(X) \arrow[l] \arrow[u, "\exp^*"] & E^W(X) \arrow[l] \arrow[u, "\pi^*"'] & \Omega E(X \setminus W) \arrow[l, "\partial"] \arrow[u, "\Omega \exp^{0*}"] \end{tikzcd}$$

where the rows are the evident homotopy fiber sequences. It follows by the definition of $\exp^0$ and exp that this diagram is just the "outside" of the diagram in Corollary 4.1.4(3), extended to make the distinguished triangles explicit. Thus the diagram (5.2.1) commutes, which finishes the proof. □

*Remark* 5.2.2. The exponential maps exp and $\exp^0$ are natural with respect to maps of closed immersions $f : (W' \xrightarrow{i'} X') \to (W \xrightarrow{i} X)$ satisfying the cartesian condition of remark 4.1.5(2). This follows from the naturality of the isomorphisms $\phi_E$, $\phi^0_E$ described in Remark 4.1.5, and the functoriality of the (punctured) tubular neighborhood construction. □

## 6. NEIGHBORHOODS OF NORMAL CROSSING SCHEMES

We extend our results to the case of a strict normal crossing divisor $W \subset X$ by using a Mayer-Vietoris construction.

6.1. NORMAL CROSSING SCHEMES. Let $D$ be a reduced effective Cartier divisor on a smooth $k$-scheme $X$ with irreducible components $D_1,\ldots, D_m$. For each $I \subset \{1, \ldots, m\}$, we set

$$D_I := \cap_{i\in I} D_i$$

We let $i : D \to X$ the inclusion. For each $I \neq \emptyset$, we let $\iota_I : D_I \to D$, $i_I : D_I \to X$ be the inclusions; for $I \subset J \subset \{1, \ldots, m\}$ we have as well the inclusion $\iota_{I,J} : D_J \to D_I$.

Recall that $D$ is a *strict normal crossing divisor* if for each $I$, $D_I$ is smooth over $k$ and $\mathrm{codim}_X D_I = |I|$.

We extend this notion a bit: We call a closed subscheme $D \subset X$ a *strict normal crossing subscheme* if $X$ is in $\mathbf{Sm}/k$ and, locally on $X$, there is a smooth locally closed subscheme $Y \subset X$ containing $D$ such that $D$ is a strict normal crossing divisor on $Y$

6.2. THE TUBULAR NEIGHBORHOOD. Let $D \subset X$ be a strict normal crossing subscheme with irreducible components $D_1, \ldots, D_m$. For each $I \subset \{1, \ldots, m\}$, $I \neq \emptyset$, we have the tubular neighborhood co-presheaf $\tau^{\hat{X}}_\epsilon(D_I)$ on $D_I$. The various inclusions $\iota_{I,J}$ give us the maps of co-presheaves

$$\hat{\iota}_{I,J} : \iota_{I,J*}(\tau^{\hat{X}}_\epsilon(D_J)) \to \tau^{\hat{X}}_\epsilon(D_I);$$

pushing forward by the maps $\iota_I$ yields the diagram of co-presheaves on $D_{\mathrm{Zar}}$ (with values in cosimplicial pro-objects of $\mathbf{Sm}/k$)

$$I \mapsto \iota_{I*}(\tau_\epsilon^{\hat{X}}(D_I)) \tag{6.2.1}$$

indexed by the non-empty $I \subset \{1, \ldots, m\}$. We have as well the diagram of identity co-presheaves

$$I \mapsto \iota_{I*}(D_{I\mathrm{Zar}}) \tag{6.2.2}$$

as well as the diagram

$$I \mapsto \iota_{I*}(\Delta^*_{D_{I\mathrm{Zar}}}) \tag{6.2.3}$$

We denote these diagrams by $\tau_\epsilon^{\hat{X}}(D)$, $D_\bullet$ and $\Delta^*_{D_\bullet}$, respectively. The projections $p_I : \Delta^*_{D_{I\mathrm{Zar}}} \to D_{I\mathrm{Zar}}$ and the closed immersions $\hat{\iota}_{D_I} : \Delta^*_{D_{I\mathrm{Zar}}} \to \tau_\epsilon^{\hat{X}}(D_I)$ yield the natural transformations

$$D_\bullet \xleftarrow{p_\bullet} \Delta^*_{D_\bullet} \xrightarrow{\hat{\iota}_{D_\bullet}} \tau_\epsilon^{\hat{X}}(D).$$

Now take $E \in \mathbf{Spt}(k)$. Applying $E$ to the diagram (6.2.1) yields the diagram of presheaves on $D_{\mathrm{Zar}}$

$$I \mapsto \iota_{I*}(E(\tau_\epsilon^{\hat{X}}(D_I)))$$

Similarly, applying $E$ to (6.2.2) and (6.2.3) yields the diagrams of presheaves on $D_{\mathrm{Zar}}$

$$I \mapsto \iota_{I*}(E(D_{I\mathrm{Zar}}))$$

and

$$I \mapsto \iota_{I*}(E(\Delta^*_{D_{I\mathrm{Zar}}})).$$

Definition 6.2.1. For $D \subset X$ a strict normal crossing subscheme and $E \in \mathbf{Spt}(k)$, set

$$E(\tau_\epsilon^{\hat{X}}(D)) := \operatorname*{holim}_{I \neq \emptyset} \iota_{I*}(E(\tau_\epsilon^{\hat{X}}(D_I))).$$

Similarly, set

$$E(D_\bullet) := \operatorname*{holim}_{I \neq \emptyset} \iota_{I*}(E(D_I))$$

$$E(\Delta^*_{D_\bullet}) := \operatorname*{holim}_{I \neq \emptyset} \iota_{I*}(E(\Delta^*_{D_I}))$$

□

The natural transformations $\hat{\iota}_D$ and $p_\bullet$ yield the maps of presheaves on $D_{\mathrm{Zar}}$

$$E(D_\bullet) \xrightarrow{p_\bullet^*} E(\Delta^*_{D_\bullet}) \xleftarrow{\hat{\iota}_D^*} E(\tau_\epsilon^{\hat{X}}(D)).$$

Proposition 6.2.2. *Suppose $E \in \mathbf{Spt}(\mathbf{Sm}/k)$ is homotopy invariant and satisfies Nisnevich excision. Then the maps $\hat{\iota}_D^*$ and $p_\bullet^*$ are Zariski-local weak equivalences.*

*Proof.* The maps $p_I^*$ are pointwise weak equivalences by homotopy invariance. By Theorem 3.2.1, the maps $\hat{\iota}_{D_I}$ are Zariski-local weak equivalences. Since the homotopy limits are finite, the stalk of each homotopy limit is weakly equivalence to the homotopy limit of the stalks. By [8] this suffices to conclude that the map on the homotopy limits is a Zariski-local weak equivalence. □

*Remark* 6.2.3. One could also attempt a more direct definition of $\tau_\epsilon^{\hat{X}}(D)$ by just using our definition in the smooth case $i : W \to X$ and replacing the smooth $W$ with the normal crossing scheme $D$, in other words, the co-presheaf on $D_{\mathrm{Zar}}$

$$D \setminus F \mapsto \hat{\Delta}^*_{X\setminus F, D\setminus F}.$$

Labeling this choice $\tau_\epsilon^{\hat{X}}(D)_{\mathrm{naive}}$, and considering $\tau_\epsilon^{\hat{X}}(D)_{\mathrm{naive}}$ as a constant diagram, we have the evident map of diagrams

$$\phi : \tau_\epsilon^{\hat{X}}(D) \to \tau_\epsilon^{\hat{X}}(D)_{\mathrm{naive}}$$

We were unable to determine if $\phi$ induces a weak equivalence after evaulation on $E \in \mathbf{Spt}(k)$, even assuming that $E$ is homotopy invariant and satisfies Nisnevich excision. We were also unable to construct such an $E$ for which $\phi$ fails to be a weak equivalence. □

6.3. The punctured tubular neighborhood. To define the punctured tubular neighborhood $\tau_\epsilon^{\hat{X}}(D)^0$, we proceed as follows: Fix a subset $I \subset \{1, \ldots, m\}$, $I \neq \emptyset$. Let $p : X' \to X$, $s : D_I \to X'$ be a Nisnevich neighborhood of $D_I$ in $X$, and let $D_{X'} = p^{-1}(D)$. Sending $X' \to X$ to $\Delta^n_{D_{X'}}$ gives us the pro-scheme $\hat{\Delta}^n_{D\subset X, D_I}$, and the closed immersion $\hat{\Delta}^n_{D\subset X, D_I} \to \hat{\Delta}^n_{X, D_I}$. Varying $n$, we have the cosimplicial pro-scheme $\hat{\Delta}^*_{D\subset X, D_I}$, and the closed immersion $\hat{\Delta}^*_{D\subset X, D_I} \to \hat{\Delta}^*_{X, D_I}$.

Take a closed subset $F \subset D_I$, and let $U := D_I \setminus F$. As in the definition of the punctured tubular neighborhood of a smooth closed subscheme in section 4.1, we pass to the appropriate cofinal subcategory of Nisnevich neighborhoods to show that the open complements $\hat{\Delta}^n_{X\setminus F, U} \setminus \hat{\Delta}^n_{D\setminus F\subset X\setminus F, U}$ for varying $n$ form a cosimplicial pro-scheme

$$n \mapsto \hat{\Delta}^n_{X\setminus F, U} \setminus \hat{\Delta}^n_{D\setminus F\subset X\setminus F, U}.$$

Similarly, we set

$$\tau_\epsilon^{\hat{X}}(D, D_I)^0(U) := \hat{\Delta}^*_{X\setminus F, U} \setminus \hat{\Delta}^*_{D\setminus F\subset X\setminus F, U}.$$

This forms the co-presheaf $\tau_\epsilon^{\hat{X}}(D, D_I)^0$ on $D_{I\mathrm{Zar}}$. The open immersions

$$\hat{j}_I(U)^n : \hat{\Delta}^n_{X\setminus F, U} \setminus \hat{\Delta}^n_{D\setminus F\subset X\setminus F, U} \to \hat{\Delta}^n_{X\setminus F, U}$$

define the map

$$\hat{j}_I(U) : \tau_\epsilon^{\hat{X}}(D, D_I)^0(U) \to \tau_\epsilon^{\hat{X}}(D_I)(U),$$

giving the map of co-presheaves

$$\hat{j}_I : \tau_\epsilon^{\hat{X}}(D, D_I)^0 \to \tau_\epsilon^{\hat{X}}(D_I).$$

For $J \subset I$, we have the map $\hat{\iota}_{J,I} : \hat{\Delta}^*_{X,D_I} \to \hat{\Delta}^*_{X,D_J}$ and

$$\hat{\iota}^{-1}_{J,I}(\hat{\Delta}^*_{D\subset X,D_J}) = \hat{\Delta}^*_{D\subset X,D_I}.$$

Thus we have the map $\hat{\iota}^0_{J,I} : \tau^{\hat{X}}_\epsilon(D, D_I)^0 \to \tau^{\hat{X}}_\epsilon(D, D_J)^0$ and the diagram of co-presheaves on $D_{\mathrm{Zar}}$

$$I \mapsto \iota_{I*}(\tau^{\hat{X}}_\epsilon(D, D_I)^0) \tag{6.3.1}$$

which we denote by $\tau^{\hat{X}}_\epsilon(D)^0$. The maps $\hat{j}_I$ define the map

$$\hat{j} : \tau^{\hat{X}}_\epsilon(D)^0 \to \tau^{\hat{X}}_\epsilon(D).$$

The projection maps $\pi_I : \tau^{\hat{X}}_\epsilon(D_I) \to X$ (where we consider $X$ as the constant co-presheaf on $D_{I\mathrm{Zar}}$) restrict to maps $\pi^0_I : \tau^{\hat{X}}_\epsilon(D, D_I)^0 \to X \setminus D$, which in turn induce the map

$$\pi^0 : \tau^{\hat{X}}_\epsilon(D)^0 \to X \setminus D,$$

where we consider $X \setminus D$ the constant diagram of constant co-presheaves on $D_{\mathrm{Zar}}$.

Definition 6.3.1. For $E \in \mathbf{Spt}(k)$, let $E(\tau^{\hat{X}}_\epsilon(D)^0)$ be the presheaf on $D_{\mathrm{Zar}}$,

$$E(\tau^{\hat{X}}_\epsilon(D)^0) := \operatorname*{holim}_{\emptyset\neq I\subset\{1,\ldots,m\}} \iota_{I*}E(\tau^{\hat{X}}_\epsilon(D, D_I)^0).$$

□

The map $\hat{j}$ defines the map of presheaves

$$\hat{j}^* : E(\tau^{\hat{X}}_\epsilon(D)) \to E(\tau^{\hat{X}}_\epsilon(D)^0).$$

We let $E^{D_{\mathrm{Zar}}}(\tau^{\hat{X}}_\epsilon(D))$ denote the homotopy fiber of $\hat{j}^*$. Via the commutative diagram

$$\begin{tikzcd} E(X \setminus F) \arrow[r, "j^*"] \arrow[d, "\pi^*"'] & E(X \setminus D) \arrow[d, "\pi^{0*}"] \\ E(\tau^{\hat{X}}_\epsilon(D))(D \setminus F) \arrow[r, "\hat{j}^*"'] & E(\tau^{\hat{X}}_\epsilon(D))^0(D \setminus F) \end{tikzcd}$$

we have the canonical map

$$\pi^*_D : E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\tau^{\hat{X}}_\epsilon(D)).$$

We want to show that the map $\pi^*_D$ is a weak equivalence, assuming that $E$ is homotopy invariant and satisfies Nisnevich excision. We first consider a simpler situation. We begin by noting the following

Lemma 6.3.2. *Let $\square^n_0$ denote the category of non-empty subsets of $\{1,\ldots,n\}$ with maps the inclusions, let $\mathcal{C}$ be a small category and let $F : \mathcal{C} \times \square^n_0 \to \mathbf{Spt}^{\mathbf{Ord}^{\mathrm{op}}}$ be a functor. Let $\operatorname{holim}_{\square^n_0} F : \mathcal{C} \to \mathbf{Spt}^{\mathbf{Ord}^{\mathrm{op}}}$ be the functor with* *value the simplicial spectrum* $m \mapsto \operatorname{holim}_{\square_0^n} F(i \times [m])$ *at* $i \in \mathcal{C}$. *There is a isomorphism*

$$\mathrm{Tot}(\operatorname*{holim}_{\square_0^n} F) \to \operatorname*{holim}_{\square_0^n} \mathrm{Tot}(F).$$

*in* $\mathcal{H}\mathbf{Spt}(\mathcal{C}^{\mathrm{op}})$.

*Proof.* Letting $\square^n$ be the category of all subsets of $\{1,\ldots,n\}$ (including the empty set), we may extend $F$ to $F* : \square^n \to \mathbf{Spt}(\mathbf{Ord}^{\mathrm{op}})$ by $F * (\emptyset) = 0$. Similarly, given a functor $G : \square^n \to \mathbf{Spt}$, we may extend $G$ to $G_\natural : \square_0^{n+1} \to \mathbf{Spt}$ by $G_\natural(I) = 0$, $G_\natural(I \cup \{n+1\}) = G(I)$ for $I \subset \{1,\ldots,n\}$. We define the *iterated homotopy fiber* of $G$, $\mathrm{fib}_n G \in \mathbf{Spt}$, by

$$\mathrm{hofib}_n(G) := \operatorname*{holim}_{\square_0^{n+1}} G_\natural.$$

One easily checks that for a map $g : A \to B$ of spectra, considered in the evident manner as a functor $g_1 : \square^1 \to \mathbf{Spt}$, we have $\mathrm{hofib} g = \mathrm{hofib}_1 g_1$. More generally, if we let $i_-, i_+ : \square^{n-1} \to \square^n$ be the inclusions

$$i_-(I) := I,\ i_+(I) := I \cup \{n\}$$

we have the evident natural transformation $\omega : i_- \to i_+$ and for $G : \square^n \to \mathbf{Spt}$ a functor, we have a natural isomorphism

$$\mathrm{hofib}(\mathrm{hofib}_{n-1} G \circ i_- \xrightarrow{\mathrm{hofib}_{n-1} G(\omega)} \mathrm{hofib}_{n-1} G \circ i_+) \cong \mathrm{hofib}_n G,$$

hence the name iterated homotopy fiber. Finally, one has the natural isomorphism

$$\mathrm{hofib}_n G* \cong \Omega \operatorname*{holim}_{\square_0^n} G$$

for $G : \square_0^n \to \mathbf{Spt}$.

Since Tot is compatible with suspension we may replace our original functor $F$ with $\Sigma F \cong \Omega^{-1} F$; using induction on $n$, it suffices to show that there is a natural isomorphism in $\mathcal{H}\mathbf{Spt}(\mathcal{C}^{\mathrm{op}})$

$$\mathrm{Tot}(\mathrm{hofib} F) \to \mathrm{hofib}\mathrm{Tot}(F)$$

for $F : A \to B$ a map in $\mathbf{Spt}^{\mathcal{C} \times \mathbf{Ord}^{\mathrm{op}}}$.

For this, note that for $f : X \to Y$ a map of spectra, there is a natural weak equivalence

$$a(f) : \Sigma \mathrm{hofib} f \to \mathrm{hocofib} f$$

Since Tot commutes with suspension and preserves weak equivalences, it suffices to define a natural weak equivalence

$$\mathrm{Tot}(\mathrm{hocofib} f) \to \mathrm{hocofib}(\mathrm{Tot} f).$$

In fact, since Tot preserves cofiber squares and is compatible with the wedge action of pointed simplicial sets on $\mathbf{Spt}^{\mathbf{Ord}^{\mathrm{op}}}$ and $\mathbf{Spt}$, there is a natural isomorphism $\mathrm{Tot}(\mathrm{hocofib} f) \to \mathrm{hocofib}(\mathrm{Tot} f)$, completing the proof. $\square$

This lemma allows us to define a simplicial model for $E^{D_{\rm Zar}}(\tau_\epsilon^{\hat{X}}(D))$, induced by the cosimplicial structure on the co-presheaves $\tau_\epsilon^{\hat{X}}(D_I)$ and $\tau_\epsilon^{\hat{X}}(D_I)^0$. In fact, let

$$E(\tau_\epsilon^{\hat{X}}(D))_n := \operatorname*{holim}_{I\neq\emptyset} \iota_{I*}E(\hat{\Delta}^n_{\hat{X},D_{I\rm Zar}})$$
$$E(\tau_\epsilon^{\hat{X}}(D)^0)_n := \operatorname*{holim}_{I\neq\emptyset} \iota_{I*}E(\hat{\Delta}^n_{\hat{X},D_{I\rm Zar}} \setminus \hat{\Delta}^n_{D\subset X,D_{I\rm Zar}})$$

and set

$$E^{D_{\rm Zar}}(\tau_\epsilon^{\hat{X}}(D))_n := \mathrm{hofib}(\hat{j}_n^* : E(\tau_\epsilon^{\hat{X}}(D))_n \to E(\tau_\epsilon^{\hat{X}}(D)^0)_n).$$

It follows from lemma 6.3.2 that $E(\tau_\epsilon^{\hat{X}}(D))$, $E(\tau_\epsilon^{\hat{X}}(D)^0)$ and $E^{D_{\rm Zar}}(\tau_\epsilon^{\hat{X}}(D))$ are isomorphic in the homotopy category to the total presheaves of spectra associated to the simplical presheaves

$$n \mapsto E(\tau_\epsilon^{\hat{X}}(D))_n$$
$$n \mapsto E(\tau_\epsilon^{\hat{X}}(D)^0)_n$$
$$n \mapsto E^{D_{\rm Zar}}(\tau_\epsilon^{\hat{X}}(D))_n$$

respectively. The map $\pi_D^*$ is defined by considering $E^{D_{\rm Zar}}(X)$ as a constant simplicial object. Let

$$\pi_{D,0}^* : E^{D_{\rm Zar}}(X) \to E^{D_{\rm Zar}}(\tau_\epsilon^{\hat{X}}(D))_0$$

be the map of $E^{D_{\rm Zar}}(X)$ to the 0-simplices of $E^{D_{\rm Zar}}(\tau_\epsilon^{\hat{X}}(D))$.

Proposition 6.3.3. *Suppose that $E$ satisfies Nisnevich excision and $D$ is a strict normal crossing subscheme of $X$. Then $\pi_{D,0}^*$ is a weak equivalence.*

Before we give the proof of this result, we prove two preliminary lemmas.

Lemma 6.3.4. *Let $x$ be a point on a finite type $k$-scheme $X$, let $Y = \operatorname{Spec}\mathcal{O}_{X,x}$ and $Z$ and $W$ be closed subschemes of $Y$. Then $\hat{Y}_Z^h \times_Y W \cong \hat{W}^h_{Z\cap W}$.*

*Proof.* Since $Y$ and $W$ are local, the pro-schemes $\hat{Y}_Z^h$ and $\hat{W}_{Z\cap W}$ are represented by local $Y$-schemes. If $Y' \to Y, s : Z \to Y'$ is a Nisnevich neighborhood of $Z$ in $Y$, and $i : Z\cap W \to W$ is the inclusion, then $Y' \times_Y W \to W$, $(s_{|Z\cap W}, i) : Z\cap W \to Y'\times_Y W$ is a Nisnevich neighborhood of $Z\cap W$ in $W$, giving us the $W$-morphism

$$f : \hat{W}^h_{Z\cap W} \to \hat{Y}^h_Z \times_Y W.$$

As $W$ is local, we have a co-final family in the category of all finite type étale morphisms $W' \to W$ of the form $W' = \operatorname{Spec}(\mathcal{O}_W[T]/F)_G$, i.e., the localization of $\mathcal{O}_W[T]/F$ with respect to some $G \in \mathcal{O}[T]$, where $(\partial F/\partial T, F)$ is the unit ideal in $\mathcal{O}_W[T]_G$. Those $W' \to W$ of this form which give a Nisnevich neighborhood of $Z\cap W$ are those for which $F$ contains a linear factor, modulo the ideal $I_{Z\cap W}$ of $Z\cap W$. Each such pair $(F,G)$ lifts to a pair $(\tilde{F},\tilde{G})$ of elements in $\mathcal{O}_Y[T]$

such that $\mathrm{Spec}\,(\mathcal{O}_Y[T]/\tilde{F})_{\tilde{G}} \to Y$ is étale, and such that the linear factor in $F$ mod $I_{Z\cap W}$ lifts to a linear factor of $\tilde{F}$ mod $I_Z$. This easily implies that $f$ is an isomorphism. $\square$

Let $i : W \to Y$ be a closed immersion of finite type $k$-schemes, $E \in \mathbf{Spt}(Y_{\mathrm{Zar}})$. Define the functor

$$i^! : \mathbf{Spt}(Y_{\mathrm{Zar}}) \to \mathbf{Spt}(W_{\mathrm{Zar}})$$

by

$$(i^! E)(W \setminus F) := \mathrm{hofib}(E(Y \setminus F) \to E(Y \setminus W))$$

for each $F \subset W$ closed.

For each $I \subset \{1, \ldots, m\}$, let $\iota_I : D_I \to D$ be the inclusion. For $J \subset I$, and $F \subset D$ closed, the diagram of restriction maps

$$\begin{tikzcd} E(D \setminus (D_I \cap F)) \arrow[r] \arrow[d] & E(D \setminus D_I) \arrow[d] \\ E(D \setminus (D_J \cap F)) \arrow[r] & E(D \setminus D_J) \end{tikzcd}$$

gives the map

$$\iota_{I*}\iota_I^! E \to \iota_{J*}\iota_J^! E$$

Lemma 6.3.5. *Suppose $E \in \mathbf{Spt}(D_{\mathrm{Zar}})$ is satisfies Zariski excision. Then the evident map*

$$\operatorname*{hocolim}_{I \in \square_0^{n\,\mathrm{op}}} \iota_{I*}\iota_I^! E \to E$$

*is a pointwise weak equivalence.*

*Proof.* Suppose temporarily that $D$ is an arbitrary finite type $k$-scheme, written as a union of two closed subschemes: $D = D^1 \cup D^2$, and take an $E \in \mathbf{Spt}(D_{\mathrm{Zar}})$ which is additive. Let $D^{12} := D^1 \cap D^2$, with inclusions $\iota^j : D^j \to D$, $\iota^{12} : D^{12} \to D$, $\iota^{j,12} : D^{12} \to D^j$. We have the natural map

$$\mathrm{hocolim}\left[\begin{tikzcd} \iota_{12*}\iota^{!12}E \arrow[r, "\iota_*^{1,12}"] \arrow[d, "\iota_*^{2,12}"'] & \iota_*^1\iota^{1!}E \\ \iota_*^2\iota^{2!}E & \end{tikzcd}\right] \xrightarrow{\alpha} E$$

We first show that $\alpha$ is a pointwise weak equivalence. It suffices to show that $\alpha$ is a weak equivalence on global sections, equivalently, that the diagram

$$\begin{tikzcd} E^{D^{12}}(D) \arrow[r] \arrow[d] & E^{D^1}(D) \arrow[d] \\ E^{D^2}(D) \arrow[r] & E(D) \end{tikzcd}$$

is homotopy cocartesian.

The homotopy cofiber of $E^{D^1}(D) \to E(D)$ is homotopy equivalence to $E(D \setminus D^1)$ and the homotopy cofiber of $E^{D^{12}}(D) \to E^{D^2}(D)$ is homotopy equivalent to $E^{D^2 \setminus D^{12}}(D \setminus D^{12})$. Since

$$D \setminus D^{12} = D^1 \setminus D^{12} \amalg D^2 \setminus D^{12}$$

and $E$ is additive, the map on the homotopy cofibers is a weak equivalence, as desired.

The proof of the lemma now follows easily by induction on the number $m$ of irreducible components of $D = \cup_{i=1}^m D_i$. Indeed, write $D = D^1 \cup D^2$, with $D^1 = D_1$ and $D^2 = \cup_{i=2}^m D_i$. Note that the Zariski excision property is preserved by the functor $i^!$ and that a presheaf that satisfies Zariski excision is additive. By induction the maps

$$\operatorname*{hocolim}_{\emptyset \neq I \subset \{2,\ldots,m\}} \iota_{I*}\iota_I^! E \to \iota_*^2 \iota^{2!} E$$

$$\operatorname*{hocolim}_{\{1\} \subsetneq I \subset \{1,\ldots,m\}} \iota_{I*}\iota_I^! E \to \iota_*^{12} \iota^{12!} E$$

are pointwise weak equivalences. Thus the map

$$\operatorname*{hocolim}_{I \in \square_0^{nop}} \iota_{I*}\iota_I^! E \to \operatorname{hocolim} \left[ \begin{array}{ccc} \iota_{12*}\iota^{!12}E & \xrightarrow{\iota_*^{1,12}} & \iota_*^1 \iota^{1!} E \\ {\scriptstyle \iota_*^{2,12}}\downarrow & & \\ \iota_*^2 \iota^{2!} E & & \end{array} \right]$$

is a pointwise weak equivalence; combined with our previous computation, this proves the lemma. □

*Proof of proposition 6.3.3.* Write $D$ as a sum, $D = \sum_{i=1}^m D_i$ with each $D_i$ smooth (but not necessarily irreducible), and with $m$ minimal. We proceed by induction on $m$.

For $m = 1$, Nisnevich excision implies that the natural map

$$E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\hat{X}_D^h)$$

is a weak equivalence in $\mathbf{Spt}(D_{\mathrm{Zar}})$. Since $D$ is smooth, the map $E^{D_{\mathrm{Zar}}}(\hat{X}_D^h) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_0$ is an isomorphism, which proves the result in this case.

By lemma 6.3.5 it suffices to show that $\iota_I^!(\pi_{D,0}^*)$ is a weak equivalence for all $I$. More generally, let $\iota_{I,J} : D_I \to D_J$ be the inclusion for $I \subset J$. If $E$ satisfies Zariski excision on $D_{\mathrm{Zar}}$, the same holds for $\iota_I^! E$ on $D_{I,\mathrm{Zar}}$ and there is a natural weak equivalence

$$\iota_{J,I}^!(\iota_I^! E) \to \iota_J^! E$$

Thus it suffices to show that $\iota_i^!(\pi_{D,0}^*)$ is a weak equivalence for all $i \in \{1, \ldots, m\}$, e.g., for $i = m$. In what follows, we will only apply $\iota_I^!$ to presheaves $E$ which satisfy Zariski excision, which suffices for the proof.

We use the following notation: for $W \subset D_I$ a closed subset, we let $E^{W_{\mathrm{Zar}}}(X)$ denote the presheaf on $D_I$

$$E^{W_{\mathrm{Zar}}}(X)(D_I \setminus F) := E^{W \setminus F}(X \setminus F).$$

We use the same notation for the presheaf

$$D \setminus F \mapsto E^{W \setminus F}(X \setminus F)$$

on $D_{\mathrm{Zar}}$, relying on the context to make the meaning clear.
Clearly $\iota_m^! \iota_{m*} E^{D_{m\mathrm{Zar}}}(X) \to E^{D_{m\mathrm{Zar}}}(X)$ is a weak equivalence and the map $E^{D_{m\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(X)$ induces a weak equivalence $\iota_m^! E^{D_{m\mathrm{Zar}}}(X) \to \iota_m^! E^{D_{\mathrm{Zar}}}(X)$, so we need to show that

$$E^{D_{m\mathrm{Zar}}}(X) \to \iota_m^! E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_0 = \iota_m^! (\operatorname*{holim}_{I \neq \emptyset} E^{\hat{X}_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}_{D_I})$$

is a weak equivalence.
For this, we decompose the set of non-empty $I \subset \{1, \ldots, m\}$ into three sets:

1. $I = \{m\}$,
2. $I$ with $m \notin I$,
3. $I$ with $\{m\} \subsetneqq I$.

Let

$$E_1 := \iota_m^! E^{\hat{X}^h_{D_m} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_m})$$

$$E_2 := \operatorname*{holim}_{m \notin I} \iota_m^! E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I})$$

$$E_3 := \operatorname*{holim}_{\{m\} \subsetneqq I} \iota_m^! E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I})$$

We thus have the isomorphism

$$\iota_m^! \left( \operatorname*{holim}_{I \neq \emptyset} E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I}) \right) \cong \operatorname{holim} \left[ \begin{array}{ccc} & & E_1 \\ & & \downarrow \\ E_2 & \longrightarrow & E_3 \end{array} \right]$$

For $I$ of type 2, lemma 6.3.4 says that the natural map

$$\hat{X}^h_{D_{I \cup \{m\}}} \times_X D_m \to \hat{X}^h_{D_I} \times_X D_m$$

is an isomorphism. Since the restriction map

$$\iota_m^! E^{\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I}) \to \iota_m^! E^{\hat{X}^h_{D_{I \cup \{m\}}} \times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_{I \cup \{m\}}})$$

identifies itself with the pull-back

$$E^{\hat{X}^h_{D_I} \times_X D_{m\mathrm{Zar}}}(\hat{X}^h_{D_I}) \to E^{\hat{X}^h_{D_{I \cup \{m\}}} \times_X D_{m\mathrm{Zar}}}(\hat{X}^h_{D_{I \cup \{m\}}})$$

the Nisnevich excision property of $E$ implies that $E_2 \to E_3$ is a weak equivalence. Thus

$$E_1 \to \operatorname{holim}\left[\begin{array}{ccc} & & E_1 \\ & & \downarrow \\ E_2 & \longrightarrow & E_3 \end{array}\right]$$

is a weak equivalence, and

$$E^{D_m\mathrm{Zar}}(X) \to \iota_m^! E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))_0 = \iota_m^! \left(\operatorname*{holim}_{I\neq\emptyset} E^{\hat{X}^h_{D_I}\times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_I})\right)$$

is identified with

$$E^{D_m\mathrm{Zar}}(X) \to \iota_m^! E^{\hat{X}^h_{D_m}\times_X D_{\mathrm{Zar}}}(\hat{X}^h_{D_m}) = E^{\hat{X}^h_{D_m}\times_X D_m\mathrm{Zar}}(\hat{X}^h_{D_m}),$$

which is a weak equivalence by Nisnevich excision. □

Proposition 6.3.6. *Suppose that $E$ is homotopy invariant and satisfies Nisnevich excision, and $D$ is a strict normal crossing subscheme of $X$. Then*

$$\pi_D^* : E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))$$

*is a weak equivalence in* $\mathbf{Spt}(D_{\mathrm{Zar}})$.

*Proof.* Let $p_n : \Delta^n_D \to D$ be the projection. Applying Proposition 6.3.3 to the strict normal crossing subscheme $\Delta^n_D$ of $\Delta^n_X$, the map

$$\pi_{\Delta^n_D,0} : p_{n*}E^{\Delta^n_{D_{\mathrm{Zar}}}}(\Delta^n_X) \to p_{n*}E^{\Delta^n_{D_{\mathrm{Zar}}}}(\tau_\epsilon^{\widehat{\Delta^n_X}}(\Delta^n_D))_0$$

is a weak equivalence for each $n$. Thus

$$\pi_{\Delta^*_D} : p_*E^{\Delta^*_{D_{\mathrm{Zar}}}}(\Delta^*_X) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D))$$

is a weak equivalence. Indeed, $E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D)$ is a simplicial object with $n$-simplices $p_{n*}E^{\Delta^n_{D_{\mathrm{Zar}}}}(\tau_\epsilon^{\widehat{\Delta^n_X}}(\Delta^n_D))_0$. Since $E$ is homotopy invariant, the map

$$p^* : E^{D_{\mathrm{Zar}}}(X) \to p_*E^{\Delta^*_{D_{\mathrm{Zar}}}}(\Delta^*_X)$$

is a weak equivalence, whence the result. □

We can now state and prove the main result for strict normal crossing schemes.

Theorem 6.3.7. *Let $D$ be a strict normal crossing scheme on some $X \in \mathbf{Sm}/k$ and take $E \in \mathbf{Spt}(k)$ which is homotopy invariant and satisfies Nisnevich excision. Then there is a natural distinguished triangle in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$*

$$E^{D_{\mathrm{Zar}}}(X) \xrightarrow{\alpha_D} E(D_\bullet) \xrightarrow{\beta_D} E(\tau_\epsilon^{\hat{X}}(D)^0)$$

*Proof.* By proposition 6.3.6, we have the weak equivalence

$$\pi_D^* : E^{D_{\mathrm{Zar}}}(X) \to E^{D_{\mathrm{Zar}}}(\tau_\epsilon^{\hat{X}}(D)).$$

By Proposition 6.2.2, we have the isomorphism

$$(p_\bullet^*)^{-1}\hat{i}_D^* : E(\tau_\epsilon^{\hat{X}}(D)) \to E(D_\bullet).$$

in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$. Since $E^D(\tau_\epsilon^{\hat{X}}(D))$ is by definition the homotopy fiber of the restriction map $E(\tau_\epsilon^{\hat{X}}(D)) \to E(\tau_\epsilon^{\hat{X}}(D)^0)$, the result is proved. □

## 7. Comparison isomorphisms

We give a comparison of our tubular neighborhood construction with the categorical version $Li^*Rj_*$ of Morel-Voevodsky.

7.1. Model structure and cross functors. Fix a noetherian separated scheme $S$ of finite Krull dimension, and let $\mathbf{Sch}_S$ denote the category of finite type $S$-schemes (for our application, we will take $S = \operatorname{Spec} k$ for a field $k$). Morel-Voevodsky show how to make the catgory $\mathcal{SH}_{\mathbb{A}^1}(X)$ functorial in $X \in \mathbf{Sch}_S$, defining an adjoint pair of exact functors $Lf^*, Rf_*$ for each morphism $f : Y \to X$ in $\mathbf{Sch}_S$. Roendigs shows in [39] how to achieve this on the model category level and in addition that this structure extends to give cross functors $(f_*, f^*, f^!, f_!)$ as defined by Voevodsky and investigated in detail by Ayoub [3]. We begin by describing the model structure used by Roendigs, which is different from the one we have used up to now, and recalling his main results. For $B \in \mathbf{Sch}_S$, we denote by $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ the model structure on $\mathbf{Spc}_*(\mathbf{Sm}/B)$ described by Roendigs in [39]. To describe this model structure, we first recall the projective model structure $\mathbf{Spc}(\mathbf{Sm}/B)_{\mathrm{proj}}$ on $\mathbf{Spc}(\mathbf{Sm}/B)$. Here the weak equivalences and fibrations are the pointwise ones and the cofibrations are generated by the maps

$$Z \times \partial\Delta^n \to Z \times \Delta^n,$$

with $Z \in \mathbf{Sm}/B$. This induces a model structure $\mathbf{Spc}_*(\mathbf{Sm}/B)_{\mathrm{proj}}$ on $\mathbf{Spc}_*(\mathbf{Sm}/B)$ by forgetting/adjoining a base-point. One has a functorial cofibrant replacement $E^c \to E$ defined as in [34, Lemma 1.16].

The model structure $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ is defined by Bousfield localization: the cofibrations are the same as in $\mathbf{Spc}_*(\mathbf{Sm}/B)_{\mathrm{proj}}$. $E$ is fibrant if $E(\emptyset)$ is contractible, $E$ is a fibrant in $\mathbf{Spc}_*(\mathbf{Sm}/B)_{\mathrm{proj}}$, $E$ transforms elementary Nisnevich squares to homotopy fiber squares and transforms $Z \times \mathbb{A}^1 \to Z$ to a weak equivalence. A map $A \to B$ is a weak equivalence if $\mathcal{H}om(B^c, E) \to \mathcal{H}om(A^c, E)$ is a weak equivalence for each fibrant $E$. The fibrations in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ are determined by having the right lifting property with respect to trivial cofibrations.

Let $f : X \to Y$ be a morphism in $\mathbf{Sch}_S$. We have the functor

$$f_* : \mathbf{Spc}_*(\mathbf{Sm}/X) \to \mathbf{Spc}_*(\mathbf{Sm}/Y)$$

defined by pre-composition with the pull-back functor $- \times_Y X$, i.e.

$$f_*E(Y' \to Y) := E(Y' \times_Y X \to X).$$

$f_*$ has the left adjoint $f^*$ defined as the Kan extension, and characterized by $f^*(Y'_+) = Y' \times_Y X_+$ for $Y' \to Y \in \mathbf{Sm}/Y$. In case $f$ is a smooth morphism, $f^*$ is given by precomposition with the functor

$$f \circ - : \mathbf{Sm}/X \to \mathbf{Sm}/Y,$$

and thus has the left adjoint $f_\sharp$ characterized by

$$f_\sharp(Z \xrightarrow{p} X) = Z \xrightarrow{fp} Y$$

on the representable presheaves. We have

Proposition 7.1.1 (proposition 2.18 of [39]). *Let $f : X \to Y$ be a morphism in $\mathbf{Sch}_S$. Then $(f^*, f_*)$ is a Quillen adjoint pair $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/X) \leftrightarrow \mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/Y)$. If $f$ is smooth, then $(f_\sharp, f^*)$ is a Quillen adjoint pair $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/Y) \leftrightarrow \mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/X)$.*

For spectra, the projective model structure $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)_{\mathrm{proj}}$ on $\mathbf{Spt}(\mathbf{Sm}/B)$ is defined as follows: For $\phi : E \to F$ a morphism in $\mathbf{Spt}(\mathbf{Sm}/B)$, $\phi : E \to F$ is a cofibration if $\phi_0 : E_0 \to F_0$ is a cofibration in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ and if for each $n \geq 1$, the map

$$\phi_n \cup \Sigma\phi_{n-1} : E_n \cup_{\Sigma E_{n-1}} \Sigma F_{n-1} \to F_n$$

is a cofibration in $\mathbf{Spc}_*(\mathbf{Sm}/B)_{\mathrm{proj}}$. Weak equivalences (resp. fibrations) are maps $\phi$ such that $\phi_n$ is a weak equivalence (resp. fibration) in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ for all $n$. There is a functorial cofibrant replacement $E^c \to E$.
Now for the motivic model structure $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)$: The cofibrations are the same as in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)_{\mathrm{proj}}$. $\phi$ is a fibration if $\phi_n$ is a fibration in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ for all $n$ and the diagram

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, "\phi_n"'] & \Omega E_{n+1} \arrow[d, "\Omega\phi_{n+1}"] \\ F_n \arrow[r] & \Omega F_{n+1} \end{tikzcd}$$

is homotopy cartesian in $\mathbf{Spc}_{*\mathrm{mot}}(\mathbf{Sm}/B)$ for all $n$. There is a fibrant replacement functor $E \to E^f$; $\phi : E \to F$ is a weak equivalence if $\phi^f : E^f \to F^f$ is a weak equivalence in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)_{\mathrm{proj}}$.
Given $f : X \to Y$ in $\mathbf{Sch}_S$, define the functors $f_* : \mathbf{Spt}(\mathbf{Sm}/X) \to \mathbf{Spt}(\mathbf{Sm}/Y)$ and $f^* : \mathbf{Spt}(\mathbf{Sm}/Y) \to \mathbf{Spt}(\mathbf{Sm}/X)$ by $f_*(E)_n := f_*(E_n)$, $f^*(F)_n := f^*(F_n)$. If $f$ is smooth, we have $f_\sharp : \mathbf{Spt}(\mathbf{Sm}/X) \to \mathbf{Spt}(\mathbf{Sm}/Y)$ defined similarly by $f_\sharp(E)_n := f_\sharp(E_n)$.
We have the following result from [39]:

Proposition 7.1.2 (proposition 2.23 of [39]). *Let $f : X \to Y$ be a morphism in $\mathbf{Sch}_S$. Then $(f_*, f^*)$ is a Quillen adjoint pair $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X) \leftrightarrow \mathbf{Spt}(\mathbf{Sm}/Y)$. If $f$ is smooth, then $(f_\sharp, f^*)$ is a Quillen adjoint pair $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/Y) \leftrightarrow \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$. In particular:*

1. *$f^*$ preserves cofibrations and trivial cofibration and $f_*$ preserves fibrations and trivial fibrations.*
2. *if $f$ is smooth, then $f^*$ preserves fibrations and $f_\sharp$ preserves cofibrations*

It is clear that a cofibration in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ is pointwise a cofibration in $\mathbf{Spt}$, hence a cofibration in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$. As mentioned in [39] a fibrant object in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ satisfies both Nisnevich excision and is $\mathbb{A}^1$-local, hence the weak equivalences between fibrant objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ are weak equivalences in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$ and are in fact pointwise weak equivalences in $\mathbf{Spt}(\mathbf{Sm}/X)$; similarly one shows that each fibration in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$ is a fibration in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ and each (trivial) cofibration in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ is a (trivial) cofibration in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$. Thus the identity on $\mathbf{Spt}(\mathbf{Sm}/X)$ defines a (left) Quillen equivalence $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X) \to \mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$. In particular, we have the equivalence of the homotopy categories

$$\mathcal{H}\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}}) \cong \mathcal{H}\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X).$$

We write $\mathcal{SH}_{\mathbb{A}^1}(X)$ for either $\mathcal{H}\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$ or $\mathcal{H}\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/X_{\mathrm{Nis}})$, depending on the context.
One main result of [39] is

Theorem 7.1.3 ([39, corollary 3.17]). *Sending $f : Y \to X$ in $\mathbf{Sch}_S$ to $Lf^* : \mathcal{SH}_{\mathbb{A}^1}(X) \to \mathcal{SH}_{\mathbb{A}^1}(X)$ satisfies the conditions of* [3, definition 1.4.1]. *In particular, the properties of a "2-foncteur homotopique stable" described in* [3] *are satisfied for $X \mapsto \mathcal{SH}_{\mathbb{A}^1}(X)$.*

*Remark* 7.1.4. Let $i : D \to X$ be a closed immersion in $\mathbf{Sch}_S$ with open complement $j : U \to X$. We have the functor

$$Li^*Rj_* : \mathcal{SH}_{\mathbb{A}^1}(X \setminus D) \to \mathcal{SH}_{\mathbb{A}^1}(D),$$

We would like to view our construction $E(\tau_\epsilon^{\hat{X}}(D)^0)$ as a weak version of $Li^*Rj_*$, in case $D$ is a normal crossing divisor on a smooth $k$ scheme $X$, the input $E$ is the pull-back from $\mathbf{Spt}(\mathbf{Sm}/k)$, and the output $E(\tau_\epsilon^{\hat{X}}(D)^0)$ is in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$. In particular, $E(\tau_\epsilon^{\hat{X}}(D)^0)$ is only defined on Zariski open subsets of $D$, rather than on all of $\mathbf{Sm}/D$. In this section, we make this statement precise, defining an isomorphism of $E(\tau_\epsilon^{\hat{X}}(D)^0)$ with the restriction of $Li^*Rj_*(E)$ to $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$. □

7.2. The smooth case. Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_S$ with open complement $j : U \to X$. Let

$$\Theta : \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U) \to \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/W)$$

be the functor representing $Li^*Rj_*$, i.e.

$$\Theta(E) := i^*(j_*(E^f)^c)^f.$$

*Remark* 7.2.1. Even for $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$ bifibrant, one cannot simplify this expression for $Li^*Rj_*E$ beyond replacing $E^f$ with $E$. The inexplicit nature of the cofibrant and fibrant replacement functors make a concrete determination of $Li^*Rj_*E$ difficult, which is one advantage of our approach using the punctured tubular neighborhood. □

Lemma 7.2.2. *Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_S$ with open complement $j : U \to X$.*

(1) *For fibrant $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$ all the maps in the square*

$$\begin{tikzcd} j_*(E)^c \arrow[r] \arrow[d] & Rj_*(E)^c \arrow[d] \\ j_*(E) \arrow[r] & Rj_*(E) \end{tikzcd}$$

*are pointwise weak equivalences*

(2) *Let $X' \to X$ be in $\mathbf{Sm}/X$, let $W' := W \times_X X'$. There is a canonical map*

$$\nu^0_{X'} : Rj_*(E)^c(X') \to \Theta E(W')$$

*natural in $X'$.*

*Proof.* (1) Since $E$ is fibrant, the canonical map $E \to E^f$ is a trivial cofibration of fibrant objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$, hence a homotopy equivalence. Thus $j_*E \to Rj_*E := j_*(E^f)$ is a homotopy equivalence of fibrant objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$, hence a pointwise weak equivalence. Applying the cofibrant replacement functor, we see that $(j_*E)^c \to (Rj_*E)^c$ is also a homotopy equivalance and a pointwise weak equivalence. Also the cofibrant replacement maps $(j_*E)^c \to j_*E$, $(Rj_*E)^c \to Rj_*E$ are trivial fibrations between fibrant objects of $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$, hence are both pointwise weak equivalences.

For (2), the unit $\mathrm{id} \to i_*i^*$ for the adjunction applied to $(Rj_*E)^c$ gives us the map

$$\nu^0_{X'} : (Rj_*E)^c(X') \to i_*i^*(Rj_*E)^c(X')$$

natural in $X'$. As $i_*i^*(Rj_*E)^c(X') = i^*(Rj_*E)^c(W')$, we have the natural transformation

$$\nu^0_{X'} : (Rj_*E)^c(X') \to i^*(Rj_*E)^c(W')$$

Composing with the canonical map $i^*(Rj_*E)^c \to (i^*(Rj_*E)^c)^f = \Theta(E)$ gives us the map we want. □

For $E \in \mathbf{Spt}(\mathbf{Sm}/B)$ or in $\mathbf{Spt}(B_{\mathrm{Nis}})$, we let $E_{\mathrm{Zar}}$ denote the restriction to $\mathbf{Spt}(B_{\mathrm{Zar}})$. Identifying $\mathcal{SH}_{\mathbb{A}^1}(B)$ with the homotopy category of bifibrant objects in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/B)$, we have the similarly defined restriction functor $\mathcal{SH}_{\mathbb{A}^1}(B) \to \mathcal{H}\mathbf{Spt}(B_{\mathrm{Zar}})$ sending $E$ to $E_{\mathrm{Zar}}$.

Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ with open complement $j : U \to X$. We note that the "evaluation" maps

$$E \mapsto E(\tau^{\hat{X}}_\epsilon(W)),\ E \mapsto E(\tau^{\hat{X}}_\epsilon(W)^0)$$

are in fact defined for $E \in \mathbf{Spt}(\mathbf{Sm}/X)$. Similarly, the evaluation map $E \mapsto E(\tau^{\hat{X}}_\epsilon(W)^0)$ is defined for $E \in \mathbf{Spt}(\mathbf{Sm}/U)$. In addition, for $E \in \mathbf{Spt}(\mathbf{Sm}/U)$ we have a canonical isomorphism

$$E(\tau^{\hat{X}}_\epsilon(W)^0) \cong (j_*E)(\tau^{\hat{X}}_\epsilon(W)) \tag{7.2.1}$$

since $\hat{\Delta}^n_{X,W} \setminus \Delta^n_W \cong \hat{\Delta}^n_{X,W} \times_X U$ (as a pro-scheme).

Lemma 7.2.3. *Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ with open complement $j : U \to X$, and let $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$ be fibrant.*

(1) *There is a map*

$$\eta^c_E : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(W)) \to \Theta(E)_{\mathrm{Zar}}$$

*in* $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$*, natural in* $E$*.*

(2) *Let*

$$\begin{tikzcd} & j_*(E)^c(\tau^{\hat{X}}_\epsilon(W)) \arrow[r] \arrow[d] & Rj_*(E)^c(\tau^{\hat{X}}_\epsilon(W)) \arrow[d] \\ E(\tau^{\hat{X}}_\epsilon(W)^0) \arrow[r, "\sim", "\phi"'] & j_*(E)(\tau^{\hat{X}}_\epsilon(W)) \arrow[r] & Rj_*(E)(\tau^{\hat{X}}_\epsilon(W)) \end{tikzcd}$$

*be the diagram in* $\mathbf{Spt}(W_{\mathrm{Zar}})$ *formed by evaluating the diagram of lemma 7.2.2(1) at* $\tau^{\hat{X}}_\epsilon(W)$*, and adding the isomorphism* (7.2.1)*. Then all the maps in this diagram are pointwise weak equivalences.*

*Proof.* By lemma 7.2.2, we have maps

$$\eta^0_{X'} : (Rj_*E)^c(X') \to \Theta(E)(X' \times_X W)$$

natural in $X' \in \mathbf{Sm}/X$. For each open subscheme $U = D \setminus F \subset W$, the maps $\eta_{\hat{\Delta}^n_{X\setminus F,U}}$ define the map

$$\eta^0_{\Delta^*}(U) : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(W)(U)) \to \Theta(E)(\Delta^*_U).$$

Since $\Theta(E)$ is $\mathbb{A}^1$-local, the canonical map $\Theta(E)(U) \to \Theta(E)(\Delta^*_U)$ is a weak equivalence. This gives us the natural map in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\eta^0 : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(W)) \to \Theta(E)_{\mathrm{Zar}},$$

proving (1).
(2) follows immediately from lemma 7.2.2(1). □

Combining the morphism (1) with the diagram (2) gives us the canonical morphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\eta^0_E : E(\tau^{\hat{X}}_\epsilon(W)^0) \to \Theta(E)_{\mathrm{Zar}}.$$

Let $i : D \to X$ be a closed immersion in $\mathbf{Sch}_S$. We have the exact functor $i^! : \mathcal{SH}_{\mathbb{A}^1}(X) \to \mathcal{SH}_{\mathbb{A}^1}(D)$ which is characterized by the identity for fibrant $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$:

$$i_*i^!E(X' \to X) := \mathrm{hofib}(E(X') \to E(X' \times_X (X \setminus D)).$$

In fact, this operation gives the distinguished triangle, natural in fibrant $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$:

$$Ri_*i^!E \to E \to Rj_*j^*E \to i_*i^!E[1].$$

Applying $Li^*$ (and noting that the counit $Li^*Ri_* \to \mathrm{id}$ is an isomorphism [3, definition 1.4.1]) gives the distinguished triangle in $\mathcal{SH}_{\mathbb{A}^1}(D)$

$$i^!E \to Li^*E \to \Theta(j^*E) \to i^!E[1] \tag{7.2.2}$$

We refer the reader to [3, proposition 1.4.9] for the construction of this triangle in the abstract setting.

Proposition 7.2.4. *Let $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/k)$ be fibrant, let $f : X \to \operatorname{Spec} k$ be in $\mathbf{Sm}/k$ and let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$ with open complement $j : U \to X$. Then*

$$\eta^0_E : E(\tau^{\hat{X}}_\epsilon(W)^0) \to \Theta(j^*f^*E)_{\mathrm{Zar}}$$

*is an isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$.*

*Proof.* Let $f_W : W \to \operatorname{Spec} k$ be the structure morphism. Since $f$ and $fi = f_W$ are smooth, we have $Lf^* \cong f^*$, $f_W^* \cong L(fi)^* \cong Li^*f^*$, so $Li^*f^*$ is isomorphic to the restriction functor for $f_W \circ - : \mathbf{Sm}/W \to \mathbf{Sm}/k$. The definition of $i^!$ gives the commutative diagram for each $X' \to X$ in $\mathbf{Sm}/X$ (with $W' := X' \times_X W$)

$$\begin{tikzcd}
E^{W'}(X') \arrow[r] \arrow[d, "\phi_{X'}"'] & E(X') \arrow[d, "\eta_{X'}"] & \\
i^!f^*E(W') \arrow[r] & Li^*f^*E(W') \arrow[r, phantom, "\cong"] & E(W')
\end{tikzcd}$$

where $\eta_{X'}$ is just the restriction map $E(X') \to E(W')$ and $\phi_{X'}$ is the canonical isomorphism given by the definition of $i^!$. Using lemma 7.2.3, this gives us the map of distinguished triangles in $\mathcal{SH}$

$$\begin{tikzcd}
E^{W'}(X') \arrow[r] \arrow[d, "\phi_{X'}"'] & E(X') \arrow[r] \arrow[d, "\eta_{X'}"] & E(X' \setminus W') \arrow[r] \arrow[d, "\eta^0_{X'}"] & E^{W'}(X')[1] \arrow[d] \\
i^!f^*E(W') \arrow[r] & E(W') \arrow[r] & Li^*Rj_*j^*f^*E(W') \arrow[r] & i^!f^*E(W')[1]
\end{tikzcd}$$

Just as for $\eta^0_E$, these give rise to the natural map in $\mathcal{H}\mathbf{Spt}(\mathbf{Sm}/W_{\mathrm{Zar}})$

$$\eta_E : E(\tau^{\hat{X}}_\epsilon(W)) \to Li^*f^*E_{\mathrm{Zar}}$$

and the commutative diagram in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$

$$\begin{tikzcd}
E^{W_{\mathrm{Zar}}}(X) \arrow[r] \arrow[d, "\phi"'] & E(\tau^{\hat{X}}_\epsilon(W)) \arrow[r] \arrow[d, "\eta"] & E(\tau^{\hat{X}}_\epsilon(W)^0) \arrow[r] \arrow[d, "\eta^0"] & E^{W_{\mathrm{Zar}}}(X)[1] \arrow[d] \\
i^!f^*E_{\mathrm{Zar}} \arrow[r] & Li^*f^*E_{\mathrm{Zar}} \arrow[r] & Li^*Rj_*j^*f^*E_{\mathrm{Zar}} \arrow[r] & i^!f^*E_{\mathrm{Zar}}[1]
\end{tikzcd}$$

The bottom row is the distinguished triangle (7.2.2) for $f^*E$, restricted to $W_{\mathrm{Zar}}$, and the top row is the distinguished triangle of corollary 4.1.3, after applying theorem 3.2.1. Similarly, theorem 3.2.1 shows that $\eta$ is an isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$. Since $\phi$ is an isomorphism in $\mathcal{H}\mathbf{Spt}(W_{\mathrm{Zar}})$ $\eta^0$ is an isomorphism as well. $\square$

7.3. The normal crossing case. We fix a reduced strict normal crossing divisor $i : D \to X$ on some $X \in \mathbf{Sm}/k$. Write $D = \sum_{i=1}^m D_i$ with the $D_i$ smooth. For $X' \to X$ in $\mathbf{Sm}/X$, we write $D'$ for $X' \times_X D$ and $D'_I$ for $X' \times_X D_I$ and for $I \subset \{1, \ldots, m\}$. As in the previous section, we note that our definition of $E(\tau_\epsilon^{\hat{X}}(D))$ extends without change to $E \in \mathbf{Spt}(\mathbf{Sm}/X)$, and similarly, the construction of $E(\tau_\epsilon^{\hat{X}}(D)^0)$ extends without change to $E \in \mathbf{Spt}(\mathbf{Sm}/X \setminus D)$. The extension of proposition 7.2.4 to the normal crossing case follows essentially the same outline as before, with some additional patching results for the operation $Li^*Rj_*$ that allow us give a description of $Li^*Rj_*$ as a homotopy limit, matching our definition of $E(\tau_\epsilon^{\hat{X}}(W)^0)$.

Lemma 7.3.1. *Suppose that $F \in \mathbf{Spt}(\mathbf{Sm}/D)$ satisfies Nisnevich excision. For $I \subset \{1, \ldots, m\}$, $I \neq \emptyset$, let $F_I$ be the presheaf on $\mathbf{Sm}/X$*

$$F_I(X') := F(\hat{X}'^h_{D'_I} \times_X D).$$

*Then the canonical map*

$$i_*F \to \operatorname*{holim}_{I \neq \emptyset} F_I$$

*is a weak equivalence in $\mathbf{Spt}(\mathbf{Sm}/X)$.*

*Proof.* Let $\{U_j \to D' \mid j \in M\}$ be a Nisnevich cover of $D'$, with $M$ a finite set. For $J \subset M$, set $U_J := \prod_{j \in J} U_j$, where the product is $\times_{D'}$. Since $F$ satisfies Nisnevich excision, the canonical map

$$F(D') \to \operatorname*{holim}_{J \neq \emptyset} F(U_J)$$

is a weak equivalence. An argument similar to that of lemma 6.3.2 shows that one can replace the $U_i$ with a pro-system of Nisnevich covers (with $M$ fixed). Similarly, the Zariski stalk of $\operatorname{holim}_{I \neq \emptyset} F_I$ at $x \in X' \in \mathbf{Sm}/X$ is weakly equivalent to $\operatorname{holim}_{I \neq \emptyset} F(\hat{X}'^h_{x, D'_I} \times_X D)$, where $X'_x = \operatorname{Spec} \mathcal{O}_{X', x}$. Thus we need only show that for $X' \to X$ smooth, with $X'$ local, the schemes $U_i := \hat{X}'^h_{D'_i} \times_X D$ form a pro-Nisnevich cover of $D'$, and that

$$\prod_{i \in I} U_i \cong \hat{X}'^h_{D'_I} \times_X D$$

for each non-empty $I \subset \{1, \ldots, m\}$.
In fact the pro-schemes $\hat{X}'^h_{D'_i} \times_X D$, $i = 1, \ldots, m$, obvioiusly form a pro-Nisnevich cover of $D'$; it follows from lemma 6.3.4 that for each $I \subset \{1, \ldots, m\}$, $I \neq \emptyset$, we have natural isomorphisms (where $\prod$ is $\times_{X'}$)

$$\prod_{i \in I} \hat{X}'^h_{D'_i} \cong \hat{X}'^h_{D'_I}.$$

Thus (with the product over $D'$)

$$\prod_{i \in I} \hat{X}'^h_{D'_i} \times_X D \cong \hat{X}'^h_{D'_I} \times_X D.$$

∎

Lemma 7.3.2. *Let $i : D \to X$ be a strict normal crossing divisor on some $f : X \to \operatorname{Spec} k$ in $\mathbf{Sm}/k$, and let $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/U)$ be fibrant. Then there is a canonical map in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$,*

$$\eta^0_E : E(\tau^{\hat{X}}_\epsilon(W)^0) \to \Theta(E)_{\mathrm{Zar}},$$

*natural in $E$.*

*Proof.* As in the smooth case, we construct $\eta^0_E$ using lemmas 7.2.2 and 7.3.1. Indeed, let $j : X \setminus D \to X$ be the inclusion. Let $\Theta(E)_{I\mathrm{Zar}}$ denote the pull-back of $\Theta(E)$ to $\mathbf{Spt}(\hat{X}^h_{D_I} \times_X D_{\mathrm{Zar}})$. Let $\Theta(E)^*_{I\mathrm{Zar}}$ be the presheaf

$$\Theta(E)^*_{I\mathrm{Zar}}(U) := \Theta(E)_{I\mathrm{Zar}}(\Delta^*_U).$$

Similarly, let $\Theta(E)^*_{\mathrm{Zar}}$ denote the presheaf on $D_{\mathrm{Zar}}$

$$\Theta(E)^*_{\mathrm{Zar}}(U) := \Theta(E)(\Delta^*_U)$$

and let $\Theta(E)_{\mathrm{Zar}}$ denote the restriction of $\Theta(E)$ to $D_{\mathrm{Zar}}$.
The construction of lemma 7.2.3 gives us the diagram of maps

$$\tilde{\eta}^0_{E,I} : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(D_I)) \to \Theta(E)^*_{I\mathrm{Zar}}$$

and thus the map

$$\tilde{\eta}^0_E : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(D)) \to \operatorname*{holim}_{I \neq \emptyset}(I \mapsto \Theta(E)^*_{I\mathrm{Zar}})$$

By lemma 7.3.1 we have the canonical isomorphism in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$

$$\operatorname*{holim}_{I \neq \emptyset}(I \mapsto \Theta(E)^*_{I\mathrm{Zar}}) \cong \Theta(E)^*_{\mathrm{Zar}}.$$

Since $\Theta(E)$ is $\mathbb{A}^1$-homotopy invariant, the canonical map $\Theta(E)_{\mathrm{Zar}} \to \Theta(E)^*_{\mathrm{Zar}}$ is a pointwise weak equivalence, giving us the map in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$

$$\tilde{\eta}^0_E : (Rj_*E)^c(\tau^{\hat{X}}_\epsilon(D)) \to \Theta(E)_{\mathrm{Zar}}$$

Using the diagram of lemma 7.2.3, with $W = D_I$, and then taking the appropriate homotopy limit, we arrive at a canonical isomorphism in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$

$$(Rj_*E)^c(\tau^{\hat{X}}_\epsilon(D)) \cong E(\tau^{\hat{X}}_\epsilon(D)^0).$$

Combining $\tilde{\eta}^0_E$ with this isomorphism gives us the desired map $\eta^0_E$. □

Lemma 7.3.3. *Let $i : W \to X$ be a closed immersion in $\mathbf{Sch}_S$. Suppose $W$ is a union of closed subschemes, $W = W_1 \cup W_2$. Let $W_{12} := W_1 \cap W_2$ and let $i_j : W_j \to X$, $j = 1, 2$, $i_{12} : W_{12} \to X$ be the inclusions. Then for $E \in \mathcal{SH}_{\mathbb{A}^1}(\mathbf{Sm}/X)$ there is a canonical homotopy cartesian diagram in $\mathcal{SH}_{\mathbb{A}^1}(\mathbf{Sm}/X)$*

$$\begin{array}{ccc} Ri_*Li^*E & \longrightarrow & Ri_{1*}Li_1^*E \\ \downarrow & & \downarrow \\ Ri_{2*}Li_2^*E & \longrightarrow & Ri_{12*}L^*_{12}E \end{array}$$

*Proof.* Throughout the proof we use the canonical lifting of $Li^*$, $Ri_*$, etc., to functors on $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/-)$ by taking the appropriate cofibrant/fibrant replacement, but we use the same notation to denote these liftings.
Let $\iota : W_1 \to W$ be the inclusion. The unit $\mathrm{id} \to R\iota_* L\iota^*$ gives the map

$$Li^*E \to R\iota_* L\iota^* Li^*E \cong R\iota_* Li_1^*E$$

in $\mathcal{SH}_{A^1}(\mathbf{Sm}/W)$; applying $Ri_*$ gives the map $Ri_*Li^*E \to Ri_{1*}Li_1^*E$. The other maps in the square are defined similarly; as the two compositions $Ri_*Li^*E \to Ri_{12*}L_{12}^*E$ are likewise defined by the adjoint property, these agree and the diagram commutes.
To show that the diagram is homotopy cartesian, let $j : U \to X$ be the complement of $W$, $j_1 : U_1 \to X$ the complement of $W_1$ and $j' : U \to U_1$, $i_2' : D_2 \cap U_1 \to U_1$ the inclusions.
We have the distinguished triangles (see [3, Lemme 1.4.6])

$$\begin{gathered} Lj_!j^*E \to E \to Ri_*Li^*E \to Lj_!j^*E[1] \\ Lj_{1!}j_1^*E \to E \to Ri_{1*}Li_1^*E \to Lj_{1!}j_1^*E[1] \\ Lj_!'j^*E \to j_1^*E \to Ri_{2*}'Li_2'^*j_1^*E \to Lj_!'j^*E[1] \end{gathered}$$

Applying $Lj_{1!}$ to the last line gives us the distinguished triangle

$$Lj_!j^*E \to Lj_{1!}j_1^*E \to Lj_{1!}Ri_{2*}'Li_2'^*j_1^*E \to Lj_!j^*E[1]$$

Thus we have the distinguished triangle

$$Lj_{1!}Ri_{2*}'Li_2'^*j_1^*E \to Ri_*Li^*E \to Ri_{1*}Li_1^*E \to Lj_{1!}Ri_{2*}'Li_2'^*j_1^*E[1]$$

The same argument applied to the complement $j_2 : U_2 \to X$ of $W_2$, the map $j'' : U_2 \to U'' := U \setminus W_{12}$, $j_1' : U'' \to X$ and the inclusion $i_2'' : D_2 \cap U_1 \to U''$ gives the distinguished triangle

$$Lj_{1!}'Ri_{2*}''Li_2''^*j_1'^*E \to Ri_{2*}Li_2^*E \to Ri_{12*}Li_{12}^*E \to Lj_{1!}'Ri_{2*}''Li_2''^*j_1'^*E[1]$$

Since $D_2 \cap U_1$ is closed in $U_1$ and in $U''$, the natural map

$$Lj_{1!}Ri_{2*}'Li_2'^*j_1^*E \to Lj_{1!}'Ri_{2*}''Li_2''^*j_1'^*E$$

is an isomorphism. This shows that the diagram is homotopy cartesian. □

Given a strict normal crossing divisor $i : D \to X$, $D = \sum_{i=1}^m D_i$, we have the inclusions $\iota_I : D_I \to D$, $\iota_{I,J} : D_J \to D_I$ for $I \subset J$ and $i_I : D_I \to X$. For $E \in \mathbf{Spt}(\mathbf{Sm}/X)$ we thus have the presheaves $i_I^*E \in \mathbf{Spt}(\mathbf{Sm}/D_I)$. The isomorphism $\iota_{I,J}^*i_I^*E \cong i_J^*E$ gives us the canonical maps $i_I^*E \to \iota_{I,J*}i_J^*E$; applying $\iota_{I*}$ to this map gives us the natural maps $\alpha_{J,I} : \iota_{I*}i_I^*E \to \iota_{J*}i_J^*E$. For $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X)$, using the cofibrant replacement of $E$, we see that the same procedure gives us the functor

$$I \mapsto \iota_{I*}(i_I^*E^c)^f \in \mathbf{Spt}(\mathbf{Sm}/D)$$

together with the natural map

$$\alpha : i^*(E^c)^f \to \operatorname*{holim}_{I \neq \emptyset} \iota_{I*}(i_I^*E^c)^f.$$

Lemma 7.3.4. *The map $\alpha$ is an isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(D)$.*

*Proof.* As the co-unit $Li^*Ri_* \to \mathrm{id}$ is an isomorphism, $Ri_*$ is faithful, so it suffices to show that $Ri_*(\alpha)$ is an isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(X)$. This follows from lemma 7.3.3 and induction on $m$. □

Recall that for $E \in \mathbf{Spt}(\mathbf{Sm}/k)$ and $i : D \to X$ a strict normal crossing divisor, $D = \sum_{i=1}^m D_i$, we have the presheaf $E(D_{\mathrm{Zar}})$ on $D_{\mathrm{Zar}}$ defined by

$$E(D_{\mathrm{Zar}})(U) := \operatorname*{holim}_{I \neq \emptyset} E(D_I \cap U).$$

Proposition 7.3.5. *Let $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/k)$ be fibrant, $i : D \to X$ a strict normal crossing divisor on $X \in \mathbf{Sm}/k$, $f : X \to \operatorname{Spec} k$ the structure morphism. Then we have a natural isomorphism in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$*

$$E(D_{\mathrm{Zar}}) \cong Li^*(f^*E)_{\mathrm{Zar}}$$

*Proof.* Let $i_I : D_I \to X$ be the inclusion, $f_I : D_I \to \operatorname{Spec} k$ the structure morphism. By theorem 3.2.1, the canonical map

$$\eta_E : E(\tau_\epsilon^{\hat{X}}(D_I)) \to (f_I^*E)_{\mathrm{Zar}} \cong Li_I^*(f^*E)_{\mathrm{Zar}}$$

is an isomorphism in $\mathcal{H}\mathbf{Spt}(D_{I\mathrm{Zar}})$. By lemma 7.3.4 the induced map on the holim gives the desired isomorphism. □

Theorem 7.3.6. *Let $i : D \to X$ be a strict normal crossing divisor on $f : X \to \operatorname{Spec} k$ in $\mathbf{Sm}/k$, and let $E$ be a fibrant object in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/k_{\mathrm{Nis}})$. Then the map*

$$\eta_E^0 : E(\tau_\epsilon^{\hat{X}}(D)^0) \to \Theta(f^*E)_{\mathrm{Zar}} = [Li^*Rj_*(f^*E)]_{\mathrm{Zar}}$$

*is an isomorphism in $\mathcal{H}\mathbf{Spt}(D_{\mathrm{Zar}})$.*

*Proof.* The proof is the same as the proof of proposition 7.2.4, using the distinguished triangle of theorem 6.3.7 together with the isomorphism of proposition 7.3.5 instead of the triangle of corollary 4.1.3. □

*Remark* 7.3.7. Fix a fibrant $E \in \mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/k)$. Let $D' \to D$ be in $\mathbf{Sm}/D$ and suppose we have an $X' \to X$ in $\mathbf{Sm}/X$ and a $D$-isomorphism $D' \cong X' \times_X D$. Then we can replace $i : D \to X$ with $i' : D' \to X'$ and use theorem 7.3.6 to show that our tubular neighborhood construction gives the model $E(\tau_\epsilon^{\hat{X}'}(D')^0)$ for the restriction of $Li^*Rj_*(f^*E)$ to $\mathbf{Sm}/D'_{\mathrm{Zar}}$.
If $D$ and $D'$ are affine, then the theorem of [2] gives the existence of an $X'$ as above, so our result gives at least a "local" description of the entire presheaf $Li^*Rj_*(f^*E)$. □

## 8. Limit objects

Let $p : X \to C$ be a morphism in $\mathbf{Sm}/k$, with $C$ a smooth curve. Fix a $k$-point $0 \in C(k)$ and a parameter $t \in \mathcal{O}_{C,0}$. Ayoub combines the functor $Li^*Rj_*$ with a cosimplicial version of the classical path space (i.e., the universal cover) construction to define the *unipotent specialization functor*

$$\mathrm{sp} : \mathcal{SH}(X \setminus p^{-1}(0)) \to \mathcal{SH}(p^{-1}(0))$$

Replacing $Li^*Rj_*$ with the punctured tubular neighborhood, the same construction gives a model of this construction as a Zariski presheaf on $X_0$. In particular, we give a description of the "limiting values" $\lim_{t\to 0} E(X_t)$ for a semi-stable degeneration $\mathcal{X} \to (C, 0)$. As we mentioned in the introduction, we expect that this construction, applied to a suitable version of the de Rham complex (with weight and Hodge filtrations) as in [4] would yield the classical limit mixed Hodge structure of a semi-stable degeneration.

*Remark* 8.0.8. In [3, chapter 3] Ayoub describes a general theory of specialization structures; we concentrate on the unipotent structure, which Ayoub denotes $\Upsilon$, and describes in [3, §3.4]. □

8.1. Path spaces. Before defining the cosimplicial models for various path spaces and homotopy fibers, we recall some basic operations of simplicial sets on schemes. We let $\mathbf{Spc}_f$ denote the full subcategory of $\mathbf{Spc}$ consisting of simplicial sets $S$ with $S([n])$ finite for each $n$.

Let $Y$ be a $k$-scheme. For a finite set $S$, let $Y^S := \prod_{s\in S} Y$, with the product being over $\operatorname{Spec} k$. This defines the contravariant functor $S \mapsto Y^S$ from finite sets to $k$-schemes. In particular, for $S \in \mathbf{Spc}_f$ we have the cosimplicial scheme $Y^S$ with $Y^S([n]) := Y^{S([n])}$, giving the functor

$$Y^? : \mathbf{Spc}_f^{\mathrm{op}} \to \mathbf{Sch}_k^{\mathbf{Ord}}.$$

Similarly, if $T$ is a simplicial set, we have the cosimplicial-simplicial set (cosimplicial space) $T^S$ and the functor

$$T^? : \mathbf{Spc}^{\mathrm{op}} \to \mathbf{Spc}^{\mathbf{Ord}}.$$

Setting $Y \times S := \amalg_{s\in S} Y$, we have the functor $S \mapsto Y \times S$ from finite sets to $k$-schemes; if $S$ is a simplicial set as above, we thus have the simplicial scheme $Y \times S$, giving the functor

$$Y \times ? : \mathbf{Spc}_f \to \mathbf{Sch}_k^{\mathbf{Ord}^{\mathrm{op}}}.$$

The adjunction

$$\mathrm{Hom}_{\mathbf{Sch}_k}(X \times S, Y) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(X, Y^S)$$

for $S$ a finite set extends to $S$ a simplicial set as above, giving the adjunction

$$\mathrm{Hom}_{\mathbf{Sch}_k^{\mathbf{Ord}^{\mathrm{op}}}}(X \times S, Y) \cong \mathrm{Hom}_{\mathbf{Sch}_k^{\mathbf{Ord}}}(X, Y^S)$$

where on the left, we consider $Y$ as a constant simplicial scheme and on the right, $X$ as a constant cosimplicial scheme. This is an analog of the adjunction for spaces

$$\mathrm{Hom}_{\mathbf{Spc}}(A \times S, T) \cong \mathrm{Hom}_{\mathbf{Spc}^{\mathbf{Ord}^{\mathrm{op}}}}(A \times S, T) \cong \mathrm{Hom}_{\mathbf{Spc}^{\mathbf{Ord}}}(A, T^S)$$

where the first isomorphism is the well-known identity relating maps of bisimplicial sets with maps of the corresponding diagonal simplicial sets.

For $E \in \mathbf{Spc}(k)$ and $Y$ a simplicial object in $\mathbf{Sm}/k$, we have the cosimplicial space $E(Y)$ with $n$ cosimplices $E(Y([n]))$. For $s$ an element of a finite set $S$, and a scheme $Y \in \mathbf{Sm}/k$, we have the inclusion $i_s : Y = Y \times s \to Y \times S$; the inclusions $i_s : Y \to Y \times S$, $s \in S$ induce the canonical natural map

$$E(Y \times S) \to E(Y)^S$$

which is an isomorphism if $E$ is additive: $E(Y \amalg Y') \cong E(Y) \times E(Y')$. This isomorphsim extends immediately to finite simplicial sets $S \in \mathbf{Spc}_f$ and additive $E$.

*Examples* 8.1.1. (1) For a $k$-scheme $Y$, the *free path space* $\mathcal{P}_Y$ on $Y$ is $Y^{[0,1]}$, where $[0,1]$ is just the 1-simplex $\Delta[1] := \mathrm{Hom}_{\mathbf{Ord}}(-,[1])$. Explicitly, $\mathcal{P}_Y$ has $n$-cosimplices $Y^{n+2}$, with structure maps as follows: Label the factors in $Y^{n+2}$ from 0 to $n+1$. Send $\delta_i^n : [n] \to [n+1]$ to the diagonal

$$(y_0, \ldots, y_{n+1}) \mapsto (y_0, \ldots, y_{i-1}, y_i, y_i, y_{i+1}, \ldots, y_{n+1})$$

and send $s_i^n : [n] \to [n-1]$ to the projection

$$(y_0, \ldots, y_{n+1}) \mapsto (y_0, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n+1}).$$

The inclusion $\{0,1\} \to [0,1]$ gives rise to the projection $Y^{[0,1]} \to Y^{\{0,1\}}$, i.e. $\pi : \mathcal{P}_Y \to Y \times_k Y$; we thus have two structures of a cosimplicial $Y$-scheme on $\mathcal{P}_Y$: $\pi_1 : \mathcal{P}_Y \to Y$ and $\pi_2 : \mathcal{P}_Y \to Y$, with $\pi_i := p_i \circ \pi$.

(2) For a pointed $k$-scheme $(Y, y : \operatorname{Spec} k \to Y)$, we have the *pointed path space*

$$\mathcal{P}_Y(y) := \mathcal{P}_Y \times_{(\pi_2, y)} \operatorname{Spec} k.$$

(3) Now let $p : \mathcal{Y} \to Y$ be a $Y$-scheme,, $y : \operatorname{Spec} k \to Y$ a point. We have the *cosimplicial homotopy fiber of $p$ over $y$*:

$$\mathcal{P}_{\mathcal{Y}/Y}(y) := \mathcal{Y} \times_{(p,\pi_1)} \mathcal{P}_Y(y)$$

We extend this definition to cosimplicial $Y$-schemes in the evident manner: if $\mathcal{Y}^\bullet \to Y$ is a cosimplicial $Y$-scheme, we have the bi-cosimplicial $Y$-scheme $\mathcal{P}_{\mathcal{Y}^\bullet/Y}(y)$; the extension to functors from some small category to cosimplicial $Y$-schemes is done in the same way. □

Denoting the pointed $k$-scheme $(Y, y)$ by $Y_*$, we sometimes write $\mathcal{P}_{Y_*}$ for $\mathcal{P}_Y(y)$ and $\mathcal{P}_{\mathcal{Y}^\bullet/Y_*}$ for $\mathcal{P}_{\mathcal{Y}^\bullet/Y}(y)$. For $E \in \mathbf{Spt}(k)$, we have the simplicial spectrum $E(\mathcal{P}_{\mathcal{Y}/Y_*})$.

The pointed path space $\mathcal{P}_Y(y)$ is contractible in the following sense:

Lemma 8.1.2. *Let $(Y, y)$ be a pointed smooth $k$-scheme, $U$ a smooth $k$-scheme. Then for $E \in \mathbf{Spt}(k)$, the projection $U \times \mathcal{P}_Y(y) \to U$ induces a weak equivalence*

$$E(U) \to E(U \times \mathcal{P}_Y(y)).$$

*Proof.* To prove the lemma, it suffices to show that, for $E \in \mathbf{Spc}(k)$, the projection $U \times \mathcal{P}_Y(y) \to U$ induces a homotopy equivalence

$$E(U) \to E(U \times \mathcal{P}_Y(y)).$$

We first show that $U \times \mathcal{P}_Y(y) \to U$ induces a homotopy equivalance of cosimplicial schemes.

The projection $[0,1]\to pt$ gives the map of cosimplicial schemes $s : Y = Y^{pt}\to Y^{[0,1]}$; composing with the $k$-point $y\to Y$ gives the point $s_y : \operatorname{Spec} k\to Y^{[0,1]}$ and thus the section $(y, s_y) : \operatorname{Spec} k\to \mathcal{P}_Y(y)$ to the projection $\mathcal{P}_Y(y)\to \operatorname{Spec} k$. This induces the section $s_U : U\to U\times\mathcal{P}_Y(y)$ to the projection $p_U : U\times\mathcal{P}_Y(y)\to U$.
We proceed to construct a homotopy between $p_U\circ s_U$ and the identity on $U\times\mathcal{P}_Y(y)$; it suffices to construct the homotopy for $U = \operatorname{Spec} k$.
For this, let $\sigma : Y^{[0,1]}\to Y^{[0,1]}$ be the map induced by the map of simplicial sets $[0,1]\to[0,1]$ sending $[0,1]$ to 1. Then $p_{\operatorname{Spec} k}\circ s_{\operatorname{Spec} k} : \mathcal{P}_Y(y)\to\mathcal{P}_Y(y)$ is the map $(\mathrm{id}_{\operatorname{Spec} k},\sigma)$.
Let $p_0, p_1 : Y^{[0,1]\times[0,1]}\to Y^{[0,1]}$ be the maps induced by the inclusions $i_0, i_1 : [0,1]\to[0,1]\times[0,1]$, $i_0(x) = x\times 0$, $i_1(x) = x\times 1$, and let $\pi : Y\to Y^{[0,1]\times[0,1]}$ be the map induced by $[0,1]\times[0,1]\to pt$. Let $h : ([0,1]\times[0,1], 1\times[0,1])\to([0,1],1)$ be any map of pairs of simplicial sets which is the identity on $[0,1]\times 0$ and the map to $1\in[0,1]$ on $[0,1]\times 1$. Then $h$ defines a map

$$H : Y^{[0,1]}\to Y^{[0,1]\times[0,1]}$$

with

$$\begin{aligned}
&p_0\circ H = \mathrm{id}_{Y^{[0,1]}}\\
&p_1\circ H = \sigma\\
&H\circ s = \pi.
\end{aligned}$$

From these identities, it follows that $(H, \mathrm{id}_y)$ induces a co-homotopy

$$H_y : \mathcal{P}_Y(y)\to\mathcal{P}_Y(y)^{[0,1]}$$

with $p_0\circ H_y = \mathrm{id}$, $p_1\circ H_y = p_{\operatorname{Spec} k}\circ s_{\operatorname{Spec} k}$. Taking the adjoint of $H_y$, we have the homotopy

$$h_y : \mathcal{P}_Y(y)\times[0,1]\to\mathcal{P}_Y(y);\ \ h_y\circ i_0 = \mathrm{id}, h_y\circ i_1 = p_{\operatorname{Spec} k}\circ s_{\operatorname{Spec} k},$$

where $\mathcal{P}_Y(y)\times[0,1]$ and $\mathcal{P}_Y(y)$ are to be considered as cosimplicial-simplicial schemes, with $\mathcal{P}_Y(y)$ constant in the simplicial direction.
Applying $E$ to $\mathrm{id}_U\times h_y$ and composing with the canonical map

$$E(U\times\mathcal{P}_Y(y)\times[0,1])\to E(U\times\mathcal{P}_Y(y))^{[0,1]}$$

gives us the co-homotopy

$$E(\mathrm{id}_U\times h_y) : E(U\times\mathcal{P}_Y(y))\to E(U\times\mathcal{P}_Y(y))^{[0,1]}$$

between the identity and $E(p_U\circ s_U)$. Thus $E(U)\to E(U\times\mathcal{P}_Y(y))$ is a homotopy equivalence, as desired. □

8.2. Limit structures. For our purposes, a semi-stable degeneration need not be proper, so even if this is somewhat non-standard terminology, we use the following definition:

Definition 8.2.1. A *semi-stable degeneration* is a flat morphism $p : \mathcal{X} \to (C, 0)$, where $(C, 0)$ is a smooth pointed local curve over $k$, $C = \operatorname{Spec} \mathcal{O}_{C,0}$, $\mathcal{X}$ is a smooth irreducible $k$-scheme, $p$ is smooth over $C \setminus 0$ and $X_0 := p^{-1}(0)$ is a reduced strict normal crossing divisor on $\mathcal{X}$. □

For the rest of this section, we fix a semi-stable degeneration $\mathcal{X} \to (C, 0)$. We denote the open complement of $X_0$ in $\mathcal{X}$ by $\mathcal{X}^0$. We write $\mathbb{G}_m$ for the pointed $k$-scheme $(\mathbb{A}^1_k \setminus \{0\}, 1)$.

Fix a uniformizing parameter $t \in \mathcal{O}_{C,0}$, giving the morphism $t : (C, 0) \to (\mathbb{A}^1_k, 0)$, which restricts to $t : C \setminus 0 \to \mathbb{G}_m$. Let $p[t] : \mathcal{X} \to \mathbb{A}^1$ be the composition $t \circ p$, and let $p[t]^0 : \mathcal{X}^0 \to \mathbb{G}_m$ be the restriction of $p[t]$. Composing $p[t]$ with the canonical morphism $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \to \mathcal{X}^0$ yields the map

$$\hat{p}[t]^0 : \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \to \mathbb{G}_m.$$

Let $X_0^1, \ldots, X_0^m$ be the irreducible components of $X_0$. Recalling the construction of $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0$ as a diagram (see (6.3.1)), let us denote, for $I \subset \{1, \ldots, m\}$, the co-presheaf $\iota_{I*}(\tau_\epsilon^{\hat{\mathcal{X}}}(X_{0I})^0)$ by $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I$. The map $\hat{p}[t]^0$ makes $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0$ into a diagram of co-presheaves (on $X_{0\mathrm{Zar}}$) of cosimplicial pro-schemes over $\mathbb{G}_m$. We thus have the diagram of cosimplicial co-presheaves on $X_{0\mathrm{Zar}}$:

$$I \mapsto \mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}.$$

We denote this diagram by

$$\lim_{t \to 0} X_t. \tag{8.2.1}$$

Now let $E$ be in $\mathbf{Spt}(k)$. For each $I \subset \{1, \ldots, m\}$, we have the presheaf of bisimplicial spectra on $X_{0\mathrm{Zar}}$, $E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m})$, giving us the functor

$$I \mapsto \tilde{E}(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}).$$

where $\tilde{}$ means fibrant model. Taking the homotopy limit over $I$ of the associated diagram of presheaves of total spectra gives us the fibrant presheaf of spectra

$$E(\lim_{t \to 0} X_t) := \operatorname*{holim}_{I \neq \emptyset} \operatorname{Tot} \tilde{E}(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}).$$

Taking the global sections gives us the spectrum $E(\lim_{t \to 0} X_t)(X_0)$, which we denote by $\lim_{t \to 0} E(X_t)$.

*Remark* 8.2.2. Suppose $E \in \mathbf{Spt}(k)$ is homotopy invariant and satisfies Nisnevich excision. We can form the homotopy limit $\bar{E}(\lim_{t \to 0} X_t)$ of the diagram of presheaves

$$I \mapsto E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}).$$

Since $E$ is quasi-fibrant (see remark 2.3.2) the map

$$E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m}) \to \tilde{E}(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0_I/\mathbb{G}_m})$$

is a pointwise weak equivalence, hence the map $\bar{E}(\lim_{t \to 0} X_t) \to E(\lim_{t \to 0} X_t)$ is a pointwise weak equivalence. In particular, $\bar{E}(\lim_{t \to 0} X_t)(X)_0) \to$

$\lim_{t\to 0} E(X_t)$ is a weak equivalence. In short, if $E$ is homotopy invariant and satisfies Nisnevich excision, then it is not necessary to take the fibrant model $\tilde{E}$ in the construction of $E(\lim_{t\to 0} X_t)$ or $\lim_{t\to 0} E(X_t)$. □

We remind the reader of the presheaves $E(X_{0\bullet})$ and $E(\Delta_{X_{0\bullet}})$ on $X_{0\mathrm{Zar}}$ described in definition 6.2.1.

Proposition 8.2.3. *Suppose $E$ is homotopy invariant and satisfies Nisnevich excision. Then*
*(1) There is a canonical map in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$:*

$$E(X_{0\bullet}) \xrightarrow{\gamma_X} E(\lim_{t\to 0} X_t).$$

*(2) If $X_0$ is smooth, then $E(X_{0\bullet}) = E(X_{0\mathrm{Zar}})$ and $\gamma_X$ is an isomorphism.*

*Proof.* We have the maps

$$E(X_{0\bullet}) \xrightarrow{p^*_\bullet} E(\Delta^*_{X_{0\bullet}}) \xleftarrow{\hat{\iota}^*} E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)).$$

which by proposition 6.2.2 are Zariski-local weak equivalences. Similarly, we have the diagram of open immersions

$$\hat{j} : \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \to \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)$$

inducing

$$\hat{j}^* : E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)) \to E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0).$$

Thus we have the map

$$p_0^* : E(X_{0\bullet}) \to E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0);$$

$p_0^* := \hat{j}^*(\hat{\iota}^*)^{-1}p^*_\bullet$.
Similarly, we have the projection

$$\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m} \to \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0,$$

giving the map

$$q^* : E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0) \to E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m});$$

we set $\gamma_X := q^* \circ p^*$.
For (2), the diagram $X_\bullet$ is just the identity copresheaf $X_{0\mathrm{Zar}}$, hence $E(X_{0\bullet}) = E(X_{0\mathrm{Zar}})$. To show $\gamma_X$ is an isomorphism, fix a point $x \in X_0$. There is a Zariski neighborhood $U$ of $x$ in $X_0$ and a Nisnevich neighborhood $\mathcal{X}' \to \mathcal{X}$ of $U$ in $\mathcal{X}$ which is isomorphic to a Nisnevich neighborhood of $U$ in $U \times \mathbb{A}^1$. Thus it suffices to prove the result in the case $\mathcal{X} = X_0 \times \mathbb{A}^1$, $(C, 0) = (\mathbb{A}^1, 0)$ and $p = p_2 : \mathcal{X} \to \mathbb{A}^1$.
For each smooth $k$-scheme $T$, it follows from homotopy invariance and theorem 3.2.1 that the canonical map $p : \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0 \times 0) \to X_0 \times \mathbb{A}^1$ induces a weak equivalence

$$p^* : E(T \times_k X_{0\mathrm{Zar}} \times \mathbb{A}^1) \to E(T \times_k \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0 \times 0))$$

The Morel-Voevodsky purity theorem [34, theorem 2.23] plus Nisnevich excision and the homotopy property for $E$ implies that $p$ induces a weak equivalence

$$p^*: E^{T\times X_{0\mathrm{Zar}}\times 0}(T\times_k X_{0\mathrm{Zar}}\times\mathbb{A}^1)\to E^{T\times\Delta^*_{X_{0\mathrm{Zar}}\times 0}}(T\times_k \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)).$$

This gives us the map of homotopy fiber sequences

$$\begin{tikzcd}
E^{T\times X_{0\mathrm{Zar}}\times 0}(T\times_k X_{0\mathrm{Zar}}\times\mathbb{A}^1) \arrow[r, "p^*"] \arrow[d] & E^{T\times\Delta^*_{X_{0\mathrm{Zar}}\times 0}}(T\times_k \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)) \arrow[d] \\
E(T\times_k X_{0\mathrm{Zar}}\times\mathbb{A}^1) \arrow[r, "p^*"] \arrow[d, "j^*"'] & E(T\times_k \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)) \arrow[d, "\hat{j}^*"] \\
E(T\times_k X_{0\mathrm{Zar}}\times\mathbb{G}_m) \arrow[r, "p^{0*}"] & E(T\times_k \tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)^0)
\end{tikzcd}$$

with $p^{0*}$ induced by the restriction of $p$,

$$p^0:\tau_\epsilon^{\widehat{X_0\times\mathbb{A}^1}}(X_0\times 0)^0\to X_0\times\mathbb{G}_m.$$

Thus $p^{0*}$ is a weak equivalence.
Applying these term-by-term with respect to the cosimplicial schemes defining the respective path spaces, we have the weak equivalence (assuming $\mathcal{X}=X_0\times\mathbb{A}^1$)

$$E(U\times\mathcal{P}_{\mathbb{G}_m})\to E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m})(U).$$

Thus we need only show that the projection $U\times\mathcal{P}_{\mathbb{G}_m}\to U$ induces a weak equivalence

$$E(U)\to E(U\times\mathcal{P}_{\mathbb{G}_m})$$

for all smooth $k$-schemes $U$. This is lemma 8.1.2 □

8.3. Comparison. We conclude this section by connecting our construction with the specialization functor sp for the specialization structure $\Upsilon$ defined by Ayoub [3, chapter 3].
Let $E\in\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/k)$ be fibrant, let $p:\mathcal{X}\to(C,0)$ be a semi-stable degeneration and choose a parameter $t\in\mathcal{O}_{C,0}$. In this setting, Ayoub's functor sp applied to some $E\in\mathbf{Spt}(\mathbf{Sm}/\mathcal{X}^0)$ is defined as follows: First form the presheaf $E(\mathcal{P}_{-/\mathbb{G}_m})$ on $\mathbf{Sm}/\mathcal{X}^0$ by taking the total spectrum

$$E(\mathcal{P}_{-/\mathbb{G}_m})(X'\to\mathcal{X}^0):=\mathrm{Tot}(E(\mathcal{P}_{X'/\mathbb{G}_m})).$$

where we use the composition $X'\to\mathcal{X}^0\xrightarrow{t}\mathbb{G}_m$ as structure morphism. Then $\mathrm{sp}(E)\in\mathcal{SH}_{\mathbb{A}^1}(\mathbf{Sm}/X_0)$ is represented by the presheaf

$$\mathrm{sp}(E):=i^*\left(j_*\left(E(\mathcal{P}_{-/\mathbb{G}_m})^f\right)^c\right).$$

Similarly, we have the simplicial presheaf on $\mathbf{Sm}/X_0$ with $n$-simplices

$$\mathrm{sp}(E)_n:=i^*\left(j_*\left(E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f\right)^c\right).$$

Let $\mathrm{Tot}(\mathrm{sp}(E)_*)$ denote the presheaf formed by taking the total spectrum of $n \mapsto \mathrm{sp}(E)_n$.

Lemma 8.3.1. *Suppose $E$ is fibrant. Then there is a natural isomorphism in* $\mathcal{SH}_{\mathbb{A}^1}(X_0)$

$$\mathrm{Tot}(\mathrm{sp}(E)_*) \cong \mathrm{sp}(E)$$

*Proof.* Since $E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f$ is fibrant, the presheaf $E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f$ on $\mathcal{X}^0$ satisfies Nisnevich excision and is $\mathbb{A}^1$ homotopy invariant. Thus the same holds for the total spectrum of the simplicial spectrum $n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f$, hence

$$\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f) \to \left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f)\right)^f$$

is a pointwise weak equivalence in $\mathbf{Spt}_{\mathbb{A}^1}(\mathbf{Sm}/\mathcal{X}^0)$, and thus we still have a pointwise weak equivalence after applying $j_*$. Similarly, the evident map

$$\left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])\right)^f \to \left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f\right)^f$$

is a pointwise weak equivalence. Taking cofibrant models and applying $i^*$ gives the isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(X_0)$

$$\mathrm{sp}(E) \cong i^*\left(\left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f\right)^c\right).$$

On the other hand, taking the total complex commutes with taking the cofibrant model, and with the functor $i^*$, so we have the isomorphism in $\mathbf{Spt}_{\mathrm{mot}}(\mathbf{Sm}/X_0)$

$$\begin{aligned}\mathrm{sp}(E) = i^*&\left(\left(\mathrm{Tot}(n \mapsto E(\mathcal{P}_{-/\mathbb{G}_m}[n]))^f\right)^c\right)\\ &\cong \mathrm{Tot}\left(n \mapsto i^*\left(\left(E(\mathcal{P}_{-/\mathbb{G}_m}[n])^f\right)^c\right)\right) = \mathrm{Tot}(\mathrm{sp}_*(E)).\end{aligned}$$

□

Using the diagram of lemma 7.2.3 for the $n$-cosimplices $\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \times \mathbb{G}_m^n$ of $\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m}$, and taking the total spectrum, we arrive at a natural map

$$E(\lim_{t\to 0} X_t) \to \mathrm{Tot}(\mathrm{sp}(E)_*)_{\mathrm{Zar}}$$

in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$; combining this with lemma 8.3.1 gives us the comparison map

$$\theta_E : E(\lim_{t\to 0} X_t) \to \mathrm{sp}(E)_{\mathrm{Zar}}$$

in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$.

Proposition 8.3.2. *The map $\theta_E : E(\lim_{t\to 0} X_t) \to \mathrm{sp}(E)_{\mathrm{Zar}}$ is an isomorphism in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$.*

*Proof.* By theorem 7.3.6, the map

$$\theta_E(n) : E(\mathcal{P}_{\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0/\mathbb{G}_m}([n])) \to \mathrm{sp}_n(E)$$

is an isomorphism in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$ for each $n$, thus the map $\theta_E$ on the total spectra is also an isomorphism in $\mathcal{H}\mathbf{Spt}(X_{0\mathrm{Zar}})$. □

## 9. The monodromy sequence

In this section, we construct the monodromy sequence for the limit object $E(\lim_{t\to 0} X_t)$ (see corollary 9.3.5). As pointed out to us by Ayoub, one needs to restrict $E$ quite a bit. We give here a theory valid for presheaves of complexes of $\mathbb{Q}$-vector space on $\mathbf{Sm}/k$ which are homotopy invariant and satisfy Nisnevich excision, and satisfy an additional "alternating" property (definition 9.2.2). Ayoub [3, Chap. 3] constructs the monodromy sequence in a more general setting; our construction is based on his ideas applied to our tubular neighborhood construction. In particular, our mondromy sequence agrees with the monodromy sequence of *loc. cite*.

9.1. Presheaves of complexes. For a noetherian ring $R$, we let $C_R$ denote the category of (unbounded) homological complexes of $R$-modules, $C_{R\geq 0}$ the full subcategory of $C_R$ consisting of complexes which are zero in strictly negative degrees.
By the Dold-Kan equivalence, we may identify $C_{R\geq 0}$ with the category of simplicial $R$-modules $\mathbf{Spc}_R$. The forgetful functor $\mathbf{Spc}_R \to \mathbf{Spc}_*$ allows us to use the standard model structure on $\mathbf{Spc}_*$ to induce a model structure on $\mathbf{Spc}_R$, i.e., cofibrations are degreewise monomorphisms, weak equivalences are homotopy equivalences on the geometric realization and fibrations are maps with the RLP for trivial cofibrations. This induces a model structure on $C_{R\geq 0}$ with weak equivalence the quasi-isomorphisms; the suspension functor is the usual (homological) shift operator: $\Sigma C := C[1]$, $C[1]_n := C_{n-1}$, $d_{C[1],n} = -d_{C,n-1}$. This model structure is extended to $C_R$ by identifying $C_R$ with the category of "spectra in $C_{R\geq 0}$", i.e., sequences $(C^0, C^1, \ldots)$ with bonding maps $\epsilon_n : C^n[1] \to C^{n+1}$. Following Hovey [17], the model structure on $\mathbf{Spt}$ induces a model structure on spectra of simplicial $R$-modules, and thus a model structure on $C_R$, with weak equivalences the quasi-isomorphisms. In particular, the homotopy category $\mathcal{H}C_R$ is just the unbounded derived category $D_R$.
Similarly, for a category $\mathcal{C}$, the model structure for the presheaf category $\mathbf{Spt}(\mathcal{C})$ gives a model structure for presheaves of complexes on $\mathcal{C}$, $C_R(\mathcal{C})$ with weak equivalences the pointwise quasi-isomorphisms, and homotopy category the derived category $D_R(\mathcal{C})$. We may introduce a topology (e.g., the Zariski or Nisnevich topology), giving the model categories $C_R(X_{\mathrm{Zar}})$, $C_R(\mathbf{Sm}/B_{\mathrm{Zar}})$, $C_R(X_{\mathrm{Nis}})$, $C_R(\mathbf{Sm}/B_{\mathrm{Nis}})$. These have homotopy categories equivalent to the derived categories (on the small or big sites) $D_R(X_{\mathrm{Zar}})$, $D_R(\mathbf{Sm}/S_{\mathrm{Zar}})$, $D_R(X_{\mathrm{Nis}})$, $D_R(\mathbf{Sm}/S_{\mathrm{Nis}})$, respectively. Finally, we may consider the $\mathbb{A}^1$-localization, giving the Nisnevich-local $\mathbb{A}^1$-model structure $C_{R,\mathbb{A}^1}(\mathbf{Sm}/B_{\mathrm{Nis}})$ with homotopy category $D_{R\mathbb{A}^1}(B)$.
Let $I$ be a small category, $F : I \to C_R$ a functor. Since we can consider $F$ as a spectrum-valued functor by the various equivalences described above, we may form the complex $\mathrm{holim}_I F$. Explicitly, this is the following complex: One first

forms the cosimplicial complex $\underline{\mathrm{holim}}_I F$ with $n$-cosimplices

$$\underline{\mathrm{holim}}_I F^n := \prod_{\sigma=(\sigma_0\to\ldots\to\sigma_n)\in\mathcal{N}(I)_n} F(\sigma_n).$$

For $g : [m] \to [n]$, with $g(m) = m' \leq n$, the $\sigma$-component of the map $\underline{\mathrm{holim}}_I F^n(g)$ sends $\prod x_\tau$ to $F(\sigma_{m'} \to \sigma_n)(x_{g^*(\sigma)})$. The complex $\mathrm{holim}_I F$ is then the total complex of the double complex $n \mapsto \underline{\mathrm{holim}}_I F^n$, with second differential the alternating sum of the coface maps. This construction being functorial and preserving quasi-isomorphisms, it passes to the derived category $D_R(\mathcal{C})$. If $I$ is a finite category, the construction commutes with filtered colimits, hence passes to the Zariski- and Nisnevich-local derived categories, as well as the $\mathbb{A}^1$-local versions.

*Remarks* 9.1.1. (1) For a set $S$, let $RS$ denote the free $R$-module on $S$. Sending a pointed space $(S,*)$ to the simplicial $R$-module $RS$, with $RS(n) := RS_n/R\{*\}$ defines the *$R$-localization functor* $\mathbf{Spc}_* \to \mathbf{Spc}_R$. This extends to the spectrum categories, and gives us the exact $R$-localization functor on homotopy category $\otimes R : \mathcal{SH} \to D_R$. The $R$-localization functor $\otimes R$ extends to all the model categories we have been considering, in particular, we have the $R$-localization

$$\otimes R : \mathcal{SH}_{\mathbb{A}^1}(B) \to D_{R,\mathbb{A}^1}(B),$$

For $R = \mathbb{Q}$, we can also take the $\mathbb{Q}$-localization of $\mathcal{SH}$ by performing a Bousfield localization, i.e., define $Z \in \mathbf{Spt}$ to be $\mathbb{Q}$-local if $\pi_n(Z)$ is a $\mathbb{Q}$-vector space for all $n$, and $E \to F$ a $\mathbb{Q}$ weak equivalence if $\mathrm{Hom}_{\mathbf{Spt}}(F,Z) \to \mathrm{Hom}_{\mathbf{Spt}}(E,Z)$ is an isomorphism for all $\mathbb{Q}$-local $Z$. Inverting the $\mathbb{Q}$-weak equivalences defines the $\mathbb{Q}$-local homotopy category $\mathcal{SH}_{\mathbb{Q}}$, and $\otimes\mathbb{Q} : \mathcal{SH} \to D_{\mathbb{Q}}$ identifies $\mathcal{SH}_{\mathbb{Q}}$ with $D_{\mathbb{Q}}$. This passes to the other homotopy categories we have defined, in particular, $\otimes\mathbb{Q} : \mathcal{SH}_{\mathbb{A}^1}(B) \to D_{\mathbb{Q},\mathbb{A}^1}(B)$ identifies $SH_{\mathbb{A}^1}(B)_{\mathbb{Q}}$ with $D_{\mathbb{Q},\mathbb{A}^1}(B)$.

(2) $D_{\mathbb{Q},\mathbb{A}^1}(k)$ is *not* the same as the ($\mathbb{Q}$-localized) big category of motives over $k$, $DM(k)_{\mathbb{Q}}$; the $\mathbb{Q}$-localization does not give rise to transfers. □

9.2. The log complex. Let $\mathrm{sgn} : S_n \to \{\pm 1\}$ be the sign representation of the symmetric group $S_n$. Consider a presheaf of $\mathbb{Q}$-vector spaces $E$ on $\mathbf{Sm}/k$. For $X, Y \in \mathbf{Sm}/k$, let $\mathrm{alt}_n : E(Y \times X^n) \to E(Y \times X^n)$ be the alternating projector

$$\mathrm{alt}_n = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)(\mathrm{id}_Y \times \sigma)^*,$$

with $\sigma$ operating on $X^n$ by permuting the factors. Let $E(Y \times X^n)^{\mathrm{alt}} \subset E(Y \times X^n)$ be the image of $\mathrm{alt}_n$ and $E(Y, X^n)^{\mathrm{alt}}_{\perp}$ the kernel. We extend these constructions to presheaves of complexes $E$ by operating degreewise.
If $(X,*)$ is a pointed $k$-scheme, we have the inclusions $i_j : Y \times X^{n-1} \to Y \times X^n$ inserting the point $*$ in the $j$th factor. For $E$ a presheaf of $\mathbb{Q}$-vector spaces, we let $E(Y \wedge X^{\wedge n})$ be the intersection of the kernels of the restriction maps

$$(\mathrm{id}_Y \times i_j)^* : E(Y \times X^n) \to E(Y \times X^{n-1}).$$

Letting $p_j : X^n \to X^{n-1}$ be the projection omitting the $j$th factor, the composition $(\mathrm{id} - p_n^* i_n^*) \circ \ldots \circ (\mathrm{id} - p_1^* i_1^*)$ gives a splitting

$$\pi_n : E(Y \times X^n) \to E(Y \wedge X^{\wedge n})$$

to the inclusion $E(Y \wedge X^{\wedge n}) \to E(Y \times X^n)$.
Clearly $S_n$ acts on $E(Y \wedge X^{\wedge n})$ through its action on $X^n$; we let $E(Y, X^{\wedge n})^{\mathrm{alt}}$ and $E(Y \wedge X^{\wedge n})^{\mathrm{alt}}_{\perp}$ be the image and kernel of $\mathrm{alt}_n$ on $E(Y \wedge X^{\wedge n})$.
Let $f : X \to \mathbb{G}_m$ be a morphism, $E$ a presheaf of $\mathbb{Q}$-vector spaces on $\mathbf{Sm}/k$. Let $f_n : X \times \mathbb{G}_m^n \to X \times \mathbb{G}_m^{n+1}$ be the morphism

$$f_n(x, t_1, \ldots, t_n) := (x, f(x), t_1, \ldots, t_n).$$

Denote the map $\mathrm{alt}_n \circ \pi_n \circ f_n^* : E(X \wedge \mathbb{G}_m^{\wedge n+1})^{\mathrm{alt}} \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$ by

$$\cup f : E(X \wedge \mathbb{G}_m^{\wedge n+1})^{\mathrm{alt}} \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

One checks that

Lemma 9.2.1. $(\cup f)^2 = 0$.

*Proof.* We work in the $\mathbb{Q}$-linear category $\mathbb{Q}\mathbf{Sm}/k$, with the same objects as $\mathbf{Sm}/k$, disjoint union being direct sum, and, for $X$, $Y$ connected, $\mathrm{Hom}_{\mathbb{Q}\mathbf{Sm}/k}(X, Y)$ is the $\mathbb{Q}$-vector space freely generated by the set $\mathrm{Hom}_{\mathbf{Sm}/k}(X, Y)$. Product over $k$ makes $\mathbb{Q}\mathbf{Sm}/k$ a tensor category. The map $\cup f$ is gotten by applying $E$ to the map $\cup f^\vee : X \times \mathbb{G}_m^n \to X \times \mathbb{G}_m^{n+1}$ in $\mathbb{Q}\mathbf{Sm}/k$:

$$(x, t_1, \ldots, t_{n-1}) \mapsto \mathrm{alt}[((x, f(x)) - (x, 1)) \otimes t_1 - 1 \otimes \ldots \otimes t_n - 1)]$$

and restricting to $E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$. But $(\cup f^\vee)^2$ is

$$\begin{aligned}&(x, t_1, \ldots, t_n) \mapsto \\ &\quad \mathrm{alt}[((x, f(x), f(x)) - (x, 1, f(x)) - (x, f(x), 1) + (x, 1, 1)) \otimes \ldots \otimes t_n - 1)\end{aligned}$$

which is evidently the zero map. □

Form the complex $E(\log_f)$ by

$$E(\log_f)_n := E(X \wedge \mathbb{G}_m^{\wedge n}))^{\mathrm{alt}}$$

with differential $\cup f$. Since $E(\log_f)_0 = E(X)$, we have the canonical map $\iota_X : E(X) \to E(\log_f)$.
We extend this definition to an $I$-diagram of schemes over $\mathbb{G}_m$, $f^\bullet : X^\bullet \to \mathbb{G}_m$ (with the $X^n \in \mathbf{Sm}/k$) by

$$E(\log_{f^\bullet}) := \operatorname*{holim}_{i \in I} E(\log_{f^i});$$

similarly, we extend to $E$ a presheaf of complexes on $\mathbf{Sm}/k$ by taking the total complex of the double complex $n \mapsto E_n(\log_{f^\bullet})$. The map $\iota_X$ extends to

$$\iota_{X^\bullet} : E(X^\bullet) \to E(\log_{f^\bullet}),$$

where

$$E(X^\bullet) := \operatorname*{holim}_{i \in I} E(X^i).$$

We consider as well a truncation of $E(\log_f)$. Recall that the stupid truncation $\sigma_{\geq n}C$ of a homological complex $C$ is the quotient complex of $C$ with

$$\sigma_{\geq n}C_m := \begin{cases} C_m & \text{for } m \geq n \\ 0 & \text{for } m < n. \end{cases}$$

For $E$ a presheaf of abelian groups and $f : X \to \mathbb{G}_m$ a morphism in $\mathbf{Sm}/k$, set

$$E(\sigma_{\geq 1}\log_f) := \sigma_{\geq 1}E(\log_f).$$

We have the quotient map $N : E(\log_f) \to E(\sigma_{\geq 1}\log_f)$, natural in $f$ and $E$. We extend to $I$-diagrams $f^\bullet : X^\bullet \to \mathbb{G}_m$ and to presheaves of complexes as for $E(\log_f)$. The quotient map $N$ defined above extends to the natural map

$$N : E(\log_{f^\bullet}) \to E(\sigma_{\geq 1}\log_{f^\bullet}).$$

for $f^\bullet : X^\bullet \to \mathbb{G}_m$ an $I$-diagram of morphisms in $\mathbf{Sm}/k$, and $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$. Finally, for $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$, define $E(-1)$ to be the presheaf of complexes

$$E(-1)(X) := E(X \wedge \mathbb{G}_m)[1] := \ker\left(E(X \times \mathbb{G}_m) \xrightarrow{i_1^*} E(X)\right)[1].$$

Definition 9.2.2. Let $E$ be be in $C_{\mathbb{Q}}(\mathbf{Sm}/k)$. Call $E$ *alternating* if for every $X \in \mathbf{Sm}/k$ and every $n \geq 0$, the alternating projection

$$\mathrm{alt}_n : E(X \wedge \mathbb{G}_m^{\wedge n}) \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

is a quasi-isomorphism. □

*Remarks* 9.2.3. (1) Clearly, $E$ is alternating if and only if $S_n$ acts via the sign representation on $H_pE(X \wedge \mathbb{G}_m^{\wedge n})$ for all $X$, $n$ and $p$.

(2) Fix integers $1 \leq i \leq n$. We have the split injection $\iota_{i,i+1} : E(X \wedge \mathbb{G}_m^{\wedge n}) \to E((X \times \mathbb{G}_m^{n-2}) \wedge \mathbb{G}_m^{\wedge 2})$ by shuffling the $i, i+1$ coordinates to position $n-1, n$. In particular, we have the injection

$$H_p(\iota_{i,i+1}) : H_pE(X \wedge \mathbb{G}_m^{\wedge n}) \to H_pE((X \times \mathbb{G}_m^{n-2}) \wedge \mathbb{G}_m^{\wedge 2}).$$

Since $S_n$ is generated by simple transpositions, this shows that $E$ is alternating if and only if the exchange of factors in $\mathbb{G}_m \wedge \mathbb{G}_m$ acts by -1 on $H_pE(X \wedge \mathbb{G}_m \wedge \mathbb{G}_m)$ for all $X$ and $p$.

(3) Suppose that $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ is homotopy invariant and satisfies Nisnevich excision. Consider $\mathbb{P}^1$ as pointed by $\infty$. Then $E(X \wedge \mathbb{P}^1)$ is quasi-isomorphic to the suspension $E(X \wedge \mathbb{G}_m)[-1]$, hence $E$ is alternating if and only if the exchange of factors in $\mathbb{P}^1 \wedge \mathbb{P}^1$ induces the identity on $H_pE(X \wedge \mathbb{P}^1 \wedge \mathbb{P}^1)$ for all $X$ and $p$.
The homotopy invariance and Nisnevich excision properties of $E$ give a natural quasi-isomorphism of $E(X \wedge \mathbb{P}^1 \wedge \mathbb{P}^1)$ with $E(X \wedge (\mathbb{A}^2/\mathbb{A}^2 \setminus \{0\}))$, with the exchange of factors in $\mathbb{P}^1 \wedge \mathbb{P}^1$ going over to the linear transformation $(x, y) \mapsto (y, x)$. If the characterstic of $k$ is different from 2, this transformation is conjugate to $(x, y) \mapsto (-x, y)$. Thus $E$ is alternating if and only if the map

$[-1] : \mathbb{P}^1 \to \mathbb{P}^1$, $[-1](x_0, x_1) = (x_0, -x_1)$, acts by the identity on $H_pE(X \wedge \mathbb{P}^1)$ for all $X$ and $p$.

(4) Call $E$ *oriented* if $E$ is an associative graded-commutative ring:

$$\mu : E \otimes_{\mathbb{Q}} E \to E$$

and (roughly speaking) $E$ admits a natural Chern class transformation

$$c_1 : \mathrm{Pic} \to H^2E$$

satisfying the projective bundle formula: For $\mathcal{E} \to X$ a rank $r$ vector bundle with associated projective space bundle $\mathbb{P}(\mathcal{E}) \to X$ and tautological line bundle $\mathcal{O}(1)$, $H^*E(\mathbb{P}(\mathcal{E}))$ is a free $H^*E(X)$-module with basis $1, \xi, \ldots, \xi^{r-1}$, where $\xi = c_1(\mathcal{O}(1)) \in H^2E(\mathbb{P}(\mathcal{E}))$. We do not assume that $c_1$ is a group homomorphism. The projective bundle formula and the fact that $[-1]^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ implies that an oriented $E$ is alternating. In particular, rational motivic cohomology, $\mathbb{Q}_\ell(*)$ étale cohomlogy, $\mathbb{Q}$-singular cohomology (with respect to a chosen embedding $k \to \mathbb{C}$) and rational algebraic cobordism $\mathrm{MGL}^{**}_{\mathbb{Q}}$ are all alternating.
On the other hand, rational motivic co-homotopy is alternating if $-1$ is a square in $k$, but is not alternating for $k = \mathbb{R}$. This is pointed out in [31]: if $-1 = i^2$, $[-1]$ is represented by the $2 \times 2$ matrix with diagonal entries $i$ and $-i$. As this is a product of elementary matrices, one has an $\mathbb{A}^1$-homotopy connecting $[-1]$ and id. To see the non-triviality of $[-1]$ for $k = \mathbb{R}$, let $[X, Y]$ denote the set of morphisms $X \to Y$ in $\mathcal{H}_{\mathbb{A}^1}\mathbf{Spc}_*(k)$. Morel defines a map (of sets) $[\mathbb{P}^1, \mathbb{P}^1] \to \phi(k)$, where $\phi(k)$ is the set of isomorphism classes of quadratic forms over $k$, and notes that the map $[u]$,

$$[u](x_0, x_1) := (x_0, ux_1),$$

goes to the class of the form $ux^2$. This map extends to a ring homomorphism

$$\mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\mathbb{P}^1, \mathbb{P}^1) \to \mathrm{GW}(k),$$

where $\mathrm{GW}(k)$ is the Grothendieck-Witt ring (see also [32, Lemma 3.2.4] for details). Identifying $\mathrm{GW}(\mathbb{R})$ with $\mathbb{Z} \times \mathbb{Z}$ by rank and signature, we see that $[-1]$ goes to the non-torsion element $(1, -1)$.
The example of motivic (co)homotopy is in fact universal for this phenomenon, so if $[-1]$ vanishes in $[\mathbb{P}^1, \mathbb{P}^1]$, then every $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ satisfying homotopy invariance and Nisnevich excision is alternating.
We are grateful to F. Morel for explaining the computation of the transposition action on $\mathbb{P}^1 \wedge \mathbb{P}^1$ in terms of quadratic forms and the Grothendieck-Witt group.
(4) Looking at the $\mathbb{A}^1$-stable homotopy category of $T$-spectra over $k$, $\mathcal{SH}(k)$, one can decompose the $\mathbb{Q}$-linearization $\mathcal{SH}(k)_{\mathbb{Q}}$ into the symmetric part $\mathcal{SH}(k)_+$ and alternating part $\mathcal{SH}(k)_-$ with respect to the exchange of factors on $\mathbb{G}_m \wedge \mathbb{G}_m$. Morel [33] has announced a result stating that $\mathcal{SH}(k)_-$ is in general equivalent to Voevodsky's big motivic category $DM(k)_{\mathbb{Q}}$, and that $\mathcal{SH}(k)_+$ is zero if -1 is a sum of squares. This suggests that the alternating part

$\mathcal{SH}_{S^1}(k)$ of the category of rational $S^1$-spectra $\mathcal{SH}_{S^1}(k)_\mathbb{Q}$ is closely related to the big category of effective motives (with $\mathbb{Q}$-coefficients) $DM^{\rm eff}(k)_\mathbb{Q}$. □

PROPOSITION 9.2.4. *Let $E$ be in $C_\mathbb{Q}(\mathbf{Sm}/k)$, $f : X \to \mathbb{G}_m$ an $I$-diagram of morphisms in $\mathbf{Sm}/k$.*

(1) *The sequence*

$$E(X) \xrightarrow{\iota_X} E(\log_f) \xrightarrow{N} E(\sigma_{\geq 1}\log_f)$$

*identifies $E(\sigma_{\geq 1}\log_f)$ with the quotient complex $E(\log_f)/E(X)$.*

(2) *Suppose $E$ is alternating. Then there is a natural quasi-isomorphism* $\mathrm{alt} : E(-1)(\log_f) \to E(\sigma_{\geq 1}\log_f)$*.*

*Proof.* It suffices to prove (1) for $E$ a presheaf of $\mathbb{Q}$-vector spaces, and $f : X \to \mathbb{G}_m$ a morphism in $\mathbf{Sm}/k$, where the assertion is obvious. Similarly, it suffices to construct a natural map $\theta_{E,X} : E(-1)(\log_f) \to E(\sigma_{\geq 1}\log_f)$ for $E$ a presheaf of $\mathbb{Q}$-vector spaces, extend as above to a map in general, and show that $\theta_{E,X}$ is a quasi-isomorphism for $E \in C_\mathbb{Q}(\mathbf{Sm}/k)$ alternating and $f : X \to \mathbb{G}_m$ a morphism in $\mathbf{Sm}/k$.

In fact, for $E$ a presheaf of $\mathbb{Q}$-vector spaces and $n \geq 1$,

$$E(-1)(\log_f)_n = \ker[(\mathrm{id}_X \times i)^* : E(X \times \mathbb{G}_m, \mathbb{G}_m^{\wedge n-1})^{\rm alt} \to E(X, \mathbb{G}_m^{\wedge n-1})^{\rm alt}]$$

so $E(-1)(\log_f)_n$ is a subspace of $E(X, \mathbb{G}_m^{\wedge n})$; thus $\mathrm{alt}_n$ defines a map $E(-1)(\log_f)_n \to E(\log_f)_n$. One easily checks that this defines a map of complexes

$$\mathrm{alt}_* : E(-1)(\log_f) \to E(\sigma_{\geq 1}\log_f),$$

as desired.

Now suppose that $E$ is alternating, i.e., that

$$(a) \qquad E(X \wedge \mathbb{G}_m^{\wedge n})^{\rm alt} \to E(X \wedge \mathbb{G}_m^{\wedge n})$$

is a quasi-isomorphism for all $n$ and $X$. This implies that the maps

$$E(X \times \mathbb{G}_m, \mathbb{G}_m^{\wedge n-1})^{\rm alt} \to E(X \times \mathbb{G}_m, \mathbb{G}_m^{\wedge n-1})$$
$$E(X \wedge \mathbb{G}_m^{\wedge n-1})^{\rm alt} \to E(X \wedge \mathbb{G}_m^{\wedge n-1})$$

are quasi-isomorphisms, hence

$$(b) \quad \mathrm{id}_{X\wedge\mathbb{G}_m} \times \mathrm{alt}_{n-1} : E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1}) \to E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1})^{\rm alt}$$

is a quasi-isomorphism. Since $E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1}) = E(X \wedge \mathbb{G}_m^{\wedge n})$, $(a)$ and $(b)$ imply that

$$\mathrm{id}_X \times \mathrm{alt}_n : E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1})^{\rm alt} \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\rm alt}$$

is a quasi-isomorphism. As $\mathrm{alt}_* : E(-1)(\log_f) \to E(\sigma_{\geq 1}\log_f)$ is the map on the total complex of the double complexes

$$n \mapsto \mathrm{id}_X \times \mathrm{alt}_n : E((X \wedge \mathbb{G}_m) \wedge \mathbb{G}_m^{\wedge n-1})^{\rm alt} \to E(X \wedge \mathbb{G}_m^{\wedge n})^{\rm alt}$$

we see that $\mathrm{alt}_*$ is a quasi-isomorphism. □

9.3. The log complex and path spaces. Let $f : X \to \mathbb{G}_m$ be a morphism in $\mathbf{Sm}/k$ arising from a semi-stable degeneration $\mathcal{X} \to (C, 0)$ and choice of parameter in $\mathcal{O}_{C,0}$. The monodromy sequence for $E(\lim_{t\to 0} X_t)$ arises from the sequence of Proposition 9.2.4 by comparing the path space $E(\mathcal{P}_{X/\mathbb{G}_m})$ with $E(\log_f)$.

We use the Dold-Kan correspondence to rewrite $E(\mathcal{P}_{X/\mathbb{G}_m})$ as a complex, namely: take for each $p$ the associated complex $E_p(\mathcal{P}^*_{X/\mathbb{G}_m})$ of the simplicial abelian group $n \mapsto E_p(\mathcal{P}^n_{X/\mathbb{G}_m})$, with differential the alternating sum of the face maps, and then take the total complex of the double complex

$$p \mapsto E_p(\mathcal{P}^*_{X/\mathbb{G}_m}).$$

We write this complex as $E(\mathcal{P}_{X/\mathbb{G}_m})$.

We also have the normalized subcomplex $NE(\mathcal{P}_{X/\mathbb{G}_m})$ of $E(\mathcal{P}_{X/\mathbb{G}_m})$, quasi-isomorphic to $E(\mathcal{P}_{X/\mathbb{G}_m})$ via the inclusion. Recall that, for a simplicial abelian group $n \mapsto A_n$, the normalized complex $NA_*$ has

$$NA_n := \cap_{i=1}^n \ker d_i : A_n \to A_{n-1}$$

with differential $d_0 : NA_n \to NA_{n-1}$. We define $NE(\mathcal{P}_{X/\mathbb{G}_m})$ by first taking the normalized subcomplex $NE_p(\mathcal{P}_{X/\mathbb{G}_m})$ of $E_p(\mathcal{P}^*_{X/\mathbb{G}_m})$ for each $p$, and then forming the total complex of the double complex $p \mapsto NE_p(\mathcal{P}_{X/\mathbb{G}_m})$.

In particular, we have the inclusion of double complexes

$$NE_*(\mathcal{P}^*_{X/\mathbb{G}_m}) \subset E_*(\mathcal{P}^*_{X/\mathbb{G}_m});$$

which gives for each $n$ the inclusion of single complexes

$$NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \subset E_*(\mathcal{P}^n_{X/\mathbb{G}_m});$$

Recalling that $\mathcal{P}^n_{X/\mathbb{G}_m} = X \times \mathbb{G}_m^n$, we thus have for each $n$ the inclusion of complexes

$$NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \subset E_*(X \times \mathbb{G}_m^n),$$

We may therefore apply the projections $\pi_n : E_*(X \times \mathbb{G}_m^n) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})$ and $\mathrm{alt}_n$, giving the map

$$\mathrm{alt}_n \circ \pi_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}.$$

Lemma 9.3.1. *Suppose that $E$ is alternating. Then*

$$\mathrm{alt}_n \circ \pi_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

*is a quasi-isomorphism.*

*Proof.* The map $p_1^* : E(X) \to E(X \times \mathbb{G}_m)$ splits $i_1^* : E(X \times \mathbb{G}_m) \to E(X)$, so we have the natural splitting

$$E(X \times \mathbb{G}_m) = E(X) \oplus E(X \wedge \mathbb{G}_m).$$

Extending this to $E(X \times \mathbb{G}_m^n)$ by using the maps $i_j^*$ and $p_j^*$, we have the natural splitting

$$E(X \times \mathbb{G}_m^n) = \oplus_{m=0}^n \oplus_{\substack{I \subset \{1,\ldots,n\} \\ |I|=m}} E(X \wedge \mathbb{G}_m^{\wedge I}). \tag{9.3.1}$$

To explain the notation: For $I \subset \{1,\ldots,n\}$, $E(X \wedge \mathbb{G}_m^{\wedge I}) = E(X \wedge \mathbb{G}_m^{\wedge|I|})$, included in $E(X \times \mathbb{G}_m^n)$ by the composition

$$E(X \wedge \mathbb{G}_m^{\wedge|I|}) \subset E(X \times \mathbb{G}_m^{|I|}) \xrightarrow{(\mathrm{id}_X \times p_I)^*} E(X \times \mathbb{G}_m^n)$$

where $p_I : \mathbb{G}_m^n \to \mathbb{G}_m^{|I|}$ is the projection on the factors $i_1,\ldots,i_m$ if $I = \{i_1,\ldots,i_m\}$ with $i_1 < \ldots < i_m$.
The action of $S_n$ on $E(X \times \mathbb{G}_m^n)$ preserves this decomposition, with $\sigma \in S_n$ mapping $E(X \wedge \mathbb{G}_m^{\wedge I})$ to $E(X \wedge \mathbb{G}_m^{\wedge\sigma^{-1}(I)})$ in the evident manner.
Now, for a simplicial abelian group $A$, the inclusion $NA_n \to A_n$ is split by universal expressions in the face and degeneracy maps. If $n \mapsto C_{*n}$ is a simplicial complex, we can form the complex of normalized subgroups (with respect to the simplicial variable) $N_{*_2}(C_{*_1,*_2})$ and take the homology $H_p(N_{*_2}(C_{*_1,*_2}), d_1)$, or we can form the simplicial abelian group $n \mapsto H_p(C_{*n})$ and take the normalized subgroup $N_{*_2}H_p(C_{*_1,*_2}, d_1) \subset H_p(C_{*_1,n}, d_1)$. Using the universal spitting mentioned above, we see that the two are the same:

$$H_p(N_{*_2}(C_{*_1,*_2}), d_1) = N_{*_2}H_p(C_{*_1,*_2}, d_1)$$

Since $S_n$ acts by the sign representation on $H_pE(X \wedge \mathbb{G}_m^{\wedge n})$, it follows that, for $1 \le j < n$, the diagonal map

$$\begin{aligned}&\delta_j : \mathbb{G}_m^{n-1} \to \mathbb{G}_m^n\\ &(t_1,\ldots,t_{n-1}) \mapsto (t_1,\ldots t_j, t_j, t_{j+1},\ldots t_{n-1})\end{aligned}$$

induces the zero map on $H_pE(X \wedge \mathbb{G}_m^{\wedge n})$. Similarly, the inclusion

$$\begin{aligned}&i_n : \mathbb{G}_m^{n-1} \to \mathbb{G}_m^n\\ &(t_1,\ldots,t_{n-1}) \mapsto (t_1,\ldots t_{n-1}, 1)\end{aligned}$$

is the zero map on $H_pE(X \wedge \mathbb{G}_m^I)$ if $n \in I$.
From this, it is not hard to see that

$$NH_pE_*(NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) = H_pE_*(X \wedge \mathbb{G}_m^{\wedge n}),$$

with respect to the decomposition of $E_*(\mathcal{P}^n_{X/\mathbb{G}_m}) = E_*(X \times \mathbb{G}_m^n)$ given by (9.3.1). Indeed,

$$\begin{aligned}\ker(H_p(d_n)) &= \ker(i_n^* : H_pE_*(X \times \mathbb{G}_m^n) \to H_pE_*(X \times \mathbb{G}_m^{n-1})\\ &= \oplus_{\substack{I\subset\{1,\ldots,n\}\\ n\in I}} H_pE_*(X \wedge \mathbb{G}_m^{\wedge I})\end{aligned}$$

It is then easy to show by descending induction on $i$ that

$$\cap_{j=i}^n \ker H_p(d_j) = \oplus_{\substack{I\subset\{1,\ldots,n\}\\ \{i,\ldots,n\}\subset I}} H_pE_*(X \wedge \mathbb{G}_m^{\wedge I})$$

from which our claim follows taking $i = 1$. Thus the projection

$$p_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})$$

is a quasi-isomorphism for each $n$. As $E$ is alternating, the alternating projection

$$\mathrm{alt}_n : E_*(X \wedge \mathbb{G}_m^{\wedge n}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$$

is a quasi-isomorphism as well, completing the proof. □

Lemma 9.3.2. *Let $E$ be in $C_{\mathbb{Q}}(\mathbf{Sm}/k)$. Let $\delta_0 : E(X \times \mathbb{G}_m^n) \to E(X \times \mathbb{G}_m^{n-1})$ be the map $[(\mathrm{id}_X, f), \mathrm{id}_{\mathbb{G}_m^{n-1}}]^*$. Then the diagram*

$$\begin{array}{ccc} E(X \times \mathbb{G}_m^n) & \xrightarrow{\delta_0} & E(X \times \mathbb{G}_m^{n-1}) \\ \downarrow{\scriptstyle \mathrm{alt}_n \circ \pi_n} & & \downarrow{\scriptstyle \mathrm{alt}_{n-1} \circ \pi_{n-1}} \\ E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}} & \xrightarrow[\cup f]{} & E(X \wedge \mathbb{G}_m^{\wedge n-1})^{\mathrm{alt}} \end{array}$$

*commutes.*

*Proof.* This follows directly from the definition of $\cup f$ and the fact that $\mathrm{alt}_n \circ \pi_n$ is the identity on $E(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$. □

Proposition 9.3.3. *Let $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ be alternating. Then the maps $\mathrm{alt}_n \circ \pi_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \times \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$ define a quasi-isomorphism of total complexes*

$$\mathrm{alt} \circ \pi : NE(\mathcal{P}_{X/\mathbb{G}_m}) \to E(\log_f)$$

*Proof.* That the maps $\mathrm{alt}_n \circ \pi_n : NE_*(\mathcal{P}^n_{X/\mathbb{G}_m}) \to E_*(X \wedge \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}$ define a map of total complex $NE(\mathcal{P}_{X/\mathbb{G}_m}) \to E(\log_f)$ follows from Lemma 9.3.2 and the fact that, for each $n$, the differential $d_0$ on $NE(\mathcal{P}^n_{X/\mathbb{G}_m})$ is the restriction of $\delta_0 : E(X \times \mathbb{G}_m^n) \to E(X \times \mathbb{G}_m^{n-1})$. Lemma 9.3.1 implies that $\mathrm{alt} \circ \pi$ is a quasi-isomorphism. □

We collect our results in

Theorem 9.3.4. *Let $E \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ be alternating, $f^\bullet : X^\bullet \to \mathbb{G}_m$ an $I$-diagram of morphisms in $\mathbf{Sm}/k$. Consider the diagram*

$$\begin{array}{ccccccccc} & & & & E(\mathcal{P}_{X/\mathbb{G}_m}) & & E(-1)(\mathcal{P}_{X/\mathbb{G}_m}) & & \\ & & & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} & & \\ & & E(X^\bullet) & \xrightarrow{\iota_{X^\bullet}} & NE(\mathcal{P}_{X/\mathbb{G}_m}) & & NE(-1)(\mathcal{P}_{X/\mathbb{G}_m}) & & \\ & & \| & & \downarrow{\scriptstyle \mathrm{alt}\circ\pi} & & \downarrow{\scriptstyle \mathrm{alt}\circ\pi} & & \\ & & \| & & \downarrow & & E(-1)(\log_{f^\bullet}) & & \\ & & \| & & \downarrow & & \uparrow{\scriptstyle \mathrm{alt}} & & \\ 0 & \longrightarrow & E(X^\bullet) & \xrightarrow[\iota_{X^\bullet}]{} & E(\log_{f^\bullet}) & \xrightarrow[N]{} & E(\sigma_{\geq 1}\log_{f^\bullet}) & \longrightarrow & 0 \end{array}$$

*Here the maps $i$ are the canonical inclusions and the maps $\iota_{X^\bullet}$ are the canonical maps given by the identities $E_n(\mathcal{P}_{X^i/\mathbb{G}_m})_0 = E_n(\log_{f^i})_0 = E_n(X^i)$. Then*

(1) *The diagram commutes and is natural in $E$ and $f^\bullet$.*
(2) *All the maps in the diagram are maps of complexes.*

(3) *All the vertical maps are quasi-isomorphisms*
(4) *The bottom sequence is termwise exact.*

*Proof.* The first point follows by construction, the remaining assertions follow from the Dold-Kan correspondence, Proposition 9.3.3 and Proposition 9.2.4. □

Corollary 9.3.5 (Monodromy sequence). *Let $W \in C_{\mathbb{Q}}(\mathbf{Sm}/k)$ be alternating, $p : \mathcal{X} \to (C, 0)$ a semi-stable degeneration, $t \in \mathcal{O}_{C,0}$ a uniformizing parameter. Then there is a distinguished triangle in $D(X_{0\mathrm{Zar}})$*

$$E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0) \to E(\lim_{t\to 0} X_t) \xrightarrow{N} E(-1)(\lim_{t\to 0} X_t) \to E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0)[1],$$

*natural in $(p, t)$ and in $E$.*

*Proof.* The commutative diagram of Theorem 9.3.4 being natural in the choice of $I$-diagram and in $E$, one can extend the diagram directly to the case of a co-presheaf of cosimplicial $I$-diagrams

$$U \mapsto f^\bullet(U) : X^\bullet(U) \to \mathbb{G}_m.$$

If we take $I$ to be finite, we can extend further to a co-presheaf of cosimplicial $I$-diagrams $f^\bullet : X^\bullet \to \mathbb{G}_m$, with $X^\bullet(U)$ a pro-scheme smooth over $k$, and still preserve the quasi-isomorphisms and exactness. Feeding the $I$-diagram

$$t \circ p : \tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0 \to \mathbb{G}_m$$

to this machine and taking the distinguished triangle induced by the exact sequence of log complexes at the bottom of the diagram completes the proof. □

*Remark* 9.3.6. If we splice together the long exact homotopy sequence for the monodromy distinguished triangle

$$E(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0) \to E(\lim_{t\to 0} X_t) \xrightarrow{N} E(-1)(\lim_{t\to 0} X_t)$$

with the localization distinguished triangle of Theorem 6.3.7

$$E^{D_{\mathrm{Zar}}}(X) \xrightarrow{\alpha_D} E(D_{\mathrm{Zar}}) \xrightarrow{\beta_D} E(\tau_\epsilon^{\hat{X}}(D)^0)$$

(both evaluated on $D = X_0$), we have the complex

$$(9.3.2) \quad \ldots \to E_n^{X_{0\mathrm{Zar}}}(X) \to E_n(X_{0\mathrm{Zar}}) \to E_n(\lim_{t\to 0} X_t) \xrightarrow{N} E(-1)_n(\lim_{t\to 0} X_t) \to E_{n-2}^{X_{0\mathrm{Zar}}}(X) \to E_{n-2}(X_{0\mathrm{Zar}}) \to \ldots$$

If $k = \mathbb{C}$ and $E$ represents singular cohomology (for the classical topology)

$$E_n(Y) = H^{-n}(Y(\mathbb{C}), \mathbb{Q})$$

then Steenbrink's theorem [44] states that the above sequence is exact. The argument uses the mixed Hodge structure on all the terms together with a weight argument.

One should be able to define a natural geometric "weight filtration" on $E(\lim_{t\to 0} X_t)$ by using the stratification of $X_0$ by faces. However, for general $E$, this additional structure might not suffice to force the exactness of the above sequence. It would be interesting to give a general additional structure on $E$ that would imply this exactness. □

9.4. Ayoub's monodromy sequence. The monodromy sequence of corollary 9.3.5 agrees with the monodromy sequence constructed by Ayoub in [3, section 3.6] after making the identification described in proposition 8.3.2 and working throughout in the category of rational motives $DM(k)_{\mathbb{Q}}$. Indeed, it is easy to check that our complex $E(\log_{f^\bullet})$ agrees with the contruction $E\otimes f_\eta^*\mathcal{L}og^\vee$ of [3, section 3.6.3], and that our isomorphism $E(\mathcal{P}_{X/\mathbb{G}_m}) \cong E(\log_{f^\bullet})$ agrees with the map $E\otimes f_\eta^*\mathcal{L}og^\vee \to E\otimes f_\eta^*\mathcal{U}$ induced by the map $\ell : \mathcal{L}og^\vee \to \mathcal{U}$ of [3, definition 3.6.42]. From there, one can easily compare with Ayoub's monodromy sequence [3, definition 3.6.37]. We give a sketch of these comparisons.

Ayoub's construction begins with the *Kummer motive* $\mathcal{K}$. We denote the object in $DM(S)_{\mathbb{Q}}$ represented by a smooth $S$-scheme $X$ as $m_S(X)$ and write $1_S$ for $m_S(S)$, the unit for the tensor structure in $DM(S)_{\mathbb{Q}}$; we delete the subscript $S$ from the notation for $S = \operatorname{Spec} k$. The 1-section $i_1 : S \to \mathbb{G}_{mS}$ induces the splitting

$$m_S(\mathbb{G}_{mS}) = 1_S \oplus 1_S(1)[1]$$

and thus the projection $\pi : m_S(\mathbb{G}_{mS}) \to 1_S(1)[1]$
We take $S = \mathbb{G}_m$. The diagonal $\Delta : \mathbb{G}_m \to \mathbb{G}_m \times_k \mathbb{G}_m = \mathbb{G}_{mS}$ induces $m_S(\Delta) : 1_S \to m_S(\mathbb{G}_{mS})$; composing with $\pi$ and twisting and shifting gives the map

$$\cup t_* : 1_{\mathbb{G}_m}(-1)[-1] \to 1_{\mathbb{G}_m}$$

The Kummer motive $\mathcal{K} \in DM(\mathbb{G}_m)_{\mathbb{Q}}$ is defined as the "cone" of $\cup t_*$: Ayoub shows there is a canonial distinguished triangle in $DM(\mathbb{G}_m)_{\mathbb{Q}}$

$$1_{\mathbb{G}_m}(-1)[-1] \xrightarrow{\cup t_*} 1_{\mathbb{G}_m} \to \mathcal{K} \to 1_{\mathbb{G}_m}(-1)$$

Next, Ayoub defines the object $\mathcal{L}og^\vee$ of $DM(\mathbb{G}_m)_{\mathbb{Q}}$. Viewing $\mathcal{K}$ as the two-term complex $[m_S(\mathbb{G}_{mS})(-1)[-1] \xrightarrow{\cup t_*} m_S(\mathbb{G}_{mS})]$ with $m_S(\mathbb{G}_{mS})$ in degree zero, one sees that the $n$th symmetric product $\mathrm{Sym}^n\mathcal{K}$ is the complex

$$\begin{aligned} 1_{\mathbb{G}_m}(-n)[-n] \xrightarrow{\cup t_*} 1_{\mathbb{G}_m}(-n+1)[-n+1] \xrightarrow{\cup t_*} \ldots \\ \xrightarrow{\cup t_*} 1_{\mathbb{G}_m}(-1)[-1] \xrightarrow{\cup t_*} 1_{\mathbb{G}_m}, \end{aligned}$$

where we write the map $\cup t_*(-i)[-i]$ as $\cup t_*$ for short. The map $1_{\mathbb{G}_m} \to \mathcal{K}$ gives rise to the map $\mathrm{Sym}^n\mathcal{K} \to \mathrm{Sym}^{n+1}\mathcal{K}$. We can take the limit $\mathcal{L}og^\vee$ in $DM(\mathbb{G}_m)_{\mathbb{Q}}$

$$\mathcal{L}og^\vee := \lim_n \mathrm{Sym}^n\mathcal{K}$$

As a complex, $\mathcal{L}og^\vee$ is just

$$\ldots \xrightarrow{\cup t_*} 1_{\mathbb{G}_m}(-n)[-n] \xrightarrow{\cup t_*} 1_{\mathbb{G}_m}(-n+1)[-n+1] \xrightarrow{\cup t_*} \ldots$$
$$\xrightarrow{\cup t_*} 1_{\mathbb{G}_m}(-1)[-1] \xrightarrow{\cup t_*} 1_{\mathbb{G}_m}.$$

Now suppose we have a semi-stable degeneration $f : \mathcal{X} \to \mathbb{A}^1$ and an object $E \in DM(\mathcal{X}^0)_\mathbb{Q}$. Let $f^0 : \mathcal{X}^0 \to \mathbb{G}_m$ be the restriction of $f$; since $f^0$ is smooth, we have $Lf^{0*} = f^*$. Let $i : X_0 \to \mathcal{X}$, $j : \mathcal{X}^0 \to \mathcal{X}$ be the inclusions. The *logarithmic* specialization functor $\log_f$ is defined by

$$\log_f(E) := Li^*Rj_*(E \otimes f^{0*}\mathcal{L}og^\vee)$$

*Remark* 9.4.1. If we replace $\mathcal{X}$ with $\mathcal{X} \times_{\mathbb{A}^1} \operatorname{Spec} \mathcal{O}_{\mathbb{A}^1,0}$, we have the canonical identification of $\mathcal{X}^0$ with the generic fiber $\mathcal{X}_\eta$ and $f^0$ with $f_\eta$. We avoid doing this to keep with the notation of our earlier sections. □

The first step in our comparison is

Lemma 9.4.2. *Take $E \in DM(\mathcal{X}^0)_\mathbb{Q}$, represented by a fibrant object $\tilde{E} \in C_\mathbb{Q}(\mathbf{Sm}/\mathcal{X}^0)$. Then $E \otimes f^{0*}\mathcal{L}og^\vee$ is represented by $\tilde{E}(\log_{f^\bullet})$.*

*Proof.* Note that we may assume that $\tilde{E}$ is alternating, since $E$ is a motive. Letting $\mathcal{H}om$ denote the internal Hom in $DM(\mathcal{X}^0)_\mathbb{Q}$. We have the distinguished triangle

$$E(-1)[-1] \to \mathcal{H}om(\mathbb{G}_{m\mathcal{X}_0}, E) \xrightarrow{i_1^*} \mathcal{H}om(1, E) = E \to E(-1)$$

Thus $E(-1)[-1]$ is represented by the presheaf

$$X' \mapsto \mathrm{fib}[\tilde{E}(X' \times_k \mathbb{G}_m) \xrightarrow{\mathrm{id} \times i_1^*} \tilde{E}(X')]$$

Similarly, for $n \geq 1$, $E(-n)[-n]$ is represented by the presheaf

$$X' \mapsto \tilde{E}(X' \wedge \mathbb{G}_m^{\wedge n}).$$

SInce $\tilde{E}$ is alternating, this latter presheaf is equivalent to

$$X' \mapsto \tilde{E}(X', \mathbb{G}_m^{\wedge n})^{\mathrm{alt}}.$$

Finally, the map $\cup f : \tilde{E}(X'_+ \wedge \mathbb{G}_m) \to \tilde{E}(X')$ is just the map induced by the pull-back by $f$ and $f \times \mathrm{id}$ of the diagonal map $\mathbb{G}_m \to \mathbb{G}_m \times_k \mathbb{G}_m$, hence $\cup f$ represents the map $f^{0*}(\cup t_*)$. The comparison follows easily from this. □

Next, Ayoub considers the object $\mathcal{U}$ of $DM(\mathbb{G}_m)_\mathbb{Q}$. Interpreting his general construction in the case of $DM(\mathbb{G}_m)_\mathbb{Q}$, $\mathcal{U}$ is the motive associated to the simplicial object

$$n \mapsto \mathcal{H}om_{DM(\mathbb{G}_m)_\mathbb{Q}}(\mathcal{P}^n_{\mathbb{G}_m}, 1_{\mathbb{G}_m}),$$

i.e., the homological complex which is $\mathcal{H}om_{DM(\mathbb{G}_m)_\mathbb{Q}}(\mathcal{P}^n_{\mathbb{G}_m}, 1_{\mathbb{G}_m})$ in degree $n$, and with differential the alternating sum of the maps induced by the coface maps in $\mathcal{P}_{\mathbb{G}_m}$. Naturally, to make sense of this, we need to lift this construction to the appropriate category of complexes. In any case, the same proof as for lemma 9.4.2 gives us a canonical isomorphism of $E \otimes f^{0*}\mathcal{U}$ with $E(\mathcal{P}_{\mathcal{X}^0/\mathbb{G}_m})$. Similarly, the uni-potent specialization functor $\Upsilon$ is given by

$$\Upsilon(E) = Li^*Rj_*(E \otimes f^{0*}\mathcal{U}).$$

Finally, we have

$$\mathcal{H}om_{DM(\mathbb{G}_m)_\mathbb{Q}}(\mathcal{P}^n_{\mathbb{G}_m}, 1_{\mathbb{G}_m}) \cong (1_{\mathbb{G}_m}(-1)[-1] \oplus 1_{\mathbb{G}_m})^{\otimes n}$$

and the first differential $\mathcal{U}_1 \to \mathcal{U}_0$ is

$$1_{\mathbb{G}_m}(-1)[-1] \oplus 1_{\mathbb{G}_m} \xrightarrow{\cup t_* + \mathrm{id}} 1_{\mathbb{G}_m}$$

Thus we have the evident map $\mathcal{K} \to \mathcal{U}$. The diagonal map on $\mathcal{P}_{\mathbb{G}_m}$ dualizes to make $\mathcal{U}$ a commutive ring object in $DM(\mathbb{G}_m)_\mathbb{Q}$. Ayoub notes that $\mathcal{K} \to \mathcal{L}og^\vee$ is universal for maps of $\mathcal{K}$ to a commutative ring in $DM(\mathbb{G}_m)_\mathbb{Q}$, hence there is a unique ring map $\ell : \mathcal{L}og^\vee \to \mathcal{U}$ making

$$\begin{tikzcd} \mathcal{K} \arrow[r] \arrow[dr] & \mathcal{L}og^\vee \arrow[d, "\ell"] \\ & \mathcal{U} \end{tikzcd}$$

commute. It is not hard to see that our map $\mathrm{alt} \circ \pi$ is induced by a map of complexes in $\mathbb{Q}\mathbf{Sm}/\mathbb{G}_m$ which commutes with the co-multiplications dual to the ring multiplications for $\mathcal{U}$ and $\mathcal{L}og^\vee$. Since both

$$E \otimes f^{0*}\mathcal{L}og^\vee \xrightarrow{\mathrm{id}\otimes\ell} E \otimes f^{0*}\mathcal{U}$$

and

$$E(\mathcal{P}_{\mathcal{X}^0/\mathbb{G}_m}) \xrightarrow{\mathrm{alt}\circ\pi} E(\log_{f^\bullet})$$

are isomorphisms in $DM(\mathcal{X}^0)_\mathbb{Q}$, it follows that these maps are inverse to each other.

Once we have the pair of compatible isomorphisms $E \otimes f^{0*}\mathcal{L}og^\vee \cong E(\log_{f^\bullet})$ and $E \otimes \mathcal{U} \cong E(\mathcal{P}_{\mathcal{X}^0/\mathbb{G}_m})$, it is easy to see that Ayoub's construction of the monodromy sequence and ours are compatible: Ayoub's construction follows from the obvious identification of $\mathcal{L}og^\vee(-1)$ with $\sigma_{\leq -1}\mathcal{L}og^\vee$ (cohomological notation) giving the distinguished triangle

$$1_{\mathbb{G}_m} \to \mathcal{L}og^\vee \xrightarrow{N} \mathcal{L}og^\vee(-1) \to 1_{\mathbb{G}_m}[1]$$

which clearly passes over to our identification $E(\sigma_{\geq 1}\log_{f^\bullet}) \cong E(-1)$ and the monodromy distinguished triangle of corollary 9.3.5.

## 10. Limit motives

We use our construction of limit cohomology, slightly modified, to give a construction of the limit motive of a semi-stable degeneration, as an object in the big category of motives $DM(k)$.

10.1. THE BIG CATEGORY OF MOTIVES. Voevodsky has defined the category of effective motives as the full subcategory $DM_-^{\mathrm{eff}}(k)$ of the derived category of Nisnevich sheaves with transfer $D_-(\mathbf{NST}(k))$ consisting of those complexes with strictly homotopy invariant cohomology sheaves.

In his thesis, Spitzweck [43] defines a "big" category of motives over a field $k$. Other constructions of a big category of motives over a noetherian base scheme $S$ have been given by Østvar-Röndigs [40] and also by Cisinski-Deglise [9]. To give the reader the main idea of all these constructions, we quote from a recent letter from Röndigs [41]:

"One may construct a model category of simplicial presheaves with transfers on $\mathbf{Sm}/k$, in which the weak equivalences and fibrations are defined via the functor forgetting transfers. Via the Dold-Kan correspondence, there is an induced model structure on nonnegative chain complexes of presheaves with transfers. Both may be stabilized with respect to $T$ or $\mathbb{P}^1$, in the sense of [18]. The Dold-Kan correspondence extends accordingly. Since $T$ is a suspension already, one can then pass to a model category of $\mathbb{G}_m$-spectra of integer-indexed chain complexes as well. For k a perfect field, results from [46] show that the homotopy category of the latter model category contains Voevodsky's $DM_{gm}$ as a full subcategory. "

We will use the $\mathbb{P}^1$-spectrum model. For details, we refer the reader to [40] and [35].

10.2. THE COHOMOLOGICAL MOTIVE. We start with the category of presheaves with transfer $\mathbf{PST}(k)$ on $\mathbf{Sm}/k$, which is defined as in [46] as the category of presheaves on the correspondence category $\mathbf{Cor}(k)$. We let $C_{\geq 0}(\mathbf{PST}(k))$ denote the model category of non-negative chain complexes in $\mathbf{PST}(k)$, with model structure induced from simplicial presheaves on $\mathbf{Sm}/k$, as described above. For $P \in C_{\geq 0}(\mathbf{PST}(k))$, let $P(-1)$ denote the presheaf

$$Y \mapsto \ker[P(Y \times \mathbb{P}^1) \xrightarrow{i_\infty^*} P(Y \times \infty)][2].$$

where "ker" means the termwise kernel of the termwise split surjection $i_\infty^*$. Let $\mathbb{Z}_X^{tr}$ denote the presheaf on $\mathbf{Cor}(k)$ represented by $X \in \mathbf{Sm}/k$, and let

$$\tilde{\mathbb{Z}}_{\mathbb{P}^1}^{tr} := \mathrm{coker}(\mathbb{Z}_{\mathrm{Spec}\, k}^{tr} \xrightarrow{i_{\infty *}} \mathbb{Z}_{\mathbb{P}^1}^{tr}).$$

One has the adjoint isomorphism

$$\mathrm{Hom}_{C_{\geq 0}(\mathbf{PST}(k))}(C \otimes \tilde{\mathbb{Z}}_{\mathbb{P}^1}^{tr}, C') \cong \mathrm{Hom}_{C_{\geq 0}(\mathbf{PST}(k))}(C, C'(-1)[-2])$$

so the bonding maps for $\mathbb{P}^1$-spectra in $C_{\geq 0}(\mathbf{PST}(k))$ can be just as well defined via maps

$$C_n \to C_{n+1}(-1)[-2].$$

We will use this normalization of the bonding morphisms from now on.

For an integer $q \geq 0$, we have the (homological) Friedlander-Suslin presheaf $\mathbb{Z}_{FS}(q)$. To define this, one starts with the presheaf with transfers of quasi-finite cycles $z_{q.fin}(\mathbb{A}^q)$, with value on $Y \in \mathbf{Sm}/k$ the cycles on $Y \times \mathbb{A}^q$ which are quasi-finite over $Y$. One then forms the Suslin complex $C_*(z_{q.fin}(\mathbb{A}^q))$ and reindexes:

$$\mathbb{Z}_{FS}(q)(Y)_n := C_{n-2q}(z_{q.fin}(\mathbb{A}^q))(Y) := z_{q.fin}(\mathbb{A}^q)(Y \times \Delta^{n-2q}).$$

(see [26, §2.4] for a precise definition). This represents motivic cohomology Zariski-locally:

$$H^p(X, \mathbb{Z}(q)) = \mathbb{H}^p(X_{\mathrm{Zar}}, \mathbb{Z}_{FS}(q)).$$

More generally, for $X \in \mathbf{Sm}/k$, define $\mathbb{Z}^X_{FS}(q)$ by

$$\mathbb{Z}^X_{FS}(q)(Y) := \mathbb{Z}_{FS}(q)(X \times Y).$$

We define

$$\delta_n : \mathbb{Z}^X_{FS}(n) \to \mathbb{Z}^X_{FS}(n+1)(-1)$$

by sending a cycle $W$ on $X \times Y \times \Delta^m \times \mathbb{A}^n$ to $W \times \Delta$, where $\Delta \subset \mathbb{A}^1 \times \mathbb{P}^1$ is the graph of the inclusion $\mathbb{A}^1 \subset \mathbb{P}^1$, and then reordering the factors to yield a cycle on $X \times Y \times \mathbb{P}^1 \times \Delta^m \times \mathbb{A}^{n+1}$.

Definition 10.2.1. Let $X$ be in $\mathbf{Sm}/k$. The *cohomological motive* of $X$ is the sequence

$$\tilde{h}(X) := (\mathbb{Z}^X_{FS}(0), \mathbb{Z}^X_{FS}(1)[2], \ldots, \mathbb{Z}^X_{FS}(n)[2n], \ldots)$$

with the bonding morphisms $\delta_n[2n]$. □

*Remark* 10.2.2. One can also define the cohomological motive $h(X) \in DM_{gm}(k)$ as the dual of the usual (homological) motive $m(X) := C^{\mathrm{Sus}}(\mathbb{Z}^{\mathrm{tr}}_X)$. For $X$ of dimension $d$, $h(X)(n)$ is actually in $DM^{\mathrm{eff}}_-(k)$ for all $n \geq d$, and is represented by $\mathbb{Z}^X_{FS}(n)$. From this, one sees that the image of $\tilde{h}(X)$ in $DM(k)$ is canonically isomorphic to $h(X)$.
Also, one can work in $DM^{\mathrm{eff}}_-(k)$ if one wants to define the cohomological motive of a diagram in $\mathbf{Sm}/k$ if the varieties involved have uniformly bounded dimension. Since our construction of limit cohomology uses varieties of arbitrarily large dimension, we need to work in $DM(k)$. □

10.3. The limit motive. It is now an easy matter to define the limit motive for a semi-stable degeneration. Let $\mathcal{X} \to (C, 0)$ be a semi-stable degeneration with parameter $t$ at 0; suppose the special fiber $X_0$ has irreducible components $X_0^1, \ldots, X_0^m$. We have the diagram (8.2.1) of cosheaves on $X_{0\mathrm{Zar}}$, $\lim_{t\to 0} X_t$, indexed by the non-empty subsets $I \subset \{1, \ldots, m\}$, which we write as

$$I \mapsto [\lim_{t\to 0} X_t]_I.$$

Taking global sections on $X_0$ yields the diagram of cosimplicial schemes

$$I \mapsto [\lim_{t\to 0} X_t]_I(X_0).$$

Applying $\tilde{h}$ gives us the diagram of $\mathbb{P}^1$-spectra in $C_{\geq 0}(\mathbf{PST}(k))$

$$I \mapsto \tilde{h}([\lim_{t\to 0} X_t]_I(X_0)).$$

We then take the homotopy limit over this diagram forming the complex

$$\lim_{t\to 0} \tilde{h}(X_t) := \operatorname*{holim}_I \{I \mapsto \tilde{h}([\lim_{t\to 0} X_t]_I(X_0))\}.$$

Definition 10.3.1. Let $\mathcal{X} \to (C,0)$ be a semi-stable degeneration with parameter $t$ at 0. The limit cohomological motive $\lim_{t\to 0} h(X_t)$ is the image of $\lim_{t\to 0} \tilde{h}(X_t)$ in $DM(k)$. □

Using the same procedure, we have, for $D \subset X$ a normal crossing scheme, the motive of the tubular neighborhood $h(\tau_\epsilon^{\hat{X}}(D))$ and the motive of the punctured tubular neighborhood $h(\tau_\epsilon^{\hat{X}}(D)^0)$. All the general results now apply for these cohomological motives. In particular, from corollary 9.3.5 we have the monodromy distinguished triangle (for the $\mathbb{Q}$-motive)

$$h(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0)_{\mathbb{Q}} \to \lim_{t\to 0} h(X_t)_{\mathbb{Q}} \to \lim_{t\to 0} h(X_t)_{\mathbb{Q}}(-1)$$

and theorem 6.3.7 gives the localization distinguished triangle

$$h^{X_0}(\mathcal{X}) \to h(X_0) \to h(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0),$$

where $h^{X_0}(\mathcal{X})$ is represented by

$$\mathrm{Cone}(\tilde{h}(\mathcal{X}) \xrightarrow{j^*} \tilde{h}(\mathcal{X} \setminus X_0))[-1].$$

From this latter triangle, we see that $h(\tau_\epsilon^{\hat{\mathcal{X}}}(X_0)^0)$ is in $DM_{gm}(k)$.

## 11. Gluing smooth curves

We use the exponential map defined in §5 to define an algebraic version of gluing smooth curves along boundary components. We begin by recalling the construction of the moduli space of smooth curves with boundary components; for details we refer the reader to the article by Hain [15].

11.1. Curves with boundary components. For a $k$-scheme $Y$, a *smooth curve over* $Y$ is a smooth proper morphism of finite type $p : \mathcal{C} \to Y$ with geometrically irreducible fibers of dimension one. We say that $\mathcal{C}$ has genus $g$ if all the geometric fibers of $p$ are curves of genus $g$. A *boundary component* of $\mathcal{C} \to Y$ consists of a section $x : Y \to \mathcal{C}$ together with an isomorphism $v : \mathcal{O}_Y \to x^*T_{\mathcal{C}/Y}$, where $T_{\mathcal{C}/Y}$ is the relative tangent bundle on $\mathcal{C}$. Equivalently, $v$ is a nowhere vanishing section of $T_{\mathcal{C}/Y}$ along $x$. A smooth curve with $n$ boundary components is $(\mathcal{C} \to Y, (x_1, v_1), \ldots, (x_n, v_n))$ with all the $x_i$ disjoint. One has the evident notion of isomorphism of such tuples, so we can consider the functor $\mathcal{M}_g^n$ on $\mathbf{Sch}_k$:

$$\mathcal{M}_g^n(Y) := \{\text{smooth genus } g \text{ curves over } Y \text{ with } n \text{ boundary components}\}/\cong$$

For $n = 0$, this is just the well-know functor of moduli of smooth curves, which admits the coarse moduli space $M_g$. For $n \geq 1$ and $g \geq 1$, it is easy to show that a smooth curve over $Y$ with $n$ boundary components admits no non-identity automorphisms (over $Y$), from which it follows that $\mathcal{M}_g^n$ is representable; we denote the representing scheme by $\mathcal{M}_g^n$ as well. The same holds for genus 0 if $n \geq 2$; in fact the data of a genus zero curve $C$ with two points $0, \infty$ together with a tangent vector $v \neq 0$ in $T_0(C)$ has no non-identity automorphisms.

One can form a partial compactification of $\mathcal{M}_g^n$ by allowing *stable* curves with boundary components. As we will not require the full extent of this theory, we restrict ourselves to connected curves $C$ with a single singularity, this being an ordinary double point $p$. We require that the boundary components are in the smooth locus of $C$. If $C$ is reducible, then $C$ has two irreducible components $C_1$, $C_2$; we also require that both $C_1$ and $C_2$ have at least one boundary component. As above, such data has no non-trivial automorphisms, which leads to the existence of a fine moduli space $\bar{\mathcal{M}}_g^n$. We let $\mathcal{C}_g^n \to \mathcal{M}_g^n$ be the universal curve with universal boundary components $(x_1, v_1), \ldots, (x_n, v_n)$, and $\bar{\mathcal{C}}_g^n \to \bar{M}_g^n$ the extended universal curve.
The boundary $\partial\bar{\mathcal{M}}_g^n := \bar{\mathcal{M}}_g^n \setminus \mathcal{M}_g^n$ is a disjoint union of divisors

$$\partial\bar{\mathcal{M}}_g^n := D_{(g,n)} \amalg \coprod_{(g_1,g_2),(n_1,n_2)} D_{(g_1,g_2),(n_1,n_2)},$$

where $D_{(g_1,g_2),(n_1,n_2)}$ consists of the curves $C_1 \cup C_2$ with $g(C_i) = g_i$, and with $C_i$ having $n_i$ boundary components (we specify which component is $C_1$ by requiring $C_1$ to contain the boundary component $(x_1, v_1)$) and $D_{(g,n)}$ is the locus of irreducible singular curves.
Let $(C, (x_1, v_1), \ldots)$ be a curve in $\partial\bar{\mathcal{M}}_g^n$ with singular point $p$. By standard deformation theory, it follows that $\partial\bar{\mathcal{M}}_g^n$ is a smooth divisor in $\bar{\mathcal{M}}_g^n$; let $N_{(g_1,g_2),(n_1,n_2)}$ denote the normal bundle of $D_{(g_1,g_2),(n_1,n_2)}$. Deformation theory gives a canonical identification of the fiber of the punctured normal bundle $N^0_{g_1,g_2,n_1,n_2} := N_{(g_1,g_2),(n_1,n_2)} \setminus 0$ at $(C, (x_1, v_1), \ldots)$ with $\mathbb{G}_m$-torsor of isomorphisms

$$\Lambda^2 T_{C,p} \cong k(p).$$

11.2. Algebraic gluing. We can now describe our algebraic construction of gluing curves. Fix integers $g_1, g_2, n_1, n_2 \geq 1$. We define the morphism

$$\bar{\mu} : \mathcal{M}_{g_1,n_1} \times \mathcal{M}_{g_2,n_2} \to D_{g_1,g_2,n_1-1,n_2-1}.$$

by gluing $(C_1, (x_1, v_1), \ldots, (x_{n_1}, v_{n_1}))$ and $(C_2, (y_1, w_1), \ldots, (y_{n_2}, w_{n_2}))$ along $x_{n_1}$ and $y_1$, forming the curve $C := C_1 \cup C_2$ with boundary components $(x_1, v_1), \ldots, (x_{n_1-1}, v_{n_1-1})$, $(y_2, w_2), \ldots, (y_{n_2}, w_{n_2})$ and singular point $p$. We lift $\bar{\mu}$ to

$$\mu : \mathcal{M}_{g_1,n_1} \times \mathcal{M}_{g_2,n_2} \to N^0_{g_1,g_2,n_1,n_2}$$

using the isomorphism $\Lambda^2 T_{C,p} \to k(p)$ which sends $v_{n-1} \wedge w_1$ to 1 and the identification of $(N^0_{g_1,g_2,n_1,n_2})_{C_1 \cup C_2, \ldots}$ described above.
We now pass to the category $\mathcal{SH}_{\mathbb{A}^1}(k)$. Taking the infinite suspension, the map $\mu$ defines the map

$$\Sigma^\infty \mu : \Sigma^\infty \mathcal{M}_{g_1,n_1+} \wedge \Sigma^\infty \mathcal{M}_{g_2,n_2+} \to \Sigma^\infty N^0_{g_1,g_2,n_1,n_2+}.$$

Composing with our exponential map defined in §5 gives us our gluing map

$$\oplus : \Sigma^\infty \mathcal{M}_{g_1,n_1+} \wedge \Sigma^\infty \mathcal{M}_{g_2,n_2+} \to \Sigma^\infty \mathcal{M}_{g_1+g_2,n_1+n_2-2+}.$$

*Remarks* 11.2.1. (1) If one fixes a curve $\mathcal{E} := (E, (x_1, v_1), (x_2, v_2)) \in \mathcal{M}_{1,2}$, one can form the tower under $\mathcal{E}\oplus$

$$\ldots \to \Sigma^\infty \mathcal{M}_{g,n} \to \Sigma^\infty \mathcal{M}_{g+1,n} \to \ldots,$$

and form the homotopy colimit $\Sigma^\infty \mathcal{M}_{\infty,n}$. If $E$ is an object of $\mathcal{SH}_{\mathbb{A}^1}(k)$, one thus has the $E$-cohomology $E^*(\mathcal{M}_{\infty,n})$. For intance, this gives a possible definition of stable motivic cohomology or algebraic $K$-theory of smooth curves. However, it is not at all clear if this limit is independent of the choice of $\mathcal{E}$. In the topological setting, one notes that the space $\mathcal{M}_{1,2}(\mathbb{C})$ is connected, so the limit cohomology, for example, is independent of the choice of $\mathcal{E}$. On the contrary, $\mathcal{M}_{1,2}(\mathbb{R})$ is not connected (the number of connected components in the real points of the curve corresponding to a real point of $\mathcal{M}_{1,2}$ splits $\mathcal{M}_{1,2}(\mathbb{R})$ into disconnected pieces), so even there, the choice of $\mathcal{E}$ plays a role. It is also not clear if $\mathcal{M}_{\infty,n}$ is independent of $n$ (up to isomorphism in $\mathcal{SH}_{\mathbb{A}^1}(k)$).
(2) In the topological setting, the map $\oplus$ is the infinite suspension of a map

$$\phi : \mathcal{M}_{g_1,n_1}(\mathbb{C}) \times \mathcal{M}_{g_2,n_2}(\mathbb{C}) \to \mathcal{M}_{g_1+g_2,n_1+n_2-2}(\mathbb{C}),$$

making $\amalg_{g,n}\mathcal{M}_{g,n+2}(\mathbb{C})$ into a topological monoid; the group completion is homotopy equivalent to the plus construction on the stable moduli space $\lim_{g\to\infty} \mathcal{M}_{g,1}(\mathbb{C})$ formed as in (1). Letting $\mathcal{M}_\infty(\mathbb{C})^+$ denote this group completion, the group structure induces on $\Sigma^\infty\mathcal{M}_\infty(\mathbb{C})^+$ the structure of a Hopf algebra (this was pointed out to me by Fabian Morel), the co-algebra structure being the canonical one on a suspension spectrum. The functoriality of the exponential map $\exp^0$ as described in Remark 5.2.2 shows that the maps $\oplus$ make $\bigvee_{g,n} \Sigma^\infty \mathcal{M}_{g,n+2}$ into a biaglebra object in $\mathcal{SH}_{\mathbb{A}^1}(k)$. It is not clear if there is an analogous “Hopf algebra completion” of $\bigvee_{g,n} \Sigma^\infty \mathcal{M}_{g,n+2}$ in $\mathcal{SH}_{\mathbb{A}^1}(k)$. □

## 12. Tangential base-points

Since, by work of Østvar-Röndigs [35], motivic cohomology is represented in $\mathcal{SH}_{\mathbb{A}^1}(k)$, our methods are applicable to this theory. However, one can simplify the construction somewhat, since we are dealing with complexes of abelian groups rather than spectra. One can also achieve a refinement incorporating the multiplicative structure; this allows for a motivic definition of tangential base-points for the category of mixed Tate motives from the point of view of cycle algebras. Of course, the unipotent specialization functor of Ayoub [3], when restricted to the triangulated category of Tate motives in $DM(-)$ also gives tangential base-points for mixed Tate motives, but we hope our construction will be useful for applications of this operation.

12.1. Cubical complexes. If we work with presheaves of complexes rather than presheaves of spectra, we can replace all our simplicial constructions with cubical versions. This enables an easy extension to the setting of differential graded algebras (d.g.a.'s), or even graded-commutative d.g.a.'s (c.d.g.a.'s) if we work with complexes of $\mathbb{Q}$-vector spaces. We list the main results without proof here; the methods discussed in [26, §2.5] carry over without difficulty.

For a commutative ring $R$, we denote the model category of complexes of $R$-modules on the big Nisnevich site, $\mathbf{C}_R(\mathbf{Sm}/S_{\mathrm{Nis}})$ by $\mathbf{C}_{R,\mathrm{Nis}}(S)$ and the derived category by $\mathbf{D}_{R,\mathrm{Nis}}(S)$.

The *cubical category* **Cube** has objects $\underline{n}$, $n = 0, 1, \ldots$. **Cube** is a subcategory of the category of finite sets, with $\underline{n}$ standing for the set $\{0,1\}^n$, with morphisms making **Cube** the smallest subcategory of finite sets containing the following maps:

(1) all inclusions $s_{i,n,\epsilon} : \{0,1\}^n \to \{0,1\}^{n+1}$, $\epsilon \in \{0,1\}$, $i = 1, \ldots, n+1$, where $s_{i,n,\epsilon}$ is the inclusion inserting $\epsilon$ in the $i$th factor.

(2) all projections $p_{i,n} : \{0,1\}^n \to \{0,1\}^{n-1}$, $i = 1, \ldots, n$, where $p_{i,n}$ is the projection deleting the $i$th factor.

(3) all maps $q_{i,n} : \{0,1\}^n \to \{0,1\}^{n-1}$, $i = 1, \ldots, n-1$, $n \geq 2$, defined by

$$q_{i,n}(\epsilon_1, \ldots, \epsilon_n) := (\epsilon_1, \ldots, \epsilon_{i-1}, \delta, \epsilon_{i+2}, \ldots, \epsilon_n)$$

with

$$\delta := \begin{cases} 0 & \text{if } (\epsilon_i, \epsilon_{i+1}) = (0,0) \\ 1 & \text{else.} \end{cases}$$

A cubical object in a category $\mathcal{C}$ is a functor $\mathbf{Cube} \to \mathcal{C}$.

The basic cubical object in **Sch** is the sequence of $n$-cubes $\square^* : \mathbf{Cube} \to \mathbf{Sm}/k$. The operations of the projections $p_{i,n}$ and inclusions $s_{i,n}$ are the evident ones; $q_{i,n}$ acts by

$$q_{i,n}(x_1, \ldots, x_n) := (x_1, \ldots, x_{i-1}, 1 - (x_i - 1)(x_{i+1} - 1), x_{i+2}, \ldots, x_n).$$

Now let $P : \mathbf{Cube} \to \mathbf{Mod}_R$ be a cubical $R$-module. We have the *cubical realization* $|P|^c \in \mathbf{C}_R$ with

$$|P|^c_n := P(\underline{n}) / \sum_{i=1}^{n} p^*_{i,n}(P(\underline{n-1})).$$

The differential $d^c_n : |P|^c_n \to |P|^c_{n-1}$ is

$$d^c_n := \sum_{i=1}^{n} (-1)^i s^*_{i,1} - \sum_{i=1}^{n} (-1)^i s^*_{i,0}.$$

$|-|^c$ is clearly a functor from the $R$-linear category of cubical $R$-modules to $\mathbf{C}(R)$; in particular, if we apply $|-|^c$ to a complex of $R$-modules, we end up with a double complex. For a complex $C$, also write $|C|^c$ for the total complex of this double complex, letting the context make the meaning clear.

*Example* 12.1.1. For a presheaf of abelian groups $P$ on $\mathbf{Sm}/k$, we have cubical presheaf $\mathcal{C}^c(P)$ with

$$\mathcal{C}^c(P)(Y) := P(Y \times \square^*).$$

Taking the cubical realization yields the *cubical Suslin complex* $C_*(P)^c$ with

$$C_*(P)^c(Y) := |\mathcal{C}^c(P)(Y)|^c.$$

□

The symmetric group $S_n$ acts on $C_n(P)^c$, we let $C_n(P)^c_{\mathrm{alt}}$ denote the subpresheaf of alternating sections. If $P$ is a presheaf of $\mathbb{Q}$-vector spaces, $C_n(P)^c_{\mathrm{alt}}$ is a canonical summand of $C_n(P)^c$, with projection given by the idempotent $\mathrm{Alt}_n := \frac{1}{n!}\sum_g \mathrm{sgn}(g)g$; one checks that the $C_n(P)^c_{\mathrm{alt}}$ form a subcomplex of $C_*(P)^c$.
The main result on these constructions is

PROPOSITION 12.1.2. *(1) There is a canonical homotopy equivalence of functors*

$$C_* \to C_*^c : \mathbf{C}(k) \to \mathbf{C}(k)$$

*(2) If $P$ is a complex of presheaves of $\mathbb{Q}$-vector spaces, the inclusion*

$$C_*(P)^c_{\mathrm{alt}} \to C_*(P)^c$$

*is a quasi-isomorphism*

*Sketch of proof; see* [27, §5] *for details.* For (1), one uses the algebraic maps

$$\square^n \to \Delta^n$$

which collapse the faces $x_i = 1$ to the vertex $(0, \ldots, 0, 1)$ to get a map $C_* \to C_*^c$. The homotopy inverse is given by triangulating the $\square^n$. For (2) one checks that $S_n$ acts by the sign representation on the homology sheaves of $C_*(P)^c$. The projections $\mathrm{Alt}_n$ define a map of complexes $\mathrm{Alt}_* : C_*(P)^c \to C_*(P)^c_{\mathrm{alt}}$ which thus gives the inverse in homology. □

12.2. CUBICAL TUBULAR NEIGHBORHOODS. For a closed immersion $i : W \to X$ in $\mathbf{Sm}/k$, set $\hat{\square}^n_{X,W} := (\widehat{\square^n_X})^h_{\square^n_W}$, giving us the cubical pro-scheme

$$\hat{\square}^*_{X,W} : \mathbf{Cube} \to \mathbf{Pro\text{-}Sm}/k$$

We use the same notation for morphisms in the cubical setting as in the simplicial version, e.g., $\hat{i}_W : \square^*_W \to \hat{\square}^*_{X,W}$. We have as well the co-presheaf on $W_{\mathrm{Zar}}$

$$\hat{\square}^n_{X,W_{\mathrm{Zar}}}(W \setminus F) := \hat{\square}^n_{X\setminus,W\setminus F}$$

and the cubical co-presheaf

$$\tau_\epsilon^{\hat{X}}(W)^c := \hat{\square}^*_{X,W_{\mathrm{Zar}}}.$$

Now let $P$ be in $\mathbf{C}(k)$. We define $P(\tau_\epsilon^{\hat{X}}(W)^c)_*$ to be the complex of presheaves

$$P(\tau_\epsilon^{\hat{X}}(W)^c)_* := |P(\tau_\epsilon^{\hat{X}}(W)^c)|^c.$$

We also have the alternating subcomplex $P(\tau_\epsilon^{\hat{X}}(W)^c)^{\mathrm{alt}} \subset P(\tau_\epsilon^{\hat{X}}(W)^c)$.
We have as well the punctured tubular neighborhood in cubical form

$$\tau_\epsilon^{\hat{X}}(W)^{0c} := \tau_\epsilon^{\hat{X}}(W)^c \setminus \square^*_{W_{\mathrm{Zar}}}$$

on which we can evaluate $P$:

$$P(\tau_\epsilon^{\hat{X}}(W)^{0c})_* := |P(\tau_\epsilon^{\hat{X}}(W)^{0c})|^c.$$

Let $P(\tau_\epsilon^{\hat{X}}(W)^{0c})^{\mathrm{alt}} \subset P(\tau_\epsilon^{\hat{X}}(W)^{0c})$ be the alternating subcomplex.

We let $EM : \mathbf{C} \to \mathbf{Spt}$ be a choice of the Eilenberg-Maclane spectrum functor. Our main comparison result is

Theorem 12.2.1. *(1) Let $i : W \to X$ be a closed immersion in $\mathbf{Sm}/k$. For $P \in \mathbf{C}(k)$, there are natural isomorphisms in $\mathcal{SH}(W_{\mathrm{Zar}})$*

$$EM(P(\tau_\epsilon^{\hat{X}}(W)^c)) \cong EM(P)(\tau_\epsilon^{\hat{X}}(W))$$
$$EM(P(\tau_\epsilon^{\hat{X}}(W)^{0c})) \cong EM(P)(\tau_\epsilon^{\hat{X}}(W)^0)$$

*(2) If $P$ is a presheaf of complexes of $\mathbb{Q}$-vector spaces, then the inclusion*

$$P(\tau_\epsilon^{\hat{X}}(W)^c)^{\mathrm{alt}} \to P(\tau_\epsilon^{\hat{X}}(W)^c)$$

*is a quasi-isomorphism.*

*Proof.* Define $P(\tau_\epsilon^{\hat{X}}(W))$ to be the total complex of the double complex associated to the simplicial complex $n \mapsto P(\tau_\epsilon^{\hat{X}}(W)^n)$. The homotopy equivalence used in Proposition 12.1.2(1) extends, by the functoriality of the Nisnevich neighborhood, to a homotopy equivalence

$$P(\tau_\epsilon^{\hat{X}}(W))^c \sim P(\tau_\epsilon^{\hat{X}}(W))$$

This yields a weak equivalence on the associated Eilenberg-Maclane spectra. Since the functor $EM$ passes to the homotopy category, we have a canonical isomorphism

$$EM(P)(\tau_\epsilon^{\hat{X}}(W))) \cong EM(P(\tau_\epsilon^{\hat{X}}(W))).$$

Putting these isomorphisms together completes the proof of the first assertion for the tubular neighborhood. The proof for the punctured tubular neighborhood is essentially the same. The second assertion follows from Proposition 12.1.2(2). □

12.3. The motivic c.d.g.a. There are a number of different complexes which represent motivic cohomology; we will use the strictly functorial one of Friedlander-Suslin, $\mathbb{Z}_{FS}(q)$ (see the description in §10.2) reindexed as a cohomological complex:

$$\mathbb{Z}_{FS}(q)^n := \mathbb{Z}_{FS}(q)_{-n}.$$

We will use the cubical version $\mathbb{Z}_{FS}(q)^c$:

$$\mathbb{Z}_{FS}(q)^{c,n}(Y) := C_{2q-n}(z_{q.fin}(\mathbb{A}^q))^c(Y).$$

By Proposition 12.1.2, $\mathbb{Z}_{FS}(q)^c$ is quasi-isomorphic to $\mathbb{Z}_{FS}(q)$.
Passing to $\mathbb{Q}$-coefficients, we have the quasi-isomorphic alternating subcomplex $\mathbb{Q}_{FS}(q)^c_{\mathrm{alt}} \subset \mathbb{Q}_{FS}(q)^c$. We may also symmetrize with respect to the coordinates in the $\mathbb{A}^q$ in $z_{q.fin}(\mathbb{A}^q)$; it is shown in [26] that the inclusion

$$\mathbb{Q}_{FS}(q)^c_{\mathrm{alt,sym}} \subset \mathbb{Q}_{FS}(q)^c_{\mathrm{alt}}$$

is also a quasi-isomorphism.
The product map

$$z_{q.fin}(\mathbb{A}^q)(\square^n \times Y) \otimes z_{q.fin}(\mathbb{A}^{q'})(\square^{n'} \times Y) \to z_{q.fin}(\mathbb{A}^{q+q'})(\square^{n+n'} \times Y)$$

makes the graded complex

$$\tilde{\mathcal{N}}_{\mathbb{Z}} := \oplus_{q\geq 0}\mathbb{Z}_{FS}(q)^c$$

into a presheaf of Adams-graded d.g.a.'s on $\mathbf{Sm}/k$ (with Adams grading $q$). Passing to $\mathbb{Q}$-coefficients, and following the product with the alternating and symmetric projections makes

$$\mathcal{N} := \oplus_{q\geq 0}\mathbb{Q}_{FS}(q)^c_{\mathrm{alt,sym}}$$

a presheaf of Adams-graded c.d.g.a.'s, the *motivic* c.d.g.a. on $\mathbf{Sm}/k$.
We let $\mathcal{N} \to \mathcal{N}^{\mathrm{fib}}$ denote a fibrant model of $\mathcal{N}$ in the model category of (Adams-graded) c.d.g.a.'s on $\mathbf{Sm}/k$, where the weak equivalences are Adams-graded quasi-isomorphisms of c.d.g.a.'s for the Zariski topology.

*Remarks* 12.3.1. (1) Since $\mathcal{N}$ is strictly homotopy invariant [46, Theorem 4.2], $\mathcal{N}^{\mathrm{fib}}$ is homotopy invariant.
(2) In case $k$ admits resolution of singularities (i.e., char$k = 0$) the canonical map $\mathbb{Z}_{FS}(q) \to \mathbb{Z}_{FS}(q)^{\mathrm{fib}}$ is a pointwise weak equivalence [46, Theorem 7.4]. Thus, in this case, we can use $\mathcal{N}$ instead of $\mathcal{N}^{\mathrm{fib}}$. □

12.4. The specialization map. We consider the situation of a smooth curve $C$ over our base-field $k$ with a $k$-point x. We let $\mathcal{O}$ denote the local ring of $x$ in $C$, $K$ the quotient field of $\mathcal{O}$ and choose a uniformizing parameter $t$, which we view as giving a map

$$t : \operatorname{Spec}\mathcal{O} \to \mathbb{A}^1.$$

sending $x$ to 0.
Letting $i_x : x \to \operatorname{Spec}\mathcal{O}$ be the inclusion, we have the restriction map

$$i_x^* : \mathcal{N}(\mathcal{O}) \to \mathcal{N}(k(x)),$$

which is a morphism of Adams-graded c.d.g.a.'s. In this section, we extend $i_x^*$ to a map

$$sp_t : \mathcal{N}(K) \to \mathcal{N}(k(x))$$

in the homotopy category of Adams-graded c.d.g.a.'s over $\mathbb{Q}$ (denoted $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$). This is essentially our construction of the tubular neighborhood, where we use cubical constructions throughout to keep track of the multiplication.
First, if we apply $\mathcal{N}$ to $\square^* \times Y$ and take the alternating projection again, we have the presheaf of c.d.g.a.'s $\mathcal{N}(\square^*_{\mathrm{alt}})$ and the quasi-isomorphism of presheaves of c.d.g.a.'s

$$\iota : \mathcal{N} \to \mathcal{N}(\square^*_{\mathrm{alt}}).$$

Next, write $\hat{\square}^{m0}_{C,x}$ for $\hat{\square}^m_{C,x} \setminus \square^m_x$, and consider the cubical punctured tubular neighborhood $\mathbb{Z}_{FS}(q)^c(\tau^{\hat{C}}_\epsilon(x)^{0c})$. The product map

$$\begin{aligned} z_{q.fin}(\mathbb{A}^q)(\square^n \times \hat{\square}^{m0}_{C,x}) \otimes z_{q.fin}(\mathbb{A}^{q'})(\square^{n'} \times \hat{\square}^{m'0}_{C,x}))\\ \to z_{q.fin}(\mathbb{A}^{q+q'})(\square^{n+n'} \times \hat{\square}^{m+m'0}_{C,x})) \end{aligned}$$

makes $\oplus_{q\geq 0}\mathbb{Z}_{FS}(q)^c(\tau_\epsilon^{\widehat{\mathrm{Spec}\,\mathcal{O}}}(x)^{0c})$ into an Adams-graded d.g.a.; taking the alternating projection in both the $\square^n$ and $\hat{\square}^{m0}_{C,x}$ variables, and the symmetric projection in $\mathbb{A}^q$ and applying the fibrant model gives a presheaf of Adams-graded c.d.g.a.'s, denoted $\mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\widehat{\mathrm{Spec}\,\mathcal{O}}}(x)^{0c}_{\mathrm{alt}})$.
Similarly, we perform this construction using the full tubular neighborhood, giving the presheaf $\mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^c_{\mathrm{alt}})$, and the commutative diagram of Adams-graded c.d.g.a.'s:

$$\begin{tikzcd}
\mathcal{N}(k(x)) \arrow[d, "\iota"'] & \mathcal{N}(\mathcal{O}) \arrow[l, "i_x^*"'] \arrow[r, "\mathrm{res}"] \arrow[d, "\pi_{\mathcal{O}}^*"'] & \mathcal{N}(K) \arrow[d, "\pi_K^*"] \\
\mathcal{N}^{\mathrm{fib}}(\square^*_{\mathrm{alt}})(k(x)) & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^c_{\mathrm{alt}}) \arrow[l, "i_x^*"] \arrow[r, "\mathrm{res}"'] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^{0c}_{\mathrm{alt}})
\end{tikzcd}$$

Replacing $(C,x)$ with $(\mathbb{A}^1,0)$ and using $\mathbb{A}^1$ and $\mathbb{G}_m$ instead of $\mathrm{Spec}\,\mathcal{O}$ and $\mathrm{Spec}\,K$ yields the commutative diagram of Adams-graded c.d.g.a.'s

$$\begin{tikzcd}
\mathcal{N}(k(0)) \arrow[d, "\iota"'] & \mathcal{N}^{\mathrm{fib}}(\mathbb{A}^1) \arrow[l, "i_0^*"'] \arrow[r, "\mathrm{res}"] \arrow[d, "\pi_{\mathbb{A}^1}^*"'] & \mathcal{N}^{\mathrm{fib}}(\mathbb{G}_m) \arrow[d, "\pi_{\mathbb{G}_m}^*"] \\
\mathcal{N}^{\mathrm{fib}}(\square^*_{\mathrm{alt}})(k(x)) & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{\mathbb{A}^1}}(0)^c_{\mathrm{alt}}) \arrow[l, "i_0^*"] \arrow[r, "\mathrm{res}"'] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{\mathbb{A}^1}}(0)^{0c}_{\mathrm{alt}}).
\end{tikzcd}$$

By Corollary 3.3.3 and Corollary 4.1.4, the maps $\pi_{\mathbb{A}^1}^*$ and $\pi_{\mathbb{G}_m}^*$ are quasi-isomorphisms of complexes, hence quasi-isomorphisms of Adams-graded c.d.g.a.'s. Since $\mathcal{N}^{\mathrm{fib}}$ is homotopy invariant, the maps $\iota$ are quasi-isomorphisms of Adams-graded c.d.g.a.'s.
Finally, the map $t$ induces the commutative diagram of Adams-graded c.d.g.a.'s

$$\begin{tikzcd}
\mathcal{N}(k(x)) & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^c_{\mathrm{alt}}) \arrow[l, "i_x^*"'] \arrow[r, "\mathrm{res}"] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{C}}(x)^{0c}_{\mathrm{alt}}) \\
\mathcal{N}(k(0)) \arrow[u, "t^*"] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{\mathbb{A}^1}}(x)^c_{\mathrm{alt}}) \arrow[l, "i_0^*"] \arrow[r, "\mathrm{res}"'] \arrow[u, "t^*"] & \mathcal{N}^{\mathrm{fib}}(\tau_\epsilon^{\hat{\mathbb{A}^1}}(x)^{0c}_{\mathrm{alt}}). \arrow[u, "t^*"']
\end{tikzcd}$$

Since $t : (C,x) \to (\mathbb{A}^1,0)$ is a Nisnevich neighborhood of 0 in $\mathbb{A}^1$, all three maps $t^*$ are isomorphisms. Putting these diagrams together and inverting the quasi-isomorphisms $\iota$, $t^*$, $\pi_{\mathbb{A}^1}^*$ and $\pi_{\mathbb{G}_m}^*$ yields the commutative diagram in $\mathcal{H}(\mathrm{c.d.g.a}_{\mathbb{Q}})$:

(12.4.1)
$$\begin{tikzcd}
\mathcal{N}(k(x)) & \mathcal{N}(\mathcal{O}) \arrow[l, "i_x^*"'] \arrow[r, "\mathrm{res}"] \arrow[d, "\phi_{\mathcal{O}}^*"'] & \mathcal{N}(K) \arrow[d, "\phi_K^*"] \\
\mathcal{N}(k(0)) \arrow[u, "t^*"] & \mathcal{N}^{\mathrm{fib}}(\mathbb{A}^1) \arrow[l, "i_0^*"] \arrow[r, "\mathrm{res}"'] & \mathcal{N}^{\mathrm{fib}}(\mathbb{G}_m)
\end{tikzcd}$$

Definition 12.4.1. Let $i_1 : \operatorname{Spec} k \to \mathbb{G}_m$ be the inclusion. The map $sp_t : \mathcal{N}(K) \to \mathcal{N}(k(x))$ in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$ is defined to be the composition

$$\mathcal{N}(K) \xrightarrow{\phi_K^*} \mathcal{N}^{\text{fib}}(\mathbb{G}_m) \xrightarrow{i_1^*} \mathcal{N}^{\text{fib}}(k) \cong \mathcal{N}(k) = \mathcal{N}(k(0)) \xrightarrow{t^*} \mathcal{N}(k(x)).$$

□

Proposition 12.4.2. *The diagram in* $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$

$$\begin{tikzcd} \mathcal{N}(\mathcal{O}) \arrow[r, "\text{res}"] \arrow[dr, "i_x^*"'] & \mathcal{N}(K) \arrow[d, "sp_t"] \\ & \mathcal{N}(k(x)) \end{tikzcd}$$

*commutes.*

*Proof.* Since $\mathcal{N}^{\text{fib}}$ is homotopy invariant, the maps

$$i_0^*, i_1^* : \mathcal{N}^{\text{fib}}(\mathbb{A}^1) \to \mathcal{N}(k)$$

are equal in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$. The proposition follows directly from this and a chase of the commutative diagrams defined above. □

*Remark* 12.4.3. In the situation we are considering, we already have the following diagram:

$$\mathcal{N}(K) \to \mathcal{N}(\lim_{t\to 0} \operatorname{Spec} K) \cong \mathcal{N}(k(0)).$$

However, the above diagram is only a diagram in the homotopy category of complexes of $\mathbb{Q}$-vector spaces, which is thus equivalent to the same diagram for cohomology of the complexes involved. We have gone to the trouble of redoing our theory using cubes throughout because we need to keep track of the multiplication, i.e. our construction lifts the above diagram in $D^b(\mathbb{Q})$ to one in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$. □

12.5. The specialization functor. For a field $k$, we have the triangulated category $\mathrm{DMT}(k)$ of *mixed Tate motives* over $k$, this being the full triangulated subcategory of Voevodsky's triangulated category of motives (with $\mathbb{Q}$-coefficients), $DM_{gm}(k)_{\mathbb{Q}}$, generated by the Tate objects $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.

We will also use in this section the derived category of *finite cell modules* over an Adams-graded c.d.g.a. $\mathcal{A}$, $\mathrm{DCM}(\mathcal{A})$. This construction was introduced in [23]; we refer the reader to the discussion in [26, §5] for the properties of DCM we will be using below.

Let $\mathcal{O}$ be as in the previous section the local ring of a $k$-point $x$ on a smooth curve $C$ over $k$, with quotient field $K$. The map $sp_t : \mathcal{N}(K) \to \mathcal{N}(k(x))$ yields an exact tensor functor

$$sp_t : \mathrm{DMT}(K) \to \mathrm{DMT}(k(x))$$

Indeed, as discussed in [26, §5.5], Spitzweck's representation theorem gives a natural equivalence of $\mathrm{DMT}(k)$ with the derived category $\mathrm{DCM}(\mathcal{N}(k))$ of finite cell modules over the Adams-graded c.d.g.a. $\mathcal{N}(k)$, as triangulated tensor $\mathbb{Q}$-tensor categories.

The functor DCM associating to an Adams-graded $\mathbb{Q}$-c.d.g.a. $\mathcal{A}$ the triangulated tensor category DCM($\mathcal{A}$) takes quasi-isomorphisms to triangulated tensor equivalences, hence DCM descends to a well-defined pseudo-functor on $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$. Thus, we may make the following

Definition 12.5.1. Let $\mathcal{O}$ be the local ring of a $k$-point $x$ on a smooth curve $C$ over $k$, with quotient field $K$ and uniformizing parameter $t$. Let $sp_t : \mathrm{DMT}(K) \to \mathrm{DMT}(k(x))$ be the exact tensor functor induced by $\mathrm{DCM}(sp_t) : \mathrm{DCM}(\mathcal{N}(K)) \to \mathrm{DCM}(\mathcal{N}(k(x))$, using Spitzweck's representation theorem to identify the derived categories of cell modules with the appropriate category of mixed Tate motives. □

*Remark* 12.5.2. (1) The discussion in [26, §5.5], in particular, the statement and proof of Spitzweck's representation theorem, is in the setting of motives over a field. However, we now have available a reasonable triangulated category $DM(S)$ of motives over an arbitrary base-scheme $S$ (see [48]), and we can thus define the triangulated category of mixed Tate motives over $S$, DMT($S$), as in the case of a field.
Furthermore, if $S$ is in $\mathbf{Sm}/k$ for $k$ a field of characteristic zero, then $\mathcal{N}(S)$ has the correct cohomology, i.e.

$$H^n(\mathcal{N}(S)) = \oplus_{q\geq 0} H^n(S, \mathbb{Q}(q)),$$

and one has the isomorphism

$$H^n(S, \mathbb{Z}(q)) \cong \mathrm{Hom}_{DM(S)}(\mathbb{Z}, \mathbb{Z}(q)).$$

This is all that is required for the argument in [26, §5.5] to go through, yielding the equivalence of the triangulated tensor category of cell modules DCM($\mathcal{N}(S)$) with DMT($S$).

(2) Joshua [20] has *defined* the triangulated category of $\mathbb{Q}$ mixed Tate motives over $S$ as DCM($\mathcal{N}(S)$); the discussion in (1) shows that this agrees with the definition as a subcategory of $DM(S)_{\mathbb{Q}}$. □

With these remarks, we can now state the main compatibility property of the functor $sp_t : \mathrm{DMT}(K) \to \mathrm{DMT}(k(x))$.

Proposition 12.5.3. *Let $\mathcal{O}$ be the local ring of a $k$-point $x$ on a smooth curve $C$ over $k$, with quotient field $K$ and uniformizing parameter $t$. Let $i_x^* : \mathrm{DMT}(\mathcal{O}) \to \mathrm{DMT}(k)$ and $j^* : \mathrm{DMT}(\mathcal{O}) \to \mathrm{DMT}(K)$ be the functors induced by the inclusions $i_x : \mathrm{Spec}\, k \to \mathrm{Spec}\, \mathcal{O}$ and $j : \mathrm{Spec}\, K \to \mathrm{Spec}\, \mathcal{O}$, respectively. Then the diagram*

$$\begin{tikzcd} \mathrm{DMT}(\mathcal{O}) \arrow[r, "j^*"] \arrow[dr, "i_x^*"'] & \mathrm{DMT}(K) \arrow[d, "sp_t"] \\ & \mathrm{DMT}(k(x)) \end{tikzcd}$$

*commutes up to natural isomorphism.*

*Proof.* This follows from Proposition 12.4.2 and the functoriality (up to natural isomorphism) of the equivalence $\mathrm{DCM}(\mathcal{N}(S)) \sim \mathrm{DMT}(S)$. □

12.6. COMPATIBILITY WITH SPECIALIZATION ON MOTIVIC COHOMOLOGY. As above, let $\mathcal{O}$ be the local ring of a closed point $x$ on a smooth curve $C$ over $k$, with quotient field $K$ and uniformizing parameter $t$. We have the localization sequence for motivic cohomology

$$\ldots \to H^n(\mathcal{O}, \mathbb{Z}(q)) \xrightarrow{j^*} H^n(K, \mathbb{Z}(q)) \xrightarrow{\partial} H^{n-1}(k(x), \mathbb{Z}(q-1)) \xrightarrow{i_{x*}} \ldots$$

In addition, the parameter $t$ determines the element $[t] \in H^1(K, \mathbb{Z}(1))$. One defines the *specialization homomorphism*

$$\widetilde{\mathrm{sp}}_t : H^n(K, \mathbb{Z}(q)) \to H^n(k(x), \mathbb{Z}(q))$$

by the formula

$$\widetilde{\mathrm{sp}}_t(\alpha) := \partial([t] \cup \alpha).$$

On the other hand, if $k(x) = k$, we have the specialization functor

$$sp_t : \mathrm{DMT}(K) \to \mathrm{DMT}(k(x))$$

and the natural identifications

$$H^n(K, \mathbb{Q}(q)) \cong \mathrm{Hom}_{\mathrm{DMT}(K)}(\mathbb{Q}, \mathbb{Q}(q)[n])$$
$$H^n(k, \mathbb{Q}(q)) \cong \mathrm{Hom}_{\mathrm{DMT}(k)}(\mathbb{Q}, \mathbb{Q}(q)[n]).$$

Thus the functor $sp_t$ induces the homomorphism

$$sp_t : \mathrm{Hom}_{\mathrm{DMT}(K)}(\mathbb{Q}, \mathbb{Q}(q)[n]) \to \mathrm{Hom}_{\mathrm{DMT}(k)}(\mathbb{Q}, \mathbb{Q}(q)[n])$$

and hence a new homomorphism

$$sp'_t : H^n(K, \mathbb{Q}(q)) \to H^n(k, \mathbb{Q}(q)).$$

PROPOSITION 12.6.1. *$sp'_t$ agrees with the $\mathbb{Q}$-extension of $\widetilde{\mathrm{sp}}_t$.*

*Proof.* Using the equivalence $\mathrm{DMT}(K) \sim \mathrm{DCM}(\mathcal{N}(K))$ and the canonical identifications

$$\mathrm{Hom}_{\mathrm{DCM}(K)}(\mathbb{Q}, \mathbb{Q}(q)[n]) \cong H^n(\mathcal{N}(K)) \cong \oplus_{q \geq 0} H^n(K, \mathbb{Q}(q))$$

(and similarly for $k$) we need to show that the $\mathbb{Q}$-linear extension of $\widetilde{\mathrm{sp}}_t$ agrees with the map

$$H^n(sp_t) : H^n(\mathcal{N}(K)) \to H^n(\mathcal{N}(k))$$

induced by $sp_t : \mathcal{N}(K) \to \mathcal{N}(k)$.
For this, take an element $\alpha \in H^n(K, \mathbb{Z}(q))$ and set

$$\bar{\beta} := \partial\alpha \in H^{n-1}(k, \mathbb{Z}(q-1)).$$

Since $i_x : x \to \mathrm{Spec}\, \mathcal{O}$ is split by the structure morphism $\pi : \mathrm{Spec}\, \mathcal{O} \to \mathrm{Spec}\, k$, we can lift $\bar{\beta}$ to $\beta : \pi^*(\bar{\beta}) \in H^{n-1}(\mathcal{O}, \mathbb{Z}(q-1))$. Then

$$\partial([t] \cup \beta) = \partial([t]) \cup i_x^* \beta = \bar{\beta},$$

the first identity following from the Leibniz rule and the second from the fact that $\partial([t]) = 1 \in H^0(k, \mathbb{Z}(0))$. Thus

$$\partial(\alpha - [t] \cup \beta) = 0,$$

hence there is a class $\gamma \in H^n(\mathcal{O}, \mathbb{Z}(q))$ with

$$j^*\gamma = \alpha - [t] \cup \beta.$$

We consider $\gamma$ as an element of $H^n(\mathcal{N}(\mathcal{O}))$.
By Proposition 12.4.2, we have

$$H^n(i_x^*)(\gamma) = H^n(sp_t)(\alpha - [t] \cup \beta).$$

By the functoriality of the identification

$$H^n(\mathcal{N}(-)) \cong \oplus_{q\geq 0}\mathrm{Hom}_{\mathrm{DCM}}(\mathcal{N}(-))(\mathbb{Q}, \mathbb{Q}(q))$$

and Proposition 12.4.2 it follows that

$$\widetilde{\mathrm{sp}}_t(j^*\gamma) = H^n(i_x^*)(\gamma) = H^n(sp_t)(j^*\gamma)$$

so we reduce to showing

$$\widetilde{\mathrm{sp}}_t([t] \cup \beta) = 0 = H^n(sp_t)([t] \cup \beta).$$

The first identity follows from $[t] \cup [t] = 0$ in $H^2(K, \mathbb{Q}(2))$. For the second, because $sp_t$ is a morphism in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$, the map $H^*(sp_t)$ is multiplicative, hence it suffices to show that $H^1(sp_t)([t]) = 0$.
For this, it follows from the constuction of the map $sp_t : \mathcal{N}(K) \to \mathcal{N}(k(x))$ in $\mathcal{H}(\text{c.d.g.a}_{\mathbb{Q}})$ that $sp_t$ is natural with respect to Nisnevich neighborhoods $f : (C', x') \to (C, x)$ of $x$, i.e.,

$$sp_{f^*(t)} \circ f^* = sp_t.$$

Now, the map $t : (C, x) \to (\mathbb{A}^1, 0)$ is clearly a Nisnevich neighborhood of 0 (after shrinking $C$ if necessary) and

$$[t] = t^*([T])$$

where $\mathbb{A}^1 = \mathrm{Spec}\, k[T]$. Thus, we may assume that $C = \mathbb{A}^1$ and $t = T$. But then $[T]$ is a well-defined element of $H^1(\mathcal{N}(\mathbb{G}_m))$ hence

$$H^1(sp_t)([T]) = i_1^*([T]) = [1] = 0$$

by definition of $sp_t : \mathcal{N}(\mathcal{O}_{\mathbb{A}^1,0}) \to \mathcal{N}(k)$. This completes the proof. □

*Remark* 12.6.2. Since $sp'_t$ is multiplicative, as we have already remarked, Proposition 12.6.1 gives a rather long-winded re-proof of the multiplicativity of the specialization homomorphism $\widetilde{\mathrm{sp}}_t$ □

12.7. Tangential base-points. As shown in [29], the category $\mathrm{DMT}(k)$ carries a canonical exact *weight filtration*. For an Adams-graded c.d.g.a. $\mathcal{A}$, the derived category of cell modules $\mathrm{DCM}(\mathcal{A})$ carries a natural weight filtration as well; the equivalence $\mathrm{DCM}(\mathcal{N}(k)) \sim \mathrm{DMT}(k)$ given by Spitzweck's representation theorem is compatible with the weight filtrations [26, Theorem 5.24].
If $\mathcal{A}$ is cohomologically connected ($H^n(\mathcal{A}) = 0$ for $n < 0$ and $H^0(\mathcal{A}) = \mathbb{Q} \cdot \mathrm{id}$), then $\mathrm{DCM}(\mathcal{A})$ carries a $t$-structure, natural among cohomologically connected $\mathcal{A}$. Finally, if $\mathcal{A}$ is *1-minimal* then $\mathrm{DCM}(\mathcal{A})$ is equivalent to the derived category of the heart of this $t$-structure (see [26, §5]).
Thus, if $\mathcal{N}(F)$ is cohomologically connected, then $\mathrm{DMT}(F)$ has a $t$-structure; the heart is called the category of mixed Tate motives over $F$, denoted $\mathrm{MT}(F)$. In fact, $\mathrm{MT}(F)$ is a Tannakian category , with natural fiber functor given by the weight filtration; let $\mathrm{Gal}_\mu(F)$ denote the pro-algebraic group scheme over $\mathbb{Q}$ associated to $\mathrm{MT}(F)$ by the Tannakian formalism. If $\mathcal{N}(F)$ is 1-minimal, then $\mathrm{DMT}(F)$ is equivalent to $D^b(\mathrm{MT}(F))$, but we won't be using this.
Now let $x$ be a $k$-point on a smooth curve $C$ over $k$, and $t$ a parameter in $\mathcal{O}_{C,x}$. The specialization functor

$$sp_t : \mathrm{DMT}(k(C)) \to \mathrm{DMT}(k(x))$$

arises from the map $sp_t : \mathcal{N}(k(C)) \to \mathcal{N}(k(x))$ in $\mathcal{H}(\text{c.d.g.a.}_{\mathbb{Q}})$, hence $sp_t$ is compatible with the weight filtrations. When $\mathcal{N}(k(C))$ and $\mathcal{N}(k(x))$ are cohomologically connected, $sp_t$ is compatible with the $t$-structures, hence induces an exact functor of Tannakian categories

$$sp_t : \mathrm{MT}(k(C)) \to \mathrm{MT}(k(x))$$

compatible with the fiber functors $\mathrm{gr}^W$. By Tannakian duality, $sp_t$ is equivalent to a homomorphism

$$\frac{\partial}{\partial t}_* : \mathrm{Gal}_\mu(k(x)) \to \mathrm{Gal}_\mu(k(C)),$$

which is the *tangential base-point* associated to the parameter $t$. This gives a purely "motivic" construction of the tangential base-point construction of Deligne-Goncharov [10]; the construction in [10] relies on realization functors.

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# De Rham-Witt Cohomology and Displays

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Abstract. Displays were introduced to classify formal $p$-divisible groups over an arbitrary ring $R$ where $p$ is nilpotent. We define a more general notion of display and obtain an exact tensor category. In many examples the crystalline cohomology of a smooth and proper scheme $X$ over $R$ carries a natural display structure. It is constructed from the relative de Rham-Witt complex. For this we refine the comparison between crystalline cohomology and de Rham-Witt cohomology of [LZ]. In the case where $R$ is reduced the display structure is related to the strong divisibility condition of Fontaine [Fo].

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## 1 Introduction

Displays of formal p-divisible groups were introduced in [Z2]. They are one possible extension of classical Dieudonné theory to more general ground rings. In [LZ] we gave a direct construction of a display for an abelian scheme by the relative de Rham-Witt complex. In the case where the p-divisible group of the abelian scheme is local the construction leads to the display of [Z2].
We define here a more general notion of display over a ring $R$, where a given prime number $p$ is nilpotent. If $R$ is a perfect field a display is just a finitely generated free $W(R)$-module $M$ endowed with an injective Frobenius linear map $F : M \to M$, while a display of [Z2] is a Dieudonné module, where $V$ acts topologically nilpotent. Our category of displays is an exact tensor category which contains the displays of [Z2] as a full subcategory. There is also a good notion of base change for displays with respect to arbitrary ring morphisms $R \to R'$. Neither the construction of the tensor product nor the construction of base change is straightforward. Special types of tensor products are related

in [Z2] to biextensions of formal groups. Other operations of linear algebra as exterior products and duals up to Tate twist may be performed but we don't discuss them here, since we don't use them essentially and their construction requires just the same ideas. We add that the exact category of displays is Karoubian [T] and has a derived category.

In many examples we have a display structure on the cohomology of a projective and smooth scheme which arises as follows: Let $p$ be a fixed prime number and let $R$ be a ring such that $p$ is nilpotent in $R$. We denote by $W(R)$ the ring of Witt vectors and we set $I_R = VW(R)$. Let $X$ be a projective and smooth scheme over $R$. Let $W\Omega^{\cdot}_{X/R}$ be the de Rham-Witt complex. We define for $m \geq 0$ the Nygaard complex $\mathcal{N}^m W\Omega^{\cdot}_{X/R}$ of sheaves of $W(R)$-modules:

$$(W\Omega^0_{X/R})_{[F]} \xrightarrow{d} \dots \xrightarrow{d} (W\Omega^{m-1}_{X/R})_{[F]} \xrightarrow{dV} W\Omega^m_{X/R} \xrightarrow{d} W\Omega^{m+1}_{X/R} \xrightarrow{d} \dots .$$

Here $F$ indicates restriction of scalars with respect to the Frobenius $F : W(R) \to W(R)$. We remark that $\mathcal{N}^0 W\Omega^{\cdot}_{X/R} = W\Omega^{\cdot}_{X/R}$. These complexes were considered by Nygaard, Illusie and Raynaud [I-R], and Kato [K] if $R$ is a perfect field.

Let $m$ be a nonnegative integer and consider the hypercohomology groups

$$P_i = \mathbb{H}^m(X, \mathcal{N}^i W\Omega_{X/R})$$

for $i \geq 0$. The structure of the de Rham-Witt complex gives naturally three sets of maps (compare: Definition 2.2):

1) A chain of morphisms of $W(R)$-modules
$$\dots \to P_{i+1} \xrightarrow{\iota_i} P_i \to \dots \to P_1 \xrightarrow{\iota_0} P_0.$$

2) For each $i \geq 0$ a $W(R)$-linear map
$$\alpha_i : I_R \otimes_{W(R)} P_i \to P_{i+1}.$$

3) For each $i \geq 0$ a Frobenius linear map
$$F_i : P_i \to P_0.$$

The composition of $\iota$ and $\alpha$ is the multiplication $I_R \otimes P_i \to P_i$. Moreover we have the equation:

$$F_{i+1}(\alpha_i({}^V\eta \otimes x)) = \eta F_i x, \quad \eta \in I_R,\ x \in P_i \tag{1}$$

We will call a set of data $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ with the properties above a predisplay. The predisplays form an abelian category. The equation (1) implies:

$$F_i(\iota_i(y)) = pF_{i+1}(y)$$

i.e. the Frobenius $F_0$ becomes more and more divisible by $p$ if it is restricted to the Nygaard complexes.

We are interested in predisplays, which are obtained by the following construction. We start with a set of data which are called standard:

A sequence $L_0, \ldots, L_d$ of finitely generated projective $W(R)$-modules.

A sequence of Frobenius linear maps for $i = 0, \ldots d$:

$$\Phi_i : L_i \to L_0 \oplus \ldots \oplus L_d$$

We require that the map $\oplus_i \Phi_i$ is a Frobenius linear automorphism of $L_0 \oplus \ldots \oplus L_d$.
From these data one defines a predisplay $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$, with

$$P_i = (I_R \otimes L_0) \oplus \ldots \oplus (I_R \otimes L_{i-1}) \oplus L_i \oplus \ldots \oplus L_d$$

for $i \in \mathbb{Z}, i \geq 0$. The definition of the maps $\iota_i, \alpha_i, F_i$ (compare Definition 2.2) is not obvious, but we skip it for the moment. We should warn the reader that the $P_i$ for $i > d$ are obviously isomorphic, but these isomorphisms are not canonical, i.e. they depend on our construction and not only on the predisplay $\mathcal{P}$.

Definition: *A predisplay is called a display if it is isomorphic to a predisplay associated to standard data.*

A decomposition $P_0 = L_0 \oplus L_1 \oplus \ldots \oplus L_d$ which is given by standard data is called a *normal decomposition*.
If we start with standard data for $d = 1$ we obtain exactly the $3n$-displays of [Z2], which are called displays in [Me]. In this work we call them 1-displays.
If we assume that the $L_i$ are free the map $\oplus\Phi_i$ is represented by a block matrix $(A_{ij})$, where $A_{ij}$ is the matrix of the Frobenius linear map $L_j \to L_i$ induced by $\oplus\Phi_i$, where $0 \leq i, j \leq d$. Conversely any block matrix $(A_{ij})$ from $\mathrm{GL}(W(R))$ defines standard data for a display. Over a local ring $R$ it would be possible to define the category of displays in terms of matrices.
We note that the maps $\iota_i$ for a display $\mathcal{P}$ are generally not injective unless the ring $R$ is reduced. In this case the whole display is uniquely determined by the Frobenius module $(P_0, F_0)$. Indeed the display property implies that:

$$P_i = \{x \in P_0 \mid F_0(x) \in p^i P_0\} \tag{2}$$

One has $F_i = (1/p^i)F_0$. This makes sense because $p$ is not a zero divisor in $W(R)$ if $R$ is reduced. Therefore over a reduced ring a display is a special kind of Frobenius module.
If $R = k$ is a perfect field a display is just the same as a Frobenius module $(P_0, F_0)$. Indeed, consider the map $F_0 : P_0 \otimes \mathbb{Q} \to P_0 \otimes \mathbb{Q}$. We obtain inclusions of $W(k)$-modules:

$$P_0 \subset F_0^{-1} P_0 \subset P_0 \otimes \mathbb{Q}.$$

By the theory of elementary divisors we find a decomposition by $W(R)$-modules $P_0 = L_0 \oplus L_1 \oplus \ldots \oplus L_d$, such that

$$F_0^{-1} P_0 = L_0 \oplus p^{-1} L_1 \oplus \ldots \oplus p^{-d} L_d.$$

Therefore the restriction of $p^{-i}F_0$ to $L_i$ defines a map $\Phi_i : L_i \to P_0$, for $i = 0, \ldots, d$. These are the standard data for the display associated to the Frobenius module $(P_0, F_0)$.
If $pR = 0$ Moonen and Wedhorn [MW] introduced the structure of an $F$-zip. It is defined in terms of the de Rham cohomology of the scheme $X/R$. As one should expect any display gives rise to an $F$-zip (compare the remark after Definition 2.6.).
For an arbitrary projective and smooth variety $X/R$ we can't expect that the crystalline cohomology $H^m_{crys}(X/W(R))$ has a display structure. Therefore we consider the following assumptions: There is a compatible system of smooth liftings $\tilde{X}_n/W_n(R)$ for $n \in \mathbb{N}$ of $X/R$ such that the following properties hold:

(*) The cohomology groups $H^j(\tilde{X}_n, \Omega^i_{\tilde{X}_n/W_n(R)})$ are for each $n$, $i$ and $j$ locally free $W_n(R)$-modules of finite type.

(**) The de Rham spectral sequence degenerates at $E_1$

$$E_1^{ij} = H^j(\tilde{X}_n, \Omega^i_{\tilde{X}_n/W_n(R)}) \Rightarrow \mathbb{H}^{i+j}(\tilde{X}_n, \Omega^{\cdot}_{\tilde{X}_n/W_n(R)}).$$

Theorem: *Let $X$ be smooth and projective over a reduced ring $R$, such that the assumptions (*) and (**) are satisfied. Let $d$ be an integer $0 \leq m < p$. Consider the Frobenius module $P_0 = H^m_{crys}(X/W(R))$ and define $P_i$ by the formula (2).*
*Then the $P_i$ form a display and $P_i$ coincides with the hypercohomology of the Nygaard complex $\mathcal{N}^i W\Omega^{\cdot}_{X/R}$.*
It would follow from the general conjecture made below that this theorem holds without the restriction $m < p$.
Finally we indicate how to proceed if the ring $R$ is not reduced. In order to overcome the problem with the $p$-torsion in $W(R)$ we use frames [Z1]. A frame for $R$ is a triple $(A, \sigma, \alpha)$, where $A$ is a $p$-adic ring without $p$-torsion, $\sigma : A \to A$ is an endomorphism which lifts the Frobenius on $A/pA$, and $\alpha : A \to R$ is a surjective ring homomorphism whose kernel has divided powers. Let us assume that $X$ admits a lifting to a smooth formal scheme $\mathcal{Y}$ over $\operatorname{Spf} A$, which satisfies assumptions analogous to (*) and (**). We define "displays" relative to $A$ which we call windows (see [Z1]). Theorem 5.5 says that under the conditions made $H^m_{crys}(X/A, \mathcal{O}_{X/A})$ has a window structure for $m < p$ . There is a morphism $A \to W(A) \to W(R)$ which allows to pass from windows to displays. We remark that because of this morphism the assumptions (*) and (**) for $A$ are stronger than the original assumption for $W(R)$. In equal characteristic we obtain e.g. the following:

Theorem *Let $X$ be smooth and projective over a ring $R$, such that $pR = 0$. Let us assume that there is a frame $A \to R$ and a smooth $p$-adic lifting $\mathcal{Y}/\operatorname{Spf} A$ of $X$, which satisfies the conditions analogous to (*) and (**).*
*Then there is a canonical display structure on $H^m_{crys}(X/W(R))$ for $m < p$, which does not depend on the lifting $\mathcal{Y}$ nor on the frame $A$.*

We discuss three examples where the assumptions $(*)$ and $(**)$ hold. In these examples the assumptions made on $X$ in the two preceding theorems are fullfilled.

Let $X$ be a $K3$-surface over $R$. We assume without restriction of generality that $R$ is noetherian. We denote by $\mathcal{T}_{X/R}$ the tangent bundle of $X$. The cohomology group $H^2(X, \mathcal{T}_{X/R})$ commutes with base change by [M] §5 Cor.3. From the case where $R$ is an algebraically closed field, we deduce that this cohomology group vanishes. It follows that $X$ has a formal lifting over $\operatorname{Spf} W(R)$ resp. $\operatorname{Spf} A$. From the Hodge numbers of a K3-surface over an algebraically closed field [De1] one deduces that $H^1(X, \mathcal{O}_X) = 0$, $H^0(X, \Omega^1_{X/R}) = 0$, $H^2(X, \Omega^1_{X/R}) = 0$, $H^1(X, \Omega^2_{X/R}) = 0$. It follows that the cohomology of $X$ commutes with arbitrary base change and is therefore locally free [M] loc.cit.. The degeneration of the de Rham spectral sequence follows now because the Hodge numbers above are zero, because there is no room for non-zero differentials.

Let $X$ be an abelian variety over $R$. In this case the assumptions $(*)$ and $(**)$ are fullfilled by [BBM] 2.5.2.

Finally let $X$ be a smooth relative complete intersection in a projective space over $R$. Then the conditions $(*)$ and $(**)$ are fullfilled by [De2] Thm.1.5.

Let $p$ be a prime number. Let $R$ be a ring such that $p$ is nilpotent in $R$. In [LZ] Thm. 3.5 we proved a comparison between the crystalline cohomology and the hypercohomology of the de Rham-Witt complex extending a result of Illusie [I] if $R$ is a perfect field. We show here a filtered version of this comparison, which is the key to the display structure. We conjecture a more precise comparison, which would lead to a wide generalization of the theorems above.

Let $W_n(R)$ be the truncated Witt vectors. We set $I_{R,n} = VW_{n-1}(R)$. This ideal is 0 for $n = 1$.

Let $X/R$ be a smooth and projective scheme. We consider the crystalline site $Crys(X/W_n(R))$ with its structure sheaf $\mathcal{O}_{X/W_n(R)}$. Let us denote by $\mathcal{J}_{X/W_n(R)} \subset \mathcal{O}_{X/W_n(R)}$ the sheaf of pd-ideals. We denote by $\mathcal{J}^{[m]}_{X/W_n(R)}$ its $m$-th divided power. Let

$$u_n : Crys(X/W_n(R))^{\sim} \longrightarrow X^{\sim}_{zar}$$

be the canonical morphism of topoi.

The comparison isomorphism [LZ] is an isomorphism in the derived category $D(X_{zar})$ of sheaves of $W_n(R)$-modules on $X_{zar}$:

$$Ru_{n*}\mathcal{O}_{X/W_n(R)} \longrightarrow W_n\Omega^{\cdot}_{X/R}$$

We will prove a filtered version of this. Let $m$ be a natural number. Let $\mathcal{I}^m W_n\Omega^{\cdot}_{X/R}$ be the following subcomplex of the de Rham-Witt complex:

$$p^{m-1}VW_{n-1}\Omega^0_{X/R} \xrightarrow{d} p^{m-2}VW_{n-1}\Omega^1_{X/R} \ldots \xrightarrow{d} VW_{n-1}\Omega^{m-1}_{X/R} \xrightarrow{d} W_n\Omega^m_{X/R} \ldots$$

The filtered comparison Theorem 4.6 says that for $m < p$ we have an isomorphism in the derived category

$$Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)} \longrightarrow \mathcal{I}^m W_n\Omega^{\cdot}_{X/R} \tag{3}$$

We would like to have a similar comparison theorem for the truncated Nygaard complex $\mathcal{N}^m W_n\Omega^{\cdot}_{X/R}$ instead of $\mathcal{I}^m W_n\Omega^{\cdot}_{X/R}$:

$$(W_{n-1}\Omega^0_{X/R})_{[F]} \xrightarrow{d} \ldots \xrightarrow{d} (W_{n-1}\Omega^{m-1}_{X/R})_{[F]} \xrightarrow{dV} W_n\Omega^m_{X/R} \xrightarrow{d} W_n\Omega^{m+1}_{X/R} \xrightarrow{d} \ldots$$

The advantage of the Nygaard complex is that the restriction of the Frobenius from $W\Omega^{\cdot}_{X/R}$ to $\mathcal{N}^m W\Omega^{\cdot}_{X/R}$ is in a natural way divisible by $p^m$ even if $p$ is a zero divisor. For a reduced ring $R$ the Nygaard complex $\mathcal{N}^m W\Omega^{\cdot}_{X/R}$ is quasi-isomorphic to $\mathcal{I}^m W\Omega^{\cdot}_{X/R}$. Unfortunately in general we don't know a definition for the Nygaard complex in terms of crystalline cohomology. Nevertheless we make the conjecture 4.1:

Conjecture: *Assume that $\tilde{X}/W_n(R)$ is a smooth lifting of $X$. Then the Nygaard complex is in the derived category canonically isomorphic to the following complex $\mathcal{F}^m\Omega^{\cdot}_{\tilde{X}/W_n(R)}$:*

$$I_{R,n} \otimes_{W_n(R)} \Omega^0_{\tilde{X}/W_n(R)} \xrightarrow{pd} \ldots \xrightarrow{pd} I_{R,n} \otimes_{W_n(R)} \Omega^{m-1}_{\tilde{X}/W_n(R)} \xrightarrow{d} \Omega^m_{\tilde{X}/W_n(R)} \xrightarrow{d} \ldots .$$

Assume that we have for varying $n$ a compatible system of smooth liftings $\tilde{X}_n/W_n(R)$. We obtain a formal scheme $\mathcal{X} = \varinjlim \tilde{X}_n$. We set:

$$\mathcal{F}^m\Omega^{\cdot}_{\mathcal{X}/W(R)} = \varprojlim_n \mathcal{F}^m\Omega^{\cdot}_{\tilde{X}_n/W_n(R)} \qquad \mathcal{N}^m W\Omega^{\cdot}_{X/R} = \varprojlim_n \mathcal{N}^m W_n\Omega^{\cdot}_{X/R}$$

We show the following weak form of the conjecture (Corollary 4.7):

Theorem: *Assume that $R$ is reduced and that $m < p$. Then there is a natural isomorphism in the derived category of $W(R)$-modules on $X_{zar}$:*

$$\mathcal{N}^m W\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m\Omega^{\cdot}_{\mathcal{X}/W(R)}$$

Moreover we can show in support of our conjecture, that the complexes $\mathcal{N}^m W_n\Omega^{\cdot}_{X/R}$ and $\mathcal{F}^m\Omega^{\cdot}_{\tilde{X}_n/W_n(R)}$ are always locally quasi-isomorphic on $X_{zar}$. The last theorem is closely related to strong divisibility in the sense of [Fo] 1.3: Assume the assumptions $(*)$ and $(**)$ are satisfied. By the last theorem the splitting of the Hodge filtration of the formal scheme $\mathcal{X}$ defines a normal decomposition:

$$\mathbb{H}^m(X, \mathcal{F}^j\Omega^{\cdot}_{\mathcal{X}/W(R)}) = I_R L_0 \oplus \ldots \oplus I_R L_{j-1} \oplus L_j \oplus \ldots \oplus L_d$$

It is obvious from Definition 2.2 that the Frobenius $F_j : H^m(X, \mathcal{N}^j W\Omega^{\cdot}_{X/R}) \to \mathbb{H}^m(X, W\Omega^{\cdot}_{X/R})$ is bijective if $j$ is bigger than the dimension. Therefore $F_0 \oplus F_1 \oplus \ldots \oplus F_d$: induces a bijection:

$$I_R L_0 \oplus \ldots \oplus I_R L_d \to L_0 \oplus \ldots \oplus L_d$$

This is what strong divisibility asserts.

## 2 The Category of Displays

Let $R$ be a ring, and let $W(R)$ be the ring of Witt vectors. We set $I_R = VW(R)$. If no confusion is possible we sometimes use the abbreviation $I = I_R$. Let $\Phi : M \to N$ a Frobenius-linear homomorphism of $W(R)$-modules. We define a Frobenius-linear homomorphism $\tilde{\Phi}$:

$$\begin{array}{rcl} \tilde{\Phi} : I_R \otimes_{W(R)} M & \to & N \\ {}^V\xi \otimes m & \mapsto & \xi\Phi(m) \end{array} \tag{4}$$

Definition 2.1 *A predisplay over $R$ consists of the following data:*

*1) A chain of morphisms of $W(R)$-modules*

$$\ldots \to P_{i+1} \xrightarrow{\iota_i} P_i \to \ldots \to P_1 \xrightarrow{\iota_0} P_0.$$

*2) For each $i \geq 0$ a $W(R)$-linear map*

$$\alpha_i : I_R \otimes_{W(R)} P_i \to P_{i+1}.$$

*3) For each $i \geq 0$ a Frobenius linear map*

$$F_i : P_i \to P_0.$$

*The following axioms should be fulfilled*

*(D1) For $i \geq 1$ the diagram below is commutative and its diagonal $I_R \otimes P_i \to P_i$ is the multiplication.*

$$\begin{array}{ccc} I_R \otimes P_i & \xrightarrow{\alpha_i} & P_{i+1} \\ {\scriptstyle I_R \otimes \iota_{i-1}}\downarrow & & {\scriptstyle \iota_i}\downarrow \\ I_R \otimes P_{i-1} & \xrightarrow{\alpha_{i-1}} & P_i \end{array}$$

*For $i = 0$ the following map is the multiplication:*

$$I_R \otimes P_0 \xrightarrow{\alpha_0} P_1 \xrightarrow{\iota_0} P_0$$

*(D2) $F_{i+1}\alpha_i = \tilde{F}_i : I_R \otimes P_i. \to P_0$*

We will denote a predisplay as follows:

$$\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i), \quad i \in \mathbb{Z}_{\geq 0}.$$

Let $X$ be a smooth and proper scheme over a scheme $S$. Then we obtain a predisplay structure on the crystalline cohomology through the Nygaard complexes $\mathcal{N}^m W_n\Omega_{X/S}$ which are built from the de Rham-Witt complex as follows:

$$(W_{n-1}\Omega^0_{X/S})_{[F]} \xrightarrow{d} \ldots \xrightarrow{d} (W_{n-1}\Omega^{m-1}_{X/S})_{[F]} \xrightarrow{dV} W_n\Omega^m_{X/S} \xrightarrow{d} W_n\Omega^{m+1}_{X/S} \ldots$$

This is considered as a complex of $W_n(\mathcal{O}_S)$-modules. The index $[F]$ means that we consider this term as a $W_n(\mathcal{O}_S)$-module via restriction of scalars $F : W_n(\mathcal{O}_S) \to W_{n-1}(\mathcal{O}_S)$.
Let $I_{S,n} = VW_{n-1}(\mathcal{O}_S) \subset W_n(\mathcal{O}_S)$ be the sheaf of ideals. We define three sets of maps:

$$\begin{array}{rccc} \hat{\alpha}_m : & I_{S,n} \otimes_{W_n(\mathcal{O}_S)} \mathcal{N}^m W_n\Omega^{\cdot}_{X/S} & \to & \mathcal{N}^{m+1} W_n\Omega^{\cdot}_{X/S} \\ \hat{\iota}_m : & \mathcal{N}^{m+1} W_n\Omega^{\cdot}_{X/S} & \to & \mathcal{N}^m W_n\Omega^{\cdot}_{X/S} \\ \hat{F}_m : & \mathcal{N}^m W_n\Omega^{\cdot}_{X/S} & \to & W_{n-1}\Omega^{\cdot}_{X/S} \end{array} \tag{5}$$

These maps are given in this order by the maps between the following vertically written procomplexes (the index $n$ is omitted):

$$\begin{array}{ccccccc}
I_S \otimes (W\Omega^0_{X/S})_{[F]} & \longrightarrow & (W\Omega^0_{X/S})_{[F]} & \xrightarrow{p} & (W\Omega^0_{X/S})_{[F]} & \xrightarrow{id} & W\Omega^0_{X/S} \\
{\scriptstyle I_S \otimes d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
\cdots & & \cdots & & \cdots & & \cdots \\
I_S \otimes (W\Omega^{m-1}_{X/S})_{[F]} & \longrightarrow & (W\Omega^{m-1}_{X/S})_{[F]} & \xrightarrow{p} & (W\Omega^{m-1}_{X/S})_{[F]} & \xrightarrow{id} & W\Omega^{m-1}_{X/S} \\
{\scriptstyle \mathrm{id} \otimes dV}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle dV}\downarrow & & {\scriptstyle d}\downarrow \\
I_S \otimes (W\Omega^m_{X/S}) & \xrightarrow{\tilde{F}} & (W\Omega^m_{X/S})_{[F]} & \xrightarrow{V} & W\Omega^m_{X/S} & \xrightarrow{F} & W\Omega^m_{X/S} \\
{\scriptstyle \mathrm{id} \otimes d}\downarrow & & {\scriptstyle dV}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
I_S \otimes W\Omega^{m+1}_{X/S} & \xrightarrow{mult} & W\Omega^{m+1}_{X/S} & \xrightarrow{id} & W\Omega^{m+1}_{X/S} & \xrightarrow{pF} & W\Omega^{m+1}_{X/S} \\
{\scriptstyle \mathrm{id} \otimes d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
I_S \otimes W\Omega^{m+2}_{X/S} & \xrightarrow{mult} & W\Omega^{m+2}_{X/S} & \xrightarrow{id} & W\Omega^{m+2}_{X/S} & \xrightarrow{p^2F} & W\Omega^{m+2}_{X/S} \\
\cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

The first unlabeled arrows on the left hand side denote the maps ${}^V\xi \otimes \omega \mapsto \xi\omega$, where the product is taken in $W\Omega^i_{X/S}$ (without restriction of scalars).

DEFINITION 2.2 *Let $S = \operatorname{Spec} R$ be an affine scheme. Let $X/S$ be a smooth and proper scheme. Then we associate a predisplay. We set:*

$$P_i = \mathbb{H}^d(X, \mathcal{N}^i W\Omega_{X/S})$$

*The predisplay structure on the $P_i$ is easily obtained by taking the cohomology of the maps (5).*

Here we write $\mathcal{N}^m W\Omega^{\cdot}_{X/R} = \varprojlim_n \mathcal{N}^m W_n\Omega^{\cdot}_{X/R}$. The $P_i$ coincide with the cohomology of $R\varprojlim_n R\Gamma(X, \mathcal{N}^i W_n\Omega_{X/S})$ by the proof of [LZ] Prop. 1.13 (compare [BO] Appendix).

Remark: Let $S = \operatorname{Spec} k$ be the spectrum of a perfect field. Then $I(k)$ is isomorphic to $W(k)$ as $W(k)$-module. The maps of complexes which define $\hat{\alpha}_i$ and $\hat{\iota}_i$ are in this case the maps $\tilde{F}$ and $\tilde{V}$ used by Kato in his definition of the $F$-gauges $GH^d(X/S)$.

Let $A/S$ be an abelian scheme. Then the predisplay structure on the crystalline cohomology $H^1(A/W(R), \mathcal{O}_{A/W(R)})$ is in fact a 3n-display structure in the sense of [Z2]. We will introduce additional properties of predisplay structures which arise in the crystalline cohomology of smooth and proper varieties.
Let $\mathcal{P}$ be a predisplay. Then we have a commutative diagram:

$$\begin{array}{ccc} P_i & \xrightarrow{F_i} & P_0 \\ {\scriptstyle \iota_i}\uparrow & & {\scriptstyle p}\uparrow \\ P_{i+1} & \xrightarrow{F_{i+1}} & P_0 \end{array} \tag{6}$$

Indeed, let $y \in P_{i+1}$. Then we obtain from (D1) that

$$\alpha_i({}^V1 \otimes \iota_i(y)) = {}^V1y$$

If we apply $F_{i+1}$ to the last equation and use (D2), we obtain:

$$F_i(\iota_i(y)) = pF_{i+1}(y)$$

Definition 2.3 *A predisplay $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ is called separated if the map of $P_{i+1}$ to the fibre product induced by the commutative diagram (6) is injective.*

Remark: Predisplays form obviously an abelian category. To each predisplay $\mathcal{P}$ we can associate a separated predisplay $\mathcal{P}^{sep}$ and a canonical surjection $\mathcal{P} \to \mathcal{P}^{sep}$. This is defined inductively: $P_0^{sep} = P_0$ and $P_{i+1}^{sep}$ is the image of $P_{i+1}$ in the fibre product of:

$$P_i^{sep} \xrightarrow{F_i^{sep}} P^0 \xleftarrow{\;p\;} P^0$$

The functor $\mathcal{P} \mapsto \mathcal{P}^{sep}$ to the category of separated displays is left adjoint to the forgetful functor, but it is not exact.

It is not difficult to prove that a separated predisplay has the following property: Consider the iteration of the maps $\alpha$:

$$I^{\otimes k} \otimes P_i \xrightarrow{\alpha_i} I^{\otimes k-1} \otimes P_{i+1} \xrightarrow{\alpha_{i+1}} \dots \xrightarrow{\alpha_{i+k-1}} P_{i+k} \tag{7}$$

Here the maps $\alpha$ pick up the last factor of $I^{\otimes}$. The following map is called the "Verjüngung":

$$\begin{array}{cccc} \nu^{(k)}: & I^{\otimes k} & \to & I \\ & {}^V\xi_1 \otimes \dots \otimes {}^V\xi_k & \mapsto & {}^V(\xi_1 \cdot \ldots \cdot \xi_k) \end{array} \tag{8}$$

For a separated display the iteration (7) factors through the Verjüngung:

$$I^{\otimes k} \otimes P_i \xrightarrow{\nu^{(k)}} I \otimes P_i \longrightarrow P_{i+k}$$

The last arrow will be called $\alpha_i^{(k)}$. In particular this shows that the iteration (7) is independent of the factors we picked up, when forming $\alpha_j$.
For a separated display the data $\alpha_i$, $i \geq 0$ are uniquely determined by the remaining data. This is seen by the following commutative diagram:

$$\begin{tikzcd}
& P_i \arrow[r, "F_i"] & P_0 & \\
& & & (*) \\
& P_{i+1} \arrow[uu, "\iota_i"'] \arrow[r, "F_{i+1}"] & P_0 \arrow[uu, "p"'] & \\
I \otimes P_i \arrow[uuur] \arrow[ur, "\alpha_i"] \arrow[urr, "\tilde{F}_i"'] & & &
\end{tikzcd}$$

For a predisplay $\mathcal{P}$ the cokernel $E_{i+1} := \mathit{Coker}\ \alpha_i$ is annihilated by $I$. It is therefore an $R$-module.

Definition 2.4 *We say that a predisplay is of degree $d$ (or a $d$-predisplay), if the maps $\alpha_i$ are surjective for $i \geq d$.*

A separated predisplay of degree $d$ is already uniquely determined by the data:

$$P_0, \ldots P_d, \iota_0, \ldots \iota_{d-1}, F_0, \ldots, F_d, \alpha_0, \ldots, \alpha_{d-1} \tag{9}$$

For this consider the diagram $(*)$ above for $i = d$. The data already given determine a map of $I \otimes P_d$ to the fibre product. This map is $\alpha_d$ and the image is $P_{d+1}$. Thus inductively all data of the display are uniquely determined.
Conversely assume that we have data (9) which satisfy all predisplay axioms reasonable for these data. Then we define $P_{d+1}$ by the diagram $(*)$ above. We obtain also the maps $\alpha_d$, $\iota_d$, and $F_{d+1}$. The axioms for the extended data are trivially satisfied, except for the requirement that

$$I \otimes P_{d+1} \to I \otimes P_d \to P_{d+1}$$

is the multiplication. But this follows easily by composing the diagram ($*$) for $i = d$, with the arrow $\mathrm{id} \otimes \iota_d : I \otimes P_{i+1} \to I \otimes P_i$. Inductively we see that a set of data (9) satisfying the predisplay axioms may be extended uniquely to a predisplay of degree $d$.

We may define the twist of a predisplay. Let

$$\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$$

be a predisplay. Then we define its *Tate-twist*

$$\mathcal{P}(1) = (P'_i, \iota'_i, \alpha'_i, F'_i) \tag{10}$$

as follows: For $i \geq 1$ we set $P'_i = P_{i-1}, \iota'_i = \iota_{i-1}, \alpha'_i = \alpha_{i-1}, F'_i = F_{i-1}$. We set $P'_0 = P_0 = P'_1, F'_0 = pF_0, \iota'_0 = \mathrm{id}_{P_0}$. Finally $\alpha'_0 : I \otimes P_0 \to P_0$ is defined to be the multiplication. If we repeat this $n$-times we write $\mathcal{P}(n)$.

We define a predisplay $\mathcal{U} = (P_i, \iota_i, \alpha_i, F_i)$ called the *unit display* as follows: $P_0 = W(R)$, $P_i = I$ for $i \geq 1$. The chain of the maps $\iota$ is as follows:

$$\ldots I \xrightarrow{p} I \ldots \xrightarrow{p} I \to W(R), \tag{11}$$

where the last map $\iota_0$ is the natural inclusion.
The maps $F_i : I = P_i \to W(R)$ for $i \geq 1$ coincide with the map

$$V^{-1} : I \to W(R), \qquad {}^V\xi \mapsto \xi.$$

The map $F_0$ is the Frobenius on $W(R)$. The map $\alpha_0 : I \otimes W(R) \to I$ is the multiplication. The maps $\alpha_i : I \otimes I \to I$ are the Verjüngung $\nu^{(2)}$. Since the "Verjüngung" is surjective the unit display has degree zero.

A 3n-display $(P, Q, F, V^{-1})$ as defined in [Z2] gives naturally rise to data of type (9) with $P_0 = P$, $P_1 = Q$, $F_0 = F$, $F_1 = V^{-1}$ and therefore extends naturally to a predisplay of degree 1 as we explained above. We will make this explicit later on.

Let $R$ be a reduced ring. Then the multiplication by $p$ is injective on $W(R)$. Let $M$ be a projective $W(R)$-module, and $F : M \to M$ be a Frobenius linear map. Then we set:

$$P_i = \{x \in M \mid F(x) \in p^i M\}$$

We obtain maps

$$F_i = (1/p^i)F : P_i \to P_0 = M$$

For $\iota_i$ we take the natural inclusion $P_{i+1} \to P_i$. For $\alpha_i$ we take the maps $I \otimes P_i \to IP_i \subset P_{i+1}$ induced by multiplication. The data $(P_i, \iota_i, \alpha_i, F_i)$ constructed in this way are a separated predisplay.

The predisplays we are interested in arise from a construction which we explain now.

Definition 2.5 *The following set of data we will call standard data for a display of degree $d$.*

*A sequence $L_0, \dots, L_d$ of finitely generated projective $W(R)$-modules.*

*A sequence of Frobenius linear maps for $i = 0, \dots d$:*

$$\Phi_i : L_i \to L_0 \oplus \dots \oplus L_d$$

*We require that the map $\oplus_i \Phi_i$ is a Frobenius linear automorphism of $L_0 \oplus \dots \oplus L_d$*

From these data we obtain a predisplay in the following manner: We set:

$$P_i = (I \otimes L_0) \oplus \dots \oplus (I \otimes L_{i-1}) \oplus L_i \oplus \dots \oplus L_d$$

for $i \in \mathbb{Z}, i \geq 0$.

We note that $P_i = P_{d+1}$ for $i > d$. But these identifications are not part of the predisplay structure we are going to define. They depend on the standard data!

We define "divided" Frobenius maps:

$$F_i : P_i \to P_0$$

The restriction of $F_i$ to $I \otimes L_k$ for $k < i$ is $\tilde{\Phi}_k$, and to $L_{i+j}$ for $j \geq 0$ is $p^j \Phi_{i+j}$.

The map $\iota_i : P_{i+1} \to P_i$ is given by the following diagram:

$$\begin{array}{ccccccccc}
(I \otimes L_0) & \oplus \dots \oplus & (I \otimes L_{i-1}) & \oplus & (I \otimes L_i) & \oplus & L_{i+1} & \oplus \dots \oplus & L_d \\
{\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow & & {\scriptstyle mult}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\
(I \otimes L_0) & \oplus \dots \oplus & (I \otimes L_{i-1}) & \oplus & L_i & \oplus & L_{i+1} & \oplus \dots \oplus & L_d
\end{array} \tag{12}$$

The map $\alpha_i : I \otimes P_i \to P_{i+1}$ is given by the following diagram:

$$\begin{array}{ccccccccc}
I \otimes (I \otimes L_0) & \oplus \dots \oplus & I \otimes (I \otimes L_{i-1}) & \oplus & I \otimes L_i & \oplus & I \otimes L_{i+1} & \oplus \dots \oplus & I \otimes L_d \\
{\scriptstyle \nu}\big\downarrow & & {\scriptstyle \nu}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle mult}\big\downarrow & & {\scriptstyle mult}\big\downarrow \\
(I \otimes L_0) & \oplus \dots \oplus & (I \otimes L_{i-1}) & \oplus & (I \otimes L_i) & \oplus & L_{i+1} & \oplus \dots \oplus & L_d
\end{array} \tag{13}$$

Here $\nu = \nu^{(2)}$ is the Verjüngung. We leave the verification that $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ is a separated predisplay to the reader.

DEFINITION 2.6 *A predisplay is called a display if it is isomorphic to a predisplay associated to standard data.*

Remark: Let us assume that $pR = 0$. There is the notion of an $F$-zip by Moonen and Wedhorn. The relation to displays is as follows. Let $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ be a display over $R$. We define an $F$-zip structure on $M = P_0/I_R P_0$ by the following two filtrations. Let $C^i$ as the image of $P_i$ in $P_0/I_R P_0$ given by the composite of the maps $\iota_k$. This gives the decreasing "Hodge filtration":

$$\ldots \subset C^d \subset C^{d-1} \subset \ldots \subset C^1 \subset C^0 = M.$$

We set $D_i = W(R)F_i P_i + I_R P_0/I_R P_0$ and obtain an increasing filtration, called the "conjugate filtration":

$$0 = D_{-1} \subset D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_d \subset \ldots \subset M.$$

The morphisms $F_i$ for $i \geq 0$ induce Frobenius linear morphisms:

$$F_i : C_i/C_{i+1} \to D_i/D_{i-1} \tag{14}$$

These are Frobenius linear isomorphisms of $R$-modules. Indeed, if we choose a normal decomposition $\{L_i\}$ we obtain identification:

$$C^i/C^{i+1} \cong L_i/I_R L_i \quad \text{and} \quad D_i/D_{i-1} \cong W(R)F_i L_i/I_R W(R)F_i L_i$$

The two filtrations $C^{\cdot}$ and $D_{\cdot}$ together with the operators (14) form an $F$-zip [MW] Def. 1.5.

Let $\mathcal{P}$ be the display associated to the standard data $(L_i, \Phi_i)$ as above. Let $\mathcal{Q} = (Q_i, \iota_i, \alpha_i, F_i)$ be a predisplay. Assume we are given homomorphisms $\rho_i : L_i \to Q_i$. Then we define maps $\tau_i$:

$$P_i = (I \otimes L_0) \oplus \ldots \oplus (I \otimes L_{i-1}) \oplus L_i \oplus \ldots \oplus L_d \longrightarrow Q_i$$

On the summand $(I \otimes L_{i-k})$ the map $\tau_i$ is the composite:

$$I \otimes L_{i-k} \xrightarrow{\mathrm{id} \otimes \rho_{i-k}} I \otimes Q_{i-k} \xrightarrow{\alpha^{(k)}} Q_k$$

On the summand $L_{i+j}$ the map $\tau_i$ is the composite:

$$L_{i+j} \xrightarrow{\rho_{i+j}} Q_{i+j} \xrightarrow{\iota^{(j)}} Q_i,$$

where the last arrow is a compositions of $\iota' s$.

Proposition 2.7 *The maps $\tau_i$ define a homomorphism of predisplays $\mathcal{P} \to \mathcal{Q}$ if and only if the following diagrams are commutative:*

$$\begin{array}{ccc} L_i & \xrightarrow{\rho_i} & Q_i \\ {\scriptstyle \Phi_i}\downarrow & & \downarrow{\scriptstyle F_i} \\ P_0 & \xrightarrow{\tau_0} & Q_0 \end{array}$$

We omit the verification.

If $\mathcal{P} = (P, Q, F, V^{-1})$ is 3n-display in the sense of [Z2], then any normal decomposition $P = L_0 \oplus L_1$, $Q = IL_0 \oplus L_1$ defines standard data, which determine this display.

We will now define the tensor product of displays: Assume that $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ and $\mathcal{P}' = (P'_i, \iota'_i, \alpha'_i, F'_i)$ are displays over $R$.

A tensor product $\mathcal{T} = (T_i, \overset{\circ}{\iota}_i, \overset{\circ}{\alpha}_i, \overset{\circ}{F}_i)$ may be constructed as follows. We choose normal decompositions

$$P_0 = \underset{n \geq 0}{\oplus} L_n, \quad P'_0 = \underset{n \geq 0}{\oplus} L'_n.$$

More precisely this means, that we fix isomorphisms of $\mathcal{P}$ resp. $\mathcal{P}'$ with standard displays. We obtain:

$$P_i = I \otimes L_0 \oplus \cdots \oplus I \otimes L_{i-1} \oplus L_i \oplus \ldots$$

We denote the restriction of $F_i : P_i \longrightarrow P_0$ to the direct summand $L_i$ by $\Phi_i$. We obtain data for a standard display $K_l, \overset{\circ}{\Phi}_l, l \geq 0$, if we set

$$K_l = \underset{n+m=l}{\oplus} (L_n \otimes L'_m).$$

Then $\oplus_l K_l = P_0 \otimes P'_0$, and we define Frobenius linear maps

$$\overset{\circ}{\Phi}_l : K_l \longrightarrow P_0 \otimes P'_0,$$

by

$$\overset{\circ}{\Phi}_l = \sum_{n+m=l} \Phi_n \otimes \Phi'_m$$

From the standard data $K_l, \overset{\circ}{\Phi}_l$ we obtain a display

$$\mathcal{T} = (T_i, \overset{\circ}{\iota}_i, \overset{\circ}{\alpha}_i, \overset{\circ}{F}_i) \tag{15}$$

We will show that $\mathcal{T}$ is up to canonical isomorphism independent of the normal decompositions of $\mathcal{P}$ resp. $\mathcal{P}'$.

In order to do this we define bilinear forms of displays. Let $\mathcal{T}$ be an arbitrary predisplay. A bilinear form

$$\lambda : \mathcal{P} \times \mathcal{P}' \longrightarrow \mathcal{T}.$$

consists of the following data.
$\lambda$ is a sequence of maps of $W(R)$-modules

$$\lambda_{ij} : P_i \otimes P'_j \longrightarrow T_{i+j}.$$

We require that the following diagrams are commutative:

$$\begin{tikzcd}
P_i \otimes P'_j \arrow[r] \arrow[d, "F_i \otimes F'_j"'] & T_{i+j} \arrow[d, "\mathring{F}_{i+j}"] \\
P_0 \otimes P'_0 \arrow[r, "\mathrm{id}"'] & T_0
\end{tikzcd}$$

$$\begin{tikzcd}
P_i \otimes P'_j \arrow[r] & T_{i+j} \\
P_{i+1} \otimes P'_j \arrow[r] \arrow[u, "\iota \otimes \mathrm{id}"] & T_{i+j+1} \arrow[u, "\mathring{\iota}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P_i \otimes P'_j \arrow[r] & T_{i+j} \\
P_i \otimes P'_{j+1} \arrow[r] \arrow[u, "\mathrm{id} \otimes \iota'"] & T_{i+j+1} \arrow[u, "\mathring{\iota}"']
\end{tikzcd}$$

$$\begin{tikzcd}
I_R \otimes P_i \otimes P'_j \arrow[r] \arrow[d, "\alpha_i \otimes \mathrm{id}"'] & I_R \otimes T_{i+j} \arrow[d, "\mathring{\alpha}_{i+j}"] \\
P_{i+1} \otimes P'_j \arrow[r] & T_{i+j+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
I_R \otimes P_i \otimes P'_j \arrow[r] \arrow[d, "\mathrm{id} \otimes \alpha'_j"'] & I \otimes T_{i+j} \arrow[d, "\mathring{\alpha}_{i+j}"] \\
P_i \otimes P'_{j+1} \arrow[r] & T_{i+j+1}.
\end{tikzcd}$$

Remark: We will consider also the maps

$$P_i \otimes P_j \longrightarrow T_k, \ \text{ for } \ i + j \geq k,$$

which are the compositions of $\lambda_{ij}$ and $T_{i+j} \longrightarrow T_k$, the iteration of $\iota$.
If $i + j > k$ we obtain a commutative diagram:

$$\begin{array}{cccc}
 & P_{i-1} \otimes P_j & \longrightarrow & T_k \\
\iota \otimes \mathrm{id} & \uparrow & & \uparrow \\
 & P_i \otimes P_j & \longrightarrow & T_{k+1}.
\end{array} \tag{16}$$

We will denote the set of bilinear forms of displays in this sense by

$$Bil(\mathcal{P} \times \mathcal{P}', \mathcal{T}).$$

We return to the display $\mathcal{T}$ given by the standard data $K_l, \mathring{\Phi}_l$. We will now define maps $\lambda_{ij} : P_i \otimes P'_j \longrightarrow T_{i+j}$. For this we write $P_i \otimes P'_j$ according to the normal decompositions:

$$
\begin{aligned}
P_i \otimes P'_j = (\bigoplus_{\substack{n<i\\ m<j}} I \otimes I \otimes L_n \otimes L'_m) \oplus (\bigoplus_{\substack{n<i\\ m\geq j\\ m+n<i+j}} I \otimes L_n \otimes L'_m)\\
\oplus(\bigoplus_{\substack{n\geq i\\ m<j\\ n+m<i+j}} (I \otimes L_n \otimes L'_m)) \oplus (\bigoplus_{\substack{n<i\\ m\geq j\\ n+m\geq i+j}} I \otimes L_n \otimes L'_m)\\
\oplus(\bigoplus_{\substack{m\geq i\\ m\leq j\\ n+m\geq i+j}} (I \otimes L_n \otimes L'_m) \oplus (\bigoplus_{\substack{n\geq i\\ m\geq j}} L_n \otimes L'_m).
\end{aligned} \tag{17}
$$

We have six direct sums in brackets, which we denote by $Z_i$, $i = 1, \dots, 6$ in the order as above.
By definition $T_{i+j}$ has the decomposition

$$T_{i+j} = (\bigoplus_{n+m<i+j} I \otimes L_n \otimes L'_m) \oplus (\bigoplus_{n+m\geq i+j} L_n \otimes L'_m). \tag{18}$$

We define $\lambda_{ij} : P_i \otimes P'_j \longrightarrow T_{i+j}$ as a bigraded map with respect to $n, m \geq 0$, which is on the homogeneous components as follows.
Case $Z_1$: $n < i, m < j$

$$
\begin{aligned}
I \otimes I \otimes L_n \otimes L'_m &\longrightarrow I \otimes L_n \otimes L'_m\\
{}^V\xi \otimes {}^V\eta \otimes l_n \otimes l'_m &\longmapsto {}^V(\xi\eta) \otimes l_n \otimes l'_m
\end{aligned}
$$

Case $Z_2$: $n < i,\ m \geq j,\ n + m < i + j$

$$p^{m-j}\,\mathrm{id} : I \otimes L_n \otimes L'_m \longrightarrow I \otimes L_n \otimes L'_m$$

Case $Z_3$: $n \geq i,\ m < j,\ n + m < i + j$

$$p^{n-i}\,\mathrm{id} : I \otimes L_n \otimes L'_m \longrightarrow I \otimes L_n \otimes L'_m$$

Case $Z_4$: $n < i,\ m \geq j,\ n + m \geq i + j$

$$p^{i-n-1}\,\mathrm{id} : I \otimes L_n \otimes L'_m \longrightarrow I \otimes L_n \otimes L'_m$$

Case $Z_5$: $n \geq i,\ m < j,\ n + m \geq i + j$

$$p^{j-m-1}\,\mathrm{id} : I \otimes L_n \otimes L'_m \longrightarrow I \otimes L_n \otimes L'_m$$

Case $Z_6$: $n \geq i,\ m \geq j$

$$\mathrm{id} : L_n \otimes L'_m \longrightarrow L_n \otimes L'_m.$$

Proposition 2.8 *The homomorphism $\lambda_{ij} : P_i \otimes P'_j \longrightarrow T_{i+j}$ defined by $Z_1 - Z_6$ above define a bilinear form of displays.*

Proof: We omit the tedious but simple verification.

Lemma 2.9 *The homomorphism*

$$\oplus_{i+j=k} P_i \otimes P'_j \longrightarrow T_k$$

*given by the sum of $\lambda_{ij}$ is surjective.*

Proof: We have to show that all summand of (18) are in the image. Consider the submodule $L_n \otimes L'_m \subset T_k$ where $n + m \geq k$. We set $i = n$ and $j = k - i = k - n \leq m$. By $Z_6$ this submodule is in the image of $P_i \otimes P'_j \longrightarrow T_k$. Next we consider a submodule $I \otimes L_n \otimes L'_m \subset T_k$, where $n + m < k$. We set $i = n$ and $j = k - i = k - n > m$. Thus we are in the case $Z_3$ with factor $p^{n-i} = 1$. Again the submodule is in the image of $P_i \otimes P'_j \longrightarrow T_k$. *Q.E.D.*

Proposition 2.10 *Let $\mathcal{P}$ and $\mathcal{P}'$ be displays. Let $\mathcal{T} = (T_i, \mathring{\iota}_i, \mathring{\alpha}_i, \mathring{F}_i)$ be the display (15). Let $\mathcal{Q}$ be a separated predisplay. There is a canonical isomorphism of abelian groups*

$$Bil(\mathcal{P} \times \mathcal{P}', \mathcal{Q}) \cong \operatorname{Hom}(\mathcal{T}, \mathcal{Q}).$$

Proof: Assume that we are given a bilinear form. We set $\mathcal{T} = \mathcal{P} \otimes \mathcal{P}'$. The maps $T_i \longrightarrow Q_i$ are constructed inductively. For $i = 0$ this map is $\lambda_{00}$, where $\lambda$ denotes the bilinear form. For the induction step to $i + 1$ we consider the diagram

$$\begin{tikzcd} T_i \arrow[r] & Q_i \arrow[r, "F_i"] & Q_0 \\ T_{i+1} \arrow[u] \arrow[r, "F_{i+1}"] & T_0 \arrow[r] & Q_0 \arrow[u, "p"'] \end{tikzcd} \tag{19}$$

We claim that (19) is commutative. By Lemma 2.9 it suffices to show the commutativity if we compose the diagram with the maps $P_s \otimes P'_r \longrightarrow T_{i+1}$, for $s + r = i + 1$. This amounts to the commutativity of the following diagram

$$\begin{tikzcd}
 & Q_i \arrow[r, "F_i"] & Q_0 \\
 & Q_{i+1} \arrow[u, "\iota"'] \arrow[r] & Q_0 \arrow[u, "p"'] \\
P_r \otimes P'_s \arrow[uur] \arrow[ur] \arrow[urr, "F_r \otimes F_s"'] & &
\end{tikzcd}$$

But the diagram is commutative by the definition of a bilinear form. Now the commutativity of (19) gives a map: $T_{i+1} \longrightarrow Q_i \times_{F_i, Q_0, p} Q_0$. It is clear from the diagram above and Lemma 2.9 that this map factors through $Q_{i+1}$. *Q.E.D.*

COROLLARY 2.11 *The display (15)*

$$\mathcal{T} = (T_i, \overset{\circ}{\iota}_i, \overset{\circ}{\alpha}_i, \overset{\circ}{F}_i)$$

*does not depend up to canonical isomorphism on the normal decompositions of* $\mathcal{P}$ *and* $\mathcal{P}'$. *We write*

$$\mathcal{T} = \mathcal{P} \otimes \mathcal{P}'$$

This is clear because of the universal property of $\mathcal{T}$ proved in the last proposition. *Q.E.D.*

REMARK: Assume that $\mathcal{P}$ and $\mathcal{P}'$ are given by standard data $(L_i, \Phi_i)$ and $(L'_i, \Phi'_i)$. Assume we are given bilinear forms of $W(R)$-modules:

$$\beta_{ij} : L_i \otimes L'_j \to Q_{i+j}.$$

Composing this with the iteration of $\iota$, $Q_{i+j} \to Q_0$, we obtain a bilinear form

$$P_0 \otimes P'_0 = (\oplus_i L_i) \otimes (\oplus_j L'_j) \to Q_0$$

Let us assume that the following diagrams are commutative:

$$\begin{tikzcd}
L_i \otimes L'_j \arrow[r, "\Phi_i \otimes \Phi'_j"] \arrow[d, "\beta_{ij}"'] & P_0 \otimes P'_0 \arrow[d] \\
Q_{i+j} \arrow[r, "F_{i+j}"] & Q_0
\end{tikzcd}$$

Then the $\beta_{ij}$ extend uniquely to a bilinear form

$$\mathcal{P} \times \mathcal{P}' \to \mathcal{Q}$$

In [Z2] Definition 18 the notion of a bilinear form of 1-displays was defined. It is obvious from the formulas there, that a bilinear form on two 1-displays in the sense of loc.cit. is the same as a bilinear form

$$\mathcal{P} \times \mathcal{P}' \to \mathcal{U}(1),$$

where the right hand side is the twisted unit display (11).
Starting from the normal decomposition of a display $\mathcal{P}$ it is easy to write down the standard data of a candidate for the exterior power $\bigwedge^k \mathcal{P}$. It comes with an alternating map $\otimes^k \mathcal{P} \to \bigwedge^k \mathcal{P}$. One proves as above that $\bigwedge^k \mathcal{P}$ has the universal property.
We will now define the base change for displays. Let $R \longrightarrow S$ be a homomorphism of rings. Let $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ be a display over $R$. We will define a display $\mathcal{P}_S = (Q_i, \iota_i, \alpha_i, F_i)$ over $S$, with the following properties. There are $W(R)$-linear maps

$$P_i \longrightarrow Q_i,$$

such that the following diagrams are commutative

$$\begin{array}{ccc} P_i & \longrightarrow & Q_i \\ {\scriptstyle \iota_i}\uparrow & & \uparrow{\scriptstyle \iota_i} \\ P_{i+1} & \longrightarrow & Q_{i+1} \end{array} \qquad \begin{array}{ccc} Q_i & \xrightarrow{F_i} & Q_0 \\ \uparrow & & \uparrow \\ P_i & \xrightarrow{F_i} & P_0 \end{array} \qquad \begin{array}{ccc} I_R \otimes Q_i & \xrightarrow{\alpha_i} & Q_{i+1} \\ \uparrow & & \uparrow \\ I_R \otimes P_i & \xrightarrow{\alpha_i} & P_{i+1} \end{array} \tag{20}$$

Proposition 2.12 *There is a unique display $\mathcal{P}_S$ as above which enjoys the following universal property.*
*If $\mathcal{T} = (T_i, \iota_i, \alpha_i, F_i)$ is an arbitrary display over $S$ and*

$$P_i \longrightarrow T_i$$

*is a set of $W(R)$-linear morphisms, such that the diagrams above, with $Q_i$ replaced by $T_i$ are commutative, then there is a unique morphism of displays over $S$*

$$\mathcal{P}_S \longrightarrow \mathcal{T},$$

*such that the following diagrams are commutative:*

$$\begin{tikzcd} Q_i \arrow[rr] & & T_i \\ & P_i \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The display $\mathcal{P}_S$ may be constructed using a normal decomposition of $\mathcal{P}$. Let $P_0 = \oplus L_i$ be such a decomposition, and let $\Phi_i : L_i \longrightarrow P_0$ be the maps induced by $F_i$. Then $L_i, \Phi_i$ are standard data for a display over $R$. We can define $\mathcal{P}_S$ to be the display over $S$ associated to the standard data $W(S) \otimes_{W(R)} L_i$, with the Frobenius linear maps $F \otimes_{W(R)} \Phi_i = \Phi'_i$.
We will now see that this definition is up to canonical isomorphism independent of the normal decomposition chosen. It suffices to see that $\mathcal{P}_S$ has the universal property Proposition 2.12.
The obvious maps $P_i \longrightarrow Q_i$ make the diagrams (20) commutative.

Lemma 2.13 *The following $W(S)$-module homomorphism is surjective*

$$W(S) \otimes_{W(R)} P_i \oplus I_S \otimes_{W(S)} Q_{i-1} \longrightarrow Q_i.$$

Proof: This is clear from the definitions.
Assume that $P_i \longrightarrow T_i$ is a set of maps as in Proposition 2.12. We construct inductively maps $Q_i \longrightarrow T_i$, which are compatible with $F_i, \iota_i, \alpha_i$. Therefore we obtain the desired morphism of displays $\mathcal{P}_S \longrightarrow \mathcal{T}$. Since $P_0 \longrightarrow T_0$ is $W(R)$-linear, we obtain a map

$$Q_0 = W(S) \otimes_{W(R)} P_0 \longrightarrow T_0.$$

Assume we have already constructed $W(S)$-module homomorphisms

$$Q_j \longrightarrow T_j,$$

which are compatible with $F, \iota$ and $\alpha$ for $j \leq i$.
Consider the diagram

$$\begin{array}{ccc} T_i & \xrightarrow{F_i} & T_0 \\ \uparrow & & \uparrow p \\ Q_{i+1} & \longrightarrow & T_0. \end{array} \tag{21}$$

The arrow $Q_{i+1} \longrightarrow T_i$ is the composition $Q_{i+1} \xrightarrow{\iota} Q_i \longrightarrow T_i$ and the arrow $Q_{i+1} \longrightarrow T_0$ is the composition $Q_{i+1} \xrightarrow{F_{i+1}} Q_0 \longrightarrow T_0$. By Lemma 2.13 we deduce that (21) is commutative. Thus it induces a map

$$Q_{i+1} \longrightarrow T_i \times_{F_i, T_0, p} T_0. \tag{22}$$

It suffices to show that the last map factors through $T_{i+1}$. This is seen easily by composing (22) with the morphism of the lemma.
The uniqueness of the constructed morphism $\mathcal{P}_S \longrightarrow \mathcal{T}$ is obvious. This proves the proposition. *Q.E.D.*

## 3 Degeneracy of some Spectral Sequences

Proposition 3.1 *Let $\pi : X \to Y$ be a separated and quasicompact morphism. Let $K^{\cdot}$ be a complex of of flat $\pi^{-1}\mathcal{O}_Y$-modules on $X$ which is bounded above. We assume that each $K^i$ is a quasicoherent $\mathcal{O}_X$-module. Then for each $m$ the hypercohomology groups $\mathbb{R}^m\pi_*K^{\cdot}$ are quasicoherent $\mathcal{O}_Y$-modules. If $M$ is a quasicoherent $\mathcal{O}_Y$-module there is a canonical isomorphism*

$$\mathbb{R}\pi_*(K^{\cdot} \otimes^{\mathbb{L}}_{\pi^{-1}(\mathcal{O}_Y)} \pi^{-1}M) \cong \mathbb{R}\pi_*K^{\cdot} \otimes^{\mathbb{L}}_{\mathcal{O}_Y} M \tag{23}$$

Proof: We may assume that $Y$ is affine. Let $\mathcal{U} = \{U_i\}$ be a finite affine covering of $X$. Let $F^{\cdot} = C^{\cdot}(\mathcal{U}, K^{\cdot})$ be the Czech complex. It is the complex of global sections of a sheafified Czech complex on $Y$: $\mathcal{F}^{\cdot} = \mathcal{C}^{\cdot}(\mathcal{U}, K^{\cdot})$. The sheaves in this complex are acyclic with respect to $\pi_*$ because the cohomology of an affine scheme vanishes. One concludes [EGA III] Prop. 1.4.10 that $\mathbb{R}\pi_*K^m$ are quasicoherent $\mathcal{O}_Y$-modules namely the sheaves associated to the cohomology of $F^{\cdot}$. Since the modules and sheaves involved are flat with respect to $Y$ the projection formula reduces to the trivial equation:

$$C^{\cdot}(\mathcal{U}, K^{\cdot} \otimes_{\mathcal{O}_Y} M) \cong F^{\cdot} \otimes_{\Gamma(Y,\mathcal{O}_Y)} \Gamma(Y, M)$$

*Q.E.D.*

Let $\pi : X \to S$ be a proper morphism of schemes, such that $S$ is affine. In this section we consider a bounded complex $K^{\cdot}$ of flat $\pi^{-1}(\mathcal{O}_S)$-modules. We assume that each $K^i$ is a quasicoherent $\mathcal{O}_X$-module. Moreover we assume that the following conditions are satisfied:

(i) $R^j\pi_*K^i$ is a locally free $\mathcal{O}_S$-module of finite type for any $i$ and $j$.

(ii) the spectral sequence of hypercohomology degenerates:

$$E_1^{ij} = R^j\pi_*K^i \Rightarrow \mathbb{R}^n\pi_*K^{\cdot}$$

One can easily see that with these assumptions the simple complex associated to $C^{\cdot}(\mathcal{U}, K^{\cdot})$ as above is quasi-isomorphic to the direct sum of its cohomology groups. It follows that $\mathbb{R}^m\pi_*K^{\cdot}$ commutes with arbitrary base change for any $m$. For the same reason the cohomology groups $\mathbb{R}^j\pi_*K^i$ commute with arbitrary base change.

The degeneration of this spectral sequence may be reformulated as follows. Let us denote the by $\sigma^{\geq m}K^{\cdot}$ and $\sigma^{<m}K^{\cdot}$ the truncated complexes with respect to the naive truncation. Then the cohomology sequence of

$$0 \to \sigma^{\geq m}K^{\cdot} \to K^{\cdot} \to \sigma^{<m}K^{\cdot} \to 0,$$

splits into short exact sequences:

$$0 \to \mathbb{R}^q\pi_*(\sigma^{\geq m}K^{\cdot}) \to \mathbb{R}^q\pi_*K^{\cdot} \to \mathbb{R}^q\pi_*(\sigma^{<m}K^{\cdot}) \to 0. \tag{24}$$

Indeed, take a Cartan-Eilenberg resolution $K^{\cdot} \to I^{\cdot}$ by injective sheaves of abelian groups. Let $L^{\cdot} = \pi_* I^{\cdot}$. This complex comes with a filtration $Fil^m L^{\cdot}$ which is induced by the naive filtration of $K$. The spectral sequence in question is the spectral sequence of this filtered complex. The condition (24) is equivalent to the requirement that the maps

$$H^q(Fil^{m+1}L^{\cdot}) \to H^q(Fil^m L^{\cdot})$$

are injective for each $q$ and $m$, as one may see easily from the exact cohomology sequence. This injectivity may be restated as follows:

$$d(Fil^m L^{q-1}) \cap Fil^{m+1}L^q = d(Fil^{m+1}L^{q-1}).$$

We conclude by [De3] Prop. 1.3.2.
The observation shows that the spectral sequences of hypercohomology of the truncated complexes $\sigma^{\geq m}K^{\cdot}$ and $\sigma^{<m}K^{\cdot}$ degenerate too.

Proposition 3.2 *Let $\pi : X \to S$ and $K^{\cdot}$ be as in Proposition 3.1. Let $\ldots \to M_0 \to M_1 \to M_2 \to \ldots$ be a sequence of $\mathcal{O}_S$-modules (not necessarily a complex). We consider the complex*

$$L^{\cdot} : \ldots \to K^0 \otimes_{\mathcal{O}_S} M_0 \to K^1 \otimes_{\mathcal{O}_S} M_1 \to K^2 \otimes M_2 \to \ldots$$

*Then the spectral sequence*

$$E_1^{ij} : R^j\pi_* L^i \Rightarrow \mathbb{R}^{p+q}\pi_* L^{\cdot}$$

*degenerates.*

Proof: We assume without loss of generality that $K^i = 0$ for $i < 0$. We say that a sequence $M_0 \to M_1 \to \ldots$ is $m$-stationary if it is isomorphic to a sequence of the form:

$$M_0 \to \ldots \to M_{m-1} \to M_m = M_m = \ldots$$

Because $K^{\cdot}$ is bounded it suffices to show the theorem for $m$-stationary sequences. We argue by induction. For $m = 0$ this is clear from the projection formula (23). Assume that the proposition holds for $r$-stationary sequences with $r < m$. For an $m$-stationary sequence we consider the following morphism of complexes:

$$L^{\cdot} \to I^{\cdot} \tag{25}$$

$$\begin{array}{ccccc}
K^0 \otimes M^0 \ldots K^{m-2} \otimes M^{m-2} & \longrightarrow & K^{m-1} \otimes M^{m-1} & \longrightarrow & K^m \otimes M^m \ldots \\
\mathrm{id}\downarrow \qquad\qquad \mathrm{id}\downarrow & & \downarrow & & \mathrm{id}\downarrow \\
K^0 \otimes M^0 \ldots K^{m-2} \otimes M^{m-2} & \longrightarrow & K^{m-1} \otimes M^m & \longrightarrow & K^m \otimes M^m \ldots
\end{array}$$

If we apply the induction assumption to $I^{\cdot}$ we obtain an exact sequence for each $q$ and the given $m$.

$$0 \to \mathbb{R}^q\pi_*(\sigma^{\geq m}I^{\cdot}) \to \mathbb{R}^q\pi_* I^{\cdot} \to \mathbb{R}^q\pi_*(\sigma^{<m}I^{\cdot}) \to 0. \tag{26}$$

The morphism of complexes (25) induces a commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^q\pi_*\sigma^{\geq m}L^{\cdot} & \longrightarrow & \mathbb{R}^q\pi_*L^{\cdot} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ \mathbb{R}^q\pi_*\sigma^{\geq m}I^{\cdot} & \longrightarrow & \mathbb{R}^q\pi_*I^{\cdot} \end{array}$$

By our induction assumption (26) it follows that the upper horizontal arrow is injective.
We have to prove that the following sequences are exact for arbitrary integers $q$ and $n$.

$$0 \to \mathbb{R}^q\pi_*(\sigma^{\geq n}L^{\cdot}) \to \mathbb{R}^q\pi_*L^{\cdot} \to \mathbb{R}^q\pi_*(\sigma^{<n}L^{\cdot}) \to 0.$$

We have seen this for $n = m$. For $n > m$ we have to consider the maps.

$$\mathbb{R}^q\pi_*(\sigma^{\geq n}L^{\cdot}) \to \mathbb{R}^q\pi_*(\sigma^{\geq m}L^{\cdot}) \to \mathbb{R}^q\pi_*L^{\cdot}$$

It suffices to show that the first arrow is injective. But this follows from the beginning of our induction.
Finally we consider the case $n < m$. By the cohomology sequence it is sufficient to see that the map

$$\mathbb{R}^q\pi_*L^{\cdot} \to \mathbb{R}^q\pi_*(\sigma^{<n}L^{\cdot})$$

is surjective. But this map factors as:

$$\mathbb{R}^q\pi_*L^{\cdot} \to \mathbb{R}^q\pi_*(\sigma^{<m}L^{\cdot}) \to \mathbb{R}^q\pi_*(\sigma^{<n}L^{\cdot})$$

We need to show that the second map is surjective. But the complex $\sigma^{<m}L^{\cdot}$ is the tensor product of $\sigma^{<m}K^{\cdot}$ with an $(m-1)$-stationary sequence of modules. Therefore the map is surjective by induction assumption and we are done. *Q.E.D.*

Proposition 3.3 *Let $T : \mathcal{C} \to \mathcal{D}$ be a left exact functor of abelian categories such that $\mathcal{C}$ has enough injective objects. Let $K^{\cdot}$ be a complex in $\mathcal{C}$ which is bounded below. We assume that the spectral sequences in hypercohomology*

$$E_1^{ij} = R^jTK^i \Rightarrow \mathbb{R}^{i+j}TK^{\cdot}$$

*degenerates. Let $f^{\cdot} : K^{\cdot} \to K^{\cdot}$ be a homomorphism of complexes. Then for each integer $m$ the corresponding spectral sequence of hypercohomology associated to the complex*

$$K(m,f): \quad \xrightarrow{d} K^{m-2} \xrightarrow{d} K^{m-1} \xrightarrow{f^m d} K^m \xrightarrow{d} K^{m+1} \to \dots$$

*degenerates.*

We omit the proof because it uses exactly the same arguments as above.

# 4 Filtered Comparison Theorems for the de Rham-Witt complex

Let $R$ be a ring such that $p$ is nilpotent in $R$. We consider a smooth scheme $X$ over $R$. We will fix a natural number $n$. Assume we are given a smooth lifting $\tilde{X}/W_n(R)$. If $\tilde{X}$ admits a Witt-lift ([LZ] Def.3.3) $\mathcal{O}_{\tilde{X}} \longrightarrow W_n(\mathcal{O}_X)$ we obtain a morphism of complexes

$$\Omega^{\cdot}_{\tilde{X}/W_n(R)} \longrightarrow \Omega^{\cdot}_{W_n(X)/W_n(R)} \longrightarrow W_n\Omega^{\cdot}_{X/R}. \tag{27}$$

It is shown in [LZ] 3.2 and 3.3, that even if $\tilde{X}$ admits no Witt lift, there is a natural isomorphism in the derived category $D^+(X_{zar}, W_n(R))$ of sheaves of $W_n(R)$-modules on $X$:

$$\Omega^{\cdot}_{\tilde{X}/W_n(R)} \longrightarrow W_n\Omega^{\cdot}_{X/R}.$$

The aim of this section is to prove a filtered version of this isomorphism.
For typographical reasons we use the abbreviations:

$$\tilde{\Omega}^{\cdot}_n = \Omega^{\cdot}_{\tilde{X}/W_n(R)}, \qquad W_n\Omega^{\cdot} = W_n\Omega^{\cdot}_{X/R}.$$

Let us denote by $\mathcal{F}^m\Omega^{\cdot}_{\tilde{X}/W(R)}$ the complex

$$I_{R,n} \otimes_{W_n(R)} \tilde{\Omega}^0_n \xrightarrow{pd} \ldots \xrightarrow{pd} I_{R,n} \otimes_{W_n(R)} \tilde{\Omega}^{m-1}_n \xrightarrow{d} \tilde{\Omega}^m_n \xrightarrow{d} \tilde{\Omega}^{m+1}_n \to \ldots. \tag{28}$$

Conjecture 4.1 *There is a canonical isomorphism in the derived category $D^+(X_{zar}, W_n(R))$ between the Nygaard complex and the complex* (28)*:*

$$\mathcal{N}^m W_n\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m\Omega^{\cdot}_{\tilde{X}/W_n(R)}$$

This question is closely related to the work of Deligne and Illusie [DI]. We will now see that the complexes in question are always locally isomorphic.
Let us assume we are given a Witt-lift. It induces a map

$$\kappa : \tilde{\Omega}^{\cdot}_n \longrightarrow W_n\Omega^{\cdot}.$$

By composition with the Frobenius $F : W_n\Omega^{\cdot} \to W_{n-1}\Omega^{\cdot}_{[F]}$ we obtain a map

$$\begin{aligned} \tilde{F} :& I_{R,n} \otimes_{W_n(R)} \tilde{\Omega}^{\cdot}_n \longrightarrow W_{n-1}\Omega_{[F]}. \\ & {}^V\xi \otimes \omega \longmapsto \xi\ {}^F\kappa(\omega) \end{aligned}$$

Using $\tilde{F}$ we obtain a morphism of complexes of $\mathcal{F}^m\tilde{\Omega}^{\cdot} \longrightarrow \mathcal{N}^m W_n\Omega$:

$$\begin{tikzcd} I_R \otimes \tilde{\Omega}^0_n \arrow[r, "pd"] \arrow[d, "\tilde{F}"] & \ldots \arrow[r, "pd"] & I_R \otimes \tilde{\Omega}^{m-1}_n \arrow[r, "d"] \arrow[d, "\tilde{F}"] & \tilde{\Omega}^m_n \arrow[r, "d"] \arrow[d] & \ldots \\ W_{n-1}\Omega^0_{[F]} \arrow[r, "d"] & \ldots \arrow[r, "d"] & W_{n-1}\Omega^{m-1}_{[F]} \arrow[r, "dV"] & W_n\Omega^m \arrow[r, "d"] & \ldots \end{tikzcd} \tag{29}$$

Let us consider the morphism (29) in the following simple situation:
Let $A = R[T_1, \dots, T_d]$ and $X = \operatorname{Spec} A$. We set $\tilde{A} = W_n(R)[T_1, \dots T_d]$ and $\tilde{X} = \operatorname{Spec} \tilde{A}$. We consider the Witt-lift:

$$\begin{aligned} \tilde{A} &\longrightarrow W_n(A) \\ T_i &\longrightarrow [T_i]. \end{aligned} \tag{30}$$

It is the unique map of $W_n(R)$-algebras, which maps $T_i$ to its Teichmüller representative in $W_n(A)$.

Proposition 4.2 *For the Witt-lift* (30) *the induced morphism*

$$\mathcal{F}^m \Omega^{\cdot}_{\tilde{X}/W_n(R)} \longrightarrow \mathcal{N}^m W_n \Omega^{\cdot}_{X/R} \tag{31}$$

*is for any $m \geq 0$ a quasi-isomorphism.*

Proof: We use the $W_n(R)$-basis of $\Omega^l_{\tilde{A}/W_n(R)}$ given by $p$-basic differential forms. For each weight function $k : [1, d] \to \mathbb{Z}_{\geq 0}$ we fix an order on the set

$$\begin{aligned} &\mathit{Supp}\, k = \{i_1, \dots, i_r\}, \text{ such that} \\ &\mathit{ord}_p k_{i_1} \leq \dots \leq \mathit{ord}_p k_{i_r}. \end{aligned}$$

For any ascending partition of $\mathit{Supp}\, k$ into disjoint intervals

$$\mathcal{P} : \mathit{Supp}\, k = I_0 \sqcup I_1 \sqcup \dots \sqcup I_l,$$

such that $I_t \neq \emptyset$ for $1 \leq t \leq l$, we have the $p$-basic differential

$$\tilde{e}(k, \mathcal{P}) = T^{k_{I_0}} \left(p^{-\mathit{ord}_p k_{I_1}} dT^{k_{I_1}}\right) \cdot \dots \cdot \left(p^{-\mathit{ord}_p k_{I_l}} dT^{I_l}\right). \tag{32}$$

The order on $\mathit{Supp}\, k$ is fixed once for all and therefore not indicated in the notation (compare [LZ] 2.1).
In [LZ] 2.2 we have defined the basic Witt differentials

$$e_n(\xi, k, \mathcal{P}) \in W_n \Omega^l_{A/R}.$$

They are defined for functions $k : [1, d] \to \mathbb{Z}_{\geq 0}[\frac{1}{p}]$, and $\xi \in V^{u(k)} W_{n-u(k)}(R)$, where $u(k)$ is the minimal nonnegative integer, such that the weight $p^{u(k)} k$ takes integral values.
In our case the map (27) is the unique $W_n(R)$-linear map given by

$$\begin{aligned} \Omega^l_{\tilde{A}/W_n(R)} &\longrightarrow W_n \Omega^l_{A/R}. \\ \tilde{e}(k, \mathcal{P}) &\longmapsto e_n(1, k, \mathcal{P}). \end{aligned} \tag{33}$$

The map $\tilde{F}$ looks as follows

$$\begin{aligned} \tilde{F} : I_R \otimes_{W_n(R)} \Omega^l_{\tilde{A}/W_n(R)} &\longrightarrow W_{n-1} \Omega^l_{A/R,[F]} \\ {}^V\xi \otimes \tilde{e}(k, \mathcal{P}) &\longmapsto e_{n-1}(\xi, pk, \mathcal{P}). \end{aligned}$$

For each weight $k : [1, d] \longrightarrow \mathbb{Z}_{\geq 0}[\frac{1}{p}]$, we consider the subgroup $W_n\Omega^l_{A/R}(k)$ of $W_n\Omega^l_{A/R}$, which is generated by basic Witt-differentials $e_n(\xi, k, \mathcal{P})$ of fixed weight $k$. The complex $\mathcal{N}^m W_n\Omega$ splits into a direct sum of subcomplexes $\mathcal{N}^m(k)$:

$$W_{n-1}\Omega^0_{[F]}(pk) \xrightarrow{d} \cdots \xrightarrow{d} W_{n-1}\Omega^{m-1}_{[F]}(pk) \xrightarrow{dV} W_n\Omega^m_{[F]}(k) \rightarrow \cdots .$$

Similarly let $\Omega^l_{\tilde{A}/W_n(R)}(k) \subset \Omega^l_{\tilde{A}/W_n(R)}$ the $W_n(R)$-submodule generated by the $p$-basic differentials $\tilde{e}(k, \mathcal{P})$ of fixed integral weight $k$. Then $\mathcal{F}^m\tilde{\Omega}^{\cdot}$ is the direct sum of the following subcomplexes $\mathcal{F}^m(k)$:

$$I_R \otimes_{W_n(R)} \tilde{\Omega}^0_n(k) \xrightarrow{pd} \cdots \xrightarrow{pd} I_R \otimes_{W_n(R)} \tilde{\Omega}^{m-1}_n(k) \xrightarrow{d} \tilde{\Omega}^m_n(k) \rightarrow \cdots .$$

It is obvious that for integral weight $k$ the map

$$\mathcal{F}^m(k) \longrightarrow \mathcal{N}^m(k) \tag{34}$$

is an isomorphism of complexes. Therefore the proposition follows if we show that for $k$ not integral the complexes $\mathcal{N}^m(k)$ are acyclic. This follows in degrees not equal to $m-1$ or $m$ from the corresponding statement for the de Rham-Witt complex (see [LZ] Proof of thm. 3.5).

For non-integral $k$ consider a cycle $\omega \in W_{n-1}\Omega^{m-1}_{[F]}(k)$, i.e. $dV\omega = 0$. Because of the relation $FdV = d$, it follows that $\omega$ is also a cycle in the de Rham-Witt complex $W_{n-1}\Omega^{\cdot}$ and consequently a boundary, because $k$ is not integral.

Finally consider a cycle $\omega \in W_n\Omega^m(k)$. It may be uniquely written as a sum

$$\omega = \sum_{\mathcal{P}} e_n(\xi_{\mathcal{P}}, k, \mathcal{P}).$$

By [LZ] Prop. 2.6 $\omega$ is a cycle, iff $\mathcal{P} = \emptyset \sqcup \mathcal{P}'$, i.e. iff the first interval $I_0$ of the partition $\mathcal{P}$ is empty, for all $e_n(\xi_{\mathcal{P}}, k, \mathcal{P}) \neq 0$ which appear in the sum. Since $k$ is not integral the coefficient $\xi_{\mathcal{P}}$ is of the form $\xi_{\mathcal{P}} = {}^V\tau_{\mathcal{P}}$ and

$$d\,{}^V e_{n-1}(\tau_{\mathcal{P}}, pk, \mathcal{P}) = e_n(\xi_{\mathcal{P}}, k, \mathcal{P}).$$

*Q.E.D.*

We make $n$ variable. We set $A = R[T_1, \ldots, T_d]$, $A_n = W_n(R)[T_1 \ldots T_d]$. We extend the Frobenius homomorphism $F : W_n(R) \longrightarrow W_{n-1}(R)$ to a map

$$\begin{aligned} \phi_n : &A_n \longrightarrow A_{n-1}, \\ &T_i \longmapsto T_i^p. \end{aligned} \tag{35}$$

We denote $\delta_n : A_n \longrightarrow W_n(A)$ the $W_n(R)$-algebra homomorphism, such that $\delta_n(T_i) = [T_i]$.

Assume we are given an étale homomorphism $A \longrightarrow B$ of $R$-algebras. Then we find a unique set of lifting $B_n$ of $B$, which are étale over $A_n$ and morphisms

$$\psi_n : B_n \longrightarrow B_{n-1} \text{ and } \varepsilon_n : B_n \longrightarrow W_n(B),$$

which are compatible with $\phi_n$ and $\delta_n$, compare [LZ] Prop. 3.2.

Corollary 4.3 *The map $\varepsilon_n$ defines a quasi-isomorphism of complexes:*

$$\begin{array}{ccccccc}
I_R \otimes \Omega^0_{B_n/W_n(R)} & \xrightarrow{pd} & \dots & \xrightarrow{pd} & I_R \otimes \Omega^{m-1}_{B_n/W_n(R)} & \xrightarrow{d} & \Omega^m_{B_n/W_n(R)} \cdots \\
{\scriptstyle \tilde{F}}\downarrow & & & & {\scriptstyle \tilde{F}}\downarrow & & \downarrow \\
W_{n-1}\Omega^0_{B/R,[F]} & \xrightarrow{d} & \dots & \xrightarrow{d} & W_{n-1}\Omega^{m-1}_{B/R,[F]} & \xrightarrow{dV} & W_n\Omega^m_{B/R} \cdots
\end{array}$$

Proof: For the given number $n$, we find a number $m$ such that $p^m W_n(R) = 0$. Let us denote by $\phi^m : A_{m+n} \longrightarrow A_n$ the composite of $m$ morphisms of type (35). It is clear from the definition that

$$d\phi^m : A_{m+n} \longrightarrow \Omega^1_{A_n/W_n(R)}$$

is zero. Consider the commutative diagram

$$\begin{array}{ccc}
B_{m+n} & \overset{d\psi^m}{\to} & \Omega^1_{B_n/W_n(R)} \\
\uparrow & & \uparrow \\
A_{m+n} & \overset{d\phi^m}{\to} & \Omega^1_{A_n/W_n(R)}.
\end{array}$$

The derivation $A_{m+n} \longrightarrow \Omega^1_{B_n/W_n(R)}$ is zero. Since $B_{m+n}/A_{m+n}$ is étale, the extension $d\psi^m$ is zero too.

Consider the commutative diagram

$$\begin{array}{ccc}
B_{m+n} & \overset{\psi^m}{\longrightarrow} & B_m \\
\uparrow & & \uparrow \\
A_{m+n} & \overset{\phi^m}{\to} & A_n.
\end{array}$$

It induces a morphism of algebras which are étale over $A_n$:

$$B_{m+n} \otimes_{A_{m+n},\phi^m} A_n \longrightarrow B_n. \tag{36}$$

This is an isomorphism. Indeed since $A_n \longrightarrow A/pA$ has nilpotent kernel it is enough to show that (36) becomes an isomorphism after tensoring with $\otimes_{A_n} A/pA$. But then we obtain the well-known isomorphism

$$\begin{array}{rcl}
B/pB \otimes_{A/pA, Frob^m} A/pA & \longrightarrow & B/pB \\
b \otimes a & \longmapsto & b^{p^m} \cdot a.
\end{array}$$

From the isomorphism (36) we deduce an isomorphism

$$\begin{array}{rcl}
B_{m+n} \otimes_{A_{m+n},\phi^m} \Omega^{\cdot}_{A_n/R} & \overset{\sim}{\longrightarrow} & \Omega^{\cdot}_{B_n/R} \\
b \quad \otimes \quad \omega & \longmapsto & \psi^m(b) \cdot \omega.
\end{array} \tag{37}$$

We note that (37) becomes an isomorphism of complexes if we take $1 \otimes d$ as a differential on the left hand side. Hence the first row of (4.3) is obtained by tensoring the corresponding complex for $B_n = A_n$ with $B_{n+m}$.
Let us consider the complex

$$W_{n-1}\Omega^0_{A/R,[F]} \xrightarrow{d} \cdots \xrightarrow{d} W_{n-1}\Omega^{m-1}_{A/R,[F]} \xrightarrow{dV} W_n\Omega^m_{A/R} \xrightarrow{d} \cdots . \tag{38}$$

We consider it as a complex of $W_{n+m}(A)$-modules via $F^m : W_{n+m}(A) \longrightarrow W_n(A)$. Then all differentials become linear (compare [LZ] Remark 1.8).
This shows that we obtain the second row of diagram of Corollary 4.3 if we tensorize (38) with $W_{n+m}(B)\otimes_{W_{n+m}(A),F^m}$. Because of the obvious isomorphism ([LZ] (3.2))

$$B_{n+m} \otimes_{A_{n+m},\delta} W_{n+m}(A) \tilde{\to} W_{n+m}(B),$$

the result is the same if we tensorize (38) by

$$B_{n+m} \otimes_{A_{n+m},\delta\phi^m} .$$

Therefore the whole diagram of Corollary 4.3 is obtained from the corresponding diagram for $B = A$ by tensoring with $B_{n+m}\otimes_{A_{n+m},\phi^m}$. Since this tensor product is an exact functor we obtain the corollary from the proposition. *Q.E.D.*

Let $X/R$ be a smooth scheme. We assume that $R$ is reduced and $p\cdot R = 0$. Then we consider still another complex derived from the de Rham-Witt complex. We set $W\Omega^l = W\Omega^l_{X/R}$ and define $\mathcal{I}^m W_n\Omega_{X/R}$ starting in degree 0.

$$p^{m-1}VW_{n-1}\Omega^0 \xrightarrow{d} p^{m-2}VW_{n-1}\Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} VW_{n-1}\Omega^{m-1} \xrightarrow{d} W_n\Omega \ldots . \tag{39}$$

We recall the relation $pd\ {}^V\omega = {}^V d\omega$ of [LZ] 1.17. For varying $n$ we obtain a procomplex $\mathcal{I}^m W.\Omega_{X/R}$.

Proposition 4.4 *Let $R$ be a reduced ring of char $p$. The procomplexes $\mathcal{I}^m W.\Omega$ and $\mathcal{N}^m W.\Omega$ are isomorphic in the pro-category of the category of complexes of abelian sheaves on $X_{zar}$.*

Proof: We have an obvious morphism of procomplexes

$$\mathcal{N}^m W.\Omega \longrightarrow \mathcal{I}^m W.\Omega \tag{40}$$

$$\begin{array}{ccccccccc}
W_{n-1}\Omega^0_{[F]} & \xrightarrow{d} & W_{n-1}\Omega^1_{[F]} & \ldots & W_{n-1}\Omega^{m-1}_{[F]} & \xrightarrow{dV} & W_n\Omega^m & \xrightarrow{d} \ldots \\
p^{m-1}V\downarrow & & p^{m-2}V\downarrow & & V\downarrow & & id\downarrow & \\
p^{m-1}VW_{n-1}\Omega^0 & \xrightarrow{d} & p^{m-2}VW_{n-1}\Omega^1 & \ldots & VW_{n-1}\Omega^{m-1} & \xrightarrow{d} & W_n\Omega^m & \xrightarrow{d} \ldots .
\end{array}$$

We have to prove that this induces an isomorphism of proobjects. We set $W\Omega = \varprojlim W_n\Omega$. On $W\Omega$ the multiplication by $p$ and the Verschiebung are injective. Therefore we have an inverse $p^iVW\Omega \overset{p^{-i}V^{-1}}{\longrightarrow} W\Omega_{[F]}$.

Lemma 4.5 *Let $n > k \geq i+1$. Then there is a map $p^i V W_n \Omega^l \longrightarrow W_{n-k}\Omega^l$, which makes the following diagram commutative*

$$\begin{array}{ccc} p^i V W_n \Omega^l_{X/R} & \longrightarrow & W_{n-k}\Omega^l_{X/R,[F]} \\ \uparrow & & \uparrow \\ p^i V W \Omega^l_{X/R} & \overset{p^{-i}V^{-1}}{\longrightarrow} & W\Omega^l_{X/R,[F]}. \end{array} \tag{41}$$

Proof of the lemma: Let $n > k \geq i$. For $\xi \in W_n(R)$ we denote by $\bar{\xi}$ its restriction to $W_{n-k}(R)$. Then we have a well-defined map

$$\begin{array}{rcl} p^i V W_n(R) \longrightarrow & W_{n-k}(R) \\ p^i \ {}^V\xi \longmapsto & \bar{\xi}. \end{array} \tag{42}$$

Indeed, write $\xi = (x_0, \dots, x_{n-1})$. Then

$$p^i \ {}^V\xi = (0, \dots, 0, x_0^{p^i}, \dots, x_{n-i-1}^{p^i}) \in W_{n+1}(R).$$

Therefore the vector $(x_0, \dots, x_{n-i-1}) \in W_{n-i}(R)$ is uniquely determined by $p^i \ {}^V\xi$. We view $W_{n-i}(R)$ as a $W_{n+1}(R)$-module via

$$W_{n+1}(R) \xrightarrow{F} W_n(R) \xrightarrow{Res} W_{n-i}(R).$$

Then we obtain a morphism of $W_{n+1}(R)$-modules because of the following commutative diagram

$$\begin{array}{ccc} p^i V W(R) & \overset{p^{-i}V^{-1}}{\longrightarrow} & W(R) \\ \downarrow & & \downarrow \\ p^i V W_n(R) & \longrightarrow & W_{n-i}(R). \end{array}$$

The existence of the diagram (41) is clearly local for the Zariski-topology on $X$.

We begin with the case, where $X = \operatorname{Spec} A$ and $A = R[T_1, \dots, T_d]$ is a polynomial algebra. In this case an element of $p^i V W \Omega^l_{A/R}$ may be expressed, in terms of basic Witt-differentials:

$$\omega = \sum p^i \ {}^V e_n(\xi_{\mathcal{P},k}, k, \mathcal{P}), \quad \xi_{\mathcal{P},k} \in V^{u(k)} W_{n-u(k)}(R). \tag{43}$$

Note that $e_n(\xi_{\mathcal{P},k}, k, \mathcal{P}) = 0$, when $u(k) \geq n$.

The terms of the sum (43) are uniquely determined by [LZ] Prop.2.5 because of the direct decomposition

$$W_{n+1}\Omega^l_{A/R} = \oplus_{k,\mathcal{P}} W_{n+1}\Omega^l_{A/R}(\frac{k}{p}, \mathcal{P}).$$

Using loc. cit. we find:

$$p^i \ {}^V e(\xi_{\mathcal{P},k}, k, \mathcal{P}) = p^i \ {}^V e(\xi'_{\mathcal{P},k}, k, \mathcal{P}), \tag{44}$$

iff $p^i\ {}^V\xi_{\mathcal{P},k} = p^i\ {}^V\xi'_{\mathcal{P},k}$, except in the case where $k/p$ is not integral and $I_0 = \emptyset$. In the latter case the equality (44) holds, iff $p^{i+1}\ {}^V\xi_{\mathcal{P},k} = p^{i+1}\ {}^V\xi'_{\mathcal{P},k}$.
With the lemma above this shows that the following map is well-defined:

$$\begin{array}{rc} p^iVW_n\Omega^l \longrightarrow & W_{n-(i+1)}\Omega^l \\ \omega \longmapsto & \sum e_{n-(i+1)}(\bar{\xi}_{k,\mathcal{P}}, k, \mathcal{P}). \end{array}$$

This proves the lemma in the case of a polynomial algebra $A$. Assume now that $A \longrightarrow B$ is a étale morphism.
The image of the canonical injection

$$W_{n+1}(B) \otimes_{W_{n+1}(A)} p^iVW_n\Omega_{A/R} \to W_{n+1}(B) \otimes_{W_{n+1}(A)} W_n\Omega_{A/R} \simeq W_{n+1}\Omega_{B/R}$$

coincides with $p^iVW_n\Omega_{B/R}$. This follows from the following commutative diagram

$$\begin{array}{ccc} W_{n+1}(B) \otimes_{W_{n+1}(A),F} W_n\Omega_{A/R} & \xrightarrow{\sim} & W_n\Omega_{B/R} \\ {\scriptstyle id\otimes p^iV}\Big\downarrow & & {\scriptstyle p^iV}\Big\downarrow \\ W_{n+1}(B) \otimes_{W_{n+1}(A)} W_{n+1}\Omega_{A/R} & \xrightarrow{\sim} & W_{n+1}\Omega_{B/R}. \end{array}$$

The top horizontal arrow is given by $\xi \otimes \omega \mapsto^F \xi\omega$ and the lower horizontal arrow is multiplication.
Now we find the desired map by tensoring $p^iVW_n\Omega_{A/R} \longrightarrow W_{n-(i+1)}\Omega_{A/R}$:

$$\begin{array}{ccc} W_{n+1}(B) \otimes_{W_{n+1}(A)} p^iVW_n\Omega_{A/R} & \longrightarrow & W_{n-i}(B) \otimes_{W_{n-i}(A),F} W_{n-(i+1)}\Omega_{A/R} \\ \wr\downarrow & & \wr\downarrow \\ p^iVW_n\Omega_{B/R} & \longrightarrow & W_{n-(i+1)}\Omega_{B/R}. \end{array}$$

The composition of the last map with $p^iV : W_{n-(i+1)}\Omega_{B/R} \longrightarrow W_{n-i}\Omega_{B/R}$ is just the restriction. This proves the lemma. *Q.E.D.*
The proposition follows immediately because we obtain an inverse to the map (40):

$$\begin{array}{ccccccc} p^{m-1}VW_{n-1}\Omega^0 & \xrightarrow{d} & p^{m-2}VW_{n-1}\Omega^1 \dots & VW_{n-1}\Omega^{m-1} & \xrightarrow{d} & W_n\Omega^m \dots \\ \downarrow & & \downarrow & \downarrow & & \downarrow{\scriptstyle Res} \\ W_{n-m-1}\Omega^0_{[F]} & \xrightarrow{d} & W_{n-m-1}\Omega^1_{[F]} \dots & W_{n-m-1}\Omega^{m-1}_{[F]} & \xrightarrow{dV} & W_{n-m-1}\Omega^m \dots \end{array}$$

The first $m$ vertical maps defined by the lemma are equivariant with respect to

$$W_n(R) \overset{Res}{\to} W_{n-m}(R) \overset{F}{\to} W_{n-m-1}(R)$$

The remaining maps are equivariant with respect to $W_n(R) \to W_{n-m}(R)$. The commutativity of the diagram follows, since it is a homomorphic image of a corresponding diagram for $W\Omega$ without level. This proves the proposition. *Q.E.D.*

Let $X/R$ be a smooth scheme. Let us denote by $\mathcal{J}_{X/W_n(R)} \subset \mathcal{O}_{X/W_n(R)}$ the sheaf of pd-ideals. We denote by $\mathcal{J}^{[m]}_{X/W_n(R)}$ its $m$-th divided power. Let

$$u_n : Crys(X/W_n(R)) \longrightarrow X_{zar}$$

be the canonical morphism of sites. We are going to define a morphism in $D(X_{zar})$ the derived category of abelian sheaves on $X_{zar}$ for $m < p$:

$$Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)} \longrightarrow \mathcal{I}^m W_n\Omega^{\cdot}_{X/R} \tag{45}$$

In order to define (45) we begin with the case, where $X$ admits an embedding in a smooth scheme $Y/R$, such that $Y$ has a Witt-lift: $\tilde{Y}/W_n(R)$ and $\mathcal{O}_{\tilde{Y}} \longrightarrow W_n(\mathcal{O}_Y)$.

The left hand side of (45) may be computed with the filtered Poincaré lemma [BO] Theorem 7.2: Let $D$ be the divided power hull of $X$ in $\tilde{Y}$. Let $I_D \subset \mathcal{O}_D$ be the pd-ideal. The pd-de Rham-complex $\breve{\Omega}_{D/W_n(R)}$ has the following subcomplex $Fil^m\breve{\Omega}_{D/W_n(R)}$:

$$I_D^{[m]}\breve{\Omega}^{\circ}_{D/W_n(R)} \xrightarrow{d} I_D^{[m-1]}\breve{\Omega}^1_{D/W_n(R)} \xrightarrow{d} \ldots I_D\breve{\Omega}^{m-1}_{D/W_n(R)} \xrightarrow{d} \breve{\Omega}_{D/W_n(R)} \ldots \tag{46}$$

Then the left hand side of (45) is isomorphic to the hypercohomology of (46). The Witt-lift defines a morphism

$$\mathcal{O}_{\tilde{Y}} \longrightarrow W_n(\mathcal{O}_Y) \longrightarrow W_n(\mathcal{O}_X).$$

It maps the ideal sheaf of $X \subset \tilde{Y}$ to the ideal sheaf $I_X = VW_{n-1}(\mathcal{O}_X) \subset W_n(\mathcal{O}_X)$. Since $I_X$ is endowed with divided powers, we obtain

$$\mathcal{O}_D \longrightarrow W_n(\mathcal{O}_X), \tag{47}$$

mapping $I_D$ to $I_X$. The homomorphism (47) induces a map of the pd-de Rham complexes

$$\breve{\Omega}_{D/W_n(R)} \longrightarrow \breve{\Omega}_{W_n(X)/W_n(R)} \longrightarrow W_n\Omega_{X/R}.$$

Taking into account that $I_X^{[h]} = p^{h-1}I_X$ for $h < p$, we obtain the desired morphism from (46) to the complex $\mathcal{I}^m W_n\Omega$ if $m < p$:

$$p^{m-1}I_X W_n\Omega^0_{X/R} \longrightarrow \ldots \longrightarrow I_X W_n\Omega^{m-1}_{X/R} \xrightarrow{d} W_n\Omega_{X/R} \to \ldots.$$

We note that $I_X W_n\Omega^l_{X/R} = VW_{n-1}\Omega^l_{X/R}$ follows from the formula

$$^V(\eta d\omega_1 \ldots d\omega_r) = {}^V\eta d\,{}^V\omega_1 \ldots d\,{}^V\omega_r.$$

Hence we obtain a morphism

$$Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)} \tilde{\rightarrow} Fil^m \breve{\Omega}_{D/W_n(R)} \longrightarrow \mathcal{I}^m W_n\Omega_{X/R}. \tag{48}$$

The independence of the last arrow from the embedding of $X$ into a Witt lift $(Y, \tilde{Y})$ is proved in a standard manner: Let $X \hookrightarrow Y'$ be an embedding into a second Witt lift $(Y', \tilde{Y}')$. Then we obtain a Witt lift of the product $Y \times_{\operatorname{Spec} R} Y'$ : Indeed, $\tilde{Y} \times_{\operatorname{Spec} W_n(R)} \tilde{Y}'$ is a lifting of $Y \times Y'$ and the two given Witt lifts induce a morphism:

$$\mathcal{O}_{\tilde{Y}} \otimes_{W_n(R)} \mathcal{O}_{\tilde{Y}'} \longrightarrow W_n(\mathcal{O}_Y) \otimes_{W_n(R)} W_n(\mathcal{O}_{Y'}) \longrightarrow W_n(\mathcal{O}_Y \otimes \mathcal{O}_{Y'}).$$

If $P$ denotes the pd-hull of $X$ in $\tilde{Y} \times_{\operatorname{Spec} W_n(R)} \tilde{Y}'$. We obtain a commutative diagram

$$\begin{tikzcd}
Fil^m \breve{\Omega}_{D/W_n(R)} \arrow[d] \arrow[dr] & \\
 & \mathcal{I}^m W_n \Omega_{X/R} \\
Fil^m \breve{\Omega}_{P/W_n(R)} \arrow[ur] &
\end{tikzcd}$$

Since the vertical arrow induces by [BO] the identity on $Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)}$ the independence of (45) of the chosen Witt lift follows.
If $X$ admits no embedding in a smooth scheme $Y$ which has a Witt lift, one can proceed by simplicial methods [I] or [LZ] §3.2, but we omit the details here.

Theorem 4.6 *For each $m < p$ and $n$ the map in $D^+(X_{zar}, W_n(R))$*

$$Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)} \longrightarrow \mathcal{I}^m W_n\Omega_{X/R} \tag{49}$$

*is a quasi-isomorphism.*

Proof: Clearly the question is local for the Zariski-topology on $X$. We may therefore assume that $X = \operatorname{Spec} B$, where the $R$-algebra $B$ is étale over $R[T_1, \ldots, T_d]$. From the discussion above we know that any Witt-lift of $B$ leads to the same morphism (49). We choose a Frobenius lift $\{B_n\}_{n\in\mathbb{N}}$ of the algebra $B$ as in the corollary 4.3. We begin with the reduction to the case $B = R[T_1, \ldots, T_d]$. Let $J$ be the kernel of $B_n \longrightarrow B$. Then $\mathcal{J}^{[i]} = p^{i-1} I_R B_n$, where $I_R = VW_{n-1}(R) \subset W_n(R)$. Hence we have to show that the following morphism of complexes induces a quasi-isomorphism:

$$\begin{array}{ccccccc}
p^{m-1}I_R\Omega^0_{B_n/W_n(R)} & \xrightarrow{d} & \ldots I_R\Omega^{m-1}_{B_n/W_n(R)} & \xrightarrow{d} & \Omega^m_{B_n/W_n(R)} & \xrightarrow{d} \\
\downarrow & & \downarrow & & \downarrow & \\
p^{m-1}VW_{n-1}\Omega^0_{B/R} & \xrightarrow{d} & \ldots VW_{n-1}\Omega^{m-1}_{B/R} & \xrightarrow{d} & W_n\Omega^m_{B/R} & \xrightarrow{d}
\end{array} \tag{50}$$

We choose a number $s$, such that $p^sW_n(R) = 0$. We consider the groups in the first complex as $B_{n+s}$ modules via $\psi^s : B_{n+s} \to B_n$. As shown in the proof of Corollary 4.3 we obtain a complex of $B_{n+s}$-modules. The same is true if we consider the groups in the second complex as $B_{n+s}$-modules by $\psi^s : B_{n+s} \to B_n \to W_n(B)$.

We obtain the diagram above from the corresponding diagram for $B = A$ by tensoring with $B_{n+s}\otimes_{A_{n+s}}$. Since $B_{n+s}$ is étale over $A_{n+s}$, we have reduced our statement to the case where $B = R[T_1, \ldots, T_d]$ and where the Witt-lift is a standard one.

In the case of a polynomial algebra we have a decomposition of the de Rham Witt complex according to weights [LZ] 2.17.

Because the operator $V$ is homogeneous, we have a similar decomposition for the complex $\mathcal{I}^mW_n\Omega_{A/R}$. In fact, by [LZ] Prop. 2.5 an element of $p^{m-l-1}VW_{n-1}\Omega^l$, for $l \leq m-1$ may be uniquely written as a sum of elements of the following types

$e_n(p^{m-l-1}\ {}^V\xi, k, I_0, \ldots, I_l)$ for $k$ integral
$e_n(p^{m-l-1}\ {}^V\xi, k, I_0, \ldots, I_l)$ for $I_0 \neq \emptyset$, $k$ not integral
$e_n(p^{m-l}\ {}^V\xi, k, I_0, \ldots, I_l)$ for $I_0 = \emptyset$, $k$ not integral.

Here $\xi \in W_{n-1}(R)$ for $k$ integral and $\xi \in V^{u(k)-1}W_{n-u(k)}(R)$ for $k$ nonintegral. Clearly the elements of the first type span a subcomplex of $\mathcal{I}^mW_n\Omega_{A/R}$ which is isomorphic to the complex in the first row of (50). Indeed, the $p$-basic differentials of this complex are mapped to basic Witt-differentials of the first type above. The last two types of Witt-differentials above span an acyclic subcomplex because of the formula

$$de_n(p^{m-l-1}\ {}^V\xi, k, I_0, \ldots, I_l) = e_n(p^{m-l-1}\ {}^V\xi, k, \phi, I_0, \ldots, I_l),$$

for $I_0 \neq \emptyset$ and $k$ not integral. The exactness of the non integral part at $W_n\Omega^m_{B/R}$ follows in the same way. *Q.E.D.*

Let $X_n/W_n(R)$ be a compatible system of smooth liftings of $X/R$ for $n \in \mathbb{N}$. The Theorem 4.6 provides an isomorphism in the derived category between $\mathcal{I}^mW_n\Omega_{X/R}$ and

$$p^{m-1}I_R\Omega^0_{X_n/W_n(R)} \to p^{m-2}I_R\Omega^1_{X_n/W_n(R)} \to \ldots I_R\Omega^{m-1}_{X_n/W_n(R)} \to \Omega^{m-1}_{X_n/W_n(R)} \cdots. \tag{51}$$

We know by Proposition 4.4 that $\{\mathcal{I}^mW_n\Omega_{X/R}\}$ is isomorphic to the procomplex $\{\mathcal{N}^mW_n\Omega_{X/R}\}$. The same argument shows that the procomplex (51) is quasi-isomorphic to $\{\mathcal{F}^m\Omega^{\cdot}_{X_n/W_n(R)}\}_{n\in\mathcal{N}}$. Passing to the projective limit we obtain:

Corollary 4.7 *Let $R$ be a reduced ring. Let $X/R$ be a smooth and proper scheme. Assume that $X_n/W_n(R)$ is a compatible system of smooth liftings of $X$. Then there is for each number $m < p$ a natural isomorphism in the derived category $D^+(X_{zar,W(R)})$:*

$$\mathcal{N}^m W\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m \Omega^{\cdot}_{\mathcal{X}/W(R)},$$

*where $\mathcal{X} = \varinjlim X_n$ in the sense of EGA I Prop. 10.6.3.*

This is a weak form of the Conjecture 4.1 which asserts this for every level separately.

## 5 Display Structure on crystalline cohomology

Let $R$ be a ring such that $p$ is nilpotent in $R$. Let $(A, \sigma, \alpha)$ be a frame for $R$ [Z1]. This means that $A$ is a torsion free a $p$-adic ring with an endomorphism $\sigma : A \to A$, which induces the Frobenius endomorphism $A/pA \to A/pA$. The map $\alpha : A \to R$ is a surjective ring homomorphism, such that the ideal $\mathfrak{a} = \operatorname{Ker} \alpha$ has divided powers.

Definition 5.1 *An $A$-window consists of*

1) *A finitely generated projective $A$-module $P_0$.*

2) *A descending filtration of $P_0$ by $A$-submodules*

$$\dots P_{i+1} \subset P_i \subset \dots \subset P_2 \subset P_1 \subset P_0. \tag{52}$$

3) *$\sigma$-linear homomorphisms*

$$F_i : P_i \to P_0.$$

*The following conditions are required.*

(i) *$\mathfrak{a}P_i \subset P_{i+1}$ and the factor module $P_{i+1}/\mathfrak{a}P_i$ is a finitely generated projective $R$-module $E_{i+1}$ for $i \geq 0$. We set $E_0 = P_0/\mathfrak{a}P_0$.*

(ii) *The inclusions $P_{i+1} \to P_i$ induce injective $R$-module morphisms*

$$\dots \to E_{i+1} \to E_i \to \dots \to E_0,$$

*such that $E_{i+1}$ is a direct summand of $E_i$.*

(iii) *$\mathfrak{a}P_i = P_{i+1}$ if $i$ is big enough.*

(iv) *$F_i(x) = pF_{i+1}(x)$ for $x \in P_{i+1}$.*

(v) *The union of the images $F_i(P_i)$ for $i \in \mathbb{Z}_{\geq 0}$ generate $P_0$ as an $A$-module.*

A window is called standard if it arises in the following way. Let $L_0, \dots, L_d$ be finitely generated projective $A$-modules. Let

$$\Phi_i : L_i \to \bigoplus_{j=0}^{d} L_j$$

be $\sigma$-linear homomorphisms, such that the determinant of $\Phi_0 \oplus \cdots \oplus \Phi_d$ is a unit. Then we set for $i \geq 0$

$$P_i = \mathfrak{a}^i L_0 \oplus \mathfrak{a}^{i-1} L_1 \oplus \ldots \oplus \mathfrak{a} L_{i-1} \oplus L_i \oplus \cdots \oplus L_d.$$

We define $F_i$ on this direct sum as follows: The restriction of $F_i$ to $\mathfrak{a}^{i-k} L_k$ for $k < i$ resp. $L_k$ for $k \geq i$ to is defined by

$$\begin{array}{lcll} F_i(ax) & = & \frac{\sigma(a)}{p^{i-k}} \Phi_k(x) & \text{for } 0 \leq k < i, \ x \in L_k, \ a \in \mathfrak{a}^{i-k} \\ F_i(x) & = & p^{k-i} \Phi_k(x) & \text{for } i \leq k \ x \in L_k. \end{array}$$

It is clear that $(P_i, F_i)$ form a window.
Each window is isomorphic to a standard window. Indeed, let $E_0 = \oplus \bar{L}_j$ be a splitting of the filtration (52) in the definition:

$$E_i = \oplus_{j \geq i} \bar{L}_j.$$

Let $L_i$ be a finitely generated projective $A$-module which lifts $\bar{L}_i$. We find homomorphisms $L_i \to P_i$ which make the following diagrams commutative:

$$\begin{array}{ccc} \bar{L}_i & \longrightarrow & E_i \\ \uparrow & & \uparrow \\ L_i & \longrightarrow & P_i. \end{array}$$

It follows from the lemma of Nakayama that $\oplus L_i \to P_0$ is an isomorphism, since it is modulo $\mathfrak{a}$. By induction we obtain

$$P_i = \mathfrak{a}^i L_0 \oplus \cdots \oplus \mathfrak{a} L_{i-1} \oplus L_i \oplus \cdots \oplus L_d. \tag{53}$$

We set $\Phi_i = F_i | L_i$. The condition (v) implies that $\oplus \Phi_i : \oplus L_j \to \oplus L_j$ is a $\sigma$-linear epimorphism and therefore an isomorphism.
Remark: A window $(P_i)$ is of degree $d$, if $P_{i+1} = \mathfrak{a} P_i$ for $i \geq d$. To give a window of degree $d$ it is enough to give only the modules $P_0, \dots, P_d$. The axioms may be formulated in the same way for this finite chain of modules. The axiom (v) then requires that the union of $F_0(P_0), F_1(P_1) \dots, F_d(P_d)$ generates $P_0$ as an $A$-module.
We will now see that an $A$-window induces a display over $R$. There is a natural ring homomorphism $\delta : A \to W(A)$, such that for the Witt-polynomials $\mathbf{w}_n$ there is the identity

$$\mathbf{w}_n(\delta(a)) = \sigma^n(a), \quad a \in A.$$

Consider the composite ring homomorphism.

$$\varkappa : A \to W(A) \to W(R).$$

We have by [Z1] Prop. 1.5:

$$\begin{aligned} \varkappa(\sigma(a)) &= {}^{F}\varkappa(a) \quad \text{for } a \in A \\ \varkappa(\tfrac{\sigma(a)}{p}) &= {}^{V^{-1}}\varkappa(a) \quad \text{for } a \in \mathfrak{a}. \end{aligned}$$

The last equation makes sense because $\varkappa(a) \in VW(R)$ for $a \in \mathfrak{a}$.
It is clear that a datum $(L_i, \Phi_i)$ for a standard window over $A$ induces the datum $(W(R) \otimes_{W(A)} L_i, F \otimes \Phi_i)$ for a standard display over $R$. We will show that the resulting display does not depend on the decomposition $P_0 = \oplus L_i$ we have used.
We give an invariant construction of a display $(Q_i, \iota_i, \alpha_i, F_i)$ from a window $(P_i, F_i)$. The display comes with morphisms $\tau_i : P_i \to Q_i$ such that the following diagrams commute

$$\begin{tikzcd} P_i \arrow[r, "\tau_i"] & Q_i \\ P_{i+1} \arrow[u] \arrow[r, "\tau_{i+1}"] & Q_{i+1} \arrow[u, "\iota_i"'] \end{tikzcd} \qquad \begin{tikzcd} P_{i+1} \arrow[r, "\tau_{i+1}"] & Q_{i+1} \\ \mathfrak{a} \otimes P_i \arrow[u] \arrow[r] & I_R \otimes_{W(R)} Q_i \arrow[u, "\alpha_i"'] \end{tikzcd} \qquad \begin{tikzcd} P_0 \arrow[r, "\tau_0"] & Q_0 \\ P_{i+1} \arrow[u, "F_i"] \arrow[r, "\tau_i"] & Q_i. \arrow[u, "F_i"'] \end{tikzcd} \tag{54}$$

We construct $Q_i$ and $\tau_i$ inductively, such that the diagrams (54) commute. We set $Q_0 = W(R) \otimes_{\varkappa, A} P_0$ and we let $\tau_0 : P_0 \to Q_0$ be the canonical map.
Assume that $\tau_k : P_k \to Q_k$ was constructed for $k \leq i$. Then we consider the following commutative diagrams:

$$\begin{tikzcd} P_i \arrow[r, "\tau_i"] & Q_i \arrow[r, "F_i"] & Q_0 \\ & & Q_0 \arrow[u, "p"] \\ P_{i+1} \arrow[uu, "\iota"] \arrow[rr, "F_{i+1}"] & & P_0 \arrow[u, "\tau_0"] \end{tikzcd} \qquad \begin{tikzcd} Q_i \arrow[r, "F_i"] & Q_0 \\ I_R \otimes Q_i \arrow[u] \arrow[r, "\tilde{F}_i"] & Q_0 \arrow[u, "p"'] \end{tikzcd}$$

We obtain a morphism to the fibre product

$$(W(R) \otimes_A P_{i+1}) \oplus (I_R \otimes Q_i) \to Q_i \times_{F_i, Q_0, p} Q_0. \tag{55}$$

We define $Q_{i+1}$ as the image of (55). This gives a map $P_{i+1} \overset{\tau_{i+1}}{\to} Q_{i+1}$. We define $\iota : Q_{i+1} \to Q_i$ and $F_{i+1} : Q_{i+1} \to Q_0$ and $\alpha_i : I_R \otimes Q_i \to Q_{i+1}$ as the canonical maps determined by these data. A routine verification shows that this construction gives the same result as the construction via standard windows.

Moreover the following universal property holds. Let $(Q'_i, \iota'_i, \alpha'_i, F'_i)$ be a display over $R$ and let $\tau'_i : P_i \to Q'_i$ be maps such that the diagrams (54) for $\tau'_i$ commute. Then the maps $\tau'_i$ are the composition of $\tau_i$ and a morphism of displays $(Q_i, \iota_i, \alpha_i F_i) \to (Q'_i, \iota'_i, \alpha'_i, F'_i)$.

Let $A \xrightarrow{\alpha} R, \sigma, \mathfrak{a}$ as before. Let $X \to \operatorname{Spec} R$ be a scheme which is projective and smooth. Let $\mathcal{Y} \xrightarrow{f} \operatorname{Spf} A$ be a smooth $pA$-adic formal scheme, which lifts $X$. We set $A_n = A/p^n$ and $Y_n = \mathcal{Y} \times_{\operatorname{Spf} A} \operatorname{Spec} A_n$. For big $n$ the map $\alpha$ factors through $A_n \xrightarrow{\alpha_n} R$. The kernel $\mathfrak{a}_n$ inherits a pd-structure. We consider the crystalline topos $(X/A)_{crys}$. Let $\mathcal{J}_{X/A_n} \subset \mathcal{O}_{X/A_n}$ be the pd-ideal sheaf. We are interested in the cohomology groups:

$$H^i(X, \mathcal{J}^{[m]}_{X/A}) = \varprojlim_n H^i_{crys}(X/A_n, \mathcal{J}^{[m]}_{X/A_n}). \tag{56}$$

Remark: It would be more accurate to consider the cohomology groups of $R \varprojlim_n R\Gamma(X/A_n, \mathcal{J}^{[m]}_{X/A_n})$. But under the Assumptions 5.2 and 5.3 we are going to make these groups will coincide.

By [BO] 7.2 the groups $H^i_{crys}(X/A_n, \mathcal{J}^{[m]}_{X/A_n})$ are the hypercohomology groups of the following complex $Fil^{[m]}\Omega^{\cdot}_{Y_n/A_n}$:

$$\mathfrak{a}_n^{[m]} \otimes_{A_n} \Omega^0_{Y_n/A_n} \to \mathfrak{a}_n^{[m-1]} \otimes_{A_n} \Omega^1_{Y_n/A_n} \cdots \to \mathfrak{a}_n \otimes_{A_n} \Omega^{m-1}_{Y_n/A_n} \to \Omega^m_{Y_n/A_n} \cdots \tag{57}$$

We will make the following assumptions:

Assumption 5.2 *The cohomology groups $H^q(Y_n, \Omega^p_{Y_n/A_n})$ are for each $n$ locally free $A_n$-modules of finite type.*

Assumption 5.3 *The de Rham spectral sequence degenerates at $E_1$*

$$E_1^{pq} = H^q(Y_n, \Omega^p_{Y_n/A_n}) \Rightarrow \mathbb{H}^{p+q}(Y_n, \Omega^{\cdot}_{Y_n/A_n}).$$

Since $Y_n$ is quasicompact and separated by assumption the cohomology sheafs $\mathbb{R}^m f_{n*}\Omega^{\cdot}_{Y_n/A_n}$ are quasicoherent. From the assumption we see that these sheaves are locally free of finite type. Hence the complex $\mathbb{R} f_{n*}\Omega^{\cdot}_{Y_n/A_n}$ is quasi-isomorphic to the direct sum of its cohomology groups. This implies that the cohomology groups $\mathbb{R}^m f_{n*}\Omega^{\cdot}_{Y_n/A_n}$ commute with arbitrary base change. The same applies to the cohomology groups $R^q f_{n*}\Omega^p_{Y_n/A_n}$. By Proposition 3.2 and the projection formula (Proposition 3.1) we obtain a degenerating spectral sequence

$$\begin{array}{rl} E_1^{ij} = H^j(Y_n, \Omega^i_{Y_n/A_n}) \otimes_{A_n} \mathfrak{a}^{[m-i]} \Rightarrow & \mathbb{H}^{i+j}(Y_n, Fil^{[m]}\Omega^{\cdot}_{Y_n/A_n}) \\ & \qquad \| \\ & H^{i+j}_{crys}(X/A_n, \mathcal{J}^{[m]}_{X/A_n}) \end{array}$$

If we pass to the projective limit we obtain a degenerating spectral sequence

$$E_1^{ij} = \mathfrak{a}^{[m-i]} \otimes H^j(\mathcal{Y}, \Omega^i_{\mathcal{Y}/A}) \Rightarrow H^{i+j}_{crys}(X/A, \mathcal{J}^{[m]}_{X/A}). \tag{58}$$

The groups involved have no $p$-torsion.
We set $\bar{X} = X \times_{\operatorname{Spec} R} \operatorname{Spec} \bar{R}$, where $\bar{R} = R/pR$. By [BO] 5.17 there is a canonical isomorphism

$$H^i_{crys}(X/A, \mathcal{O}_{X/A}) \simeq H^i_{crys}(\bar{X}/A, \mathcal{O}_{\bar{X}/A}). \tag{59}$$

The absolute Frobenius on $\bar{X}$ and $\sigma$ on $A$ induce an endomorphism on the right hand side of (59) and therefore an endomorphism

$$F : H^i_{crys}(X/A, \mathcal{O}_{X/A}) \to H^i_{crys}(X/A, \mathcal{O}_{X/A}).$$

Lemma 5.4 *Let $p^{[m]}$ be the maximal power of $p$ which divides $p^m/m!$ Then the image of the following composition*

$$H^i_{crys}(X/A, \mathcal{J}^{[m]}_{X/A}) \to H^i_{crys}(X/A), \mathcal{O}_{X/A}) \xrightarrow{F} H^i_{crys}(X/A, \mathcal{O}_{X/A})$$

*is contained in $p^{[m]} H^i_{crys}(X/A, \mathcal{O}_{X/A})$.*

Proof: The argument is well known [K], but we repeat it in the generality we need. We may replace $A$ by $A_n$. We embed $X$ into a smooth and projective $A_n$-scheme $Z$, such that there is an endomorphism $\sigma : Z \to Z$ which lifts the absolute Frobenius modulo $p$ and which is compatible with $\sigma$ on $A_n$. We may take for $Z$ the projective space. Consider the $pd$-hull $D$ of $X$ in $Z$. It is also the $pd$-hull of $\bar{X}$ in $Z$. Therefore $\sigma$ extends to $D/A_n$ and to the $pd$-differentials $\breve{\Omega}_{D/A_n}$. We obtain by [BO] an isomorphism

$$\mathbb{H}^i(X, \breve{\Omega}^{\cdot}_{D/A_n}) \xrightarrow{\sim} H^i_{crys}(X/A, \mathcal{O}_{X/A_n}),$$

which is equivariant with respect to the action of $\sigma$ on the left hand side and $F$ on the right hand side.
Consider the morphisms

$$\bar{X} \to D \to Z.$$

Let $I(\bar{X})$ be the ideal of $\bar{X}$ in $Z$ and $\bar{\mathcal{J}}_D$ be the ideal of $\bar{X}$ in $D$. Consider the diagram

$$\begin{tikzcd} (\mathcal{O}_Z, I(\bar{X})) \arrow[r] \arrow[d, "\sigma"] \arrow[dr, "\kappa"] & (\mathcal{O}_D, \bar{\mathcal{J}}_D) \arrow[d, "\sigma_D"] \\ (\mathcal{O}_Z, I(\bar{X})) \arrow[r] & (\mathcal{O}_D, \bar{\mathcal{J}}_D) \end{tikzcd}$$

The composite $\kappa$ maps $I(\bar{X})$ to $p \cdot \mathcal{O}_D$. This follows because

$$\sigma(z) \equiv z^p \mod p \text{ for } z \in \mathcal{O}_Z. \tag{60}$$

If $z \in I(\bar{X})$ the image of $z^p$ in $\bar{\mathcal{J}}_D$ becomes divisible by $p$, because we have divided powers. Therefore the induced map $\sigma_D$ on the divided power envelope maps $\bar{\mathcal{J}}_D$ to $p\mathcal{O}_D$. Therefore

$$\sigma(\bar{\mathcal{J}}_D^{[m]}) \subset p^{[m]}\mathcal{O}_D.$$

For $z \in \mathcal{O}_Z$ we find from (60) that in $\breve{\Omega}^1_{D/A_n}$:

$$d\sigma(z) \equiv 0 \mod p.$$

The composite map of the lemma is induced by a map of complexes:

$$\begin{array}{ccccccc} \mathcal{J}_D^{[m]}\breve{\Omega}^\circ_{D/A_n} & \longrightarrow & \dots & \longrightarrow & \mathcal{J}_D^{[m-i]}\breve{\Omega}^i_{D/A_n} & \longrightarrow & \dots \\ \sigma\big\downarrow & & & & \sigma\big\downarrow & & \\ \breve{\Omega}^\circ_{D/A_n} & \longrightarrow & \dots & \longrightarrow & \breve{\Omega}^i_{D/A_n} & \longrightarrow & \dots . \end{array} \tag{61}$$

The image of this map lies in $p^{[m]} \cdot \breve{\Omega}^{\cdot}_{D/A_n} = p^{[m]}A_n \otimes^{\mathbb{L}}_{A_n} \breve{\Omega}_{D/A_n}$. The last equality follows since by [BO] 3.32 the sheaf $\mathcal{O}_D$ is flat over $A_n$. The hypercohomology of the last complex is by the projection formula

$$\begin{aligned} p^{[m]}A_n \otimes^{\mathbb{L}} R\Gamma(X, \breve{\Omega}_{D/A_n}) &= p^{[m]}A_n \otimes^{\mathbb{L}} \mathbb{R}\Gamma_{crys}(X/A_n, \mathcal{O}_{X/A_n}) \\ &= p^{[m]}A_n \otimes^{\mathbb{L}} R\Gamma(Y_n, \Omega^{\cdot}_{Y_n/A_n}) \end{aligned}$$

But the cohomology of the last complex is $p^{[m]}\mathbb{H}^i(Y_n, \Omega^{\cdot}_{Y_n/A_n})$, since we assumed that the cohomology is locally free. This shows that (61) factors on the hypercohomology through $p^{[m]}\mathbb{H}_{crys}(X/A_n, \mathcal{O}_{X/A_n}) = p^{[m]}\mathbb{H}^i(Y_n, \Omega^{\cdot}_{Y_n/A_n})$. *Q.E.D.*

Theorem 5.5 *Let $R$ be a ring, such that $p$ is nilpotent in $R$. Let $X$ be a scheme which is projective and smooth over $R$. Let $A \to R$ be a frame. We assume that $X$ lifts to a projective and smooth $p$-adic formal scheme $\mathcal{Y}/\operatorname{Spf} A$ such that the assumptions 5.2 and 5.3 are fullfilled. Then for each number $n < p$ the canonical maps*

$$H^n_{crys}(X/A, \mathcal{J}^{[m]}_{X/A}) \to H^n_{crys}(X/A, \mathcal{J}^{[m-1]}_{X/A}) \to \dots \to H^n_{crys}(X/A, \mathcal{O}_{X/A})$$

*are injective. The $A$-modules $P_m = H^n_{crys}(X/A, \mathcal{J}^{[m]}_{X/A})$ for $m \leq n$ together with the maps*

$$\frac{1}{p^m}F = F_m : P_m \to P_0$$

*given by Lemma 5.4 form a window of degree $n$.*

Proof: We consider a number $m \leq n$. Then we have $\mathcal{J}^m_{X/A} = \mathcal{J}^{[m]}_{X/A}, \mathfrak{a}^m = \mathfrak{a}^{[m]}$. We write $Fil^{[m]}\Omega^{\cdot}_{\mathcal{Y}/A} = \varprojlim_n Fil^{[m]}\Omega^{\cdot}_{Y_n/A_n}$. Then we find a canonical isomorphism

$$P_m = \mathbb{H}^n(X, Fil^{[m]}\Omega_{\mathcal{Y}/A}) \cong H^n_{crys}(X/A, \mathcal{J}^m_{X/A}) \tag{62}$$

From the degenerating spectral sequence (58) we obtain the injectivity of $P_m \to P_{m-1}$, since we have injectivity on the associated graded groups.
In the following considerations $m, n$ can be arbitrary natural number, without the restriction $m \leq n < p$. Then $Fil^{[m]}_{\mathcal{Y}/A}$ will be the complex $Fil^m_{\mathcal{Y}/A}$

$$\mathfrak{a}^m\Omega^0_{\mathcal{Y}/A} \to \mathfrak{a}^{m-1}\Omega^1_{\mathcal{Y}/A} \to \cdots \to \mathfrak{a}\Omega^{m-1}_{\mathcal{Y}/A} \to \Omega^m_{\mathcal{Y}/A} \to \ldots$$

Consider the following morphism:

$$\mathfrak{a} \otimes \mathbb{H}^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, \mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}). \tag{63}$$

We have for $\mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}$ a degenerating spectral sequence as (58). Therefore the right hand side of (63) is a subgroup of $H^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A})$.
We claim that the induced inclusion is an equality

$$\mathfrak{a}\mathbb{H}^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) = \mathbb{H}^n(X, \mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}). \tag{64}$$

This equality holds for $m = 0$ by the projection formula. Indeed, consider the canonical map:

$$Fil^m\Omega_{\mathcal{Y}/A} \to \mathfrak{a}^m\Omega^0_{\mathcal{Y}/A} \to 0.$$

The kernel is the following complex $C$:

$$0 \to \mathfrak{a}^{m-1}\Omega^1_{\mathcal{Y}/A} \to \cdots \to \mathfrak{a}\Omega^{m-1}_{\mathcal{Y}/A} \to \Omega^m_{\mathcal{Y}/A} \to \ldots.$$

This complex $C$ is of the same nature as $Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}$ but with less ideals involved. By an induction we may assume that

$$\mathfrak{a}\mathbb{H}^n(X, C) = \mathbb{H}^n(X, \mathfrak{a}C).$$

By the projection formula we find

$$\mathfrak{a}H^n(X, \mathfrak{a}^m\Omega^0_{\mathcal{Y}/A}) = \mathfrak{a}^{m+1}H^n(X, \Omega^0_{\mathcal{Y}/A}).$$

The assertion (64) follows from the diagram

$$\begin{array}{ccccc}
\mathbb{H}^n(X, \mathfrak{a}C) & \longrightarrow & \mathbb{H}^n(X, \mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) & \longrightarrow & H^n(X, \mathfrak{a}^{m+1}\Omega^0_{\mathcal{Y}/A}) \\
\| \uparrow & & \cup \uparrow & & \| \uparrow \\
\mathfrak{a}\mathbb{H}^n(X, C) & \longrightarrow & \mathfrak{a}\mathbb{H}^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) & \longrightarrow & \mathfrak{a}H^n(X, \mathfrak{a}^m\Omega^0_{\mathcal{Y}/A})
\end{array} \tag{65}$$

The upper line is a short exact sequence by a spectral sequence argument as above. The lower line is a complex. The first arrow is injective and the second surjective but it is a priori not exact in the middle term. One sees that the upper and lower line in (63) must be isomorphic. This proves (65).
We have already seen that the following maps are injective

$$\mathbb{H}^n(X, \mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, Fil^{m+1}\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}).$$

Therefore we obtain an exact sequence

$$0 \to \mathbb{H}^n(X, \mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, Fil^{m+1}\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, \sigma^{\geq m+1}\Omega^{\cdot}_{X/R}) \to 0.$$

Since by (64) the map $\mathfrak{a} \otimes \mathbb{H}^n(X, Fil^m\Omega_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, \mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A})$ is surjective, we see that

$$P_m = \mathbb{H}^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) \text{ and } E_m = \mathbb{H}^n(X, \sigma^{\geq m}\Omega^{\cdot}_{X/R})$$

fulfill the conditions (i)-(iii) for a window without any restriction on $m$ and $n$. We note that for fixed $n$ we have $P_{m+1} = \mathfrak{a}P_m$ for $m \geq n$. As explained after the definition of a window, we can obtain a decomposition

$$P_m = \mathfrak{a}^m L_0 \oplus \mathfrak{a}^{m-1}L_1 \oplus \cdots \oplus \mathfrak{a}^{m-n}L_n,$$

with the convention that $\mathfrak{a}^k = A$ if $k \leq 0$.
Concretely we can find the liftings $L_i$ as follows. We consider the maps:

$$\mathbb{H}^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, \sigma^{\geq m}\Omega^{\cdot}_{\mathcal{Y}/A}) \to H^{(n-m)}(X, \Omega^m_{\mathcal{Y}/A})$$

Then $L_m$ is obtained by splitting the last surjection. This construction gives isomorphisms:

$$L_m \cong H^{(n-m)}(X, \Omega^m_{\mathcal{Y}/A})$$

We now impose the condition $m \leq n < p$ of the theorem. By lemma 5.4 and (62) the Frobenius endomorphism $F : P_0 \to P_0$ is divisible by $p^m$ when restricted to $P_m$. We set

$$\Phi_m = \frac{1}{p^m} F_{|L_m}.$$

The assertion that $\{P_m\}$ is a window is then equivalent with the condition that

$$\oplus^n_{i=0}\Phi_i : \oplus^n_{i=0}L_i \to \oplus^u_{i=0}L_i$$

is a $\sigma$-linear isomorphism, or in other words that $\det(\oplus^n_{i=0}\Phi_i)$ is a unit in $W(A)$. Clearly it suffices to show that for any homomorphism $R \to k$ to a perfect field $k$ the image of $\det(\oplus\Phi_i)$ by the morphism

$$A \xrightarrow{\varkappa} W(R) \to W(k) \to k$$

is a nonzero. The compositum map $A \to W(k)$ respects the Frobenius and induces a map on crystalline cohomology

$$H^n_{crys}(X/A, \mathcal{O}_{X/A}) \to H^n_{crys}(X_k/W(k), \mathcal{O}_{X_k/W(k)})$$

which respects the Frobenius. It is induced by the base change map for de Rham cohomology.

$$\mathbb{H}^n(X, \Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X_k, \Omega^{\cdot}_{Y \otimes_A W(k)/W(k)}).$$

The special decomposition we have chosen

$$\mathbb{H}^n(X, \Omega^{\cdot}_{\mathcal{Y}/A}) = \oplus L_i,$$

induces a similar decomposition

$$\mathbb{H}^n(X_k, \Omega^{\cdot}_{\mathcal{Y}_{W(k)}/W(k)}) = \mathbb{H}^n(X, \Omega^{\cdot}_{\mathcal{Y}/A}) \otimes_A W(k) = \oplus L_i \otimes_A W(k).$$

Therefore we have reduced our assertion to the case $R = k$ a perfect field and $A = W(k)$. This case was proved by Mazur (Compare [Fo] p.91 and Kato [K] Prop.2.5). We give an argument in the case $n < p - 2$ which is based on the comparison Corollary 4.7 but doesn't use gauges.
For any complex $\mathcal{A}^{\cdot}$ of abelian sheaves on $X$ consider the exact sequence induced by the naive filtration.

$$0 \to \sigma_{>i}\mathcal{A}^{\cdot} \to \mathcal{A}^{\cdot} \to \sigma_{\leq i}\mathcal{A}^{\cdot} \to 0,$$

where $i$ is an arbitrary integer. If $n + 1 \leq i$ we obtain an isomorphism

$$\mathbb{H}^n(X, \mathcal{A}^{\cdot}) \cong \mathbb{H}^n(X, \sigma_{\leq i}\mathcal{A}^{\cdot}).$$

We apply this to the Nygaard complex $\mathcal{N}^m W\Omega^{\cdot}_{X/k}$ and to the de Rham-Witt complex $W\Omega^{\cdot}_{X/k}$. For $i \leq m-1$ the operator $\hat{F}_m$ (5) induces clearly a bijection of the truncated complexes

$$\hat{F}_m : \sigma_{\leq i}\mathcal{N}^m W\Omega^{\cdot}_{X/k} \to \sigma_{\leq i} W\Omega^{\cdot}_{X/k}$$

Therefore if $n + 1 \leq i \leq m - 1$ we obtain a bijection

$$F_m : \mathbb{H}^n(X, \mathcal{N}^m W\Omega^{\cdot}_{X/k}) \to \mathbb{H}^n(X, W\Omega^{\cdot}_{X/k})$$

We set $m = n + 2$. Since $m < p$ by assumption (and because $k$ is reduced) there are canonical isomorphisms in the derived category:

$$\mathcal{N}^m W\Omega^{\cdot}_{X/k} \cong \mathcal{F}^m\Omega_{\mathcal{Y}/W(k)} \cong Fil^m\Omega_{\mathcal{Y}/W(k)}$$

But since $m > n$ the map $F_m$ is identified with the linearization of $\oplus\Phi_i$. This says that the last map is a Frobenius linear isomorphism. *Q.E.D.*
Remark: The proof shows that $H^n_{DR}(\mathcal{Y})$ with its Hodge filtration is strongly divisible (compare [Fo] 1.2 Prop.) for $n < p - 2$. If we knew that $\mathcal{N}^m W\Omega^{\cdot}_{X/k}$ and $\mathcal{F}^m\Omega_{\mathcal{Y}/W(k)}$ are quasi-isomorphic, the last argument would imply that $H^n_{DR}(\mathcal{Y})$ is strongly divisible without restriction on $n$. We note also that the last argument works directly over any reduced ring $k$.

Corollary 5.6 *Let $X$ be a smooth and projective scheme over a ring $R$ such that $p$ is nilpotent in $R$.*
*Let us assume that there is a frame $A \to R$ and a smooth and projective $p$-adic lifting $\mathcal{Y}/\operatorname{Spf} A$ of $X$, which satisfies the conditions of the theorem.*
*Then we obtain for $n < p$ by base change a display structure of degree $n$ on $H^n_{crys}(X/W(R), \mathcal{O}_{X/W(R)})$. This display structure is independent of the frame $A$ and the formal lifting $\mathcal{Y}$ we have chosen if $p \cdot R = 0$.*

Proof: For a given frame $A$ the independence of the lifting $\mathcal{Y}$ is clear, because the window structure is purely defined in terms of the crystalline cohomology of $X/A$.

If we have a morphism of frames $B \to A$ and a formal lifting $\mathcal{Z}$ of $X$ to $B$, then we set $\mathcal{Y} = \mathcal{Z}_A$. Then the window associated to $\mathcal{Y}$ is obtained from the window associated to $\mathcal{Z}$ by base change (one should think in terms of decompositions (53)). Therefore the induced displays are the same.

If $p \cdot R = 0$ and $A'$ and $A''$ are 2 frames, we obtain a new frame $A' \times_R A'' \to R$. Then $\sigma' \times \sigma''$ is an endomorphism of $A' \times_R A''$ because $\sigma'$ and $\sigma''$ induce the same endomorphism on $R$. If $\mathcal{Y}'/\operatorname{Spf} A'$ and $\mathcal{Y}''/\operatorname{Spf} A''$ are formal liftings, we obtain a formal lifting $\mathcal{Y}' \times_\kappa \mathcal{Y}''$ of $X$ over $A' \times_R A''$. Therefore we obtain the same display structure by base change.

Theorem 5.7 *Let $R$ be a reduced ring of characteristic $p$. Let $X/R$ be a smooth projective scheme. Assume that there is a compatible system of smooth and projective liftings $Y_n/W_n(R)$. We assume that the assumptions 5.2 and 5.3 are satisfied with $A_n = W_n(R)$*

*Then there is a display structure on $H^n_{crys}(X/W(R), \mathcal{O}_{X/W(R)})$ for $n < p$, where*

$$P_m = \mathbb{H}^n(X, \mathcal{N}^m W\Omega_{X/R}) = H^n_{crys}(X/W(R), \mathcal{J}^{[m]}_{X/W(R)}).$$

Proof: The second equality is the filtered comparison theorem. If we had a $p$-adic lifting $\mathcal{Y}/\operatorname{Spf} W(R)$, the theorem would follow from the last one because $W(R) \to R$ is a frame. The slightly more general statement follows by the same reasoning as the last theorem. *Q.E.D.*

We make the following conjecture:

Conjecture 5.8 *Let $R$ be a ring such that $p$ is nilpotent in $R$. Let $X/R$ be a smooth projective scheme. Let us assume that the crystalline cohomology groups $H^i_{crys}(X/W_n(R))$ are locally free $W_n(R)$-modules for $i \geq 0$ and $n > 1$, and that the de Rham spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega^p_{X/R}) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^{\cdot}_{X/R})$$

*degenerates.*

*Then the canonical predisplay structure on $P_m = \mathbb{H}^n(X, \mathcal{N}^m W\Omega_{X/R})$ is a display structure.*

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# On the Structure of Calabi-Yau Categories with a Cluster Tilting Subcategory

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Abstract. We prove that for $d \geq 2$, an algebraic $d$-Calabi-Yau triangulated category endowed with a $d$-cluster tilting subcategory is the stable category of a DG category which is perfectly $(d+1)$-Calabi-Yau and carries a non degenerate $t$-structure whose heart has enough projectives.

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Contents

## 1. Introduction

In this article, we propose a description of a class of Calabi-Yau categories using the formalism of DG-categories and the notion of 'stabilization', as used for the description of triangulated orbit categories in section 7 of [21]. For $d \geq 2$, let $\mathcal{C}$ be an algebraic $d$-Calabi-Yau triangulated category endowed with a $d$-cluster tilting subcategory $\mathcal{T}$, *cf.* [23] [18] [19], see also [3] [13] [14]. Such categories occur for example,

- in the representation-theoretic approach to Fomin-Zelevinsky's cluster algebras [12], *cf.* [6] [9] [15] and the references given there,
- in the study of Cohen-Macaulay modules over certain isolated singularities, *cf.* [17] [23] [16], and the study of non commutative crepant resolutions [36], *cf.* [17].

From $\mathcal{C}$ and $\mathcal{T}$ we construct an exact dg category $\mathcal{B}$, which is perfectly $(d+1)$-Calabi-Yau, and a non-degenerate aisle $\mathcal{U}$, *cf.* [25], in $\mathrm{H}^0(\mathcal{B})$ whose heart has enough projectives. We prove, in theorem 7.1, how to recover the category $\mathcal{C}$ from $\mathcal{B}$ and $\mathcal{U}$ using a general procedure of stabilization defined in section 7. This extends previous results of [24] to a more general framework.
It follows from [30] that for $d = 2$, up to derived equivalence, the category $\mathcal{B}$ only depends on $\mathcal{C}$ (with its enhancement) and not on the choice of $\mathcal{T}$. In the appendix, we show how to naturally extend a $t$-structure, *cf.* [2], on the compact objects of a triangulated category to the whole category.

Example Let $k$ be a field, $A$ a finite-dimensional hereditary $k$-algebra and $\mathcal{C} = \mathcal{C}_A$ the cluster category of $A$, see [7] [8], i.e. the quotient of the bounded derived category of finitely generated modules over $A$ by the functor $F = \tau^{-1}[1]$, where $\tau$ denotes the $\mathsf{AR}$-translation and [1] denotes the shift functor.
Then $\mathcal{B}$ is given by the dg algebra, see section 7 of [21],

$$B = A \oplus (DA)[-3]$$

and theorem 7.1 reduces to the equivalence

$$\mathcal{D}^b(\mathcal{B})/\mathsf{per}(\mathcal{B}) \xrightarrow{\sim} \mathcal{C}_A$$

of section 7.1 of [21].

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## 3. Preliminaries

Let $k$ be a field. Let $\mathcal{E}$ be a $k$-linear Frobenius category with split idempotents. Suppose that its stable category $\mathcal{C} = \underline{\mathcal{E}}$, with suspension functor $S$, has finite-dimensional Hom-spaces and admits a Serre functor $\Sigma$, see [4]. Let $d \geq 2$ be an integer. We suppose that $\mathcal{C}$ is Calabi-Yau of CY-dimension $d$, *i.e.* [27] there is an isomorphism of triangle functors

$$S^d \xrightarrow{\sim} \Sigma\,.$$

We fix such an isomorphism once and for all. See section 4 of [23] for several examples of the above situation.
For $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, we put

$$\mathrm{Ext}^n(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X, S^n Y)\,.$$

We suppose that $\mathcal{C}$ is endowed with a $d$-cluster tilting subcategory $\mathcal{T} \subset \mathcal{C}$, *i.e.*

a) $\mathcal{T}$ is a $k$-linear subcategory,
b) $\mathcal{T}$ is functorially finite in $\mathcal{C}$, i.e. the functors $\mathrm{Hom}_{\mathcal{C}}(?,X)|\mathcal{T}$ and $\mathrm{Hom}_{\mathcal{C}}(X,?)|\mathcal{T}$ are finitely generated for all $X \in \mathcal{C}$,
c) we have $\mathrm{Ext}^i(T,T') = 0$ for all $T, T' \in \mathcal{T}$ and all $0 < i < d$ and
d) if $X \in \mathcal{C}$ satisfies $\mathrm{Ext}^i(T,X) = 0$ for all $0 < i < d$ and all $T \in \mathcal{T}$, then $T$ belongs to $\mathcal{T}$.

Let $\mathcal{M} \subset \mathcal{E}$ be the preimage of $\mathcal{T}$ under the projection functor. In particular, $\mathcal{M}$ contains the subcategory $\mathcal{P}$ of the projective-injective objects in $\mathcal{M}$. Note that $\mathcal{T}$ equals the quotient $\underline{\mathcal{M}}$ of $\mathcal{M}$ by the ideal of morphisms factoring through a projective-injective.
We dispose of the following commutative square:

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{T} \arrow[r, hook] & \underline{\mathcal{E}} = \mathcal{C}\,. \end{tikzcd}$$

We use the notations of [20]. In particular, for an additive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$ (resp. $\mathcal{C}^-(\mathcal{A})$, $\mathcal{C}^b(\mathcal{A})$, $\ldots$) the category of unbounded (resp. right bounded, resp. bounded, $\ldots$) complexes over $\mathcal{A}$ and by $\mathcal{H}(\mathcal{A})$ (resp. $\mathcal{H}^-(\mathcal{A})$, $\mathcal{H}^b(\mathcal{A})$, $\ldots$) its quotient modulo the ideal of nullhomotopic morphisms. By [26], *cf.* also [31], the projection functor $\mathcal{E} \to \underline{\mathcal{E}}$ extends to a canonical triangle functor $\mathcal{H}^b(\mathcal{E})/\mathcal{H}^b(\mathcal{P}) \to \underline{\mathcal{E}}$. This induces a triangle functor $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \underline{\mathcal{E}}$. It is shown in [30] that this functor is a localization functor. Moreover, the projection functor $\mathcal{H}^b(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ induces an equivalence from the subcategory $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ of bounded $\mathcal{E}$-acyclic complexes with components in $\mathcal{M}$ onto its kernel. Thus, we have a short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \longrightarrow \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{C} \longrightarrow 0\,.$$

Let $\mathcal{B}$ be the dg (=differential graded) subcategory of the category $\mathcal{C}^b(\mathcal{M})_{dg}$ of bounded complexes over $\mathcal{M}$ whose objects are the $\mathcal{E}$-acyclic complexes. We denote by $G : \mathcal{H}^-(\mathcal{M}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ the functor which takes a right bounded complex $X$ over $\mathcal{M}$ to the dg module

$$B \mapsto \mathrm{Hom}^\bullet_{\mathcal{M}}(X, B)\,,$$

where $B$ is in $\mathcal{B}$.

*Remark* 3.1. By construction, the functor $G$ restricted to $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ establishes an equivalence

$$G : \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \xrightarrow{\sim} \mathrm{per}(\mathcal{B}^{op})^{op} .$$

Recall that if $P$ is a right bounded complex of projectives and $A$ is an acyclic complex, then each morphism from $P$ to $A$ is nullhomotopic. In particular, the complex $\mathrm{Hom}^{\bullet}_{\mathcal{M}}(P, A)$ is nullhomotopic for each $P$ in $\mathcal{H}^-(\mathcal{P})$. Thus $G$ takes $\mathcal{H}^-(P)$ to zero, and induces a well defined functor (still denoted by $G$)

$$G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{D}(\mathcal{B}^{op})^{op} .$$

## 4. Embedding

Proposition 4.1. *The functor $G$ is fully faithful.*

For the proof, we need a number of lemmas.
It is well-known that the category $\mathcal{H}^-(\mathcal{E})$ admits a semiorthogonal decomposition, *cf.* [5], formed by $\mathcal{H}^-(\mathcal{P})$ and its right orthogonal $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{E})$, the full subcategory of the right bounded $\mathcal{E}$-acyclic complexes. For $X$ in $\mathcal{H}^-(\mathcal{E})$, we write

$$\mathbf{p}X \to X \to \mathbf{a}_p X \to S\mathbf{p}X$$

for the corresponding triangle, where $\mathbf{p}X$ is in $\mathcal{H}^-(\mathcal{P})$ and $\mathbf{a}_pX$ is in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{E})$. If $X$ lies in $\mathcal{H}^-(\mathcal{M})$, then clearly $\mathbf{a}_pX$ lies in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$ so that we have an induced semiorthogonal decomposition of $\mathcal{H}^-(\mathcal{M})$.

Lemma 4.1. *The functor $\Upsilon : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$ which takes $X$ to $\mathbf{a}_pX$ is fully faithful.*

*Proof.* By the semiorthogonal decomposition of $\mathcal{H}^-(\mathcal{M})$, the functor $X \mapsto \mathbf{a}_pX$ induces a right adjoint of the localization functor

$$\mathcal{H}^-(\mathcal{M}) \longrightarrow \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P})$$

and an equivalence of the quotient category with the right orthogonal $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$.

$$\begin{tikzcd}
 & \mathcal{H}^-(\mathcal{P}) \arrow[d, hook, shift right] & \\
 & \mathcal{H}^-(\mathcal{M}) \arrow[u, shift right] \arrow[d, shift right] & \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M}) = \mathcal{H}(\mathcal{P})^{\perp} \arrow[l, hook'] \\
\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r, hook] & \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P}) \arrow[u, hook, shift right] \arrow[ur, dashed, "\sim"] &
\end{tikzcd}.$$

Moreover, it is easy to see that the canonical functor

$$\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}^-(\mathcal{M})/\mathcal{H}^-(\mathcal{P})$$

is fully faithful so that we obtain a fully faithful functor

$$\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \longrightarrow \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$$

taking $X$ to $\mathbf{a}_p X$. $\surd$

*Remark* 4.1. Since the functor $G$ is triangulated and takes $\mathcal{H}^-(\mathcal{P})$ to zero, for $X$ in $\mathcal{H}^b(\mathcal{M})$, the adjunction morphism $X \to \mathbf{a}_p X$ yields an isomorphism

$$G(X) \xrightarrow{\sim} G(\mathbf{a}_p X) = G(\Upsilon X)\,.$$

Let $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ be the full subcategory of the derived category $\mathcal{D}(\mathcal{M})$ formed by the right bounded complexes whose homology modules lie in the subcategory $\operatorname{Mod}\underline{\mathcal{M}}$ of $\operatorname{Mod}\mathcal{M}$. The Yoneda functor $\mathcal{M} \to \operatorname{Mod}\mathcal{M}$, $M \mapsto M^\wedge$, induces a full embedding

$$\Psi : \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M}) \hookrightarrow \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})\,.$$

We write $\mathcal{V}$ for its essential image. Under $\Psi$, the category $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ is identified with $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Let $\Phi : \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ be the functor which takes $X$ to the dg module

$$B \mapsto \operatorname{Hom}^\bullet(X_c, \Psi(B))\,,$$

where $B$ is in $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$ and $X_c$ is a cofibrant replacement of $X$ for the projective model structure on $\mathcal{C}(\mathcal{M})$. Since for each right bounded complex $M$ with components in $\mathcal{M}$, the complex $M^\wedge$ is cofibrant in $\mathcal{C}(\mathcal{M})$, it is clear that the functor $G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ is isomorphic to the composition $\Phi \circ \Psi \circ \Upsilon$. We dispose of the following commutative diagram

$$\begin{tikzcd}
\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r, hook, "\Upsilon"] & \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[rr, "\Psi"] \arrow[dr, "\sim"] & & \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \arrow[r, "\Phi"] & \mathcal{D}(\mathcal{B}^{op})^{op} \\
& & \mathcal{V} \arrow[ur, hook] \arrow[urr, dashed] & & \\
\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[uu, hook] \arrow[r, equal] & \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[uu, hook] \arrow[rr, "\sim"] & & \operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M}) \arrow[uu, hook] \arrow[ul, hook] \arrow[r, "\sim"] & \operatorname{per}(\mathcal{B}^{op})^{op} \arrow[uu, hook]
\end{tikzcd}$$

Lemma 4.2. *Let $Y$ be an object of $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$.*

a) *$Y$ lies in $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ iff $\mathrm{H}^p(Y)$ is a finitely presented $\underline{\mathcal{M}}$-module for all $p \in \mathbb{Z}$ and vanishes for all but finitely many $p$.*

b) *$Y$ lies in $\mathcal{V}$ iff $\mathrm{H}^p(Y)$ is a finitely presented $\underline{\mathcal{M}}$-module for all $p \in \mathbb{Z}$ and vanishes for all $p \gg 0$.*

*Proof.* a) Clearly the condition is necessary. For the converse, suppose first that $Y$ is a finitely presented $\underline{\mathcal{M}}$-module. Then, as an $\mathcal{M}$-module, $Y$ admits a resolution of length $d+1$ by finitely generated projective modules by theorem 5.4 b) of [23]. It follows that $Y$ belongs to $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Since $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ is triangulated, it also contains all shifts of finitely presented $\underline{\mathcal{M}}$-modules and all extensions of shifts. This proves the converse.

b) Clearly the condition is necessary. For the converse, we can suppose without loss of generality that $Y^n = 0$, for all $n \geq 1$ and that $Y^n$ belongs to $\mathrm{proj}\,\mathcal{M}$, for $n \leq 0$. We now construct a sequence

$$\cdots \to P_n \to \cdots \to P_1 \to P_0$$

of complexes of finitely generated projective $\mathcal{M}$-modules such that $P_n$ is quasi-isomorphic to $\tau_{\geq -n}Y$ for each $n$ and that, for each $p \in \mathbb{Z}$, the sequence of $\mathcal{M}$-modules $P_n^p$ becomes stationary. By our assumptions, we have $\tau_{\geq 0}Y \xrightarrow{\sim} \mathrm{H}^0(Y)$. Since $\mathrm{H}^0(Y)$ belongs to $\mathrm{mod}\,\underline{\mathcal{M}}$, we know by theorem 5.4 c) of [23] that it belongs to $\mathrm{per}(\mathcal{M})$ as an $\mathcal{M}$-module. We define $P_0$ to be a finite resolution of $\mathrm{H}^0(Y)$ by finitely generated $\mathcal{M}$-modules. For the induction step, consider the following truncation triangle associated with $Y$

$$S^{i+1}\mathrm{H}^{-i-1}(Y) \to \tau_{\geq -i-1}Y \to \tau_{\geq -i}Y \to S^{i+2}\mathrm{H}^{-i-1}(Y)\,,$$

for $i \geq 0$. By the induction hypothesis, we have constructed $P_0, \ldots, P_i$ and we dispose of a quasi-isomorphism $P_i \xrightarrow{\sim} \tau_{\geq -i}Y$. Let $Q_{i+1}$ be a finite resolution of $S^{i+2}\mathrm{H}^{-i-1}(Y)$ by finitely presented projective $\mathcal{M}$-modules. We dispose of a morphism $f_i : P_i \to Q_{i+1}$ and we define $P_{i+1}$ as the cylinder of $f_i$. We define $P$ as the limit of the $P_i$ in the category of complexes. We remark that $Y$ is quasi-isomorphic to $P$ and that $P$ belongs to $\mathcal{V}$. This proves the converse. $\surd$

Let $X$ be in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$.

*Remark* 4.2. Lemma 4.2 shows that the natural $t$-structure of $\mathcal{D}(\mathcal{M})$ restricts to a $t$-structure on $\mathcal{V}$. This allows us to express $\Psi(X)$ as

$$\Psi(X) \xrightarrow{\sim} \operatorname*{holim}_i \tau_{\geq -i}\Psi(X)\,,$$

where $\tau_{\geq -i}\Psi(X)$ is in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$.

Lemma 4.3. *We dispose of the following isomorphism*

$$\Phi(\Psi(X)) = \Phi(\operatorname*{holim}_i \tau_{\geq -i}\Psi(X)) \xrightarrow{\sim} \operatorname*{holim}_i \Phi(\tau_{\geq -i}\Psi(X))\,.$$

*Proof.* It is enough to show that the canonical morphism induces a quasi-isomorphism when evaluated at any object $B$ of $\mathcal{B}$. We have

$$\Phi(\operatorname*{holim}_i \tau_{\geq -i}\Psi(X))(B) = \mathrm{Hom}^\bullet(\operatorname*{holim}_i \tau_{\geq -i}\Psi(X), B)\,,$$

but since $B$ is a bounded complex, for each $n \in \mathbf{Z}$, the sequence

$$i \mapsto \mathrm{Hom}^n(\tau_{\geq -i}\Psi(X), B)$$

stabilizes as $i$ goes to infinity. This implies that

$$\mathrm{Hom}^\bullet(\operatorname*{holim}_i \tau_{\geq -i}\Psi(X), B) \xleftarrow{\sim} \operatorname*{holim}_i \Phi(\tau_{\geq -i}\Psi(X))(B)\,.$$

$\surd$

Lemma 4.4. *The functor $\Phi$ restricted to the category $\mathcal{V}$ is fully faithful.*

*Proof.* Let $X, Y$ be in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$. The following are canonically isomorphic :

$$
\begin{array}{ll}
 & \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi\Psi X, \Phi\Psi Y) \\
 & \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\Psi Y, \Phi\Psi X) \\
(4.1) & \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\operatorname{hocolim}_i \Phi\tau_{\geq -i}\Psi Y, \operatorname{hocolim}_j \Phi\tau_{\geq -j}\Psi X) \\
 & \operatorname{holim}_i \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\tau_{\geq -i}\Psi Y, \operatorname{hocolim}_j \Phi\tau_{\geq -j}\Psi X) \\
(4.2) & \operatorname{holim}_i \operatorname{hocolim}_j \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi\tau_{\geq -i}\Psi Y, \Phi\tau_{\geq -j}\Psi X) \\
 & \operatorname{holim}_i \operatorname{hocolim}_j \mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(\tau_{\geq -j}\Psi X, \tau_{\geq -i}\Psi Y) \\
(4.3) & \operatorname{holim}_i \mathrm{Hom}_{\mathcal{V}}(\operatorname{holim}_j \tau_{\geq -j}\Psi X, \tau_{\geq -i}\Psi Y) \\
 & \mathrm{Hom}_{\mathcal{V}}(\Psi(X), \Psi(Y))\,.
\end{array}
$$

Here (4.1) is by the lemma 4.3 seen in $\mathcal{D}(\mathcal{B}^{op})$, (4.2) is by the fact that $\Phi\tau_{\geq -i}\Psi Y$ is compact and (4.3) is by the fact that $\tau_{\geq -i}\Psi Y$ is bounded. $\surd$

It is clear now that lemmas 4.1, 4.3 and 4.4 imply the proposition 4.1.

## 5. Determination of the image of $G$

Let $L_\rho : \mathcal{D}^-(\underline{\mathcal{M}}) \to \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ be the restriction functor induced by the projection functor $\mathcal{M} \to \underline{\mathcal{M}}$. $L_\rho$ admits a left adjoint $L : \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}^-(\underline{\mathcal{M}})$ which takes $Y$ to $Y \otimes^{\mathbb{L}}_{\mathcal{M}} \underline{\mathcal{M}}$. Let $\mathcal{B}^-$ be the dg subcategory of $\mathcal{C}^-(\mathrm{Mod}\,\mathcal{M})_{dg}$ formed by the objects of $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ that are in the essential image of the restriction of $\Psi$ to $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M})$. Let $\mathcal{B}'$ be the DG quotient, *cf.* [11], of $\mathcal{B}^-$ by its quasi-isomorphisms. It is clear that the dg categories $\mathcal{B}'$ and $\mathcal{B}$ are quasi-equivalent, *cf.* [22], and that the natural dg functor $\mathcal{M} \to \mathcal{C}^-(\mathrm{Mod}\,\mathcal{M})_{dg}$ factors through $\mathcal{B}^-$. Let $R' : \mathcal{D}(\mathcal{B}^{op})^{op} \to \mathcal{D}(\underline{\mathcal{M}}^{op})^{op}$ be the restriction functor induced by the dg functor $\underline{\mathcal{M}} \to \mathcal{B}'$. Let $\Phi' : \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}(\mathcal{B}'^{op})^{op}$ be the functor which takes $X$ to the dg module

$$B' \mapsto \mathrm{Hom}^\bullet(X_c, B')\,,$$

where $B'$ is in $\mathcal{B}'$ and $X_c$ is a cofibrant replacement of $X$ for the projective model structure on $\mathcal{C}(\mathrm{Mod}\,\mathcal{M})$. Finally let $\Gamma : \mathcal{D}(\underline{\mathcal{M}}) \to \mathcal{D}(\underline{\mathcal{M}}^{op})^{op}$ be the functor that sends $Y$ to

$$M \mapsto \mathrm{Hom}^\bullet(Y_c, \underline{\mathcal{M}}(?, M))\,,$$

where $Y_c$ is a cofibrant replacement of $Y$ for the projective model structure on $\mathcal{C}(\mathrm{Mod}\,\underline{\mathcal{M}})$ and $M$ is in $\underline{\mathcal{M}}$.

We dispose of the following diagram :

$$\begin{tikzcd}
 & & & \mathcal{D}(\mathcal{B}^{op})^{op} & \mathcal{B} \arrow[d] \\
\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r, hook, "\Upsilon"] & \mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[r, "\Psi"] & \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \arrow[ur, "\Phi"] \arrow[r, "\Phi'"] \arrow[d, "L"'] & \mathcal{D}(\mathcal{B}'^{op})^{op} \arrow[u, "\sim"'] \arrow[d, "R'"] & \mathcal{B}' \\
 & & \mathcal{D}^-(\underline{\mathcal{M}}) \arrow[r, "\Gamma"'] & \mathcal{D}(\underline{\mathcal{M}}^{op})^{op} & \underline{\mathcal{M}} \arrow[u]
\end{tikzcd}$$

LEMMA 5.1. *The following square*

$$\begin{tikzcd}
\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \arrow[r, "\Phi'"] \arrow[d, "L"'] & \mathcal{D}(\mathcal{B}'^{op})^{op} \arrow[d, "R'"] & \mathcal{B}' \\
\mathcal{D}^-(\underline{\mathcal{M}}) \arrow[r, "\Gamma"'] & \mathcal{D}(\underline{\mathcal{M}}^{op})^{op} & \underline{\mathcal{M}} \arrow[u]
\end{tikzcd}$$

*is commutative.*

*Proof.* By definition $(R' \circ \Phi')(X)(M)$ equals $\mathrm{Hom}^\bullet(X_c, \underline{\mathcal{M}}(?, \mathcal{M}))$. Since $\underline{\mathcal{M}}(?, M)$ identifies with $L_\rho M^\wedge$ and by adjunction, we have

$$\mathrm{Hom}^\bullet(X_c, \underline{\mathcal{M}}(?, M)) \xrightarrow{\sim} \mathrm{Hom}^\bullet(X_c, L_\rho M^\wedge) \xrightarrow{\sim} \mathrm{Hom}^\bullet((LX)_c, \underline{\mathcal{M}}(?, M)) ,$$

where the last member equals $(\Gamma \circ L)(X)(M)$. $\surd$

LEMMA 5.2. *The functor $L$ reflects isomorphisms .*

*Proof.* Since $L$ is a triangulated functor, it is enough to show that if $L(Y) = 0$, then $Y = 0$. Let $Y$ be in $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ such that $L(Y) = 0$. We can suppose, without loss of generality, that $\mathrm{H}^p(Y) = 0$ for all $p > 0$. Let us show that $\mathrm{H}^0(Y) = 0$. Indeed, since $\mathrm{H}^0(Y)$ is an $\underline{\mathcal{M}}$-module, we have $\mathrm{H}^0(Y) \cong L^0\mathrm{H}^0(Y)$, where $L^0 : \mathrm{Mod}\,\mathcal{M} \to \mathrm{Mod}\,\underline{\mathcal{M}}$ is the left adjoint of the inclusion $\mathrm{Mod}\,\underline{\mathcal{M}} \to \mathrm{Mod}\,\mathcal{M}$. Since $\mathrm{H}^p(Y)$ vanishes in degrees $p > 0$, we have

$$L^0\mathrm{H}^0(Y) = \mathrm{H}^0(LY) .$$

By induction, one concludes that $\mathrm{H}^p(Y) = 0$ for all $p \leq 0$. $\surd$

PROPOSITION 5.1. *An object $Y$ of $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ lies in the essential image of the functor $\Psi \circ \Upsilon : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ iff $\tau_{\geq -n}Y$ is in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, for all $n \in \mathbb{Z}$ and $L(Y)$ belongs to $\mathrm{per}(\underline{\mathcal{M}})$.*

*Proof.* Let $X$ be in $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$. By lemma 4.2 a), $\tau_{\geq -n}\Psi\Upsilon(X)$ is in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, for all $n \in \mathbb{Z}$. Since $X$ is a bounded complex, there exists an $s \ll 0$ such that for all $m < s$ the $m$-components of $\Upsilon(X)$ are in $\mathcal{P}$, which implies that $L\Psi\Upsilon(X)$ belongs to $\mathrm{per}(\underline{\mathcal{M}})$.
Conversely, suppose that $Y$ is an object of $\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$ which satisfies the conditions. By lemma 4.2, $Y$ belongs to $\mathcal{V}$. Thus we have $Y = \Psi(Y')$ for some $Y'$

in $\mathcal{H}^-_{\mathcal{E}\text{-}ac}(\mathcal{M})$. We now consider $Y'$ as an object of $\mathcal{H}^-(\mathcal{M})$ and also write $\Psi$ for the functor $\mathcal{H}^-(\mathcal{M}) \to \mathcal{D}^-(\mathcal{M})$ induced by the Yoneda functor. We can express $Y'$ as

$$Y' \xleftarrow{\sim} \operatorname*{hocolim}_i \sigma_{\geq -i} Y' \,,$$

where the $\sigma_{\geq -i}$ are the naive truncations. By our assumptions on $Y'$, $\sigma_{\geq -i}Y'$ belongs to $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$, for all $i \in \mathbb{Z}$. The functors $\Psi$ and $L$ clearly commute with the naive truncations $\sigma_{\geq -i}$ and so we have

$$L(Y) = L(\Psi Y') \xleftarrow{\sim} \operatorname*{hocolim}_i L(\sigma_{\geq -i}\Psi Y') \xrightarrow{\sim} \operatorname*{hocolim}_i \sigma_{\geq -i} L(\Psi Y') \,.$$

By our hypotheses, $L(Y)$ belongs to $\operatorname{per}(\underline{\mathcal{M}})$ and so there exists an $m \gg 0$ such that

$$L(Y) = L(\Psi Y') \xleftarrow{\sim} \sigma_{\geq -m} L(\Psi Y') = L(\sigma_{\geq -m}\Psi Y') \,.$$

By lemma 5.2, the inclusion

$$\Psi(\sigma_{\geq -m}Y)' = \sigma_{\geq -m}\Psi Y' \longrightarrow \Psi(Y') = Y$$

is an isomorphism. But since $\sigma_{\geq -m}Y'$ belongs to $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$, $Y$ identifies with $\Psi(\sigma_{\geq -m}Y')$. $\surd$

*Remark* 5.1. It is clear that if $X$ belongs to $\operatorname{per}(\underline{\mathcal{M}})$, then $\Gamma(X)$ belongs to $\operatorname{per}(\underline{\mathcal{M}}^{op})^{op}$. We also have the following partial converse.

Lemma 5.3. *Let $X$ be in $\mathcal{D}^-_{\operatorname{mod}\underline{\mathcal{M}}}(\underline{\mathcal{M}})$ such that $\Gamma(X)$ belongs to $\operatorname{per}(\underline{\mathcal{M}}^{op})^{op}$. Then $X$ is in $\operatorname{per}(\underline{\mathcal{M}})$.*

*Proof.* By lemma 4.2 b) we can suppose, without loss of generality, that $X$ is a right bounded complex with finitely generated projective components. Applying $\Gamma$, we get a perfect complex $\Gamma(X)$. In particular $\Gamma(X)$ is homotopic to zero in high degrees. But since $\Gamma$ is an equivalence

$$\operatorname{proj}\underline{\mathcal{M}} \xrightarrow{\sim} (\operatorname{proj}\underline{\mathcal{M}}^{op})^{op} \,,$$

it follows that $X$ is already homotopic to zero in high degrees. $\surd$

*Remark* 5.2. The natural right aisle on $\mathcal{D}(\mathcal{M})$ is the full subcategory of the objects $X$ such that $\mathsf{H}^n(X) = 0$ for all $n < 0$. The associated truncation functor $\tau_{\geq 0}$ takes $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ to itself. Therefore, the natural right aisle on $\mathcal{D}(\mathcal{M})$ restricts to a natural right aisle $\mathcal{U}^{op}$ on $\mathsf{per}_{\underline{\mathcal{M}}}(\mathcal{M})$.

Definition 5.1. *Let $\mathcal{U}$ be the natural left aisle in $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$ associated with $\mathcal{U}^{op}$.*

Lemma 5.4. *The natural left aisle $\mathcal{U}$ on $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op} \xrightarrow{\sim} \operatorname{per}(\mathcal{B}^{op})$ satisfies the conditions of proposition A.1 b).*

*Proof.* Clearly the natural left aisle $\mathcal{U}$ in $\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$ is non-degenerate. We need to show that for each $C \in \operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$, there is an integer $N$ such that $\operatorname{Hom}(C, S^N U) = 0$ for each $U \in \mathcal{U}$. We dispose of the following isomorphism

$$\operatorname{Hom}_{\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}}(C, S^N\mathcal{U}) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(S^{-N}\mathcal{U}^{op}, C) \,,$$

where $\mathcal{U}^{op}$ denotes the natural right aisle on $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Since by theorem 5.4 c) of [23] an $\underline{\mathcal{M}}$-module admits a projective resolution of length $d+1$ as an $\mathcal{M}$-module and $C$ is a bounded complex, we conclude that for $N \gg 0$

$$\mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(S^{-N}\mathcal{U}^{op}, C) = 0\,.$$

This proves the lemma. $\surd$

We denote by $\tau_{\leq n}$ and $\tau_{\geq n}$, $n \in \mathbb{Z}$, the associated truncation functors on $\mathcal{D}(\mathcal{B}^{op})^{op}$.

LEMMA 5.5. *The functor $\Phi : \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ restricted to the category $\mathcal{V}$ is exact with respect to the given $t$-structures.*

*Proof.* We first prove that $\Phi(\mathcal{V}_{\leq 0}) \subset \mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$. Let $X$ be in $\mathcal{V}_{\leq 0}$. We need to show that $\Phi(X)$ belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$. The following have the same classes of objects :

$$\mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$$

$$\mathcal{D}(\mathcal{B}^{op})_{>0}$$

$$(\mathrm{per}(\mathcal{B}^{op})_{\leq 0})^{\perp} \tag{5.1}$$

$$^{\perp}(\mathrm{per}(\mathcal{B}^{op})^{op})_{>0}\,, \tag{5.2}$$

where in (5.1) we consider the right orthogonal in $\mathcal{D}(\mathcal{B}^{op})$ and in (5.2) we consider the left orthogonal in $\mathcal{D}(\mathcal{B}^{op})^{op}$. These isomorphisms show us that $\Phi(X)$ belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$ iff

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X), \Phi(P)) = 0\,,$$

for all $P \in \mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})_{>0}$. Now, by lemma 4.4 the functor $\Phi$ is fully faithful and so

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X), \Phi(P)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(X, P)\,.$$

Since $X$ belongs to $\mathcal{V}_{\leq 0}$ and $P$ belongs to $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})_{>0}$, we conclude that

$$\mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(X, P) = 0\,,$$

which implies that $\Phi(X) \in \mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$. Let us now consider $X$ in $\mathcal{V}$. We dispose of the truncation triangle

$$\tau_{\leq 0}X \to X \to \tau_{>0}X \to S\tau_{\leq 0}X\,.$$

The functor $\Phi$ is triangulated and so we dispose of the triangle

$$\Phi\tau_{\leq 0}X \to X \to \Phi\tau_{>0}X \to S\Phi\tau_{\leq 0}X\,,$$

where $\Phi\tau_{\leq 0}X$ belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_{\leq 0}$. Since $\Phi$ induces an equivalence between $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ and $\mathrm{per}(\mathcal{B}^{op})^{op}$ and $\mathrm{Hom}(P, \tau_{>0}X) = 0$, for all $P$ in $\mathcal{V}_{\leq 0}$, we conclude that $\Phi\tau_{>0}X$ belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_{>0}$. This implies the lemma. $\surd$

DEFINITION 5.2. *Let $\mathcal{D}(\mathcal{B}^{op})^{op}_{f}$ denote the full triangulated subcategory of $\mathcal{D}(\mathcal{B}^{op})^{op}$ formed by the objects $Y$ such that $\tau_{\geq -n}Y$ is in $\mathrm{per}(\mathcal{B}^{op})^{op}$, for all $n \in \mathbb{Z}$, and $R(Y)$ belongs to $\mathrm{per}(\underline{\mathcal{M}}^{op})^{op}$.*

Proposition 5.2. *An object $Y$ of $\mathcal{D}(\mathcal{B}^{op})^{op}$ lies in the essential image of the functor $G : \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \to \mathcal{D}(\mathcal{B}^{op})^{op}$ iff it belongs to $\mathcal{D}(\mathcal{B}^{op})^{op}_f$.*

*Proof.* Let $X$ be in $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$. It is clear that the $\tau_{\geq -n}G(X)$ are in $\mathrm{per}(\mathcal{B}^{op})^{op}$ for all $n \in \mathbb{Z}$. By proposition 5.1 we know that $L\Psi\Upsilon(X)$ belongs to $\mathrm{per}(\underline{\mathcal{M}})$. By lemma 5.1 and remark 5.1 we conclude that $RG(X)$ belongs to $\mathrm{per}(\underline{\mathcal{M}}^{op})^{op}$. Let now $Y$ be in $\mathcal{D}(\mathcal{B}^{op})^{op}_f$. We can express it, by the dual of lemma A.2 as the homotopy limit of the following diagram

$$\cdots \to \tau_{\geq -n-1}Y \to \tau_{\geq -n}Y \to \tau_{\geq -n+1}Y \to \cdots ,$$

where $\tau_{\geq -n}Y$ belongs to $\mathrm{per}(\mathcal{B}^{op})^{op}$, for all $n \in \mathbb{Z}$. But since $\Phi$ induces an equivalence between $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$ and $\mathrm{per}(\mathcal{B}^{op})^{op}$, this last diagram corresponds to a diagram

$$\cdots \to M_{-n-1} \to M_{-n} \to M_{-n+1} \to \cdots$$

in $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. Let $p \in \mathbb{Z}$. The relations among the truncation functors imply that the image of the above diagram under each homology functor $\mathrm{H}^p$, $p \in \mathbb{Z}$, is stationary as $n$ goes to $+\infty$. This implies that

$$\mathrm{H}^p \operatorname*{holim}_n M_{-n} \xrightarrow{\sim} \lim_n \mathrm{H}^p M_{-n} \cong \mathrm{H}^p M_j ,$$

for all $j < p$. We dispose of the following commutative diagram

$$\begin{tikzcd}
\operatorname*{holim}_n M_{-n} \arrow[rr] \arrow[d] & & \operatorname*{holim}_n \tau_{\geq -i} M_{-n} \cong M_{-i} \\
\tau_{\geq -i} \operatorname*{holim}_n M_{-n} \arrow[urr, "\sim"] & &
\end{tikzcd}$$

which implies that

$$\tau_{\geq -i} \operatorname*{holim}_n M_{-n} \xrightarrow{\sim} M_{-i} ,$$

for all $i \in \mathbb{Z}$. Since $\operatorname*{holim}_n M_{-n}$ belongs to $\mathcal{V}$, lemma 4.3 allows us to conclude that $\Phi(\operatorname*{holim}_n M_{-n}) \cong Y$. We now show that $\operatorname*{holim}_n M_{-n}$ satisfies the conditions of proposition 5.1. We know that $\tau_{\geq -i} \operatorname*{holim}_n M_{-n}$ belongs to $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, for all $i \in \mathbb{Z}$. By lemma 5.1 $(\Gamma \circ L)(\operatorname*{holim}_n M_{-n})$ identifies with $R(Y)$, which is in $\mathrm{per}(\underline{\mathcal{M}}^{op})^{op}$. Since $\operatorname*{holim}_n M_{-n}$ belongs to $\mathcal{V}$, its homologies lie in $\mathrm{mod}\, \underline{\mathcal{M}}$ and so we are in the conditions of lemma 5.1, which implies that $L(\operatorname*{holim}_n M_{-n})$ belongs to $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$. This finishes the proof. $\surd$

## 6. Alternative description

In this section, we present another characterization of the image of $G$, which was identified as $\mathcal{D}(\mathcal{B}^{op})^{op}_f$ in proposition 5.2. Let $M$ denote an object of $\mathcal{M}$ and also the naturally associated complex in $\mathcal{H}^b(\mathcal{M})$. Since the category $\mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P})$ is generated by the objects $M \in \mathcal{M}$ and the functor $G$ is fully faithful, we remark that $\mathcal{D}(\mathcal{B}^{op})^{op}_f$ equals the triangulated subcategory of $\mathcal{D}(\mathcal{B}^{op})^{op}$ generated by the objects $G(M)$, $M \in \mathcal{M}$. The rest of this section is concerned with the problem of characterizing the objects $G(M)$, $M \in \mathcal{M}$. We denote by $P_M$ the projective $\underline{\mathcal{M}}$-module $\underline{\mathcal{M}}(?, M)$ associated with $M \in \mathcal{M}$ and by $X_M$ the image of $M$ under $\Psi \circ \Upsilon$.

Lemma 6.1. *We dispose of the following isomorphism*

$$\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, Y) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{mod}\,\underline{\mathcal{M}}}(P_M, \mathrm{H}^0(Y))\,,$$

*for all* $Y \in \mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})$.

*Proof.* Clearly $X_M$ belongs to $\mathcal{D}_{\underline{\mathcal{M}}}(\mathcal{M})_{\leq 0}$ and is of the form

$$\cdots \to P_n^\wedge \to \cdots \to P_1^\wedge \to P_0^\wedge \to M^\wedge \to 0\,,$$

where $P_n \in \mathcal{P}$, $n \geq 0$. Now Yoneda's lemma and the fact that $\mathrm{H}^m(Y)(P_n) = 0$, for all $m \in \mathbb{Z}$, $n \geq 0$, imply the lemma. $\surd$

*Remark* 6.1. Since the functor $\Phi$ restricted to $\mathcal{V}$ is fully faithful and exact, we have

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(G(M), \Phi(Y)) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})^{op}}(\Phi(P_M), \mathrm{H}^0(\Phi(Y)))\,,$$

for all $Y \in \mathcal{V}$.

We now characterize the objects $G(M) = \Phi(X_M)$, $M \in \mathcal{M}$, in the triangulated category $\mathcal{D}(\mathcal{B}^{op})$. More precisely, we give a description of the functor

$$R_M := \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(?, \Phi(X_M)) : \mathcal{D}(\mathcal{B}^{op})^{op} \to \mathrm{Mod}\, k$$

using an idea of M. Van den Bergh, *cf.* lemma 2.13 of [10]. Consider the following functor

$$F_M := \mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\mathrm{H}^0(?), \Phi(P_M)) : \mathrm{per}(\mathcal{B}^{op})^{op} \to \mathrm{mod}\, k\,.$$

*Remark* 6.2. Remark 6.1 shows that the functor $R_M$ when restricted to $\mathrm{per}(\mathcal{B}^{op})$ coincides with $F_M$.

Let $DF_M$ be the composition of $F_M$ with the duality functor $D = \mathrm{Hom}(?, k)$. Note that $DF_M$ is homological.

Lemma 6.2. *We dispose of the following isomorphism of functors on* $\mathrm{per}(\mathcal{B}^{op})$

$$DF_M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(X_M), ?[d+1])\,.$$

*Proof.* The following functors are canonically isomorphic to $DF\Phi$ :

$$D\mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\mathrm{H}^0\Phi(?), \Phi(P_M))$$

$$D\mathrm{Hom}_{\mathrm{per}(\mathcal{B}^{op})}(\Phi\mathrm{H}^0(?), \Phi(P_M)) \tag{6.1}$$

$$D\mathrm{Hom}_{\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})}(P_M, \mathrm{H}^0(?)) \tag{6.2}$$

$$D\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, ?) \tag{6.3}$$

$$\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(?[-d-1], X_M) \tag{6.4}$$

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(?)[-d-1], \Phi(X_M)) \tag{6.5}$$

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})^{op}}(\Phi(X_M), \Phi(?)[d+1]) \tag{6.6}$$

Step (6.1) follows from the fact that $\Phi$ is exact. Step (6.2) follows from the fact that $\Phi$ is fully faithful and we are considering the opposite category. Step (6.3) is a consequence of lemma 6.1. Step (6.4) follows from the $(d+1)$-Calabi-Yau property and remark 4.2. Step (6.5) is a consequence of $\Phi$ being fully faithful and step (6.6) is a consequence of working in the opposite category. Since the functor $\Phi^{op}$ establish an equivalence between $\mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})^{op}$ and $\mathrm{per}(\mathcal{B}^{op})$ the lemma is proven. $\surd$

Now, since the category $\mathrm{Mod}\, k$ is cocomplete, we can consider the left Kan extension, *cf.* [28], $E_M$ of $DF_M$ along the inclusion $\mathrm{per}(\mathcal{B}^{op}) \hookrightarrow \mathcal{D}(\mathcal{B}^{op})$. We dispose of the following commutative square :

$$\begin{tikzcd} \mathrm{per}(\mathcal{B}^{op}) \arrow[r, "DF_M"] \arrow[d, hook] & \mathrm{mod}\, k \arrow[d, hook] \\ \mathcal{D}(\mathcal{B}^{op}) \arrow[r, dashed, "E_M"'] & \mathrm{Mod}\, k\,. \end{tikzcd}$$

For each $X$ of $\mathcal{D}(\mathcal{B}^{op})$, the comma-category of morphisms $P \to X$ from a perfect object $P$ to $X$ is filtered. Therefore, the functor $E_M$ is homological. Moreover, it preserves coproducts and so $DE_M$ is cohomological and transforms coproducts into products. Since $\mathcal{D}(\mathcal{B}^{op})$ is a compactly generated triangulated category, the Brown representability theorem, *cf.* [29], implies that there is a $Z_M \in \mathcal{D}(\mathcal{B}^{op})$ such that

$$DE_M \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(?, Z_M)\,.$$

*Remark* 6.3. Since the duality functor $D$ establishes and anti-equivalence in $\mathrm{mod}\, k$, the functor $DE_M$ restricted to $\mathrm{per}(\mathcal{B}^{op})$ is isomorphic to $F_M$.

Theorem 6.1. *We dispose of an isomorphism*

$$G(M) \xrightarrow{\sim} Z_M\,.$$

*Proof.* We now construct a morphism of functors from $R_M$ to $DE_M$. Since $R_M$ is representable, by Yoneda's lemma it is enough to construct an element in $DE_M(\Phi(X_M))$. Let $\mathcal{C}$ be the category $\mathrm{per}(\mathcal{B}^{op}) \downarrow \Phi(X_M)$, whose objects are the morphisms $Y' \to \Phi(X_M)$ and let $\mathcal{C}'$ be the category $X_M \downarrow \mathrm{per}_{\underline{\mathcal{M}}}(\mathcal{M})$, whose objects are the morphisms $X_M \to X'$. The following are canonically isomorphic :

$$DE_M(\Phi(X_M))$$

$$D\,\operatorname*{colim}_{\mathcal{C}} \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^{op})}(\Phi(X_M), Y'[d+1]) \tag{6.7}$$

$$D\,\operatorname*{colim}_{\mathcal{C}'} \mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X'[-d-1], X_M) \tag{6.8}$$

$$\mathrm{D}\,\operatorname*{colim}_{i} \mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}((\tau_{\geq -i}X_M)[-d-1], X_M) \tag{6.9}$$

$$\lim_i \mathrm{DHom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}((\tau_{\geq -i}X_M)[-d-1], X_M)$$

$$\lim_i \mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, \tau_{\geq -i}X_M) \tag{6.10}$$

Step (6.7) is a consequence of the definition of the left Kan extension and lemma 6.2. Step (6.8) is obtained by considering the opposite category. Step (6.9) follows from the fact that the system $(\tau_{\geq -i}X_M)_{i\in\mathbb{Z}}$ forms a cofinal system for the index system of the colimit. Step (6.10) follows from the $(d+1)$-Calabi-Yau property. Now, the image of the identity by the canonical morphism

$$\mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, X_M) \longrightarrow \lim_i \mathrm{Hom}_{\mathcal{D}^-_{\underline{\mathcal{M}}}(\mathcal{M})}(X_M, \tau_{\geq -i}X_M)\,,$$

gives us an element of $(DE_M)(\Phi(X_M))$ and so a morphism of functors from $R_M$ to $DE_M$. We remark that this morphism is an isomorphism when evaluated at the objects of $\mathrm{per}(\mathcal{B}^{op})$. Since both functors $R_M$ and $DE_M$ are cohomological, transform coproducts into products and $\mathcal{D}(\mathcal{B}^{op})$ is compactly generated, we conclude that we dispose of an isomorphism

$$G(M) \xrightarrow{\sim} Z_M\,.$$

$\surd$

## 7. The main theorem

Consider the following commutative square as in section 3:

$$\begin{tikzcd} \mathcal{M} \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathcal{T} \arrow[r, hook] & \underline{\mathcal{E}} = \mathcal{C}\,. \end{tikzcd}$$

In the previous sections we have constructed, from the above data, a dg category $\mathcal{B}$ and a left aisle $\mathcal{U} \subset \mathrm{H}^0(\mathcal{B})$, see [25], satisfying the following conditions :

- $\mathcal{B}$ is an exact dg category over $k$ such that $\mathrm{H}^0(\mathcal{B})$ has finite-dimensional Hom-spaces and is Calabi-Yau of CY-dimension $d+1$,
- $\mathcal{U} \subset \mathrm{H}^0(\mathcal{B})$ is a non-degenerate left aisle such that :
  - for all $B \in \mathcal{B}$, there is an integer $N$ such that $\mathrm{Hom}_{\mathrm{H}^0(\mathcal{B})}(B, S^N U) = 0$ for each $U \in \mathcal{U}$,
  - the heart $\mathcal{H}$ of the $t$-structure on $\mathrm{H}^0(\mathcal{B})$ associated with $\mathcal{U}$ has enough projectives.

Let now $\mathcal{A}$ be a dg category and $\mathcal{W} \subset \mathrm{H}^0(\mathcal{A})$ a left aisle satisfying the above conditions. We can consider the following general construction : Let $\mathcal{Q}$ denote the category of projectives of the heart $\mathcal{H}$ of the $t$-structure on $\mathrm{H}^0(\mathcal{A})$ associated with $\mathcal{W}$. We claim that the following inclusion

$$\mathcal{Q} \hookrightarrow \mathcal{H} \hookrightarrow \mathrm{H}^0(\mathcal{A})\,,$$

lifts to a morphism $\mathcal{Q} \xrightarrow{j} \mathcal{A}$ in the homotopy category of small dg categories, *cf.* [20] [32] [33] [34] [35]. Indeed, recall the following argument from section 7 of [22]: Let $\tilde{\mathcal{Q}}$ be the full dg subcategory of $\mathcal{A}$ whose objects are the same as those of $\mathcal{Q}$. Let $\tau_{\leq 0}\tilde{\mathcal{Q}}$ denote the dg category obtained from $\tilde{\mathcal{Q}}$ by applying the truncation functor $\tau_{\leq 0}$ of complexes to each Hom-space. We dispose of the following diagram in the category of small dg categories

$$\begin{tikzcd} & \tilde{\mathcal{Q}} \arrow[r, hook] & \mathcal{A} \\ & \tau_{\leq 0}\tilde{\mathcal{Q}} \arrow[u] \arrow[d] & \\ \mathcal{Q} \arrow[r, equal] & \mathrm{H}^0(\tilde{\mathcal{Q}}) & \end{tikzcd} .$$

Let $X$, $Y$ be objects of $\mathcal{Q}$. Since $X$ and $Y$ belong to the heart of a $t$-structure in $\mathrm{H}^0(\mathcal{A})$, we have

$$\mathrm{Hom}_{\mathrm{H}^0(\mathcal{A})}(X, Y[-n]) = 0\,,$$

for $n \geq 1$. The dg category $\mathcal{A}$ is exact, which implies that

$$\mathrm{H}^{-n}\mathrm{Hom}^{\bullet}_{\tilde{\mathcal{Q}}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{H}^0(\mathcal{A})}(X, Y[-n]) = 0\,,$$

for $n \geq 1$. This shows that the dg functor $\tau_{\leq 0}\tilde{\mathcal{Q}} \to \mathrm{H}^0(\tilde{\mathcal{Q}})$ is a quasi-equivalence and so we dispose of a morphism $\mathcal{Q} \xrightarrow{j} \mathcal{A}$ in the homotopy category of small dg categories. We dispose of a triangle functor $j^* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{Q})$ given by restriction. By proposition A.1, the left aisle $\mathcal{W} \subset \mathrm{H}^0(\mathcal{A})$ admits a smallest extension to a left aisle $\mathcal{D}(\mathcal{A}^{op})^{op}_{\leq 0}$ on $\mathcal{D}(\mathcal{A}^{op})^{op}$. Let $\mathcal{D}(\mathcal{A}^{op})^{op}_f$ denote the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{op})^{op}$ formed by the objects $Y$ such that $\tau_{\geq -n}Y$ is in $\mathrm{per}(\mathcal{A}^{op})^{op}$, for all $n \in \mathbb{Z}$, and $j^*(Y)$ belongs to $\mathrm{per}(\mathcal{Q}^{op})^{op}$.

Definition 7.1. *The stable category of $\mathcal{A}$ with respect to $\mathcal{W}$ is the triangle quotient*

$$\mathrm{stab}(\mathcal{A}, \mathcal{W}) = \mathcal{D}(\mathcal{A}^{op})^{op}_f / \mathrm{per}(\mathcal{A}^{op})^{op}\,.$$

We are now able to formulate the main theorem. Let $\mathcal{B}$ be the dg category and $\mathcal{U} \subset \mathrm{H}^0(\mathcal{B})$ the left aisle constructed in sections 1 to 5.

Theorem 7.1. *The functor $G$ induces an equivalence of categories*

$$\tilde{G} : \mathcal{C} \xrightarrow{\sim} \mathrm{stab}(\mathcal{B}, \mathcal{U})\,.$$

*Proof.* We dispose of the following commutative diagram :

$$\begin{tikzcd} \mathcal{C} \arrow[r, dashed, "\tilde{G}", "\sim"'] & \mathrm{stab}(\mathcal{B}, \mathcal{U}) \\ \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[u] \arrow[r, "G", "\sim"'] & \mathcal{D}(\mathcal{B}^{op})^{op}_f \arrow[u] \\ \mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \arrow[u] \arrow[r, "\sim"'] & \mathrm{per}(\mathcal{B}^{op})^{op} \arrow[u] \end{tikzcd} .$$

The functor $G$ is an equivalence since it is fully faithful by proposition 4.1 and essentially surjective by proposition 5.2. Since we dispose of an equivalence $\mathcal{H}^b_{\mathcal{E}\text{-}ac}(\mathcal{M}) \xrightarrow{\sim} \mathrm{per}(\mathcal{B}^{op})^{op}$ by construction of $\mathcal{B}$ and the columns of the above diagram are short exact sequences of triangulated categories, the theorem is proved. $\surd$

## Appendix A. Extension of $t$-structures

Let $\mathcal{T}$ be a compactly generated triangulated category with suspension functor $S$. We denote by $\mathcal{T}_c$ the full triangulated sub-category of $\mathcal{T}$ formed by the compact objects, see [29]. We use the terminology of [25]. Let $\mathcal{U} \subseteq \mathcal{T}_c$ be a left aisle on $\mathcal{T}_c$, i.e. a full additive subcategory $\mathcal{U}$ of $\mathcal{T}_c$ which satisfies:

a) $S\mathcal{U} \subset \mathcal{U}$,

b) $\mathcal{U}$ is stable under extensions, i.e. for each triangle

$$X \to Y \to Z \to SX$$

of $\mathcal{T}_c$, we have $Y \in \mathcal{U}$ whenever $X, Z \in \mathcal{U}$ and

c) the inclusion functor $\mathcal{U} \hookrightarrow \mathcal{T}_c$ admits a right adjoint.

As shown in [25], the concept of aisle is equivalent to that of $t$-structure.

Proposition A.1. a) *The left aisle $\mathcal{U}$ admits a smallest extension to a left aisle $\mathcal{T}_{\leq 0}$ on $\mathcal{T}$.*

b) *If $\mathcal{U} \subseteq \mathcal{T}_c$ is non-degenerate (*i.e.*, $f : X \to Y$ is invertible iff $\mathrm{H}^p(f)$ is invertible for all $p \in \mathbb{Z}$) and for each $X \in \mathcal{T}_c$, there is an integer $N$ such that $\mathrm{Hom}(X, S^N U) = 0$ for each $U \in \mathcal{U}$, then $\mathcal{T}_{\leq 0}$ is also non-degenerate.*

*Proof.* a) Let $\mathcal{T}_{\leq 0}$ be the smallest full subcategory of $\mathcal{T}$ that contains $\mathcal{U}$ and is stable under infinite sums and extensions. It is clear that $\mathcal{T}_{\leq 0}$ is stable by $S$ since $\mathcal{U}$ is. We need to show that the inclusion functor $\mathcal{T}_{\leq 0} \hookrightarrow \mathcal{T}$ admits a right adjoint. For completeness, we include the following proof, which is a variant of the 'small object argument', *cf.* also [1]. We dispose of the following recursive procedure. Let $X = X_0$ be an object in $\mathcal{T}$. For the initial step consider all morphisms from any object $P$ in $\mathcal{U}$ to $X_0$. This forms a set $I_0$ since $\mathcal{T}$ is compactly generated and so we dispose of the following triangle

$$\coprod_{f \in I_0} P \longrightarrow X_0 \longrightarrow X_1 \rightsquigarrow \coprod_{f \in I_0} P \,.$$

For the induction step consider the above construction with $X_n$, $n \geq 1$, in the place of $X_{n-1}$ and $I_n$ in the place of $I_{n-1}$. We dispose of the following diagram

$$\begin{tikzcd}
X = X_0 \arrow[r] & X_1 \arrow[r] \arrow[dl, squiggly] & X_2 \arrow[r] \arrow[dl, squiggly] & X_3 \arrow[r] \arrow[dl, squiggly] & \cdots \arrow[r] & X' \\
\coprod_{f \in I_0} P \arrow[u] & \coprod_{f \in I_1} P \arrow[u] & \coprod_{f \in I_2} P \arrow[u] & \coprod_{f \in I_3} P \arrow[u] & &
\end{tikzcd},$$

where $X'$ denotes the homotopy colimit of the diagram $(X_i)_{i\in\mathbb{Z}}$. Consider now the following triangle

$$S^{-1}X' \to X'' \to X \to X'\,,$$

where the morphism $X \to X'$ is the transfinite composition in our diagram. Let $P$ be in $\mathcal{U}$. We remark that since $P$ is compact, $\mathrm{Hom}_{\mathcal{T}}(P, X') = 0$. This also implies, by construction of $\mathcal{T}_{\leq 0}$, that $\mathrm{Hom}_{\mathcal{T}}(R, X') = 0$, for all $R$ in $\mathcal{T}_{\leq 0}$. The long exact sequence obtained by applying the functor $\mathrm{Hom}_{\mathcal{T}}(R, ?)$ to the triangle above shows that

$$\mathrm{Hom}(R, X'') \xrightarrow{\sim} \mathrm{Hom}(R, X)\,.$$

Let $X''_{n-1}$, $n \geq 1$, be an object as in the following triangle

$$X = X_0 \to X_n \to X''_{n-1} \to S(X)\,.$$

A recursive application of the octahedron axiom implies that $X''_{n-1}$ belongs to $S(\mathcal{T}_{\leq 0})$, for all $n \geq 1$. We dispose of the isomorphism

$$\operatorname*{hocolim}_n X''_{n-1} \xrightarrow{\sim} S(X'')\,.$$

Since $\operatorname*{hocolim}_n X''_{n-1}$ belongs to $S(\mathcal{T}_{\leq 0})$, we conclude that $X''$ belongs to $\mathcal{T}_{\leq 0}$. This shows that the functor that sends $X$ to $X''$ is the right adjoint of the inclusion functor $\mathcal{T}_{\leq 0} \hookrightarrow \mathcal{T}$. This proves that $\mathcal{T}_{\leq 0}$ is a left aisle on $\mathcal{T}$. We now show that the $t$-structure associated to $\mathcal{T}_{\leq 0}$, *cf.* [25], extends, from $\mathcal{T}_c$ to $\mathcal{T}$, the one associated with $\mathcal{U}$. Let $X$ be in $\mathcal{T}_c$. We dispose of the following truncation triangle associated with $\mathcal{U}$

$$X_{\mathcal{U}} \to X \to X^{\mathcal{U}^{\perp}} \to SX_{\mathcal{U}}\,.$$

Clearly $X_{\mathcal{U}}$ belongs to $\mathcal{T}_{\leq 0}$. We remark that $\mathcal{U}^{\perp} = \mathcal{T}_{\leq 0}^{\perp}$, and so $X^{\mathcal{U}^{\perp}}$ belongs to $\mathcal{T}_{>0} := \mathcal{T}_{\leq 0}^{\perp}$.
We now show that $\mathcal{T}_{\leq 0}$ is the smallest extension of the left aisle $\mathcal{U}$. Let $\mathcal{V}$ be an aisle containing $\mathcal{U}$. The inclusion functor $\mathcal{V} \hookrightarrow \mathcal{T}$ commutes with sums, because it admits a right adjoint. Since $\mathcal{V}$ is stable under extensions and suspensions, it contains $\mathcal{T}_{\leq 0}$.

b) Let $X$ be in $\mathcal{T}$. We need to show that $X = 0$ iff $\mathrm{H}^p(X) = 0$ for all $p \in \mathbb{Z}$. Clearly the condition is necessary. For the converse, suppose that $\mathrm{H}^p(X) = 0$ for all $p \in \mathbb{Z}$. Let $n$ be an integer. Consider the following truncation triangle

$$\mathrm{H}^{n+1}(X) \to \tau_{>n}X \to \tau_{>n+1}X \to S\mathrm{H}^{n+1}(X)\,.$$

Since $\mathrm{H}^{n+1}(X) = 0$ we conclude that

$$\tau_{>n}X \in \bigcap_{m\in\mathbb{Z}} \mathcal{T}_{>m}\,,$$

for all $n \in \mathbb{Z}$. Now, let $C$ be a compact object of $\mathcal{T}$. We know that there is a $k \in \mathbb{Z}$ such that $C \in \mathcal{T}_{\leq k}$. This implies that

$$\mathrm{Hom}_{\mathcal{T}}(C, \tau_{>n}X) = 0$$

for all $n \in \mathbb{Z}$, since $\tau_{>n}X$ belongs to $(\mathcal{T}_{\leq k})^{\perp}$. The category $\mathcal{T}$ is compactly generated and so we conclude that $\tau_{>n}X = 0$, for all $n \in \mathbb{Z}$. The following truncation triangle

$$\tau_{\leq n}X \to X \to \tau_{>n}X \to S\tau_{\leq n}X\,,$$

implies that $\tau_{\leq n}X$ is isomorphic to $X$ for all $n \in \mathbb{Z}$. This can be rephrased as saying that

$$X \in \bigcap_{n \in \mathbb{N}} \mathcal{T}_{\leq -n}\,.$$

Now by our hypothesis there is an integer $N$ such that

$$\mathrm{Hom}_{\mathcal{T}}(C, \mathcal{U}_{\leq -N}) = 0\,.$$

Since $C$ is compact and by construction of $\mathcal{T}_{\leq -N}$, we have

$$\mathrm{Hom}_{\mathcal{T}}(C, \mathcal{T}_{\leq -N}) = 0\,.$$

This implies that $\mathrm{Hom}_{\mathcal{T}}(C, X) = 0$, for all compact objects $C$ of $\mathcal{T}$. Since $\mathcal{T}$ is compactly generated, we conclude that $X = 0$. This proves the converse. $\surd$

Lemma A.1. *Let $(Y_p)_{p \in \mathbb{Z}}$ be in $\mathcal{T}$. We dispose of the following isomorphism*

$$\mathrm{H}^n\,(\coprod_p Y_p) \xleftarrow{\sim} \coprod_p \mathrm{H}^n(Y_p)\,,$$

*for all $n \in \mathbb{Z}$.*

*Proof.* By definition $\mathrm{H}^n := \tau_{\geq n}\,\tau_{\leq n}\,, n \in \mathbb{Z}$. Since $\tau_{\geq n}$ admits a right adjoint, it is enough to show that $\tau_{\leq n}$ commute with infinite sums. We consider the following triangle

$$\coprod_p \tau_{\leq n}Y_p \to \coprod_p Y_p \to \coprod_p \tau_{>n}Y_p \to S(\coprod_p \tau_{\leq n}Y_p)\,.$$

Here $\coprod_p \tau_{\leq n}Y_p$ belongs to $\mathcal{T}_{\leq n}$ since $\mathcal{T}_{\leq n}$ is stable under infinite sums. Let $P$ be an object of $S^n\mathcal{U}$. Since $P$ is compact, we have

$$\mathrm{Hom}_{\mathcal{T}}(P, \coprod_p \tau_{>n}Y_p) \xleftarrow{\sim} \coprod_p \mathrm{Hom}_{\mathcal{T}}(P, \tau_{>n}Y_p) = 0\,.$$

Since $\mathcal{T}_{\leq n}$ is generated by $S^n\mathcal{U}$, $\coprod_i \tau_{>n}Y_p$ belongs to $\mathcal{T}_{>n}$. Since the truncation triangle of $\coprod_p Y_p$ is unique, this implies the following isomorphism

$$\coprod_p \tau_{\leq n}Y_p \xrightarrow{\sim} \tau_{\leq n}(\coprod_p Y_p)\,.$$

This proves the lemma. $\surd$

Proposition A.2. *Let $X$ be an object of $\mathcal{T}$. Suppose that we are in the conditions of proposition A.1 b). We dispose of the following isomorphism*

$$\operatorname*{hocolim}_i \tau_{\leq i}X \xrightarrow{\sim} X\,.$$

*Proof.* We need only show that

$$\mathrm{H}^n(\operatorname{hocolim}_i \tau_{\leq i} X) \xrightarrow{\sim} \mathrm{H}^n(X)\,,$$

for all $n \in \mathbb{Z}$. We dispose of the following triangle, *cf.* [29],

$$\coprod_p \tau_{\leq p} X \to \coprod_q \tau_{\leq q} X \to \operatorname{hocolim}_i \tau_{\leq i} X \to S(\coprod_p \tau_{\leq p} X)\,.$$

Since the functor $\mathrm{H}^n$ is homological, for all $n \in \mathbb{Z}$ and it commutes with infinite sums by lemma A.1, we obtain a long exact sequence

$$\cdots \to \coprod_p \mathrm{H}^n(\tau_{\leq p} X) \to \coprod_q \mathrm{H}^n(\tau_{\leq q} X) \to \mathrm{H}^n\,(\operatorname{hocolim}_i \tau_{\leq i} X) \to$$
$$\to \coprod_p \mathrm{H}^n\, S(\tau_{\leq p} X) \to \coprod_q \mathrm{H}^n\, S(\tau_{\leq q} X) \to \cdots$$

We remark that the morphism $\coprod_p \mathrm{H}^n\, S(\tau_{\leq p} X) \to \coprod_q \mathrm{H}^n\, S(\tau_{\leq q} X)$ is a split monomorphism and so we obtain

$$\mathrm{H}^n(X) = \operatorname{colim}_i \mathrm{H}^n(\tau_{\leq i} X) \xrightarrow{\sim} \mathrm{H}^n(\operatorname{hocolim}_i \tau_{\leq i} X)\,.$$

$\surd$

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# The Hirzebruch-Mumford Volume for the Orthogonal Group and Applications

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Abstract. In this paper we derive an explicit formula for the Hirzebruch-Mumford volume of an indefinite lattice $L$ of rank $\geq 3$. If $\Gamma \subset \mathrm{O}(L)$ is an arithmetic subgroup and $L$ has signature $(2, n)$, then an application of Hirzebruch-Mumford proportionality allows us to determine the leading term of the growth of the dimension of the spaces $S_k(\Gamma)$ of cusp forms of weight $k$, as $k$ goes to infinity. We compute this in a number of examples, which are important for geometric applications.

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## 0 Introduction

In [Hi1] and [Hi2] Hirzebruch considered compact quotients of a homogeneous domain by an arithmetic group. He observed that the Chern numbers of such quotients are proportional to the Chern numbers of the compact duals of the homogeneous domains, and he also showed how the proportionality factor can be used to compute the dimension of spaces of automorphic forms. Later Mumford [Mum] extended Hirzebruch's approach to the case where the quotient is no longer compact, but only of finite volume. In this case the space of cusp forms of weight $k$ with respect to some arithmetic group $\Gamma$ grows asymptotically proportional to the dimension of the space of sections of the $(1-k)$-th power of the canonical bundle of the compact dual (for a precise formulation see Theorem 1.1). We call the proportionality constant the *Hirzebruch-Mumford volume.* Thus a computation of the Hirzebruch-Mumford volume for a given group $\Gamma$ gives the leading term of the Hilbert polynomial of forms of weight $k$

with respect to $\Gamma$. Knowledge of this term is essential for many geometric applications, in particular when one considers the Kodaira dimension of modular varieties.

The subject starts with the seminal work of Siegel [Sie1] on the volume of the orthogonal group. Very many authors have taken up his theory and generalised it in many different directions, including Harder [Ha], Serre [Se], Prasad [Pr] and many others. Our specific interest lies in indefinite orthogonal groups (see the work by Shimura [Sh], Gross [Gr], Gan, Hanke and Yu [GHY], as well as Belolipetsky and Gan [BG], to name some important recent work in this direction). Motivated by possible applications (cf. [GHS1], [GHS2]) concerning moduli spaces of K3 surfaces and similar modular varieties we started to investigate the volume of certain arithmetic subgroups of orthogonal groups $\mathrm{O}(L)$ of even indefinite lattices of signature $(2, n)$. All our groups are defined over the rational numbers, but for the applications we have in mind we cannot restrict ourselves to unimodular or maximal lattices. To our knowledge there exist no results in the literature that allow an easy calculation of the Hirzebruch-Mumford volume for the groups we treat in this paper.

In order to compute these volumes we therefore decided to return to Siegel's work. Let $L$ be an even indefinite lattice of signature $(2, n)$ and let $\mathrm{O}(L)$ be its group of isometries. The lattice $L$ defines a domain

$$\Omega_L = \{[\mathbf{w}] \in \mathbb{P}(L \otimes \mathbb{C});\ (\mathbf{w}, \mathbf{w})_L = 0,\ (\mathbf{w}, \overline{\mathbf{w}})_L > 0\}.$$

This domain has two connected components $\mathcal{D}_L$ and $\mathcal{D}'_L$, which are interchanged by complex conjugation, where $\mathcal{D}_L = \mathrm{O}(2, n)/\mathrm{O}(2) \times \mathrm{O}(n)$. Let $\mathrm{O}^+(L)$ be the index 2 subgroup of $\mathrm{O}(L)$ which fixes $\mathcal{D}_L$. The fundamental problem of our paper is to determine the Hirzebruch-Mumford volume of this group. For this one has to compare the volume of the quotient $\mathrm{O}^+(L)\backslash\mathcal{D}_L$ to the volume of the compact dual $\mathcal{D}_L^{(c)} = \mathrm{O}(2 + n)/\mathrm{O}(2) \times \mathrm{O}(n)$. To do so correctly, one has to choose volume forms on the domain $\mathcal{D}_L$ and the compact dual $\mathcal{D}_L^{(c)}$ that coincide at the common point of both domains given by a maximal compact subgroup. This is in fact a problem which does not depend on the complex structure of the domains, but can be considered in greater generality for indefinite lattices of signature $(r, s)$. We use the volume form on $\mathcal{D}_L$ which was introduced by Siegel. It then turns out that this must be compared to the volume form on $\mathcal{D}_L^{(c)}$ which is given by $1/2$ of the volume form induced by the Killing form on the Lie algebra of the group $\mathrm{SO}(r + s)$. Comparing these two volumes gives us the main formula for the Hirzebruch-Mumford volume of $\mathrm{O}^+(L)$. This formula involves the Tamagawa (Haar) measure of the group $\mathrm{O}(L)$. However, again using a result of Siegel, the computation of the Tamagawa measure can be reduced to computing the local densities $\alpha_p(L)$ of the lattice $L$ over the $p$-adic integers. Our main formula for any indefinite lattice $L$ of rank $\rho \geq 3$ is

$$\mathrm{vol}_{HM}(\mathrm{O}(L)) = \frac{2}{g^+_{sp}}\,|\det L|^{(\rho+1)/2} \prod_{k=1}^{\rho} \pi^{-k/2}\Gamma(k/2) \prod_p \alpha_p(L)^{-1}$$

where $g_{sp}^+$ is the number of the proper spinor genera in the genus of $L$ (see Theorem 2.1). Since everything is defined over the rationals, one can use Kitaoka's book [Ki] on quadratic forms to compute the local densities in question.
In order to illustrate our results, and particularly in view of applications, we compute the Hirzebruch-Mumford volume for several examples. The lattices and the groups which we consider are mostly related to moduli problems. We start with a series of even unimodular lattices, namely the lattices $II_{2,2m+8} = 2U \oplus mE_8(-1)$, where $U$ denotes the hyperbolic plane and $E_8$ is the positive definite root lattice associated to $E_8$. The next series of examples consists of the lattices $L_{2d}^{(m)} = 2U \oplus mE_8(-1) \oplus \langle -2d \rangle$, which are closely related to well known moduli problems. Let

$$\mathcal{F}_{2d}^{(m)} = \widetilde{\mathrm{O}}^+(L_{2d}^{(m)}) \backslash \mathcal{D}_{L_{2d}^{(m)}}$$

where $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$ is the subgroup of $\mathrm{O}^+(L_{2d}^{(m)})$ which acts trivially on the discriminant group. For $m = 0$ and $d$ a prime number, $\mathcal{F}_{2d}^{(0)}$ is a moduli space of Kummer surfaces (see [GH]). The spaces $\mathcal{F}_{2d}^{(1)}$ parametrise certain lattice-polarised K3 surfaces and if $m = 2$, then $\mathcal{F}_{2d} = \mathcal{F}_{2d}^{(2)}$ is the moduli space of K3 surfaces of degree $2d$. We compute the Hirzebruch-Mumford volumes of the groups $\mathrm{O}^+(L_{2d}^{(m)})$ and $\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})$ and obtain as a corollary the leading term controlling the growth behaviour of the dimension of the spaces of cusp forms for these groups. As a specialisation of this example we recover known formulae for the Siegel modular group in genus 2 and the paramodular group. Other series of examples considered in this paper, namely the even indefinite unimodular lattices (Section 3.3), their sublattices $T$ (Section 3.4) and some lattices of signature $(2, 8m+2)$ (Section 3.6), are closely related to moduli of K3 surfaces and related quotients of homogeneous varieties of type IV. The volumes of these lattices determine the part of the obstruction for extending pluricanonical differential forms on $\mathcal{F}_{2d}^{(m)}$ to a smooth compactification of this variety which comes from the ramification divisor.
In [GHS2] we use these results to obtain information about the Kodaira dimension of two series of modular varieties, including effective bounds on the degree $d$ which guarantee that the varieties $\mathcal{F}_{2d}^{(m)}$ are of general type. The case of polarised K3 surfaces is considered in [GHS1].
The paper is organised as follows: in Section 1 we recall Hirzebruch-Mumford proportionality and the Hirzebruch-Mumford volume in the form in which we need it (see Theorem 1.1 and Corollary 1.2). In Section 2 we perform the necessary volume computations and derive the main formula (see Theorem 2.1). In Section 3 we treat in some detail several lattices which appear naturally in moduli problems.

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## 1 Hirzebruch-Mumford proportionality

In this section we consider an indefinite even lattice $L$ of signature $(2, n)$. Let $\mathrm{O}(L)$ be its group of isometries. We denote by $( \ , \ )_L$ the form defined on $L$, extended bilinearly to $L \otimes \mathbb{R}$ and $L \otimes \mathbb{C}$. The domain

$$\Omega_L = \{[\mathbf{w}] \in \mathbb{P}(L \otimes \mathbb{C});\ (\mathbf{w}, \mathbf{w})_L = 0,\ (\mathbf{w}, \overline{\mathbf{w}})_L > 0\}$$

has two connected components, say $\Omega_L = \mathcal{D}_L \cup \mathcal{D}'_L$, which are interchanged by complex conjugation. By $\mathcal{D}_L^\bullet$ we denote the affine cone over $\mathcal{D}_L$ in $L \otimes \mathbb{C}$. Let $\Gamma \subset \mathrm{O}(L)$ be an arithmetic group which leaves the domain $\mathcal{D}_L$ invariant. A *modular form* of *weight* $k$ with respect to the group $\Gamma$ and with a (finite order) *character* $\chi : \Gamma \to \mathbb{C}^*$ is a holomorphic map

$$f : \mathcal{D}_L^\bullet \to \mathbb{C}$$

which has the two properties

$$\begin{array}{lcll} f(tz) & = & t^{-k} f(z) & \text{for } t \in \mathbb{C}^*, \\ f(g(z)) & = & \chi(g) f(z) & \text{for } g \in \Gamma. \end{array}$$

If $n \leq 2$ the function $f(z)$ must also be required to be holomorphic at infinity. A *cusp form* is a modular form which vanishes on the boundary.
We denote the spaces of modular forms and of cusp forms of weight $k$, with respect to the group $\Gamma$ and character $\chi$, by $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ respectively. These are finite dimensional vector spaces. Note that if $-\mathrm{id} \in \Gamma$ and $(-1)^k \neq \chi(-\mathrm{id})$ then obviously $M_k(\Gamma, \chi) = 0$.
Modular forms can be interpreted as sections of suitable line bundles. For this, we first assume that the group $\Gamma$ is neat, in which case it acts freely on $\mathcal{D}_L$, and we also assume that the character $\chi$ is trivial. Then the transformation rules of modular forms of weight 1 define a line bundle $\mathcal{L}$ on the quotient $\Gamma \backslash \mathcal{D}_L$ and modular forms of weight $k$ with trivial character become sections in $\mathcal{L}^{\otimes k}$. The line bundle $\mathcal{L}$, and its sections, extend to the Baily-Borel compactification $\overline{\Gamma \backslash \mathcal{D}_L}$. In fact, the Baily-Borel compactification is the normal projective variety associated to $\mathrm{Proj}\left(\bigoplus_k H^0(\mathcal{L}^{\otimes k})\right)$. In general, modular forms of weight $k$ and with a character $\chi$ define sections of a line bundle $\mathcal{L}_{k,\chi}$ which differs from $\mathcal{L}^{\otimes k}$ only by torsion.

Every toroidal compactification $(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}}$ has a morphism $(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}} \to \overline{\Gamma\backslash\mathcal{D}_L}$ which is the identity on $\Gamma\backslash\mathcal{D}_L$. Via this morphism, we shall also consider $\mathcal{L}$ and $\mathcal{L}_{k,\chi}$ as line bundles on $(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}}$, using the same symbol by abuse of notation. If $\Gamma$ is not neat then the above remains true, as long as we consider $\mathcal{L}$ and $\mathcal{L}_{k,\chi}$ as $\mathbb{Q}$-line bundles or only consider weights $k$ that are sufficiently divisible.
The connection with pluricanonical forms is as follows. There is an $n$-form $dZ$ on $\mathcal{D}_L$ such that if $f$ is a modular form of weight $n = \dim \mathcal{D}_L$ with character det, then $\omega = f dZ$ is a $\Gamma$-invariant $n$-form on $\mathcal{D}_L$. Hence, if the action of $\Gamma$ on $\mathcal{D}_L$ is free, $\omega$ descends to an $n$-form on $\Gamma\backslash\mathcal{D}_L$. Similarly, modular forms of weight $kn$ with character $\det^k$ define $k$-fold pluricanonical forms on $\Gamma\backslash\mathcal{D}_L$. If $\Gamma$ does not act freely, then this is still true outside the ramification locus of the quotient map $\mathcal{D}_L \to \Gamma\backslash\mathcal{D}_L$. These forms will, in general, not extend to compactifications of $\Gamma\backslash\mathcal{D}_L$. If $\Gamma$ is a neat group, then let $(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}}$ be a smooth toroidal compactification (which always exists by [AMRT]). Let $D$ be the boundary of such a toroidal compactification. If $\mathcal{L}_{n,\det}$ is the line bundle of modular forms of weight $n$ and character det, then the canonical bundle is given by $\omega_{(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}}} = \mathcal{L}_{n,\det} \otimes \mathcal{O}_{(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}}}(-D)$. Hence, if $f$ is a weight $n$ form with character det, not vanishing at the boundary, then $f dZ$ defines an $n$-form on $(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}}$ with poles along the boundary. However, if $f$ is a cusp form, then $f dZ$ does define an $n$-form on $(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}}$, and similarly forms of weight $kn$ and character $\det^k$ that vanish along the boundary of order $k$ define $k$-fold pluricanonical forms on $(\Gamma\backslash\mathcal{D}_L)^{\mathrm{tor}}$. It should be pointed out that some authors define automorphic forms a priori as those functions that give rise to pluricanonical forms. In our context, this means a restriction to forms of weight $kn$. Moreover, the weight of these forms is sometimes defined as $k$. We shall refer to the latter as the *geometric* weight, in contrast to the *arithmetic* weight of our definition. This difference accounts for the fact that some of our formulae differ from corresponding formulae in the literature by powers of $n$.
The Hirzebruch-Mumford proportionality principle, which works very generally for quotients of a homogeneous domain $\mathcal{D}$ by an arithmetic group $\Gamma$, allows us to estimate the growth behaviour of spaces of cusp forms as a function of the weight $k$ in terms of a suitably defined volume. This was first discovered by Hirzebruch [Hi1], [Hi2] in the case where the quotient $\Gamma\backslash\mathcal{D}$ is compact, and was generalised by Mumford [Mum] to the case where $\Gamma\backslash\mathcal{D}$ has finite volume. We denote the compact dual of $\mathcal{D}$ by $\mathcal{D}^{(c)}$. Let $\overline{X}$ be the Baily-Borel compactification of $X = \Gamma\backslash\mathcal{D}$ and let $X^{\mathrm{tor}}$ be some smooth toroidal compactification of $X$.

Theorem 1.1 *Let $\Gamma$ be a neat arithmetic group which acts on a bounded symmetric domain $\mathcal{D}$. Let $S_k^{\mathrm{geom}}(\Gamma) = S_{nk}(\Gamma, \det^k)$ be the space of cusp forms of geometric weight $k$ with respect to $\Gamma$. Then*

$$\dim S_k^{\mathrm{geom}}(\Gamma) = \mathrm{vol}_{HM}(\Gamma) h^0(\omega_{\mathcal{D}^{(c)}}^{(1-k)}) + P_1(k)$$

*where $P_1(k)$ is a polynomial whose degree is at most the dimension of $\overline{X}\backslash X$.*

*Proof.* This is [Mum, Corollary 3.5]. □

Here $\mathrm{vol}_{HM}(\Gamma\backslash\mathcal{D})$ denotes a suitably normalised volume of the quotient $\Gamma\backslash\mathcal{D}$, which we shall refer to as the *Hirzebruch-Mumford volume*. If $\Gamma$ acts freely, then the Hirzebruch-Mumford volume is a quotient of Euler numbers

$$\mathrm{vol}_{HM}(\Gamma) = \mathrm{vol}_{HM}(\Gamma\backslash\mathcal{D}) = \frac{e(\Gamma\backslash\mathcal{D})}{e(\mathcal{D}^{(c)})}.$$

If $\Gamma$ does not act freely, then choose a normal subgroup $\Gamma' \lhd \Gamma$ of finite index which does act freely. Then

$$\mathrm{vol}_{HM}(\Gamma) = \frac{\mathrm{vol}_{HM}(\Gamma')}{[\mathrm{P}\Gamma : \Gamma']}$$

where $\mathrm{P}\Gamma$ is the image of $\Gamma$ in $\mathrm{Aut}(\mathcal{D})$, i.e. the group $\Gamma$ modulo its centre. This value is independent of the choice of the subgroup $\Gamma'$.

Hirzebruch [Hi1] first formulated his result in the case where the group is co-compact, i.e., where the quotient $X = \Gamma\backslash\mathcal{D}$ is compact. Since the Chern numbers of $X$ and that of the compact dual are proportional and the factor of proportionality is given by the volume, one can use Riemann-Roch to compute the exact dimension of the space of modular foms (in this case it does not make sense to talk about cusp forms).

We shall now apply this to orthogonal groups.

Proposition 1.2 *Let $L$ be an indefinite even lattice of signature $(2,n)$ and let $\Gamma$ be an arithmetic subgroup which acts on the domain $\mathcal{D}_L$. Fix a positive integer $k$ and a character $\chi$. If $-\mathrm{id} \in \Gamma$, then we restrict to those $k$ for which $(-1)^k = \chi(-\mathrm{id})$. Then the dimension of the space $S_k(\Gamma,\chi)$ of cusp forms of arithmetic weight $k$ grows as*

$$\dim S_k(\Gamma,\chi) = \frac{2}{n!}\,\mathrm{vol}_{HM}(\Gamma\backslash\mathcal{D}_L)k^n + O(k^{n-1}).$$

*Proof.* We shall first assume that $\Gamma$ is neat (in which case automatically $-\mathrm{id} \notin \Gamma$) and that $\chi$ is trivial. We consider $\mathcal{L}$ as a line bundle on a smooth toroidal compactification $X^{\mathrm{tor}}$ of $X = \Gamma\backslash\mathcal{D}_L$. It follows from the definition of cusp forms that $H^0(X^{\mathrm{tor}},\mathcal{L}^{\otimes k}(-D)) = S_k(\Gamma)$. Since $\mathcal{L}$ is big and nef and $K_{X^{\mathrm{tor}}} = \mathcal{L}^{\otimes n}(-D)$, it follows from Kawamata-Viehweg vanishing that $h^i(X^{\mathrm{tor}},\mathcal{L}^{\otimes k}(-D)) = 0$ for $i \geq 1$ and $k \gg 0$ and hence $\chi(X^{\mathrm{tor}},\mathcal{L}^{\otimes k}(-D)) = h^0(X^{\mathrm{tor}},\mathcal{L}^{\otimes k}(-D))$ for $k \gg 0$. The leading term of the Riemann-Roch polynomial as a function of $k$ is given by $c_1^n(\mathcal{L})/n!$. The same argument goes through for $\mathcal{L}_{k,\chi}$. Since $\mathcal{L}^{\otimes k}$ and $\mathcal{L}_{k,\chi}$ only differ by torsion they have the same leading coefficients.

In order to apply Theorem 1.1 we consider the line bundle $\mathcal{L}_{n,\det}$ of modular forms of weight $n$ and character det. Note that $\mathcal{L}^k_{n,\det} = \mathcal{L}^{nk}$ for suitably divisible $k$. Also recall that in the orthogonal case the compact dual $\mathcal{D}^{(c)}$

is the complex $n$-dimensional quadric $Q_n \subset \mathbb{P}^{n+1}$ whose canonical bundle is $\omega_{Q_n} = \mathcal{O}_{Q_n}(-n)$ and it follows from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(n(k-1)-2) \to \mathcal{O}_{\mathbb{P}^{n+1}}(n(k-1)) \to \omega_{Q_n}^{(1-k)} \to 0$$

that the leading term of $h^0(\omega_{\mathcal{D}^{(c)}}^{(1-k)})$ is equal to $2n^n/n!$. It then follows from Hirzebruch-Mumford proportionality that

$$\frac{c_1^n(\mathcal{L}_{n,\det}^n)}{n!} = \frac{2n^n}{n!} \operatorname{vol}_{HM}(X)$$

and hence

$$\frac{c_1^n(\mathcal{L})}{n!} = \frac{2}{n!} \operatorname{vol}_{HM}(X)$$

which gives the claim in the case of a neat group.

We now consider a group $\Gamma$ which is not necessarily neat and choose $\Gamma' \lhd \Gamma$ neat and of finite index. The group $\Gamma$ acts on the total space of the line bundle $\mathcal{L}$, and if $-\operatorname{id} \in \Gamma$ then it follows from our assumptions on $k$ that this element acts trivially. We can now apply the Lefschetz fixed point formula (cf. [T, Appendix to §2]), from which we obtain

$$\begin{aligned}
\dim S_k(\Gamma) &= \dim S_k(\Gamma')^{\Gamma} \\
&= \frac{1}{[\mathrm{P}\Gamma : \Gamma']} \cdot \sum_{\gamma \in \mathrm{P}\Gamma/\Gamma'} \operatorname{tr}\left(\gamma|_{S_k(\Gamma')}\right) \\
&= \frac{1}{[\mathrm{P}\Gamma : \Gamma']} \dim S_k(\Gamma') + O(k^{n-1}) \\
&= \frac{1}{[\mathrm{P}\Gamma : \Gamma']} \operatorname{vol}_{HM}(\Gamma' \backslash \mathcal{D}_L) \frac{2}{n!} k^n + O(k^{n-1}) \\
&= \frac{2}{n!} \operatorname{vol}_{HM}(\Gamma \backslash \mathcal{D}_L) k^n + O(k^{n-1}).
\end{aligned}$$

□

Note that the growth behaviour of the space of modular forms of weight $k$ and that of the space of cusp forms are the same. This follows from the exact sequence

$$0 \to \mathcal{L}^{\otimes k}(-D) \to \mathcal{L}^{\otimes k} \to \mathcal{L}^{\otimes k}|_D \to 0.$$

## 2 Computation of volumes

In order to compute the leading coefficient that determines the growth of the dimension of spaces of cusp forms, we have to compare the volume of a fundamental domain of an arithmetic group $\Gamma$ to the volume of the compact dual. For this, the complex structure is not important and we therefore consider, more generally, an indefinite integral lattice $L$ of signature $(r, s)$.

As before, we denote the group of isometries of the lattice $L$ by $\mathrm{O}(L)$. The lattice $L$ defines a homogeneous domain $\mathcal{D}_{rs}$. In terms of groups the domain $\mathcal{D}_{rs}$ is the quotient of the orthogonal group $\mathrm{O}(L\otimes\mathbb{R})$ by a maximal compact subgroup, i.e.,

$$\mathcal{D}_{rs}=\mathcal{D}_L=\mathrm{O}(r,s)/\,\mathrm{O}(r)\times\mathrm{O}(s)=\mathrm{SO}(r,s)_0/\,\mathrm{SO}(r)\times\mathrm{SO}(s)$$

where all groups are real Lie groups and $\mathrm{SO}(r,s)_0$ is the connected component of the identity of $\mathrm{SO}(r,s)$.
The domain $\mathcal{D}_{rs}$ can be realised as a bounded domain in the form

$$\mathcal{D}_{rs}=\{X\in\mathrm{Mat}_{r\times s}(\mathbb{R})\,;\,I_r-X\,{}^tX>0\}$$

where $I_r\in\mathrm{Mat}_{r\times r}(\mathbb{R})$ is the identity matrix and the action of the orthogonal group is given in the usual form, namely by

$$M(X)=(AX+B)(CX+D)^{-1}$$

for

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{O}(r,s),\qquad A\in\mathrm{Mat}_{r\times r}(\mathbb{R}),\ D\in\mathrm{Mat}_{s\times s}(\mathbb{R}).$$

We consider the $\mathrm{O}(r,s)$-invariant metric given by

$$ds^2=\mathrm{tr}\big((I_r-X\,{}^tX)^{-1}\,dX\,(I_s-{}^tXX)^{-1}\,d\,{}^tX\big).$$

Since

$$\det((I_r-X\,{}^tX)^{-1})^s\cdot\det((I_s-{}^tXX)^{-1})^r=\det((I_r-X\,{}^tX)^{-1})^{r+s}$$

the corresponding volume form is given by

$$dV=(\det(I_r-X\,{}^tX)^{-1})^{\frac{r+s}{2}}\prod_{i,j}dx_{ij}.$$

Siegel computed the volume of $\mathcal{D}_{rs}$ with respect to this volume form in [Sie2] (see also [Sie3, Theorem 7, p. 155]). His result is

$$\mathrm{vol}_S(\mathrm{O}(L))=\mathrm{vol}_S(\mathrm{O}(L)\backslash\mathcal{D}_{rs})=2\alpha_\infty(L)|\det L|^{(r+s+1)/2}\,\gamma_r^{-1}\gamma_s^{-1}\,,\qquad(1)$$

where

$$\gamma_m=\prod_{k=1}^{m}\pi^{k/2}\,\Gamma(k/2)^{-1}\qquad(2)$$

and $\alpha_\infty(L)$ is the real Tamagawa (Haar) measure of the lattice $L$. Formula (1) is valid for any indefinite lattice $L$ of rank $\geq 3$. As indicated by the subscript, we shall refer to this volume as the *Siegel volume* of the group $\mathrm{O}(L)$.

We want to understand the Siegel metric in terms of Lie algebras. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the indefinite orthogonal group $\mathrm{O}(r,s)$ and its maximal compact subgroup $\mathrm{O}(r)\times\mathrm{O}(s)$ respectively. Then

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. By [He, p. 239] this is isomorphic to

$$\mathfrak{p}=\left\{\begin{pmatrix}0 & U\\ {}^tU & 0\end{pmatrix};\quad U\in\mathrm{Mat}_{r\times s}(\mathbb{R})\right\}.$$

The space $\mathfrak{p}$ is isomorphic to the tangent space of $\mathcal{D}_{rs}$ at 0. A straightforward calculation shows that the $\mathrm{O}(r,s)$-invariant metric $ds^2$ is induced by the Killing functional $\mathrm{tr}(U_1{}^tU_2)$ on the tangent space at 0.
We now want to compare this to a suitable volume form on the compact dual. Recall that the general situation is as follows. Let $H$ be a bounded homogeneous domain and $G=\mathrm{Aut}(H)_0$ be the connected component of the identity of the group of automorphisms of $H$. In particular, $H=G/K$ where $K=G_{z_0}$ is the stabiliser of some point $z_0$. There exists a unique compact real form $G_u$ of the complex group $G_{\mathbb{C}}$ such that $G\cap G_u=K$ and the symmetric domain $H=G/K$ can be embedded into the compact manifold $\mathcal{D}^{(c)}=G_u/K$ as an open submanifold. In our situation

$$\mathcal{D}^{(c)}_{rs}=\mathrm{SO}(r+s)/\,\mathrm{SO}(r)\times\mathrm{SO}(s).$$

Again by [He, p.239] the tangent space of $\mathcal{D}^{(c)}_{rs}$ at the point $I_{r+s}$ is given by the subspace

$$\mathfrak{p}'=\left\{\begin{pmatrix}0 & U\\ -{}^tU & 0\end{pmatrix};\quad U\in\mathrm{Mat}_{r\times s}(\mathbb{R})\right\}$$

of the Lie algebra of $\mathrm{SO}(r+s)$. The Killing form $\mathrm{tr}(W_1{}^tW_2)$ of the Lie algebra of the compact group $\mathrm{SO}(r+s)$ induces the form $2\,\mathrm{tr}(U_1{}^tU_2)$ on the tangent space $\mathfrak{p}'$. In order to compare the volumes of $\mathcal{D}_{rs}$ and its compact dual $\mathcal{D}^{(c)}_{rs}$ we have to normalise this form in such a way that it coincides with the Siegel metric in the common base point $K\in\mathcal{D}_{rs}\subset\mathcal{D}^{(c)}_{rs}$, i.e. we have to use the form $\frac{1}{2}\,\mathrm{tr}(W_1{}^tW_2)$. Since the dimension of $\mathrm{SO}(n)$ is $\frac{1}{2}n(n-1)$, we get a factor $2^{-(r+s)(r+s-1)/4}$ in front of the volume of the compact group, calculated in terms of the volume form induced by the Killing functional on $\mathrm{SO}(r+s)$. The latter volume is computed in [Hua, §3.7]. Taking the above normalisation into account we find

$$\mathrm{vol}_S(\mathrm{SO}(m))=2^{m-1}\,\gamma_m \tag{3}$$

and we shall again refer to this volume as the Siegel volume. For the compact dual this gives

$$\mathrm{vol}_S(\mathcal{D}^{(c)}_{rs})=2\,\gamma_{r+s}\gamma_r^{-1}\gamma_s^{-1}. \tag{4}$$

Our aim is to compute the Hirzebruch-Mumford volume

$$\mathrm{vol}_{HM}(\mathrm{O}(L)) = \frac{\mathrm{vol}_S(\mathrm{O}(L)\backslash\mathcal{D}_{rs})}{\mathrm{vol}_S(\mathcal{D}_{rs}^{(c)})}. \tag{5}$$

To make the above equation effective, we have to determine the Tamagawa measure

$$\alpha_\infty(L) = \alpha_\infty(\mathrm{O}(L)\backslash \mathrm{O}(L\otimes\mathbb{R})) = \alpha_\infty(\mathrm{SO}(L)\backslash \mathrm{SO}(L\otimes\mathbb{R})).$$

The genus of the indefinite lattice $L$ contains a finite number $g_{sp}^+(L)$ of (proper) spinor genera (for a definition see [Ki, §6.3]). (We consider only proper classes and proper spinor genera.) This number is always a power of two and can be calculated effectively. It is well known that the spinor genus of an indefinite lattice of rank $\geq 3$ coincides with the class. As was proved by M. Kneser (see [Kn]) the weight of the representations of a given number $m$ by a spinor genus is the same for all genera in the genus of $L$. The same arguments show that all spinor genera in the genus have the same mass. (We are grateful to R. Schulze-Pillot for drawing our attention to this fact.) It is easy to see this in adelic terms. A spinor genus corresponds to a double class $\mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)\,b\,\mathrm{SO}_{\mathbb{A}}(L)$ in the adelic group $\mathrm{SO}_{\mathbb{A}}(V)$, where $V = L\otimes\mathbb{Q}$ is the rational quadratic space and

$$\mathrm{SO}'_{\mathbb{A}}(V) = \ker \mathrm{sn}\colon\ \mathrm{SO}_{\mathbb{A}}(V) \to \mathbb{Q}_{\mathbb{A}}^\times/(\mathbb{Q}_{\mathbb{A}}^\times)^2$$

is the kernel of the spinor norm. We note that the genus of $L$ is given by $\mathrm{SO}_{\mathbb{A}}(V)L$. It follows from the definition that the group $\mathrm{SO}'_{\mathbb{A}}(V)$ contains the commutator of $\mathrm{SO}_{\mathbb{A}}(V)$, therefore

$$\mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)\,b\,\mathrm{SO}_{\mathbb{A}}(L) = \mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)\,\mathrm{SO}_{\mathbb{A}}(L)\,b.$$

The mass of a spinor genus

$$\tau(\mathrm{SO}(V)\setminus \mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)b\,\mathrm{SO}_{\mathbb{A}}(L)) = \tau(\mathrm{SO}(V)\setminus \mathrm{SO}(V)\,\mathrm{SO}'_{\mathbb{A}}(V)\,\mathrm{SO}_{\mathbb{A}}(L))$$

depends only on the genus, since the Tamagawa measure is invariant. The Tamagawa number of the orthogonal group is 2 (see [Sie1], [W], [Sh]), i.e., $\tau(\mathrm{SO}(V)\backslash\mathrm{SO}_{\mathbb{A}}(V)) = 2$. Then the Tamagawa measure $\alpha_\infty(L)$ can be computed via the local densities of the lattices $L\otimes\mathbb{Z}_p$ over the $p$-adic integers $\mathbb{Z}_p$ (the local Tamagawa measures). More precisely,

$$\alpha_\infty(L) = \alpha_\infty(\mathrm{SO}(L)\setminus \mathrm{SO}(L\otimes\mathbb{R})) = \frac{2}{g_{sp}^+(L)}\prod_p \alpha_p(L)^{-1}, \tag{6}$$

where $p$ runs through all prime numbers and $g_{sp}^+(L)$ is the number of spinor genera in the genus of $L$. The local densities can be computed, at least for quadratic forms over $\mathbb{Q}$ and its quadratic extensions: see [Ki]. In order to find $\alpha_p(L)$ it is enough to know the Jordan decomposition of $L$ over the $p$-adic integers.

We can now summarise our results as follows.

THEOREM 2.1 (MAIN FORMULA) *Let $L$ be an indefinite lattice of rank $\rho \geq 3$. Then the Hirzebruch-Mumford volume of* $\mathrm{O}(L)$ *equals*

$$\mathrm{vol}_{HM}(\mathrm{O}(L)) = \frac{2}{g_{sp}^{+}(L)} \cdot |\det L|^{(\rho+1)/2} \prod_{k=1}^{\rho} \pi^{-k/2}\Gamma(k/2) \prod_{p} \alpha_p(L)^{-1} \qquad (7)$$

*where the $\alpha_p(L)$ are the local densities of the lattice $L$ and $g_{sp}^{+}(L)$ is the number of spinor genera in the genus of $L$.*

*Proof.* This follows immediately from formulae (1), (2), (5) and (6). □

## 3 APPLICATIONS

In this section we want to apply the above results to compute the asymptotic behaviour of the dimension of spaces of cusp forms for a number of specific groups. The main applications have to do with locally symmetric varieties. In [GHS1] we prove general type results for the moduli spaces $\mathcal{F}_{2d}$ of K3 surfaces of degree $2d$, but in that special case we can use a different method. The results we have here are used in [GHS2] to prove similar results in greater generality.

### 3.1 GROUPS

We first have to clarify the various groups which will play a role. In this section, $L$ will be an even indefinite lattice of signature $(2, n)$, containing at least one hyperbolic plane as a direct summand. By a classical result of Kneser we know that if the genus of an indefinite lattice $L$ contains more than one class, then there is a prime $p$ such that the quadratic form of $L$ can be diagonalised over the $p$-adic numbers and the diagonal entries all involve distinct powers of $p$ (see [CS, Chapter 15]). Therefore the genus of any indefinite lattice with one hyperbolic plane contains only one class.
As an immediate corollary of Theorem 2.1 we obtain

THEOREM 3.1 *Let $L$ be a lattice of signature $(2, n)$ ($n \geq 1$) containing at least one hyperbolic plane. Let $\Gamma$ be an arithmetic subgroup of* $\mathrm{O}(L)$. *Then*

$$\mathrm{vol}_{HM}(\Gamma) = 2 \cdot [\mathrm{PO}(L) : \mathrm{P}\Gamma] |\det L|^{(n+3)/2} \prod_{k=1}^{n+2} \pi^{-k/2}\Gamma(k/2) \prod_{p} \alpha_p(L)^{-1}. \quad (8)$$

REMARK. In many interesting cases a subgroup $\Gamma$ is given in terms of the orthogonal group of some sublattice $L_1$ of $L$. In this case one can use the volume in order to calculate the index (see Section 3.4 below).
We shall now discuss the various groups which are of importance to us and compute their indices in $\mathrm{O}(L)$. The group $\mathrm{O}(L)$ interchanges the two connected components of the domain $\Omega_L$ and we define $\mathrm{O}^{+}(L)$ as the index 2 subgroup which fixes each of these components (as sets). This group can also be described

using the $(-1)$-*spinor norm* on the group $\mathrm{O}(L\otimes\mathbb{R})$ which is defined as follows. Every element $g$ can be represented as a product of reflections

$$g = \sigma_{v_1}\cdot\dots\cdot\sigma_{v_m}$$

and, following Brieskorn [Br], we define

$$\mathrm{sn}_{-1}(g) = \begin{cases} +1 & \text{if } (v_k, v_k) > 0 \;\text{ for an even number of } v_k \\ -1 & \text{otherwise.} \end{cases}$$

This is independent of the representation of $g$ as a product of reflections. It is well known that

$$\mathrm{O}^+(L) = \mathrm{Ker}(\mathrm{sn}_{-1}) \cap \mathrm{O}(L).$$

To see this, note that any reflection with respect to a vector of negative square has $(-1)$-spinor norm equal to 1, and any reflection with respect to a vector of positive square has $(-1)$-spinor norm equal to $-1$ and interchanges the two components. The Hirzebruch–Mumford volume of $\mathrm{O}^+(L)$ is twice that of $\mathrm{O}(L)$. Let $L^\vee = \mathrm{Hom}(L,\mathbb{Z})$ be the dual lattice and $A_L = L^\vee/L$. The finite group $A_L$ carries a discriminant quadratic form $q_L$ with values in $\mathbb{Q}/2\mathbb{Z}$ [Ni, 1.3]. By $\mathrm{O}(q_L)$ we denote the corresponding group of isometries and the group $\widetilde{\mathrm{O}}(L)$, called the *stable orthogonal group*, is defined as the kernel of the natural homomorphism $\mathrm{O}(L)\to\mathrm{O}(q_L)$. Since $L$ contains a hyperbolic plane, it follows from [Ni, Theorem 1.14.2] that this map is surjective. Set

$$\widetilde{\mathrm{O}}^+(L) = \widetilde{\mathrm{O}}(L)\cap\mathrm{O}^+(L).$$

Finally the groups $\mathrm{SO}^+(L)$ and $\widetilde{\mathrm{SO}}^+(L)$ are defined as the corresponding groups of isometries of determinant 1.

Lemma 3.2 *Let $D = |\,\mathrm{O}(q_L)|$. Then we have the following diagram of groups with indices as indicated:*

$$\begin{array}{ccc} \widetilde{\mathrm{O}}(L) & \overset{D:1}{\subset} & \mathrm{O}(L) \\ \bigcup\, {\scriptstyle 2:1} & & \bigcup\, {\scriptstyle 2:1} \\ \widetilde{\mathrm{O}}^+(L) & \overset{D:1}{\subset} & \mathrm{O}^+(L) \\ \bigcup\, {\scriptstyle 2:1} & & \bigcup\, {\scriptstyle 2:1} \\ \widetilde{\mathrm{SO}}^+(L) & \overset{D:1}{\subset} & \mathrm{SO}^+(L). \end{array}$$

*Proof.* We shall first prove that the indices of the vertical inclusions are all 2. To do this, we choose a hyperbolic plane $U$ in $L$, which exists by assumption. Let $e_1, e_2$ be a basis of $U$ with $e_1^2 = e_2^2 = 0$ and $e_1.e_2 = 1$. If $u = e_1 - e_2$, $v = e_1 + e_2$, then $u^2 = -2$, $v^2 = 2$ and the two reflections $\sigma_u$ and $\sigma_v$ belong to $\widetilde{\mathrm{O}}(L)$, since they act trivially on the orthogonal complement of $U$. Moreover $\mathrm{sn}_{-1}(\sigma_v) = -1$ and $\mathrm{sn}_{-1}(\sigma_u) = 1$. Hence we can use $\sigma_v$ to conclude that the

top two vertical inclusions are of index 2, whereas $\sigma_u$ shows the same for the bottom two vertical inclusions.
We have already observed that the natural map $\mathrm{O}(L) \to \mathrm{O}(q_L)$ is surjective, which shows that the top horizontal inclusion has index $D$. Taking into account that the reflections $\sigma_u$ and $\sigma_v$ act trivially on the discriminant form, we obtain that

$$D = [\mathrm{O}(L) : \widetilde{\mathrm{O}}(L)] = [\mathrm{O}^+(L) : \widetilde{\mathrm{O}}^+(L)] = [\mathrm{SO}^+(L) : \widetilde{\mathrm{SO}}^+(L)].$$

□

Finally, we want to consider the projective groups $\mathrm{PO}(L)$, $\mathrm{PO}^+(L)$ and $\mathrm{P}\widetilde{\mathrm{O}}^+(L)$, i.e., the corresponding groups modulo their centres. It follows immediately from the above diagram that

$$[\mathrm{PO}(L) : \mathrm{P}\widetilde{\mathrm{O}}^+(L)] = \begin{cases} D & \text{if } -\mathrm{id} \notin \widetilde{\mathrm{O}}^+(L) \\ 2D & \text{if } -\mathrm{id} \in \widetilde{\mathrm{O}}^+(L). \end{cases} \tag{9}$$

Note that $-\mathrm{id} \in \widetilde{\mathrm{O}}^+(L)$ if and only if $A_L$ is a 2-group.

### 3.2 Local densities

Siegel's definition of local densities of a quadratic form over a number field $K$ given by a matrix $S \in \mathrm{Mat}_{n\times n}(K)$ is

$$\alpha_p(S) = \frac{1}{2} \lim_{r\to\infty} p^{-\frac{rn(n-1)}{2}} \left| \left\{ X \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p) \bmod p^r;\ {}^tXSX \equiv S \bmod p^r \right\} \right|.$$

The local densities can be calculated explicitly, at least in the cases where $K = \mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$ (see chapter 5 of the book [Ki] and references there). For the convenience of the reader we include the formulae over $\mathbb{Q}$ in the present paper. To calculate $\alpha_p(L)$ one should know the Jordan decomposition of the lattice $L$ over the local ring $\mathbb{Z}_p$ of $p$-adic integers. The main difficulties arise for $p = 2$: see [Ki, Theorem 5.6.3].
Let us introduce some notation. Let $L$ be a $\mathbb{Z}_p$-lattice in a regular (i.e. non-degenerate) quadratic space over $\mathbb{Q}_p$ of rank $n$, and let $(\mathbf{v}_i)$ be a basis of $L$. There are two invariants of $L$: the scale

$$\mathrm{scale}(L) = \{(\mathbf{x}, \mathbf{y})_L\, ;\ \mathbf{x}, \mathbf{y} \in L\}$$

and the norm

$$\mathrm{norm}(L) = \{\textstyle\sum a_{\mathbf{x}} (\mathbf{x}, \mathbf{x})_L\, ;\ \mathbf{x} \in L,\ a_{\mathbf{x}} \in \mathbb{Z}_p\}.$$

We have $2\,\mathrm{scale}(L) \subset \mathrm{norm}(L) \subset \mathrm{scale}(L)$. In fact, over $\mathbb{Z}_p$ ($p \neq 2$) we have $\mathrm{norm}(L) = \mathrm{scale}(L)$, whereas over $\mathbb{Z}_2$ we have either $\mathrm{norm}(L) = \mathrm{scale}(L)$ or $\mathrm{norm}(L) = 2\,\mathrm{scale}(L)$.

$L$ is called $p^r$-modular, for $r \in \mathbb{Z}$, if the matrix $p^{-r}(\mathbf{v}_i, \mathbf{v}_j)_L$ belongs to $\mathrm{GL}_n(\mathbb{Z}_p)$. In this case we can write $L$ as the scaling $N(p^r)$ of a unimodular lattice $N$. By a hyperbolic space we mean a (possibly empty) orthogonal sum of hyperbolic planes.

A regular lattice $L$ decomposes as the orthogonal sum of lattices $\bigoplus_{j\in\mathbb{Z}} L_j$, where $L_j$ is a $p^j$-modular lattice of rank $n_j \in \mathbb{Z}_{\geq 0}$. Put

$$w = \sum_j jn_j\big((n_j+1)/2 + \sum_{k>j} n_k\big)$$

and

$$P_p(n) = \prod_{i=1}^{n}(1-p^{-2i}).$$

For a regular quadratic space $W$ over the finite field $\mathbb{Z}/p\mathbb{Z}$ one puts

$$\chi(W) = \begin{cases} 0 & \text{if } \dim W \text{ is odd,} \\ 1 & \text{if } W \text{ is a hyperbolic space,} \\ -1 & \text{otherwise.} \end{cases}$$

For a unimodular lattice $N$ over $\mathbb{Z}_2$ with $\mathrm{norm}(N) = 2\,\mathrm{scale}(N)$ we define $\chi(N) = \chi(N/2N)$, where $N/2N$ is given the structure of a regular quadratic space over $\mathbb{Z}/2\mathbb{Z}$ via the quadratic form $Q(\mathbf{x}) = \frac{1}{2}(\mathbf{x}, \mathbf{x})_N \bmod 2$.

For the local density $\alpha_p(L)$ for $p \neq 2$ we have the formula

$$\alpha_p(L) = 2^{s-1}p^w P_p(L)E_p(L) \tag{10}$$

where $s$ is the number of non-zero $p^j$-modular terms $L_j$ in the orthogonal decomposition of $L$, and

$$P_p(L) = \prod_j P_p([n_j/2]), \qquad E_p(L) = \prod_{j,\, L_j\neq 0}\big(1+\chi(N_j)p^{-n_j/2}\big)^{-1}$$

where $L_j$ is the $p^j$-scaling of the unimodular lattice $N_j$ and $[n_j/2]$ denotes the integer part.

The local density $\alpha_2(L)$ is given by

$$\alpha_2(L) = 2^{n-1+w-q}P_2(L)E_2(L). \tag{11}$$

In this formula $q = \sum_j q_j$ where

$$q_j = \begin{cases} 0 & \text{if } N_j \text{ is even,} \\ n_j & \text{if } N_j \text{ is odd and } N_{j+1} \text{ is even,} \\ n_j+1 & \text{if } N_j \text{ and } N_{j+1} \text{ are odd.} \end{cases}$$

A unimodular lattice $N$ over $\mathbb{Z}_2$ is even if it is trivial or if $\operatorname{norm}(N) = 2\mathbb{Z}_2$, and odd otherwise. Any unimodular lattice can be represented as the orthogonal sum $N = N^{\text{even}} \oplus N^{\text{odd}}$ of even and odd sublattices such that $\operatorname{rank} N^{\text{odd}} \leq 2$. Then we put

$$P_2(L) = \prod_j P_2(\operatorname{rank} N_j^{\text{even}}/2).$$

The second factor is $E_2(L) = \prod_j E_j^{-1}$, where $E_j$ is defined by

$$E_j = \frac{1}{2}(1 + \chi(N_j^{\text{even}}) 2^{-\operatorname{rank} N_j^{\text{even}}/2})$$

if both $N_{j-1}$ and $N_{j+1}$ are even, unless $N_j^{\text{odd}} \cong \langle \epsilon_1 \rangle \oplus \langle \epsilon_2 \rangle$ with $\epsilon_1 \equiv \epsilon_2 \mod 4$: in all other cases we put $E_j = 1/2$.
We note that $E_j$ depends on $N_{j-1}$, $N_j$ and $N_{j+1}$ and $E_j = 1$ if all of them are trivial. Also $q_j$ depends on $N_j$ and $N_{j+1}$ and $q_j = 0$ if $N_j$ is trivial.

### 3.3 The even unimodular lattices $II_{2,8m+2}$

We start with the example

$$II_{2,8m+2} = 2U \oplus mE_8(-1), \qquad \text{where} \quad m \geq 0$$

which is a natural series of even unimodular lattices of signature $(2, 8m+2)$. Note that $II_{2,26} \cong 2U \oplus \Lambda$, where $\Lambda$ is the Leech lattice.
The local densities are easy to calculate, since for every prime $p$ the lattice $II_{2,8m+2} \otimes \mathbb{Z}_p$ over the $p$-adic integers is a direct sum of hyperbolic planes. Then using (10) and (11) we obtain

$$\alpha_p(II_{2,8m+2}) = 2^{\delta_{2,p}(8m+4)} P_p(4m+2)(1 + p^{-(4m+2)})^{-1}$$

where $\delta_{2,p}$ is the Kronecker delta. By our main formula (8) from Theorem 3.1 we obtain

$$\operatorname{vol}_{HM}(\mathrm{O}^+(II_{2,8m+2})) = 2^{-(8m+2)} \gamma_{8m+4}^{-1} \zeta(2)\zeta(4) \cdot \ldots \cdot \zeta(8m+2)\zeta(4m+2)$$

where $\gamma_{8m+4}$ is as in formula (2). In order to simplify this expression we use the $\zeta$-identity

$$\pi^{-\frac{1}{2}-2k}\Gamma(k)\Gamma\left(k + \frac{1}{2}\right)\zeta(2k) = (-1)^k \zeta(1-2k) = (-1)^{k+1}\frac{B_{2k}}{2k}. \tag{12}$$

Together with

$$\begin{aligned} &\pi^{-(4m+2)}\Gamma(4m+2)\zeta(4m+2) \\ &\quad = \quad 2^{4m+1}\pi^{-\frac{1}{2}-(4m+2)}\Gamma(2m+1)\Gamma(\frac{4m+3}{2})\zeta(4m+2) \\ &\quad = \quad 2^{4m+1}\frac{B_{4m+2}}{4m+2} \end{aligned}$$

where the first equality comes from the Legendre duplication formula of the $\Gamma$-function, and the second equality is again a consequence of the $\zeta$-identity, we obtain

$$\mathrm{vol}_{HM}(\mathrm{O}^+(II_{2,8m+2})) = 2^{-(4m+1)} \frac{B_2 \cdot B_4 \cdot \ldots \cdot B_{8m+2}}{(8m+2)!!} \cdot \frac{B_{4m+2}}{4m+2}.$$

Here $(2n)!! = 2 \cdot 4 \cdot \ldots \cdot 2n$. Since the discriminant group of the lattice $II_{2,8m+2}$ is trivial, we have the equality

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(II_{2,8m+2})) = \mathrm{vol}_{HM}(\mathrm{O}^+(II_{2,8m+2})).$$

In a similar way one can derive a formula for any indefinite unimodular lattice of signature $(r,s)$. For example, for the odd unimodular lattice $M$ defined by $x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$ we have to take into account that the even $(M \otimes \mathbb{Z}_2)^{\mathrm{even}}$ and odd $(M \otimes \mathbb{Z}_2)^{\mathrm{odd}}$ parts of the lattice $M$ over 2-adic numbers depend on $r+s \mod 2$ and $r-s \mod 8$ (see [BG] for a different approach in this special case).

We can now use this to compute dimensions of cusp forms for this group and we obtain

$$\dim S_k(\widetilde{\mathrm{O}}^+(II_{2,8m+2}), \det{}^{\varepsilon}) = \frac{2^{-4m}}{(8m+2)!} \cdot \frac{B_2 \cdot B_4 \cdot \ldots \cdot B_{8m+2}}{(8m+2)!!} \cdot \frac{B_{4m+2}}{4m+2} k^{8m+2} + O(k^{8m+1}).$$

Here $\varepsilon = \pm 1$ and we must assume that $k$ is even, since otherwise there are no forms for trivial reasons.

### 3.4 The lattices $T_{2,8m+2}$

The orthogonal group of the lattice $II_{2,8m+2}$ for $m = 2$ defines an irreducible component of the branch divisor of the modular variety $\mathcal{F}_{2d}^{(m)}$. The same branch divisor contains another component defined by the lattice

$$T_{2,8m+2} = U \oplus U(2) \oplus mE_8(-1)$$

of discriminant 4. We note that this lattice is not maximal. For a prime number $p \neq 2$ the $p$-local densities of the lattices $T$ and $M$ coincide. Let us calculate $\alpha_2(T)$. Over the 2-adic ring we have $T_{2,8m+2} \otimes \mathbb{Z}_2 \cong (4m+1)U \oplus U(2)$. We have (see (11))

$$N_0 = N_0^{\mathrm{even}} = (4m+1)U, \quad N_1 = N_1^{\mathrm{even}} = U, \quad w = 3, \quad q = 0,$$

$$E_0 = \frac{1}{2}(1 + 2^{-(4m+1)}), \qquad E_1 = \frac{1}{2}(1 + 2^{-1}).$$

Thus

$$\alpha_2(T_{2,8m+2}) = 2^{8m+7}(1-2^{-2}) \cdot \cdots \cdot (1-2^{-8m})(1-2^{-(4m+1)}).$$

We note that $[\mathrm{PO}^+(T_{2,8m+2}) : \mathrm{P}\widetilde{\mathrm{O}}^+(T_{2,8m+2})] = 2$ since the finite orthogonal discriminant group of $T_{2,8m+2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. As a result we get

$$\begin{aligned}
&\mathrm{vol}_{MH}(\widetilde{\mathrm{O}}^+(T_{2,8m+2})) = \\
&\qquad 2\,\gamma_{8m+4}^{-1}\zeta(2)\cdot\dots\cdot\zeta(8m+2)\zeta(4m+2)(1+2^{-(4m+1)})(1-2^{-(4m+2)}).
\end{aligned}$$

Using the formula for the volume of $II_{2,8m+2}$ we see that

$$\frac{\mathrm{vol}_{MH}\,\widetilde{\mathrm{O}}^+(T_{2,8m+2})}{\mathrm{vol}_{MH}\,\widetilde{\mathrm{O}}^+(II_{2,8m+2})} = (2^{4m+1}+1)(2^{4m+2}-1). \tag{13}$$

If $L_1$ is a sublattice of finite index of a lattice $L$ then $\widetilde{\mathrm{O}}^+(L)$ is a subgroup of $\widetilde{\mathrm{O}}^+(L_1)$. One can use the formula of Theorem 2.1 to calculate easily the index $[\widetilde{\mathrm{O}}^+(L_1) : \widetilde{\mathrm{O}}^+(L)]$. For example, formula (13) above gives the index of $\widetilde{\mathrm{O}}^+(II_{2,8m+2})$ in $\widetilde{\mathrm{O}}^+(T_{2,8m+2})$. This method is much shorter than the calculation in terms of finite geometry over $\mathbb{Z}/2\mathbb{Z}$.

### 3.5 The lattices $L_{2d}^{(m)}$

We consider the lattice

$$L_{2d}^{(m)} = 2U \oplus mE_8(-1) \oplus \langle -2d\rangle$$

of signature $(2, 8m+3)$. The lattice $L_{2d}^{(m)}$ is not maximal if $d$ is not square free. This lattice is of particular interest, as the lattice $L_{2d}^{(2)}$ is closely related to the moduli space of polarised K3 surfaces of degree $2d$. More precisely, the quotient space

$$\mathcal{F}_{2d} = \widetilde{\mathrm{O}}^+(L_{2d}^{(2)})\backslash\mathcal{D}_{L_{2d}^{(2)}}$$

is the moduli space of K3 sufaces of degree $2d$. As we shall see, there is also a relation to Siegel modular forms for both the group $\mathrm{Sp}(2,\mathbb{Z})$ and the paramodular group.

Again, the lattices over the $p$-adic integers are easy to understand, since $E_8(-1)\otimes\mathbb{Z}_p$ is the direct sum of four copies of a hyperbolic plane. By (10) and (11) we find

$$\begin{aligned}
\alpha_p(L_{2d}^{(m)}) &= P_p(4m+2) && \text{if } p\nmid 2d\\
\alpha_p(L_{2d}^{(m)}) &= 2p^s P_p(4m+2)(1+p^{-(4m+2)})^{-1} && \text{if } p \text{ is odd, } p^s\| d\\
\alpha_2(L_{2d}^{(m)}) &= 2^{8m+6}P_2(4m+2) && \text{if } d \text{ is odd}\\
\alpha_2(L_{2d}^{(m)}) &= 2^{8m+7+s}P_2(4m+2)(1+2^{-(4m+2)})^{-1} && \text{if } d \text{ is even, } 2^s\| d
\end{aligned}$$

where the expression $p^s \| d$ means that $p^s$ is the highest power of $p$ which divides $d$. Therefore

$$\prod_p \alpha_p(L_{2d}^{(m)})^{-1} = \zeta(2)\zeta(4)\ldots\zeta(8m+4)\,(2d)^{-1}2^{-\rho(d)-8m-5}\prod_{p|d}(1+p^{-(4m+2)})$$

where $\rho(d)$ denotes the number of prime divisors of $d$.
We shall need the following.

Lemma 3.3 *Let $R = \langle -2d\rangle$. Then the order of the discriminant group* $\mathrm{O}(q_R)$ *is* $2^{\rho(d)}$.

*Proof.* Let $g$ be the standard generator of $A_R = \mathbb{Z}/2d\mathbb{Z}$, given by the equivalence class of 1. Then $q_R(g) = -1/2d \mod 2\mathbb{Z}$. If $\varphi \in \mathrm{O}(q_R)$, then $\varphi(g) = xg$ for some $x$ with $(x, 2d) = 1$. Hence $\varphi$ is orthogonal if and only if

$$-\frac{x^2}{2d} \equiv -\frac{1}{2d} \mod 2\mathbb{Z},$$

or equivalently

$$x^2 \equiv 1 \mod 4d\mathbb{Z}.$$

This equation has $2^{\rho(d)+1}$ solutions modulo $4d\mathbb{Z}$, and hence $2^{\rho(d)}$ solutions modulo $2d\mathbb{Z}$. □

From this it follows also that the discriminant group of the lattice $L_{2d}^{(m)}$ also has order $2^{\rho(d)}$.
From (9) it follows that

$$[\mathrm{PO}(L_{2d}^{(m)}) : \mathrm{P}\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})] = 2^{\rho(d)} \quad \text{if} \quad d > 1$$

and 2 if $d = 1$. We first assume that $d > 1$. We put $n = 8m + 3$, which is the dimension of the homogeneous domain. It follows from Corollary 3.1 that

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) = 2^{\rho(d)+1}(2d)^{\frac{n+3}{2}}\gamma_{n+2}^{-1}\prod_p \alpha_p(L_{2d}^{(m)})^{-1}.$$

If $d = 1$ we have to multiply the right hand side by a factor 2. Using the $\zeta$ identity, a straightforward calculation gives (again for $d > 1$ and $n = 8m + 3$)

$$\mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(L_{2d}^{(m)})) = \left(\frac{d}{2}\right)^{\frac{n+1}{2}}\prod_{p|d}(1+p^{-\frac{n+1}{2}})\cdot\frac{|B_2\cdot B_4\cdot\cdots\cdot B_{n+1}|}{(n+1)!!}.$$

We want to apply this to the moduli space of K3 surfaces of degree $2d$. This is the case $m = 2$: the dimension of the domain is $n = 19$. Using Hirzebruch-Mumford proportionality and specialising the above volume computation to this case, we compute the dimension of the spaces of cusp forms:

$$\dim S_k(\widetilde{\mathrm{O}}^+(L_{2d}^{(2)}), \det{}^\varepsilon) = \\ \frac{2^{-9}}{19!} d^{10} \cdot \prod_{p|d}(1+p^{-10}) \frac{|\, B_2 \cdot B_4 \cdot \ldots \cdot B_{20}\,|}{20!!} \cdot k^{19} + O(k^{18})$$

which holds for $d > 1$, with an additional factor 2 for $d = 1$. In the latter case we must assume that $k$ and $\varepsilon$ have the same parity. For $d > 1$ there is no restriction since $-\operatorname{id} \notin \widetilde{\mathrm{O}}^+(L_{2d}^{(2)})$. This should be compared to Kondo's formula [Ko] where, however, the Hirzebruch-Mumford volume has not been computed explicitly. It should also be noted that Kondo uses the geometric, rather than the arithmetic, weight.

#### 3.5.1 Siegel modular forms

The case $m = 0$ gives applications to Siegel modular forms. We shall first consider the case $d = 1$. Recall that

$$\widetilde{\mathrm{SO}}^+(L_2^{(0)}) \cong \widetilde{\mathrm{O}}^+(L_2^{(0)})/\{\pm \operatorname{id}\} \cong \mathrm{Sp}(2,\mathbb{Z})/\{\pm \operatorname{id}\}.$$

From our previous computation we obtain that

$$\operatorname{vol}_{HM}(\widetilde{\mathrm{SO}}^+(L_2^{(0)})) = \operatorname{vol}_{HM}(\widetilde{\mathrm{O}}^+(L_2^{(0)})) = 2^{-4}|B_2 B_4|$$

and by Hirzebruch-Mumford proportionality this gives

$$\dim S_k(\mathrm{Sp}(2,\mathbb{Z})) = 2^{-4}3^{-1}|B_2B_4|k^3 + O(k^2).$$

Note that this coincides with [T, p. 428], taking into account that Tai's formula refers to modular forms of weight $3k$. Tai uses Siegel's computation of the volume of the group $\mathrm{Sp}(2,\mathbb{Z})$, rather than the orthogonal group.

#### 3.5.2 The paramodular group

Finally, we consider the case $m = 0$ and $d > 1$. This is closely related to the so-called *paramodular* group $\Gamma_d^{(\mathrm{Sp})}$, which gives rise to the moduli space of $(1, d)$-polarised abelian surfaces. In fact

$$\widetilde{\mathrm{SO}}^+(L_{2d}^{(0)}) \cong \mathrm{P}\Gamma_d^{(\mathrm{Sp})}$$

by [GH, Proposition 1.2]. We note that in this case

$$[\widetilde{\mathrm{O}}^+(L_{2d}^{(0)}) : \widetilde{\mathrm{SO}}^+(L_{2d}^{(0)})] = 2$$

and that $-\operatorname{id}$ is in neither of these groups. Hence

$$\begin{aligned} \operatorname{vol}_{HM}(\widetilde{\mathrm{SO}}^+(L_{2d}^{(0)})) &= 2\operatorname{vol}_{HM}(\widetilde{\mathrm{O}}^+(L_{2d}^{(0)})) \\ &= 2^{-4}d^2\prod_{p|d}(1+p^{-2})|B_2B_4| \end{aligned}$$

and by Hirzebruch-Mumford proportionality

$$\dim S_k(\Gamma_d^{(\mathrm{Sp})}) = \frac{d^2}{3\cdot 2^4}\prod_{p|d}(1+p^{-2})|B_2B_4|k^3 + O(k^2).$$

This agrees with [Sa, Proposition 2.2], where this formula was derived for $d$ a prime.

### 3.6 Lattices associated to Heegner divisors

We shall conclude this section by computing the volume of two lattices of rank $8m+4$. Both of these lattices $K_{2d}^{(m)}$ and $N_{2d}^{(m)}$ arise from the $(-2)$-reflective part of the ramification divisor of the quotient map

$$\mathcal{D}_{L_{2d}^{(m)}} \to \widetilde{\mathrm{O}}^+_{L_{2d}^{(m)}}\backslash \mathcal{D}_{L_{2d}^{(m)}} = \mathcal{F}_{2d}^{(m)}.$$

For $m=2$ this is the moduli space of K3 surfaces of degree $2d$. For $m=0$ and a prime $d$ we get the moduli of Kummer surfaces associated to $(1,d)$-polarised abelian surfaces (see [GH]). Since the branch locus of the quotient map gives rise to obstructions for extending pluricanonical forms defined by modular forms, knowledge of their volumes is important for the computation of the Kodaira dimension of $\mathcal{F}_{2d}^{(m)}$ (see [GHS2]).

#### 3.6.1 The lattices $K_{2d}^{(m)}$

We consider the lattice

$$K_{2d}^{(m)} = U \oplus mE_8(-1)\oplus\langle 2\rangle\oplus\langle -2d\rangle$$

where $d$ is a positive integer. We first have to determine the local densities for this lattice. Since $\det(K_{2d}^{(m)}) = 4d$, this lattice is equivalent to the following lattices over the $p$-adic integers for odd primes $p$:

$$K_{2d}^{(m)}\otimes\mathbb{Z}_p \cong (4m+1)\,U\oplus\begin{cases} U & \text{if } \left(\frac{4d}{p}\right)=1\\ x^2-4dy^2 & \text{if } \left(\frac{4d}{p}\right)=-1.\end{cases}$$

For the local densities we obtain from equations (10) and (11)

$$\begin{aligned}
\alpha_p(K_{2d}^{(m)}) &= P_p(4m+1)\Big(1-\left(\frac{4d}{p}\right)p^{-(4m+2)}\Big) && \text{if } p\not| d\\
\alpha_p(K_{2d}^{(m)}) &= 2p^sP_p(4m+1) && \text{if } p^s\|d\\
\alpha_2(K_{2d}^{(m)}) &= 2^{8m+v(d)}P_2(4m+1)
\end{aligned}$$

where $v(d)=6$ if $d\equiv 1 \mod 4$, $v(d)=7$ if $d\equiv -1 \mod 4$, $v(d)=8$ if $d\equiv 2 \mod 4$, and $v(d)=8+s$ if $d\equiv 0 \mod 4$ and $2^s\|d$.

From this we obtain that

$$\prod_p \alpha_p(K_{2d}^{(m)})^{-1} = A_2(d)d^{-1}\zeta(2)\zeta(4)\dots\zeta(8m+2)L(4m+2,\left(\frac{4d}{*}\right)), \tag{14}$$

where

$$A_2(d) = \begin{cases} 2^{-\rho(d)-8m-6} & \text{if } d \equiv 1,2 \mod 4 \\ 2^{-\rho(d)-8m-7} & \text{if } d \equiv 0,3 \mod 4. \end{cases}$$

Application of our main formula (7) then gives

$$\mathrm{vol}_{HM}(\mathrm{O}^+(K_{2d}^{(m)})) = 4 \cdot (4d)^{\frac{8m+5}{2}} \cdot \prod_{k=1}^{8m+4} \pi^{-\frac{k}{2}}\Gamma(\frac{k}{2}) \cdot \prod_p \alpha_p(K_{2d})^{-1}. \tag{15}$$

Combining formulae (14) and (15) and the $\zeta$-identity (12) leads to

$$\mathrm{vol}_{HM}(\mathrm{O}^+(K_{2d}^{(m)})) = \\ C_2(d)d^{\frac{8m+3}{2}}\pi^{-(4m+2)}\Gamma(4m+2)L(4m+2,\left(\frac{4d}{*}\right))\frac{B_2B_4\dots B_{8m+2}}{(8m+2)!!}$$

where

$$C_2(d) = \begin{cases} 2^{-\rho(d)+1} & \text{if } d \equiv 1,2 \mod 4 \\ 2^{-\rho(d)} & \text{if } d \equiv 0,3 \mod 4. \end{cases}$$

For applications it is also important to compute the volume with respect to the group $\widetilde{\mathrm{O}}^+(K_{2d}^{(m)})$. For this, we have to know the order of the group of isometries of the discriminant group.

Lemma 3.4 *Let $S = \langle 2\rangle \oplus \langle -2d\rangle$. The order of the discriminant group is*

$$|\,\mathrm{O}(q_S)| = \begin{cases} 2^{1+\rho(d)} & \textit{if } d \equiv -1 \mod 4 \textit{ or } d \textit{ is divisible by } 8 \\ 2^{\rho(d)} & \textit{for all other } d. \end{cases}$$

*Proof.* We denote the standard generators of $\mathbb{Z}/2d\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ by $g$ and $h$ respectively. We shall first consider automorphisms $\varphi$ with $\varphi(g) = xg$. Then orthogonality implies $x^2 \equiv 1 \mod 4d\mathbb{Z}$ which means, in particular, that $x$ is odd and $(x, 2d) = 1$. We then have $\varphi(dg) = dg$. We cannot have $\varphi(h) = dg + h$, because orthogonality implies that for the bilinear form $B_q$, defined by the quadratic form $q = q_S$, we have $B_q(xg, dg + h) = B_q(g, h) = 0$ and hence $-x/2 \equiv 0 \mod \mathbb{Z}$, which shows that $x$ is even, a contradiction. Hence $\varphi(h) = h$ and $\varphi = \varphi' \times \mathrm{id}$ where $\varphi' \in \mathrm{O}(q_R)$ (with $R = \langle -2d\rangle$). In this way we obtain $2^{\rho(d)}$ elements in $\mathrm{O}(q_S)$.

We shall now investigate automorphisms with $\varphi(g) = xg + h$. Then $q(g) = q(\varphi(g))$ implies the condition

$$x^2 \equiv 1 + d \mod 4d\mathbb{Z}.$$

It is not hard to check that this only has solutions if either $d \equiv -1 \mod 4$ or $d$ is divisible by 8. We shall distinguish between the cases $d$ even and $d$ odd. In the first case $x$ must be odd and $(x, 2d) = 1$. Moreover $\varphi(dg) = dg$ and the only possibility for an orthogonal automorphism is $\varphi(h) = dg + h$ and indeed this gives rise to another $2^{\rho(d)}$ orthogonal automorphisms. Now assume $d$ is odd. Then $x$ is even and $(x, d) = 1$. In this case $\varphi(dg) = h$ and the only possibility to obtain an orthogonal automorphism is $\varphi(h) = dg$. Once more, this gives another $2^{\rho(d)}$ orthogonal automorphisms and this proves the lemma. □

By formula (9) it then follows that

$$[\mathrm{PO}(K_{2d}^{(m)}) : \mathrm{P}\widetilde{\mathrm{O}}^+(K_{2d}^{(m)})] = \begin{cases} 2 & \text{if } d = 1 \\ 2^{\rho(d)} & \text{if } d \equiv 1,\, 2 \mod 4,\ d > 1 \\ 2^{\rho(d)+1} & \text{if } d \equiv 3 \mod 4. \end{cases}$$

Therefore

$$\begin{aligned} \mathrm{vol}_{HM}(\widetilde{\mathrm{O}}^+(K_{2d}^{(m)})) &= 2^{\delta_{1,d} - \delta_{4,d(8)}} \frac{B_2 B_4 \dots B_{8m+2}}{(8m+2)!!} \\ &\cdot d^{\frac{8m+3}{2}} \pi^{-(4m+2)} \Gamma(4m+2) L(4m+2, \left(\frac{4d}{*}\right)) \end{aligned} \tag{16}$$

where $d(8)$ denotes $d \mod 8$ and $\delta_{*,*}$ is the Kronecker symbol.
We want to reformulate this result in terms of generalised Bernoulli numbers. In order to avoid too many different cases, we restrict here to $d \not\equiv 0 \mod 4$ (but it is clear how to remove this restriction). If $d = d_0 t^2$, with $d_0$ a positive and square-free integer, then the discriminant of the real quadratic field $\mathbb{Q}(\sqrt{d})$ is equal to

$$D = \begin{cases} d_0 & \text{if } d \equiv 1 \mod 4 \\ 4d_0 & \text{if } d \equiv 2,\, 3 \mod 4. \end{cases}$$

Note that

$$d^{\frac{8m+3}{2}} = t^{8m+3} D^{\frac{8m+3}{2}} \cdot \begin{cases} 1 & \text{if } d \equiv 1 \mod 4 \\ 2^{-(8m+3)} & \text{if } d \equiv 2,\, 3 \mod 4. \end{cases} \tag{17}$$

Let $\chi_D$ be the quadratic character of this field. Then

$$L(s, \left(\frac{4d}{*}\right)) = L(s, \chi_D) \prod_{p|2t} (1 - \chi_D(p) p^{-s}). \tag{18}$$

The character $\chi_D$ is an even primitive character modulo $D$, and the Dirichlet $L$-function $L(s, \chi_D)$ satisfies the functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) D^s L(s, \chi_D) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) D^{\frac{1}{2}} L(1-s, \chi_D). \tag{19}$$

Moreover

$$L(1-k,\chi_D) = -\frac{B_{k,\chi_D}}{k}$$

where $B_{k,\chi_D}$ is the corresponding generalised Bernoulli number. Using the functional equation (19) we obtain

$$\begin{aligned}\pi^{-(4m+2)}\Gamma(4m+2)D^{\frac{8m+3}{2}}L(4m+2,\chi_D) &= -2^{4m+1}L(1-(4m+2),\chi_D)\\ &= 2^{4m+1}\frac{B_{4m+2,\chi_D}}{4m+2}. \qquad (20)\end{aligned}$$

Combining (16), (17), (18), (20) and the result of Lemma 3.4 then gives the result

$$\operatorname{vol}_{HM}(\widetilde{\mathrm{O}}^+(K_{2d}^{(m)})) = F_2(d)t^{8m+3}\frac{B_2B_4\dots B_{8m+2}}{(8m+2)!!}\frac{B_{4m+2,\chi_D}}{4m+2}\prod_{p|2t}(1-\chi_D(p)p^{-(4m+2)}) \quad (21)$$

where

$$F_2(d) = \begin{cases} 2^{4m+2} & \text{if } d\equiv 1 \mod 4\\ 2^{-4m-1} & \text{if } d \equiv 2,\ 3 \mod 4.\end{cases}$$

Using this, together with Hirzebruch-Mumford proportionality, we finally find that $\dim S_k(\widetilde{\mathrm{O}}^+(K_{2d}^{(m)}))$ grows as

$$\frac{G_2(d)}{(8m+2)!}\frac{B_2\cdot B_4\dots B_{8m+2}}{(8m+2)!!}\cdot\frac{B_{4m+2,\chi_D}}{4m+2}t^{8m+3}\prod_{p|2t}(1-\chi_D(p)p^{-(4m+2)})\,k^{8m+2}$$

where

$$G_2(d) = \begin{cases} 2^{4m+2+\delta_{1,d}} & \text{if } d\equiv 1 \mod 4,\\ 2^{-(4m+1)} & \text{if } d \equiv 2,\ 3 \mod 4.\end{cases}$$

### 3.6.2 THE LATTICES $N_{2d}^{(m)}$

We assume that $d\equiv 1 \mod 4$ and consider the even lattice

$$N_{2d}^{(m)} = U \oplus mE_8(-1)\oplus\begin{pmatrix}2 & 1\\ 1 & \frac{1-d}{2}\end{pmatrix}.$$

We first have to understand this lattice over the $p$-adic integers. If $p>2$ then 2 is a $p$-adic integer and we have the following equality for the anisotropic binary form in $N_{2d}^{(m)}$:

$$\frac{1-d}{2}x^2+2xy+2y^2 = -\frac{d}{2}x^2+2(y+\frac{x}{2})^2.$$

Depending on whether $d$ is a square in $\mathbb{Z}_p^*$ or not, we then obtain from the classification theory of quadratic forms over $\mathbb{Z}_p$ that

$$N_{2d}^{(m)} \otimes \mathbb{Z}_p \cong (4m+1)\,U \oplus \begin{cases} U & \text{if } \left(\frac{d}{p}\right) = 1 \\ -dx^2 + y^2 & \text{if } \left(\frac{d}{p}\right) = -1. \end{cases}$$

We now turn to $p = 2$. Recall that there are only two even unimodular binary forms over $\mathbb{Z}_2$, namely the hyperbolic plane and the form given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. This implies that

$$N_{2d}^{(m)} \otimes \mathbb{Z}_2 \cong (4m+1)\,U \oplus \begin{cases} U & \text{if } d \equiv 1 \mod 8 \\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \text{if } d \equiv 5 \mod 8. \end{cases}$$

Once again by (10) and (11) we find for the local densities that

$$\begin{aligned} \alpha_p(N_{2d}^{(m)}) &= P_p(4m+1)(1 - \left(\frac{d}{p}\right) p^{-(4m+2)}) && \text{if } p \not| d \\ \alpha_p(N_{2d}^{(m)}) &= 2p^s P_p(4m+1) && \text{if } p^s \| d \\ \alpha_2(N_{2d}^{(m)}) &= 2^{8m+4} P_2(4m+1)(1 - \left(\frac{d}{2}\right) 2^{-(4m+2)}). \end{aligned}$$

We are interested mainly in the group $\widetilde{\mathrm{O}}^+(N_{2d}^{(m)})$. For this we need the next lemma.

Lemma 3.5 *Let*

$$T = \begin{pmatrix} 2 & 1 \\ 1 & \frac{1-d}{2} \end{pmatrix}$$

*Then $A_T \cong \mathbb{Z}/2d\mathbb{Z}$ and*

$$|\,\mathrm{O}(q_T)| = 2^{\rho(d)}.$$

*Proof.* Since $\det(T) = -d$, the discriminant group has order $d$. In fact, it is cyclic of order $d$. To see this, let $e$ and $f$ be the basis with respect to which the form is given by the matrix $T$. Then $(e - 2f)/d$ is in the dual lattice and its class, say $h$, generates the group $A_T$. Every homomorphism of $A_T$ is of the form $\varphi(h) = xh$, and it is an isometry if and only if $x^2 \equiv 1 \mod 2d$. This equation has $2^{\rho(d)}$ solutions modulo $d\mathbb{Z}$. $\square$

It now follows from (9) that

$$[\mathrm{PO}(N_{2d}^{(m)}) : \mathrm{P}\widetilde{\mathrm{O}}^+(N_{2d}^{(m)})] = \begin{cases} 2^{\rho(d)} & \text{if } d \equiv 1 \mod 4 \text{ and } d \neq 1 \\ 2 & \text{if } d = 1. \end{cases}$$

By the same calculation as in the preceding example we find now that

$$\begin{aligned}\operatorname{vol}_{HM}(\widetilde{\mathrm{O}}^+(N_{2d}^{(m)})) &= 2^{\delta_{1,d}-8m-3}\frac{B_2B_4\dots B_{8m+2}}{(8m+2)!!}\cdot\\ &d^{\frac{8m+3}{2}}\pi^{-(4m+2)}\Gamma(4m+2)L(4m+2,\left(\frac{d}{*}\right)).\end{aligned} \tag{22}$$

As above we can use generalised Bernoulli numbers. Hence by Hirzebruch-Mumford proportionality we obtain for $d>1$ that $\dim S_k(\widetilde{\mathrm{O}}^+(N_{2d}^{(m)}))$ grows as

$$\frac{2^{-4m-1}}{(8m+2)!}\frac{B_2\cdot B_4\dots B_{8m+2}}{(8m+2)!!}\cdot\frac{B_{4m+2,\chi_D}}{4m+2}\,t^{8m+3}\prod_{p|t}(1-\chi_D(p)p^{-(4m+2)})\,k^{8m+2}.$$

Here, as before, $d=d_0t^2$, with $d_0$ square-free, and $D=d_0$ is the discriminant of the quadratic extension $\mathbb{Q}(\sqrt{d})$. For $d=1$ we have an extra factor 2, $t=1$, $\chi_D\equiv 1$ and $B_{4m+2,\chi_D}=B_{4m+2}$. In this case the lattice $N_{2d}^{(m)}$ is unimodular and the formula again agrees with our previous computations in Section 3.3.

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# On the Parity of Ranks of Selmer Groups III

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Abstract. We show that the parity conjecture for Selmer groups is invariant under deformation in $p$-adic families of self-dual pure Galois representations satisfying Pančiškin's condition at all primes above $p$.

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## 0. Introduction

(0.0) Let $F, L$ be number fields contained in a fixed algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$; let $M$ be a motive over $F$ with coefficients in $L$. The $L$-function of $M$ (assuming it is well-defined) is a Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ with coefficients in $L$. For each embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, the complex-valued $L$-function

$$L(\iota M, s) = \sum_{n\geq 1} \iota(a_n) n^{-s}$$

is absolutely convergent for $\mathrm{Re}(s) >> 0$. It is expected to admit a meromorphic continuation to $\mathbf{C}$ and a functional equation of the form

$$(C_{FE}) \qquad (L \cdot L_\infty)(\iota M, s) \stackrel{?}{=} \varepsilon(\iota M, s)\, (L \cdot L_\infty)(\iota M^*(1), -s),$$

where

$$L_\infty(\iota M, s) = \prod_{v\mid\infty} L_v(\iota M, s)$$

is a product of appropriate $\Gamma$-factors (independent of $\iota$) and

$$\varepsilon(\iota M, s) = \iota(\varepsilon(M))\, \mathrm{cond}(M)^{-s}, \qquad \varepsilon(M) \in \overline{\mathbf{Q}}^*.$$

(0.1) Let $p$ be a prime number and $\mathfrak{p} \mid p$ a prime of $L$ above $p$. The $\mathfrak{p}$-adic realization $M_{\mathfrak{p}}$ of $M$ is a finite-dimensional $L_{\mathfrak{p}}$-vector space equipped with a continuous action of the Galois group $G_{F,S} = \mathrm{Gal}(F_S/F)$, where $F_S \subset \overline{\mathbf{Q}}$ is the maximal extension of $F$ unramified outside a suitable finite set $S \supset S_p \cup S_\infty$ of primes of $F$. According to the conjectures of Bloch and Kato [Bl-Ka] (generalized by Fontaine and Perrin-Riou [Fo-PR]),

$$(C_{BK}) \qquad \begin{aligned} \mathrm{ord}_{s=0} L(\iota M, s) &\stackrel{?}{=} \dim_{L_{\mathfrak{p}}} H^1_f(F, M^*_{\mathfrak{p}}(1)) - \dim_{L_{\mathfrak{p}}} H^0(F, M^*_{\mathfrak{p}}(1)) = \\ &= h^1_f(F, M^*_{\mathfrak{p}}(1)) - h^0(F, M^*_{\mathfrak{p}}(1)), \end{aligned}$$

where $H^1_f(F, V) \subseteq H^1(G_{F,S}, V)$ is the generalized Selmer group defined in [Bl-Ka].

(0.2) Consider the special case when the motive $M$ is SELF-DUAL (i.e., when there exists a skew-symmetric isomorphism $M \xrightarrow{\sim} M^*(1)$) and PURE (necessarily of weight $-1$). In this case $H^0(F, M_{\mathfrak{p}}) = 0$ and $\mathrm{ord}_{s=0} L_\infty(\iota M, s) = 0$, which means that the global $\varepsilon$-factor $\varepsilon(M)$ determines the parity of $\mathrm{ord}_{s=0} L(\iota M, s)$ (assuming the validity of $(C_{FE})$):

$$(-1)^{\mathrm{ord}_{s=0} L(\iota M, s)} = \varepsilon(M). \tag{0.2.1}$$

In this article we concentrate on the PARITY CONJECTURE FOR SELMER GROUPS, namely on the conjecture

$$(C_{BK} \pmod 2) \qquad \mathrm{ord}_{s=0} L(\iota M, s) \stackrel{?}{\equiv} h^1_f(F, M_{\mathfrak{p}}) \pmod 2.$$

In view of (0.2.1), this conjecture can be reformulated (assuming $(C_{FE})$) as follows:

$$(-1)^{h^1_f(F, M_{\mathfrak{p}})} \stackrel{?}{=} \varepsilon(M) \tag{0.2.2}$$

(0.3) The advantage of the formulation (0.2.2) is that the global $\varepsilon$-factor

$$\varepsilon(M) = \prod_v \varepsilon_v(M), \qquad \varepsilon_v(M) = \varepsilon_v(M_{\mathfrak{p}})$$

is a product of local $\varepsilon$-factors, which can be expressed in terms of the Galois representation $M_{\mathfrak{p}}$ alone: for $v \nmid p\infty$ (resp., $v \mid p$), $\varepsilon_v(M)$ is the local $\varepsilon$-factor of the representation of the Weil-Deligne group of $F_v$ attached to the action of $\mathrm{Gal}(\overline{F}_v/F_v)$ on $M_{\mathfrak{p}}$ (resp., attached to the corresponding Fontaine module $D_{pst}(M_{\mathfrak{p}})$ over $F_v$). For $v \mid \infty$, $\varepsilon_v(M)$ depends on the Hodge numbers of the de Rham realization $M_{dR}$ of $M$, which can be read off from $D_{dR}(M_{\mathfrak{p}})$ over $F_v$, for any $v \mid p$.

It makes sense, therefore, to rewrite the conjecture (0.2.2) as

$$(-1)^{h^1_f(F, V)} \stackrel{?}{=} \varepsilon(V) = \prod_v \varepsilon_v(V), \tag{0.3.1}$$

for any symplectically self-dual ($V \xrightarrow{\sim} V^*(1)$) representation of $G_{F,S}$ which is geometric (= potentially semistable at all primes above $p$) and pure (of weight $-1$).

In the present article we consider the following question: is the conjecture (0.3.1) invariant under deformation in $p$-adic families of representations of $G_{F,S}$? In other words, if $V, V'$ are two representations of $G_{F,S}$ (self-dual, geometric and pure) belonging to the same $p$-adic family (say, in one parameter) of representations of $G_{F,S}$, is it true that

$$(-1)^{h^1_f(F,V)}/\,\varepsilon(V) \stackrel{?}{=} (-1)^{h^1_f(F,V')}/\,\varepsilon(V') \qquad ? \tag{0.3.2}$$

The main result of this article (Thm. 5.3.1) implies that (0.3.2) holds for families satisfying the Pančiškin condition at all primes $v \mid p$. The proof follows the strategy employed in [Ne 2, ch. 12] in the context of Hilbert modular forms [(1)] : multiplying both sides of (0.3.1) by a common sign (the contribution of the "trivial zeros"), we rewrite (0.3.1) as

$$(-1)^{\widetilde{h}^1_f(F,V)} \stackrel{?}{=} \widetilde{\varepsilon}(V) = \prod_v \widetilde{\varepsilon}_v(V), \tag{0.3.3}$$

where $\widetilde{h}^1_f(F,V) = \dim_{L_\mathfrak{p}} \widetilde{H}^1_f(F,V)$ is the dimension of the extended Selmer group (defined in 4.2 below) and $\widetilde{\varepsilon}_v(V) = \varepsilon_v(V)$, unless $v \mid p$ and the local Euler factor at $v$ admits a "trivial zero". The goal is to show that both sides of (0.3.3) remain constant in the family [(2)].

The variation of $\widetilde{H}^1_f(F,V)$ in the family is controlled by the torsion submodule of a suitable $\widetilde{H}^2_f$. The generalized Cassels-Tate pairing constructed in [Ne 2, ch. 10] defines a skew-symmetric form on this torsion submodule, which implies that the parity of $\widetilde{h}^1_f(F,V)$ is constant in family:

$$(-1)^{\widetilde{h}^1_f(F,V)} = (-1)^{\widetilde{h}^1_f(F,V')}.$$

The Pančiškin condition allows us to compute explicitly the local terms $\widetilde{\varepsilon}_v(V)$ for all $v \mid p$, which yields

$$\prod_{v \mid p\infty} \widetilde{\varepsilon}_v(V) = \prod_{v \mid p\infty} \widetilde{\varepsilon}_v(V').$$

Finally, it follows from general principles (and the purity assumption) that

$$\forall v \nmid p\infty \qquad \varepsilon_v(V) = \varepsilon_v(V'),$$

hence $\widetilde{\varepsilon}(V) = \widetilde{\varepsilon}(V')$.

---

(1) In [loc. cit.] we worked with automorphic $\varepsilon$-factors, but they coincide with the Galois-theoretical $\varepsilon$-factors ([Ne 2], 12.4.3, 12.5.4(iii)).

(2) Morally, $\widetilde{\varepsilon}(V)$ should be the sign in the functional equation of a $p$-adic $L$-function attached to the family.

## 1. Representations of the Weil-Deligne group

(1.1) The general setup ([De 1, §8], [De 2, 3.1], [Fo-PR, I.1.1-2])

(1.1.1) We use the notation of [Fo-PR, ch.I]. For a field $L$, denote by $L^{sep}$ a separable closure of $L$ and by $G_L = \mathrm{Gal}(L^{sep}/L)$ the absolute Galois group of $L$. Throughout this article, $K$ will be a complete discrete valuation field of characteristic zero with finite residue field $k$ of cardinality $q = q_k$; denote by $f = f_k \in G_k$ the GEOMETRIC Frobenius element ($f(x) = x^{1/q}$). We identify $G_k \xrightarrow{\sim} \widehat{\mathbf{Z}}$ via $f \mapsto 1$ and denote by $\nu : G_K \xrightarrow{\mathrm{can}} G_k \xrightarrow{\sim} \widehat{\mathbf{Z}}$ the canonical surjection whose kernel $\mathrm{Ker}(\nu) = I_K = I$ is the inertia group of $K$. The Weil group (of $K$) $W_K = \nu^{-1}(\mathbf{Z}) = \coprod_{n\in\mathbf{Z}} \widetilde{f}^n I$ ($\widetilde{f} \in \nu^{-1}(1)$) is equipped with the topology of a disjoint union of countably many pro-finite sets. The homomorphism

$$|\cdot| : W_K \longrightarrow q^{\mathbf{Z}}, \qquad |w| = q^{-\nu(w)}$$

corresponds to the normalized valuation $|\cdot| : K^* \longrightarrow q^{\mathbf{Z}}$ via the reciprocity isomorphism $\mathrm{rec}_K : K^* \xrightarrow{\sim} W_K^{ab}$ (normalized using the geometric Frobenius element).

(1.1.2) Let $E$ be a field of characteristic zero.

An object of $\mathrm{Rep}_E(W_K)$ (= a representation of the Weil group of $K$ over $E$) is a finite-dimensional $E$-vector space $\Delta$ equipped with a continuous homomorphism $\rho = \rho_\Delta : W_K \longrightarrow \mathrm{Aut}_E(\Delta)$ (with respect to the discrete topology on the target). As $\mathrm{Ker}(\rho)$ is open, $\rho(I)$ is finite and $\rho|_I$ is semi-simple.

An object of $\mathrm{Rep}_E('W_K)$ (= a representation of the Weil-Deligne group of $K$ over $E$) is a pair $(\rho, N)$, where $\rho = \rho_\Delta \in \mathrm{Rep}_E(W_K)$ and $N \in \mathrm{End}_E(\Delta)$ is a nilpotent endomorphism satisfying

$$\forall w \in W_K \qquad \rho(w)N\rho(w)^{-1} = |w|\, N.$$

Morphisms in $\mathrm{Rep}_E(W_K)$ (resp., in $\mathrm{Rep}_E('W_K)$) are $E$-linear maps commuting with the action of $W_K$ (resp., with the action of $W_K$ and $N$). We consider $\mathrm{Rep}_E(W_K)$ as a full subcategory of $\mathrm{Rep}_E('W_K)$ via the full embedding $\rho \mapsto (\rho, 0)$. Tensor products and duals in $\mathrm{Rep}_E('W_K)$ are defined in the usual way: $N_{\Delta\otimes\Delta'} = N_\Delta \otimes 1 + 1 \otimes N_{\Delta'}$, $N_{\Delta^*} = -(N_\Delta)^*$. The Tate twist of $\Delta \in \mathrm{Rep}_E('W_K)$ by an integer $m \in \mathbf{Z}$ is defined as $\Delta|\cdot|^m = \Delta \otimes E|\cdot|^m$, where $w \in W_K$ acts on the one-dimensional representation $E|\cdot|^m \in \mathrm{Rep}_E(W_K)$ by $|w|^m$.

The Frobenius semi-simplification

$$\Delta = (\rho, N) \mapsto \Delta^{f-ss} = (\rho^{ss}, N)$$

is an exact tensor functor $\mathrm{Rep}_E('W_K) \longrightarrow \mathrm{Rep}_E('W_K)$. The "forget the monodromy" functor

$$\Delta = (\rho, N) \mapsto \Delta^{N-ss} = (\rho, 0)$$

is an exact tensor functor $\mathrm{Rep}_E('W_K) \longrightarrow \mathrm{Rep}_E(W_K)$.

Following [Fo-PR, I.1.2.1], we put, for each $\Delta \in \mathrm{Rep}_E('W_K)$,

$$\Delta_g = \Delta^{\rho(I)}, \qquad \Delta_f = \mathrm{Ker}(N)^{\rho(I)} \subset \Delta_g, \qquad P_K(\Delta, u) = \det(1 - fu \mid \Delta_f) \in E[u].$$

We also set

$$H^0(\Delta) = \mathrm{Ker}\,(\Delta_f \xrightarrow{f-1} \Delta_f)\,.$$

(1.1.3) In the special case when $E$ is a finite extension of $\mathbf{Q}_p$ ($p \neq \mathrm{char}(k)$) and when $V \in \mathrm{Rep}_E(G_K)$ is a representation of $G_K$ over $E$ (finite-dimensional and continuous with respect to the topology on $E$ defined by the $p$-adic valuation), then $V$ gives rise to a representation $WD(V) = \Delta = (\rho_\Delta, N) \in \mathrm{Rep}_E('W_K)$ acting on $V$, which is defined as follows ([De 1, 8.4]): there exists an open subgroup $J$ of $I$ which acts on $V$ unipotently, and through the map $J \hookrightarrow I \twoheadrightarrow I(p)$, where $I(p)$ is the maximal pro-$p$-quotient of $I$ (isomorphic to $\mathbf{Z}_p$). Fixing a topological generator $t$ of $I(p)$ and an integer $a \geq 1$ such that $t^a$ lies in the image of $J$, the nilpotent endomorphism

$$N = \frac{1}{a} \log \rho_V(t^a) \in \mathrm{End}_E(V)$$

(where $\rho_V : G_K \longrightarrow \mathrm{Aut}_E(V)$ denotes the action of $G_K$ on $V$) is independent of $a$. Fix a lift $\widetilde{f} \in \nu^{-1}(1) \subset W_K$ of $f$ and define

$$\rho_\Delta : W_K \longrightarrow \mathrm{Aut}_E(V)$$

by

$$\rho_\Delta(\widetilde{f}^n u) := \rho_V(\widetilde{f}^n u) \exp(-bN) \qquad\qquad (n \in \mathbf{Z},\ u \in I),$$

where $b \in \mathbf{Z}_p$ is such that the image of $u$ in $I(p)$ is equal to $t^b$. The pair $(\rho_\Delta, N)$ defines an object $\Delta = WD(V)$ of $\mathrm{Rep}_E('W_K)$, the isomorphism class of which is independent of the choices of $\widetilde{f}$ and $t$ ([De 1], Lemma 8.4.3), and which satisfies

$$\Delta_f = V^{\rho_V(I)}, \qquad H^0(\Delta) = V^{\rho_V(G_K)}.$$

(1.2) Self-dual representations

(1.2.1) Definition. *Let $\omega : W_K \longrightarrow E^*$ be a one-dimensional object of $\mathrm{Rep}_E(W_K)$. We say that $\Delta \in \mathrm{Rep}_E('W_K)$ is $\omega$-*orthogonal *(resp., $\omega$-*symplectic*) if there exists a morphism in $\mathrm{Rep}_E('W_K)$ $\Delta \otimes \Delta \longrightarrow \omega$ which is non-degenerate (i.e., which induces an isomorphism $\Delta \xrightarrow{\sim} \Delta^* \otimes \omega$) and *symmetric *(resp., *skew-symmetric*). If $\omega = 1$, we say that $\Delta$ is *orthogonal *(resp., *symplectic*).*

(1.2.2) (1) If $\Delta$ is $\omega$-orthogonal, then $\det(\Delta)^2 = \omega^{\dim(\Delta)}$.
(2) If $\Delta$ is $\omega$-symplectic, then $2 \mid \dim(\Delta)$ and $\det(\Delta) = \omega^{\dim(\Delta)/2}$.

(1.2.3) Example: For $m \geq 1$, define $sp(m) \in \mathrm{Rep}_E('W_K)$ by

$$sp(m) = \bigoplus_{i=0}^{m-1} Ee_i, \qquad N(e_i) = e_{i+1}, \qquad \forall w \in W_K \quad w(e_i) = |w|^i e_i.$$

Up to a scalar multiple, there is a unique non-degenerate morphism $sp(m) \otimes sp(m) \longrightarrow E|\cdot|^{m-1}$ in $\mathrm{Rep}_E('W_K)$, namely

$$sp(m) \otimes sp(m) \longrightarrow E|\cdot|^{m-1}, \qquad e_i \otimes e_j \mapsto \begin{cases} (-1)^i, & i+j = m-1 \\ 0, & i+j \neq m-1. \end{cases}$$

This morphism is $|\cdot|^{m-1}$-symplectic (resp., $|\cdot|^{m-1}$-orthogonal) if $2 \mid m$ (resp., if $2 \nmid m$).

(1.2.4) According to [De 2, 3.1.3(ii)], indecomposable $f$-semi-simple objects of $\mathrm{Rep}_E('W_K)$ are of the form $\rho \otimes sp(m)$, where $\rho \in \mathrm{Rep}_E(W_K)$ is irreducible and $m \geq 1$. This implies that, for each $|\cdot|$-symplectic representation $\Delta \xrightarrow{\sim} \Delta^*|\cdot| \in \mathrm{Rep}_E('W_K)$, the $f$-semi-simplification $\Delta^{f-ss}$ is a direct sum of $|\cdot|$-symplectic representations of the following type:

(1) $X \oplus X^*|\cdot|$ $(X \in \mathrm{Rep}_E('W_K))$ with the standard symplectic form $(x, x^*) \otimes (y, y^*) \mapsto x^*(y) - y^*(x)$;
(2) $\rho \otimes sp(m)$, where $m \geq 1$, $\rho \in \mathrm{Rep}_E(W_K)$ is irreducible and $|\cdot|^{2-m}$-symplectic (resp., $|\cdot|^{2-m}$-orthogonal) if $2 \nmid m$ (resp., if $2 \mid m$).

(1.3) THE MONODROMY FILTRATION

(1.3.1) For each $\Delta = (\rho, N) \in \mathrm{Rep}_E('W_K)$, the monodromy filtration

$$M_n\Delta := \sum_{i-j=n+1} \ker(N^i) \cap \mathrm{Im}(N^j) \qquad (n \in \mathbf{Z})$$

is the unique increasing filtration of $\Delta$ by $E$-vector subspaces satisfying

$$\bigcap_n M_n\Delta = 0, \qquad \bigcup_n M_n\Delta = \Delta, \qquad N(M_n\Delta) \subseteq M_{n-2}\Delta,$$
$$\forall r \geq 0 \quad N^r : \mathrm{gr}_r^M\Delta \xrightarrow{\sim} \mathrm{gr}_{-r}^M\Delta.$$

(1.3.2) EXAMPLES: (1) $N = 0 \iff M_{-1}\Delta = 0,\ M_0\Delta = \Delta$.
(2) If $N^r \neq 0 = N^{r+1}$ $(r \geq 0)$, then $M_{-r-1}\Delta = 0$, $M_{-r}\Delta = \mathrm{Im}(N^r) \neq 0$, $M_{r-1}\Delta = \mathrm{Ker}(N^r) \neq \Delta$, $M_r\Delta = \Delta$.

(1.3.3) More precisely, the endomorphism $N \in \mathrm{End}_E(\Delta)$ defines a morphism in $\mathrm{Rep}_E(W_K)$

$$N : \Delta \longrightarrow \Delta|\cdot|^{-1},$$

which implies that each $M_n\Delta$ is a sub-object of $\Delta^{N-ss}$ in $\mathrm{Rep}_E(W_K)$,

$$N : M_n\Delta \longrightarrow (M_{n-2}\Delta)|\cdot|^{-1}$$

and, for each $r \geq 0$, the endomorphism $N^r$ induces an isomorphism in $\mathrm{Rep}_E(W_K)$

$$N^r : \mathrm{gr}_r^M \Delta \xrightarrow{\sim} (\mathrm{gr}_{-r}^M \Delta)|\cdot|^{-r}.$$

(1.3.4) The monodromy filtration on the dual representation $\Delta^* = (\rho^*, -N^*)$ satisfies $M_n\Delta^* = (M_{-1-n}\Delta)^\perp$ ($n \in \mathbf{Z}$), which yields canonical isomorphisms in $\mathrm{Rep}_E(W_K)$

$$\forall m \leq n \qquad M_n\Delta^*/M_m\Delta^* \xrightarrow{\sim} (M_{-1-m}\Delta/M_{-1-n}\Delta)^*.$$

(1.3.5) If $\langle\ ,\ \rangle : \Delta \otimes \Delta \longrightarrow E \otimes \omega$ is an $\omega$-symplectic form on $\Delta$, then, for each $r \geq 0$, the formula $\langle x, y\rangle_r = \langle N^r x, y\rangle$ defines an $\omega|\cdot|^{-r}$-symplectic (resp., $\omega|\cdot|^{-r}$-orthogonal) form on $\mathrm{gr}_r^M \Delta \in \mathrm{Rep}_E(W_K)$ if $2 \mid r$ (resp., if $2 \nmid r$).

(1.3.6) Dimensions. The dimensions

$$d_r = d_r(\Delta) = \dim\, \mathrm{gr}_r^M \Delta = d_{-r} \qquad\qquad (r \in \mathbf{Z})$$

can be interpreted as follows. By the Jacobson-Morozov theorem, there exists a (non-unique) representation

$$\rho : sl(2) = sl(2, E) \longrightarrow \mathrm{End}_E(\Delta)$$

such that $\rho(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = N$. Putting $H = \rho(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ and $\Delta_m = \{x \in \Delta \mid Hx = mx\}$ ($m \in \mathbf{Z}$), then

$$M_n\Delta = \sum_{m \leq n} \Delta_m.$$

Decomposing $\Delta$ as a representation of $sl(2)$

$$\Delta \xrightarrow{\sim} \bigoplus_{j \geq 0} \left(S^j E^2\right)^{\oplus m_j(\Delta)},$$

then the multiplicities $m_j = m_j(\Delta)$ are related to other numerical invariants of $\Delta$ as follows:

$$\dim(\Delta) = \sum_{j \geq 0} (j+1) m_j, \quad (\forall r \geq 0) \quad d_{-r} = \sum_{i \geq 0} m_{r+2i}, \quad m_r = d_{-r} - d_{-r-2},$$

$$\dim\, \mathrm{Im}(N^r) = d_r + 2\sum_{j > r} d_j, \quad \dim\, \mathrm{Ker}(N^{r+1}) = d_0 + 2\sum_{j=1}^{r} d_j + d_{r+1}. \tag{1.3.6.1}$$

(1.4) PURITY

(1.4.1) DEFINITION. *Let $E'$ be a field containing $E$ and $a \in \mathbf{Z}$. We say that $\alpha \in E'$ is a $q^a$-WEIL NUMBER OF WEIGHT $n \in \mathbf{Z}$ if $\alpha$ is algebraic over $\mathbf{Q}$, there exists $N \in \mathbf{Z}$ such that $q^N \alpha$ is integral over $\mathbf{Z}$, and for each embedding $\sigma : \mathbf{Q}(\alpha) \hookrightarrow \mathbf{C}$, the usual archimedean absolute value of $\sigma(\alpha)$ is equal to $|\sigma(\alpha)|_\infty = q^{an/2}$.*

(1.4.2) DEFINITION. *We say that $\Delta \in \mathrm{Rep}_E('W_K)$ is STRICTLY PURE OF WEIGHT $n \in \mathbf{Z}$ if $\Delta = \rho \in \mathrm{Rep}_E(W_K)$ and if for each $w \in W_K$ all eigenvalues of $\rho(w)$ are $q^{\nu(w)}$-Weil numbers of weight $n \in \mathbf{Z}$.*

(1.4.3) DEFINITION. *We say that $\Delta \in \mathrm{Rep}_E('W_K)$ is PURE OF WEIGHT $n \in \mathbf{Z}$ if, for each $r \in \mathbf{Z}$, $\mathrm{gr}_r^M \Delta \in \mathrm{Rep}_E(W_K)$ is strictly pure of weight $n + r$.*

(1.4.4) (1) Each representation $\rho \in \mathrm{Rep}_E(W_K)$ with finite image is strictly pure of weight 0.
(2) If $\Delta, \Delta' \in \mathrm{Rep}_E('W_K)$ are (strictly) pure of weights $n$ and $n'$, respectively, then $\Delta \otimes \Delta'$ is (strictly) pure of weight $n + n'$, and $\Delta^*$ is (strictly) pure of weight $-n$.
(3) For each $m \in \mathbf{Z}$, $E|\cdot|^m$ is strictly pure of weight $-2m$.
(4) For each $\rho \in \mathrm{Rep}_E(W_K)$ and $m \geq 1$,

$$\begin{aligned} \Delta = \rho \otimes sp(m) \text{ is pure of weight } n \iff & \rho \text{ is strictly pure of weight } n + m - 1 \\ \implies \Delta_f = \rho^I |\cdot|^{m-1} & \text{ is strictly pure of weight } n + 1 - m. \end{aligned}$$

(5) If $\Delta \in \mathrm{Rep}_E('W_K)$ is pure of weight $n < 0$, then all eigenvalues of $\rho(\widetilde{f})$ (for any $\widetilde{f} \in \nu^{-1}(1)$) on $\mathrm{Ker}(N) \subseteq M_0\Delta$ are $q$-Weil numbers of weights $\leq n < 0$, hence $H^0(\Delta) = 0$.
(6) If $\Delta \in \mathrm{Rep}_E('W_K)$ is pure of weight $n$ (but not necessarily $f$-semi-simple), then $\Delta \xrightarrow{\sim} \bigoplus \rho_j \otimes sp(m_j)$, where each $\rho_j \in \mathrm{Rep}_E(W_K)$ is strictly pure of weight $n + m_j - 1$.

(1.4.5) DEFINITION. *In the situation of 1.1.3, we say that $V \in \mathrm{Rep}_E(G_K)$ is PURE OF WEIGHT $n \in \mathbf{Z}$ if $WD(V) \in \mathrm{Rep}_E('W_K)$ is pure of weight $n \in \mathbf{Z}$ in the sense of 1.4.3.*

(1.5) SPECIALIZATION OF REPRESENTATIONS OF THE WEIL-DELIGNE GROUP

(1.5.1) Let $\mathcal{O}$ be a discrete valuation ring containing $\mathbf{Q}$; denote by $E$ (resp., $k_{\mathcal{O}}$) the field of fractions (resp., the residue field) of $\mathcal{O}$.

(1.5.2) An object of $\mathrm{Rep}_{\mathcal{O}}('W_K)$ (= a representation of the Weil-Deligne group of $K$ over $\mathcal{O}$) consists of a free $\mathcal{O}$-module of finite type $T$, a continuous homomorphism $\rho = \rho_T : W_K \longrightarrow \mathrm{Aut}_{\mathcal{O}}(T)$ (with respect to the discrete topology on the target) and a nilpotent endomorphism $N = N_T \in \mathrm{End}_{\mathcal{O}}(T)$ satisfying

$$\forall w \in W_K \qquad \rho(w) N \rho(w)^{-1} = |w|\, N.$$

The GENERIC FIBRE (resp., the SPECIAL FIBRE) of $T$ is the representation $T_\eta = T \otimes_{\mathcal{O}} E \in \mathrm{Rep}_E('W_K)$ (resp., the representation $T_s = T \otimes_{\mathcal{O}} k_{\mathcal{O}} \in \mathrm{Rep}_{k_{\mathcal{O}}}('W_K)$). We denote by $N_\eta$ (resp., $N_s$) the monodromy operator $N_T \otimes 1$ on $T_\eta$ (resp., on $T_s$).

(1.5.3) For $T \in \mathrm{Rep}_{\mathcal{O}}('W_K)$, we denote by $T^*$ the representation $T^* = \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O})$ (equipped with the dual action of $W_K$ and the monodromy operator $N_{T^*} = -(N_T)^*$). Given a representation $\omega : W_K \longrightarrow \mathcal{O}^*$, we say that $T$ is $\omega$-ORTHOGONAL (resp., $\omega$-SYMPLECTIC) if there exists an isomorphism $j : T \xrightarrow{\sim} T^* \otimes \omega$ in $\mathrm{Rep}_{\mathcal{O}}('W_K)$ satisfying $j^* \otimes \omega = j$ (resp., $j^* \otimes \omega = -j$).

(1.5.4) PROPOSITION. *Assume that $T \in \mathrm{Rep}_{\mathcal{O}}('W_K)$ is $|\cdot|$-symplectic (hence so are $T_\eta$ and $T_s$) and that $T_s \in \mathrm{Rep}_{k_{\mathcal{O}}}('W_K)$ is pure (necessarily of weight $-1$). Then:*

(1) $\forall j \geq 0 \qquad m_j(T_\eta) = m_j(T_s)$.

(2) $\forall j \geq 0 \qquad \dim_E \mathrm{Ker}(N_\eta^j) = \dim_{k_{\mathcal{O}}} \mathrm{Ker}(N_s^j)$.

(3) *For each $j \geq 0$, the natural injective map $\big(\mathrm{Ker}(N_\eta^j) \cap T\big) \otimes_{\mathcal{O}} k_{\mathcal{O}} \longrightarrow \mathrm{Ker}(N_s^j)$ is an isomorphism.*

*Proof.* It is enough to prove (1), since (2) is a consequence of (1) and the formulas (1.3.6.1), and (2) is equivalent to (3) for trivial reasons. We prove (1) by induction on $r = \min\{j \geq 0 \mid N_T^{j+1} = 0\}$. If $r = 0$, then there is nothing to prove. Assume that $r \geq 1$ and that (1) holds whenever $N_T^r = 0$. Recall from 1.3.2(2) and 1.3.5 that

$$M_{-r-1}(T_\eta) = 0 \neq M_{-r}(T_\eta) = \mathrm{Im}(N_\eta^r), \quad M_{r-1}(T_\eta) = \mathrm{Ker}(N_\eta^r) \neq T_\eta = M_r(T_\eta),$$
$$M_{-r-1}(T_s) = 0, \quad M_{-r}(T_s) = \mathrm{Im}(N_s^r), \quad M_{r-1}(T_s) = \mathrm{Ker}(N_s^r), \quad M_r(T_s) = T_s$$

and that $M_{-r}(T_\eta)$ is $|\cdot|^{r+1}$-symplectic (resp., $|\cdot|^{r+1}$-orthogonal) if $2 \mid r$ (resp., if $2 \nmid r$). The latter property implies that, for any eigenvalue $\alpha \in \overline{k_{\mathcal{O}}}$ of any lift $\widetilde{f} \in \nu^{-1}(1)$ of $f$ acting on $(M_{-r}(T_\eta) \cap T) \otimes_{\mathcal{O}} k_{\mathcal{O}}$ there exists another eigenvalue $\alpha'$ such that $\alpha\alpha' = q^{-r-1}$. On the other hand, $(M_{-r}(T_\eta) \cap T) \otimes_{\mathcal{O}} k_{\mathcal{O}} \in \mathrm{Rep}_{k_{\mathcal{O}}}(W_K)$ is a sub-object of $T_s$ in $\mathrm{Rep}_{k_{\mathcal{O}}}('W_K)$, and all eigenvalues of $\widetilde{f}$ on $T_s$ are $q$-Weil numbers of weights contained in $\{-r-1, -r, \ldots, r-1\}$; thus both $\alpha$ and $\alpha'$ are $q$-Weil numbers of weight $-r-1$. In other words, $(\mathrm{Im}(N_\eta^r) \cap T) \otimes_{\mathcal{O}} k_{\mathcal{O}} = (M_{-r}(T_\eta) \cap T) \otimes_{\mathcal{O}} k_{\mathcal{O}}$ is strictly pure of weight $-r-1$, hence is contained in $M_{-r}(T_s) = \mathrm{Im}(N_s^r) = (\mathrm{Im}(N_T^r)) \otimes_{\mathcal{O}} k_{\mathcal{O}}$. The opposite inclusion being trivial, we deduce that $\mathrm{Im}(N_T^r)$ is equal to $\mathrm{Im}(N_\eta^r) \cap T$, hence is a direct summand of $T$ (as an $\mathcal{O}$-module); it follows that

$$m_r(T_s) = \dim_{k_{\mathcal{O}}} \mathrm{Im}(N_s^r) = \dim_E \mathrm{Im}(N_\eta^r) = m_r(T_\eta).$$

The representation $T' = (M_{r-1}(T_\eta) \cap T)/(M_{-r}(T_\eta) \cap T) \in \mathrm{Rep}_{\mathcal{O}}('W_K)$ is also $|\cdot|$-symplectic, satisfies $N_{T'}^r = 0$, and $T'_s$ is pure of weight $-1$. By induction hypothesis, we have

$$\forall j \geq 0 \qquad m_j(T'_s) = m_j(T'_\eta).$$

The relations

$$m_j(T'_?) = \begin{cases} m_j(T_?), & j \neq r, r-2 \\ m_{r-2}(T_?) + m_r(T_?), & j = r-2 \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad (? = \eta, s)$$

then imply

$$\forall j \geq 0 \qquad m_j(T_s) = m_j(T_\eta).$$

## 2. Local $\varepsilon$-factors

(2.1) General facts

(2.1.1) Fix an algebraically closed field $E' \supset E$. Let $\psi : K \longrightarrow E'^*$ be a non-trivial continuous homomorphism (with respect to the discrete topology on the target); it always exists. If $\psi' : K \longrightarrow E'^*$ is another non-trivial continuous homomorphism, then there exists unique $a \in K^*$ such that $\psi' = \psi_a$, where $\psi_a(y) = \psi(ay)$. Denote by $\mu_\psi$ the unique $E'$-valued Haar measure on $K$ which is self-dual with respect to $\psi$; then

$$\forall a \in K^* \qquad \mu_{\psi_a} = |a|^{1/2}\, \mu_\psi, \tag{2.1.1.1}$$

and every non-zero $E'$-valued Haar measure $\mu$ on $K$ is a scalar multiple of $\mu_\psi$: $\mu = b\, \mu_\psi$, for some $b \in E'^*$.

(2.1.2) Deligne [De 1] associated to each triple $(\Delta, \psi, \mu)$, where $\Delta \in \mathrm{Rep}_E('W_K)$ and $\psi, \mu$ are as in 2.1.1, the local $\varepsilon$-factor $\varepsilon(\Delta, \psi, \mu) \in E'^*$ satisfying the following properties.

(2.1.2.1) $\varepsilon(\Delta, \psi, \mu) = \varepsilon(\Delta^{f-ss}, \psi, \mu)$.

(2.1.2.2) If $0 \longrightarrow \rho' \longrightarrow \rho \longrightarrow \rho'' \longrightarrow 0$ is an exact sequence in $\mathrm{Rep}_E(W_K)$, then $\varepsilon(\rho, \psi, \mu) = \varepsilon(\rho', \psi, \mu)\varepsilon(\rho'', \psi, \mu)$.

(2.1.2.3) $\varepsilon_0(\Delta, \psi, \mu) = \varepsilon(\Delta, \psi, \mu)\det(-f \mid \Delta_f)$ depends only on $\Delta^{N-ss} \in \mathrm{Rep}_E(W_K)$. As $(\Delta^{N-ss})_f = \Delta_g$, it follows that

$$\varepsilon(\Delta, \psi, \mu) = \varepsilon(\Delta^{N-ss}, \psi, \mu)\det(-f \mid \Delta_g/\Delta_f).$$

(2.1.2.4) $\forall a \in K^* \qquad \varepsilon(\Delta, \psi_a, \mu) = (\det \Delta)(a)\, |a|^{-\dim(\Delta)}\, \varepsilon(\Delta, \psi, \mu)$.

(2.1.2.5) $\forall b \in E'^* \qquad \varepsilon(\Delta, \psi, b\,\mu) = b^{\dim(\Delta)}\, \varepsilon(\Delta, \psi, \mu)$.

(2.1.2.6) If $\Delta = \rho \in \mathrm{Rep}_E(W_K)$, then $\varepsilon(\rho, \psi, \mu)\, \varepsilon(\rho^* | \cdot |, \psi_{-1}, \mu^*) = 1$ (where $\mu^*$ is the measure dual to $\mu$ with respect to $\psi$).

(2.1.2.7) If $\Delta = \rho \in \mathrm{Rep}_E(W_K)$, and if $\chi : W_K/I \longrightarrow E^*$ is an unramified one-dimensional representation, then

$$\varepsilon(\rho \otimes \chi, \psi, \mu) = \varepsilon(\rho, \psi, \mu)\, \chi(\pi)^{a(\rho) + \dim(\rho) n(\psi)},$$

where $\pi$ is a prime element of $\mathcal{O}_K$ and $a(\rho)$ (resp., $n(\psi)$) is the conductor exponent of $\rho$ (resp., of $\psi$).

(2.1.2.8) ([Fo-PR, I.1.2.3]) For an exact sequence in $\mathrm{Rep}_E('W_K)$

$$(\beta) \qquad 0 \longrightarrow \Delta' \longrightarrow \Delta \longrightarrow \Delta'' \longrightarrow 0,$$

set $P_K(\beta) = P_K(\Delta, u)/P_K(\Delta', u)P_K(\Delta'', u)$, $a(\beta) = \dim \Delta'_f + \dim \Delta''_f - \dim \Delta_f$, $\varepsilon(\beta) = \varepsilon(\Delta, \psi, \mu)/\varepsilon(\Delta', \psi, \mu)\,\varepsilon(\Delta'', \psi, \mu)$, and similarly for the dual exact sequence

$$(\beta^*|\cdot|) \qquad 0 \longrightarrow \Delta''^*|\cdot| \longrightarrow \Delta^*|\cdot| \longrightarrow \Delta'^*|\cdot| \longrightarrow 0;$$

then

$$P_K(\beta^*|\cdot|, u^{-1}) = \varepsilon(\beta)\, u^{a(\beta)}\, P_K(\beta, u).$$

(2.1.3) Lemma. *If $\Delta \in \mathrm{Rep}_E('W_K)$, then $\varepsilon(\Delta, \psi, \mu)\,\varepsilon(\Delta^*|\cdot|, \psi_{-1}, \mu^*) = 1$ (where $\mu^*$ is the measure dual to $\mu$ with respect to $\psi$).*

*Proof.* Thanks to (2.1.2.1-2), we can assume that $\Delta$ is $f$-semi-simple and indecomposable: $\Delta = \rho \otimes sp(m)$, $\rho \in \mathrm{Rep}_E(W_K)$, $m \geq 1$. In this case

$$\Delta_g = \bigoplus_{j=0}^{m-1} \rho^I |\cdot|^j, \qquad \Delta_g/\Delta_f = \bigoplus_{j=0}^{m-2} \rho^I |\cdot|^j, \qquad \Delta^*|\cdot| = \rho^* \otimes sp(m)|\cdot|^{2-m}$$

$$(\Delta^*|\cdot|)_g/(\Delta^*|\cdot|)_f = \bigoplus_{j=0}^{m-2} (\rho^*)^I |\cdot|^{2-m+j} = (\Delta_g/\Delta_f)^*$$

(as $\rho(I)$ is finite, we have $(\rho^*)^I = (\rho^I)^*$), hence

$$\det(-f \mid \Delta_g/\Delta_f)\, \det(-f \mid (\Delta^*|\cdot|)_g/(\Delta^*|\cdot|)_f) = 1;$$

we deduce that

$$\varepsilon(\Delta, \psi, \mu)\,\varepsilon(\Delta^*|\cdot|, \psi_{-1}, \mu^*) = \varepsilon(\Delta^{N-ss}, \psi, \mu)\,\varepsilon((\Delta^*|\cdot|)^{N-ss}, \psi_{-1}, \mu^*),$$

which is equal to 1, by (2.1.2.6).

(2.2) $|\cdot|$-symplectic representations

(2.2.1) Proposition. *Let $\Delta \xrightarrow{\sim} \Delta^*|\cdot| \in \mathrm{Rep}_E('W_K)$ be $|\cdot|$-symplectic. Then:*
(1) *$\varepsilon(\Delta) := \varepsilon(\Delta, \psi, \mu_\psi)$ does not depend on $\psi$.*
(2) *$\varepsilon(\Delta) = \pm 1$; more precisely:*
(3) *If $\rho \xrightarrow{\sim} \rho^*|\cdot| \in \mathrm{Rep}_E(W_K)$ is $|\cdot|$-symplectic, then $\varepsilon(\rho) = \pm 1$.*
(4) *If $\Delta = X \oplus X^*|\cdot|$ is as in 1.2.4(1), then $\varepsilon(\Delta) = \varepsilon(\Delta^{N-ss}) = (\det X)(-1)$.*
(5) *If $\Delta = \rho \otimes sp(2n+1)$ ($\rho \in \mathrm{Rep}_E(W_K)$, $n \geq 0$), then $\rho|\cdot|^n \in \mathrm{Rep}_E(W_K)$ is $|\cdot|$-symplectic and $\varepsilon(\Delta) = \varepsilon(\Delta^{N-ss}) = \varepsilon(\rho|\cdot|^n)$.*

(6) *If* $\Delta = \rho \otimes sp(2n)$ $(\rho \in \mathrm{Rep}_E(W_K),\ n \geq 1)$, *then* $\rho|\cdot|^{n-1} \in \mathrm{Rep}_E(W_K)$ *is orthogonal, there is an exact sequence in* $\mathrm{Rep}_E('W_K)$

$$0 \longrightarrow \Delta^+ \longrightarrow \Delta \longrightarrow \Delta^- \longrightarrow 0$$
$$\Delta^+ = \rho \otimes sp(n)|\cdot|^n, \qquad \Delta^- = \rho \otimes sp(n),$$

$H^0(\Delta^-) = H^0(\rho|\cdot|^{n-1})$ *and*

$$\varepsilon(\Delta) = (-1)^{\dim_E H^0(\Delta^-)}(\det \Delta^+)(-1), \qquad \varepsilon(\Delta^{N-ss}) = (\det \Delta^+)(-1).$$

*Proof.* (1) For each $a \in K^*$,

$$\begin{aligned} \varepsilon(\Delta, \psi_a, \mu_{\psi_a}) &= \varepsilon(\Delta, \psi_a, |a|^{1/2}\mu_\psi) && \text{(by (2.1.1.1))} \\ &= |a|^{\dim(\Delta)/2}\,\varepsilon(\Delta, \psi_a, \mu_\psi) && \text{(by (2.1.2.5))} \\ &= (\det \Delta)(a)\,|a|^{-\dim(\Delta)/2}\,\varepsilon(\Delta, \psi, \mu_\psi) && \text{(by (2.1.2.4))} \\ &= \varepsilon(\Delta, \psi, \mu_\psi). && \text{(by 1.2.2(2))} \end{aligned}$$

(2) Writing $\Delta^{f-ss}$ as a direct sum of $|\cdot|$-symplectic representations of the form 1.2.4(1) or 1.2.4(2), the statement follows from the explicit formulas (4)-(6) and (3), proved below.
(3) Combining (2.1.2.6), (2.1.2.4) and 1.2.2(2), we obtain

$$\begin{aligned} \varepsilon(\rho, \psi, \mu_\psi)^2 = \varepsilon(\rho, \psi, \mu_\psi)\,(\det \rho)(-1)\varepsilon(\rho, \psi, \mu_\psi) = \varepsilon(\rho, \psi, \mu_\psi)\,\varepsilon(\rho, \psi_{-1}, \mu_\psi) = \\ = \varepsilon(\rho, \psi, \mu_\psi)\,\varepsilon(\rho^*|\cdot|, \psi_{-1}, \mu_\psi) = 1. \end{aligned}$$

(4) As in the proof of (3), Lemma 2.1.3 together with (2.1.2.4) yield

$$\begin{aligned} \varepsilon(\Delta) = \varepsilon(X, \psi, \mu_\psi)\,\varepsilon(X^*|\cdot|, \psi, \mu_\psi) = (\det X)(-1)\,\varepsilon(X, \psi_{-1}, \mu_\psi)\,\varepsilon(X^*|\cdot|, \psi, \mu_\psi) = \\ = (\det X)(-1). \end{aligned}$$

Replacing $X$ by $X^{N-ss}$, we obtain $\varepsilon(\Delta^{N-ss}) = (\det X^{N-ss})(-1) = (\det X)(-1) = \varepsilon(\Delta)$.
(5) As $\Delta = \rho \otimes sp(2n+1)$ is $|\cdot|$-symplectic, the representation $\rho|\cdot|^n$ is also $|\cdot|$-symplectic, by 1.2.3-4 (in particular, $\det(\rho) = |\cdot|^{(1-2n)\dim(\rho)/2}$). The same calculation as in the proof of Lemma 2.1.3 yields

$$\Delta_g/\Delta_f = \bigoplus_{j=0}^{2n-1} \rho^I|\cdot|^j, \qquad (\rho^I|\cdot|^j)^* = (\rho^*|\cdot|^{-j})^I = \rho^I|\cdot|^{2n-1-j},$$
$$\Delta_g/\Delta_f = \bigoplus_{j=0}^{n-1} \rho^I|\cdot|^j \oplus \left(\rho^I|\cdot|^j\right)^*,$$

which implies that $\det(-f \mid \Delta_g/\Delta_f) = 1$, hence

$$\varepsilon(\Delta) = \varepsilon(\Delta^{N-ss}) = \prod_{j=0}^{2n} \varepsilon(\rho|\cdot|^j, \psi, \mu_\psi) = \varepsilon(\rho|\cdot|^n) \prod_{j=0}^{n-1} \varepsilon(\rho|\cdot|^j \oplus (\rho|\cdot|^j)^*|\cdot|) \overset{(4)}{=} \varepsilon(\rho|\cdot|^n).$$

(6) As $\Delta = \rho \otimes sp(2n)$ is $|\cdot|$-symplectic, the representation $\rho|\cdot|^{n-1}$ is orthogonal, by 1.2.3. The same calculation as in the proof of (5) shows that

$$\varepsilon(\Delta^{N-ss}) = \prod_{j=0}^{n-1} \varepsilon(\rho|\cdot|^j \oplus (\rho|\cdot|^j)^*|\cdot|) \overset{(4)}{=} \prod_{j=0}^{n-1} \det(\rho|\cdot|^j)(-1) = (\det\Delta^+)(-1)$$

and

$$\Delta_g/\Delta_f = \rho^I|\cdot|^{n-1} \oplus \bigoplus_{j=0}^{n-2} \rho^I|\cdot|^j \oplus \left(\rho^I|\cdot|^j\right)^*, \quad \det(-f \mid \Delta_g/\Delta_f) = \left(-f \mid \rho^I|\cdot|^{n-1}\right).$$

As $\rho(I)$ acts semi-simply, the (unramified) representation $V = \rho^I|\cdot|^{n-1} \in \mathrm{Rep}_E(W_K)$ is also orthogonal; applying Lemma 2.2.2 below to $u = f$ acting on $V$, we obtain

$$\varepsilon(\Delta)/\varepsilon(\Delta^{N-ss}) = \det(-f \mid \Delta_g/\Delta_f) = (-1)^{\dim_E \mathrm{Ker}(f-1:V\longrightarrow V)}.$$

Finally,

$$\mathrm{Ker}(V \xrightarrow{f-1} V) = H^0(\rho|\cdot|^{n-1}) = H^0(\rho \otimes sp(n)) = H^0(\Delta^-).$$

(2.2.2) Lemma. *Let $(V, q)$ be a non-degenerate quadratic space over a field $L$ of characteristic not equal to 2. If $u \in O(V, q)$, then*

$$\det(-u) = (-1)^{\dim_L \mathrm{Ker}(u-1)}, \qquad \det(u) = (-1)^{\dim_L \mathrm{Im}(u-1)}.$$

*Proof.* The following short argument is due to J. Oesterlé. The two formulas being equivalent, it is enough to prove the second one. Let $a \in V$, $q(a) \neq 0$; denote by $s \in O^-(V, q)$ the reflection with respect to the hyperplane $\mathrm{Ker}(s-1) = a^\perp$. A short calculation shows that

$$\mathrm{Ker}(su-1) = \begin{cases} \mathrm{Ker}(u-1) \oplus Lb, & a = (u-1)b,\ b \in V \\ \mathrm{Ker}(u-1) \cap a^\perp \subsetneq \mathrm{Ker}(u-1), & a \notin \mathrm{Im}(u-1), \end{cases}$$

hence

$$\dim_L \mathrm{Im}(su-1) = \dim_L \mathrm{Im}(u-1) \mp 1. \tag{2.2.2.1}$$

Writing $u$ as a product of $r \geq 1$ reflections, we deduce from (2.2.2.1), by induction, that $\dim_L \mathrm{Im}(u-1) \equiv r \pmod 2$, as claimed.

(2.2.3) Proposition. *Let* $\Delta \xrightarrow{\sim} \Delta^*|\cdot| \in \mathrm{Rep}_E('W_K)$ *be* $|\cdot|$*-symplectic and pure (of weight* $-1$*). Assume that there exists an exact sequence in* $\mathrm{Rep}_E('W_K)$

$$(\beta) \qquad 0 \longrightarrow \Delta^+ \longrightarrow \Delta \longrightarrow \Delta^- \longrightarrow 0$$

*such that the isomorphism* $\Delta \xrightarrow{\sim} \Delta^*|\cdot|$ *induces isomorphisms* $\Delta^\pm \xrightarrow{\sim} (\Delta^\mp)^*|\cdot|$. *Assume, in addition, that there exists a direct sum decomposition* $\Delta = \Delta_1 \oplus \Delta_2$ *in* $\mathrm{Rep}_E('W_K)$ *compatible with the isomorphism* $\Delta \xrightarrow{\sim} \Delta^*|\cdot|$ *and the exact sequence* $(\beta)$*, and such that* $H^0(\Delta_2^-) = 0$*, while*

$$(\beta_1) \qquad 0 \longrightarrow \Delta_1^+ \longrightarrow \Delta_1 \longrightarrow \Delta_1^- \longrightarrow 0$$

*is a direct sum of exact sequences of the type considered in Proposition 2.2.1(6). Then*

$$\varepsilon(\Delta) = (-1)^{\dim_E H^0(\Delta^-)}(\det \Delta^+)(-1), \qquad \varepsilon(\Delta^{N-ss}) = (\det \Delta^+)(-1).$$

*Proof.* It is enough to treat separately $\Delta_1$ and $\Delta_2$. For $\Delta = \Delta_1$, the statement follows from Proposition 2.2.1(6). For $\Delta = \Delta_2$, the assumption $H^0(\Delta^-) = 0$ implies that $P_K(\Delta^-, 1) \neq 0$. As $\Delta$ is pure of weight $-1 < 0$, we also have $H^0(\Delta^+) \subseteq H^0(\Delta) = 0$, by 1.4.4(5), hence $P_K(\Delta^+, 1)P_K(\Delta, 1) \neq 0$. Letting $u \longrightarrow 1$ in (2.1.2.8), we obtain $\varepsilon(\beta) = 1$, hence

$$\varepsilon(\Delta) = \varepsilon(\Delta^+, \psi, \mu_\psi)\, \varepsilon(\Delta^-, \psi, \mu_\psi) = \varepsilon(\Delta^+ \oplus (\Delta^+)^*|\cdot|) = (\det \Delta^+)(-1).$$

Finally,

$$\begin{aligned} \varepsilon(\Delta^{N-ss}) &= \varepsilon((\Delta^+)^{N-ss}, \psi, \mu_\psi)\, \varepsilon((\Delta^-)^{N-ss}, \psi, \mu_\psi) = \\ &= \varepsilon((\Delta^+)^{N-ss} \oplus ((\Delta^+)^{N-ss})^*|\cdot|) = (\det(\Delta^+)^{N-ss})(-1) = (\det \Delta^+)(-1). \end{aligned}$$

(2.2.4) Proposition. *In the situation of 1.5.4,* $\varepsilon(T_s) = \varepsilon(T_\eta) \in \{\pm 1\}$.

*Proof.* For any $\mathcal{O}$-module $X$, denote by $red : X \longrightarrow X \otimes_{\mathcal{O}} k_{\mathcal{O}}$ the canonical surjection. Proposition 1.5.4 implies that

$$red\left(\frac{T \cap (T_\eta)_g}{T \cap (T_\eta)_f}\right) = (T_s)_g/(T_s)_f,$$

hence

$$\begin{aligned} red\left(\varepsilon(T_\eta)/\varepsilon(T_\eta^{N-ss})\right) &= red\,(\det(-f \mid (T_\eta)_g/(T_\eta)_f)) = \\ &= (\det(-f \mid (T_s)_g/(T_s)_f) = \varepsilon(T_s)/\varepsilon(T_s^{N-ss}). \end{aligned}$$

As $\varepsilon(T_\eta), \varepsilon(T_\eta^{N-ss}), \varepsilon(T_s), \varepsilon(T_s^{N-ss}) \in \{\pm 1\}$, we are reduced to showing that

$$red\left(\varepsilon(T_\eta^{N-ss})\right) \stackrel{?}{=} \varepsilon(T_s^{N-ss}).$$

The following argument is based on a suggestion of T. Saito. We can replace $(\rho_T, N_T)$ by $(\rho_T, 0)$ and assume that $N_T = 0$. Furthermore, after replacing $E$ by a finite extension, we can assume (see [De 1, 4.10]) that

$$T_\eta^{f-ss} = \bigoplus_\alpha \rho_\alpha \otimes \omega_\alpha,$$

where $\rho_\alpha \in \mathrm{Rep}_L(W_K)$ for a subfield $L \subset \mathcal{O}$ of finite degree over $\mathbf{Q}$, and $\omega_\alpha : W_K/I \longrightarrow \mathcal{O}^*$ is an unramified representation of rank 1. We have

$$\forall w \in W_K \qquad \mathrm{Tr}(w \mid T_s) = red\left(\mathrm{Tr}(w \mid T_\eta)\right),$$

hence

$$T_s^{f-ss} = \bigoplus_\alpha \rho_\alpha \otimes red(\omega_\alpha).$$

Applying (2.1.2.7) to each direct summand, we obtain

$$red\left(\varepsilon(T_\eta)\right) = \prod_\alpha red\left(\varepsilon(\rho_\alpha \otimes \omega_\alpha, \psi, \mu_\psi)\right) = \prod_\alpha \varepsilon(\rho_\alpha \otimes red(\omega_\alpha), red \circ \psi, red \circ \mu_\psi) = \\ = \varepsilon(T_s).$$

(2.3) The archimedean case

Let $L = \mathbf{R}$ or $\mathbf{C}$. If $H$ is a pure $\mathbf{R}$-Hodge structure over $L$ ([Fo-PR, III.1]) of weight $-1$, then

$$H = \bigoplus_{r>0} H_r(r)^{\oplus m_r},$$

where $H_r$ is a two-dimensional pure $\mathbf{R}$-Hodge structure over $L$ of Hodge type $(2r-1, 0)$, $(0, 2r-1)$. The standard formulas ([De 3, 5.3], [Fo-PR, III.1.1.10, III.1.2.7]) yield

$$\varepsilon(H_r(r)) = (-1)^{[L:\mathbf{R}]\,r} \times \begin{cases} 1, & L = \mathbf{R} \\ -1, & L = \mathbf{C}. \end{cases}$$

As

$$\forall p < 0 \qquad h^{p,-1-p}(H) = m_{-p},$$

we obtain

$$\varepsilon(H) = (-1)^{[L:\mathbf{R}]\,d^-(H)} \times \begin{cases} 1, & L = \mathbf{R} \\ (-1)^{(\dim_{\mathbf{R}} H)/2}, & L = \mathbf{C}, \end{cases} \quad d^-(H) = \sum_{p<0} p\, h^{p,q}(H). \tag{2.3.1}$$

## 3. Local $p$-adic Galois representations

(3.1) General facts

(3.1.1) Notation. Let $p$ be the characteristic of the residue field $k$ of $K$; then $q = p^h$ and $K$ is a finite extension of $\mathbf{Q}_p$. Denote by $\sigma \in \mathrm{Gal}(\mathbf{Q}_p^{ur}/\mathbf{Q}_p) \xrightarrow{\sim} G_{\mathbf{F}_p}$ the lift of the arithmetic Frobenius element $x \mapsto x^p$. Let $L$ be another finite extension of $\mathbf{Q}_p$.

We use the standard notation

$$\mathrm{Rep}_{cris,L}(G_K) \subset \mathrm{Rep}_{st,L}(G_K) \subset \mathrm{Rep}_{pst,L}(G_K) = \mathrm{Rep}_{dR,L}(G_K) \subset \mathrm{Rep}_L(G_K)$$

for Fontaine's hiearchy of (finite-dimensional, $L$-linear) representations of $G_K$ ([Fo]), and

$$D_{cris}(V) = (V \otimes_{\mathbf{Q}_p} B_{cris})^{G_K}, \qquad D_{st}(V) = (V \otimes_{\mathbf{Q}_p} B_{st})^{G_K},$$
$$D_{pst}(V) = \varinjlim_{K'} (V \otimes_{\mathbf{Q}_p} B_{st})^{G_{K'}},$$
$$D_{dR}^i(V) = (V \otimes_{\mathbf{Q}_p} t^i B_{dR}^+)^{G_K} \subset D_{dR}(V) = (V \otimes_{\mathbf{Q}_p} B_{dR})^{G_K}$$

for various Fontaine's functors (above, $V \in \mathrm{Rep}_L(G_K)$, and $K'$ runs through all finite extensions of $K$ contained in $\overline{K}$). As in [Bl-Ka], put $H^i(K, -) = H^i_{\mathrm{cont}}(G_K, -)$ and, for $* = e, f, st, g$,

$$H^1_*(K, V) = \mathrm{Ker}\left(H^1(K, V) \longrightarrow H^1(K, V \otimes_{\mathbf{Q}_p} B_*)\right),$$
$$B_e = B_{cris}^{\varphi=1}, \qquad B_f = B_{cris}, \qquad B_g = B_{dR}.$$

If $K'/K$ is a finite Galois extension, then

$$H^1_*(K, V) = H^1_*(K', V)^{\mathrm{Gal}(K'/K)}, \qquad (* = \emptyset, e, f, st, g) \qquad (3.1.1.1)$$

(as both $H^1(-, V)$ and $H^1(-, V \otimes_{\mathbf{Q}_p} B_*)$ satisfy Galois descent w.r.t. the extension $K'/K$, and the functor of $\mathrm{Gal}(K'/K)$-invariants is exact on the category of $\mathbf{Q}[\mathrm{Gal}(K'/K)]$-modules).

(3.1.2) For $V \in \mathrm{Rep}_{dR,L}(G_K)$ and $i \in \mathbf{Z}$, define

$$d_L^i(V) := \dim_L \left(D_{dR}^i(V)/D_{dR}^{i+1}(V)\right), \qquad d_L^-(V) := \sum_{i<0} i\, d_L^i(V),$$
$$d_L(V) := \sum_{i \in \mathbf{Z}} i\, d_L^i(V).$$

(3.1.3) If $V \in \mathrm{Rep}_{pst,L}(G_K)$, then $D = D_{pst}(V)$ is a free $(\mathbf{Q}_p^{ur} \otimes_{\mathbf{Q}_p} L)$-module of rank equal to $\dim_L(V)$, which is equipped (among others) with the following structure ([Fo], [Fo-PR, I.2.2]):

(1) An $L$-linear action $\rho_{sl} : W_K \longrightarrow \mathrm{Aut}_L(D)$, which is $\mathbf{Q}_p^{ur}$-semi-linear in the following sense:

$$\forall w \in W_K \ \forall \lambda \in \mathbf{Q}_p^{ur} \ \forall x \in D \qquad \rho_{sl}(w)(\lambda x) = f_k^{\nu(w)}(\lambda)\, \rho_{sl}(w)(x).$$

(2) An $L$-linear, $\sigma$-semi-linear map $\varphi : D \longrightarrow D$ commuting with $\rho_{sl}(w)$ (for all $w \in W_K$):

$$\forall w \in W_K \ \forall \lambda \in \mathbf{Q}_p^{ur} \ \forall x \in D \qquad \varphi(\lambda x) = \sigma(\lambda)\, \varphi(x).$$

(3) A $(\mathbf{Q}_p^{ur} \otimes_{\mathbf{Q}_p} L)$-linear nilpotent endomorphism $N : D \longrightarrow D$ commuting with $\rho_{sl}(w)$ (for all $w \in W_K$) and satisfying $N\varphi = p\varphi N$.

(4) An isomorphism of $(K \otimes_{\mathbf{Q}_p} L)$-modules

$$(D \otimes_{\mathbf{Q}_p^{ur}} \overline{K})^{G_K} \xrightarrow{\sim} D_{dR}(V).$$

## (3.2) Potentially semistable representations and representations of the Weil-Deligne group

We recall how, for each $V \in \mathrm{Rep}_{pst,L}(G_K)$, the structure 3.1.3(1)-(3) can be used to define a representation of the Weil-Deligne group of $K$ ([Fo], [Fo-PR, I.1.3.2]).

(3.2.1) Fix a field $E \supset \mathbf{Q}_p^{ur}$ for which there exists an embedding $\tau : L \hookrightarrow E$, and define

$$WD_\tau(V) := D_{pst}(V) \otimes_{\mathbf{Q}_p^{ur} \otimes_{\mathbf{Q}_p} L, \mathrm{id} \otimes \tau} E,$$

which is an $E$-vector space of dimension $\dim_E(WD_\tau(V)) = \dim_L(V)$. We define an $E$-linear action of $W_K$ on $WD_\tau(V)$ by

$$\rho(w) := \rho_{sl}(w) \circ \varphi^{h\nu(w)} \otimes \mathrm{id} \qquad\qquad (w \in W_K)$$

and a monodromy operator $N = N \otimes \mathrm{id} \in \mathrm{End}_E(WD_\tau(V))$. This defines a representation

$$WD_\tau(V) = (\rho, N) \in \mathrm{Rep}_E({}'W_K),$$

whose isomorphism class does not depend on $\tau$. Furthermore,

$$WD_\tau : \mathrm{Rep}_{pst,L}(G_K) \longrightarrow \mathrm{Rep}_E({}'W_K)$$

is an exact tensor functor.

(3.2.2) Examples: (1) If $V$ is potentially unramified, then $WD_\tau(V) = V \otimes_{L,\tau} E \in \mathrm{Rep}_E(W_K)$.

(2) If $V$ is semistable, then $WD(V) = D_{st}(V) \otimes_{K_0 \otimes_{\mathbf{Q}_p} L, \mathrm{id} \otimes \tau} E$ ($K_0 = K \cap \mathbf{Q}_p^{ur}$), with $\rho(I)$ acting trivially, $N = N \otimes \mathrm{id}$ and $\rho(f_k) = \varphi^h \otimes \mathrm{id}$. Conversely, if $\rho(I)$ acts trivially, then $V$ is semistable.
(3) If $V = L(n) = L \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(n)$ ($n \in \mathbf{Z}$), then $WD_\tau(V) = E|\cdot|^n = E \otimes |\cdot|^n$.
(4) (Lubin-Tate theory) Fix a prime element $\pi \in \mathcal{O}_K$. The reciprocity map $\mathrm{rec}_K : K^* \longrightarrow G_K^{ab}$ (normalized using the geometric Frobenius element) defines a one-dimensional representation $V_\pi \in \mathrm{Rep}_{cris,K}(G_K)$

$$\chi_\pi : G_K \longrightarrow G_K^{ab} \xrightarrow{\sim} \widehat{K}^* = \pi^{\widehat{\mathbf{Z}}} \times \mathcal{O}_K^* \twoheadrightarrow \mathcal{O}_K^* \hookrightarrow K^*,$$

which arises in the $\pi$-adic Tate module of any Lubin-Tate group over $K$ associated to $\pi$. In this case

$$\begin{gathered} D_{pst}(V_\pi) = (\mathbf{Q}_p^{ur} \otimes_{\mathbf{Q}_p} K)u, \qquad \varphi^h(u) = (1 \otimes \pi)^{-1} u, \qquad Nu = 0, \\ \forall w \in W_K \quad \rho_{sl}(w)(u) = u. \end{gathered}$$

If $E \supset \mathbf{Q}_p^{ur}$ is a field and $\tau : K \hookrightarrow E$ an embedding of fields, then $WD_\tau(V_\pi) \in \mathrm{Rep}_E(W_K)$ is an unramified one-dimensional representation of $W_K$, on which $f = f_k$ acts by $\tau(\pi)^{-1}$. For $K = \mathbf{Q}_p$ and $\pi = p$ we recover Example (3) for $n = 1$.

(3.2.3) Definition. *We say that $V \in \mathrm{Rep}_{pst,L}(G_K)$ is* pure of weight $n \in \mathbf{Z}$ *if $WD_\tau(V) \in \mathrm{Rep}_E('W_K)$ is pure of weight $n$, in the sense of 1.4.3.*

(3.2.4) Lemma. *For each $V \in \mathrm{Rep}_{pst,L}(G_K)$ and each $\tau : L \hookrightarrow E \supset \mathbf{Q}_p^{ur}$,*

$$WD_\tau(V)_g^{f_k=1} = D_{st}(V)^{\varphi=1} \otimes_{L,\tau} E, \qquad H^0(WD_\tau(V)) = D_{cris}(V)^{\varphi=1} \otimes_{L,\tau} E.$$

*Proof.* As $D_{cris}(V) = D_{st}(V)^{N=0}$, it is enough to prove the first equality. As both sides satisfy Galois descent with respect to finite Galois extensions $K'/K$, we can assume that $V$ is semistable. In this case, 3.2.2(2) implies that

$$WD_\tau(V)_g^{f_k=1} = WD_\tau(V)^{f_k=1} = D_{st}(V)^{\varphi^h=1} \otimes_{K_0 \otimes_{\mathbf{Q}_p} L, \mathrm{id} \otimes \tau} E \qquad (K_0 = K \cap \mathbf{Q}_p^{ur}).$$

As

$$D_{st}(V)^{\varphi^h=1} = D_{st}(V)^{\varphi=1} \otimes_{\mathbf{Q}_p} K_0$$

(thanks to Hilbert's Theorem 90 for $H^1(K_0/\mathbf{Q}_p, GL_n(K_0))$), it follows that

$$WD_\tau(V)_g^{f_k=1} = D_{st}(V)^{\varphi=1} \otimes_{L,\tau} E.$$

(3.2.5) Corollary. *If $V \in \mathrm{Rep}_{pst,L}(G_K)$ is pure of weight $n < 0$, then $D_{cris}(V)^{\varphi=1} = 0$.*

(3.2.6) Proposition. *For each $V \in \mathrm{Rep}_{pst,L}(G_K)$,*

$$(\det_E(WD_\tau(V)))(-1) = (-1)^{d_L(V)} (\det_L V)(-1).$$

*Proof.* As $WD_\tau$ is a tensor functor and $d_L(V) = d_L(\det_L(V))$, we can replace $V$ by $\det_L(V)$, hence assume that $\dim(V) = 1$; denote by $\chi_V : G_K \longrightarrow K^*$ the character by which $G_K$ acts on $V$. After replacing $L$ by a finite extension, we can assume that $L$ contains the Galois closure of $K$ over $\mathbf{Q}_p$. As $V$ is potentially semistable, there exists a one-dimensional representation

$$\chi : G_K \longrightarrow L^*$$

with finite image and integers $n_\sigma$ ($\sigma : K \hookrightarrow L$) such that

$$\chi_V = \chi \prod_{\sigma : K \hookrightarrow L} (\sigma \circ \chi_\pi)^{-n_\sigma},$$

where $\chi_\pi : G_K \longrightarrow K^*$ is as in 3.2.2(4). It follows from 3.2.2 that $WD_\tau(V) = (\tau \circ \chi)\alpha$, where $\alpha : W_K/I \longrightarrow E^*$ is the one-dimensional unramified representation satisfying

$$\alpha(f) = \prod_{\sigma : K \hookrightarrow L} \tau(\sigma(\pi))^{n_\sigma}.$$

This implies that

$$(\det_E(WD_\tau(V)))(-1) = \chi(-1), \quad (\det_L V)(-1) = (-1)^n \chi(-1), \quad n = \sum_{\sigma : K \hookrightarrow L} n_\sigma.$$

On the other hand,

$$d_L^i(V) = |\{\sigma : K \hookrightarrow L \mid n_\sigma = i\}|,$$

hence $n = d_L(V)$.

(3.3) Representations satisfying Pančiškin's condition

We recall a few basic facts from [Ne 1].

(3.3.1) Definition. *We say that $V \in \mathrm{Rep}_L(G_K)$ satisfies* Pančiškin's condition *if there exists an exact sequence in $\mathrm{Rep}_L(G_K)$*

$$0 \longrightarrow V^+ \longrightarrow V \longrightarrow V^- \longrightarrow 0$$

*such that $V^\pm \in \mathrm{Rep}_{pst,L}(G_K)$ and $D^0_{dR}(V^+) = 0 = D_{dR}(V^-)/D^0_{dR}(V^-)$. If this is the case, then $V^\pm$ are uniquely determined ([Ne 1], 6.7), $V \in \mathrm{Rep}_{pst,L}(G_K)$ ([Ne 1], 1.28) and $V^*(1)$ also satisfies Pančiškin's condition (with $(V^*(1))^\pm = (V^\mp)^*(1)$).*

(3.3.2) Proposition. *If $V$ satisfies Pančiškin's condition, then:*
(1) $H^0(K, V^-) = D_{cris}(V^-)^{\varphi=1} = D_{st}(V^-)^{\varphi=1}$.
(2) *Assume that there exists a finite Galois extension $K'/K$ over which $V$ becomes semistable and such that $D_{cris}(V|_{G_{K'}})^{\varphi=1} = D_{cris}(V^*(1)|_{G_{K'}})^{\varphi=1} = 0$ (the latter condition holds, e.g., if $V$ is pure of weight $-1$, by 3.2.5). Then*

$$H^1_e(K, V) = H^1_f(K, V) = H^1_{st}(K, V) = H^1_g(K, V)$$

*and there is an exact sequence*

$$0 \longrightarrow H^0(K, V^-) \longrightarrow H^1(K, V^+) \longrightarrow H^1_f(K, V) \longrightarrow 0,$$

*in which* $H^1(K, V^+) = H^1_{st}(K, V^+)$.

*Proof.* (1) This is proved in [Ne 1, 1.28(3)] under the tacit assumption that $V^-$ is semistable. The general case follows by passing to a finite Galois extension over which $V^-$ becomes semistable and taking Galois invariants.
(2) Over $K'$, the statement is proved in [Ne 1, 1.32]; the general case follows by applying (3.1.1.1).

(3.3.3) PROPOSITION. *Assume that* $V$ *satisfies Pančiškin's condition, is pure (of weight* $-1$*) and that there exists an isomorphism* $j : V \xrightarrow{\sim} V^*(1)$ *in* $\mathrm{Rep}_L(G_K)$ *satisfying* $j^*(1) = -j$. *Then:*
(1) $j$ *induces isomorphisms* $V^\pm \xrightarrow{\sim} (V^\mp)^*(1)$.
(2) *Fix an embedding of fields* $\tau : L \hookrightarrow E \supset \mathbf{Q}_p^{ur}$ *and put* $\Delta = WD_\tau(V)$, $\Delta^\pm = WD_\tau(V^\pm)$. *Then* $\Delta \in \mathrm{Rep}_E('W_K)$ *is* $|\cdot|$*-symplectic and the exact sequence in* $\mathrm{Rep}_E('W_K)$

$$0 \longrightarrow \Delta^+ \longrightarrow \Delta \longrightarrow \Delta^- \longrightarrow 0$$

*satisfies the assumptions of Proposition 2.2.3.*
(3) $(\det_E \Delta^+)(-1)/(\det_L V^+)(-1) = (-1)^{d_L(V^+)} = (-1)^{d_L^-(V)}$.
(4) *The* $\varepsilon$*-factors of* $\Delta$ *and* $\Delta^{N-ss}$ *are equal to*

$$\varepsilon(\Delta) = (-1)^{\dim_L H^0(K, V^-)} (-1)^{d_L^-(V)} (\det_L V^+)(-1),$$
$$\varepsilon(\Delta^{N-ss}) = (-1)^{d_L^-(V)} (\det_L V^+)(-1).$$

*Proof.* (1) This follows from the remarks made in 3.3.1.
(2) $\Delta$ is $|\cdot|$-symplectic, since $WD_\tau$ is a tensor functor. In order to verify the assumptions of Proposition 2.2.3, we are going to decompose $\Delta$ into several components. Firstly, the functor

$$\mathrm{Rep}_E('W_K) \longrightarrow \mathrm{Rep}_E('W_K), \qquad X \mapsto X^{\rho(I)}$$

is exact and commutes with duals. In addition, $X^{\rho(I)}$ is a direct summand of $X$, with a functorial complement $X'$. Secondly, for each $\lambda \in \overline{E}$, the minimal polynomial $p_{[\lambda]}(T)$ of $\lambda$ over $E$ depends only on the $G_E$-orbit $[\lambda]$ of $\lambda$. We define

$$\Delta_1 = \bigoplus_{\lambda \in q^{\mathbf{Z}}} \bigcup_{n \geq 1} \mathrm{Ker}\left((f - \lambda)^n : \Delta^{\rho(I)} \longrightarrow \Delta^{\rho(I)}\right),$$
$$\Delta_2 = \Delta' \oplus \bigoplus_{\lambda \notin q^{\mathbf{Z}}} \bigcup_{n \geq 1} \mathrm{Ker}\left(p_{[\lambda]}(f)^n : \Delta^{\rho(I)} \longrightarrow \Delta^{\rho(I)}\right).$$

The direct sum decomposition $\Delta = \Delta_1 \oplus \Delta_2$ in $\mathrm{Rep}_E('W_K)$ is compatible with the isomorphism $\Delta \xrightarrow{\sim} \Delta^*|\cdot|$ and the exact sequence

$$0 \longrightarrow \Delta^+ \longrightarrow \Delta \longrightarrow \Delta^- \longrightarrow 0.$$

By construction, every subquotient of $\Delta_2$ in $\mathrm{Rep}_E('W_K)$ has trivial $H^0$, hence $H^0(\Delta_2^-) = 0$. As $\Delta$ is pure of weight $-1$, it follows that

$$\Delta_1 = \bigoplus_{m\geq 1} \sigma_m \otimes sp(2m),$$

where each $\sigma_m \in \mathrm{Rep}_E('W_K)$ is an unramified representation of $W_K$ on which $q^{1-m}f$ acts unipotently.
As $V$ satisfies the Pančiškin condition, weak admissibility of $V^{\pm}$ implies that all eigenvalues of $f$ on $\Delta_1^+ = \Delta^+ \cap \Delta_1$ (resp., on $\Delta_1^- = \Delta_1/\Delta_1^+$) are of the form $q^n$ with $n < 0$ (resp., with $n \geq 0$). It follows that

$$\Delta_1^+ = \bigoplus_{m\geq 1} \sigma_m \otimes sp(m)|\cdot|^m, \qquad \Delta_1^- = \bigoplus_{m\geq 1} \sigma_m \otimes sp(m),$$

which proves (2).
(3) This follows from Proposition 3.2.6 applied to $V^+$.
(4) We combine Proposition 2.2.3 (which applies to $\Delta$, thanks to (2)) with the formula (3) and the fact that

$$H^0(\Delta^-) = D_{cris}(V^-)^{\varphi=1} \otimes_{L,\tau} E = (D_{cris}(V^-)^{\varphi=1} \cap D^0_{dR}(V^-)) \otimes_{L,\tau} E = \\ = H^0(K, V^-) \otimes_{L,\tau} E.$$

## 4. Global $p$-adic Galois representations

### (4.1) Generalities

(4.1.1) Notation. Let $F$ be a number field. For each prime $l$ of $\mathbf{Q}$, let $S_l$ be the set of primes of $F$ above $l$. Fix a prime number $p$, a finite extension $L_{\mathfrak{p}}$ of $\mathbf{Q}_p$ and a finite set $S \supset S_\infty \cup S_p$ of primes of $F$. Let $F_S$ be the maximal extension of $F$ (contained in $\overline{F}$) unramified outside $S$; put $G_{F,S} = \mathrm{Gal}(F_S/F)$. For each prime $v$ of $F$ fix an embedding $\overline{F} \hookrightarrow \overline{F}_v$; this defines a morphism $G_{F_v} \longrightarrow G_F \longrightarrow G_{F,S}$. For each Galois representation $V \in \mathrm{Rep}_{L_{\mathfrak{p}}}(G_{F,S})$ (continuous and finite-dimensional over $L_{\mathfrak{p}}$), denote by $V_v \in \mathrm{Rep}_{L_{\mathfrak{p}}}(G_{F_v})$ the local Galois representation induced by the map $G_{F_v} \longrightarrow G_{F,S}$. For each $v \notin S_\infty \cup S_p$, denote by $WD(V_v) \in \mathrm{Rep}_{L_{\mathfrak{p}}}('W_{F_v})$ the associated representation of the Weil-Deligne group of $F_v$ (see 1.1.3). As in [Bl-Ka], we put

$$\forall v \notin S_\infty \cup S_p \qquad H^1_f(F_v, V) = H^1_{ur}(F_v, V) = \mathrm{Ker}\left(H^1(F_v, V) \longrightarrow H^1(F_v^{ur}, V)\right)$$

$$H^1_f(F, V) = \mathrm{Ker}\left(H^1(G_{F,S}, V) \longrightarrow \bigoplus_{v\in S-S_\infty} H^1(F_v, V)/H^1_f(F_v, V)\right).$$

The $L_{\mathfrak{p}}$-vector space $H^1_f(F,V)$ does not change if we enlarge $S$.

(4.1.2) Throughout §4, assume that $V$ satisfies the following conditions.

(1) There exists an isomorphism $j : V \xrightarrow{\sim} V^*(1)$ in $\mathrm{Rep}_{L_{\mathfrak{p}}}(G_{F,S})$ satisfying $j^*(1) = -j$.
(2) For each $v \in S_p$, $V_v \in \mathrm{Rep}_{L_{\mathfrak{p}}}(G_{F_v})$ satisfies the Pančiškin condition 3.3.1:

$$0 \longrightarrow V_v^+ \longrightarrow V_v \longrightarrow V_v^- \longrightarrow 0$$

(in particular, $V_v \in \mathrm{Rep}_{pst,L_{\mathfrak{p}}}(G_{F_v})$).
(3) For each $v \notin S_\infty \cup S_p$ (resp., $v \in S_p$), $V_v$ is pure (necessarily of weight $-1$) in the sense of 1.4.5 (resp., in the sense of 3.2.3).
(4) For each $i \in \mathbf{Z}$, the integer

$$d^i(V) := \dim_{L_{\mathfrak{p}}} \left(D^i_{dR}(V_v)/D^{i+1}_{dR}(V_v)\right)/[F_v : \mathbf{Q}_p]$$

does not depend on $v \in S_p$. This condition is satisfied if $V = M_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic realization of a motive (pure of weight $-1$) $M$ over $F$ with coefficients in a number field $L$ (of which $L_{\mathfrak{p}}$ is a completion), as

$$d^i(V) = \dim_L \left(F^i M_{dR}/F^{i+1}M_{dR}\right)$$

in this case.

Example: $F = \mathbf{Q}$ and $V = (S^{2m-1}V(f))(mk - m + 1 - k/2)$, where $m \geq 1$ and $V(f)$ is the Galois representation (pure of weight $k-1$) associated to a potentially $p$-ordinary Hecke eigenform $f \in S_k(\Gamma_0(N))$ of (even) weight $k$ and trivial character.

(4.1.3) $\varepsilon$-FACTORS. We define

$$d^-(V) = \sum_{i<0} i\, d^i(V), \tag{4.1.3.1}$$

$$\forall v \in S_\infty \qquad \varepsilon(V_v) = (-1)^{[F_v:\mathbf{R}]\, d^-(V)} \times \begin{cases} 1, & F_v = \mathbf{R} \\ (-1)^{\dim_{L_{\mathfrak{p}}}(V)/2}, & F_v = \mathbf{C} \end{cases} \tag{4.1.3.2}$$

(in view of (2.3.1), this is the correct archimedean local $\varepsilon$-factor if $V = M_{\mathfrak{p}}$ is as in 4.1.2(4)) and

$$\forall v \notin S_\infty \qquad \varepsilon(V_v) = \varepsilon(WD(V_v)). \tag{4.1.3.3}$$

For any prime $v$ of $F$, let

$$\widetilde{\varepsilon}(V_v) = \varepsilon(V_v) \times \begin{cases} (-1)^{h^0(F_v,V_v^-)}, & v \in S_p \\ 1, & v \notin S_p, \end{cases} \tag{4.1.3.4}$$

where

$$h^i(F_v, X) = \dim_{L_{\mathfrak{p}}} H^i(F_v, X) \qquad\qquad (X \in \mathrm{Rep}_{L_{\mathfrak{p}}}(G_{F_v})).$$

Finally, define

$$\varepsilon(V) = \prod_v \varepsilon(V_v), \qquad \widetilde{\varepsilon}(V) = \prod_v \widetilde{\varepsilon}(V_v) \tag{4.1.3.5}$$

(this makes sense, as $\varepsilon(V_v) = 1$ for all but finitely many $v$). It follows from Proposition 3.3.3 that

$$\forall v \in S_p \qquad \widetilde{\varepsilon}(V_v) = (-1)^{[F_v:\mathbf{Q}_p]\, d^-(V)}\, (\det V_v^+)(-1) = \varepsilon(WD(V_v)^{N-ss}), \tag{4.1.3.6}$$

hence

$$\prod_{v\in S_p} \widetilde{\varepsilon}(V_v) = (-1)^{[F:\mathbf{Q}]\, d^-(V)} \prod_{v\in S_p} (\det V_v^+)(-1).$$

As

$$\prod_{v\in S_\infty} \varepsilon(V_v) = (-1)^{[F:\mathbf{Q}]\, d^-(V)},$$

it follows that

$$\prod_{v\in S_p\cup S_\infty} \widetilde{\varepsilon}(V_v) = \prod_{v\in S_p} (\det V_v^+)(-1). \tag{4.1.3.7}$$

(4.2) Selmer complexes and extended Selmer groups

(4.2.1) For a pro-finite group $G$ and a representation $X \in \mathrm{Rep}_{L_\mathfrak{p}}(G)$ (continuous, finite-dimensional), denote by $C^\bullet(G, X)$ the standard complex of (non-homogeneous) continuous cochains of $G$ with values in $X$. Fix a set $S_p \subset \Sigma \subset S$ and define, for each $v \in S - S_\infty$, the complex

$$U_v^+(V) = \begin{cases} C^\bullet(G_{F_v}, V_v^+), & v \in S_p \\ 0, & v \in \Sigma - S_p \\ C^\bullet_{ur}(G_{F_v}, V_v) = C^\bullet(G_{F_v}/I_v, V_v^{I_v}), & v \in S - \Sigma, \end{cases}$$

where $I_v \subset G_{F_v}$ is the inertia group. As in ([Ne 2], 12.5.9.1), define the Selmer complex of $V$ associated to the local conditions $\Delta_\Sigma(V) = (U_v^+(V))_{v\in S-S_\infty}$ as

$$\widetilde{C}^\bullet_f(G_{F,S}, V; \Delta_\Sigma(V)) =$$
$$= \mathrm{Cone}\left( C^\bullet(G_{F,S}, V) \oplus \bigoplus_{v\in S-S_\infty} U_v^+(V) \longrightarrow \bigoplus_{v\in S-S_\infty} C^\bullet(G_{F_v}, V) \right)[-1].$$

(4.2.2) Proposition. (1) *For each $v \notin S_\infty \cup S_p$, the complexes $C^\bullet(G_{F_v}, V)$ and $C^\bullet_{ur}(G_{F_v}, V)$ are acyclic.*
(2) *Up to a canonical isomorphism, the image of $\widetilde{C}^\bullet_f(G_{F,S}, V; \Delta_\Sigma(V))$ in the*

*derived category $D^b_{ft}(L_\mathfrak{p} - \mathrm{Mod})$ does not depend on $\Sigma$ and $S$; denote it by $\widetilde{\mathbf{R}\Gamma}_f(F,V)$ and its cohomology by $\widetilde{H}^i_f(F,V)$ (as $L_\mathfrak{p}$ is a field, $\widetilde{\mathbf{R}\Gamma}_f(F,V) = \bigoplus_{i\in\mathbf{Z}} \widetilde{H}^i_f(F,V)[-i]$).*

(3) *There is an exact sequence*

$$0 \longrightarrow \bigoplus_{v\in S_p} H^0(F_v, V_v^-) \longrightarrow \widetilde{H}^1_f(F,V) \longrightarrow H^1_f(F,V) \longrightarrow 0.$$

(4) *If we put* $h^1_f(F,V) = \dim_{L_\mathfrak{p}} H^1_f(F,V)$, $\widetilde{h}^1_f(F,V) = \dim_{L_\mathfrak{p}} \widetilde{H}^1_f(F,V)$, *then*

$$(-1)^{h^1_f(F,V)} / \varepsilon(V) = (-1)^{\widetilde{h}^1_f(F,V)} / \widetilde{\varepsilon}(V).$$

*Proof.* (cf. [Ne 2, 12.5.9.2]) (1) The cohomology group $H^0(F_v,V) = 0$ vanishes by purity (1.4.4(5)), $H^2(F_v,V) \xrightarrow{\sim} H^0(F_v,V^*(1))^* \xrightarrow{\sim} H^0(F_v,V)^* = 0$ by duality and $H^1(F_v,V) = 0$ by the local Euler characteristic formula $\sum_{i=0}^2 (-1)^i h^i(F_v,V) = 0$. Finally, $\dim_{L_\mathfrak{p}} H^1_{ur}(F_v,V) = h^0(F_v,V) = 0$.
(2) Independence of $\Sigma$ follows from (1), independence of $S$ is a general fact ([Ne 2], Prop. 7.8.8).
(3) It follows from (1) and [Ne 2, Lemma 9.6.3] that there is an exact sequence

$$0 \longrightarrow \widetilde{H}^0_f(F,V) \longrightarrow H^0(G_{F,S},V) \longrightarrow \bigoplus_{v\in S_p} H^0(F_v,V_v^-) \longrightarrow \widetilde{H}^1_f(F,V) \longrightarrow H \longrightarrow 0,$$

where

$$H = \mathrm{Ker}\left( H^1(G_{F,S},V) \longrightarrow \bigoplus_{v\in S-S_\infty} H^1(F_v,V)/\mathrm{Im}(H^1(U_v^+(V))) \right).$$

As

$$\mathrm{Im}(H^1(U_v^+(V))) = \begin{cases} 0 = H^1_f(F_v,V), & v \notin S_p \\ H^1_f(F_v,V), & v \in S_p \end{cases}$$

by (1) and Proposition 3.3.2(2), respectively, we deduce that $H = H^1_f(F,V)$. Finally, $H^0(G_{F,S},V) = 0$ by purity.
(4) This is a consequence of (3) and (4.1.3.4).

## 5. $p$-ADIC FAMILIES OF GLOBAL $p$-ADIC GALOIS REPRESENTATIONS

### (5.1) THE GENERAL SETUP

(5.1.1) Fix a number field $F$, a prime number $p$ and a finite set $S \supset S_p \cup S_\infty$ of primes of $F$.

(5.1.2) Assume that we are given the following data.

(1) A complete local noetherian domain $R$ of dimension $\dim(R) = 2$, whose residue field is a finite extension of $\mathbf{F}_p$ and whose fraction field $\mathscr{L}$ is of characteristic zero.
(2) An $R$-module of finite type $\mathcal{T}$ equipped with an $R$-linear continuous action of $G_{F,S}$ (with respect to the pro-finite topology of $\mathcal{T}$). Set $\mathcal{V} = \mathcal{T} \otimes_R \mathscr{L}$.
(3) A skew-symmetric morphism of $R[G_{F,S}]$-modules

$$( \ , \ ) : \mathcal{T} \otimes_R \mathcal{T} \longrightarrow R(1) = R \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(1)$$

inducing an isomorphism of $\mathscr{L}[G_{F,S}]$-modules

$$\mathcal{V} \xrightarrow{\sim} \mathcal{V}^*(1) = \mathrm{Hom}_{\mathscr{L}}(\mathcal{V}, \mathscr{L})(1).$$

(4) For each $v \in S_p$ an $R[G_{F_v}]$-submodule $\mathcal{T}_v^+ \subset \mathcal{T}_v$ such that the isomorphism $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^*(1)$ induces isomorphisms of $\mathscr{L}[G_{F_v}]$-modules

$$\mathcal{V}_v^{\pm} \xrightarrow{\sim} (\mathcal{V}_v^{\mp})^*(1) = \mathrm{Hom}_{\mathscr{L}}(\mathcal{V}_v^{\mp}, \mathscr{L})(1),$$

where $\mathcal{V}_v^+ = \mathcal{T}_v^+ \otimes_R \mathscr{L}$, $\mathcal{V}_v^- = \mathcal{V}_v / \mathcal{V}_v^+$.
(5) A prime ideal $P \in \mathrm{Spec}(R)$ of height $ht(P) = 1$, which does not contain $p$ and such that $R_P$ is a discrete valuation ring. Fix a prime element $\varpi_P$ of $R_P$. The residue field $\kappa(P) = R_P/\varpi_P R_P$ is a finite extension of $\mathbf{Q}_p$. Define

$$\mathcal{T}_P = \mathcal{T} \otimes_R R_P \subset \mathcal{V}, \qquad V = \mathcal{T}_P/\varpi_P \mathcal{T}_P \in \mathrm{Rep}_{\kappa(P)}(G_{F,S})$$

and, for each $v \in S_p$,

$$(\mathcal{T}_P)_v^+ = \mathcal{T}_P \cap \mathcal{V}_v^+, \qquad (\mathcal{T}_P)_v^- = \mathcal{T}_P/(\mathcal{T}_P)_v^+, \qquad V_v^+ = (\mathcal{T}_P)_v^+/\varpi_P(\mathcal{T}_P)_v^+ \subset V_v,$$
$$V_v^- = V_v/V_v^+ \qquad\qquad (V_v^{\pm} \in \mathrm{Rep}_{\kappa(P)}(G_{F_v})).$$

(6) We assume that there exists $u \in \mathscr{L}^*$ such that $u \cdot ( \ , \ )$ induces an isomorphism of $R_P[G_{F,S}]$-modules

$$\mathcal{T}_P \xrightarrow{\sim} \mathcal{T}_P^*(1) = \mathrm{Hom}_{R_P}(\mathcal{T}_P, R_P)(1).$$

This implies that, for each $v \in S_p$, $u \cdot ( \ , \ )$ induces an isomorphism of $R_P[G_{F_v}]$-modules $(\mathcal{T}_P)_v^{\pm} \xrightarrow{\sim} ((\mathcal{T}_P)_v^{\mp})^*(1)$. Reducing $u \cdot ( \ , \ )$ modulo $P$, we obtain a non-degenerate skew-symmetric morphism of $\kappa(P)[G_{F,S}]$-modules $V \otimes_{\kappa(P)} V \longrightarrow \kappa(P)(1)$ which induces, for each $v \in S_p$, isomorphisms $V_v^{\pm} \xrightarrow{\sim} (V_v^{\mp})^*(1)$ in $\mathrm{Rep}_{\kappa(P)}(G_{F_v})$.
(7) We assume that, for each $v \in S_p$, the exact sequence

$$0 \longrightarrow V_v^+ \longrightarrow V_v \longrightarrow V_v^- \longrightarrow 0$$

satisfies the Pančiškin condition: $V_v^{\pm} \in \mathrm{Rep}_{pst,\kappa(P)}(G_{F_v})$ and $D_{dR}^0(V_v^+) = 0 = D_{dR}(V_v^-)/D_{dR}^0(V_v^-)$.

(8) We assume that, for each $v \notin S_\infty$, $V_v$ is pure of weight $-1$ (in the sense of 1.4.5 and 3.2.3, respectively).
(9) We assume that, for each $i \in \mathbf{Z}$, the integer

$$d^i(V) := \dim_{\kappa(P)} \left(D^i_{dR}(V_v)/D^{i+1}_{dR}(V_v)\right)/[F_v : \mathbf{Q}_p]$$

does not depend on $v \in S_p$; put

$$d^-(V) = \sum_{i<0} i\, d^i(V).$$

(5.1.3) This implies, in particular, that $V$ satisfies the assumptions 4.1.2(1)-(4).

(5.1.4) Fix $v \notin S_p \cup S_\infty$. As $\mathrm{Aut}_R(\mathcal{T})$ is a pro-finite group containing a pro-$p$ open subgroup, there exists an open subgroup $J$ of the inertia group $I = I_v = \mathrm{Gal}(\overline{F}_v/F_v^{ur})$ such that $J$ acts on $\mathcal{T}$ through the map $J \hookrightarrow I \twoheadrightarrow I(p)$, where $I(p)$ is the maximal pro-$p$-quotient of $I$ (isomorphic to $\mathbf{Z}_p$). Fixing a topological generator $t$ of $I(p)$ and an integer $a \geq 1$ such that $t^a$ lies in the image of $J$, then the set of eigenvalues of $t^a$ acting on $\mathcal{V}$ is stable under the map $\lambda \mapsto \lambda^{Nv}$, which implies that there exists an integer $c \geq 1$ divisible by $a$ such that $t^c$ acts unipotently on $\mathcal{V}$. Defining

$$N = \frac{1}{c} \log \rho_{\mathcal{T}}(t^c) \in \mathrm{End}_R(\mathcal{T}) \otimes \mathbf{Q}$$

(where $\rho_{\mathcal{T}} : G_K \longrightarrow \mathrm{Aut}_R(\mathcal{T})$ denotes the action of $G_{F_v}$ on $\mathcal{T}$) and (fixing a lift $\widetilde{f} \in \nu^{-1}(1) \subset W_K$ of $f$)

$$\rho_T(\widetilde{f}^n u) := \rho_{\mathcal{T}}(\widetilde{f}^n u)\exp(-bN) \in \mathrm{Aut}_{R\otimes\mathbf{Q}}(\mathcal{T} \otimes \mathbf{Q}) \subset \mathrm{Aut}_{R_P}(\mathcal{T}_P) \quad (n \in \mathbf{Z},\, u \in I)$$

(where $b \in \mathbf{Z}_p$ is such that the image of $u$ in $I(p)$ is equal to $t^b$), the pair $(\rho_T, N)$ defines an object $T = (\rho_T, N)$ of $\mathrm{Rep}_{R_P}({}'W_{F_v})$ in the sense of 1.5.2, the isomorphism class of which is independent of the choice of $\widetilde{f}$ ([De 1], 8.4.3). By construction, the special fibre of $T$ is isomorphic to

$$T_s \xrightarrow{\sim} WD(V_v) \in \mathrm{Rep}_{\kappa(P)}({}'W_{F_v}).$$

We define

$$\begin{aligned} WD(\mathcal{V}_v) &:= T_\eta = T \otimes_{R_P} \mathscr{L} \in \mathrm{Rep}_{\mathscr{L}}({}'W_{F_v}) \\ \varepsilon(\mathcal{V}_v) &:= \varepsilon(WD(\mathcal{V}_v)). \end{aligned} \tag{5.1.4.1}$$

If we choose another generator of $I(p)$, then $N$ is multiplied by a scalar $\lambda \in \mathbf{Z}_p^*$, which does not change the isomorphism class of $WD(\mathcal{V}_v)$ ([De 1], 8.4.3).

## (5.2) Selmer complexes and extended Selmer groups

(5.2.1) We equip each $R$-module $Y = \mathcal{T}, \mathcal{T}_v^+, \mathcal{T}_v^{I_v}$ with the pro-finite topology and we denote by $C^\bullet(G, Y)$ the corresponding complex of continuous cochains (for

$G = G_{F,S}, G_{F_v}, G_{F_v}/I_v$, respectively). For $R' = R_P, \mathscr{L}$, define $C^\bullet(G, Y \otimes_R R') = C^\bullet(G, Y) \otimes_R R'$. As in 4.2.1, fix a set $S_p \subset \Sigma \subset S$ and define, for $X = \mathcal{T}_P, \mathcal{V}$, $R_X = R_P, \mathscr{L}$ and each $v \in S - S_\infty$, complexes of $R_X$-modules

$$U_v^+(X) = \begin{cases} C^\bullet(G_{F_v}, X_v^+), & v \in S_p \\ 0, & v \in \Sigma - S_p \\ C^\bullet_{ur}(G_{F_v}, X) = C^\bullet(G_{F_v}/I_v, X^{I_v}), & v \in S - \Sigma, \end{cases}$$

and

$$\widetilde{C}^\bullet_f(G_{F,S}, X; \Delta_\Sigma(X)) =$$
$$= \mathrm{Cone}\left( C^\bullet(G_{F,S}, X) \oplus \bigoplus_{v \in S - S_\infty} U_v^+(X) \longrightarrow \bigoplus_{v \in S - S_\infty} C^\bullet(G_{F_v}, X) \right)[-1].$$

(5.2.2) Proposition. (1) *For each $X = \mathcal{T}_P, \mathcal{V}$ and each $v \notin S_\infty \cup S_p$, the complexes $C^\bullet(G_{F_v}, X)$ and $C^\bullet_{ur}(G_{F_v}, X)$ are acyclic.*
(2) *Up to a canonical isomorphism, the image of $\widetilde{C}^\bullet_f(G_{F,S}, X; \Delta_\Sigma(X))$ in $D^b_{ft}(R_X - \mathrm{Mod})$ does not depend on $\Sigma$ and $S$; denote it by $\widetilde{\mathbf{R}\Gamma}_f(F, X)$ and its cohomology by $\widetilde{H}^i_f(F, X)$ (as $\mathscr{L}$ is a field, $\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{V}) = \bigoplus_{i \in \mathbf{Z}} \widetilde{H}^i_f(F, \mathcal{V})[-i]$).*
(3) *There is an exact triangle in $D^b_{ft}(R_P - \mathrm{Mod})$*

$$\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \xrightarrow{\varpi_P} \widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(F, V) \longrightarrow \widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P)[1]$$

*giving rise to exact sequences*

$$0 \longrightarrow \widetilde{H}^i_f(F, \mathcal{T}_P)/\varpi_P \widetilde{H}^i_f(F, \mathcal{T}_P) \longrightarrow \widetilde{H}^i_f(F, V) \longrightarrow \widetilde{H}^{i+1}_f(F, \mathcal{T}_P)[\varpi_P] \longrightarrow 0,$$

*and an isomorphism* $\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \overset{\mathbf{L}}{\otimes}_{R_P} \mathscr{L} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{V})$ *in $D^b_{ft}(\mathscr{L} - \mathrm{Mod})$.*
(4) *There exists a skew-symmetric isomorphism in $D^b_{ft}(R_P - \mathrm{Mod})$*

$$\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{R_P}(\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P), R_P)[-3]$$

*inducing a skew-symmetric non-degenerate pairing*

$$\widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \times \widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \longrightarrow \mathscr{L}/R_P.$$

(5) *There exists an $R_P$-module $Z$ of finite length such that $\widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \xrightarrow{\sim} Z \oplus Z$.*
(6) *$\widetilde{H}^1_f(F, \mathcal{T}_P)$ is a free $R_P$-module of rank $\widetilde{h}^1_f(F, \mathcal{V}) := \dim_{\mathscr{L}} \widetilde{H}^1_f(F, \mathcal{V})$.*
(7) *$\widetilde{h}^1_f(F, V) \equiv \widetilde{h}^1_f(F, \mathcal{V}) \pmod 2$.*

*Proof.* (cf. [Ne 2, 12.7.13.4]) (1) It is enough to prove the statement for $X = \mathcal{T}_P$. By ([Ne 2], Prop. 3.4.2 and 3.4.4), there is an exact sequence of complexes

$$0 \longrightarrow C^\bullet(G_{F_v}, \mathcal{T}_P) \xrightarrow{\varpi_P} C^\bullet(G_{F_v}, \mathcal{T}_P) \longrightarrow C^\bullet(G_{F_v}, V) \longrightarrow 0,$$

which induces injections

$$H^i(G_{F_v}, \mathcal{T}_P)/\varpi_P H^i(G_{F_v}, \mathcal{T}_P) \hookrightarrow H^i(F_v, V).$$

As $H^i(F_v, V) = 0$ by Proposition 4.2.2(1), and $H^i(G_{F_v}, \mathcal{T}_P) = H^i(G_{F_v}, \mathcal{T}) \otimes_R R_P$ is an $R_P$-module of finite type (by [Ne 2], Prop. 4.2.3), it follows that $H^i(G_{F_v}, \mathcal{T}_P) = 0$. Finally, the unramified cohomology $H^1_{ur} = H^1_{ur}(G_{F_v}, \mathcal{T}_P) = \mathcal{T}_P^{I_v}/(f_v - 1)\mathcal{T}_P^{I_v}$ is an $R_P$-module of finite type and $H^1_{ur}/\varpi_P H^1_{ur}$ is a subquotient of $V^{I_v}/\varpi_P V^{I_v} = H^1_{ur}(G_{F_v}, V) = 0$; thus $H^1_{ur} = 0$.
(2) This follows from (1), as in the proof of 4.2.2(2).
(3) According to (2), we can take $\Sigma = S$, in which case the exact triangle in question follows from the exact sequences

$$0 \longrightarrow C^\bullet(G, \mathcal{T}_P) \xrightarrow{\varpi_P} C^\bullet(G, \mathcal{T}_P) \longrightarrow C^\bullet(G, V) \longrightarrow 0 \qquad (G = G_{F,S},\, G_{F_v}).$$

The isomorphism $\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \overset{\mathbf{L}}{\otimes}_{R_P} \mathscr{L} \xrightarrow{\sim} \widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{V})$ is a direct consequence of the definitions.
(4) Take again $\Sigma = S$. According to a localized version of ([Ne 2], 7.8.4.4), there exists an exact triangle in $D^b_{ft}(R_P - \mathrm{Mod})$

$$\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P) \xrightarrow{\gamma} \mathbf{R}\mathrm{Hom}_{R_P}(\widetilde{\mathbf{R}\Gamma}_f(F, \mathcal{T}_P), R_P)[-3] \longrightarrow \bigoplus_{v \in S - S_\infty} \mathrm{Err}_v,$$

in which the error terms $\mathrm{Err}_v$ vanish for $v \in S_p$ (as $(\mathcal{T}_P)^\pm \xrightarrow{\sim} ((\mathcal{T}_P)^\mp)^*(1))$, as well as for $v \notin S_p$ (by (1) and [Ne 2], Prop. 6.7.6(iv)). The map $\gamma$ (which is an isomorphism, by the previous discussion) is skew-symmetric, by ([Ne 2], Prop. 6.6.2 and 7.7.3). The skew-symmetric non-degenerate pairing

$$\widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \times \widetilde{H}^2_f(F, \mathcal{T}_P)_{R_P-\mathrm{tors}} \longrightarrow \mathscr{L}/R_P.$$

is constructed from $\gamma$ in ([Ne 2], Prop. 10.2.5).
(5) This follows from (4) and the structure theory of symplectic modules of finite length over discrete valuation rings (note that 2 is invertible in $R_P$).
(6) It is enough to show that $\widetilde{H}^1_f(F, \mathcal{T}_P)$ has no $R_P$-torsion, which si a consequence of the exact sequence from (3) (for $i = 0$).
(7) In the exact sequence from (3) for $i = 1$, the term on the left (resp., on the right), is a $\kappa(P)$-vector space of dimension $\widetilde{h}^1_f(F, \mathcal{V})$, by (6) (resp., of even dimension, by (5)); thus the dimension of the middle term ($= \widetilde{h}^1_f(F, V)$) has the same parity as $\widetilde{h}^1_f(F, \mathcal{V})$.

(5.3) The parity conjecture in $p$-adic families

(5.3.1) Theorem. *Under the assumptions 5.1.2(1)-(9), the quantity*

$$(-1)^{h^1_f(F,V)}/\,\varepsilon(V) = (-1)^{\widetilde{h}^1_f(F,V)}/\,\widetilde{\varepsilon}(V) =$$
$$= (-1)^{\widetilde{h}^1_f(F,\mathcal{V})} \prod_{v\in S_p} (\det \mathcal{V}_v^+)(-1) \prod_{v\notin S_p\cup S_\infty} \varepsilon(\mathcal{V}_v)$$

*depends only on $\mathcal{V}$ and $\mathcal{V}_v^+$ $(v \in S_p)$.*

*Proof.* We combine the equalities

$$(-1)^{h^1_f(F,V)}/\,\varepsilon(V) = (-1)^{\widetilde{h}^1_f(F,V)}/\,\widetilde{\varepsilon}(V) \qquad \text{(Prop. 4.2.2(4))}$$

$$(-1)^{\widetilde{h}^1_f(F,V)} = (-1)^{\widetilde{h}^1_f(F,\mathcal{V})} \qquad \text{(Prop. 5.2.2(7))}$$

$$\widetilde{\varepsilon}(V) = \prod_{v\in S_p\cup S_\infty} \widetilde{\varepsilon}(V_v) \prod_{v\notin S_p\cup S_\infty} \varepsilon(V_v) = \prod_{v\in S_p} (\det V_v^+)(-1) \prod_{v\notin S_\infty\cup S_p} \varepsilon(V_v)$$
(by 4.1.3.7)

$$\forall v \notin S_\infty \cup S_p \qquad \varepsilon(V_v) = \varepsilon(\mathcal{V}_v) \qquad \text{(Prop. 2.2.4)}$$

$$\forall v \in S_p \qquad (\det V_v^+)(-1) = (\det \mathcal{V}_v^+)(-1)$$

(both sides are equal to $\pm 1$, and the L.H.S. is the reduction of the R.H.S. modulo $P$).

(5.3.2) Corollary. *Under the assumptions 5.1.2(1)-(4), if $P, P' \in \mathrm{Spec}(R)$ are prime ideals satisfying 5.1.2(5)-(9), then the Galois representations $V = \mathcal{T}_P/P\mathcal{T}_P$ and $V' = \mathcal{T}_{P'}/P'\mathcal{T}_{P'}$ satisfy*

$$(-1)^{h^1_f(F,V)}/\,\varepsilon(V) = (-1)^{h^1_f(F,V')}/\,\varepsilon(V').$$

(5.3.3) Open questions. It would be of interest to generalize Corollary 5.3.2 to self-dual families of Galois representations that do not satisfy the Pančiškin condition. Is it true, in general, that

$$(-1)^{[F_v:\mathbf{Q}_p]\,d^-(V)}\,\varepsilon(WD(V_v)^{N-ss}) \qquad (v \in S_p)$$

depends only on $\mathcal{V}_v$, and that

$$(-1)^{h^1_f(F,V)} \prod_{v\in S_p} \frac{\varepsilon(WD(V_v))}{\varepsilon(WD(V_v)^{N-ss})}$$

depends only on $\mathcal{V}$?

(5.3.4) Example (dihedral Iwasawa theory). Assume that $F_0 \subset F_\infty$ are Galois extension of $F$ such that $[F_0 : F] = 2$, $\Gamma = \mathrm{Gal}(F_\infty/F_0) \xrightarrow{\sim} \mathbf{Z}_p$ and $\Gamma^+ = \mathrm{Gal}(F_\infty/F) = \Gamma \rtimes \{1, \tau\}$ is dihedral:

$$\tau \in \Gamma^+ - \Gamma, \qquad \tau^2 = 1, \qquad \forall g \in \Gamma \quad \tau g \tau^{-1} = g^{-1}.$$

Let $V \in \mathrm{Rep}_{L_\mathfrak{p}}(G_{F,S})$ be a Galois representation satisfying 4.1.2(1)-(4); fix a $G_{F,S}$-stable $\mathcal{O}_\mathfrak{p}$-lattice $T \subset V$ ($\mathcal{O}_\mathfrak{p} = \mathcal{O}_{L_\mathfrak{p}}$) such that the pairing $(\ ,\ )_V : V \times V \longrightarrow L_\mathfrak{p}(1)$ induced by $j$ maps $T \times T$ into $\mathcal{O}_\mathfrak{p}(1)$. After enlarging $S$ if necessary, we can assume that $S$ contains all primes that ramify in $F_0/F$; then $F_\infty \subset F_S$. We define the following data of the type considered in 5.1.2:

(1) Let $R = \mathcal{O}_\mathfrak{p}[[\Gamma]]$ be the Iwasawa algebra of $\Gamma$ (isomorphic to the power series ring $\mathcal{O}_\mathfrak{p}[[X]]$). The Iwasawa algebra of $\Gamma^+$ is a free (both left and right) $R$-module of rank 2:
$$\mathcal{O}_\mathfrak{p}[[\Gamma^+]] = R \oplus R\tau = R \oplus \tau R.$$
Denote by $\iota$ the standard $\mathcal{O}_\mathfrak{p}$-linear involution on $\mathcal{O}_\mathfrak{p}[[\Gamma^+]]$ ($\iota(\sigma) = \sigma^{-1}$ for all $\sigma \in \Gamma^+$).

(2) Let $\mathcal{T} = T \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}[[\Gamma^+]]$, considered as a left $R[G_{F,S}]$-module with the action given by
$$r(x \otimes a) = x \otimes ra, \quad g(x \otimes a) = g(x) \otimes a(\overline{g})^{-1} \qquad (r \in R,\ x \in T,\ a \in \mathcal{O}_\mathfrak{p}[[\Gamma^+]]),$$
where we have denoted by $\overline{g}$ the image of $g \in G_{F,S}$ in $\Gamma^+$ (cf., [Ne 2], 10.3.5.3).

(3) As in ([Ne 2], 10.3.5.10), the formula
$$(x \otimes (a_1 + \tau a_2), y \otimes (b_1 + \tau b_2)) = (x, y)_V (a_1 \iota(b_2) + \iota(a_2) b_1)$$
defines a skew-symmetric $R$-bilinear pairing $(\ ,\ ) : \mathcal{T} \times \mathcal{T} \longrightarrow R(1)$, which induces an isomorphism
$$\mathcal{T} \otimes \mathbf{Q} \xrightarrow{\sim} \mathrm{Hom}_R(\mathcal{T}, R(1)) \otimes \mathbf{Q}$$
(hence satisfies 5.1.2(3)).

(4) For each $v \in S_p$, define $\mathcal{T}_v^+ = T_v^+ \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}[[\Gamma^+]]$ (where $T_v^+ = T \cap V_v^+$).

(5) Let $\beta : \Gamma \longrightarrow L_\mathfrak{p}(\beta)^*$ be a homomorphism with finite image (where $L_\mathfrak{p}(\beta)$ is a field generated over $L_\mathfrak{p}$ by the values of $\beta$); then $P = \mathrm{Ker}(\beta : R \longrightarrow L_\mathfrak{p}(\beta)) \in \mathrm{Spec}(R)$ is as in 5.1.2(5), with $\kappa(P) = L_\mathfrak{p}(\beta)$. It follows from ([Ne 2], Lemma 10.3.5.4) that
$$\mathcal{T}_P/\varpi_P \mathcal{T}_P = \mathrm{Ind}_{G_{F_0,S}}^{G_{F,S}}(V \otimes \beta),$$
where we have denoted by $V \otimes \beta \in \mathrm{Rep}_{L_\mathfrak{p}(\beta)}(G_{F_0,S})$ the $G_{F_0,S}$-module $V \otimes_{L_\mathfrak{p}} L_\mathfrak{p}(\beta)$ on which $g \in G_{F_0,S}$ acts by $g \otimes \beta(\overline{g})$, where $\overline{g}$ is the image of $g$ in $\Gamma$. The discussion in ([Ne 2], 10.3.5.10) implies that 5.1.2(6) holds with $u = 1$. The conditions 5.1.2(7)-(9) for $\mathcal{T}_P/\varpi_P \mathcal{T}_P$ follow from the corresponding conditions 4.1.2(2)-(4) for $V$.

(5.3.5) In the situation of 5.3.4, putting $F_\beta = F_\infty^{\mathrm{Ker}(\beta)}$ and, for each $L_\mathfrak{p}[\Gamma]$-module $M$,

$$M^{(\beta)} = \{x \in M \otimes_{L_\mathfrak{p}} L_\mathfrak{p}(\beta) \mid \forall \sigma \in \Gamma \quad \sigma(x) = \beta(\sigma)x\},$$

then we have

$$H^1_f(F, \mathcal{T}_P/\varpi_P\mathcal{T}_P) = H^1_f(F_0, V \otimes \beta) = \left(H^1_f(F_\beta, V) \otimes \beta\right)^{\mathrm{Gal}(F_\beta/F_0)} = \\ = H^1_f(F_\beta, V)^{(\beta^{-1})},$$

and the action of $\tau$ induces an isomorphism of $L_\mathfrak{p}(\beta)$-vector spaces

$$\tau : H^1_f(F_\beta, V)^{(\beta^{-1})} \xrightarrow{\sim} H^1_f(F_\beta, V)^{(\beta)}.$$

Applying Corollary 5.3.2, we obtain, for any pair of characters of finite order $\beta, \beta' : \Gamma \longrightarrow \overline{L}_\mathfrak{p}^*$, that

$$(-1)^{h^1_f(F_0, V\otimes\beta)}/\,\varepsilon(F_0, V \otimes \beta) = (-1)^{h^1_f(F_0, V\otimes\beta')}/\,\varepsilon(F_0, V \otimes \beta'). \qquad (5.3.5.1)$$

In this special case one can prove Proposition 2.2.4 directly (at least if $p \neq 2$) by using (2.1.2.7).

It would be of interest to generalize (5.3.5.1) to more general dihedral characters, as in [Ma-Ru].

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# The Chow-Witt ring

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Abstract. We define a ring structure on the total Chow-Witt group of any integral smooth scheme over a field of characteristic different from 2.

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Contents

## 1 Introduction

Let $A$ be a commutative noetherian ring of Krull dimension $n$ and $P$ a projective $A$-module of rank $d$. One can ask the following question: does $P$ admit a free factor of rank one? Serre proved a long time ago that the answer is always positive when $d > n$. So in fact the first interesting case is when $P$ is projective of rank equal to the dimension of $A$. Suppose now that $X$ is an integral smooth scheme over a field $k$ of characteristic not 2. To deal with the above question, Barge and Morel introduced the Chow-Witt groups $\widetilde{CH}^j(X)$ of $X$ (called at that time *groupes de Chow des cycles orientés*, see [BM]) and associated to each vector bundle $E$ of rank $n$ an Euler class $\tilde{c}_n(E)$ in $\widetilde{CH}^n(X)$. It was proved recently that if $X = \mathrm{Spec}(A)$ we have $\tilde{c}_n(P) = 0$ if and only if $P \simeq Q \oplus A$ (see [Mo] for $n \geq 4$, [FS] for $n = 3$ and [BM] or [Fa] for the case $n = 2$). It is therefore important to provide more tools, such as a ring structure and a pull-back for regular embeddings, to compute the Chow-Witt groups and the Euler classes.
To define $\widetilde{CH}^p(X)$ consider the fibre product of the complex in Milnor K-theory

$$0 \to K_p^M(k(X)) \to \bigoplus_{x_1 \in X^{(1)}} K_{p-1}^M(k(x_1)) \to \cdots \to \bigoplus_{x_n \in X^{(n)}} K_{p-n}^M(k(x_n)) \to 0$$

and the Gersten-Witt complex restricted to the fundamental ideals

$$0 \to I^p(k(X)) \to \bigoplus_{x_1 \in X^{(1)}} I^{p-1}(\mathcal{O}_{X,x_1}) \to \cdots \to \bigoplus_{x_n \in X^{(n)}} I^{p-n}(\mathcal{O}_{X,x_n}) \to 0$$

over the quotient complex

$$0 \to I^p/I^{p+1}(k(X)) \to \cdots \to \bigoplus_{x_n \in X^{(n)}} I^{p-n}/I^{p+1-n}(\mathcal{O}_{X,x_n}) \to 0.$$

The group $\widetilde{CH}^p(X)$ is defined as the $p$-th cohomology group of this fibre product. Roughly speaking, an element of $\widetilde{CH}^p(X)$ is the class of a sum of varieties of codimension $p$ with a quadratic form defined on each variety. We oviously have a map $\widetilde{CH}^p(X) \to CH^p(X)$.
Using the functoriality of the two complexes we see that the Chow-Witt groups satisfy good functorial properties (see [Fa]). For example, we have a pull-back morphism $f^* : \widetilde{CH}^j(X) \to \widetilde{CH}^j(Y)$ associated to each flat morphism $f : Y \to X$ and a push-forward morphism $g_* : \widetilde{CH}^j(Y, L) \to \widetilde{CH}^{j+r}(X)$ associated to each proper morphism $g : Y \to X$, where $r = \dim(X) - \dim(Y)$ and $L$ is a suitable line bundle over $Y$. Using this functorial behaviour, it is possible to produce a good intersection theory. This is what we do in this paper

using the classical strategy (see for example [Fu] or [Ro]). First we define an exterior product

$$\widetilde{CH}^j(X) \times \widetilde{CH}^i(Y) \to \widetilde{CH}^{i+j}(X \times Y)$$

and then a Gysin-like homomorphism $i^! : \widetilde{CH}^d(X) \to \widetilde{CH}^d(Y)$ associated to a closed embedding $i : Y \to X$ of smooth schemes. The product is then defined as the composition

$$\widetilde{CH}^j(X) \times \widetilde{CH}^i(X) \longrightarrow \widetilde{CH}^{i+j}(X \times X) \xrightarrow{\triangle^!} \widetilde{CH}^{i+j}(X)$$

where $\triangle : X \to X \times X$ is the diagonal embedding. To define the exterior product, we first note that Rost already defined an exterior product on the homology of the complex in Milnor K-theory ([Ro]). Thus it is enough to define an exterior product on the homology of the Gersten-Witt complex and show that both exterior products coincide over the quotient complex. We use the usual product on derived Witt groups ([GN]) and show that this product passes to homology using the Leibnitz rule proved by Balmer (see [Ba2]).

The definition of the Gysin-like map is done by following the ideas of Rost ([Ro]). It uses the deformation to the normal cone to modifiy any closed embedding to a nicer closed embedding and uses also the long exact sequence associated to a triple $(Z, X, U)$ where $Z$ is a closed subset of $X$ and $U = X \setminus Z$.

The product that we obtain has the meaning of intersecting varieties with quadratic forms defined on them. It is therefore not a surprise that the natural map $\widetilde{CH}^{tot}(X) \to CH^{tot}(X)$ turns out to be a ring homomorphism. There is however a surprise: the product that we obtain is a priori neither commutative nor anticommutative. This comes from the fact that the product of triangulated Grothendieck-Witt groups $GW^i \times GW^j \to GW^{i+j}$ does not satisfy any commutativity property.

The organization of this paper is as follows: In section 2, we recall some basic results on triangular Witt groups. This includes the construction of the Gersten-Witt complex, and some results on products and consanguinity. In section 3, we construct the Chow-Witt groups, recall some results and prove some basic facts. The definition of the exterior product takes place in section 4 and the definition of the Gysin-Witt map in section 5. In this part, we also prove the functoriality of this map. Finally we put all the pieces together in section 6 and prove some basic results in section 7.

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### 1.1 Conventions

All schemes are smooth and integral over a field $k$ of characteristic different from 2, or are localizations of such schemes. For any two schemes $X$ and $Y$ we will always denote by $X \times Y$ the fibre product $X \times_{\mathrm{Spec}(k)} Y$.

## 2 Preliminaries

### 2.1 Witt groups

We recall here some basic facts on Witt groups of triangulated categories following the exposition of [Ba2]. We suppose that for any triangulated category $\mathcal{C}$ and any objects $A, B$ of $\mathcal{C}$ the group $\mathrm{Hom}(A,B)$ is uniquely 2-divisible. We also suppose that all triangulated categories are essentially small.

Definition 2.1. Let $\mathcal{C}$ be a triangulated category. A duality on $\mathcal{C}$ is a triple $(D, \delta, \varpi)$ where $\delta = \pm 1$, $D : \mathcal{C} \to \mathcal{C}$ is a $\delta$-exact contravariant functor and $\varpi : 1 \simeq D^2$ is an isomorphism of functors satisfying $D(\varpi_A) \circ \varpi_{DA} = id_{DA}$ and $T(\varpi_A) = \varpi_{TA}$ for all $A \in \mathcal{C}$. A triangulated category $\mathcal{C}$ with a duality $(D, \delta, \varpi)$ is written $(\mathcal{C}, D, \delta, \varpi)$.

*Example* 2.2. Let $X$ be a regular scheme and $\mathcal{P}(X)$ the category of locally free coherent $\mathcal{O}_X$-modules. Let $D^b(\mathcal{P}(X))$ be the triangulated category of bounded complexes of objects of $\mathcal{P}(X)$. Then the usual duality $^\vee$ on $\mathcal{P}(X)$ defined by $P^\vee = \mathrm{Hom}_{\mathcal{O}_X}(P, \mathcal{O}_X)$ induces a 1-exact duality on $D^b(\mathcal{P}(X))$. We also denote this derived duality by $^\vee$. Moreover, the canonical isomorphism $ev : P \to P^{\vee\vee}$ for any locally free module $P$ induces a canonical isomorphism $\varpi : 1 \to^{\vee\vee}$ in $D^b(\mathcal{P}(X))$. More generally, if $L$ is any invertible module over $X$, then the duality $\mathrm{Hom}_{\mathcal{O}_X}(\_, L)$ on $\mathcal{P}(X)$ also induces a duality on $D^b(\mathcal{P}(X))$.

Definition 2.3. Let $(\mathcal{C}, D, \delta, \varpi)$ be a triangulated category with duality. For any $i \in \mathbb{Z}$, define $(D^{(i)}, \delta^{(i)}, \varpi^{(i)})$ by $D^{(i)} = T^i \circ D$, $\delta^{(i)} = (-1)^i \delta$ and $\varpi^{(i)} = \delta^i (-1)^{i(i+1)/2} \varpi$. It is easy to check that $(D^{(i)}, \delta^{(i)}, \varpi^{(i)})$ is a duality on $\mathcal{C}$. It is called the $i^{th}$-shifted duality of $(D, \delta, \varpi)$.

Definition 2.4. Let $(\mathcal{C}, D, \delta, \varpi)$ be a triangulated category with duality, $A \in \mathcal{C}$ and $i \in \mathbb{Z}$. A morphism $\varphi : A \to D^{(i)}A$ is $i$-symmetric if the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\varpi_A^{(i)}"'] & D^{(i)}A \arrow[d, equal] \\ (D^{(i)})^2(A) \arrow[r, "D^{(i)}\varphi"'] & D^{(i)}A. \end{tikzcd}$$

The couple $(A, \varphi)$ is called an $i$-symmetric pair.

Definition 2.5. We denote by $Symm^i(\mathcal{C})$ the monoid of isometry classes of $i$-symmetric pairs, equipped with the orthogonal sum.

Definition 2.6. An $i$-symmetric form is an $i$-symmetric pair $(A, \varphi)$ where $\varphi$ is an isomorphism.

Theorem 2.7. *Let $(\mathcal{C}, D, \delta, \varpi)$ be a triangulated category with duality and let $(A, \phi)$ be an $i$-symmetric pair. Choose an exact triangle containing $\phi$*

$$A \xrightarrow{\phi} D^{(i)}A \xrightarrow{\alpha} C \xrightarrow{\beta} TA.$$

*Then there exists an $(i+1)$-symmetric isomorphism $\psi : C \to D^{(i+1)}C$ such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\varpi^{(i)}"'] & D^{(i)}A \arrow[r, "\alpha"] \arrow[d, equal] & C \arrow[r, "\beta"] \arrow[d, dashed, "\psi"'] & TA \arrow[d, "T\varpi^{(i)}"] \\ D^{(i)}(D^{(i)}A) \arrow[r, "D^{(i)}\phi"'] & D^{(i)}A \arrow[r, "\delta^{(i+1)}D^{(i+1)}\beta"'] & D^{(i+1)}C \arrow[r, "D^{(i+1)}\alpha"'] & T(D^{(i)}(D^{(i)}A)) \end{tikzcd}$$

*where the rows are exact triangles and the second one is the dual of the first. Moreover, the $(i+1)$-symmetric form $(C, \psi)$ is unique up to isometry. It is denoted by cone$(A, \phi)$.*

*Proof.* See [Ba1], Theorem 1.6. □

*Example* 2.8. Let $A \in \mathcal{C}$. For any $i$, the morphism $0 : A \to D^{(i)}A$ is symmetric and then cone$(A, 0)$ is well defined.

Corollary 2.9. *The above construction gives a well defined homomorphism of monoids $d^i : Symm^{(i)}(\mathcal{C}) \to Symm^{(i+1)}(\mathcal{C})$ such that $d^{i+1}d^i = 0$.*

Definition 2.10. Let $(\mathcal{C}, D, \delta, \varpi)$ be a triangulated category with duality. The Witt group $W^i(\mathcal{C})$ is defined as $\mathrm{Ker}(d^i)/\mathrm{Im}(d^{i+1})$. Remark that $\mathrm{Ker}(d^i)$ is just the monoid of isometry classes of $i$-symmetric forms.

Definition 2.11. Let $(\mathcal{C}, D, \delta, \varpi)$ be a triangulated category with duality. The Grothendieck-Witt group $GW^i(\mathcal{C})$ is defined as the quotient of $\mathrm{Ker}(d^i)$ by the submonoid generated by the elements $\mathrm{cone}(A, \phi) - \mathrm{cone}(A, 0)$ where $A \in \mathcal{C}$ and $\phi$ is $(i-1)$-symmetric ($0$ is also seen as an $(i-1)$-symmetric morphism).

*Example* 2.12. Let $(D^b(\mathcal{P}(X)),^\vee, 1, \varpi)$ be the triangulated category with duality defined in Example 2.2. Its Witt groups are the Witt groups $W^i(X)$ of the scheme $X$ as defined in [Ba1].

## 2.2 Products

Given a pairing $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{M}$ of triangulated categories with duality and assuming that this pairing satisfies some nice conditions, the authors of [GN] define a pairing of Witt groups. We briefly recall some definitions (see 1.2 and 1.11 in [GN]):

DEFINITION 2.13. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{M}$ be triangulated categories. A product between $\mathcal{C}$ and $\mathcal{D}$ with codomain $\mathcal{M}$ is a covariant bi-functor

$$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{M}$$

exact in both variables and satisfying the following condition: the functorial isomorphisms $r_{A,B} : A \otimes TB \simeq T(A \otimes B)$ and $l_{A,B} : TA \otimes B \simeq T(A \otimes B)$ make the diagram

$$\begin{tikzcd} TA \otimes TB \arrow[r, "l_{A,TB}"] \arrow[d, "r_{TA,B}"'] & T(A \otimes TB) \arrow[d, "T(r_{A,B})"] \\ T(TA \otimes B) \arrow[r, "T(l_{A,B})"'] & T^2(A \otimes B) \end{tikzcd}$$

skew-commutative.

DEFINITION 2.14. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{M}$ be triangulated categories with dualities. Where there is no possible confusion, we drop the subscripts for $D, \delta$ and $\varpi$. A *dualizing pairing between $\mathcal{C}$ and $\mathcal{D}$ with codomain $\mathcal{M}$* is a product $\otimes$ with isomorphisms

$$\eta_{A,B} : DA \otimes DB \simeq D(A \otimes B)$$

natural in $A$ and $B$ which make the following diagrams commute

1.

$$\begin{tikzcd} A \otimes B \arrow[r, "\varpi_A \otimes \varpi_B"] \arrow[d, "\varpi_{A \otimes B}"'] & D^2A \otimes D^2B \arrow[d, "\eta_{DA,DB}"] \\ D^2(A \otimes B) \arrow[r, "D(\eta_{A,B})"'] & D(DA \otimes DB) \end{tikzcd}$$

2.

$$\begin{tikzcd} T(DTA \otimes DB) \arrow[d, "\delta_{\mathcal{C}} \delta_{\mathcal{M}} T(\eta_{TA,B})"'] & DA \otimes DB \arrow[l, "l_{DTA,DB}"'] \arrow[r, "r_{DA,DTB}"] \arrow[d, "\eta_{A,B}"] & T(DA \otimes DTB) \arrow[d, "\delta_{\mathcal{L}} \delta_{\mathcal{M}} T(\eta_{A,TB})"] \\ TD(TA \otimes B) & D(A \otimes B) \arrow[l, "TD(l_{A,B})"] \arrow[r, "TD(r_{A,B})"'] & TD(A \otimes TB). \end{tikzcd}$$

THEOREM 2.15. *Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{M}$ be triangulated categories with duality. Let $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{M}$ be a dualizing pairing between $\mathcal{C}$ and $\mathcal{D}$ with codomain $\mathcal{M}$. Then $\otimes$ induces for all $i, j \in \mathbb{Z}$ a pairing*

$$\star : W^i(\mathcal{C}) \times W^j(\mathcal{D}) \to W^{i+j}(\mathcal{M}).$$

*Proof.* See [GN], Theorem 2.9. ☐

*Example* 2.16. Let $(D^b(\mathcal{P}(X)),^\vee,1,\varpi)$ be the triangulated category with duality defined in Example 2.2. The usual tensor product induces a dualizing pairing of triangulated categories and then a product $W^i(X)\times W^j(X)\to W^{i+j}(X)$. Suppose that $L$ and $N$ are invertible modules over $X$. Then $\mathrm{Hom}_{\mathcal{O}_X}(\_,L)$, $\mathrm{Hom}_{\mathcal{O}_X}(\_,N)$ and $\mathrm{Hom}_{\mathcal{O}_X}(\_,L\otimes N)$ give dualities $\sharp$, $\natural$ and $\flat$ on $D^b(\mathcal{P}(X))$. The tensor product gives a dualizing pairing

$$\otimes:(D^b(\mathcal{P}(X)),^\sharp,1,\varpi)\times(D^b(\mathcal{P}(X)),^\natural,1,\varpi)\to(D^b(\mathcal{P}(X)),^\flat,1,\varpi).$$

## 2.3 SUPPORTS

We briefly recall the notion of triangulated category with supports following [Ba2].

DEFINITION 2.17. Let $X$ be a topological space. A triangulated category defined over $X$ is a pair $(\mathcal{C},\mathrm{Supp})$ where $\mathcal{C}$ is a triangulated category and Supp assigns to each object $A\in\mathcal{C}$ a closed subset $\mathrm{Supp}(A)$ of $X$ such that the following rules are satisfied:

(S1) $\mathrm{Supp}(A)=\emptyset \iff A\simeq 0$.

(S2) $\mathrm{Supp}(A\oplus B)=\mathrm{Supp}(A)\cup\mathrm{Supp}(B)$.

(S3) $\mathrm{Supp}(A)=\mathrm{Supp}(TA)$.

(S4) For every distinguished triangle

$$A\longrightarrow B\longrightarrow C\longrightarrow TA$$

we have $\mathrm{Supp}(C)\subset\mathrm{Supp}(A)\cup\mathrm{Supp}(B)$.

When $\mathcal{I}$ is a saturated triangulated subcategory of $\mathcal{C}$ and $S$ is the multiplicative system of morphisms whose cone is in $\mathcal{I}$, then we can construct a support on the category $S^{-1}\mathcal{C}:=\mathcal{C}/\mathcal{I}$. This is done in [Ba3] when $\mathcal{C}$ has a tensor product. However we will only need some basic facts, so we prove the following lemma:

LEMMA 2.18. *let $\mathcal{C}$ be a triangulated category defined over a topological space $X$. Let $\mathcal{I}$ be a saturated triangulated subcategory of $\mathcal{C}$ and let $Supp(\mathcal{I})=\cup_{A\in\mathcal{I}}Supp(A)$. Suppose that $Supp(A)\subset Supp(\mathcal{I})$ implies $A\in\mathcal{I}$. Let $S$ be the multiplicative system in $\mathcal{C}$ of morphisms $f$ such that $cone(f)\in\mathcal{I}$ and let*

$$\mathcal{I}\longrightarrow\mathcal{C}\longrightarrow\mathcal{C}/\mathcal{I}$$

*be the exact sequence of triangulated categories obtained by inverting $S$. Then $\mathcal{C}/\mathcal{I}$ is a triangulated category defined over $X'=X\setminus Supp(\mathcal{I})$ (with the induced topology).*

*Proof.* We define $\mathrm{Supp}_S(A) := \mathrm{Supp}(A) \cap X'$ for any object $A \in \mathcal{C}/\mathcal{I}$ and show that $\mathrm{Supp}_S$ satisfies the properties of Definition 2.17. It is easy to see that the rules (S1), (S2) and (S3) are satisfied. We only have to prove (S4).
First observe that if $s : A \to B$ is a morphism in $S$ and

$$A \xrightarrow{s} B \longrightarrow C \longrightarrow TA$$

is an exact triangle in $\mathcal{C}$ containing $s$, then $\mathrm{Supp}_S(A) = \mathrm{Supp}_S(B)$ (use (S4) for the category $\mathcal{C}$). This shows that $\mathrm{Supp}_S(A) = \mathrm{Supp}_S(A')$ if $A \simeq A'$ in $\mathcal{C}/\mathcal{I}$. By definition of the triangulation of $\mathcal{C}/\mathcal{I}$, any exact triangle

$$A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow TA$$

in $\mathcal{C}/\mathcal{I}$ is isomorphic to the localization of an exact triangle in $\mathcal{C}$. This shows that $\mathrm{Supp}_S(C) \subset \mathrm{Supp}_S(A) \cup \mathrm{Supp}_S(B)$. □

*Example* 2.19. Let $D^b(\mathcal{P}(X))$ be the usual triangulated category. Define the support of an object $P \in D^b(\mathcal{P}(X))$ as the union of the support of all the cohomology groups of $P$, i.e

$$\mathrm{Supp}(P) = \bigcup_i \mathrm{Supp}(H^i(P)).$$

Then it is easy to see that $(D^b(\mathcal{P}(X)), \mathrm{Supp})$ is a triangulated category with support. Denote by $D^b(\mathcal{P}(X))^{(k)}$ the full subcategory of $D^b(\mathcal{P}(X))$ of objects whose support is of codimension $\geq k$. Then $D^b(\mathcal{P}(X))^{(k)}$ is a saturated triangulated category and the following sequence

$$D^b(\mathcal{P}(X))^{(k)} \to D^b(\mathcal{P}(X)) \to D^b(\mathcal{P}(X))/D^b(\mathcal{P}(X))^{(k)}$$

satisfies the conditions of Lemma 2.18. So $D^b(\mathcal{P}(X))/D^b(\mathcal{P}(X))^{(k)}$ is a triangulated category over $X' = \{x \in X \mid \mathrm{codim}(x) \leq k-1\}$.

The following definitions are also due to Balmer (see [Ba2]):

DEFINITION 2.20. Let $(\mathcal{C}, \mathrm{Supp})$ be a triangulated category over $X$ and assume that $\mathcal{C}$ has a structure of triangulated category with duality $(\mathcal{C}, D, \delta, \varpi)$. Then we say that $\mathcal{C}$ is a triangulated category with duality defined over $X$ if

(S5) $\mathrm{Supp}(A) = \mathrm{Supp}(DA)$ for every object $A$.

DEFINITION 2.21. Let $(\mathcal{C}, \mathrm{Supp}_{\mathcal{C}})$, $(\mathcal{D}, \mathrm{Supp}_{\mathcal{D}})$ and $(\mathcal{M}, \mathrm{Supp}_{\mathcal{M}})$ be triangulated categories defined over $X$. Suppose that

$$\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{M}$$

is a pairing of triangulated categories. The pairing $\otimes$ is defined over $X$ if

(S6) $\mathrm{Supp}_{\mathcal{M}}(A \otimes B) = \mathrm{Supp}_{\mathcal{C}}(A) \cap \mathrm{Supp}_{\mathcal{D}}(B)$.

*Example* 2.22. The triangulated category $D^b(\mathcal{P}(X))$ with the support defined in Example 2.19 and the pairing of Example 2.16 satisfy the condition (S5) and (S6).

Definition 2.23. The degeneracy locus of a symmetric pair $(A, \alpha)$ is defined to be the support of the cone of $\alpha$:

$$\mathrm{DegLoc}(\alpha) = \mathrm{Supp}(\mathrm{cone}(\alpha)).$$

Definition 2.24. Let $(\mathcal{C}, \mathrm{Supp})$ be a triangulated category with duality defined over $X$. The consanguinity of two symmetric pairs $\alpha$ and $\beta$ is defined to be the following subset of $X$:

$$\mathrm{Cons}(\alpha, \beta) = (\mathrm{Supp}(\alpha) \cap \mathrm{DegLoc}(\beta)) \cup (\mathrm{DegLoc}(\alpha) \cap \mathrm{Supp}(\beta)).$$

We are now ready to state the Leibnitz formula:

Theorem 2.25 (Leibnitz formula). *Assume that we have a dualizing pairing $\otimes : \mathcal{C} \times \mathcal{D} \to \mathcal{F}$ of triangulated categories with dualities over $X$. Let $\alpha$ and $\beta$ be two symmetric pairs such that DegLoc$(\alpha) \cap$ DegLoc$(\beta) = \emptyset$. Then we have an isometry*

$$\delta_{\mathcal{F}} \cdot d(\alpha \star \beta) = \delta_{\mathcal{C}} \cdot d(\alpha) \star \beta + \delta_{\mathcal{D}} \cdot \alpha \star d(\beta)$$

*where $\delta_{\mathcal{C}}, \delta_{\mathcal{D}}, \delta_{\mathcal{F}}$ are the signs involved in the dualities of $\mathcal{C}, \mathcal{D}$ and $\mathcal{F}$.*

*Proof.* See [Ba2], Theorem 5.2. □

## 3 Chow-Witt groups

Let $(D^b(\mathcal{P}(X)),^\vee, 1, \varpi)$ be the triangulated category with the usual duality of Example 2.2 and consider its full subcategory $D^b(\mathcal{P}(X))^{(i)}$ of objects with supports of codimension $\geq i$ (here we use the support defined in Example 2.19). Then the duality on $D^b(\mathcal{P}(X))$ induces dualities on $D^b(\mathcal{P}(X))^{(i)}$ for any $i$ ([Ba1]). It is also clear that $D^b(\mathcal{P}(X))^{(i+1)} \subset D^b(\mathcal{P}(X))^{(i)}$ for any $i$.

Definition 3.1. For all $i \in \mathbb{N}$, denote by $D^b_i(X)$ the triangulated category $D^b(\mathcal{P}(X))^{(i)}/D^b(\mathcal{P}(X))^{(i+1)}$.

Suppose that $(A, \alpha)$ is an $i$-symmetric form in $D^b_i(X)$. Then there exists an $i$-symmetric pair $(B, \beta)$ such that the localization of $(B, \beta)$ is $(A, \alpha)$ (by localization we mean the map $Symm^i(D^b(\mathcal{P}(X))^{(i)}) \to Symm^i(D^b_i(X))$ induced by the functor $D^b(\mathcal{P}(X))^{(i)} \to D^b_i(X)$). Applying 2.7, we get an $(i+1)$-symmetric form $(C, \psi)$. By construction, $C \in D^b(\mathcal{P}(X))^{(i+1)}$. Localizing this form we get a form $(C, \psi)$ in $W^{i+1}(D^b_{i+1}(X))$. At first sight, this construction depends on some choices but in fact this is not the case (see [Ba1], Corollary 4.16). Hence we get a well defined homomorphism

$$d^i : W^i(D^b_i(X)) \to W^{i+1}(D^b_{i+1}(X)).$$

THEOREM 3.2. *Let $X$ be a regular scheme of dimension $n$. Then we have a complex*

$$0 \longrightarrow W^0(D^b_0(X)) \xrightarrow{d^0} W^1(D^b_1(X)) \xrightarrow{d^1} \cdots \xrightarrow{d^n} W^n(D^b_n(X)) \longrightarrow 0.$$

*Proof.* See [BW], Theorem 3.1 and Paragraph 8. □

Let $A$ be a regular local ring. We denote by $W^{fl}(A)$ the Witt group of finite length modules over $A$ (see [QSS] for more informations about Witt groups of finite length modules). The following proposition holds:

PROPOSITION 3.3. *We have isomorphisms*

$$W^i(D^b_i(X)) \simeq \bigoplus_{x\in X^{(i)}} W^{fl}(\mathcal{O}_{X,x}).$$

*Proof.* See [BW], Theorem 6.1 and Theorem 6.2. □

*Remark* 3.4. Since we use the isomorphism of the above proposition, we briefly recall how to obtain a symmetric complex from a finite length module. For more details, see [BW] or [Fa], Chapter 3. Choose a point $x \in X^{(i)}$, a finite length $\mathcal{O}_{X,x}$-module $M$ and a symmetric isomorphism $\phi : M \to \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(M, \mathcal{O}_{X,x})$. Let $P_\bullet$ be a resolution of $M$ by locally free coherent $\mathcal{O}_{X,x}$-modules. Then $P_\bullet$ can be chosen of the form

$$0 \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Dualizing this complex and shifting $i$ times gives the following diagram

$$\begin{tikzcd}
0 \arrow[r] & P_i \arrow[r] \arrow[d, dashed, "\exists"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "\exists"'] & M \arrow[r] \arrow[d, "\phi"'] & 0 \\
0 \arrow[r] & P_0^\vee \arrow[r] & \cdots \arrow[r] & P_i^\vee \arrow[r] & \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(M, \mathcal{O}_{X,x}) \arrow[r] & 0.
\end{tikzcd}$$

Using $\phi$ we get a symmetric morphism $\varphi : P_\bullet \to (P_\bullet)^\vee$. Thus we have constructed an $i$-symmetric pair in the category $D^b(\mathcal{P}(\mathcal{O}_{X,x}))$ from the pair $(M,\phi)$. Since $D^b_i(X) \simeq \coprod_{x\in X^{(i)}} D^b(\mathcal{P}(\mathcal{O}_{X,x}))$ ([BW], Proposition 7.1), we can see the pair $(P_\bullet, \varphi)$ as a symmetric pair in $D^b_i(X)$.

DEFINITION 3.5. The complex

$$0 \to W^{fl}(k(X)) \to \bigoplus_{x_1\in X^{(1)}} W^{fl}(\mathcal{O}_{X,x_1}) \to \cdots \to \bigoplus_{x_n\in X^{(n)}} W^{fl}(\mathcal{O}_{X,x_n}) \to 0$$

is called the Gersten-Witt complex of $X$. We denote it by $C(X,W)$.

This complex is obtained by using the usual duality $^\vee$ on the triangulated category $D^b(\mathcal{P}(X))$ (Example 2.2). For any invertible module $L$ over $X$, we have a duality derived from the functor $^\sharp = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\_, L)$ and we can apply the same construction to get a Gersten-Witt complex.

DEFINITION 3.6. Let $X$ be a regular scheme and $L$ an invertible $\mathcal{O}_X$-module. We denote by $C(X,W,L)$ the Gersten-Witt complex obtained from the duality $^\sharp$.

THEOREM 3.7. *Let $A$ be a regular local $k$-algebra and $X = Spec(A)$. Then for any $i > 0$ we have $H^i(C(X,W)) = 0$.*

*Proof.* See [BGPW], Theorem 6.1. □

Let $A$ be a regular local ring of dimension $n$. Denote by $F$ the residue field of $A$. Then any choice of a generator $\xi \in \mathrm{Ext}^n_A(F,A)$ gives an isomorphism $\alpha_\xi : W(F) \to W^{fl}(A)$. Recall that $I(F)$ is the fundamental ideal of $W(F)$. If $n \leq 0$, put $I^n(F) = W(F)$.

DEFINITION 3.8. For any $n \in \mathbb{Z}$ let $I^n_{fl}(A)$ be the image of $I^n(F)$ by $\alpha_\xi$.

*Remark* 3.9. It is easily seen that $I^n_{fl}(A)$ does not depend on the choice of the generator $\xi \in \mathrm{Ext}^n_A(F,A)$.

PROPOSITION 3.10. *The differential $d$ of the Gersten-Witt complex satisfies $d(I^m_{fl}(\mathcal{O}_{X,x})) \subset I^{m-1}_{fl}(\mathcal{O}_{X,y})$ for any $m \in \mathbb{Z}$, $x \in X^{(i)}$ and $y \in X^{(i-1)}$.*

*Proof.* See [Gi3], Theorem 6.4 or [Fa], Theorem 9.2.4. □

DEFINITION 3.11. Let $L$ be an invertible $\mathcal{O}_X$-module. We denote by $C(X, I^d, L)$ the complex

$$0 \to I^d_{fl}(k(X)) \to \bigoplus_{x_1 \in X^{(1)}} I^{d-1}_{fl}(\mathcal{O}_{X,x_1}) \to \cdots \to \bigoplus_{x_n \in X^{(n)}} I^{d-n}_{fl}(\mathcal{O}_{X,x_n}) \to 0.$$

*Remark* 3.12. In particular, we have $C(X, I^0, L) = C(X, W, L)$.

THEOREM 3.13. *Let $A$ be an essentially smooth local $k$-algebra. Then for any $i > 0$ we have $H^i(C(X, I^d)) = 0$.*

*Proof.* See [Gi3], Corollary 7.7. □

Of course, there is an inclusion $C(X, I^{d+1}, L) \to C(X, I^d, L)$ and therefore we get a quotient complex.

DEFINITION 3.14. Denote by $C(X, \overline{I}^d)$ the complex $C(X, I^d, L)/C(X, I^{d+1}, L)$.

*Remark* 3.15. For any invertible module $L$ the complexes $C(X, I^d)/C(X, I^{d+1})$ and $C(X, I^d, L)/C(X, I^{d+1}, L)$ are *canonically* isomorphic (see [Fa], Corollary E.1.3), so we can drop the $L$ in $C(X, \overline{I}^d)$.

*Remark* 3.16. The complex $C(X,\overline{I}^d)$ is of the form

$$0 \to I^d_{fl}(k(X))/I^{d+1}_{fl}(k(X)) \to \bigoplus_{x_1\in X^{(1)}} I^{d-1}_{fl}(\mathcal{O}_{X,x_1})/I^d_{fl}(\mathcal{O}_{X,x_1}) \to \dots.$$

*Remark* 3.17. As a consequence of Theorem 3.13, we immediately see that $H^i(C(X,\overline{I}^d)) = 0$ for $i>0$ if $X = \operatorname{Spec}(A)$ where $A$ is an essentially smooth local $k$-algebra.

Let $F$ be a field and denote by $K^M_i(F)$ the $i$-th Milnor K-theory group of $F$. If $i<0$ it is convenient to put $K^M_i(F) = 0$.

DEFINITION 3.18. Let $X$ be a scheme. Then for any $d$ we have a complex

$$0 \to K^M_d(k(X)) \to \bigoplus_{x_1\in X^{(1)}} K^M_{d-1}(k(x_1)) \to \dots \to \bigoplus_{x_n\in X^{(n)}} K^M_{d-n}(k(x_n)) \to 0.$$

We denote it by $C(X,K^M_d)$.

*Proof.* See [Ka], Proposition 1 or [Ro], Paragraph 3. □

We also have the exactness of this complex when $X$ is the spectrum of a smooth local $k$-algebra:

THEOREM 3.19. *Let $A$ be a smooth local $k$-algebra. Then for all $i>0$ we have $H^i(C(X,K^M_d)) = 0$.*

*Proof.* See [Ro], Theorem 6.1. □

If $F$ is a field, recall that we have a homomorphism due to Milnor

$$s : K^M_j(F) \to I^j(F)/I^{j+1}(F)$$

given by $s(\{a_1,\dots,a_j\}) =< 1,-a_1 > \otimes \dots \otimes < 1,-a_j >$. The following is true:

LEMMA 3.20. *The homomorphisms $s$ induce a morphism of complexes*

$$s : C(X,K^M_d) \to C(X,\overline{I}^d).$$

*Proof.* See [Fa], Proposition 10.2.5. □

DEFINITION 3.21. Let $C(X,G^d,L)$ be the fibre product of $C(X,K^M_d)$ and $C(X,I^d,L)$ over $C(X,\overline{I}^d)$:

$$\begin{tikzcd} C(X,G^d,L) \arrow[r] \arrow[d] & C(X,I^d,L) \arrow[d, "\pi"] \\ C(X,K^M_d) \arrow[r, "s"'] & C(X,\overline{I}^d). \end{tikzcd}$$

Definition 3.22. Let $X$ be a smooth scheme and $L$ an invertible $\mathcal{O}_X$-module. The $j$-th Chow-Witt group $\widetilde{CH}^j(X,L)$ of $X$ twisted by $L$ is the group $H^j(C(X,G^j,L))$.

*Remark* 3.23. Denote by $GW^j(D^b_j(X),L)$ the $j$-th Grothendieck-Witt group of the category $D^b_j(X)$ with the duality derived from $\mathrm{Hom}_{\mathcal{O}_X}(\_,L)$ (see Definition 2.11). It is not hard to see that $C(X,G^j,L)$ is isomorphic to $GW^j(D^b_j(X),L)$ and therefore the complex $C(X,G^j,L)$ is

$$\cdots \longrightarrow C(X,G^j,L)_{j-1} \longrightarrow GW^j(D^b_j(X),L) \xrightarrow{d^j} W^{j+1}(D^b_{j+1}(X),L) \longrightarrow \cdots$$

Hence $\widetilde{CH}^j(X,L)$ is a quotient of $\mathrm{Ker}(d^j)$ and a subquotient of $GW^j(D^b_j(X),L)$.

We also have the exactness of the complex $C(X,G^d,L)$ in the local case:

Theorem 3.24. *Let $A$ be a smooth local $k$-algebra and $X = Spec(A)$. Then $H^i(C(X,G^j)) = 0$ for all $j$ and all $i > 0$.*

*Proof.* As $C(X,G^j)$ is the fibre product of the complexes $C(X,K^M_j)$ and $C(X,I^j)$ over $C(X,\overline{I}^j)$, we have an exact sequence of complexes

$$0 \longrightarrow C(X,G^j) \longrightarrow C(X,I^j) \oplus C(X,K^M_j) \longrightarrow C(X,\overline{I}^j) \longrightarrow 0$$

inducing a long exact sequence in cohomology. It follows then from Theorem 3.13 and Theorem 3.19 that $H^i(C(X,G^j)) = 0$ if $i > 1$. For $i = 1$, we have an exact sequence

$$H^0(C(X,I^j)) \oplus H^0(C(X,K^M_j)) \to H^0(C(X,\overline{I}^j)) \to H^1(C(X,G^j)) \to 0.$$

The exact sequence of complexes

$$0 \longrightarrow C(X,I^{j+1}) \longrightarrow C(X,I^j) \longrightarrow C(X,\overline{I}^j) \longrightarrow 0$$

shows that $H^0(C(X,I^j))$ maps onto $H^0(C(X,\overline{I}^j))$. □

Definition 3.25. Let $X$ be a smooth scheme and $L$ an invertible $\mathcal{O}_X$-module. We define the sheaf $\mathcal{G}^j_L$ on $X$ by $\mathcal{G}^j_L(U) = H^0(C(U,G^j,L))$.

We have:

Theorem 3.26. *Let $X$ be a smooth scheme of dimension $n$. Then for any $i$ we have*

$$H^i_{Zar}(X,\mathcal{G}^j_L) \simeq H^i(C(X,G^j,L)).$$

*Proof.* Define sheaves $\mathcal{C}_l$ by $\mathcal{C}_l(U) = C(U, G^j, L)_l$ for any $l \geq 0$. It is clear that the $\mathcal{C}_l$ are flasque sheaves. We have a complex of sheaves over $X$

$$0 \longrightarrow \mathcal{G}_L^j \longrightarrow \mathcal{C}_0 \longrightarrow \mathcal{C}_1 \longrightarrow \cdots \longrightarrow \mathcal{C}_n \longrightarrow 0.$$

Theorem 3.24 shows that this complex is a flasque resolution of $\mathcal{G}_L^j$. Thus the theorem is proved.

□

Suppose that $f : X \to Y$ is a flat morphism. Since it preserves codimensions, it induces a morphism of complexes

$$f^* : C(Y, G^j, L) \to C(X, G^j, f^*L)$$

for any $j \in \mathbb{N}$ and any line bundle $L$ over $Y$ ([Fa], Corollary 10.4.2). Hence we have:

THEOREM 3.27. *Let $f : X \to Y$ be a flat morphism and $L$ a line bundle over $Y$. Then, for any $i, j$ we have homomorphisms*

$$f^* : H^i(C(Y, G^j, L)) \to H^i(C(X, G^j, f^*L)).$$

*In particular, if $E$ is a vector bundle over $Y$ and $\pi : E \to Y$ is the projection, we have isomorphisms*

$$\pi^* : H^i(C(Y, G^j, L)) \to H^i(C(E, G^j, \pi^*L)).$$

*Proof.* We have a morphism of complexes $f^* : C(Y, G^j, L) \to C(X, G^j, f^*L)$ which gives the induced homomorphisms in cohomology. For the proof of homotopy invariance, see Corollary 11.3.2 in [Fa]. □

PROPOSITION 3.28. *Let $f : X \to Y$ and $g : Y \to Z$ be flat morphisms. Then $(gf)^* = f^*g^*$.*

*Proof.* See [Fa], Proposition 3.4.9. □

Suppose that $f : X \to Y$ is a finite morphism with $\dim(Y) - \dim(X) = r$. Consider the morphism of locally ringed spaces $\overline{f} : (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ induced by $f$. If $X$ is smooth, then $L = \overline{f}^* \mathrm{Ext}^r_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y)$ is an invertible module over $Y$ ([Gi2], Corollary 6.6) and we get a morphism of complexes (of degree r)

$$f_* : C(X, G^{j-r}, L \otimes f^*N) \to C(Y, G^j, N)$$

for any invertible module $N$ over $Y$ ([Fa], Corollary 5.3.7).

Proposition 3.29. *Let $f : X \to Y$ be a finite morphism between smooth schemes. Let $dim(Y) - dim(X) = r$ and $N$ be an invertible module over $Y$. Then the morphism of complexes $f_*$ induces a homomorphism*

$$f_* : H^{i-r}(C(X, G^{j-r}, L \otimes f^*N)) \to H^i(C(Y, G^j, N)).$$

In particular, we have ([Fa], Remark 9.3.5):

Proposition 3.30. *Let $f : X \to Y$ be a closed immersion of codimension $r$ between smooth schemes. Then $f$ induces an isomorphism*

$$f_* : H^{i-r}(C(X, G^{j-r}, L \otimes f^*N)) \to H^i_X(C(Y, G^j, N))$$

*for any $i, j$ and any invertible module $N$ over $Y$.*

*Important remark* 3.31. If $f : X \to Y$ is a closed immersion, then $f_*$ will always be the map with support:

$$f_* : H^{i-r}(C(X, G^{j-r}, L \otimes f^*N)) \to H^i_X(C(Y, G^j, N))$$

The transfer for finite morphisms is functorial ([Fa], proposition 5.3.8):

Proposition 3.32. *Let $f : X \to Y$ and $g : Y \to Z$ be finite morphisms. Then $g_*f_* = (gf)_*$.*

*Remark* 3.33. Let $X$ be a smooth scheme and $D$ be a smooth effective Cartier divisor on $X$. Let $i : D \to X$ be the inclusion and $L(D)$ be the line bundle over $X$ associated to $D$. Then there is a canonical section $s \in L(D)$ (see [Fu], Appendix B.4.5) and an exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} L(D) \longrightarrow i_*\mathcal{O}_D \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathcal{O}_X}(\_, L(D))$ and shifting, we obtain the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "s"] \arrow[d, "\simeq"'] & L(D) \arrow[r] \arrow[d, "\simeq"'] & i_*\mathcal{O}_D \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{O}_X}(L(D), L(D)) \arrow[r, "s"'] & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, L(D)) \arrow[r] & \mathrm{Ext}^1_{\mathcal{O}_X}(i_*\mathcal{O}_D, L(D)) \arrow[r] & 0
\end{tikzcd}$$

which shows that $\mathrm{Ext}^1_{\mathcal{O}_X}(i_*\mathcal{O}_D, \mathcal{O}_X) \otimes L(D) \simeq i_*\mathcal{O}_D$. Proposition 3.30 shows that we then have an isomorphism

$$i_* : H^{i-1}(C(D, G^{j-1}, i^*L(D))) \to H^i_D(C(X, G^j)).$$

Lemma 3.34. *Let $g : X \to Y$ be a flat morphism and $f : Z \to Y$ a finite morphism. Consider the following fibre product*

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "g'"'] & X \arrow[d, "g"] \\ Z \arrow[r, "f"'] & Y. \end{tikzcd}$$

*Then* $(f')_*(g')^* = g^* f_*$.

*Proof.* See [Fa], Corollary 12.2.8. □

*Remark* 3.35. Of course, in the above fibre product we suppose that $V$ is also smooth and integral. Such a strong assumption is not necessary in general, but this case is sufficient for our purposes.

*Remark* 3.36. It is possible to define a map $f_*$ when the morphism $f$ is proper (see [Fa]) but we don't use this fact here.

## 4 The exterior product

Let $X$ and $Y$ be two schemes. The fibre product $X \times Y$ comes equipped with two projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$.

Lemma 4.1. *For any* $i, j \in \mathbb{N}$, *the pairing*

$$\boxtimes : D_i^b(X) \times D_j^b(Y) \to D_{i+j}^b(X \times Y)$$

*given by* $P \boxtimes Q = p_1^* P \otimes p_2^* Q$ *is a dualizing pairing of triangulated categories with duality.*

*Proof.* Straight verification. □

Corollary 4.2. *For any* $i, j \in \mathbb{N}$, *the pairing*

$$\boxtimes : D_i^b(X) \times D_j^b(Y) \to D_{i+j}^b(X \times Y)$$

*induces a pairing*

$$\star : W^i(D_i^b(X)) \times W^j(D_j^b(Y)) \to W^{i+j}(D_{i+j}^b(X \times Y)).$$

*Proof.* Clear by Theorem 2.15. □

Corollary 4.3. *Let* $\psi \in W^j(D_j^b(Y))$. *Then we have a homomorphism*

$$\mu_\psi : W^i(D_i^b(X)) \to W^{i+j}(D_{i+j}^b(X \times Y))$$

*given by* $\mu_\psi(\varphi) = \varphi \star \psi$.

Recall that we have isomorphisms $W^i(D_i^b(X)) \simeq \bigoplus_{x \in X^{(i)}} W^{fl}(\mathcal{O}_{X,x})$ (Proposition 3.3).

Definition 4.4. For any $s \in \mathbb{Z}$, denote by $I^s(D_i^b(X))$ the preimage of $\bigoplus_{x \in X^{(i)}} I^s_{fl}(\mathcal{O}_{X,x})$ under the above isomorphism.

Proposition 4.5. *For any $m, p \in \mathbb{N}$ the product*

$$\star : W^i(D_i^b(X)) \times W^j(D_j^b(Y)) \to W^{i+j}(D_{i+j}^b(X \times Y))$$

*induces a product*

$$\star : I^m(D_i^b(X)) \times I^n(D_j^b(Y)) \to I^{m+n}(D_{i+j}^b(X \times Y)).$$

*Proof.* Let $x \in X^{(i)}$ and $y \in Y^{(j)}$. It is clear that the product can be computed locally (use [GN], Theorem 3.2). So we can suppose that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ where $A$ and $B$ are local in $x$ and $y$ respectively. Recall that we have the following diagram

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(k). \end{tikzcd}$$

Let $P$ be an $A$-projective resolution of $k(x)$ and $Q$ be a $B$-projective resolution of $k(y)$. Consider a symmetric form $\rho : k(x) \to \mathrm{Ext}^i_A(k(x), A)$ and a symmetric form $\mu : k(y) \to \mathrm{Ext}^j_B(k(y), B)$. Then $p_1^*(\rho)$ is a symmetric isomorphism supported by the complex $P \otimes_k B$ and $p_2^*(\mu)$ is a symmetric isomorphism supported by the complex $A \otimes_k Q$. The complex $(P \otimes_k B) \otimes_{A \otimes_k B} (A \otimes_k Q)$ (which is isomorphic to $P \otimes_k Q$) has its homology concentrated in degree 0, and this homology is isomorphic to $k(x) \otimes_k k(y)$. Let $u$ be a point of $\mathrm{Spec}(k(x) \otimes_k k(y))$. Then the restriction of $p_1^*\rho \otimes p_2^*\mu$ to $u$ is a finite length module $M$ whose support is on $u$ with a symmetric form

$$M \to \mathrm{Ext}^{i+j}_{(A \otimes B)_u}(M, (A \otimes B)_u).$$

Taking its class in the Witt group, we obtain a $k(u)$-vector space $V$ with a symmetric form $\psi : V \to \mathrm{Ext}^{i+j}_{(A \otimes B)_u}(V, (A \otimes B)_u)$. Now choose a unit $a \in k(x)^\times$. Consider the image $a_u$ of $a$ under the homomorphism $k(x) \to k(u)$. The class of $p_1^*(a\rho) \otimes p_2^*(\mu)$ is the symmetric form

$$a_u \psi : V \to \mathrm{Ext}^{i+j}_{(A \otimes B)_u}(V, (A \otimes B)_u).$$

As the same property holds for any unit $b \in k(y)^\times$, we conclude that

$$p_1^*(< 1, -a_1 > \otimes \ldots \otimes < 1, -a_n > \rho) \otimes p_2^*(< 1, -b_1 > \otimes \ldots \otimes < 1, -b_m > \mu)$$

is equal to $< 1, -(a_1)_u > \otimes \ldots \otimes < 1, -(b_m)_u > \psi$. $\square$

Recall that for any scheme $X$ we have a Gersten-Witt complex (Definition 3.5)

$$C(X,W): \qquad \cdots \longrightarrow W^r(D^b_r(X)) \xrightarrow{d^r_X} W^{r+1}(D^b_{r+1}(X)) \longrightarrow \cdots$$

and a complex $C(X,I^d)$:

$$\cdots \longrightarrow \bigoplus_{x_r\in X^{(r)}} I^{d-r}_{fl}(\mathcal{O}_{X,x_r}) \longrightarrow \bigoplus_{x_{r+1}\in X^{(r+1)}} I^{d-r-1}_{fl}(\mathcal{O}_{X,x_{r+1}}) \longrightarrow \cdots.$$

The above proposition gives:

COROLLARY 4.6. *The product*

$$\star : C(X,W)\times C(Y,W)\to C(X\times Y,W)$$

*induces for any $r,s\in N$ a product*

$$\star : C(X,I^r)\times C(Y,I^s)\to C(X\times Y,I^{r+s}).$$

Now we investigate the relations between $\star$ and the differentials of the complexes.

PROPOSITION 4.7. *Let $\psi\in W^j(D^b_j(Y))$ be such that $d^j_Y(\psi)=0$. Then the following diagram commutes*

$$\begin{tikzcd}
W^i(D^b_i(X)) \arrow[r, "d^i_X"] \arrow[d, "(-1)^j\mu_\psi"'] & W^{i+1}(D^b_{i+1}(X)) \arrow[d, "\mu_\psi"] \\
W^{i+j}(D^b_{i+j}(X\times Y)) \arrow[r, "d^{i+j}_{X\times Y}"'] & W^{i+j+1}(D^b_{i+j+1}(X\times Y)).
\end{tikzcd}$$

*Proof.* Let $\varphi\in W^i(D^b_i(X))$. Let $X^{(\geq i+1)}$ be the set of points of $X$ of codimension $\geq i+1$, $Y^{(\geq j+1)}$ the points of $Y$ of codimension $\geq j+1$ and $(X\times Y)^{(\geq i+j+1)}$ the set of points of $X\times Y$ of codimension $\geq i+j+1$. By Lemma 2.18, the triangulated categories $D^b_i(X), D^b_j(Y)$ and $D^b_{i+j}(X\times Y)$ are defined over the topological spaces $X\setminus X^{(\geq i+1)}$, $Y\setminus Y^{(\geq j+1)}$ and $(X\times Y)\setminus(X\times Y)^{(\geq i+j+1)}$. Let $\alpha\in Symm^i(D^b(\mathcal{P}(X))^{(i)})$ and $\beta\in Symm^j(D^b(\mathcal{P}(Y))^{(j)})$ be symmetric pairs representing $\varphi$ and $\psi$. By definition, $\mathrm{DegLoc}(\alpha)$ is of codimension $\geq i+1$, $\mathrm{DegLoc}(\beta)$ is of codimension $\geq j+1$ and $d\beta$ is neutral. It is easily seen that $\mathrm{Supp}(dp_1^*\alpha)\cap\mathrm{Supp}(dp_2^*\beta)=\emptyset$ in the topological space $(X\times Y)\setminus(X\times Y)^{(\geq i+j+1)}$. Theorem 2.25 implies that

$$(-1)^{i+j}d(p_1^*\alpha\star p_2^*\beta)=(-1)^i dp_1^*\alpha\star p_2^*\beta+(-1)^j p_1^*\alpha\star dp_2^*\beta.$$

Using Theorem 2.15, we see that we have in $W^{i+j}(D^b_{i+j}(X \times Y))$ the equality

$$(-1)^j d^{i+j}_{X\times Y}(p_1^*\varphi \star p_2^*\psi) = p_1^* d^i_X(\varphi) \star p_2^*\psi.$$

□

The following corollary is obvious.

COROLLARY 4.8. *Let $\psi \in I^m(D^b_j(Y))$ be such that $d^Y_j(\psi) = 0$. Then the following diagram commutes*

$$\begin{tikzcd} I^p(D^b_i(X)) \arrow[r, "d^i_X"] \arrow[d, "(-1)^j\mu_\psi"'] & I^{p-1}(D^b_{i+1}(X)) \arrow[d, "\mu_\psi"] \\ I^{p+m}(D^b_{i+j}(X\times Y)) \arrow[r, "d^{i+j}_{X\times Y}"'] & I^{p+m-1}(D^b_{i+j+1}(X\times Y)). \end{tikzcd}$$

We now have to deal with the complex in Milnor K-theory. Let $C(X, K^M_r)$, $C(Y, K^M_s)$ and $C(X\times Y, K^M_{r+s})$ be the complexes in Milnor K-theory associated to $X, Y$ and $X \times Y$. In [Ro], Rost defines a product

$$\odot : C(X, K^M_r)^i \times C(Y, K^M_s)^j \to C(X \times Y, K^M_{r+s})^{i+j}$$

as follows: Let $u \in (X\times Y)^{(i+j)}$, $x \in X^{(i)}$, $y \in Y^{(j)}$ be such that $x$ and $y$ are the projections of $u$. Let $\rho = \{a_1, \dots, a_{r-i}\} \in K^M_{r-i}(k(x))$ and $\mu = \{b_1, \dots, b_{s-j}\} \in K^M_{s-j}(k(y))$. Then

$$(\rho \odot \mu)_u = l((k(x)\otimes_k k(y))_u)\{(a_1)_u, \dots, (a_{r-i})_u, (b_1)_u, \dots, (b_{s-j})_u\}$$

where the $(a_l)_u$ and $(b_t)_u$ are the images of the $a_l$ and $b_t$ under the inclusions $k(x) \to k(u)$ and $k(y) \to k(u)$, and $l((k(x)\otimes_k k(y))_u)$ is the length of the module $k(x)\otimes_k k(y)$ localized in $u$.

LEMMA 4.9. *For any $\rho \in C(X, K^M_r)^i$ and $\mu \in C(Y, K^M_s)^j$ we have*

$$d(\rho \odot \mu) = d(\rho) \odot \mu + (-1)^j \rho \odot d(\mu).$$

*Proof.* See [Ro], Paragraph 14.4. □

COROLLARY 4.10. *Let $\mu \in C(Y, K^M_s)^j$ be such that $d\mu = 0$. Then the following diagram commutes:*

$$\begin{tikzcd} C(X, K^M_r)^i \arrow[r, "d^i_X"] \arrow[d, "\odot\mu"'] & C(X, K^M_r)^{i+1} \arrow[d, "\odot\mu"] \\ C(X\times Y, K^M_{r+s})^{i+j} \arrow[r, "d^{i+j}_{X\times Y}"'] & C(X\times Y, K^M_{r+s})^{i+j+1}. \end{tikzcd}$$

*Proof.* Obvious. □

Now we compare the products $\star$ and $\odot$.

PROPOSITION 4.11. *The following diagram commutes:*

$$\begin{tikzcd}
C(X,K_r^M)^i \times C(Y,K_s^M)^j \arrow[r, "\odot"] \arrow[d, "s_{(r-i)}\times s_{(s-j)}"'] & C(X\times Y, K_{r+s}^M)^{i+j} \arrow[d, "s_{(r+s-i-j)}"] \\
C(X,\overline{I}^r)^i \times C(Y,\overline{I}^s)^j \arrow[r, "\star"'] & C(X\times Y,\overline{I}^{r+s})^{i+j}.
\end{tikzcd}$$

*Proof.* Let $\{a_1,\ldots,a_{r-i}\} \in K_{r-i}^M(k(x))$ and $\{b_1,\ldots,b_{s-j}\} \in K_{s-j}^M(k(y))$. Let $\rho'$ be a symmetric isomorphism

$$\rho' : k(x) \to \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(k(x),\mathcal{O}_{X,x})$$

and $\mu'$ a symmetric isomorphism

$$\mu' : k(y) \to \mathrm{Ext}^j_{\mathcal{O}_{Y,y}}(k(y),\mathcal{O}_{Y,y}).$$

We then have $\rho := s_{(r-i)}(\{a_1,\ldots,a_{r-i}\}) =< 1,-a_1 > \otimes \ldots \otimes < 1,-a_{r-i} > \rho'$ and $\mu := s_{(s-j)}(\{b_1,\ldots,b_{s-j}\}) =< 1,-b_1 > \otimes \ldots \otimes < 1,-b_{s-j} > \mu'$. Choose a point $u$ in $(X\times Y)^{(i+j)}$ lying over $x$ and $y$. The proof of Proposition 4.5 shows that

$$(\rho\star\mu)_u = s_{(r+s-i-j)}(\{(a_1)_u,\ldots,(a_{r-i})_u,(b_1)_u,\ldots,(b_{s-j})_u\})\varphi$$

where $\varphi : M \to \mathrm{Ext}^{i+j}_{\mathcal{O}_{X\times Y,u}}(M,\mathcal{O}_{X\times Y,u})$ is a symmetric isomorphism and $M$ is a $k(u)$-vector space. But $\dim_{k(u)} M \equiv l((k(x)\otimes k(y))_u) \pmod 2$ where $l$ denotes the length. So we have in $C(X\times Y,\overline{I}^{r+s})^{i+j}$ the equality

$$(\rho\star\mu)_u = s_{(r+s-i-j)}(\{(a_1)_u,\ldots,(a_{r-i})_u,(b_1)_u,\ldots,(b_{s-j})_u\})l((k(x)\otimes k(y))_u).$$

The right hand term is equal to $s_{(r+s-i-j)}(\{a_1,\ldots,a_{r-i}\}\odot\{b_1,\ldots,b_{s-j}\})$ by definition. □

COROLLARY 4.12. *The products*

$$\star : C(X,I^r)\times C(Y,I^s) \to C(X\times Y,I^{r+s})$$

*and*

$$\odot : C(X,K_r^M)\times C(Y,K_s^M) \to C(X\times Y,K_{r+s}^M)$$

*give a product*

$$\diamond : C(X,G^r)\times C(Y,G^s) \to C(X\times Y,G^{r+s}).$$

COROLLARY 4.13. *Let $\mu \in C(Y, G^s)^j$ such that $d_Y^j \mu = 0$. Then $\mu$ induces a product*

$$\_ \diamond \mu : H^i(C(X, G^r)) \to H^{i+j}(C(X \times Y, G^{r+s})).$$

*Proof.* This a direct consequence of Proposition 4.11, Corollary 4.8 and Corollary 4.10. □

Next we have to check that $\_ \diamond \mu$ is well defined on the cohomolgy class of $\mu$.

LEMMA 4.14. *Let $\gamma \in C(Y, G^s)^{j-1}$ and $\mu = d_Y^{j-1}\gamma$. Then $\_ \diamond \mu = 0$.*

*Proof.* Suppose that $\alpha$ is such that $d_X^i \alpha = 0$. By Corollary 4.8 and Corollary 4.10 we have up to signs $d_{X\times Y}^{i+j-1}(\alpha \diamond \gamma) = \alpha \diamond d^{j-1}\gamma = \alpha \diamond \mu$. So $\alpha \diamond \mu$ is trivial in $H^{i+j}(C(X \times Y, G^{r+s}))$. □

Finally:

THEOREM 4.15. *Let $X$ and $Y$ be smooth schemes. Then for any $i, j, r, s \in \mathbb{N}$ the product*

$$\diamond : C(X, G^r) \times C(Y, G^s) \to C(X \times Y, G^{r+s})$$

*induces an exterior product*

$$\times : H^i(C(X, G^r)) \times H^j(C(Y, G^s)) \to H^{i+j}(C(X \times Y, G^{r+s})).$$

This exterior product can also be defined with complexes twisted by invertible modules.

THEOREM 4.16. *Let $X$ and $Y$ be smooth schemes. Let $L$ and $N$ be invertible modules over $X$ and $Y$ respectively. For any $i, j, r, s \in \mathbb{N}$, the pairing*

$$\diamond : C(X, G^r, L) \times C(Y, G^s, N) \to C(X \times Y, G^{r+s}, p_1^* L \otimes p_2^* N)$$

*induces an exterior product*

$$\times : H^i(C(X, G^r, L)) \times H^j(C(Y, G^s, N)) \to H^{i+j}(C(X \times Y, G^{r+s}, p_1^* L \otimes p_2^* N)).$$

*Proof.* Left to the reader. □

If $i = r$ and $j = s$, we obtain the following corollary:

COROLLARY 4.17. *Let $X$ and $Y$ be smooth schemes. Then for any $i, j \in \mathbb{N}$ the product*

$$\diamond : C(X, G^i) \times C(Y, G^j) \to C(X \times Y, G^{i+j})$$

*gives an exterior product*

$$\times : \widetilde{CH}^i(X) \times \widetilde{CH}^j(Y) \to \widetilde{CH}^{i+j}(X \times Y).$$

Next we prove some properties of this exterior product:

Proposition 4.18. *The exterior product $\times$ is associative.*

*Proof.* It clearly suffices to prove that the exterior products $\star$ and $\odot$ are associative. For $\star$ this is clear because of the associativity of the tensor product (up to isomorphism). For the second, see (14.2) in [Ro]. $\square$

Now we deal with the commutativity. Let $X$ and $Y$ be smooth schemes and let $\tau : X \times Y \to Y \times X$ be the flip. We have:

Lemma 4.19. *Let $\mu \in H^i(C(X, K_r^M))$ and $\eta \in H^j(C(Y, K_s^M))$. Then we have $\tau^*(\eta \odot \mu) = (-1)^{(r-i)(s-j)}(\mu \odot \eta)$.*

*Proof.* This is clear from the definition. $\square$

Lemma 4.20. *Let $\mu \in H^i(C(X, I^r))$ and $\eta \in H^j(C(Y, I^s))$. Then we have $\tau^*(\eta \star \mu) = (-1)^{ij}(\mu \star \eta)$.*

*Proof.* It is clear by the skew-commutativity of the product of Witt groups ([GN], Theorem 3.1). $\square$

*Remark* 4.21. Of course, the associativity and the anticommutativity of the exterior product are also true for the twisted product of Theorem 4.16.

## 5 Intersection with a smooth subscheme

### 5.1 The Gysin-Witt map

The goal of this section is to define for any closed embedding $i : Y \to X$ of smooth schemes a Gysin-Witt map $i^! : H^r(C(X, G^j)) \to H^r(C(Y, G^j))$. In order to define such a map, we adapt the ideas of Rost ([Ro], Paragraph 11).
First we briefly recall the properties of the deformation to the normal cone. For more details, see [Fu] (Chapter 5) or [Ro] (Chapter 10). Let $Y$ be a closed subscheme of a smooth scheme $X$. Then there is a smooth scheme $D(X, Y)$, a closed imbedding $j : Y \times \mathbb{A}^1 \hookrightarrow D(X, Y)$ and a flat morphism $\rho : D(X, Y) \to \mathbb{A}^1$ such that the following diagram commutes

$$\begin{tikzcd} Y \times \mathbb{A}^1 \arrow[r, "j"] \arrow[dr, "pr"'] & D(X,Y) \arrow[d, "\rho"] \\ & \mathbb{A}^1 \end{tikzcd}$$

and

(1) $\rho^{-1}(\mathbb{A}^1 - 0) = X \times (\mathbb{A}^1 - 0)$ and the restriction of $j$ is the closed imbedding $i \times Id : Y \times (\mathbb{A}^1 - 0) \hookrightarrow X \times (\mathbb{A}^1 - 0)$.

(2) $\rho^{-1}(0) = N_Y X$, where $N_Y X$ is the normal cone to $Y$ in $X$ and the restriction of $j$ is the embedding as the zero section $s_0 : Y \to N_Y X$.

The scheme $D(X,Y)$ can be obtained as follows: Consider the blow-up $M$ of $X \times \mathbb{A}^1$ along $Y \times 0$ and the blow-up $\tilde{X}$ of $X \times 0$ along $Y \times 0$. Then define $D(X,Y)$ to be $M \setminus \tilde{X}$.
If $Y$ is smooth in a smooth scheme $X$, then it is locally of complete intersection and $N_Y X$ is a vector bundle over $Y$ of rank $\dim(X) - \dim(Y)$. Moreover, $N_Y X$ is Cartier divisor on $D(X,Y)$. If $\mathbb{A}^1 = \mathrm{Spec}(k[t])$, then the projection $\rho : D(X,Y) \to \mathbb{A}^1$ gives a homomorphism $k[t] \to \mathcal{O}_{D(X,Y)}(D(X,Y))$. We still denote by $t$ the image of $t$ under this homomorphism. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{D(X,Y)} \xrightarrow{t} \mathcal{O}_{D(X,Y)} \longrightarrow \kappa_* \mathcal{O}_{N_Y X} \longrightarrow 0$$

where $\kappa : N_Y X \to D(X,Y)$ is the inclusion. Remark 3.33 shows that $\mathrm{Ext}^1_{\mathcal{O}_{D(X,Y)}}(\kappa_* \mathcal{O}_{N_Y X}, \mathcal{O}_{D(X,Y)}) \simeq \kappa_* \mathcal{O}_{N_Y X}$ with generator the Koszul complex associated to the global section $t$.
Let $U = \mathbb{A}^1 - 0$ and consider the form

$$< 1, -t >: \mathcal{O}_U^2 \to \mathcal{O}_U^2$$

in $W^0(D^b(U))$. Now let $X$ be a smooth scheme and consider the projection $\eta : X \times U \to U$. Then $\eta^*(< 1, -t >) \in W^0(D^b(X \times U))$ and we also denote it by $< 1, -t >$. Since the support of this form is $X \times U$, the tensor product gives a functor

$$< 1, -t > \otimes \_ : D^b_i(X \times U) \to D^b_i(X \times U).$$

Using the fact that $< 1, -t >$ is symmetric, we see that this functor is duality preserving (see [GN], Definition 1.8 and Lemma 1.14) and therefore induces for any $i$ a homomorphism

$$< 1, -t > \otimes \_ : W^i(D^b_i(X \times U)) \to W^i(D^b_i(X \times U)).$$

For some sign reasons that will be made clearer in Lemma 5.10, we will in fact consider for any $i$ the homomorphism

$$m_t : W^i(D^b_i(X \times U)) \to W^i(D^b_i(X \times U))$$

defined by $m_t(\alpha) = (-1)^{i+1} < 1, -t > \otimes \alpha$.

Lemma 5.1. *For any $i, j \in \mathbb{N}$ the homomorphism $m_t$ induces a homomorphism*

$$I^j(D^b_i(X \times U)) \to I^{j+1}(D^b_i(X \times U))$$

*and the following diagram commutes*

$$\begin{tikzcd} I^j(D_i^b(X\times U)) \arrow[r, "d^i"] \arrow[d, "-m_t"'] & I^{j-1}(D_{i+1}^b(X\times U)) \arrow[d, "m_t"] \\ I^{j+1}(D_i^b(X\times U)) \arrow[r, "d^i"'] & I^j(D_{i+1}^b(X\times U)). \end{tikzcd}$$

*Proof.* The first assertion is clear. Now $< 1, -t >$ is a global isomorphism and we can use Theorem 2.10 in [GN] (or Theorem 2.25 in the present paper) to see that

$$d^i(< 1, -t > \otimes\alpha) =< 1, -t > \otimes d^i\alpha$$

for any $\alpha \in I^j(D_i^b(X\times U))$. The first term is $(-1)^{i+1}d^i(m_t(\alpha))$ and the second one is $(-1)^{i+2}m_t(d^i\alpha)$. □

Now consider $t \in \mathcal{O}^*_{X\times U}$. For any $i$ and any $x \in X\times U$, we have a multiplication by $t$:

$$n_t : K_i^M(k(x)) \to K_{i+1}^M(k(x))$$

defined by $n_t(\{a_1,\dots,a_i\}) = \{t, a_1,\dots,a_i\}$.

LEMMA 5.2. *For any $i, j \in N$ the following diagram commutes*

$$\begin{tikzcd} C(X\times U, K_j^M)^i \arrow[r, "d^i"] \arrow[d, "-n_t"'] & C(X\times U, K_j^M)^{i+1} \arrow[d, "n_t"] \\ C(X\times U, K_{j+1}^M)^i \arrow[r, "d^i"'] & C(X\times U, K_{j+1}^M)^{i+1}. \end{tikzcd}$$

*Proof.* See [Ro], Proposition 4.6. □

COROLLARY-DEFINITION 5.3. *The homomorphisms $m_t$ and $n_t$ induce for any $i, j \in N$ a homomorphism*

$$\{t\} : H^i(C(X\times U, G^j)) \to H^i(C(X\times U, G^{j+1})).$$

*We call this homomorphism* multiplication by $t$.

*Proof.* It suffices to show that $m_t$ and $n_t$ give the same operation on the complex $C(X\times U, \overline{I}^j)$. It is straightforward. □

We will need the following lemma:

LEMMA 5.4. *Let $f : X \to Y$ be a flat morphism of smooth schemes. Then for any $i, j$ the following diagram commutes*

$$\begin{tikzcd} H^i(C(Y\times U,G^j)) \arrow[r, "\{t\}"] \arrow[d, "(f\times Id)^*"'] & H^i(C(Y\times U,G^{j+1})) \arrow[d, "(f\times Id)^*"] \\ H^i(C(X\times U,G^j)) \arrow[r, "\{t\}"'] & H^i(C(X\times U,G^{j+1})). \end{tikzcd}$$

*Proof.* First observe that $(f\times Id)^*(<1,-t>)=<1,-t>$ by definition. Then for any $\alpha\in I^r(D_i^b(X\times U))$ we have $(f\times Id)^*(m_t\alpha)=m_t((f\times Id)^*\alpha)$ (use [GN], Theorem 3.4). On the other hand, we have $(f\times Id)^*(n_t(\alpha))=n_t((f\times Id)^*\alpha)$ for any $\alpha\in K_r^M(k(y))$ ([Ro], Lemma 4.3). Putting this together, we get the conclusion. □

Let $Y\to X$ be a closed embedding of smooth schemes and consider the deformation to the normal cone space $D(X,Y)$. Then $N_YX$ is a Cartier divisor and its complement in $D(X,Y)$ is $X\times U$. We have a long exact sequence associated to this triple ([Fa], Corollary 10.4.9):

$$H^i(C(D(X,Y),G^{j+1}))\to H^i(C(X\times U,G^{j+1}))\xrightarrow{\partial} H^{i+1}_{N_YX}(C(D(X,Y),G^{j+1}))$$

Combining the isomorphism of Proposition 3.30 and the isomorphism

$$\mathcal{O}_{N_YX}\to\overline{\kappa}^*\mathrm{Ext}^1_{\mathcal{O}_{D(X,Y)}}(\kappa_*\mathcal{O}_{N_YX},\mathcal{O}_{D(X,Y)})$$

mapping 1 to the Koszul complex associated to the global section $t$ of $\mathcal{O}_{D(X,Y)}$, we finally get an isomorphism

$$\kappa_*:H^i(C(N_YX,G^j))\to H^{i+1}_{N_YX}(C(D(X,Y),G^{j+1})).$$

Let $q:N_YX\to Y$ and $\pi:X\times U\to X$ be the projections and consider the following composition:

$$\begin{tikzcd} H^i(C(X,G^j)) \arrow[rr, dashed] \arrow[d, "\pi^*"'] & & H^i(C(Y,G^j)) \\ H^i(C(X\times U,G^j)) \arrow[r, "\{t\}"'] & H^i(C(X\times U,G^{j+1})) \arrow[r, "(\kappa_*)^{-1}\partial"'] & H^i(C(N_YX,G^j)). \arrow[u, "(q^*)^{-1}"'] \end{tikzcd}$$

DEFINITION 5.5. Let $Y$ be a smooth subscheme of a smooth scheme $X$ with inclusion $i:Y\to X$. We denote by $i^!:H^r(C(X,G^j))\to H^r(C(Y,G^j))$ and call *Gysin-Witt map* the composition $(q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}\pi^*$.

*Remark* 5.6. Let $i:Y\to X$ be a closed immersion of smooth schemes and let $L$ be an invertible $\mathcal{O}_X$-module. Then we have a twisted version of the Gysin-Witt map:

$$i^!:H^r(C(X,G^j,L))\to H^r(C(Y,G^j,i^*L)).$$

### 5.2 Functoriality

The goal of this section is to prove that for any inclusions of smooth schemes $Z \xrightarrow{i} Y \xrightarrow{j} X$ we have $(ji)^! = i^! j^!$. The strategy is not new. We follow the exposition of the sections 11, 12 and 13 in [Ro]. First we prove some lemmas:

Lemma 5.7. *Let $i : Y \to X$ be a closed immersion and $g : V \to X$ be a flat morphism. Consider the following fibre product*

$$\begin{tikzcd} W \arrow[r, "i'"] \arrow[d, "g'"'] & V \arrow[d, "g"] \\ Y \arrow[r, "i"'] & X. \end{tikzcd}$$

*Then we have $(g')^* i^! = (i')^! g^*$.*

*Proof.* Let $D(X,Y)$ be the deformation to the normal cone associated to the inclusion $i : Y \hookrightarrow X$ and $D(V,W)$ be the deformation associated to $i' : W \hookrightarrow V$. Let $U = \mathbb{A}^1 - 0$. Because of the universal properties of blow-ups, we see that $g$ and $g'$ give a morphism $D(g) : D(V,W) \to D(X,Y)$ such that the following diagram commutes:

$$\begin{tikzcd} D(V,W) \arrow[d, "D(g)"'] & V \times U \arrow[l, "\iota'"'] \arrow[d, "g\times 1"] \\ D(X,Y) & X \times U \arrow[l, "\iota"] \end{tikzcd}$$

where $\iota$ and $\iota'$ are the inclusions of the respective open subsets. We also get a morphism $N(g) : N_W V \to N_Y X$ such that these diagrams commute:

$$\begin{tikzcd} N_W V \arrow[r, "q'"] \arrow[d, "N(g)"'] & W \arrow[d, "g'"] \\ N_Y X \arrow[r, "q"'] & Y \end{tikzcd} \qquad \begin{tikzcd} N_W V \arrow[r, "\kappa'"] \arrow[d, "N(g)"'] & D(V,W) \arrow[d, "D(g)"] \\ N_Y X \arrow[r, "\kappa"'] & D(X,Y). \end{tikzcd}$$

Now use Propositions 3.28 and 3.34, Lemma 5.4, the naturality of the connecting homomorphism $\partial$ and the diagram

$$\begin{tikzcd} W \arrow[d, "g'"'] & N_W V \arrow[l, "q'"'] \arrow[r, "\kappa'"] \arrow[d, "N(g)"] & D(V,W) \arrow[d, "D(g)"] & V \times U \arrow[l, "\iota'"'] \arrow[r, "\pi'"] \arrow[d, "g\times 1"] & V \arrow[d, "g"] \\ Y & N_Y X \arrow[l, "q"] \arrow[r, "\kappa"'] & D(X,Y) & X \times U \arrow[l, "\iota"] \arrow[r, "\pi"'] & X \end{tikzcd}$$

to conclude (observe that $D(g)$ and $N(g)$ are flat because of [Ro], Remark 10.1).

$\square$

LEMMA 5.8. *Let* $Z \xrightarrow{i} Y \xrightarrow{j} X$ *be inclusions of smooth schemes. Then we have inclusions* $a : N_Z Y \to N_Z X$, $c : i^* N_Y X \to N_Y X$ *and isomorphisms* $N_{(i^* N_Y X)}(N_Y X) \simeq N_Z Y \oplus i^* N_Y X \simeq N_{(N_Z Y)}(N_Z X)$.

*Proof.* The first two assertions are straight computations (see also [Ne]). The relation (2.1) in [Ne] shows that we have canonical isomorphisms

$$N_{(i^* N_Y X)}(N_Y X) \simeq N_Z Y \oplus i^* N_Y X \simeq N_{(N_Z Y)}(N_Z X).$$

$\square$

LEMMA 5.9. *Let* $Z \xrightarrow{i} Y \xrightarrow{j} X$ *be inclusions of smooth schemes. Let* $a : N_Z Y \to N_Z X$, $c : i^* N_Y X \to N_Y X$ *be the inclusions and* $q : N_Y X \to Y$, $r : N_Z X \to Z$, $s_1 : N_{(i^* N_Y X)}(N_Y X) \to i^* N_Y X$, $s_2 : N_{(N_Z Y)}(N_Z X) \to N_Z Y$ *the projections. Then we have* $(s_1)^* c^! q^* j^! = (s_2)^* a^! r^* (ji)^!$

*Proof.* Consider the deformation to the normal cone spaces $D(Y, Z)$ and $D(X, Z)$. Using the universal property of blow-ups, we get a map $D(Y, Z) \to D(X, Z)$ such that the following diagram commutes

$$\begin{tikzcd}
N_Z Y \arrow[r, "a"] \arrow[d] & N_Z X \arrow[d] \\
D(Y,Z) \arrow[r] & D(X,Z) \\
Y \times U \arrow[r, "j \times 1"'] \arrow[u] & X \times U \arrow[u]
\end{tikzcd}$$

where the top vertical maps are inclusions of the exceptional fiber in the deformation to the normal space and the bottom vertical maps are inclusions of open subsets. It is easy to check that the map $D(Y, Z) \to D(X, Z)$ is a closed immersion. Let $D(X, Y, Z)$ be the deformation to the normal cone space associated to this closed immersion. Using again the universal property of blow-ups, we see that the above diagram gives a sequence

$$D(N_Z X, N_Z Y) \longrightarrow D(X, Y, Z) \longleftarrow D(X, Y) \times U$$

where the first map is a closed immersion and the second one is an open immersion. Consider now the space $D(X, Y, Z)$. We have an open immersion $D(X, Z) \times U \to D(X, Y, Z)$ and a closed immersion (as the special fiber) $N_{D(Y,Z)} D(X, Z) \to D(X, Y, Z)$. In fact, this exceptional fiber is isomorphic to $D(N_Y X, i^* N_Y X)$ (see [Ne], paragraph 3.2). So we get a diagram

$$\begin{tikzcd}
N_Y X \arrow[r, "\kappa"] & D(X,Y) & X\times U \arrow[l, "\iota"'] \arrow[r, "\pi"] & X \\
N_Y X\times U \arrow[u, "\pi"] \arrow[d, "\iota"] \arrow[r, "\kappa"] & D(X,Y)\times U \arrow[u, "\pi"] \arrow[d, "\iota"] & X\times U\times U \arrow[l, "\iota"'] \arrow[u, "\pi"] \arrow[d, "\iota"] \arrow[r, "\pi"] & X\times U \arrow[u, "\pi"] \arrow[d, "\iota"] \\
D(N_Y X, i^* N_Y X) \arrow[r, "\kappa"] & D(X,Y,Z) & D(X,Z)\times U \arrow[l, "\iota"'] \arrow[r, "\pi"] & D(X,Z) \\
N_{(i^* N_Y X)} N_Y X \arrow[u, "\kappa"] \arrow[r, "\kappa"'] & D(N_Z X, N_Z Y) \arrow[u, "\kappa"] & N_Z X\times U \arrow[l, "\iota"] \arrow[u, "\kappa"] \arrow[r, "\pi"'] & N_Z X \arrow[u, "\kappa"]
\end{tikzcd}$$

where all the lines are deformations to the normal cone, the first and fourth columns are also deformations to the normal cone. This diagram is commutative (see [Ne], paragraph 3.2). The maps $\kappa$ denote inclusions of special fibers, $\iota$ denote the inclusions of the complement of these special fibers and $\pi$ denote the relevant projections. The map $q^*j^!$ is obtained by composing the operations (in cohomology) of the top row and $s_1^*b^!$ is obtained by working with the left column. Similarly, $r^*(ji)^!$ and $s_2^*a^!$ are deduced from the right column and the bottom row. Now all the squares appearing in this diagram are commutative and give commutative diagrams in cohomology (Proposition 3.28, Proposition 3.32, Lemma 3.34 and the naturality of the residual homomorphism $\partial$). Using this and Lemma 5.4, we get the result.

$\square$

Lemma 5.10. *Let $V, X$ and $W$ be smooth schemes. Consider the following commutative diagram*

$$\begin{tikzcd}
W \arrow[r, "i"] \arrow[dr, "p'"'] & V \arrow[d, "p"] \\
& X
\end{tikzcd}$$

*where $p, p'$ are flat and $i$ is a closed immersion. Suppose that the composition $N_W V \to W \to X$ is of the same relative dimension as $p$. Then $i^! p^* = (p')^*$.*

*Proof.* Let $D(V,W)$ be the deformation to the normal cone associated to $i$ and $b : D(V,W) \to V\times\mathbb{A}^1$ be the blow-down map. We have a commutative diagram

$$\begin{tikzcd}
W & N_VW \arrow[l, "q"'] \arrow[r, "\kappa"] & D(V,W) \arrow[r, "b"] & V\times\mathbb{A}^1 \arrow[r, "p\times Id"] & X\times\mathbb{A}^1 \\
 & & V\times U \arrow[u, "\iota"] \arrow[r, equal] & V\times U \arrow[u, "\iota"] \arrow[r, "p\times Id"'] \arrow[d, "\pi"'] & X\times U \arrow[u, "\iota"'] \arrow[d, "\pi'"] \\
 & & & V \arrow[r, "p"'] & X.
\end{tikzcd}$$

By definition, $i^!p^* = (q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}\pi^*p^*$. Using Proposition 3.28, we get $i^!p^* = (q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}(p\times Id)^*(\pi')^*$. By Lemma 5.4, this gives

$$(q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}(p\times Id)^*(\pi')^* = (q^*)^{-1}(\kappa_*)^{-1}\partial(p\times Id)^*\{t\}(\pi')^*.$$

Using Remark 10.1 in [Ro], we see that $f := (p\times Id)b$ is flat because the composition $N_WV\to W\to X$ is of the same relative dimension as $p$. We have a commutative diagram

$$\begin{tikzcd}
H^i(C(X\times\mathbb{A}^1,G^j)) \arrow[r] \arrow[d, "f^*"'] & H^i(C(X\times U,G^j)) \arrow[r, "\partial'"] \arrow[d, "(p\times Id)^*"'] & H^{i+1}_X(C(X\times\mathbb{A}^1,G^j)) \arrow[r] \arrow[d, "f^*"'] & {} \\
H^i(C(D(V,W),G^j)) \arrow[r] & H^i(C(V\times U,G^j)) \arrow[r, "\partial"'] & H^{i+1}_{N_VW}(C(D(V,W),G^j)) \arrow[r] & {}
\end{tikzcd}$$

where the first line is the localization long exact sequence associated to the triple $(X\times U, X\times\mathbb{A}^1, X\times 0)$ and the second line is the one associated to the triple $(V\times U, D(V,W), N_VW)$. Then

$$(q^*)^{-1}(\kappa_*)^{-1}\partial(p\times Id)^*\{t\}(\pi')^* = (q^*)^{-1}(\kappa_*)^{-1}f^*\partial'\{t\}(\pi')^*.$$

Consider next the fibre product

$$\begin{tikzcd}
N_VW \arrow[r, "\kappa"] \arrow[d, "p'q"'] & D(V,W) \arrow[d, "f"] \\
X \arrow[r, "i_0"'] & X\times\mathbb{A}^1
\end{tikzcd}$$

where $i_0: X\to X\times\mathbb{A}^1$ is the inclusion in 0. Using Lemma 3.34, we finally find $i^!p^* = (p')^*(i_0)_*^{-1}\partial'\{t\}(\pi')^*$. It remains to show that $(i_0)_*^{-1}\partial'\{t\}(\pi')^* = Id$ to finish the proof. At the level of Milnor $K$-theory, this is Lemma 4.5 in [Ro]. Thus we only have to prove this result at the level of Witt groups. Let $\alpha\in W^i(D^b_i(X))$ be such that $d\alpha = 0\in W^{i+1}(D^b_{i+1}(X))$. Now $\mathrm{DegLoc}((\pi')^*\alpha)\cap \mathrm{DegLoc}(< 1,-t >)$ is a closed subset of $X\times\mathbb{A}^1$ of codimension $\geq i+2$. Therefore we can use 2.25 to compute

$$(-1)^i d(<1,-t>\otimes\alpha) = d(<1,-t>)\otimes\alpha + (-1)^i <1,-t>\otimes d\alpha.$$

By assumption we have $d\alpha = 0$ in $W^{i+1}(D^b_{i+1}(X))$ and then

$$(-1)^i d(<1,-t>\otimes\alpha) = d(<1,-t>)\otimes\alpha = -dt\otimes\alpha$$

in $W^{i+1}(D^b_{i+1}(X))$. By definition of $m_t$, we find $d(m_t(\alpha)) = dt\otimes\alpha$. The latter is precisely $(i_0)_*\alpha$ (see [GH], Lemma 2.8).

□

Now we have all the tools to prove the following theorem:

Theorem 5.11. *Let* $Z \xrightarrow{i} Y \xrightarrow{j} X$ *be inclusions of smooth schemes. Then* $(ji)^! = i^! j^!$.

*Proof.* Let $q: N_Y X \to Y$, $p: N_Z Y \to Z$ and $r: N_Z X \to Z$ be the projections. Consider also the projections $s_1 : N_{(i^* N_Y X)}(N_Y X) \to i^* N_Y X$ and $s_2 : N_{(N_Z Y)}(N_Z X) \to N_Z Y$. Denote by $a: N_Z Y \to N_Z X$ and $c: i^* N_Y X \to N_Y X$ the inclusions. We also have a fibre product

$$\begin{tikzcd} i^* N_Y X \arrow[r, "c"] \arrow[d, "q'"'] & N_Y X \arrow[d, "q"] \\ Z \arrow[r, "i"'] & Y. \end{tikzcd}$$

Then

$$(s_1)^*(q')^* i^! j^! = (s_1)^* c^! q^* j^! = (s_2)^* a^! r^* (ji)^! = (s_2)^* p^* (ji)^!$$

where the first equality is due to Lemma 5.7, the second is due to Lemma 5.9 and the third to Lemma 5.10. As $(s_2)^* p^*$ induces an isomorphism in cohomology and $q' s_1 = p s_2$, we get the result. □

## 6 The ring structure

Let $X$ be a smooth scheme and let $\triangle : X \to X \times X$ be the diagonal inclusion. For any $i, j, r, s$ we have an exterior product (Theorem 4.15)

$$\times : H^i(C(X, G^r)) \times H^j(C(X, G^s)) \to H^{i+j}(C(X \times X, G^{r+s}))$$

and a Gysin-Witt map (Definition 5.5)

$$\triangle^! : H^{i+j}(C(X \times X, G^{r+s})) \to H^{i+j}(C(X, G^{r+s})).$$

Definition 6.1. We denote by $\cdot$ the composition $\triangle^! \circ \times$.

*Remark* 6.2. If $X$ is a smooth scheme and $L, N$ are invertible $\mathcal{O}_X$-modules, then using Theorem 4.16 and Remark 5.6 we see that there is a product

$$\cdot : H^i(C(X, G^i, L)) \times H^j(C(X, G^j, N)) \to H^{i+j}(C(X, G^{i+j}, L \otimes_{\mathcal{O}_X} N)).$$

*Remark* 6.3. In particular, we have for any $i, j \in \mathbb{N}$ a product

$$\cdot : H^i(C(X, G^i)) \times H^j(C(X, G^j)) \to H^{i+j}(C(X, G^{i+j}))$$

which by definition is a product $\widetilde{CH}^i(X) \times \widetilde{CH}^j(X) \to \widetilde{CH}^{i+j}(X)$.

*Remark* 6.4. It is clear from our construction that we also can define a product

$$\cdot : H^i(C(X, K_r^M)) \times H^j(C(X, K_s^M)) \to H^{i+j}(C(X, K_{r+s}^M)).$$

This product coincide with the one defined by Rost ([Ro], Chapter 14) and the natural projections $\pi : C(X, G^p) \to C(X, K_p^M)$ give a commutative diagram

$$\begin{tikzcd} H^i(C(X,G^r)) \times H^j(C(X,G^s)) \arrow[r, "\cdot"] \arrow[d, "\pi\times\pi"'] & H^{i+j}(C(X,G^{r+s})) \arrow[d, "\pi"] \\ H^i(C(X,K_r^M)) \times H^j(C(X,K_s^M)) \arrow[r, "\cdot"'] & H^{i+j}(C(X,K_{r+s}^M)). \end{tikzcd}$$

*Remark* 6.5. Our technique provides also a product on the cohomology of the Gersten-Witt complex of a scheme. That is, we have a product

$$\cdot : H^i(C(X, W)) \times H^j(C(X, W)) \to H^{i+j}(C(X, W)).$$

Now we prove the associativity of the product we have defined.

Proposition 6.6. *The product $\cdot$ is associative.*

*Proof.* First note that the exterior product is associative (Proposition 4.18). We consider the following fibre product diagram

$$\begin{tikzcd} X \arrow[r, "\triangle"] \arrow[d, "\triangle"'] & X \times X \arrow[d, "Id\times\triangle"] \\ X \times X \arrow[r, "\triangle\times Id"'] & X \times X \times X. \end{tikzcd}$$

We see that $((Id \times \triangle)\triangle)^! = ((\triangle \times Id)\triangle)^!$. Theorem 5.11 shows that we have in fact $\triangle^!(Id \times \triangle)^! = \triangle^!(\triangle \times Id)^!$. Since $(Id \times \triangle)^!$ is clearly $Id \times \triangle^!$ and $(\triangle \times Id)^! = \triangle^! \times Id$, the associativity is proved. $\square$

*Remark* 6.7. In general, the product does not satisfy any commutativity property. This is due to the fact that $\times$ and $\star$ do not commute with the flip $\tau : X \times X \to X \times X$ (see 4.19 and 4.20). Moreover, the product is not anti-commutative because the signs in 4.19 and 4.20 are not compatible. However, let $\alpha \in \widetilde{CH}^i(X)$ and $\beta \in \widetilde{CH}^j(X)$. Then $\alpha \cdot \beta$ is an element of $\widetilde{CH}^{i+j}(X)$ and is therefore represented by a sum $\sum(P_s, \psi_s) \in \mathrm{Ker}(d^{i+j})$ where

$$d^{i+j} : GW^{i+j}(D^b_{i+j}(X)) \to W^{i+j+1}(D^b_{i+j+1}(X))$$

(see Remark 3.23). Using 4.19 and 4.20, we see that $\beta \cdot \alpha = \sum(P_s, (-1)^{ij}\psi_s)$. For a more precise statement, the reader is referred to Theorem 7.6.

Now remark that there is a canonical class $1_X$ in $\widetilde{CH}^0(X)$ given by the symmetric form $< 1 >$ in $GW(k(X))$.

PROPOSITION 6.8. *The class $1_X$ is a left and right unit for the product $\cdot$.*

*Proof.* Let $p_2 : X \times X \to X$ be the second projection and consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\triangle"] \arrow[dr, "Id"'] & X \times X \arrow[d, "p_2"] \\ & X. \end{tikzcd}$$

By Lemma 5.10, we see that $\triangle^!(p_2)^* = (Id)^* = Id$. Consider now $\mu \in H^i(C(X, G^j))$. It is clear that $1_X \times \mu = (p_2)^*(\mu)$ and then $1_X \cdot \mu = \mu$. Replacing $p_2$ by $p_1$ shows that $1_X$ is also a right unit. $\square$

Hence we have:

THEOREM 6.9. *Let $X$ be a smooth scheme and let $\widetilde{CH}^*(X)$ be the total Chow-Witt group of $X$. Then the product $\cdot$ turns $\widetilde{CH}^*(X)$ into a graded associative ring with unit.*

Taking the twists into account, we get the following theorem:

THEOREM 6.10. *Let $X$ be a smooth scheme and let $\bigoplus_{L \in \mathrm{Pic}(X)/2} \widetilde{CH}^*(X, L)$ be the total twisted Chow-Witt group of $X$. Then the product $\cdot$ turns this group into a graded associative ring with unit.*

DEFINITION 6.11. Let $X$ be a smooth scheme. We call *Chow-Witt ring* the ring $\widetilde{CH}^*(X)$ and *twisted Chow-Witt ring* the ring $\bigoplus_{L \in \mathrm{Pic}(X)/2} \widetilde{CH}^*(X, L)$.

The following proposition is obvious:

Proposition 6.12. *Let $X$ be a smooth scheme. Then the natural homomorphism $\widetilde{CH}^*(X) \to CH^*(X)$ is a ring homomorphism.*

*Remark* 6.13. The same methods show that the product of Remark 6.5 gives a graded associative anticommutative ring structure on the total cohomology group $H^*(C(X,W))$ of the Gersten-Witt complex associated to $X$.

## 7 Basic properties

We first show that the Chow-Witt ring is a functorial construction.

Definition 7.1. Let $X$ and $Y$ be smooth schemes and $f : X \to Y$ a morphism. Consider the graph morphism $\gamma_f : X \to X \times Y$. We define

$$f^! : \widetilde{CH}^*(Y) \to \widetilde{CH}^*(X)$$

by $f^!(y) = \gamma_f^!(1_X \times y)$ for any $y \in \widetilde{CH}^*(Y)$.

Proposition 7.2. *The map $f^! : \widetilde{CH}^*(Y) \to \widetilde{CH}^*(X)$ is a ring homomorphism.*

*Proof.* We only have to check that $f^!(y \cdot z) = f^!(y) \cdot f^!(z)$ for any $y, z \in \widetilde{CH}^*(Y)$. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\gamma_f"] \arrow[d, "\triangle_X"'] & X \times Y \arrow[d, "\triangle_{X\times Y}"] \\ X \times X \arrow[r, "\gamma_f \times \gamma_f"'] & (X \times Y) \times (X \times Y). \end{tikzcd}$$

Theorem 5.11 shows that $\gamma_f^! \triangle_{X\times Y}^! = \triangle_X^!(\gamma_f \times \gamma_f)^!$. Applying this to the cycle $1_X \times y \times 1_X \times z$, we obtain the result. ∎

*Remark* 7.3. The proposition shows that $\widetilde{CH}^*(\_)$ is a functor from the category of smooth schemes to the category of rings. It is clear that the homomorphisms $\widetilde{CH}^*(X) \to CH^*(X)$ give a natural transformation $\widetilde{CH}^*(\_) \to CH^*(\_)$.

In the case where $f : X \to Y$ is a flat morphism, we can identify $f^!$ more precisely.

Proposition 7.4. *Let $f : X \to Y$ be a flat morphism. Then $f^! = f^*$.*

*Proof.* Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\gamma_f"] \arrow[dr, "f"'] & X \times Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

where $p : X \times Y \to Y$ is the projection. Since $N_X(X \times Y)$ is of rank equal to the dimension of $Y$, we see that the relative dimension of the composition $N_X(X\times Y) \to X \to Y$ is the same as the relative dimension of $p : X\times Y \to Y$. Therefore we can use Lemma 5.10 to get $\gamma_f^! p^* = f^*$. Since $p^*\beta = 1_X \times \beta$ for any cycle on $Y$, the result is proved. □

Let $Z \subset X$ be a closed subset of pure codimension $i$. As $D^b_Z(X) \subset D^b(X)^{(i)}$, we have a homomorphism $GW^i_Z(X) \to GW^i(D^b(X)^{(i)})$. Composing with the localization, we obtain a homomorphism $GW^i_Z(X) \to GW^i(D^b_i(X))$. As the composition $GW^i(D^b(X)^{(i)}) \to GW^i(D^b_i(X)) \to W^{i+1}(D^b(X)^{(i+1)})$ is zero (see [Ba1]), we finally obtain a homomorphism (Remark 3.23):

$$\alpha_Z : GW^i_Z(X) \to \widetilde{CH}^i(X).$$

*Remark* 7.5. Let $f : X \to Y$ be a flat morphism and $Z \subset Y$ be a closed subset of pure codimension $i$. The definitions of $f^*$ for the Grothendieck-Witt groups and the definition of $f^*$ for the Chow-Witt groups show that the following diagram commutes ([Fa], Theorem 3.2.2 and Corollary 10.4.2):

$$\begin{tikzcd} GW^i_Z(Y) \arrow[r, "\alpha_Z"] \arrow[d, "f^*"'] & \widetilde{CH}^i(Y) \arrow[d, "f^*"] \\ GW^i_{f^{-1}Z}(X) \arrow[r, "\alpha_{(f^{-1}Z)}"'] & \widetilde{CH}^i(X). \end{tikzcd}$$

The next theorem shows that our intersection product is the expected one:

Theorem 7.6. *Let $Z, T \subset X$ be closed subschemes of respective pure codimension $i$ and $j$. Suppose that $Z \cap T$ is of pure codimension $i + j$. Then the following diagram commutes*

$$\begin{tikzcd} GW^i_Z(X) \times GW^j_T(X) \arrow[r, "\star"] \arrow[d, "\alpha_Z\times\alpha_T"'] & GW^{i+j}_{Z\cap T}(X) \arrow[d, "\alpha_{Z\cap T}"] \\ \widetilde{CH}^i(X) \times \widetilde{CH}^j(X) \arrow[r, "\cdot"'] & \widetilde{CH}^{i+j}(X). \end{tikzcd}$$

*Proof.* Let $\gamma \in GW^i_Z(X)$ and $\delta \in GW^j_T(X)$. Consider the deformation to the normal cone space $D(X \times X, X)$ and the blow down map $b : D(X \times X, X) \to X \times X \times \mathbb{A}^1$. We have the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, equal] & X \arrow[d, "\triangle"'] & \\
N_X(X\times X) \arrow[u, "q"] \arrow[r, "b'"] \arrow[d, "\kappa"'] & X\times X \arrow[d, "i_0"] & \\
D(X\times X,X) \arrow[r, "b"'] & X\times X\times \mathbb{A}^1 \arrow[r, "\pi'"] & X\times X \\
X\times X\times U \arrow[u, "\iota"] \arrow[r, equal] & X\times X\times U \arrow[u, "\iota"] \arrow[ur, "\pi"'] &
\end{tikzcd} \qquad (1)$$

where $i_0$ is the inclusion in 0, $q$ is the projection and the two bottom squares are fibre products. By definition, we have

$$\alpha_Z(\gamma)\cdot\alpha_T(\delta)=(q^*)^{-1}(\kappa_*)^{-1}\partial\{t\}\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta)).$$

Let $F=b^{-1}(\pi')^{-1}(p_1^{-1}Z\cap p_2^{-1}T)$ in $D(X\times X,X)$ (where $p_1$ and $p_2$ are the projections of $X\times X$ onto $X$). Observe that $\iota^{-1}F=F\cap(X\times X\times U)$ is non empty and of pure codimension $i+j$ in $X\times X\times U$. Diagram (1) gives

$$F\cap N_X(X\times X)=\kappa^{-1}F=\kappa^{-1}b^{-1}(\pi')^{-1}(p_1^{-1}Z\cap p_2^{-1}T)=q^{-1}(Z\cap T).$$

As $Z\cap T$ is of codimension $i+j$ in $X$ and $q$ is flat, $q^{-1}(Z\cap T)$ is also of codimension $i+j$ in $N_X(X\times X)$ and hence is of codimension $i+j+1$ in $D(X\times X,X)$. Therefore $F$ itself is of pure codimension $i+j$ in $D(X\times X,X)$. By commutativity of the above diagram and Remark 7.5, we have (note that $b^*$ is defined at the level of the Grothendieck-Witt groups, but not at the level of the Chow-Witt groups):

$$\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta))=\alpha_{\iota^{-1}F}(\iota^*b^*(\pi')^*(p_1^*\gamma\otimes p_2^*\delta))=\iota^*\alpha_F(b^*(\pi')^*(p_1^*\gamma\otimes p_2^*\delta)).$$

We have to compute $(\kappa_*)^{-1}\partial\{t\}\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta))$. By definition of $\partial$, we have to consider any element $\nu\in C(D(X\times X,X),G^{i+j+1})_{i+j}$ having the property that $\iota^*\nu=\{t\}\pi^*(\alpha_Z(\gamma)\times\alpha_T(\delta))$ and then compute $d_G(\nu)$ where

$$d_G: C(D(X\times X,X),G^{i+j+1})_{i+j}\to C(D(X\times X,X),G^{i+j+1})_{i+j+1}$$

is the differential of the complex $C(D(X\times X,X),G^{i+j+1})$. Consider the commutative diagram

$$\begin{tikzcd}
D(X\times X,X) \arrow[r, "b"] & X\times X\times\mathbb{A}^1 \arrow[r, "pr"] & \mathbb{A}^1 \\
X\times X\times U \arrow[u, "\iota"] \arrow[r, equal] & X\times X\times U \arrow[u, "\iota"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

and recall that $N_X(X \times X)$ is the principal Cartier divisor in $D(X \times X, X)$ defined by $f := b^*pr^*(t)$.
Consider the form $b^*(\pi')^*(p_1^*\gamma \otimes p_2^*\delta)$. Its support is $F$. Localizing at the generic points of $F$ (which are on $X \times X \times U$), we obtain a form $\nu_0$ in $W^{i+j}(D^b_{i+j}(D(X \times X, X)))$. We also obtain an element $\nu_1$ in $\bigoplus\limits_{x \in F^{(0)}} K_0(k(x))$.
The above computation shows that $f$ is a unit in $k(x)$ for any generic point $x$ of $F$. We get an element

$$\nu := ((-1)^{i+j+1} < 1, -f > \otimes \nu_0, \{f\} \cdot \nu_1) \in C(D(X \times X, X), G^{i+j+1})_{i+j}$$

which satisfy $\iota^*\nu = \{t\}\pi^*(\alpha_Z(\gamma) \times \alpha_T(\delta))$. A straightforward computation (use Theorem 2.25 again) shows that $d_G(\nu) = df \otimes b^*(\pi')^*(p_1^*\gamma \otimes p_2^*\delta)$ in the group $GW^{i+j+1}(D^b_{i+j+1}(D(X \times X, X)))$. But $df = b^*dt$ and

$$b^*dt \otimes b^*(\pi')^*(p_1^*\gamma \otimes p_2^*\delta) = b^*(dt \otimes (\pi')^*(p_1^*\gamma \otimes p_2^*\delta))$$

([GN], Theorem 3.2). Since $dt \otimes (\pi')^*(p_1^*\gamma \otimes p_2^*\delta) = (i_0)_*(p_1^*\gamma \otimes p_2^*\delta)$ ([GH], Lemma 2.8), we finally obtain

$$(\kappa_*)^{-1}\partial\{t\}\pi^*(\alpha_Z(\gamma) \times \alpha_T(\delta)) = \alpha_{F \cap N_X(X \times X)}((b')^*(p_1^*\gamma \otimes p_2^*\delta)).$$

We have a commutative diagram

$$\begin{tikzcd} N_X(X \times X) \arrow[r, "b'"] \arrow[d, "q"'] & X \times X \\ X. \arrow[ur, "\triangle"'] & \end{tikzcd}$$

Now $\triangle^{-1}(p_1^{-1}Z \cap p_2^{-1}T) = Z \cap T$ and using the diagram, we see that

$$\alpha_Z(\gamma) \cdot \alpha_T(\beta) = \alpha_{Z \cap T}(\triangle^*(p_1^*\gamma \otimes p_2^*\delta)).$$

Hence it only remains to show that $\triangle^*(p_1^*\gamma \otimes p_2^*\delta) = \gamma \star \delta$ to finish the proof. This is clear by [GN], Theorem 3.2.

$\square$

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# Equivariant Local Cyclic Homology and the Equivariant Chern-Connes Character

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Abstract. We define and study equivariant analytic and local cyclic homology for smooth actions of totally disconnected groups on bornological algebras. Our approach contains equivariant entire cyclic cohomology in the sense of Klimek, Kondracki and Lesniewski as a special case and provides an equivariant extension of the local cyclic theory developped by Puschnigg. As a main result we construct a multiplicative Chern-Connes character for equivariant $KK$-theory with values in equivariant local cyclic homology.

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## 1. Introduction

Cyclic homology can be viewed as an analogue of de Rham cohomology in the framework of noncommutative geometry [1], [3]. In this framework geometric questions are studied by means of associative algebras which need not be commutative. An important feature of cyclic homology is the fact that the theory can easily be defined on a large class of algebras, including Fréchet algebras as well as algebras without additional structure. In many cases explicit calculations are possible using standard tools from homological algebra. The connection to de Rham theory is provided by a fundamental result due to Connes [1] showing that the periodic cyclic homology of the Fréchet algebra $C^\infty(M)$ of smooth functions on a compact manifold $M$ is equal to the de Rham cohomology of $M$.

However, in general the theory does not yield good results for Banach algebras or $C^*$-algebras. Most notably, the periodic cyclic cohomology of the algebra $C(M)$ of continuous functions on a compact manifold $M$ is different from de Rham cohomology. An intuitive explanation of this phenomenon is that $C(M)$

only encodes the information of $M$ as a topological space, whereas it is the differentiable structure that is needed to define de Rham cohomology.

Puschnigg introduced a variant of cyclic homology which behaves nicely on the category of $C^*$-algebras [33]. The resulting theory, called local cyclic homology, allows for the construction of a general Chern-Connes character for bivariant $K$-theory. Using the machinery of local cyclic homology, Puschnigg proved the Kadison-Kaplansky idempotent conjecture for hyperbolic groups [32]. Unfortunately, the construction of the local theory is quite involved. Already the objects for which the theory is defined in [33], inductive systems of nice Fréchet algebras, are rather complicated.

There is an alternative approach to local cyclic homology due to Meyer [24]. Based on the theory of bornological vector spaces, some features of local cyclic homology become more transparent in this approach. It is known that bornological vector spaces provide a very natural framework to study analytic and entire cyclic cohomology [21]. Originally, entire cyclic cohomology was introduced by Connes [2] in order to define the Chern character of $\theta$-summable Fredholm modules. The analytic theory for bornological algebras contains entire cyclic cohomology as a special case. Moreover, from a conceptual point of view it is closely related to the local theory. Roughly speaking, the passage from analytic to local cyclic homology consists in the passage to a certain derived category.

An important concept in local cyclic homology is the notion of a smooth subalgebra introduced by Puschnigg [29], [33]. The corresponding concept of an isoradial subalgebra [24], [26] plays a central role in the bornological account to the local theory by Meyer. One of the main results in [24] is that local cyclic homology is invariant under the passage to isoradial subalgebras. In fact, an inspection of the proof of this theorem already reveals the essential ideas behind the definition of the local theory. A basic example of an isoradial subalgebra is the inclusion of $C^\infty(M)$ into $C(M)$ for a compact manifold as above. In particular, the natural homomorphism $C^\infty(M) \to C(M)$ induces an invertible element in the bivariant local cyclic homology group $HL_*(C^\infty(M), C(M))$. Hence, in contrast to periodic cyclic cohomology, the local theory does not distinguish between $C(M)$ and $C^\infty(M)$. Let us also remark that invariance under isoradial subalgebras is responsible for the nice homological properties of the local theory.

In this paper we define and study analytic and local cyclic homology in the equivariant setting. This is based on the general framework for equivariant cyclic homology developped in [35] and relies on the work of Meyer in the nonequivariant case. In particular, a large part of the necessary analytical considerations is already contained in [26]. In addition some of the material from [24] will be reproduced for the convenience of the reader. On the other hand, as far as homological algebra is concerned, the framework of exact categories used by Meyer is not appropriate in the equivariant situation. This is due to the fact that equivariant cyclic homology is constructed using paracomplexes [35].

We should point out that we restrict ourselves to actions of totally disconnected groups in this paper. In fact, one meets certain technical difficulties in the construction of the local theory if one moves beyond totally disconnected groups. For simplicity we have thus avoided to consider a more general setting. Moreover, our original motivation to study equivariant local cyclic homology and the equivariant Chern-Connes character comes from totally disconnected groups anyway.

Noncommutative Chern characters constitute one of the cornerstones of noncommutative geometry. The first contributions in this direction are due to Karoubi and Connes, see [14] for an overview. In fact, the construction of the Chern character in $K$-homology was the motivation for Connes to introduce cyclic cohomology [1]. Bivariant Chern characters have been studied by several authors including Kassel, Wang, Nistor, Puschnigg and Cuntz [17], [37], [28], [33], [4]. As already explained above, our character is closely related to the work of Puschnigg.

Let us now describe how the paper is organized. In section 2 we review some facts about smooth representation of totally disconnected groups and anti-Yetter-Drinfeld modules. These concepts are basic ingredients in the construction of equivariant cyclic homology. For later reference we also discuss the notion of an essential module over an idempotented algebra. We remark that anti-Yetter-Drinfeld modules are called covariant modules in [35], [36]. The terminology used here was originally introduced in [10] in the context of Hopf algebras. In section 3 we discuss the concept of a primitive module over an idempotented algebra and exhibit the relation between inductive systems of primitive modules and arbitrary essential modules. This is needed for the definition of the local derived category given in section 4. From the point of view of homological algebra the local derived category is the main ingredient in the construction of local cyclic homology. In section 5 we recall the definition of the analytic tensor algebra and related material from [21]. Moreover we review properties of the spectral radius for bornological algebras and discuss locally multiplicative algebras [26]. Section 6 contains the definition of the equivariant $X$-complex of a $G$-algebra and the definition of equivariant analytic and local cyclic homology. This generalizes the constructions in [21], [24] as well as the definition of entire cyclic cohomology for finite groups given by Klimek, Kondracki and Lesniewski [19]. We also discuss briefly the connection to the original approach to local cyclic homology due to Puschnigg. In section 7 we prove homotopy invariance, stability and excision for equivariant analytic and local cyclic homology. The arguments for the analytic and the local theory are analogous since both theories are constructed in a similar way. In section 8 we study a special situation where analytic and local cyclic homology are in fact isomorphic. Section 9 is devoted to the proof of the isoradial subalgebra theorem. As in the non-equivariant case this theorem is the key to establish some nice features of the local theory. In particular, using the isoradial subalgebra theorem we study in section 10 how local cyclic homology behaves with respect to continuous homotopies and stability in the sense of $C^*$-algebras. As

a preparation for the definition of the Chern-Connes character in the odd case we consider in section 11 the equivariant $X$-complex of tensor products. In section 12 we recall the general approach to bivariant $K$-theories developped by Cuntz [4], [5]. Based on the resulting picture of equivariant $KK$-theory we define the equivariant Chern-Connes character in section 13. In the even case the existence of this transformation is an immediate consequence of the universal property of equivariant $KK$-theory [34], [22]. As in the non-equivariant case the equivariant Chern-Connes character is multiplicative with respect to the Kasparov product and the composition product, respectively. Finally, we describe an elementary calculation of the Chern-Connes character in the case of profinite groups. More detailed computations together with applications will be discussed in a separate paper.
Throughout the paper $G$ will be a second countable totally disconnected locally compact group. All bornological vector spaces are assumed to be separated and convex.
I am indebted to R. Meyer for providing me his preprint [24] and answering several questions related to local cyclic homology.

## 2. Smooth representations and anti-Yetter-Drinfeld modules

In this section we recall the basic theory of smooth representations of totally disconnected groups and the concept of an anti-Yetter-Drinfeld module. Smooth representations of locally compact groups on bornological vector spaces were studied by Meyer in [25]. The only difference in our discussion here is that we allow for representations on possibly incomplete spaces. Apart from smooth representations, anti-Yetter-Drinfeld modules play a central role in equivariant cyclic homology. These modules were called covariant modules in [35], [36].
Smooth representations and anti-Yetter-Drinfeld modules for totally disconnected groups can be viewed as essential modules over certain idempotented algebras in the following sense.

Definition 2.1. *An algebra $H$ with the fine bornology is called idempotented if for every small subset $S \subset H$ there exists an idempotent $e \in H$ such that $e \cdot x = x = x \cdot e$ for all $x \in S$.*

In other words, for every finite set $F$ of elements in $H$ there exists an idempotent $e \in H$ which acts like a unit on every element of $F$. We call a separated $H$-module $V$ essential if the natural map $H \otimes_H V \to V$ is a bornological isomorphism. Since $H$ carries the fine bornology, the completion $V^c$ of an essential $H$-module is again essential, and our notion is compatible with the concept of an essential module over a bornological algebra with approximate identity [25]. Clearly an idempotented algebra is a bornological algebra with approximate identity.
Let us now consider smooth representations. A representation of $G$ on a separated bornological vector space $V$ is a group homomorphism $\pi : G \to \mathrm{Aut}(V)$ where $\mathrm{Aut}(V)$ denotes the group of bounded linear automorphisms of $V$. A

bounded linear map between representations of $G$ is called equivariant if it commutes with the action of $G$. We write $\mathrm{Hom}_G(V, W)$ for the space of equivariant bounded linear maps between the representations $V$ and $W$. Let $F(G, V)$ be the space of all functions from $G$ to $V$. The adjoint of a representation $\pi$ is the bounded linear map $[\pi] : V \to F(G, V)$ given by $[\pi](v)(t) = \pi(t)(v)$. In the sequel we write simply $t \cdot v$ instead of $\pi(t)(v)$.
We write $\mathcal{D}(G)$ for the space of smooth functions on $G$ with compact support equipped with the fine bornology. Smoothness of a function $f$ on a totally disconnected group is equivalent to $f$ being locally constant. If $V$ is a bornological vector space then $\mathcal{D}(G) \otimes V = \mathcal{D}(G, V)$ is the space of compactly supported smooth functions on $G$ with values in $V$. The space $\mathcal{E}(G, V)$ consists of all smooth functions on $G$ with values in $V$.

Definition 2.2. *Let $G$ be a totally disconnected group and let $V$ be a separated (complete) bornological vector space. A representation $\pi$ of $G$ on $V$ is smooth if $[\pi]$ defines a bounded linear map from $V$ into $\mathcal{E}(G, V)$. A smooth representation is also called a separated (complete) $G$-module.*

Let $V$ be a separated $G$-module. Then for every small subset $S \subset V$ the pointwise stabilizer $G_S$ of $S$ is an open subgroup of $G$. Conversely, if $\pi$ is a representation of $G$ on a bornological vector space $V$ such that $G_S$ is open for every small subset $S \subset V$ then $\pi$ is smooth. In particular, if $V$ carries the fine bornology the above definition reduces to the usual definition of a smooth representation on a complex vector space. Every representation of a discrete group is smooth. Note that a representation $\pi$ of $G$ on a separated bornological vector space $V$ determines a representation $\pi^c$ of $G$ on the completion $V^c$. If $V$ is a separated $G$-module then $V^c$ becomes a complete $G$-module in this way.
As already mentioned in the beginning, smooth representations can be identified with essential modules over a certain idempotented algebra. The Hecke algebra of a totally disconnected group $G$ is the space $\mathcal{D}(G)$ equipped with the convolution product

$$(f * g)(t) = \int_G f(s) g(s^{-1}t) ds$$

where $ds$ denotes a fixed left Haar measure on $G$. Since $G$ is totally disconnected this algebra is idempotented. Every separated $G$-module $V$ becomes an essential $\mathcal{D}(G)$-module by integration, and conversely, every essential $\mathcal{D}(G)$-module is obtained in this way. This yields a natural isomorphism between the category of separated (complete) $G$-modules and the category of separated (complete) essential $\mathcal{D}(G)$-modules.
A separated (complete) $G$-algebra is a separated (complete) bornological algebra which is also a $G$-module such that the multiplication $A \otimes A \to A$ is equivariant. For every separated $G$-algebra $A$ the (smooth) crossed product $A \rtimes G$ is the space $\mathcal{D}(G, A)$ with the convolution multiplication

$$(f * g)(t) = \int_G f(s) s \cdot g(s^{-1}t) ds.$$

Note in particular that the crossed product associated to the trivial action of $G$ on $\mathbb{C}$ is the Hecke algebra of $G$.
In connection with actions on $C^*$-algebras we will have to consider representations of $G$ which are not smooth. For an arbitrary representation of $G$ on a bornological vector space $V$ the smoothing $\mathfrak{Smooth}_G(V)$ is defined by

$$\mathfrak{Smooth}_G(V) = \{f \in \mathcal{E}(G,V) | f(t) = t \cdot f(e) \text{ for all } t \in G\}$$

equipped with the subspace bornology and the right regular representation. We will usually simply write $\mathfrak{Smooth}$ instead of $\mathfrak{Smooth}_G$ in the sequel. The smoothing $\mathfrak{Smooth}(V)$ is always a smooth representation of $G$. If $V$ is complete, then $\mathfrak{Smooth}(V)$ is a complete $G$-module. There is an injective equivariant bounded linear map $\iota_V : \mathfrak{Smooth}(V) \to V$ given by $\iota_V(f) = f(e)$.

Proposition 2.3. *Let $G$ be a totally disconnected group and $\pi$ be a representation of $G$ on a separated bornological vector space $V$. The equivariant bounded linear map $\iota_V : \mathfrak{Smooth}(V) \to V$ induces a natural isomorphism*

$$\mathrm{Hom}_G(W,V) \cong \mathrm{Hom}_G(W, \mathfrak{Smooth}(V))$$

*for all separated $G$-modules $W$.*

Hence the smoothing functor $\mathfrak{Smooth}$ is right adjoint to the forgetful functor from the category of smooth representations to the category of arbitrary representations.
Assume that $A$ is a separated bornological algebra which is at the same time equipped with a representation of $G$ such that the multiplication $A \otimes A \to A$ is equivariant. Then $\mathfrak{Smooth}(A)$ is a separated $G$-algebra in a natural way. This applies in particular to actions on $C^*$-algebras. When $C^*$-algebras are viewed as bornological algebras we always work with the precompact bornology. If $A$ is a $G$-$C^*$-algebra we use the smoothing functor to obtain a complete $G$-algebra $\mathfrak{Smooth}(A)$. We will study properties of this construction in more detail in section 10.
Next we discuss the concept of an anti-Yetter-Drinfeld module. Let $\mathcal{O}_G$ be the commutative algebra of compactly supported smooth functions on $G$ with pointwise multiplication equipped with the action of $G$ by conjugation.

Definition 2.4. *Let $G$ be a totally disconnected group. A separated (complete) $G$-anti-Yetter-Drinfeld module is a separated (complete) bornological vector space $M$ which is both an essential $\mathcal{O}_G$-module and a $G$-module such that*

$$s \cdot (f \cdot m) = (s \cdot f) \cdot (s \cdot m)$$

*for all $s \in G, f \in \mathcal{O}_G$ and $m \in M$.*

A morphism $\phi : M \to N$ between anti-Yetter-Drinfeld modules is a bounded linear map which is $\mathcal{O}_G$-linear and equivariant. In the sequel we will use the terminology $\mathsf{AYD}$-module and $\mathsf{AYD}$-map for anti-Yetter-Drinfeld modules and their morphisms. Moreover we denote by $\mathfrak{Hom}_G(M,N)$ the space of $\mathsf{AYD}$-maps between $\mathsf{AYD}$-modules $M$ and $N$. Note that the completion $M^c$ of a separated $\mathsf{AYD}$-module $M$ is a complete $\mathsf{AYD}$-module.

We write $\mathsf{A}(G)$ for the crossed product $\mathcal{O}_G \rtimes G$. The algebra $\mathsf{A}(G)$ is idempotented and plays the same role as the Hecke algebra $\mathcal{D}(G)$ in the context of smooth representations. More precisely, there is an isomorphism of categories between the category of separated (complete) $\mathsf{AYD}$-modules and the category of separated (complete) essential modules over $\mathsf{A}(G)$. In particular, $\mathsf{A}(G)$ itself is an $\mathsf{AYD}$-module in a natural way. We may view elements of $\mathsf{A}(G)$ as smooth functions with compact support on $G \times G$ where the first variable corresponds to $\mathcal{O}_G$ and the second variable corresponds to $\mathcal{D}(G)$. The multiplication in $\mathsf{A}(G)$ becomes

$$(f \cdot g)(s,t) = \int_G f(s,r) g(r^{-1}sr, r^{-1}t) dr$$

in this picture. An important feature of this crossed product is that there exists an isomorphism $T : \mathsf{A}(G) \to \mathsf{A}(G)$ of $\mathsf{A}(G)$-bimodules given by

$$T(f)(s,t) = f(s,st)$$

for $f \in \mathsf{A}(G)$. More generally, if $M$ is an arbitrary separated $\mathsf{AYD}$-module we obtain an automorphism of $M \cong \mathsf{A}(G) \otimes_{\mathsf{A}(G)} M$ by applying $T$ to the first tensor factor. By slight abuse of language, the resulting map is again denoted by $T$. This construction is natural in the sense that $T\phi = \phi T$ for every $\mathsf{AYD}$-map $\phi : M \to N$.

## 3. Primitive modules and inductive systems

In this section we introduce primitive anti-Yetter-Drinfeld-modules and discuss the relation between inductive systems of primitive modules and general anti-Yetter-Drinfeld-modules for totally disconnected groups. This is needed for the definition of equivariant local cyclic homology.

Recall from section 2 that anti-Yetter-Drinfeld modules for a totally disconnected group $G$ can be viewed as essential modules over the idempotented algebra $\mathsf{A}(G)$. Since it creates no difficulties we shall work in the more general setting of essential modules over an arbitrary idempotented algebra $H$ in this section. We let $\mathcal{C}$ be either the category of separated or complete essential modules over $H$. Morphisms are the bounded $H$-module maps in both cases. Moreover we let $\mathrm{ind}(\mathcal{C})$ be the associated ind-category. The objects of $\mathrm{ind}(\mathcal{C})$ are inductive systems of objects in $\mathcal{C}$ and the morphisms between $M = (M_i)_{i \in I}$ and $(N_j)_{j \in J}$ are given by

$$\mathrm{Hom}_{\mathrm{ind}(\mathcal{C})}(M,N) = \varprojlim_{i \in I} \varinjlim_{j \in J} \mathrm{Hom}_{\mathcal{C}}(M_i, N_j)$$

where the limits are taken in the category of vector spaces. There is a canonical functor $\varinjlim$ from $\mathrm{ind}(\mathcal{C})$ to $\mathcal{C}$ which associates to an inductive system its separated inductive limit.

If $S$ is a small disk in a bornological vector space we write $\langle S \rangle$ for the associated normed space. There is a functor which associates to a (complete) bornological vector space $V$ the inductive system of (complete) normed spaces $\langle S \rangle$ where $S$ runs over the (completant) small disks in $V$. We need a similar construction in the context of $H$-modules. Let $M$ be a separated (complete) essential $H$-module and let $S \subset M$ be a (completant) small disk. We write $H\langle S\rangle$ for the image of the natural map $H \otimes \langle S\rangle \to M$ equipped with the quotient bornology and the induced $H$-module structure. By slight abuse of language we call this module the submodule generated by $S$ and write $H\langle S\rangle \subset M$.

Definition 3.1. *An object of $\mathcal{C}$ is called primitive if it is generated by a single small disk.*

In other words, a separated (complete) essential $H$-module $P$ is primitive iff there exists a (completant) small disk $S \subset P$ such that the natural map $H\langle S\rangle \to P$ is an isomorphism. Note that in the special case $H = \mathbb{C}$ the primitive objects are precisely the (complete) normed spaces.

Let us write $\mathrm{ind}(P(\mathcal{C}))$ for the full subcategory of $\mathrm{ind}(\mathcal{C})$ consisting of inductive systems of primitive modules. For every $M \in \mathcal{C}$ we obtain an inductive system of primitive modules over the directed set of (completant) small disks in $M$ by associating to every disk $S$ the primitive module generated by $S$. This construction yields a functor $\mathfrak{dis}$ from $\mathcal{C}$ to $\mathrm{ind}(P(\mathcal{C}))$ which will be called the dissection functor. Note that the inductive system $\mathfrak{dis}(M)$ has injective structure maps for every $M \in \mathcal{C}$. By definition, an injective inductive system is an inductive system whose structure maps are all injective. An inductive system is called essentially injective if it is isomorphic in $\mathrm{ind}(\mathcal{C})$ to an injective inductive system.

The following assertion is proved in the same way as the corresponding result for bornological vector spaces [21].

Proposition 3.2. *The direct limit functor $\varinjlim$ is left adjoint to the dissection functor $\mathfrak{dis}$. More precisely, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim (M_j)_{j\in J}, N) \cong \mathrm{Hom}_{\mathrm{ind}(P(\mathcal{C}))}((M_j)_{j\in J}, \mathfrak{dis}(N))$$

*for every inductive system $(M_j)_{j\in J}$ of primitive objects and every $N \in \mathcal{C}$. Moreover $\varinjlim \mathfrak{dis}$ is naturally equivalent to the identity and the functor $\mathfrak{dis}$ is fully faithful.*

In addition we have that $\mathfrak{dis} \varinjlim (M_i)_{i\in I}$ is isomorphic to $(M_i)_{i\in I}$ provided the system $(M_i)_{i\in I}$ is essentially injective. It follows that the dissection functor $\mathfrak{dis}$ induces an equivalence between $\mathcal{C}$ and the full subcategory of $\mathrm{ind}(P(\mathcal{C}))$ consisting of all injective inductive systems of primitive modules.

## 4. Paracomplexes and the local derived category

In this section we review the notion of a paracomplex and discuss some related constructions in homological algebra. In particular, in the setting of anti-Yetter-Drinfeld modules over a totally disconnected group, we define locally contractible paracomplexes and introduce the local derived category, following [24].

Let us begin with the definition of a para-additive category [35].

Definition 4.1. *A para-additive category is an additive category $\mathcal{C}$ together with a natural isomorphism $T$ of the identity functor* $\mathrm{id} : \mathcal{C} \to \mathcal{C}$.

It is explained in section 2 that every $\mathsf{AYD}$-module is equipped with a natural automorphism denoted by $T$. Together with these automorphisms the category of $\mathsf{AYD}$-modules becomes a para-additive category in a natural way. In fact, for our purposes this is the main example of a para-additive category.

Definition 4.2. *Let $\mathcal{C}$ be a para-additive category. A paracomplex $C = C_0 \oplus C_1$ in $\mathcal{C}$ is a given by objects $C_0$ and $C_1$ together with morphisms $\partial_0 : C_0 \to C_1$ and $\partial_1 : C_1 \to C_0$ such that*

$$\partial^2 = \mathrm{id} - T.$$

*A chain map $\phi : C \to D$ between two paracomplexes is a morphism from $C$ to $D$ that commutes with the differentials.*

The morphism $\partial$ in a paracomplex is called a differential although this contradicts the classical definition of a differential. We point out that it does not make sense to speak about the homology of a paracomplex in general.
However, one can define homotopies, mapping cones and suspensions as usual. Moreover, due to naturality of $T$, the space $\mathrm{Hom}_{\mathcal{C}}(P, Q)$ of all morphisms between paracomplexes $P$ and $Q$ with the standard differential is an ordinary chain complex. We write $\mathbf{H}(\mathcal{C})$ for the homotopy category of paracomplexes associated to a para-additive category $\mathcal{C}$. The morphisms in $\mathbf{H}(\mathcal{C})$ are homotopy classes of chain maps. The supension of paracomplexes yields a translation functor on $\mathbf{H}(\mathcal{C})$. By definition, a triangle

$$C \longrightarrow X \longrightarrow Y \longrightarrow C[1]$$

in $\mathbf{H}(\mathcal{C})$ is called distinguished if it is isomorphic to a mapping cone triangle. As for ordinary chain complexes one proves the following fact.

Proposition 4.3. *Let $\mathcal{C}$ be a para-additive category. Then the homotopy category of paracomplexes $\mathbf{H}(\mathcal{C})$ is triangulated.*

Let us now specialize to the case where $\mathcal{C}$ is the category of separated (complete) $\mathsf{AYD}$-modules. Hence in the sequel $\mathbf{H}(\mathcal{C})$ will denote the homotopy category of paracomplexes of $\mathsf{AYD}$-modules. We may also consider the homotopy category associated to the corresponding ind-category of paracomplexes. There is a direct limit functor $\varinjlim$ and a dissection functor $\mathfrak{dis}$ between these categories having the same properties as the corresponding functors for $\mathsf{AYD}$-modules.
A paracomplex $P$ of separated (complete) $\mathsf{AYD}$-modules is called primitive if its underlying $\mathsf{AYD}$-module is primitive. By slight abuse of language, if $P$ is a primitive paracomplex and $\iota : P \to C$ is an injective chain map of paracomplexes we will also write $P$ for the image $\iota(P) \subset C$ with the bornology induced from $P$. Moreover we call $P \subset C$ a primitive subparacomplex of $C$ in this case.

Definition 4.4. *A paracomplex $C$ is called locally contractible if for every primitive subparacomplex $P$ of $C$ the inclusion map $\iota : P \to C$ is homotopic to* *zero. A chain map $f : C \to D$ between paracomplexes is called a local homotopy equivalence if its mapping cone $C_f$ is locally contractible.*

The class of locally contractible paracomplexes forms a null system in $\mathbf{H}(\mathcal{C})$. We have the following characterization of locally contractible paracomplexes.

LEMMA 4.5. *A paracomplex $C$ is locally contractible iff $H_*(\mathrm{Hom}_{\mathcal{C}}(P, C)) = 0$ for every primitive paracomplex $P$.*

*Proof.* Let $P \subset C$ be a primitive subparacomplex. If $H_*(\mathrm{Hom}_{\mathcal{C}}(P, C)) = 0$ then the inclusion map $\iota : P \to C$ is homotopic to zero. It follows that $C$ is locally contractible. Conversely, assume that $C$ is locally contractible. If $P$ is a primitive paracomplex and $f : P \to C$ is a chain map let $f(P) \subset C$ be the primitive subparacomplex corresponding to the image of $f$. Since $C$ is locally contractible the inclusion map $f(P) \to C$ is homotopic to zero. Hence the same is true for $f$ and we deduce $H_0(\mathrm{Hom}_{\mathcal{C}}(P, C)) = 0$. Similarly one obtains $H_1(\mathrm{Hom}_{\mathcal{C}}(P, C)) = 0$ since suspensions of primitive paracomplexes are primitive. □

We shall next construct projective resolutions with respect to the class of locally projective paracomplexes. Let us introduce the following terminology.

DEFINITION 4.6. *A paracomplex $P$ is locally projective if $H_*(\mathrm{Hom}_{\mathcal{C}}(P, C)) = 0$ for all locally contractible paracomplexes $C$.*

All primitive paracomplexes are locally projective according to lemma 4.5. Observe moreover that the class of locally projective paracomplexes is closed under direct sums.

By definition, a locally projective resolution of $C \in \mathbf{H}(\mathcal{C})$ is a locally projective paracomplex $P$ together with a local homotopy equivalence $P \to C$. We say that a functor $P : \mathbf{H}(\mathcal{C}) \to \mathbf{H}(\mathcal{C})$ together with a natural transformation $\pi : P \to \mathrm{id}$ is a projective resolution functor if $\pi(C) : P(C) \to C$ is a locally projective resolution for all $C \in \mathbf{H}(\mathcal{C})$. In order to construct such a functor we proceed as follows.

Let $I$ be a directed set. We view $I$ as a category with objects the elements of $I$ and morphisms the relations $i \leq j$. More precisely, there is a morphism $i \to j$ from $i$ to $j$ in this category iff $i \leq j$. Now consider a functor $F : I \to \mathcal{C}$. Such a functor is also called an $I$-diagram in $\mathcal{C}$. We define a new diagram $L(F) : I \to \mathcal{C}$ as follows. Set

$$L(F)(j) = \bigoplus_{i \to j} F(i)$$

where the sum runs over all morphisms $i \to j$ in $I$. The map $L(F)(k) \to L(F)(l)$ induced by a morphism $k \to l$ sends the summand over $i \to k$ identically to the summand over $i \to l$ in $L(F)(l)$. We have a natural transformation $\pi(F) : L(F) \to F$ sending the summand $F(i)$ over $i \to j$ to $F(j)$ using the map $F(i \to j)$. The identical inclusion of the summand $F(j)$ over the identity $j \to j$ defines a section $\sigma(F)$ for $\pi(F)$. Remark that this section is not a natural transformation of $I$-diagrams in general.

Now let $H : I \to \mathcal{C}$ be another diagram and let $(\phi(i) : F(i) \to H(i))_{i \in I}$ be an arbitrary family of $\mathsf{AYD}$-maps. Then there exists a unique natural transformation of $I$-diagrams $\psi : L(F) \to H$ such that $\phi(j) = \psi(j)\sigma(F)(j)$ for all $j$. Namely, the summand $F(i)$ over $i \to j$ in $L(F)(j)$ is mapped under $\psi(j)$ to $H(j)$ by the map $H(i \to j)\phi(i)$. We can rephrase this property as follows. Consider the inclusion $I^{(0)} \subset I$ of all identity morphisms in the category $I$. There is a forgetful functor from the category of $I$-diagrams to the category of $I^{(0)}$-diagrams in $\mathcal{C}$ induced by the inclusion $I^{(0)} \to I$ and a natural isomorphism

$$\mathrm{Hom}_I(L(F), H) \cong \mathrm{Hom}_{I^{(0)}}(F, H)$$

where $\mathrm{Hom}_I$ and $\mathrm{Hom}_{I^{(0)}}$ denote the morphism sets in the categories of $I$-diagrams and $I^{(0)}$-diagrams, respectively. This means that the previous construction defines a left adjoint functor $L$ to the natural forgetful functor.
For every $j \in I$ we have a split extension of $\mathsf{AYD}$-modules

$$J(F)(j) \rightarrowtail^{\iota(F)(j)} L(F)(j) \xrightarrow{\pi(F)(j)} F(j)$$

where by definition $J(F)(j)$ is the kernel of the $\mathsf{AYD}$-map $\pi(F)(j)$ and $\iota(F)(j)$ is the inclusion. The $\mathsf{AYD}$-modules $J(F)(j)$ assemble to an $I$-diagram and we obtain an extension

$$J(F) \xrightarrow{\iota(F)} L(F) \xrightarrow{\pi(F)} F$$

of $I$-diagrams which splits as an extension of $I^{(0)}$-diagrams. We apply the functor $L$ to the diagram $J(F)$ and obtain a diagram denoted by $LJ(F)$ and a corresponding extension as before. Iterating this procedure yields a family of diagrams $LJ^n(F)$. More precisely, we obtain extensions

$$J^{n+1}(F) \xrightarrow{\iota(J^n(F))} LJ^n(F) \xrightarrow{\pi(J^n(F))} J^n(F)$$

for all $n \geq 0$ where $J^0(F) = F$, $J^1(F) = J(F)$ and $LJ^0(F) = L(F)$. In addition we set $LJ^{-1}(F) = F$ and $\iota(J^{-1}(F)) = \mathrm{id}$. By construction there are natural transformations $LJ^n(F) \to LJ^{n-1}(F)$ for all $n$ given by $\iota(J^{n-1}(F))\pi(J^n(F))$. In this way we obtain a complex

$$\cdots \to LJ^3(F) \to LJ^2(F) \to LJ^1(F) \to LJ^0(F) \to F \to 0$$

of $I$-diagrams. Moreover, this complex is split exact as a complex of $I^{(0)}$-diagrams, that is, $LJ^\bullet(F)(j)$ is a split exact complex of $\mathsf{AYD}$-modules for all $j \in I$.
Assume now that $F$ is an $I$-diagram of paracomplexes in $\mathcal{C}$. We view $F$ as a pair of $I$-diagrams $F_0$ and $F_1$ of $\mathsf{AYD}$-modules together with natural transformations $\partial_0 : F_0 \to F_1$ and $\partial_1 : F_1 \to F_0$ such that $\partial^2 = \mathrm{id} - T$. Let us construct a family of $I$-diagrams $(LJ(F), d^h, d^v)$ as follows. Using the same notation as above we set

$$LJ(F)_{pq} = LJ^q(F_p)$$

for $q \geq 0$ and define the horizontal differential $d^h_{pq} : LJ(F)_{pq} \to LJ(F)_{p-1,q}$ by

$$d^h_{pq} = (-1)^q LJ^q(\partial_p).$$

The vertical differential $d^v_{pq} : LJ(F)_{pq} \to LJ(F)_{p,q-1}$ is given by

$$d^v_{pq} = \iota(J^{q-1}(F_p))\pi(J^q(F_p)).$$

Then the relations $(d^v)^2 = 0$, $(d^h)^2 = \mathrm{id} - T$ as well as $d^v d^h + d^h d^v = 0$ hold. Hence, if we define $\mathrm{Tot}(LJ(F))$ by

$$(\mathrm{Tot}\, LJ(F))_n = \bigoplus_{p+q=n} LJ(F)_{pq}$$

and equip it with the boundary $d^h + d^v$ we obtain an $I$-diagram of paracomplexes. We write $\text{ho-}\varinjlim(F)$ for the inductive limit of the diagram $\mathrm{Tot}\, LJ(F)$ and call this paracomplex the homotopy colimit of the diagram $F$. There is a canonical chain map $\text{ho-}\varinjlim(F) \to \varinjlim(F)$ and a natural filtration on $\text{ho-}\varinjlim(F)$ given by

$$\text{ho-}\varinjlim(F)^{\leq k}_n = \bigoplus_{\substack{p+q=n \\ q \leq k}} \varinjlim LJ(F)_{pq}$$

for $k \geq 0$. Observe that the natural inclusion $\iota^k : \text{ho-}\varinjlim(F)^{\leq k} \to \text{ho-}\varinjlim(F)$ is a chain map and that there is an obvious retraction $\pi^k : \text{ho-}\varinjlim(F) \to \text{ho-}\varinjlim(F)^{\leq k}$ for $\iota^k$. However, this retraction is not a chain map.

PROPOSITION 4.7. *Let $F = (F_i)_{i\in I}$ be a directed system of paracomplexes. If the paracomplexes $F_j$ are locally projective then the homotopy colimit $\text{ho-}\varinjlim(F)$ is locally projective as well. If the system $(F_i)_{i\in I}$ is essentially injective then $\text{ho-}\varinjlim(F) \to \varinjlim(F)$ is a local homotopy equivalence.*

*Proof.* Assume first that the paracomplexes $F_j$ are locally projective. In order to prove that $\text{ho-}\varinjlim(F)$ is locally projective let $\phi : \text{ho-}\varinjlim(F) \to C$ be a chain map where $C$ is a locally contractible paracomplex. We have to show that $\phi$ is homotopic to zero. The composition of the natural map $\iota^0 : \text{ho-}\varinjlim(F)^{\leq 0} \to \text{ho-}\varinjlim(F)$ with $\phi$ yields a chain map $\psi^0 = \phi\iota^0 : \varinjlim LJ^0(F) \to C$. By construction of $LJ^0(F)$ we have isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim LJ^0(F), C) \cong \mathrm{Hom}_I(LJ^0(F), C) \cong \mathrm{Hom}_{I(0)}(F, C)$$

where we use the notation introduced above and $C$ is viewed as a constant diagram of paracomplexes. Hence, since the paracomplexes $F_j$ are locally projective, there exists a morphism $h^0$ of degree one such that $\partial h^0 + h^0\partial = \psi^0$. This yields a chain homotopy between $\psi^0$ and $0$. Using the retraction $\pi^0 : \text{ho-}\varinjlim(F) \to \text{ho-}\varinjlim(F)^{\leq 0}$ we obtain a chain map $\phi^1 = \phi - [\partial, h^0\pi^0]$ from $\text{ho-}\varinjlim(F)$ to $C$. This map is clearly homotopic to $\phi$ and by construction we have $\phi^1\iota^0 = 0$. Consider next the map $\psi^1$ given by the composition

$$\varinjlim LJ^1(F) \to \text{ho-}\varinjlim(F) \to C$$

where the first arrow is the natural one and the second map is $\phi^1$. Since $\phi^1$ vanishes on $\text{ho-}\varinjlim(F)^{\leq 0}$ we see that $\psi^1$ is a chain map. Observe moreover that $J^1(F)$ is a locally projective paracomplex. The same argument as before yields a homotopy $h^1 : \varinjlim LJ^1(F) \to C$ such that $\partial h^1 + h^1\partial = \psi^1$. We define

a chain map $\phi^2$ by $\phi^2 = \phi^1 - [\partial, h^1\pi^1] = \phi - [\partial, h^1\pi^1 + h^0\pi^0]$ and get $\phi^2\iota^1 = 0$. Continuing this process we obtain a family of $\mathsf{AYD}$-maps $h^n : \varinjlim LJ^n(F) \to C$ which assembles to a homotopy between $\phi$ and zero.
Let $C(\pi)$ be the mapping cone of the natural map $\pi$ : ho-$\varinjlim(F) \to \varinjlim(F)$. Moreover we write $C(j)$ for the mapping cone of $\operatorname{Tot} LJ(F)(j) \to F(j)$. It follows immediately from the constructions that $C(j)$ is contractible for every $j \in I$. Now let $P \subset C(\pi)$ be a primitive subparacomplex. If the system $F$ is essentially injective then there exists an index $j \in I$ such that $P \subset C(j)$. Consequently, the map $P \to C(\pi)$ is homotopic to zero in this case, and we conclude that $\pi$ is a local homotopy equivalence. □
Using the previous proposition we can construct a projective resolution functor with respect to the class of locally projective paracomplexes. More precisely, one obtains a functor $P : \mathbf{H}(\mathcal{C}) \to \mathbf{H}(\mathcal{C})$ by setting

$$P(C) = \text{ho-}\varinjlim \mathfrak{dis}(C)$$

for every paracomplex $C$. In addition, there is a natural transformation $P \to \mathrm{id}$ induced by the canonical chain map ho-$\varinjlim(F) \to \varinjlim(F)$ for every inductive system $F$. Since $\mathfrak{dis}(C)$ is an injective inductive system of locally projective paracomplexes for $C \in \mathbf{H}(\mathcal{C})$ it follows from proposition 4.7 that this yields a projective resolution functor as desired.
Let us now define the local derived category of paracomplexes.

Definition 4.8. *The local derived category* $\mathbf{D}(\mathcal{C})$ *is the localization of* $\mathbf{H}(\mathcal{C})$ *with respect to the class of locally contractible paracomplexes.*

By construction, there is a canonical functor $\mathbf{H}(\mathcal{C}) \to \mathbf{D}(\mathcal{C})$ which sends local homotopy equivalences to isomorphisms. Using the projective resolution functor $P$ one can describe the morphism sets in the derived category by

$$\operatorname{Hom}_{\mathbf{D}(\mathcal{C})}(C, D) \cong \operatorname{Hom}_{\mathbf{H}(\mathcal{C})}(P(C), D) \cong \operatorname{Hom}_{\mathbf{H}(\mathcal{C})}(P(C), P(D))$$

for all paracomplexes $C$ and $D$.
For the purposes of local cyclic homology we consider the left derived functor of the completion functor. This functor is called the derived completion and is given by

$$X^{\mathbb{L}c} = P(X)^c$$

for every paracomplex $X$ of separated $\mathsf{AYD}$-modules. Inspecting the construction of the homotopy colimit shows that $X^{\mathbb{L}c} \cong$ ho-$\varinjlim(\mathfrak{dis}(X)^c)$ where the completion of an inductive system is defined entrywise.

## 5. The analytic tensor algebra and the spectral radius

In this section we discuss the definition of the analytic tensor algebra as well as analytically nilpotent algebras and locally multiplicative algebras. The spectral radius of a small subset in a bornological algebra is defined and some of its basic properties are established. We refer to [21], [24], [26] for more details.
Let $G$ be a totally disconnected group and let $A$ be a separated $G$-algebra. We

write $\Omega^n(A)$ for the space of (uncompleted) noncommutative $n$-forms over $A$. As a bornological vector space one has $\Omega^0(A) = A$ and

$$\Omega^n(A) = A^+ \otimes A^{\otimes n}$$

for $n > 0$ where $A^+$ denotes the unitarization of $A$. Simple tensors in $\Omega^n(A)$ are usually written in the form $a_0 da_1 \cdots da_n$ where $a_0 \in A^+$ and $a_j \in A$ for $j > 0$. Clearly $\Omega^n(A)$ is a separated $G$-module with the diagonal action. We denote by $\Omega(A)$ the direct sum of the spaces $\Omega^n(A)$. The differential $d$ on $\Omega(A)$ and the multiplication of forms are defined in an obvious way such that the graded Leibniz rule holds.

For the purpose of analytic and local cyclic homology it is crucial to consider a bornology on $\Omega(A)$ which is coarser than the standard bornology for a direct sum. By definition, the analytic bornology on $\Omega(A)$ is the bornology generated by the sets

$$[S](dS)^\infty = S \cup \bigcup_{n=1}^{\infty} S(dS)^n \cup (dS)^n$$

where $S \subset A$ is small. Here and in the sequel the notation $[S]$ is used to denote the union of the subset $S \subset A$ with the unit element $1 \in A^+$. Equipped with this bornology $\Omega(A)$ is again a separated $G$-module. Moreover the differential $d$ and the multiplication of forms are bounded with respect to the analytic bornology. It follows that the Fedosov product defined by

$$\omega \circ \eta = \omega\eta - (-1)^{|\omega|} d\omega d\eta$$

for homogenous forms $\omega$ and $\eta$ is bounded as well. By definition, the analytic tensor algebra $\mathcal{T}A$ of $A$ is the even part of $\Omega(A)$ equipped with the Fedosov product and the analytic bornology. It is a separated $G$-algebra in a natural way. Unless explicitly stated otherwise, we will always equip $\Omega(A)$ and $\mathcal{T}A$ with the analytic bornology in the sequel.

The underlying abstract algebra of $\mathcal{T}A$ can be identified with the tensor algebra of $A$. This relationship between tensor algebras and differential forms is a central idea in the approach to cyclic homology developped by Cuntz and Quillen [6], [7], [8]. However, since the analytic bornology is different from the direct sum bornology, the analytic tensor algebra $\mathcal{T}A$ is no longer universal for all equivariant bounded linear maps from $A$ into separated $G$-algebras. In order to formulate its universal property correctly we need some more terminology.

The curvature of an equivariant bounded linear map $f : A \to B$ between separated $G$-algebras is the equivariant linear map $\omega_f : A \otimes A \to B$ given by

$$\omega_f(x, y) = f(xy) - f(x)f(y).$$

By definition, the map $f$ has analytically nilpotent curvature if

$$\omega_f(S, S)^\infty = \bigcup_{n=1}^{\infty} \omega_f(S, S)^n$$

is a small subset of $B$ for all small subsets $S \subset A$. An equivariant bounded linear map $f : A \to B$ with analytically nilpotent curvature is called an equivariant lanilcur. The analytic bornology is defined in such a way that the equivariant homomorphism $[[f]] : \mathcal{T}A \to B$ associated to an equivariant bounded linear map $f : A \to B$ is bounded iff $f$ is a lanilcur.

It is clear that every bounded homomorphism $f : A \to B$ is a lanilcur. In particular, the identity map of $A$ corresponds to the bounded homomorphism $\tau_A : \mathcal{T}A \to A$ given by the canonical projection onto differential forms of degree zero. The kernel of the map $\tau_A$ is denoted by $\mathcal{J}A$, and we obtain an extension

$$\mathcal{J}A \rightarrowtail \mathcal{T}A \twoheadrightarrow A$$

of separated $G$-algebras. This extension has an equivariant bounded linear splitting $\sigma_A$ given by the inclusion of $A$ as differential forms of degree zero. The algebras $\mathcal{J}A$ and $\mathcal{T}A$ have important properties that we shall discuss next.

A separated $G$-algebra $N$ is called analytically nilpotent if

$$S^\infty = \bigcup_{n\in\mathbb{N}} S^n$$

is small for all small subsets $S \subset N$. For instance, every nilpotent bornological algebra is analytically nilpotent. The ideal $\mathcal{J}A$ in the analytic tensor algebra of a bornological $A$ is an important example of an analytically nilpotent algebra.

A separated $G$-algebra $R$ is called equivariantly analytically quasifree provided the following condition is satisfied. If $K$ is an analytically nilpotent $G$-algebra and

$$K \rightarrowtail E \twoheadrightarrow Q$$

is an extension of complete $G$-algebras with equivariant bounded linear splitting then for every bounded equivariant homomorphism $f : R \to Q$ there exists a bounded equivariant lifting homomorphism $F : R \to E$. The analytic tensor algebra $\mathcal{T}A$ of a $G$-algebra $A$ is a basic example of an equivariantly analytically quasifree $G$-algebra. Another fundamental example is given by the algebra $\mathbb{C}$ with the trivial action. Every equivariantly analytically quasifree $G$-algebra is in particular equivariantly quasifree in the sense of [35].

We shall next discuss the concept of a locally multiplicative $G$-algebra. If $A$ is a bornological algebra then a disk $T \subset A$ is called multiplicatively closed provided $T \cdot T \subset T$. A separated bornological algebra $A$ is called locally multiplicative if for every small subset $S \subset A$ there exists a positive real number $\lambda$ and a small multiplicatively closed disk $T \subset A$ such that $S \subset \lambda T$. It is easy to show that a separated (complete) bornological algebra is locally multiplicative iff it is a direct limit of (complete) normed algebras. We point out that the group action on a $G$-algebra usually does not leave multiplicatively closed disks invariant. In particular, a locally multiplicative $G$-algebra can not be written as a direct limit of normed $G$-algebras in general.

It is clear from the definitions that analytically nilpotent algebras are locally multiplicative. In fact, locally multiplicatively algebras and analytically nilpotent algebras can be characterized in a concise way using the notion of spectral radius.

Definition 5.1. *Let $A$ be a separated bornological algebra and let $S \subset A$ be a small subset. The spectral radius $\rho(S) = \rho(S; A)$ is the infimum of all positive real numbers $r$ such that*

$$(r^{-1}S)^\infty = \bigcup_{n=1}^{\infty} (r^{-1}S)^n$$

*is small. If no such number $r$ exists set $\rho(S) = \infty$.*

A bornological algebra $A$ is locally multiplicative iff $\rho(S) < \infty$ for all small subsets $S \subset A$. Similarly, a bornological algebra is analytically nilpotent iff $\rho(S) = 0$ for all small subsets $S$ of $A$.

Let us collect some elementary properties of the spectral radius. If $\lambda$ is a positive real number then $\rho(\lambda S) = \lambda\rho(S)$ for every small subset $S$. Moreover one has $\rho(S^n) = \rho(S)^n$ for all $n > 0$. Remark also that the spectral radius does not distinguish between a small set and its disked hull. Finally, let $f : A \to B$ be a bounded homomorphism and let $S \subset A$ be small. Then the spectral radius is contractive in the sense that

$$\rho(f(S); B) \leq \rho(S; A)$$

since $f((r^{-1}S)^\infty) = (r^{-1}f(S))^\infty \subset B$ is small provided $(r^{-1}S)^\infty$ is small.

## 6. Equivariant analytic and local cyclic homology

In this section we recall the definition of equivariant differential forms and the equivariant $X$-complex and define equivariant analytic and local cyclic homology. In addition we discuss the relation to equivariant entire cyclic homology for finite groups in the sense of Klimek, Kondracki and Lesniewski and the original definition of local cyclic homology due to Puschnigg.

First we review basic properties of equivariant differential forms. The equivariant $n$-forms over a separated $G$-algebra $A$ are defined by $\Omega^n_G(A) = \mathcal{O}_G \otimes \Omega^n(A)$ where $\Omega^n(A)$ is the space of uncompleted differential $n$-forms over $A$. The group $G$ acts diagonally on $\Omega^n_G(A)$ and we have an obvious $\mathcal{O}_G$-module structure given by multiplication on the first tensor factor. In this way the space $\Omega^n_G(A)$ becomes a separated $\mathsf{AYD}$-module.

On equivariant differential forms we consider the following operators. We have the differential $d : \Omega^n_G(A) \to \Omega^{n+1}_G(A)$ given by

$$d(f(s) \otimes x_0dx_1 \cdots dx_n) = f(s) \otimes dx_0dx_1 \cdots dx_n$$

and the equivariant Hochschild boundary $b : \Omega^n_G(A) \to \Omega^{n-1}_G(A)$ defined by

$$\begin{aligned} b(f(s) \otimes x_0dx_1 \cdots dx_n) &= f(s) \otimes x_0x_1dx_2 \cdots dx_n \\ &+ \sum_{j=1}^{n-1} (-1)^j f(s) \otimes x_0dx_1 \cdots d(x_jx_{j+1}) \cdots dx_n \\ &+ (-1)^n f(s) \otimes (s^{-1} \cdot x_n)x_0dx_1 \cdots dx_{n-1}. \end{aligned}$$

Moreover there is the equivariant Karoubi operator $\kappa : \Omega^n_G(A) \to \Omega^n_G(A)$ and the equivariant Connes operator $B : \Omega^n_G(A) \to \Omega^{n+1}_G(A)$ which are given by the formulas

$$\kappa(f(s) \otimes x_0 dx_1 \cdots dx_n) = (-1)^{n-1} f(s) \otimes (s^{-1} \cdot dx_n) x_0 dx_1 \cdots dx_{n-1}$$

and

$$B(f(s) \otimes x_0 dx_1 \cdots dx_n) = \sum_{i=0}^{n} (-1)^{ni} f(s) \otimes s^{-1} \cdot (dx_{n+1-i} \cdots dx_n) dx_0 \cdots dx_{n-i},$$

respectively. All these operators are $\mathsf{AYD}$-maps, and the natural symmetry operator $T$ for $\mathsf{AYD}$-modules is of the form

$$T(f(s) \otimes \omega) = f(s) \otimes s^{-1} \cdot \omega$$

on equivariant differential forms. We shall write $\Omega_G(A)$ for the direct sum of the spaces $\Omega^n_G(A)$ in the sequel. The analytic bornology on $\Omega_G(A)$ is defined using the identification $\Omega_G(A) = \mathcal{O}_G \otimes \Omega(A)$.
Together with the operators $b$ and $B$ the space $\Omega_G(A)$ of equivariant differential forms may be viewed as a paramixed complex [35] which means that the relations $b^2 = 0$, $B^2 = 0$ and $[b, B] = bB + Bb = \mathrm{id} - T$ hold. An important purpose for which equivariant differential forms are needed is the definition of the equivariant $X$-complex of a $G$-algebra.

Definition 6.1. *Let $A$ be a separated $G$-algebra. The equivariant $X$-complex $X_G(A)$ of $A$ is the paracomplex*

$$X_G(A)\colon\ \Omega^0_G(A) \underset{b}{\overset{d}{\rightleftarrows}} \Omega^1_G(A)/b(\Omega^2_G(A)).$$

Remark in particular that if $\partial$ denotes the boundary operator in $X_G(A)$ then the relation $\partial^2 = \mathrm{id} - T$ follows from the fact that equivariant differential forms are a paramixed complex.
After these preparations we come to the definition of equivariant analytic cyclic homology.

Definition 6.2. *Let $G$ be a totally disconnected group and let $A$ and $B$ be separated $G$-algebras. The bivariant equivariant analytic cyclic homology of $A$ and $B$ is*

$$HA^G_*(A, B) = H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}(A \otimes \mathcal{K}_G))^c, X_G(\mathcal{T}(B \otimes \mathcal{K}_G))^c)).$$

The algebra $\mathcal{K}_G$ occuring in this definition is the subalgebra of the algebra of compact operators $\mathbb{K}(L^2(G))$ on the Hilbert space $L^2(G)$ obtained as the linear span of all rank-one operators $|\xi\rangle\langle\eta|$ with $\xi, \eta \in \mathcal{D}(G)$. This algebra is equipped with the fine bornology and the action induced from $\mathbb{K}(L^2(G))$. An important property of the $G$-algebra $\mathcal{K}_G$ is that it is projective as a $G$-module.
We point out that the Hom-complex on the right hand side of the definition, equipped with the usual boundary operator, is an ordinary chain complex although both entries are only paracomplexes. Remark also that for the trivial

group one reobtains the definition of analytic cyclic homology given in [21].
It is frequently convenient to replace the paracomplex $X_G(\mathcal{T}(A \otimes \mathcal{K}_G))$ in the definition of the analytic theory with another paracomplex construced using the standard boundary $B + b$ in cyclic homology. For every separated $G$-algebra $A$ there is a natural isomorphism $X_G(\mathcal{T}A) \cong \Omega_G^{\tan}(A)$ of $\mathsf{AYD}$-modules where $\Omega_G^{\tan}(A)$ is the space $\Omega_G(A)$ equipped with the transposed analytic bornology. The transposed analytic bornology is the bornology generated by the sets

$$D \otimes S \cup D \otimes [S]dS \cup \bigcup_{n=1}^{\infty} n!\, D \otimes [S][dS](dS)^{2n}$$

where $D \subset \mathcal{O}_G$ and $S \subset A$ are small. The operators $b$ and $B$ are bounded with respect to the transposed analytic bornology. It follows that $\Omega_G^{\tan}(A)$ becomes a paracomplex with the differential $B + b$. We remark that rescaling with the constants $n!$ in degree $2n$ and $2n + 1$ yields an isomorphism between $\Omega_G^{\tan}(A)$ and the space $\Omega_G(A)$ equipped with the analytic bornology.

Theorem 6.3. *Let $G$ be a totally disconnected group. For every separated $G$-algebra $A$ there exists a bornological homotopy equivalence between the paracomplexes $X_G(\mathcal{T}A)$ and $\Omega_G^{\tan}(A)$.*

*Proof.* The proof follows the one for the equivariant periodic theory [35] and the corresponding assertion in the nonequivariant situation [21]. Let $Q_n : \Omega_G(A) \to \Omega_G^n(A) \subset \Omega_G(A)$ be the canonical projection. Using the explicit formula for the Karoubi operator one checks that the set $\{C^n \kappa^j Q_n |\, 0 \leq j \leq n, n \geq 0\}$ of operators is equibounded on $\Omega_G(A)$ with respect to the analytic bornology for every $C \in \mathbb{R}$. Similarly, the set $\{\kappa^n Q_n | n \geq 0\}$ is equibounded with respect to the analytic bornology and hence $\{C^n \kappa^j Q_n |\, 0 \leq j \leq kn, n \geq 0\}$ is equibounded as well for each $k \in \mathbb{N}$. Thus an operator on $\Omega_G(A)$ of the form $\sum_{j=0}^{\infty} Q_n h_n(\kappa)$ is bounded with respect to the analytic bornology if $(h_n)_{n\in\mathbb{N}}$ is a sequence of polynomials whose degrees grow at most linearly and whose absolut coefficient sums grow at most exponentially. By definition, the absolute coefficient sum of $\sum_{j=0}^{k} a_j x^j$ is $\sum_{j=0}^{k} |a_j|$. The polynomials $f_n$ and $g_n$ occuring in the proof of theorem 8.6 in [35] satisfy these conditions. Based on this observation, a direct inspection shows that the maps involved in the definition of the desired homotopy equivalence in the periodic case induce bounded maps on $\Omega_G(A)$ with respect to the analytic bornology. This yields the assertion. □

Let $G$ be a finite group and let $A$ be a unital Banach algebra on which $G$ acts by bounded automorphisms. Klimek, Kondracki and Lesniewski defined the equivariant entire cyclic cohomology of $A$ in this situation [19]. We may also view $A$ as a bornological algebra with the bounded bornology and consider the equivariant analytic theory of the resulting $G$-algebra.

Proposition 6.4. *Let $G$ be a finite group acting on a unital Banach algebra $A$ by bounded automorphisms. Then the equivariant entire cyclic cohomology of $A$ coincides with the equivariant analytic cyclic cohomology $HA_*^G(A, \mathbb{C})$ where $A$ is viewed as a $G$-algebra with the bounded bornology.*

*Proof.* It will be shown in proposition 7.5 below that tensoring with the algebra $\mathcal{K}_G$ is not needed in the definition of $HA^G_*$ for finite groups. Let us write $C(G)$ for the space of functions on the finite group $G$. Using theorem 6.3 we see that the analytic cyclic cohomology $HA^G_*(A,\mathbb{C})$ is computed by the complex consisting of families $(\phi_n)_{n\geq 0}$ of $n+1$-linear maps $\phi_n : A^+ \times A^n \to C(G)$ which are equivariant in the sense that

$$\phi_n(t\cdot a_0, t\cdot a_1,\dots,t\cdot a_n)(s) = \phi_n(a_0,a_1,\dots,a_n)(t^{-1}st)$$

and satisfy the entire growth condition

$$[n/2]!\ \max_{t\in G} |\phi_n(a_0,a_1,\dots,a_n)(t)| \leq c_S$$

for $a_0 \in [S], a_1,\dots,a_n \in S$ and all small sets $S$ in $A$. Here $[n/2]=k$ for $n=2k$ or $n=2k+1$ and $c_S$ is a constant depending on $S$. The boundary operator is induced by $B+b$. An argument analogous to the one due to Khalkhali in the non-equivariant case [18] shows that this complex is homotopy equivalent to the complex used by Klimek, Kondracki and Lesniewski. □

Definition 6.5. *Let $G$ be a totally disconnected group and let $A$ and $B$ be separated $G$-algebras. The bivariant equivariant local cyclic homology $HL^G_*(A,B)$ of $A$ and $B$ is given by*

$$H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}(A\otimes\mathcal{K}_G))^{\mathbb{L}c}, X_G(\mathcal{T}(B\otimes\mathcal{K}_G))^{\mathbb{L}c})).$$

Recall that the derived completion $X^{\mathbb{L}c}$ of a paracomplex $X$ was introduced in section 4. In terms of the local derived category of paracomplexes definition 6.5 can be reformulated in the following way. The construction of the derived completion shows together with proposition 4.7 that the paracomplex $X_G(\mathcal{T}(A\otimes\mathcal{K}_G))^{\mathbb{L}c}$ is locally projective for every separated $G$-algebra $A$. It follows that the local cyclic homology group $HL^G_0(A,B)$ is equal to the space of morphisms in the local derived category between $X_G(\mathcal{T}(A\otimes\mathcal{K}_G))^{\mathbb{L}c}$ and $X_G(\mathcal{T}(B\otimes\mathcal{K}_G))^{\mathbb{L}c}$. Consequently, the passage from the analytic theory to the local theory consists in passing from the homotopy category of paracomplexes to the local derived category and replacing the completion functor by the derived completion.

Both equivariant analytic and local cyclic homology are equipped with an obvious composition product. Every bounded equivariant homomorphism $f : A \to B$ induces an element $[f]$ in $HA^G_*(A,B)$ and in $HL^G_*(A,B)$, respectively. In particular, the identity map $\mathrm{id} : A \to A$ defines an element in these theories which acts as a unit with respect to the composition product.

If $G$ is the trivial group then definition 6.5 reduces to the local cyclic theory defined by Meyer in [24]. Let us briefly explain how this definition of local cyclic homology is related to the original approach by Puschnigg. In [33] a Fréchet algebra $A$ is called nice if there is a neighborhood of the origin $U$ such that $S^\infty$ is precompact for all compact sets $S \subset U$. This condition is equivalent local multiplicativity if $A$ is viewed as a bornological algebra with the precompact bornology [21]. Hence the class of nice Fréchet algebras can be viewed as a particular class of locally multiplicative bornological algebras. One has the following result [24].

Proposition 6.6. *Let $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ be inductive systems of nice Fréchet algebras and let $A$ and $B$ denote their direct limits, respectively. If the systems $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ have injective structure maps then $HL_*(A,B)$ is naturally isomorphic to the bivariant local cyclic homology for $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ as defined by Puschnigg.*

*Proof.* According to the assumptions the inductive system $\mathfrak{dis}(X(\mathcal{T}A))$ is isomorphic to the formal inductive limit of $\mathfrak{dis}(X(\mathcal{T}A_i))_{i\in I}$ in the category of inductive systems of complexes. The completion of the latter is equivalent to the inductive system that is used in [33] to define the local theory. Comparing the construction of the local derived category with the definition of the derived ind-category given by Puschnigg yields the assertion. $\square$

Consequently, the main difference between the approaches is that Meyer works explicitly in the setting of bornological vector spaces whereas Puschnigg uses inductive systems and considers bornologies only implicitly.

## 7. Homotopy invariance, stability and excision

In this section we show that equivariant analytic and local cyclic homology are invariant under smooth equivariant homotopies, stable and satisfy excision in both variables.

For the proof of homotopy invariance and stability of the local theory we need some information about partial completions. A subset $\mathcal{V}$ of a bornological vector space $V$ is called locally dense if for any small subset $S \subset V$ there is a small disk $T \subset V$ such that any $v \in S$ is the limit of a $T$-convergent sequence with entries in $\mathcal{V} \cap T$. If $V$ is a metrizable locally convex vector space endowed with the precompact bornology then a subset $\mathcal{V} \subset V$ is locally dense iff it is dense in $V$ in the topological sense [26]. Let $\mathcal{V}$ be a bornological vector space and let $i : \mathcal{V} \to V$ be a bounded linear map into a separated bornological vector space $V$. Then $V$ together with the map $i$ is called a partial completion of $\mathcal{V}$ if $i$ is a bornological embedding and has locally dense range.

We will need the following property of partial completions.

Lemma 7.1. *Let $i : \mathcal{A} \to A$ be a partial completion of separated $G$-algebras. Then the induced chain map $X_G(\mathcal{T}\mathcal{A})^{\mathbb{L}c} \to X_G(\mathcal{T}A)^{\mathbb{L}c}$ is an isomorphism. If the derived completion is replaced by the ordinary completion the corresponding chain map is an isomorphism as well.*

*Proof.* Let us abbreviate $C = X_G(\mathcal{T}\mathcal{A})$ and $D = X_G(\mathcal{T}A)$. It suffices to show that the natural map $\mathfrak{dis}(C)^c \to \mathfrak{dis}(D)^c$ is an isomorphism of inductive systems. Since $i : \mathcal{A} \to A$ is a partial completion the same holds true for the induced chain map $C \to D$. By local density, for any small disk $S \subset D$ there exists a small disk $T \subset D$ such that any point in $S$ is the limit of a $T$-convergent sequence with entries in $C \cap T$. Observe that $C \cap T$ is a small disk in $C$ since the inclusion is a bornological embedding. Consider the isometry

$\langle C \cap T\rangle \to \langle T\rangle$. By construction, the space $\langle S\rangle$ is contained in the range of the isometry $\langle C \cap T\rangle^c \to \langle T\rangle^c$ obtained by applying the completion functor. Since $\langle C\cap T\rangle^c$ maps naturally into $(\mathsf{A}(G)\langle C\cap T\rangle)^c$ we get an induced $\mathsf{AYD}$-map $\mathsf{A}(G)\langle S\rangle \to (\mathsf{A}(G)\langle C \cap T\rangle)^c$. Using this observation one checks easily that the completions of the inductive systems $\mathfrak{dis}(C)$ and $\mathfrak{dis}(D)$ are isomorphic. $\square$

We refer to [26] for the definition of smooth functions with values in a bornological vector space. For metrizable locally convex vector spaces with the precompact bornology one reobtains the usual notion. Let $B$ be a separated $G$-algebra and denote by $C^\infty([0,1],B)$ the $G$-algebra of smooth functions on the interval $[0,1]$ with values in $B$. The group $G$ acts pointwise on functions, and if $B$ is complete there is a natural isomorphism $C^\infty([0,1],B) \cong C^\infty[0,1]\hat{\otimes}B$. A smooth equivariant homotopy is a bounded equivariant homomorphism $\Phi : A \to C^\infty([0,1],B)$. Evaluation at $t \in [0,1]$ yields an equivariant homomorphism $\Phi_t : A \to B$. Two equivariant homomorphisms from $A$ to $B$ are called equivariantly homotopic if they can be connected by an equivariant homotopy.

Proposition 7.2 (Homotopy invariance). *Let $A$ and $B$ be separated $G$-algebras and let $\Phi : A \to C^\infty([0,1],B)$ be a smooth equivariant homotopy. Then the induced elements $[\Phi_0]$ and $[\Phi_1]$ in $HL^G_*(A,B)$ are equal. An analogous statement holds for the analytic theory. Hence $HA^G_*$ and $HL^G_*$ are homotopy invariant in both variables with respect to smooth equivariant homotopies.*

*Proof.* For notational simplicity we shall suppress occurences of the algebra $\mathcal{K}_G$ in our notation. Assume first that the homotopy $\Phi$ is a map from $A$ into $\mathbb{C}[t]\otimes B$ where $\mathbb{C}[t]$ is viewed as a subalgebra of $C^\infty[0,1]$ with the subspace bornology. The map $\Phi$ induces a bounded equivariant homomorphism $\mathcal{T}A \to \mathbb{C}[t] \otimes \mathcal{T}B$ since the algebra $C^\infty[0,1]$ is locally multiplicative. As in the proof of homotopy invariance for equivariant periodic cyclic homology [35] we see that the chain maps $X_G(\mathcal{T}A) \to X_G(\mathcal{T}B)$ induced by $\Phi_0$ and $\Phi_1$ are homotopic. Consider in particular the equivariant homotopy $\Phi : \mathbb{C}[x] \otimes B \to \mathbb{C}[t] \otimes \mathbb{C}[x] \otimes B$ defined by $\Phi(p(x)\otimes b) = p(tx)\otimes b$. We deduce that the map $B \to C[x]\otimes B$ that sends $b$ to $b\otimes 1$ induces a homotopy equivalence between $X_G(\mathcal{T}(\mathbb{C}[x]\otimes B))$ and $X_G(\mathcal{T}B)$. It follows in particular that the chain maps $X_G(\mathcal{T}(\mathbb{C}[x]\otimes B)) \to X_G(\mathcal{T}B)$ given by evaluation at 0 and 1, respectively, are homotopic.

Let us show that $\mathbb{C}[t] \otimes B \to C^\infty([0,1],B)$ is a partial completion. It suffices to consider the corresponding map for a normed subspace $V \subset B$ since source and target of this map are direct limits of the associated inductive systems with injective structure maps. For a normed space $V$ the assertion follows from Grothendieck's description of bounded subsets of the projective tensor product $C^\infty[0,1]\hat{\otimes}_\pi V$.

Due to lemma 7.1 the chain map $X_G(\mathcal{T}(\mathbb{C}[t] \otimes B))^{\mathbb{L}c} \to X_G(\mathcal{T}C^\infty([0,1],B))^{\mathbb{L}c}$ is an isomorphism. Hence the chain maps $X_G(\mathcal{T}C^\infty([0,1],B))^{\mathbb{L}c} \to X_G(\mathcal{T}B)^{\mathbb{L}c}$ induced by evalution at 0 and 1 are homotopic as well. Now let $\Phi : A \to C^\infty([0,1],B)$ be an arbitrary homotopy. According to our previous argument, composing the induced chain map $X_G(\mathcal{T}A)^{\mathbb{L}c} \to X_G(\mathcal{T}C^\infty([0,1],B))^{\mathbb{L}c}$ with the evaluation maps at 0 and 1 yields the claim for the local theory. The assertion for the analytic theory are obtained in the same way. □

Next we study stability. Let $V$ and $W$ be separated $G$-modules and let $b : W \times V \to \mathbb{C}$ be an equivariant bounded bilinear map. Then $l(b) = V \otimes W$ is a separated $G$-algebra with multiplication

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = x_1 \otimes b(y_1, x_2) y_2$$

and the diagonal $G$-action. A particular example is the algebra $\mathcal{K}_G$ which is obtained using the left regular representation $V = W = \mathcal{D}(G)$ and the pairing

$$b(f, g) = \int_G f(s) g(s) ds$$

with respect to left Haar measure.

Let $V$ and $W$ be separated $G$-modules and let $b : W \times V$ be an equivariant bounded bilinear map. The pairing $b$ is called admissible if there exists nonzero $G$-invariant vectors $v \in V$ and $w \in W$ such that $b(w, v) = 1$. In this case $p = v \otimes w$ is an invariant idempotent element in $l(b)$ and there is an equivariant homomorphism $\iota_A : A \to A \otimes l(b)$ given by $\iota_A(a) = a \otimes p$.

Proposition 7.3. *Let $A$ be a separated $G$-algebra and let $b : W \times V \to \mathbb{C}$ be an admissible pairing. Then the map $\iota_A$ induces a homotopy equivalence $X_G(\mathcal{T}A)^{\mathbb{L}c} \simeq X_G(\mathcal{T}(A \otimes l(b)))^{\mathbb{L}c}$. If the derived completion is replaced by the ordinary completion the corresponding map is a homotopy equivalence as well.*

This result is proved in the same way as in [35] using homotopy invariance. As a consequence we obtain the following stability properties of equivariant analytic and local cyclic homology.

Proposition 7.4 (Stability). *Let $A$ be a separated $G$-algebra and let $b : W \times V \to \mathbb{C}$ be a nonzero equivariant bounded bilinear map. Moreover let $l(b, A)$ be any partial completion of $A \otimes l(b)$. Then there exist invertible elements in $HL^G_0(A, l(b, A))$ and $HA^G_0(A, l(b, A))$.*

*Proof.* For the uncompleted stabilization $A \otimes l(b)$ the argument for the periodic theory in [35] carries over. If $l(b, A)$ is a partial completion of $A \otimes l(b)$ the natural chain map $X_G(\mathcal{T}(A \otimes l(b) \otimes \mathcal{K}_G)) \to X_G(\mathcal{T}(l(b, A) \otimes \mathcal{K}_G))$ becomes an isomorphism after applying the (left derived) completion functor according to lemma 7.1. □

An application of theorem 7.3 yields a simpler description of $HA^G_*$ and $HL^G_*$ in the case that $G$ is a profinite group. If $G$ is compact the trivial one-dimensional representation is contained in $\mathcal{D}(G)$. Hence the pairing used to define the algebra $\mathcal{K}_G$ is admissible in this case. This implies immediately the following assertion.

Proposition 7.5. *Let $G$ be a compact group. Then we have a natural isomorphism*

$$HL^G_*(A, B) \cong H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}A)^{\mathbb{L}c}, X_G(\mathcal{T}B)^{\mathbb{L}c}))$$

*for all separated $G$-algebras $A$ and $B$. An analogous statement holds for the analytic theory.*

To conclude this section we show that equivariant analytic and local cyclic homology satisfy excision in both variables.

Theorem 7.6 (Excision). *Let $A$ be a separated $G$-algebra and let $\mathcal{E} : 0 \to K \to E \to Q \to 0$ be an extension of separated $G$-algebras with bounded linear splitting. Then there are two natural exact sequences*

$$\begin{tikzcd} HL_0^G(A,K) \arrow[r] & HL_0^G(A,E) \arrow[r] & HL_0^G(A,Q) \arrow[d] \\ HL_1^G(A,Q) \arrow[u] & HL_1^G(A,E) \arrow[l] & HL_1^G(A,K) \arrow[l] \end{tikzcd}$$

*and*

$$\begin{tikzcd} HL_0^G(Q,A) \arrow[r] & HL_0^G(E,A) \arrow[r] & HL_0^G(K,A) \arrow[d] \\ HL_1^G(K,A) \arrow[u] & HL_1^G(E,A) \arrow[l] & HL_1^G(Q,A) \arrow[l] \end{tikzcd}$$

*The horizontal maps in these diagrams are induced by the maps in $\mathcal{E}$ and the vertical maps are, up to a sign, given by composition product with an element* $\mathrm{ch}(\mathcal{E})$ *in $HL_1^G(Q,K)$ naturally associated to the extension. Analogous statements hold for the analytic theory.*

Upon tensoring the given extension $\mathcal{E}$ with $\mathcal{K}_G$ we obtain an extension of separated $G$-algebras with equivariant bounded linear splitting. As in [35] we may suppress the algebra $\mathcal{K}_G$ from our notation and assume that we are given an extension

$$K \rightarrowtail^{\iota} E \xrightarrow{\pi} Q$$

of separated $G$-algebras together with an equivariant bounded linear splitting $\sigma : Q \to E$ for the quotient map $\pi : E \to Q$.

We denote by $X_G(\mathcal{T}E : \mathcal{T}Q)$ the kernel of the map $X_G(\mathcal{T}\pi) : X_G(\mathcal{T}E) \to X_G(\mathcal{T}Q))$ induced by $\pi$. The splitting $\sigma$ yields a direct sum decomposition $X_G(\mathcal{T}E) = X_G(\mathcal{T}E : \mathcal{T}Q) \oplus X_G(\mathcal{T}Q)$ of $\mathsf{AYD}$-modules. Moreover there is a natural chain map $\rho : X_G(\mathcal{T}K) \to X_G(\mathcal{T}E : \mathcal{T}Q)$.

Theorem 7.6 is a consequence of the following result.

Theorem 7.7. *The map $\rho : X_G(\mathcal{T}K) \to X_G(\mathcal{T}E : \mathcal{T}Q)$ is a homotopy equivalence.*

*Proof.* The proof follows the arguments given in [21], [35]. Let $\mathfrak{L} \subset \mathcal{T}E$ be the left ideal generated by $K \subset \mathcal{T}E$. Then $\mathfrak{L}$ is a separated $G$-algebra and we obtain an extension

$$N \rightarrowtail \mathfrak{L} \xrightarrow{\tau} K$$

of separated $G$-algebras where $\tau : \mathfrak{L} \to K$ is induced by the canonical projection $\tau_E : \mathcal{T}E \to E$. As in [35] one shows that the inclusion $\mathfrak{L} \subset \mathcal{T}E$ induces a homotopy equivalence $\psi : X_G(\mathfrak{L}) \to X_G(\mathcal{T}E : \mathcal{T}Q)$. The inclusion $\mathcal{T}K \to \mathfrak{L}$ induces a morphism of extensions from $0 \to \mathcal{J}K \to \mathcal{T}K \to K \to 0$ to $0 \to$

$N \to \mathfrak{L} \to K \to 0$. The algebra $N$ is analytically nilpotent and the splitting homomorphism $v : \mathfrak{L} \to \mathcal{T}\mathfrak{L}$ for the canonical projection constructed by Meyer in [21] is easily seen to be equivariant. Using homotopy invariance it follows that the induced chain map $X_G(\mathcal{T}K) \to X_G(\mathfrak{L})$ is a homotopy equivalence. This yields the assertion. □

## 8. Comparison between analytic and local cyclic homology

In this section we study the relation between equivariant analytic and local cyclic homology. We exhibit a special case in which the analytic and local theories agree. This allows to do some elementary calculations in equivariant local cyclic homology. Our discussion follows closely the treatment by Meyer, for the convenience of the reader we reproduce some results in [24].
A bornological vector space $V$ is called subcomplete if the canonical map $V \to V^c$ is a bornological embedding with locally dense range.

Proposition 8.1. *Let $V$ be a separated bornological vector space. The following conditions are equivalent:*
*a) $V$ is subcomplete.*
*b) for every small disk $S \subset V$ there is a small disk $T \subset V$ containing $S$ such that every $S$-Cauchy sequence that converges in $V$ is already $T$-convergent.*
*c) for every small disk $S \subset V$ there is a small disk $T \subset V$ containing $S$ such that every $S$-Cauchy sequence which is a null sequence in $V$ is already a $T$-null sequence.*
*d) for every small disk $S \subset V$ there is a small disk $T \subset V$ containing $S$ such that*
$$\ker(\langle S\rangle^c \to \langle T\rangle^c) = \ker(\langle S\rangle^c \to \langle U\rangle^c)$$
*for all small disks $U$ containing $T$.*
*e) for every small disk $S \subset V$ there is a small disk $T \subset V$ containing $S$ such that*
$$\ker(\langle S\rangle^c \to \langle T\rangle^c) = \ker(\langle S\rangle^c \to V^c).$$

*Proof.* $a) \Rightarrow b)$ Let $S \subset V$ be a small disk. Then there exists a small disk $R \subset V^c$ such that every $S$-Cauchy sequence is $R$-convergent. Since $V \to V^c$ is a bornological embedding the disk $T = R \cap V$ is small in $V$. By construction, every $S$-Cauchy sequence that converges in $V$ is already $T$-convergent. $b) \Rightarrow c)$ is clear since $V$ is separated. $c) \Leftrightarrow d)$ Let $U$ be a small disk containing $S$. Then the kernel of the map $\langle S\rangle^c \to \langle U\rangle^c$ consists of all $S$-Cauchy sequences which are $U$-null sequences. Since a null sequence in $V$ is a null sequence in $U$ for some small disk $U$ the claim follows. $d) \Rightarrow e)$ Let $\mathfrak{dis}(V)$ be the inductive system of normed spaces obtained as the dissection of the bornological vector space $V$. Condition $d)$ implies that the direct limit of $\mathfrak{dis}(V)^c$ is automatically separated. That is, $V^c = \varinjlim \mathfrak{dis}(V)^c$ is equal to the vector space direct limit of the system $\mathfrak{dis}(V)^c$ with the quotient bornology. Hence $\ker(\langle S\rangle \to V^c) = \bigcup \ker(\langle S\rangle^c \to \langle U\rangle^c)$ where the union is taken over all small disks $U$ containing $S$. $e) \Rightarrow a)$ For each small disk $S$ in $V$ let us define $\langle\langle S\rangle\rangle = \langle S\rangle^c / \ker(\langle S\rangle^c \to V^c)$. According to

$e$) the resulting inductive system is isomorphic to $\mathfrak{Dis}(V)^c$ and $\varinjlim \langle\langle S\rangle\rangle \cong V^c$. Assume that $x \in \ker(\langle S\rangle \to \langle\langle S\rangle\rangle)$. Then $x \in \ker(\langle S\rangle \to \langle T\rangle^c)$ for some small disk $T$ containing $S$. This implies $x = 0$ since the maps $\langle S\rangle \to \langle T\rangle \to \langle T\rangle^c$ are injective. Hence $\langle S\rangle \to \langle\langle S\rangle\rangle$ is injective for all small disks $S$. It follows that $\iota : V \to V^c$ is a bornological embedding with locally dense range. □

We are interested in conditions which imply that the space $\Omega(A)$ for a separated bornological algebra $A$ is subcomplete. As usual, we consider $\Omega(A)$ as a bornological vector space with the analytic bornology. Given a small set $S \subset A$ we shall write $\Omega(S)$ for the disked hull of

$$S \cup \bigcup_{n=1}^{\infty} S^{\otimes n+1} \cup S^{\otimes n}$$

inside $\Omega(A)$ where we use the canonical identification $\Omega^n(A) = A^{\otimes n+1} \oplus A^{\otimes n}$ for the space of differential forms. Remark that the sets $\Omega(S)$ generate the analytic bornology.

DEFINITION 8.2. *A separated bornological algebra $A$ is called tensor subcomplete if the space $\Omega(A)$ is subcomplete.*

Let us call the tensor powers $V^{\otimes n}$ for $n \in \mathbb{N}$ of a bornological vector space $V$ uniformly subcomplete provided the following condition is satisfied. For every small disk $S \subset V$ there is a small disk $T \subset V$ containing $S$ such that, independent of $n \in \mathbb{N}$, any $S^{\otimes n}$-Cauchy sequence which is a null sequence in $V^{\otimes n}$ is already a $T^{\otimes n}$-null sequence. In particular, the spaces $V^{\otimes n}$ are subcomplete for all $n$ in this case.

LEMMA 8.3. *A separated bornological algebra $A$ is tensor subcomplete iff the tensor powers $A^{\otimes n}$ for $n \in \mathbb{N}$ are uniformly subcomplete.*

*Proof.* Assume first that the space $\Omega(A)$ is subcomplete. Let $S \subset A$ be a small disk and let $(x_k)_{k\in\mathbb{N}}$ be a $S^{\otimes n}$-Cauchy sequence which is a null sequence in $A^{\otimes n}$. We write $i_n : A^{\otimes n} \to \Omega(A)$ and $p_n : \Omega(A) \to A^{\otimes n}$ for the natural inclusion and projection onto one of the direct summands $A^{\otimes n}$ in $\Omega(A)$. The maps $i_n$ and $p_n$ are clearly bounded. In particular, the image of $(x_k)_{k\in\mathbb{N}}$ under $i_n$ is a $\Omega(S)$-Cauchy sequence which is a null sequence in $\Omega(A)$. Hence it is a $\Omega(T)$-null sequence for some $T \subset A$. Since $p_n(\Omega(T)) = T^{\otimes n}$ and $x_k = p_n i_n(x_k)$ it follows that the sequence $(x_k)_{k\in\mathbb{N}}$ is a $T^{\otimes n}$-null sequence. Moreover the choice of $T$ does not depend on $n$. This shows that the tensor powers $A^{\otimes n}$ are uniformly subcomplete.

Conversely, assume that the tensor powers $A^{\otimes n}$ are uniformly subcomplete. Let $S \subset A$ be a small disk and let $T \subset A$ be a small disk such that $S^{\otimes n}$-Cauchy which are null sequences in $A^{\otimes n}$ are $T^{\otimes n}$-null sequences. In addition we may assume $2S \subset T$. Let us write $P_n : \Omega(A) \to \Omega(A)$ for the natural projection onto the direct summand $\bigoplus_{j=1}^n A^{\otimes j} \oplus A^{\otimes j}$. Then $P_n(\Omega(S))$ is contained in $\Omega(S)$ and the projections $P_n$ are equibounded. Moreover $P_n$ converges to the identity uniformly on $\Omega(S)$ since $\mathrm{id} - P_n$ has norm $\leq 2^{-n}$ as a map from $\langle\Omega(S)\rangle$ into $\langle\Omega(2S)\rangle \subset \langle\Omega(T)\rangle$. Now let $(x_k)_{k\in\mathbb{N}}$ be a null sequence in $\Omega(A)$ which is

$\Omega(S)$-Cauchy. The components of $P_n(x_k)$ are $S^{\otimes k}$-Cauchy sequences which are null sequences in $A^{\otimes k}$ and hence $T^{\otimes k}$-null sequences by hypothesis. Hence $(P_n(x_k))_{n\in\mathbb{N}}$ is a $\Omega(T)$-null sequence for all $n$. Moreover $(P_n(x_k))_{n\in\mathbb{N}}$ converges to $x_k$ for every $k$, and this convergence is uniform in $k$. If follows that $(x_k)_{n\in\mathbb{N}}$ is a $\Omega(T)$-null sequence, and we deduce that $\Omega(A)$ is tensor subcomplete. □

Our next aim is to exhibit certain analytical conditions which are sufficient for tensor subcompleteness. Recall that a subset $S$ of a complete bornological vector space $V$ is called (relatively) compact if it is a (relatively) compact subset of the Banach space $\langle T\rangle$ for some small completant disk $T \subset V$. A complete bornological vector space $V$ is a Schwartz space if every small subset of $V$ is relatively compact. Every Fréchet space with the precompact bornology is a Schwartz space.

Let $V$ be a normed space and let $W$ be an arbitrary bornological vector space. By definition, a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded linear maps $f_n : V \to W$ converges uniformly to $f : V \to W$ if there exists a small disk $T \subset W$ such that all $f_n$ and $f$ are bounded linear maps $V \to \langle T\rangle$ and the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f$ in $\mathrm{Hom}(V, \langle T\rangle)$ in operator norm. A bounded linear map $f : V \to W$ can be approximated uniformly on compact subsets by finite rank operators if for every compact disk $S \subset V$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of finite rank operators $f_n : V \to W$ such that $f_n$ converges uniformly to $f$ in $\mathrm{Hom}(\langle S\rangle, W)$. An operator $f : V \to W$ is of finite rank if it is contained in the image of the natural map from the uncompleted tensor product $W \otimes V'$ into $\mathrm{Hom}(V, W)$ where $V' = \mathrm{Hom}(V, \mathbb{C})$ is the dual space of $V$. By definition, a complete bornological vector space $V$ satisfies the (global) approximation property if the identity map on $V$ can be approximated uniformly on compact subsets by finite rank operators.

We recall that a bornological vector space $V$ is regular if the bounded linear functionals on $V$ separate the points of $V$. Let us remark that there is also a local version of the approximation property which is equivalent to the global one if we restrict attention to regular spaces. Finally, we point out that for a Fréchet space with the precompact bornology the bornological approximation property is equivalent to Grothendieck's approximation property [26].

Proposition 8.4. *Let $A$ be a bornological algebra whose underlying bornological vector space is a Schwartz space satisfying the approximation property. Then $A$ is tensor subcomplete.*

*Proof.* According to lemma 8.3 it suffices to show that the tensor powers of $A$ are uniformly subcomplete. Let $S \subset A$ be a small disk. We may assume $S$ is compact and that there is a completant small disk $T \subset A$ containing $S$ such that the inclusion $\langle S\rangle \to \langle T\rangle$ can be approximated uniformly by finite rank operators on $A$. We will show that $\ker(\langle S\rangle^{\hat{\otimes} n} \to \langle U\rangle^{\hat{\otimes} n}) = \ker(\langle S\rangle^{\hat{\otimes} n} \to \langle T\rangle^{\hat{\otimes} n})$ for every completant small disk $U$ containing $T$. As in the proof of proposition 8.1 this statement easily implies that the tensor powers $A^{\otimes n}$ are uniformly subcomplete.

Take an element $x \in \ker(\langle S\rangle^{\hat{\otimes} n} \to \langle U\rangle^{\hat{\otimes} n})$. Then there is a compact disk

$K \subset \langle S \rangle$ such that $x \in K^{\hat{\otimes} n}$. Since $A$ is regular we find a sequence $(f_k)_{k \in \mathbb{N}}$ of finite rank operators $f_k : A \to \langle T \rangle$ approximating the inclusion map uniformly on $K$. The uniform convergence of the operators $f_k$ on $K$ implies that $f_k^{\hat{\otimes} n}$ converges uniformly towards the canonical map $\langle K \rangle^{\hat{\otimes} n} \to \langle T \rangle^{\hat{\otimes} n}$. In particular, the image of $x$ in $\langle T \rangle^{\hat{\otimes} n}$ is the limit of $f_k^{\hat{\otimes} n}(x)$. Since the finite rank maps $f_k$ are restrictions of maps defined on $\langle U \rangle$ and $x$ is in the kernel of $\langle S \rangle^{\hat{\otimes} n} \to \langle U \rangle^{\hat{\otimes} n}$ we have $f_k^{\hat{\otimes} n}(x) = 0$ for all $k$. Hence $x \in \ker(\langle S \rangle^{\hat{\otimes} n} \to \langle T \rangle^{\hat{\otimes} n})$ as desired. $\square$

It follows in particular that the algebra $A \otimes \mathcal{K}_G$ is tensor subcomplete provided $A$ is a Schwartz space satisfying the approximation property.

Proposition 8.5. *Let $G$ be a totally disconnected group and let $A$ be a $G$-algebra whose underlying bornological vector space is a Schwartz space satisfying the approximation property. Then the canonical chain map*

$$X_G(\mathcal{T}(A \otimes \mathcal{K}_G))^{\mathbb{L}c} \to X_G(\mathcal{T}(A \otimes \mathcal{K}_G))^c$$

*induces an isomorphism in the local derived category.*

*Proof.* Let us abbreviate $X = X_G(\mathcal{T}(A \otimes \mathcal{K}_G))$ and remark that the $\mathsf{AYD}$-module $X$ can be written in the form $X = \mathsf{A}(G) \otimes V$ for a separated bornological vector space $V$. Using this observation and proposition 8.4 one checks easily that the inductive system $\mathfrak{dis}(X)^c$ is essentially injective. Due to proposition 4.7 it follows that the natural map $X^{\mathbb{L}c} \cong \text{ho-}\varinjlim(\mathfrak{dis}(X)^c) \to \varinjlim(\mathfrak{dis}(X)^c) = X^c$ is a local homotopy equivalence. This yields the claim. $\square$

An analogous argument shows that $X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \to X_G(\mathcal{T}\mathbb{C})^c$ is a local homotopy equivalence. It follows that there is a chain of canonical isomorphisms

$$X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \cong X_G(\mathcal{T}\mathbb{C})^c \cong X_G(\mathbb{C}) = \mathcal{O}_G[0]$$

in the local derived category. In fact, the seond isomorphism is a consequence of the fact that $\mathbb{C}$ is analytically quasifree combined with homotopy invariance. The last equality is established in [35].

Consider in particular the case that $G$ is a compact group. Then the paracomplex $\mathcal{O}_G[0]$ is primitive. Taking into account stability, this yields

$$HL_*^G(\mathbb{C}, B) = H_*(\mathfrak{Hom}_G(\mathcal{O}_G[0], X_G(\mathcal{T}B)^{\mathbb{L}c})),$$

and analogously we have

$$HA_*^G(\mathbb{C}, B) = H_*(\mathfrak{Hom}_G(\mathcal{O}_G[0], X_G(\mathcal{T}B)^c))$$

for every $G$-algebra $B$. We conclude that there exists a natural transformation $HL_*^G(\mathbb{C}, B) \to HA_*^G(\mathbb{C}, B)$ between equivariant local and analytic cyclic homology if the group is compact.

Proposition 8.6. *Let $G$ be compact and let $B$ be a $G$-algebra whose underlying bornological vector space is a Schwartz space satisfying the approximation property. Then the natural map*

$$HL_*^G(\mathbb{C}, B) \to HA_*^G(\mathbb{C}, B)$$

*is an isomorphism. In particular, there is a canonical isomorphism*

$$HL^G_*(\mathbb{C},\mathbb{C}) \cong HA^G_*(\mathbb{C},\mathbb{C}) = \mathcal{R}(G)$$

*where $\mathcal{R}(G)$ is the algebra of conjugation invariant smooth functions on $G$.*

*Proof.* Using stability, the first assertion follows from proposition 8.5 and the fact that $\mathcal{O}_G[0]$ is primitive. For the second claim observe that $\mathcal{R}(G) = (\mathcal{O}_G)^G$ is the invariant part of $\mathcal{O}_G$. □

## 9. The isoradial subalgebra theorem

In this section we discuss the notion of an isoradial subalgebra and prove the isoradial subalgebra theorem which states that equivariant local cyclic homology is invariant under the passage to isoradial subalgebras.

Recall that a subset $\mathcal{V}$ of a bornological vector space $V$ is called locally dense if for any small subset $S \subset V$ there exists a small disk $T \subset V$ such that any $v \in S$ is the limit of a $T$-convergent sequence with entries in $\mathcal{V} \cap T$. Moreover recall that a separated (complete) bornological algebra $A$ is locally multiplicative iff it is isomorphic to an inductive limit of (complete) normed algebras. The following definition is taken from [26].

Definition 9.1. *Let $\mathcal{A}$ and $A$ be complete locally multiplicative bornological algebra. A bounded homomorphism $\iota : \mathcal{A} \to A$ between bornological algebras is called isoradial if it has locally dense range and*

$$\rho(\iota(S); A) = \rho(S; \mathcal{A})$$

*for all small subsets $S \subset \mathcal{A}$. If in addition $\iota$ is injective we say that $\mathcal{A}$ is an isoradial subalgebra of $A$.*

We will frequently identify $\mathcal{A}$ with its image $\iota(\mathcal{A}) \subset A$ provided $\iota : \mathcal{A} \to A$ is an injective bounded homomorphism. However, note that the bornology of $\mathcal{A}$ is usually finer than the subspace bornology on $\iota(\mathcal{A})$. Remark in addition that the inequality $\rho(\iota(S); A) \leq \rho(S; \mathcal{A})$ is automatic for every small subset $S \subset \mathcal{A}$. If $\mathcal{A}$ and $A$ are $G$-algebras and $\iota : \mathcal{A} \to A$ is an equivariant homomorphism defining an isoradial subalgebra we say that $\mathcal{A}$ is an isoradial $G$-subalgebra of $A$ .

Assume that $\iota : \mathcal{A} \to A$ is an equivariant homomorphism and consider the equivariant homomorphism $i : \mathcal{A} \otimes \mathcal{K}_G \to A \otimes \mathcal{K}_G$ obtained by tensoring $\iota$ with the identity map on $\mathcal{K}_G$. It is shown in [26] that isoradial homomorphisms are preserved under tensoring with nuclear locally multiplicative algebras. In particular, this yields the following statement.

Proposition 9.2. *If $\iota : \mathcal{A} \to A$ is an isoradial $G$-subalgebra then $i : \mathcal{A} \otimes \mathcal{K}_G \to A \otimes \mathcal{K}_G$ is an isoradial $G$-subalgebra as well.*

Note that the algebra $\mathcal{K}_G$ carries the fine bornology which implies that tensor products of $\mathcal{K}_G$ with complete spaces are automatically complete.

Let us now formulate and prove the isoradial subalgebra theorem.

Theorem 9.3. *Let $\iota : \mathcal{A} \to A$ be an isoradial $G$-subalgebra. Suppose that there exists a sequence $(\sigma_n)_{n\in I}$ of bounded linear maps $\sigma_n : A \to \mathcal{A}$ such that for each completant small disk $S \subset A$ the maps $\iota\sigma_n$ converge uniformly towards the inclusion map $\langle S\rangle \to A$. Then the class $[\iota] \in HL^G_*(\mathcal{A}, A)$ is invertible.*

Note that the existence of bounded linear maps $\sigma_n : A \to \mathcal{A}$ with these properties already implies that $\mathcal{A} \subset A$ is locally dense. We point out that the maps $\sigma_n$ in theorem 9.3 are not assumed to be equivariant.
In fact, as a first step in the proof we shall modify these maps in order to obtain equivariant approximations. Explicitly, let us define equivariant bounded linear maps $s_n : A \otimes \mathcal{K}_G \to \mathcal{A} \otimes \mathcal{K}_G$ by

$$s_n(a \otimes k)(r,t) = t \cdot \sigma_n(t^{-1} \cdot a)k(r,t)$$

where we view elements in $\mathcal{A} \otimes \mathcal{K}_G$ as smooth function on $G \times G$ with values in $\mathcal{A}$. As above we write $i$ for the equivariant homomorphism $\mathcal{A} \otimes \mathcal{K}_G \to A \otimes \mathcal{K}_G$ induced by $\iota$. Since the maps $\iota\sigma_n$ converge to the identity uniformly on small subsets of $A$ by assumption, the maps $is_n$ converge to the identity uniformly on small subsets of $A \otimes \mathcal{K}_G$.
We deduce that theorem 9.3 is a consequence of the following theorem.

Theorem 9.4. *Let $\iota : \mathcal{A} \to A$ be an isoradial $G$-subalgebra. Suppose that there exists a sequence $(\sigma_n)_{n\in I}$ of equivariant bounded linear maps $\sigma_n : A \to \mathcal{A}$ such that for each completant small disk $S \subset A$ the maps $\iota\sigma_n$ converge uniformly towards the inclusion map $\langle S\rangle \to A$. Then the chain map $X_G(\mathcal{T}\mathcal{A}) \to X_G(\mathcal{T}A)$ induced by $\iota$ is a local homotopy equivalence.*

The proof of theorem 9.4 is divided into several steps. Let $S \subset A$ be a small completant multiplicatively closed disk. By the definition of uniform convergence, there exists a small completant disk $T \subset A$ containing $S$ such that $\iota\sigma_n$ defines a bounded linear map $\langle S\rangle \to \langle T\rangle$ for every $n$ and the sequence $(\iota\sigma_n)_{n\in\mathbb{N}}$ converges to the natural inclusion map in $\mathrm{Hom}(\langle S\rangle, \langle T\rangle)$ in operator norm. Hence there exists a null sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\iota\sigma_n(x) - x \in \epsilon_n T$ for all $x \in S$. After rescaling with a positive scalar $\lambda$ we may assume that $T$ is multiplicatively closed and that $S \subset \lambda T$. Using the formula

$$\omega_{\iota\sigma_n}(x,y) = \iota\sigma_n(xy) - \iota\sigma_n(x)\iota\sigma_n(y)$$
$$= (\iota\sigma_n(xy) - xy) - (\iota\sigma_n(x) - x)(\iota\sigma_n(y) - y) - x(\iota\sigma_n(y) - y) - (\iota\sigma_n(x) - x)y$$

for $x, y \in S$ and that $T$ is multiplicatively closed we see that for any given $\epsilon > 0$ we find $N \in \mathbb{N}$ such that $\omega_{\iota\sigma_n}(S,S) \subset \epsilon T$ for $n \geq N$. Remark that we have $\omega_{\iota\sigma_n} = \iota\omega_{\sigma_n}$ since $\iota$ is a homomorphism. We deduce

$$\lim_{n\to\infty} \rho(\iota\omega_{\sigma_n}(S,S); A) = 0$$

using again that $T$ is multiplicatively closed. This in turn implies

$$\lim_{n\to\infty} \rho(\omega_{\sigma_n}(S,S); \mathcal{A}) = 0$$

since $\mathcal{A} \subset A$ is an isoradial subalgebra. This estimate will be used to obtain local inverses for the chain map induced by $\iota$.

We need some preparations. Let $B$ and $C$ be arbitrary separated $G$-algebras. Any equivariant bounded linear map $f : B \to C$ extends to an equivariant homomorphism $\mathcal{T}f : \mathcal{T}B \to \mathcal{T}C$. This homomorphism is bounded iff $f$ has analytically nilpotent curvature.

Lemma 9.5. *Let $C$ be a separated bornological algebra and let $S \subset \mathcal{T}C$ be small. Then*

$$\rho(\tau_C(S); C) = \rho(S; \mathcal{T}C)$$

*where $\tau_C : \mathcal{T}C \to C$ is the quotient homomorphism.*

*Proof.* Taking into account that $\tau_C$ is a bounded homomorphism it suffices to show that $\rho(\tau_C(S); C) < 1$ implies $\rho(S; \mathcal{T}C) \leq 1$. We may assume that the set $S$ is of the form

$$S = R + [T](dTdT)^\infty$$

where $R \subset C$ and $T \subset C$ are small disks. If $\rho(\tau_C(S); C) < 1$ we find $\lambda > 1$ such that $(\lambda R)^\infty \subset C$ is small. Let us choose $\mu$ such that $\lambda^{-1} + \mu^{-1} < 1$ and consider the small disk $P = \mu(\lambda R)^\infty$ in $C$. By construction we have $R \cdot [P] \subset \lambda^{-1} P$ as well as $dRd[P] \subset \mu^{-1} dPdP$. Moreover the disked hull $I$ of $P \cup [P](dPdP)^\infty$ is a small subset of $\mathcal{T}C$ which contains $R$. Now consider $x \in R$ and $[y_0]dy_1 \cdots dy_{2n} \in [P](dPdP)^n$. Since

$$x \circ [y_0]dy_1 \cdots dy_{2n} = x[y_0]dy_1 \cdots dy_{2n} + dxd[y_0]dy_1 \cdots dy_{2n}$$

the previous relations yield $\nu R \circ I \subset I$ for some $\nu > 1$. By induction we see that the multiplicative closure $Q$ of $\nu R$ in $\mathcal{T}C$ is small. Choose $\eta$ such that $\nu^{-1} + 2\eta^{-1} < 1$, set

$$K = \eta[T](dTdT)^\infty$$

and let $L$ be the multiplicative closure of $[Q] \circ K \circ [Q]$. By construction, the set $[Q] \circ K \circ [Q]$ is contained in the analytically nilpotent algebra $\mathcal{J}C$ which implies that $L \subset \mathcal{T}C$ is small. Let $J \subset \mathcal{T}C$ be the disked hull of the set $Q + L$. Then $J$ is small and we have $S \subset J$. In addition, it is straightforward to check $R \circ J \subset \nu^{-1} J$ and $[T](dTdT)^\infty \circ J \subset 2\eta^{-1} J$ which shows $S \circ J \subset J$. In the same way as above it follows that $S^\infty \subset \mathcal{T}C$ is small and deduce $\rho(S; \mathcal{T}C) \leq 1$. $\square$

Lemma 9.6. *Let $f : B \to C$ be an equivariant bounded linear map between separated $G$-algebras. Consider the induced chain map $X_G(\mathcal{T}f) : X_G(\mathcal{T}B) \to X_G(\mathcal{T}C)$. Given a small subset $S \subset X_G(\mathcal{T}B)$ there exists a small subset $T \subset B$ such that $X_G(\mathcal{T}f)$ is bounded on the primitive submodule generated by $S$ provided $\omega_f(T, T)^\infty$ is small.*

*Proof.* It suffices to show that, given a small set $S \subset \mathcal{T}B$, there exists a small set $T \subset B$ such that $\mathcal{T}f(S) \subset \mathcal{T}C$ is small provided $\omega_f(T, T)^\infty$ is small. We may assume that $S$ is of the form $[R](dRdR)^\infty$ for some small set $R \subset B$. Let $F : B \to \mathcal{T}C$ be the bounded linear map obtained by composing $f$ with the canonical bounded linear splitting $\sigma_C : C \to \mathcal{T}C$. The homomorphism $\mathcal{T}f : \mathcal{T}B \to \mathcal{T}C$ is given by

$$\mathcal{T}f([x_0]dx_1 \cdots dx_{2n}) = [F(x_0)]\omega_F(x_1, x_2) \cdots \omega_F(x_{2n-1}, x_{2n})$$

which shows that $\mathcal{T}f(S)$ is small provided $\omega_F(R,R)^\infty$ is small. Consider the natural projection $\tau_C : \mathcal{T}C \to C$. According to lemma 9.5 the homomorphism $\tau_C$ preserves the spectral radii of all small subsets in $\mathcal{T}C$. Using $\tau_C(\omega_F(R,R)) = \omega_f(R,R)$ we see that $\omega_F(R,R)^\infty$ is small provided $\rho(\omega_f(R,R)) < 1$. Setting $T = \lambda R$ for some $\lambda > 1$ yields the assertion. □

Let us come back to the proof of theorem 9.4. If $P \subset X_G(\mathcal{T}A)$ is a primitive subparacomplex then lemma 9.6 shows that $\sigma_n$ induces a bounded chain map $P \to X_G(\mathcal{T}\mathcal{A})$ provided $n$ is large enough. In fact, we will prove that the maps $\sigma_n$ can be used to define bounded local homotopy inverses to the chain map $\iota_* : X_G(\mathcal{T}\mathcal{A}) \to X_G(\mathcal{T}A)$ induced by $\iota$.

More precisely, let $k_n : A \to A \otimes \mathbb{C}[t]$ be the equivariant bounded linear map given by $k_n(x)(t) = (1-t)(\iota\sigma_n)(x) + tx$. Here $\mathbb{C}[t]$ is equipped with the bornology induced from $C^\infty[0,1]$. Observe that the maps $k_n$ converge to the homomorphism sending $x$ to $x \otimes 1$ uniformly on small subsets of $A$. The same reasoning as for the maps $\iota\sigma_n$ above shows

$$\lim_{n\to\infty} \rho(\omega_{k_n}(S,S); A \otimes \mathbb{C}[t]) = 0$$

for all small subsets $S \subset A$. Now assume that $P \subset X_G(\mathcal{T}A)$ is a primitive subparacomplex. According to lemma 9.6 there exists $N \in \mathbb{N}$ such that the induced chain map $X_G(\mathcal{T}A) \to X_G(\mathcal{T}(A \otimes \mathbb{C}[t]))$ is bounded on $P$ for all $n > N$. We compose this map with the chain homotopy between the evaluation maps at 0 and 1 arising from homotopy invariance to get a bounded $\mathsf{AYD}$-map $K_n : P \to X_G(\mathcal{T}A)$ of degree one which satisfies $\partial K_n + K_n\partial = \mathrm{id} - (\iota\sigma_n)_*$ on $P$.

Similarly, consider the equivariant bounded linear map $h_n : \mathcal{A} \to \mathcal{A} \otimes \mathbb{C}[t]$ given by $h_n(x)(t) = (1-t)(\sigma_n\iota)(x) + tx$ and observe that $(\iota \otimes \mathrm{id})h_n = k_n\iota$. Since the algebra $C^\infty[0,1]$ is nuclear the inclusion $\mathcal{A} \otimes \mathbb{C}[t] \to A \otimes \mathbb{C}[t]$ preserves the spectral radii of small subsets [26]. Hence the above spectral radius estimate for $k_n$ implies

$$\lim_{n\to\infty} \rho(\omega_{h_n}(S,S); \mathcal{A} \otimes \mathbb{C}[t]) = 0$$

for all small subsets $S \subset \mathcal{A}$. Now let $Q \subset X_G(\mathcal{T}\mathcal{A})$ be a primitive subparacomplex. For $n$ sufficiently large we obtain in the same way as above a bounded $\mathsf{AYD}$-map $H_n : Q \to X_G(\mathcal{T}\mathcal{A})$ of degree 1 such that $\partial H_n + H_n\partial = \mathrm{id} - (\sigma_n\iota)_*$ on $Q$.

Using these considerations it is easy to construct bounded local contracting homotopies for the mapping cone of the chain map $\iota_* : X_G(\mathcal{T}\mathcal{A}) \to X_G(\mathcal{T}A)$. This shows that $\iota_*$ is a local homotopy equivalence and completes the proof of theorem 9.4.

## 10. Applications of the isoradial subalgebra theorem

In this section we study some consequences of the isoradial subalgebra theorem in connection with $C^*$-algebras. This is needed to show that equivariant local cyclic homology is a continuously and $C^*$-stable functor on the category of $G$-$C^*$-algebras. Moreover, we discuss isoradial subalgebras arising from regular

smooth functions on simplicial complexes [36].
In the sequel we write $A \otimes B$ for the (maximal) tensor product of two $C^*$-algebras $A$ and $B$. We will only consider such tensor products when one of the involved $C^*$-algebras is nuclear, hence the $C^*$-tensor product is in fact uniquely defined in these situations. Moreover, our notation should not lead to confusion with the algebraic tensor product since we will not have to work with algebraic tensor products of $C^*$-algebras at all. All $C^*$-algebras are equipped with the precompact bornology when they are considered as bornological algebras.
As a technical preparation we have to examine how the smoothing of $G$-$C^*$-algebras is compatible with isoradial homomorphisms. Let us recall from [25] that a representation $\pi$ of $G$ on a complete bornological vector space $V$ is continuous if the adjoint of $\pi$ defines a bounded linear map $[\pi] : V \to C(G, V)$ where $C(G, V)$ is the space of continuous functions on $G$ with values in $V$ in the bornological sense. For our purposes it suffices to remark that the representation of $G$ on a $G$-$C^*$-algebra equipped with the precompact bornology is continuous in the bornological sense. We need the following special cases of results obtained by Meyer in [26].

Lemma 10.1. *Let $\mathcal{A}$ and $A$ be complete locally multiplicative bornological algebras on which $G$ acts continuously. If $\iota : \mathcal{A} \to A$ is an equivariant isoradial homomorphism then*

$$\mathfrak{Smooth}(\iota) : \mathfrak{Smooth}(\mathcal{A}) \to \mathfrak{Smooth}(A)$$

*is an isoradial homomorphism as well. Moreover, if $C$ is a complete nuclear locally multiplicative $G$-algebra then the natural homomorphism*

$$\mathfrak{Smooth}(A)\hat{\otimes}C \to \mathfrak{Smooth}(A\hat{\otimes}C)$$

*is isoradial.*

*Proof.* It is shown in [26] that the inclusion $\mathfrak{Smooth}(B) \to B$ is an isoradial subalgebra for every complete locally multiplicative bornological algebra $B$ on which $G$ acts continuously. This yields easily the first claim. In addition, the homomorphism $\mathfrak{Smooth}(A)\hat{\otimes}C \to A\hat{\otimes}C$ is isoradial because $C$ is nuclear [26]. Since the action on $C$ is already smooth it follows that

$$\mathfrak{Smooth}(A)\hat{\otimes}C \cong \mathfrak{Smooth}(\mathfrak{Smooth}(A)\hat{\otimes}C) \to \mathfrak{Smooth}(A\hat{\otimes}C)$$

is isoradial according to the first part of the lemma. □

Let $A$ be a $G$-$C^*$-algebra and consider the natural equivariant homomorphism $A\hat{\otimes}C^\infty[0,1] \to C([0,1], A) = A \otimes C[0,1]$. This map induces a bounded equivariant homomorphism

$$\mathfrak{Smooth}(A)\hat{\otimes}C^\infty[0,1] \cong \mathfrak{Smooth}(A\hat{\otimes}C^\infty[0,1]) \to \mathfrak{Smooth}(C([0,1], A))$$

and we have the following result.

Proposition 10.2. *The map $\mathfrak{Smooth}(A)\hat{\otimes}C^\infty[0,1] \to \mathfrak{Smooth}(A \otimes C[0,1])$ is an isoradial $G$-subalgebra and defines an invertible element in*

$$HL_0^G(\mathfrak{Smooth}(A)\hat{\otimes}C^\infty[0,1], \mathfrak{Smooth}(A \otimes C[0,1]))$$

*for every $G$-$C^*$-algebra $A$.*

*Proof.* It is shown in [26] that the natural inclusion $\iota : A\hat{\otimes}C^\infty[0,1] \to C([0,1],A)$ is isoradial. Hence the homomorphism $\mathfrak{Smooth}(A)\hat{\otimes}C^\infty[0,1] \to \mathfrak{Smooth}(C([0,1],A))$ is isoradial according to lemma 10.1.
We choose a family $\sigma_n : C([0,1],A) \to A\hat{\otimes}C^\infty[0,1]$ of equivariant smoothing operators such that the maps $\iota\sigma_n$ are uniformly bounded and converge to the identity pointwise. It follows that the maps $\iota\sigma_n$ converge towards the identity uniformly on precompact subsets of $C([0,1],A)$. The maps $\sigma_n$ induce equivariant bounded linear maps $\sigma_n : \mathfrak{Smooth}(C([0,1],A)) \to \mathfrak{Smooth}(A)\hat{\otimes}C^\infty[0,1]$ satisfying the condition of the isoradial subalgebra theorem 9.3. This yields the assertion. □
Let $\mathbb{K}_G = \mathbb{K}(L^2(G))$ be the algebra of compact operators on the Hilbert space $L^2(G)$. The $C^*$-algebra $\mathbb{K}_G$ is equipped with the action of $G$ induced by the regular representation. For every $G$-$C^*$-algebra $A$ we have a natural bounded equivariant homomorphism $A\hat{\otimes}\mathcal{K}_G \to A \otimes \mathbb{K}_G$. This gives rise to equivariant homomorphisms

$$\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G \to \mathfrak{Smooth}(A\hat{\otimes}\mathcal{K}_G) \to \mathfrak{Smooth}(A \otimes \mathbb{K}_G).$$

Similarly, let $\mathbb{K} = \mathbb{K}(l^2(\mathbb{N}))$ be the algebra of compact operators on an infinite dimensional separable Hilbert space with the trivial $G$-action. If $M_\infty(\mathbb{C})$ denotes the direct limit of the finite dimensional matrix algebras $M_n(\mathbb{C})$ we have a canonical bounded homomorphism $\mathfrak{Smooth}(A)\hat{\otimes}M_\infty(\mathbb{C}) \to \mathfrak{Smooth}(A \otimes \mathbb{K})$.

Proposition 10.3. *The homomorphism $\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G \to \mathfrak{Smooth}(A\otimes\mathbb{K}_G)$ is an isoradial $G$-subalgebra and defines an invertible element in*

$$HL_0^G(\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G, \mathfrak{Smooth}(A \otimes \mathbb{K}_G))$$

*for every $G$-$C^*$-algebra $A$. An analogous assertion holds for the homomorphism $\mathfrak{Smooth}(A)\hat{\otimes}M_\infty(\mathbb{C}) \to \mathfrak{Smooth}(A \otimes \mathbb{K})$.*

*Proof.* We will only treat the map $\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G \to \mathfrak{Smooth}(A \otimes \mathbb{K}_G)$ since the claim concerning the compact operators with the trivial action is obtained in a similar way.
Observe that a small subset of $\mathcal{K}_G$ is contained in a finite dimensional subalgebra of the form $M_n(\mathbb{C})$. Since $A \otimes M_n(\mathbb{C})$ is a bornological subalgebra of $A \otimes \mathbb{K}_G$ it follows that the homomorphism $\iota : A\hat{\otimes}\mathcal{K}_G \to A \otimes \mathbb{K}_G$ is isoradial. Due to lemma 10.1 the same is true for the induced map $\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G \to \mathfrak{Smooth}(A \otimes \mathbb{K}_G)$. Since $G$ is second countable and $\mathcal{D}(G) \subset L^2(G)$ is dense we find a countable orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(G)$ contained in $\mathcal{D}(G)$. Projecting to the linear subspace $\mathbb{C}^n \subset L^2(G)$ generated by the vectors $e_1, \dots, e_n$ defines a bounded linear map $\sigma_n : A \otimes \mathbb{K}_G \to A\hat{\otimes}\mathcal{K}_G$. The maps $\iota\sigma_n$ are uniformly bounded and converge towards the identity on $A \otimes \mathbb{K}_G$ pointwise. Hence they converge towards the identity uniformly on small subsets of $A\otimes\mathbb{K}_G$. Explicitly, if $p_n \in A^+\hat{\otimes}\mathcal{K}_G$ denotes the element given by

$$p_n = \sum_{j=1}^{n} 1 \otimes |e_j\rangle\langle e_j|$$

then $\sigma_n$ can be written as $\sigma_n(T) = p_n T p_n$. Since the vectors $e_j$ are smooth we conclude that $\sigma_n$ induces a bounded linear map $\mathfrak{Smooth}(A \otimes \mathbb{K}_G) \to \mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G$ which will again be denoted by $\sigma_n$. The maps $\iota\sigma_n$ converge towards the identity uniformly on small subsets of $\mathfrak{Smooth}(A \otimes \mathbb{K}_G)$ as well. Hence the claim follows from the isoradial subalgebra theorem 9.3. □

We conclude this section with another application of the isoradial subalgebra theorem. Recall from [36] that a $G$-simplicial complex is a simplicial complex $X$ with a type-preserving smooth simplicial action of the totally disconnected group $G$. We will assume in the sequel that all $G$-simplicial complexes have at most countably many simplices. A regular smooth function on $X$ is a function whose restriction to each simplex $\sigma$ of $X$ is smooth in the usual sense and which is constant in the direction orthogonal to the boundary $\partial\sigma$ in a neighborhood of $\partial\sigma$. The algebra $C^\infty_c(X)$ of regular smooth functions on $X$ with compact support is a $G$-algebra in a natural way.

PROPOSITION 10.4. *Let $X$ be a finite dimensional and locally finite $G$-simplicial complex. Then the natural map $\iota : C^\infty_c(X) \to C_0(X)$ is an isoradial $G$-subalgebra and defines an invertible element in*

$$HL^G_0(C^\infty_c(X), \mathfrak{Smooth}(C_0(X))).$$

*Proof.* As for smooth manifolds one checks that the inclusion homomomorphism $\iota : C^\infty_c(X) \to C_0(X)$ is isoradial. By induction over the dimension of $X$ we shall construct a sequence of bounded linear maps $\sigma_n : C_0(X) \to C^\infty_c(X)$ such that $\iota\sigma_n$ converges to the identity uniformly on small sets. For $k = 0$ this is easily achieved by restriction of functions to finite subsets and extension by zero. Assume that the maps $\sigma_n$ are constructed for all $(k-1)$-dimensional $G$-simplicial complexes and assume that $X$ is $k$-dimensional. If $X^{k-1}$ denotes the $(k-1)$-skeleton of $X$ we have a commutative diagram

$$\begin{tikzcd} C^\infty(X, X^{k-1}) \arrow[r, tail] \arrow[d] & C^\infty_c(X) \arrow[r, two heads] \arrow[d] & C^\infty_c(X^{k-1}) \arrow[d] \\ C(X, X^{k-1}) \arrow[r, tail] & C_0(X) \arrow[r, two heads] & C_0(X^{k-1}) \end{tikzcd}$$

where $C^\infty(X, X^{k-1})$ and $C(X, X^{k-1})$ denote the kernels of the canonical restriction homomorphisms and the vertical arrows are natural inclusions. It is shown in [36] that the upper extension has a bounded linear splitting, and the lower extension has a bounded linear splitting as well. Note that the $C(X, X^{k-1})$ is a $C^*$-direct sum of algebras of the form $C_0(\Delta^k \setminus \partial\Delta^k)$ where $\Delta^k$ denotes the standard $k$-simplex and $\partial\Delta^k$ is its boundary. Similarly, $C^\infty_c(X, X^{k-1})$ is the bornological direct sum of corresponding subalgebras $C^\infty_c(\Delta^k \setminus \partial\Delta^k)$. Hence, by applying suitable cutoff functions, we are reduced to construct approximate inverses to the inclusion $C^\infty_c(\Delta^k \setminus \partial\Delta^k) \to C_0(\Delta^k \setminus \partial\Delta^k)$. This is easily achieved using smoothing operators. Taking into account the isoradial subalgebra theorem 9.3 yields the assertion. □

## 11. Tensor products

In this section we study the equivariant $X$-complex for the analytic tensor algebra of the tensor product of two $G$-algebras. This will be used in the construction of the equivariant Chern-Connes character in the odd case.

Let us first recall the definition of the tensor product of paracomplexes of $\mathsf{AYD}$-modules [35]. If $C$ and $D$ are paracomplexes of separated $\mathsf{AYD}$-modules then the tensor product $C \boxtimes D$ is given by

$$(C \boxtimes D)_0 = C_0 \otimes_{\mathcal{O}_G} D_0 \oplus C_1 \otimes_{\mathcal{O}_G} D_1, \qquad (C \boxtimes D)_1 = C_1 \otimes_{\mathcal{O}_G} D_0 \oplus C_0 \otimes_{\mathcal{O}_G} D_1$$

where the group $G$ acts diagonally and $\mathcal{O}_G$ acts by multiplication. Using that $\mathcal{O}_G$ is commutative one checks that the tensor product $C \boxtimes D$ becomes a separated $\mathsf{AYD}$-module in this way. The boundary operator $\partial$ in $C \boxtimes D$ is defined by

$$\partial_0 = \begin{pmatrix} \partial \otimes \mathrm{id} & -\mathrm{id} \otimes \partial \\ \mathrm{id} \otimes \partial & \partial \otimes T \end{pmatrix} \qquad \partial_1 = \begin{pmatrix} \partial \otimes T & \mathrm{id} \otimes \partial \\ -\mathrm{id} \otimes \partial & \partial \otimes \mathrm{id} \end{pmatrix}$$

and turns $C \boxtimes D$ into a paracomplex. Remark that the formula for $\partial$ does not agree with the usual definition of the differential in a tensor product of complexes.

Now let $A$ and $B$ be separated bornological algebras. As it is explained in [6], the unital free product $A^+ * B^+$ of $A^+$ and $B^+$ can be written as

$$A^+ * B^+ = A^+ \otimes B^+ \oplus \bigoplus_{j>0} \Omega^j(A) \otimes \Omega^j(B)$$

with the direct sum bornology and multiplication given by the Fedosov product

$$(x_1 \otimes y_1) \circ (x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2 - (-1)^{|x_1|} x_1 dx_2 \otimes dy_1 y_2.$$

An element $a_0 da_1 \cdots da_n \otimes b_0 db_1 \cdots db_n$ corresponds to $a_0 b_0 [a_1, b_1] \cdots [a_n, b_n]$ in the free product under this identification where $[x, y] = xy - yx$ denotes the ordinary commutator. Note that if $A$ and $B$ are $G$-algebras then the free product is again a separated $G$-algebra in a natural way.

Consider the extension

$$I \rightarrowtail A^+ * B^+ \xrightarrow{\pi} A^+ \otimes B^+$$

where $I$ is the kernel of the canonical homomorphism $\pi : A^+ * B^+ \to A^+ \otimes B^+$. Using the description of the free product in terms of differential forms one has

$$I^k = \bigoplus_{j \geq k} \Omega^j(A) \otimes \Omega^j(B)$$

for the powers of the ideal $I$.

Analogous to the analytic bornology on tensor algebras we consider an analytic bornology on free products. By definition, the analytic bornology on $A^+ * B^+$ is the bornology generated by the sets

$$S \otimes T \cup \bigcup_{n=1}^{\infty} (S(dS)^n \cup (dS)^n) \otimes (T(dT)^n \cup (dT)^n)$$

for all small sets $S \subset A$ and $T \subset B$. This bornology turns $A^+ * B^+$ into a separated bornological algebra. We write $A^+ \star B^+$ for the free product of $A^+$ and $B^+$ equipped with the analytic bornology. Clearly the identity map $A^+ * B^+ \to A^+ \star B^+$ is a bounded homomorphism. Consequently the natural homomorphisms $\iota_A : A^+ \to A^+ \star B^+$ and $\iota_B : B^+ \to A^+ \star B^+$ are bounded.
Every unital homomorphism $f : A^+ \star B^+ \to C$ into a unital bornological algebra $C$ is determined by a pair of homomorphisms $f_A : A \to C$ and $f_B : B \to C$. Define a linear map $c_f : A \otimes B \to C$ by $c_f(a,b) = [f_A(a), f_B(b)]$. Let us call $f_A$ and $f_B$ almost commuting if

$$c_f(S)^\infty = \bigcup_{n=1}^{\infty} c_f(S)^n$$

is small for every small subset $S \subset A \otimes B$. Clearly, $c_f = 0$ iff the images of $f_A$ and $f_B$ commute. The following property of $A^+ \star B^+$ is a direct consequence of the definition of the analytic bornology.

Lemma 11.1. *Let $A$ and $B$ be separated bornological algebras. For a pair of bounded equivariant homomorphisms $f_A : A \to C$ and $f_B : B \to C$ into a unital bornological algebra $C$ the corresponding unital homomorphism $f : A^+ \star B^+ \to C$ is bounded iff $f_A$ and $f_B$ are almost commuting.*

In particular, the canonical homomorphism $\pi : A^+ \star B^+ \to A^+ \otimes B^+$ is bounded and we obtain a corresponding extension

$$I \rightarrowtail A^+ \star B^+ \xrightarrow{\pi} A^+ \otimes B^+$$

of bornological algebras with bounded linear splitting. It is straightforward to verify that the ideal $I$ with the induced bornology is analytically nilpotent. Remark that if $A$ and $B$ are $G$-algebras then all the previous constructions are compatible with the group action.
Let $I$ be a $G$-invariant ideal in a separated $G$-algebra $R$ and define the paracomplex $\mathcal{H}^2_G(R, I)$ by

$$\mathcal{H}^2_G(R, I)^0 = \mathcal{O}_G \otimes R/(\mathcal{O}_G \otimes I^2 + b(\mathcal{O}_G \otimes I dR))$$

in degree zero and by

$$\mathcal{H}^2_G(R, I)^1 = \mathcal{O}_G \otimes \Omega^1(R)/(b(\Omega^2_G(R)) + \mathcal{O}_G \otimes I\Omega^1(R))$$

in degree one with boundary operators induced from $X_G(R)$.
Now let $A$ and $B$ be separated $G$-algebras. We abbreviate $R = A^+ \star B^+$ and define an $\mathsf{AYD}$-map $\phi : X_G(A^+) \boxtimes X_G(B^+) \to \mathcal{H}^2_G(R, I)$ by

$$\phi(f(t) \otimes x \otimes y) = f(t) \otimes xy$$
$$\phi(f(t) \otimes x_0 dx_1 \otimes y_0 dy_1) = f(t) \otimes x_0(t^{-1} \cdot y_0)[x_1, t^{-1} \cdot y_1]$$
$$\phi(f(t) \otimes x \otimes y_0 dy_1) = f(t) \otimes x y_0 dy_1$$
$$\phi(f(t) \otimes x_0 dx_1 \otimes y) = f(t) \otimes x_0 dx_1 y$$

where $[x,y]=xy-yx$ denotes the commutator. The following result for the analytic free product $R=A^+\star B^+$ is obtained in the same way as the corresponding assertion in [35] for the ordinary free product.

Proposition 11.2. *The map $\phi: X_G(A^+)\boxtimes X_G(B^+)\to \mathcal{H}^2_G(R,I)$ defined above is an isomorphism of paracomplexes for all separated $G$-algebras $A$ and $B$.*

After these preparations we shall prove the following assertion.

Proposition 11.3. *Let $A$ and $B$ be separated locally multiplicative $G$-algebras. Then there exists a natural chain map*

$$X_G(\mathcal{T}(A^+\otimes B^+))^{\mathbb{L}c}\to (X_G((\mathcal{T}A)^+)^{\mathbb{L}c}\boxtimes X_G((\mathcal{T}B)^+)^{\mathbb{L}c})^{\mathbb{L}c}$$

*of paracomplexes. There is an analogous chain map if the derived completion is replaced by the ordinary completion.*

*Proof.* Let us abbreviate $Q=(\mathcal{T}A)^+\otimes(\mathcal{T}B)^+$. The canonical homomorphism $\tau: Q\to A^+\otimes B^+$ induces a bounded equivariant homomorphism $\mathcal{T}Q\to\mathcal{T}(A^+\otimes B^+)$. Conversely, the obvious splitting for $\tau$ is a lanilcur since the algebras $A$ and $B$ are locally multiplicative. It follows that there is a canonical bounded equivariant homomorphism $\mathcal{T}(A^+\otimes B^+)\to\mathcal{T}Q$ as well. As a consequence we obtain a natural homotopy equivalence

$$X_G(\mathcal{T}Q)\simeq X_G(\mathcal{T}(A^+\otimes B^+))$$

using homotopy invariance.
We have another analytically nilpotent extension of $Q$ defined as follows. Since commutators in the unital free product $(\mathcal{T}A)^+ * (\mathcal{T}B)^+$ are mapped to zero under the natural map $(\mathcal{T}A)^+ * (\mathcal{T}B)^+\to Q$ we have the extension

$$I \rightarrowtail R \xrightarrow{\pi} Q$$

where $R=(\mathcal{T}A)^+\star(\mathcal{T}B)^+$ is the analytic free product of $(\mathcal{T}A)^+$ and $(\mathcal{T}B)^+$ and $I$ is the kernel of the bounded homomorphism $\pi: R\to Q$. Since the $G$-algebra $I$ is analytically nilpotent the natural equivariant homomorphism $\mathcal{T}Q\to R$ is bounded and induces a chain map $X_G(\mathcal{T}Q)\to X_G(R)$.
Next we have an obvious chain map

$$p: X_G(R)\to\mathcal{H}^2_G(R,I)$$

and by proposition 11.2 there exists a natural isomorphism

$$X_G((\mathcal{T}A)^+)\boxtimes X_G((\mathcal{T}B)^+)\cong\mathcal{H}^2_G(R,I)$$

of paracomplexes. Assembling these maps and homotopy equivalences yields a chain map $X_G(\mathcal{T}(A^+\otimes B^+))\to X_G((\mathcal{T}A)^+)\boxtimes X_G((\mathcal{T}B)^+)$. Inspecting the construction of the derived completion we get in addition a natural chain map

$$(X_G((\mathcal{T}A)^+)\boxtimes X_G((\mathcal{T}B)^+))^{\mathbb{L}c}\to (X_G((\mathcal{T}A)^+)^{\mathbb{L}c}\boxtimes X_G((\mathcal{T}B)^+)^{\mathbb{L}c})^{\mathbb{L}c}$$

which immediately yields the assertion for the derived completion. For the ordinary completion the argument is essentially the same. □

Corollary 11.4. *Let $A$ and $B$ be separated locally multiplicative $G$-algebras. Then there exists a natural chain map*

$$X_G(\mathcal{T}(A \otimes B))^{\mathbb{L}c} \to (X_G(\mathcal{T}A)^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}B)^{\mathbb{L}c})^{\mathbb{L}c}.$$

*An analogous assertion holds if the derived completion is replaced by the ordinary completion.*

*Proof.* The claim follows easily from proposition 11.3 by applying the excision theorem 7.7 to tensor products of the extensions $0 \to A \to A^+ \to \mathbb{C} \to 0$ and $0 \to B \to B^+ \to \mathbb{C} \to 0$. □

Proposition 11.5. *Let $A$ be a separated locally multiplicative $G$-algebra. Then the natural chain map*

$$X_G(\mathcal{T}(\mathbb{C} \otimes A))^{\mathbb{L}c} \to (X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}A)^{\mathbb{L}c})^{\mathbb{L}c}$$

*is a homotopy equivalence. Similarly, one obtains a homotopy equivalence if the derived completion is replaced by the ordinary completion.*

*Proof.* Recall that the natural map $X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \to X_G(\mathcal{T}\mathbb{C})^c$ is a local homotopy equivalence and that $X_G(\mathcal{T}\mathbb{C})^c \simeq X_G(\mathbb{C}) = \mathcal{O}_G[0]$ using the projection homomorphism $\mathcal{T}\mathbb{C} \to \mathbb{C}$. As a consequence we obtain a natural homotopy equivalence $(X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}A)^{\mathbb{L}c})^{\mathbb{L}c} \to (\mathcal{O}_G[0] \boxtimes X_G(\mathcal{T}A)^{\mathbb{L}c})^{\mathbb{L}c}$. The composition of the latter with the chain map $X_G(\mathcal{T}(\mathbb{C} \otimes A))^{\mathbb{L}c} \to (X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}A)^{\mathbb{L}c})^{\mathbb{L}c}$ obtained in corollary 11.4 can be identified with the canonical homotopy equivalence $X_G(\mathcal{T}(\mathbb{C} \otimes A))^{\mathbb{L}c} \cong X_G(\mathcal{T}A)^{\mathbb{L}c} \simeq (X_G(\mathcal{T}A)^{\mathbb{L}c})^{\mathbb{L}c}$. This proves the claim for the derived completion. For the ordinary completion the argument is analogous. □

We remark that using the perturbation lemma one may proceed in a similar way as for the periodic theory [35] in order to construct a candidate for the homotopy inverse to the map $X_G(R)^c \to \mathcal{H}^2_G(R)^c$ induced by the projection $p$ occuring in the proof of proposition 11.3. The problem is that the formula thus obtained does not yield a bounded map in general. However, a more refined construction might yield a bounded homotopy inverse. For our purposes proposition 11.5 is sufficient.

## 12. Algebraic description of equivariant Kasparov theory

In this section we review the description of equivariant $KK$-theory arising from the approach developped by Cuntz [4], [5]. This approach to $KK$-theory is based on extensions and will be used in the definition of the equivariant Chern-Connes character below.

One of the virtues of the framework in [4] is that it allows to construct bivariant versions of $K$-theory in very general circumstances. Moreover, one can adapt the setup to treat equivariant versions of such theories as well. The main ingredient in the definition is a class of extensions in the underlying category of algebras which contains certain fundamental extensions. In particular one needs a suspension extension, a Toeplitz extension and a universal extension. In addition one has to specify a tensor product which preserves the given class

of extensions.
For equivariant $KK$-theory the underlying category of algebras is the category $G$-$C^*$-$\mathsf{Alg}$ of separable $G$-$C^*$-algebras. By definition, morphisms in $G$-$C^*$-$\mathsf{Alg}$ are the equivariant $*$-homomorphisms. The correct choice of extensions is the class $\mathfrak{E}$ of extensions of $G$-$C^*$-algebras with equivariant completely positive splitting. As a tensor product one uses the maximal $C^*$-tensor product.
The suspension extension of a $G$-$C^*$-algebra $A$ is

$$\mathcal{E}_s(A) : A(0,1) \rightarrowtail A(0,1] \twoheadrightarrow A$$

where $A(0,1)$ denotes the tensor product $A \otimes C_0(0,1)$, and accordingly the algebras $A(0,1]$ and $A[0,1]$ are defined. The group action on these algebras is given by the pointwise action on $A$.
The Toeplitz extension is defined by

$$\mathcal{E}_t(A) : \mathbb{K} \otimes A \rightarrowtail \mathfrak{T} \otimes A \twoheadrightarrow C(S^1) \otimes A$$

where $\mathfrak{T}$ is the Toeplitz algebra, that is, the universal $C^*$-algebra generated by an isometry. As usual $\mathbb{K}$ is the algebra of compact operators, and $\mathbb{K}$ and $\mathfrak{T}$ are equipped with the trivial $G$-action.
Finally, one needs an appropriate universal extension [5]. Given an algebra $A$ in $G$-$C^*$-$\mathsf{Alg}$ there exists a tensor algebra $TA$ in $G$-$C^*$-$\mathsf{Alg}$ together with a canonical surjective equivariant $*$-homomorphism $\tau_A : TA \to A$ such that the extension

$$\mathcal{E}_u(A) : JA \rightarrowtail TA \twoheadrightarrow A$$

is contained in $\mathfrak{E}$ where $JA$ denotes the kernel of $\tau_A$. Moreover, this extension is universal in the following sense. Given any extension $\mathcal{E} : 0 \to K \to E \to A \to 0$ in $\mathfrak{E}$ there exists a commutative diagram

$$\begin{tikzcd}
JA \arrow[r, rightarrowtail] \arrow[d] & TA \arrow[r, twoheadrightarrow] \arrow[d] & A \arrow[d, equal] \\
K \arrow[r, rightarrowtail] & E \arrow[r, twoheadrightarrow] & A
\end{tikzcd}$$

The left vertical map $JA \to K$ in this diagram is called the classifying map of $\mathcal{E}$. One should not confuse $TA$ with the analytic tensor algebra used in the construction of analytic and local cyclic homology.
One defines $J^2A = J(JA)$ and recursively $J^nA = J(J^{n-1}A)$ for $n \in \mathbb{N}$ as well as $J^0A = A$. Let us denote by $\phi_A : JA \to C(S^1) \otimes A$ the equivariant $*$-homomorphism obtained by composing the classifying map $JA \to A(0,1)$ of the suspension extension with the inclusion map $A(0,1) \to C(S^1) \otimes A$ given by viewing $A(0,1)$ as the ideal of functions vanishing in the point 1. This yields an equivariant $*$-homomorphism $\epsilon_A : J^2A \to \mathbb{K} \otimes A$ as the left vertical arrow

in the commutative diagram

$$\begin{tikzcd} J^2A \arrow[r, tail] \arrow[d, "\epsilon_A"] & TJA \arrow[r, two heads] \arrow[d] & JA \arrow[d, "\phi_A"] \\ \mathbb{K}\otimes A \arrow[r, tail] & \mathfrak{T}\otimes A \arrow[r, two heads] & C(S^1)\otimes A \end{tikzcd}$$

where the bottom row is the Toeplitz extension $\mathcal{E}_t(A)$. The classifying map $\epsilon_A$ plays an important role in the theory. If $[A,B]_G$ denotes the set of equivariant homotopy classes of morphisms between $A$ and $B$ then the previous construction induces a map $S : [J^kA, \mathbb{K}\otimes B] \to [J^{k+2}A, \mathbb{K}\otimes B]$ by setting $S[f] = [(\mathbb{K}\otimes f)\circ \epsilon_{J^{k+2}A}]$. Here one uses the identification $\mathbb{K}\otimes\mathbb{K}\otimes B \cong \mathbb{K}\otimes B$. We write $\mathbb{K}_G$ for the algebra of compact operators on the regular representation $L^2(G)$ equipped with its natural $G$-action. The equivariant stabilization $A_G$ of a $G$-$C^*$-algebra $A$ is defined by $A_G = A\otimes\mathbb{K}\otimes\mathbb{K}_G$. It has the property that $A_G\otimes\mathbb{K}(\mathcal{H}) \cong A_G$ as $G$-$C^*$-algebras for every separable $G$-Hilbert space $\mathcal{H}$.
Using this notation the equivariant bivariant $K$-group obtained in the approach of Cuntz can be written as

$$kk^G_*(A,B) \cong \varinjlim_j \, [J^{*+2j}(A_G), \mathbb{K}\otimes B_G]$$

where the direct limit is taken using the maps $S$ defined above. It follows from the results in [5] that $kk^G_*(A,B)$ is a graded abelian group and that there exists an associative bilinear product for $kk^G_*$. Let us remark that we have inserted the algebra $J^{*+2j}(\mathbb{K}_G\otimes\mathbb{K}\otimes A)$ in the formula defining $kk^G_*$ instead of $\mathbb{K}_G\otimes\mathbb{K}\otimes J^{*+2j}A$ as in [5]. Otherwise the construction of the product seems to be unclear.
We need some more terminology. A functor $F$ defined on the category of $G$-$C^*$-algebras with values in an additive category is called (continuously) homotopy invariant if $F(f_0) = F(f_1)$ whenever $f_0$ and $f_1$ are equivariantly homotopic $*$-homomorphisms. It is called $C^*$-stable if there exists a natural isomorphism $F(A) \cong F(A\otimes\mathbb{K}\otimes\mathbb{K}_G)$ for all $G$-$C^*$-algebras $A$. Finally, $F$ is called split exact if the sequence $0\to F(K)\to F(E)\to F(Q)\to 0$ is split exact for every extension $0\to K\to E\to Q\to 0$ of $G$-$C^*$-algebras that splits by an equivariant $*$-homomorpism $\sigma : Q\to E$.
Equivariant $KK$-theory [16] can be viewed as an additive category $KK^G$ with separable $G$-$C^*$-algebras as objects and $KK^G_0(A,B)$ as the set of morphisms between two objects $A$ and $B$. Composition of morphisms is given by the Kasparov product. There is a canonical functor $\iota : G\text{-}C^*\text{-}\mathsf{Alg} \to KK^G$ which is the identity on objects and sends equivariant $*$-homomorphisms to the corresponding $KK$-elements. Equivariant $KK$-theory satisfies the following universal property [34], [22].

Theorem 12.1. *An additive functor $F$ from $G$-$C^*$-$\mathsf{Alg}$ into an additive category $\mathcal{C}$ factorizes uniquely over $KK^G$ iff it is continuously homotopy invariant, $C^*$-stable and split exact. That is, given such a functor $F$ there exists a unique functor $\mathrm{ch}_F : KK^G \to \mathcal{C}$ such that $F = \mathrm{ch}_F\,\iota$.*

It follows from the theory developped in [4] that the functor $kk^G$ is homotopy invariant, $C^*$-stable and split exact. In fact, it is universal with respect to these properties. As a consequence one obtains the following theorem.

Theorem 12.2. *For all separable $G$-$C^*$-algebras $A$ and $B$ there is a natural isomorphism $KK_*^G(A,B) \cong kk_*^G(A,B)$.*

As already indicated above we will work with the description of equivariant $KK$-theory provided by $kk_*^G$ in the sequel. In other words, for our purposes we could as well take the definition of $kk_*^G$ as definition of equivariant $KK$-theory.

## 13. The equivariant Chern-Connes character

In this section we construct the equivariant Chern-Connes character from equivariant $KK$-theory into equivariant local cyclic homology. Moreover we calculate the character in a simple special case.

First let us extend the definition of equivariant local cyclic homology $HL_*^G$ to bornological algebras that are equipped with a not necessarily smooth action of the group $G$. This is done by first applying the smoothing functor $\mathfrak{Smooth}$ in order to obtain separated $G$-algebras. In particular, we may view equivariant local cyclic homology as an additive category $HL^G$ with the same objects as $G$-$C^*$-$\mathsf{Alg}$ and $HL_0^G(A,B)$ as the set of morphisms between two objects $A$ and $B$. By construction, there is a canonical functor from $G$-$C^*$-$\mathsf{Alg}$ to $HL^G$.

Theorem 13.1. *Let $G$ be a totally disconnected group. The canonical functor from $G$-$C^*$-$\mathsf{Alg}$ to $HL^G$ is continuously homotopy invariant, $C^*$-stable and split exact.*

*Proof.* Proposition 10.2 shows together with proposition 7.2 that $HL^G$ is continuously homotopy invariant. We obtain $C^*$-stability from proposition 10.3 together with proposition 7.4. Finally, if $0 \to K \to E \to Q \to 0$ is a split exact extension of $G$-$C^*$-algebras then $0 \to \mathfrak{Smooth}(K) \to \mathfrak{Smooth}(E) \to \mathfrak{Smooth}(Q) \to 0$ is a split exact extension of $G$-algebras. Hence split exactness follows from the excision theorem 7.6. □

Having established this result, the existence of the equivariant Chern-Connes character in the even case is an immediate consequence of the universal property of equivariant Kasparov theory. More precisely, according to theorem 13.1 and theorem 12.1 we obtain an additive map

$$\mathrm{ch}_0^G : KK_0^G(A,B) \to HL_0^G(A,B)$$

for all separable $G$-$C^*$-algebras $A$ and $B$. The resulting transformation is multiplicative with respect to the Kasparov product and the composition product, respectively. Remark that the equivariant Chern-Connes character $\mathrm{ch}_0^G$ is determined by the property that it maps $KK$-elements induced by equivariant $*$-homomorphisms to the corresponding $HL$-elements.

Before we extend this character to a multiplicative transformation on $KK_*^G$

we shall describe $\mathrm{ch}_0^G$ more concretely using the theory explained in section 12. Let us fix some notation. If $f : A \to B$ is an equivariant homomorphism between $G$-algebras we denote by $\mathrm{ch}(f)$ the associated class in $H_0(\mathfrak{Hom}_G(X_G(\mathcal{T}A)^{\mathbb{L}c}, X_G(\mathcal{T}B)^{\mathbb{L}c}))$. By slight abuse of notation we will also write $\mathrm{ch}(f)$ for the corresponding element in $HL_0^G(A, B)$. Similarly, assume that $\mathcal{E} : 0 \to K \to E \to Q \to 0$ is an extension of $G$-algebras with equivariant bounded linear splitting. We denote by $\mathrm{ch}(\mathcal{E})$ the element $-\delta(\mathrm{id}_K)$ where

$$\delta : H_0(\mathfrak{Hom}_G(X_G(\mathcal{T}K)^{\mathbb{L}c}, X_G(\mathcal{T}K)^{\mathbb{L}c})) \to H_1(\mathfrak{Hom}_G(X_G(\mathcal{T}Q)^{\mathbb{L}c}, X_G(\mathcal{T}K)^{\mathbb{L}c}))$$

is the boundary map in the six-term exact sequence in bivariant homology obtained from the generalized excision theorem 7.7. Again, by slight abuse of notation we will also write $\mathrm{ch}(\mathcal{E})$ for the corresponding element in $HL_1^G(Q, K)$. If $f : A \to B$ is an equivariant $*$-homomorphism between $G$-$C^*$-algebras we write simply $\mathrm{ch}(f)$ instead of $\mathrm{ch}(\mathfrak{Smooth}(f))$ for the element associated to the corresponding homomorphism of $G$-algebras. In a similar way we proceed for extensions of $G$-$C^*$-algebras with equivariant completely positive splitting.
Using theorem 13.1 one shows that $\mathrm{ch}(\epsilon_A) \in HL_*^G(J^2A, \mathbb{K} \otimes A)$ is invertible. The same holds true for the iterated morphisms $\mathrm{ch}(\epsilon_A^n) \in HL_*^G(J^{2n}A, \mathbb{K} \otimes A)$. Remark also that $\mathrm{ch}(\iota_A) \in HL_*^G(A, \mathbb{K} \otimes A)$ is invertible.
Now assume that $x \in KK_0^G(A, B)$ is represented by $f : J^{2n}A_G \to \mathbb{K} \otimes B_G$. Then the class $\mathrm{ch}_0^G(f)$ is corresponds to

$$\mathrm{ch}(\iota_{A_G}) \cdot \mathrm{ch}(\epsilon_{A_G}^n)^{-1} \cdot \mathrm{ch}(f) \cdot \mathrm{ch}(\iota_{B_G})^{-1}$$

in $HL_0^G(A_G, B_G)$, and the latter group is canonically isomorphic to $HL_0^G(A, B)$.
For the definition of $\mathrm{ch}_1^G$ we follow the discussion in [4]. We denote by $j : C_0(0,1) \to C(S^1)$ the inclusion homomorphism obtained by viewing elements of $C_0(0,1)$ as functions on the circle vanishing in 1. Moreover let $\mathbb{K}$ be the algebra of compact operators on $l^2(\mathbb{N})$ and let $\iota : \mathbb{C} \to \mathbb{K}$ be the homomorphism determined by sending 1 to the minimal projection onto the first basis vector in the canonical orthonormal basis. If $A$ is any $G$-$C^*$-algebra we write $\iota_A : A \to A \otimes \mathbb{K}$ for the homomorphism obtained by tensoring $\iota$ with the identity on $A$. In the sequel we write $\mathcal{E}_s$ instead of $\mathcal{E}_s(\mathbb{C})$ and similarly $\mathcal{E}_t$ instead of $\mathcal{E}_t(\mathbb{C})$ for the Toeplitz extension of $\mathbb{C}$.

Proposition 13.2. *With the notation as above one has*

$$\mathrm{ch}(\mathcal{E}_s) \cdot \mathrm{ch}(j) \cdot \mathrm{ch}(\mathcal{E}_t) = \frac{1}{2\pi i}\,\mathrm{ch}(\iota)$$

*in* $H_0(\mathfrak{Hom}_G(X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c}, X_G(\mathcal{T}\mathbb{K})^{\mathbb{L}c}))$.

*Proof.* First observe that the same argument as in the proof of proposition 10.3 shows that the element $\mathrm{ch}(\iota)$ is invertible. Let us write $z$ for the element in $H_0(\mathfrak{Hom}_G(X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c}, X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c}))$ given by of $(2\pi i)\,\mathrm{ch}(\mathcal{E}_s) \cdot \mathrm{ch}(j) \cdot \mathrm{ch}(\mathcal{E}_t) \cdot \mathrm{ch}(\iota)^{-1}$. It suffices to show that $z$ is equal to the identity.
We consider the smooth analogues of the extensions $\mathcal{E}_s$ and $\mathcal{E}_t$ used in [4]. The smooth version of the suspension extension is

$$\mathbb{C}^\infty(0,1) \rightarrowtail \mathbb{C}^\infty(0,1] \twoheadrightarrow \mathbb{C}$$

where $\mathbb{C}^\infty(0,1)$ denotes the algebra of smooth functions on $[0,1]$ vanishing with all derivatives in both endpoints. Similarly, $\mathbb{C}^\infty(0,1]$ is the algebra of all smooth functions $f$ vanishing with all derivatives in $0$ and vanishing derivatives in $1$, but arbitrary value $f(1)$. The smooth Toeplitz extension is

$$\mathbb{K}^\infty \rightarrowtail \mathfrak{T}^\infty \twoheadrightarrow C^\infty(S^1)$$

where $\mathbb{K}^\infty$ is the algebra of smooth compact operators and $\mathfrak{T}^\infty$ is the smooth Toeplitz algebra defined in [4]. We obtain another endomorphism $z^\infty$ of $X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c}$ by repeating the construction of $z$ using the smooth supension and Toeplitz extensions. By naturality one has in fact $z^\infty = z$, hence it suffices to show that $z^\infty$ is equal to the identity.
Recall that we have a local homotopy equivalence $X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \to X_G(\mathcal{T}\mathbb{C})^c \simeq X_G(\mathbb{C})$. Using the fact that the $G$-action is trivial on all algebras under consideration the same argument as in [4] yields that $z^\infty$ is equal to the identity. $\square$
We shall use the abbreviation $x_A = \mathrm{ch}(\mathcal{E}_u(A))$ for the element arising from the universal extension of the $G$-$C^*$-algebra $A$.

Proposition 13.3. *Let $A$ be a $G$-$C^*$-algebra and let $\epsilon_A : J^2(A) \to \mathbb{K} \otimes A$ be the canonical map. Then we have the relation*

$$x_A \cdot x_{JA} \cdot \mathrm{ch}(\epsilon_A) = \frac{1}{2\pi i}\,\mathrm{ch}(\iota_A)$$

*in* $H_0(\mathfrak{Hom}_G(X_G(\mathcal{T}(\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c}, X_G(\mathcal{T}(\mathfrak{Smooth}(A\otimes\mathbb{K})\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c}))$.

*Proof.* For an arbitrary $G$-$C^*$-algebra $A$ consider the commutative diagram

$$\begin{tikzcd}
X_G(\mathcal{T}(\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[r, "\cong"] \arrow[d, "x_A"] & X_G(\mathcal{T}(\mathfrak{Smooth}(\mathbb{C}\otimes A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[d] \\
X_G(\mathcal{T}(\mathfrak{Smooth}(JA)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[r] \arrow[d, "x_{JA}"] & X_G(\mathcal{T}(\mathfrak{Smooth}(J\mathbb{C}\otimes A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[d] \\
X_G(\mathcal{T}(\mathfrak{Smooth}(J^2A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[r] \arrow[d, "\mathrm{ch}(\epsilon_A)"] & X_G(\mathcal{T}(\mathfrak{Smooth}(J^2\mathbb{C}\otimes A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[d, "\mathrm{ch}(\epsilon_{\mathbb{C}}\otimes\mathrm{id})"] \\
X_G(\mathcal{T}(\mathfrak{Smooth}(\mathbb{K}\otimes A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[r] \arrow[d, "\mathrm{ch}(\iota_A)^{-1}"] & X_G(\mathcal{T}(\mathfrak{Smooth}(\mathbb{K}\otimes A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[d, "\mathrm{ch}(\iota_{\mathbb{C}}\otimes\mathrm{id})^{-1}"] \\
X_G(\mathcal{T}(\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c} \arrow[r, "\cong"] & X_G(\mathcal{T}(\mathfrak{Smooth}(\mathbb{C}\otimes A)\hat{\otimes}\mathcal{K}_G))^{\mathbb{L}c}
\end{tikzcd}$$

where the upper part is obtain from the morphism of extensions

$$\begin{tikzcd}
JA \arrow[r, tail] \arrow[d] & TA \arrow[r, two heads] \arrow[d] & A \arrow[d, "\cong"] \\
A\otimes J\mathbb{C} \arrow[r, tail] & A\otimes T\mathbb{C} \arrow[r, two heads] & A\otimes\mathbb{C}
\end{tikzcd}$$

and a corresponding diagram with $A$ replaced by $JA$. Observe that there is a natural homomorphism $D\hat{\otimes}\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G \to \mathfrak{Smooth}(D \otimes A)\hat{\otimes}\mathcal{K}_G$ for every trivial $G$-$C^*$-algebra $D$. For simplicity we will write $\mathfrak{Smooth}(A)$ instead of $\mathfrak{Smooth}(A)\hat{\otimes}\mathcal{K}_G$ in the following commutative diagram

$$\begin{tikzcd}
X_G(\mathcal{T}(\mathbb{C}\hat{\otimes}\mathfrak{Smooth}(A)))^{\mathbb{L}c} \arrow[r] \arrow[d] & (X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}\mathfrak{Smooth}(A))^{\mathbb{L}c})^{\mathbb{L}c} \arrow[d, "x_{\mathbb{C}}\boxtimes \mathrm{id}"] \\
X_G(\mathcal{T}(J\mathbb{C}\hat{\otimes}\mathfrak{Smooth}(A)))^{\mathbb{L}c} \arrow[r] \arrow[d] & (X_G(\mathcal{T}(J\mathbb{C}))^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}\mathfrak{Smooth}(A))^{\mathbb{L}c})^{\mathbb{L}c} \arrow[d, "x_{J\mathbb{C}}\boxtimes \mathrm{id}"] \\
X_G(\mathcal{T}(J^2\mathbb{C}\hat{\otimes}\mathfrak{Smooth}(A)))^{\mathbb{L}c} \arrow[r] \arrow[d, "\mathrm{ch}(\epsilon_{\mathbb{C}}\hat{\otimes}\,\mathrm{id})"] & (X_G(\mathcal{T}(J^2\mathbb{C}))^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}\mathfrak{Smooth}(A))^{\mathbb{L}c})^{\mathbb{L}c} \arrow[d, "\mathrm{ch}(\epsilon_{\mathbb{C}})\boxtimes \mathrm{id}"] \\
X_G(\mathcal{T}(\mathbb{K}\hat{\otimes}\mathfrak{Smooth}(A)))^{\mathbb{L}c} \arrow[r] \arrow[d, "\mathrm{ch}(\iota_A)^{-1}"] & (X_G(\mathcal{T}\mathbb{K})^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}\mathfrak{Smooth}(A))^{\mathbb{L}c})^{\mathbb{L}c} \arrow[d, "\mathrm{ch}(\iota)^{-1}\boxtimes \mathrm{id}"] \\
X_G(\mathcal{T}(\mathbb{C}\hat{\otimes}\mathfrak{Smooth}(A)))^{\mathbb{L}c} \arrow[r] & (X_G(\mathcal{T}\mathbb{C})^{\mathbb{L}c} \boxtimes X_G(\mathcal{T}\mathfrak{Smooth}(A))^{\mathbb{L}c})^{\mathbb{L}c}
\end{tikzcd}$$

obtained using corollary 11.4. According to proposition 11.5 the first and the last horizontal map in this diagram are homotopy equivalences. Moreover, we may connect the right column of the first diagram with the left column of the previous diagram. Using these observations the assertion follows from proposition 13.2 in the same way as in [4]. □

After these preparations we shall now define the Chern-Connes character in the odd case. For notational simplicity we assume that all $G$-$C^*$-algebras $A$ are replaced by their equivariant stabilizations $A_G$. We may then use the identification

$$KK^G_*(A, B) \cong \varinjlim_j\, [J^{*+2j}(A), \mathbb{K} \otimes B]$$

and obtain a canonical isomorphism $KK^G_1(A, B) \cong KK^G_0(JA, B)$. Consider an element $u \in KK^G_1(A, B)$ and denote by $u_0$ the element in $KK^G_0(JA, B)$ corresponding to $u$. Then the element $\mathrm{ch}^G_1(u) \in HL^G_1(A, B)$ is defined by

$$\mathrm{ch}^G_1(u) = \sqrt{2\pi i}\, x_A \cdot \mathrm{ch}^G_0(u_0)$$

in terms of the character in the even case obtained before. Using proposition 13.3 one concludes in the same way as in [4] that the formula

$$\mathrm{ch}^G_{i+j}(x \cdot y) = \mathrm{ch}^G_i(x) \cdot \mathrm{ch}^G_j(y)$$

holds for all elements $x \in KK^G_i(A, B)$ and $y \in KK^G_j(B, C)$.

We have now completed the construction of the equivariant Chern-Connes character and summarize the result in the following theorem.

Theorem 13.4. *Let $G$ be a second countable totally disconnected locally compact group and let $A$ and $B$ be separable $G$-$C^*$-algebras. Then there exists a transformation*

$$\mathrm{ch}^G_* : KK^G_*(A,B) \to HL^G_*(A,B)$$

*which is multiplicative with respect to the Kasparov product in $KK^G_*$ and the composition product in $HL^G_*$. Under this transformation elements in $KK^G_0(A,B)$ induced by equivariant $*$-homomorphisms from $A$ to $B$ are mapped to the corresponding elements in $HL^G_0(A,B)$.*

The transformation obtained in this way will be called the equivariant Chern-Connes character. One shows as in nonequivariant case that, up to possibly a sign and a factor $\sqrt{2\pi i}$, the equivariant Chern-Connes character is compatible with the boundary maps in the six-term exact sequences associated to an extension in $\mathfrak{E}$.
At this point it is not clear wether the equivariant Chern-Connes character is a useful tool to detect information contained in equivariant $KK$-theory. As a matter of fact, equivariant local cyclic homology groups are not easy to calculate in general. In a separate paper we will exhibit interesting situations in which $\mathrm{ch}^G_*$ becomes in fact an isomorphism after tensoring the left hand side with the complex numbers. At the same time a convenient description of the right hand side of the character will be obtained.
Here we shall at least illustrate the nontriviality of the equivariant Chern-Connes character in a simple special case. Assume that $G$ is a profinite group. The character of a finite dimensional representation of $G$ defines an element in the algebra $\mathcal{R}(G) = (\mathcal{O}_G)^G$ of conjugation invariant smooth functions on $G$. As usual we denote by $R(G)$ the representation ring of $G$.

Proposition 13.5. *Let $G$ be a profinite group. Then the equivariant Chern-Connes character*

$$\mathrm{ch}^G_* : KK^G_*(\mathbb{C},\mathbb{C}) \to HL^G_*(\mathbb{C},\mathbb{C})$$

*can be identified with the character map $R(G) \to \mathcal{R}(G)$. This identification is compatible with the products.*

*Proof.* Let $V$ be a finite dimensional representation of $G$. Then $\mathbb{K}(V)$ is a unital $G$-algebra and the element in $R(G) = KK^G_0(\mathbb{C},\mathbb{C})$ corresponding to $V$ is given by the class of the equivariant homomorphism $p_V : \mathbb{C} \to \mathbb{K}(V)$ in $KK^G_0(\mathbb{C},\mathbb{K}(V)) \cong KK^G_0(\mathbb{C},\mathbb{C})$ where $p_V$ is defined by $p_V(1) = \mathrm{id}_V$. Using stability of $HL^G_*$ and proposition 8.6 we see that the class of $\mathrm{ch}^G_0(p_V)$ in $HL^G_0(\mathbb{C},\mathbb{K}(V)) \cong HL^G_0(\mathbb{C},\mathbb{C}) = \mathcal{R}(G)$ corresponds to the character of the representation $V$. The claim follows easily from these observations. $\square$

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# Erratum for “Slope Filtrations Revisited”

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Abstract. Some corrections are given for the manuscript “Slope Filtrations Revisited”, Documenta Math., Vol. 10 (2005), 447-525

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Laurent Berger has pointed out that the construction of Teichmüller presentations in [3, Definition 2.5.1] is not valid: it fails to properly account for the nonlinearity of the Teichmüller map. This would appear to invalidate those results of [3] depending on the use of Teichmüller presentations, or on plus-minus-zero presentations. Fortunately, these can be corrected by adapting the technique of strong semiunit decompositions from [2], as follows.

Retain notation as in [3, § 2.5]. A *strong semiunit presentation* of $x \in \Gamma_I$ is a convergent sum $x = \sum_{i \in \mathbb{Z}} u_i \pi^i$ in which:

(a) each nonzero $u_i$ belongs to $\Gamma$ and satisfies $v_n(u_i) = v_0(u_i)$ for all $n \geq 0$;

(b) if $i > j$ and $u_i, u_j$ are both nonzero, then $v_0(u_i) < v_0(u_j)$.

Such a presentation always exists by the same proof as in [2, Proposition 3.14], but there is no uniqueness property. Nonetheless, in each of [3, Proposition 3.3.7(c), Proposition 4.2.2, Lemma 4.3.2], one may safely replace all references to Teichmüller presentations (including implicit references via plus-minus-zero presentations) with strong semiunit presentations. (One should also disregard the parenthetical remark about canonicality in the proof of [3, Proposition 4.2.2].)

This substitution does not suffice for the proof of surjectivity in [3, Lemma 4.3.1], which uses the uniqueness property of Teichmüller presentations. This is harmless for the rest of the paper, because this lemma is used nowhere. For completeness, we point out that the lemma is an immediate consequence of a result of Fourquaux [1, Corollaire 3.9.19] (applied with $a = 1$).

Ruochuan Liu points out that the proof of [3, Lemma 2.9.1] is incomplete: it is only valid in case $f$ has no slopes in $[s', s)$, as otherwise we cannot choose the unit $u$ in the first sentence of the proof. To complete the proof in general, first note that the existence of $g$ satisfying (a) and (b) follows from [3, Lemma 2.6.7]. To prove (c), choose $s''$ with $s' < s'' < s$ such that $f$ has no slopes in $[s'', s)$. By the proof of [3, Lemma 2.9.1] as written, $f$ is divisible by $g$ in $\Gamma_{[s'',r]}$. However, since $g$ has no slopes less than $s$, $g$ is a unit in $\Gamma_{[s',s'']}$, so $f$ is also divisible by $g$ in that ring. Since the intersection $\Gamma_{[s',s'']} \cap \Gamma_{[s'',r]}$ inside $\Gamma_{[s'',s'']}$ is equal to $\Gamma_{[s',r]}$ by [3, Corollary 2.5.7], $f$ is divisible by $g$ in $\Gamma_{[s',r]}$ as desired.

Liu also notes a gap in the proof of [3, Lemma 2.9.3]: it is necessary to ensure that $x_{i+1} \in \Gamma_r[\pi^{-1}]$. To fix this, we must replace $g_{i+1} - x_i$ wherever it appears by some $y_i \in \Gamma_r$ such that $g_{i+1} - x_i - y_i$ is divisible by $h_{i+1}$ in $\Gamma_{i+1}$; this can be carried out by an argument similar to [3, Lemma 2.9.2].

We also take this opportunity to point out two errata to [2]. First (as noted by Kevin Buzzard), in the introduction (p. 95), it is incorrectly asserted that "$\Gamma_{\mathrm{con}}$ consists of series which take integral values on some open annulus with outer radius 1." In fact, an element of $\Gamma_{\mathrm{con}}$ acquires this property only after multiplication by a large power of $u$ (and conversely). Second, in [2, Lemma 2.3], $R$ should be taken to be a Bézout domain, not merely a Bézout ring.

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# Construction of Eigenvarieties in Small Cohomological Dimensions for Semi-Simple, Simply Connected Groups

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Abstract. We study low order terms of Emerton's spectral sequence for simply connected, simple groups. As a result, for real rank 1 groups, we show that Emerton's method for constructing eigenvarieties is successful in cohomological dimension 1. For real rank 2 groups, we show that a slight modification of Emerton's method allows one to construct eigenvarieties in cohomological dimension 2.

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Throughout this paper we shall use the following standard notation:

- $k$ is an algebraic number field, fixed throughout.
- $\mathfrak{p}$, $\mathfrak{q}$ denote finite primes of $k$, and $k_{\mathfrak{p}}$, $k_{\mathfrak{q}}$ the corresponding local fields.
- $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R}$ is the product of the archimedean completions of $k$.
- $\mathbb{A}$ is the adèle ring of $k$.
- $\mathbb{A}_f$ is the ring of finite adèles of $k$.
- For a finite set $S$ of places of $k$, we let

$$k_S = \prod_{v \in S} k_v, \qquad \mathbb{A}^S = {\prod_{v \notin S}}' k_v.$$

# 1 Introduction and Statements of Results

## 1.1 Interpolation of classical automorphic representations

Let $\mathbb{G}$ be a connected, algebraically simply connected, semi-simple group over a number field $k$. We fix once and for all a maximal compact subgroup $K_\infty \subset \mathbb{G}(k_\infty)$. Our assumptions on $\mathbb{G}$ imply that $K_\infty$ is connected in the archimedean topology. This paper is concerned with the cohomology of the following "Shimura manifolds":

$$Y(K_f) = \mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})/K_\infty K_f,$$

where $K_f$ is a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$. Let $W$ be an irreducible finite dimensional algebraic representation of $\mathbb{G}$ over a field extension $E/k$. Such a representation gives rise to a local system $\mathcal{V}_W$ on $Y(K_f)$. We shall refer to the cohomology groups of this local system as the "classical cohomology groups":

$$H^\bullet_{\text{class.}}(K_f, W) := H^\bullet(Y(K_f), \mathcal{V}_W).$$

It is convenient to consider the direct limit over all levels $K_f$ of these cohomology groups:

$$H^\bullet_{\text{class.}}(\mathbb{G}, W) = \varinjlim_{K_f} H^\bullet_{\text{class.}}(K_f, W).$$

There is a smooth action of $\mathbb{G}(\mathbb{A}_f)$ on $H^\bullet_{\text{class.}}(\mathbb{G}, W)$. Since $W$ is a representation over a field $E$ of characteristic zero, we may recover the finite level cohomology groups as spaces of $K_f$-invariants:

$$H^\bullet_{\text{class.}}(K_f, W) = H^\bullet_{\text{class.}}(\mathbb{G}, W)^{K_f}.$$

It has become clear that only a very restricted class of smooth representations of $\mathbb{G}(\mathbb{A}_f)$ may occur as subquotients of the classical cohomology $H^n_{\text{class.}}(\mathbb{G}, W)$. For example, in the case $E = \mathbb{C}$, Ramanujan's Conjecture (Deligne's Theorem) gives an archimedian bound on the eigenvalues of the Hecke operators. We shall be concerned here with the case that $E$ is an extension of a non-archimedean completion of $k$.

Fix once and for all a finite prime $\mathfrak{p}$ of $k$ over which $\mathbb{G}$ is quasi-split. Fix a Borel subgroup $\mathbb{B}$ of $\mathbb{G} \times_k k_\mathfrak{p}$ and a maximal torus $\mathbb{T} \subset \mathbb{B}$. We let $E$ be a finite extension of $k_\mathfrak{p}$, large enough so that $\mathbb{G}$ splits over $E$. It follows that the irreducible algebraic representations of $\mathbb{G}$ over $E$ are absolutely irreducible (§24.5 of [8]). By the highest weight theorem (§24.3 of [8]), an irreducible representation $W$ of $\mathbb{G}$ over $E$ is determined by its highest weight $\psi_W$, which is an algebraic character $\psi_W : \mathbb{T} \times_{k_\mathfrak{p}} E \to \mathrm{GL}_1/E$.

By a *tame level* we shall mean a compact open subgroup $K^\mathfrak{p} \subset \mathbb{G}(\mathbb{A}_f^\mathfrak{p})$. Fix a tame level $K^\mathfrak{p}$, and consider the spaces of $K^\mathfrak{p}$-invariants:

$$H^\bullet_{\text{class.}}(K^\mathfrak{p}, W) = H^\bullet_{\text{class.}}(\mathbb{G}, W)^{K^\mathfrak{p}}.$$

The group $\mathbb{G}(k_\mathfrak{p})$ acts smoothly on $H^\bullet_{\text{class.}}(K^\mathfrak{p}, W)$. We also have commuting actions of the level $K^\mathfrak{p}$ Hecke algebra:

$$\mathcal{H}(K^\mathfrak{p}) := \left\{ f : K^\mathfrak{p}\backslash \mathbb{G}(\mathbb{A}_f^\mathfrak{p})/K^\mathfrak{p} \to E : f \text{ has compact support} \right\}.$$

In order to describe the representations of $\mathcal{H}(K^\mathfrak{p})$, recall the tensor product decomposition:

$$\mathcal{H}(K^\mathfrak{p}) = \mathcal{H}(K^\mathfrak{p})^{\text{ramified}} \otimes \mathcal{H}(K^\mathfrak{p})^{\text{sph}}, \tag{1}$$

where $\mathcal{H}(K^\mathfrak{p})^{\text{sph}}$ is commutative but infinitely generated, and $\mathcal{H}(K^\mathfrak{p})^{\text{ramified}}$ is non-commutative but finitely generated. Consequently the irreducible representations of $\mathcal{H}(K^\mathfrak{p})$ are finite-dimensional.
Let $\mathfrak{q} \neq \mathfrak{p}$ be a finite prime of $k$. We shall say that $\mathfrak{q}$ is unramified in $K^\mathfrak{p}$ if

(a) $\mathbb{G}$ is quasi-split over $k_\mathfrak{q}$, and splits over an unramified extension of $k_\mathfrak{q}$, and

(b) $K^\mathfrak{p} \cap \mathbb{G}(k_\mathfrak{q})$ is a hyper-special maximal compact subgroup of $\mathbb{G}(k_\mathfrak{q})$ (see [38]).

Let $S$ be the set of finite primes $\mathfrak{q} \neq \mathfrak{p}$, which are ramified in $K^\mathfrak{p}$. This is a finite set, and we have

$$K^\mathfrak{p} = K_S \times \prod_{\mathfrak{q} \text{ unramified}} K_\mathfrak{q}, \qquad K_S = K^\mathfrak{p} \cap \mathbb{G}(k_S), \quad K_\mathfrak{q} = K^\mathfrak{p} \cap \mathbb{G}(k_\mathfrak{q}).$$

This gives the tensor product decomposition (1), where we take

$$\mathcal{H}(K^\mathfrak{p})^{\text{ramified}} = \mathcal{H}(K_S), \qquad \mathcal{H}(K^\mathfrak{p})^{\text{sph}} = \bigotimes_{\mathfrak{q} \text{ unramified}}{}' \mathcal{H}(K_\mathfrak{p}).$$

For each unramified prime $\mathfrak{q}$, the Satake isomorphism (Theorem 4.1 of [12]) shows that $\mathcal{H}(K_\mathfrak{q})$ is finitely generated and commutative. Hence the irreducible representations of $\mathcal{H}(K^\mathfrak{p})^{\text{sph}}$ over $\bar{E}$ are 1-dimensional, and may be identified with elements of $(\text{Spec } \mathcal{H}(K^\mathfrak{p})^{\text{sph}})(\bar{E})$. Since the global Hecke algebra is infinitely generated, Spec $\mathcal{H}(K^\mathfrak{p})^{\text{sph}}$ is an infinite dimensional space. One might expect that the representations which occur as subquotients of $H^\bullet_{\text{class.}}(K^\mathfrak{p}, W)$ are evenly spread around this space. There is an increasing body of evidence [1, 2, 3, 10, 11, 13, 14, 15, 18, 21, 22] that this is not the case, and that in fact these representations are contained in a finite dimensional subset of Spec $\mathcal{H}(K^\mathfrak{p})^{\text{sph}}$, independent of $W$.
More precisely, let $\pi$ be an irreducible representation of $\mathbb{G}(k_\mathfrak{p}) \times \mathcal{H}(K^\mathfrak{p})$, which occurs as a subquotient of $H^n_{\text{class.}}(K^\mathfrak{p}, W) \otimes_E \bar{E}$. We may decompose $\pi$ as a tensor product:

$$\pi = \pi_\mathfrak{p} \otimes \pi^{\text{ramified}} \otimes \pi^{\text{sph}},$$

where $\pi^{\text{sph}}$ is a character of $\mathcal{H}(K^\mathfrak{p})^{\text{sph}}$; $\pi^{\text{ramified}}$ is an irreducible representation of $\mathcal{H}(K^\mathfrak{p})^{\text{ramified}}$ and $\pi_\mathfrak{p}$ is an irreducible smooth representation of $\mathbb{G}(k_\mathfrak{p})$. We can say very little about the pair $(W,\pi)$ in this generality, so we shall make another restriction. We shall write $\mathrm{Jacq}_{\mathbb{B}}(\pi_{\mathfrak{p}})$ for the Jacquet module of $\pi_{\mathfrak{p}}$, with respect to $\mathbb{B}(k_{\mathfrak{p}})$. The Jacquet module is a smooth, finite dimensional representation of $\mathbb{T}(k_{\mathfrak{p}})$. It seems possible to say something about those pairs $(\pi, W)$ for which $\pi_{\mathfrak{p}}$ has non-zero Jacquet module. Such representations $\pi_{\mathfrak{p}}$ are also said to have *finite slope*. Classically for $\mathrm{GL}_2/\mathbb{Q}$, representations of finite slope correspond to Hecke eigenforms for which the eigenvalue of $U_{\mathfrak{p}}$ is non-zero. By Frobenius reciprocity, such a $\pi_{\mathfrak{p}}$ is a submodule of a smoothly induced representation $\mathrm{ind}_{\mathbb{B}(k_{\mathfrak{p}})}^{\mathbb{G}(k_{\mathfrak{p}})}\theta$, where $\theta : \mathbb{T}(k_{\mathfrak{p}}) \to \bar{E}^{\times}$ is a smooth character. In order to combine the highest weight $\psi_W$, which is an algebraic character of $\mathbb{T}$, and the smooth character $\theta$ of $\mathbb{T}(k_{\mathfrak{p}})$, we introduce the following rigid analytic space (see [32] for background in rigid analytic geometry):

$$\hat{T}(A) = \mathrm{Hom}_{k_{\mathfrak{p}}-\mathrm{loc.an.}}(\mathbb{T}(k_{\mathfrak{p}}), A^{\times}), \qquad \begin{array}{c} A \text{ a commutative} \\ \text{Banach algebra over } E. \end{array}$$

Emerton defined the *classical point* corresponding to $\pi$ to be the pair

$$(\theta\psi_W, \pi^{\mathrm{sph}}) \in \left(\hat{T} \times \mathrm{Spec}\ \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}\right)(\bar{E}).$$

We let $E(n,K^{\mathfrak{p}})_{\mathrm{class.}}$ denote the set of all classical points. Emerton defined the *eigenvariety* $E(n,K^{\mathfrak{p}})$ to be the rigid analytic Zariski closure of $E(n,K^{\mathfrak{p}})_{\mathrm{class.}}$ in $\hat{T} \times \mathrm{Spec}\ \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$.

Concretely, this means that for every unramified prime $\mathfrak{q}$ and each generator $T_{\mathfrak{q}}^i$ for the Hecke algebra $\mathcal{H}(K_{\mathfrak{q}})$, there is a holomorphic function $t_{\mathfrak{q}}^i$ on $E(n,K^{\mathfrak{p}})$ such that for every representation $\pi$ in $H^n_{\mathrm{class.}}(K^{\mathfrak{p}},W)\otimes\bar{E}$ of finite slope at $\mathfrak{p}$, the action of $T_{\mathfrak{q}}^i$ on $\pi$ is by scalar multiplication by $t_{\mathfrak{q}}^i(x)$, where $x$ is the corresponding classical point.

One also obtains a description of the action of the ramified part of the Hecke algebra. This description is different, since irreducible representations of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$ are finite dimensional rather than 1-dimensional. Instead one finds that there is a coherent sheaf $\mathcal{M}$ of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$-modules over $E(n,K^{\mathfrak{p}})$, such that, roughly speaking, the action of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$ on the fibre of a classical point describes the action of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$ on the corresponding part of the classical cohomology. A precise statement is given in Theorem 1 below.

Emerton introduced a criterion (Definition 1 below), according to which the Eigencurve $E(n,K^{\mathfrak{p}})$ is finite dimensional. More precisely, he was able to prove that the projection $E(n,K^{\mathfrak{p}}) \to \hat{T}$ is finite. If we let $\mathfrak{t}$ denote the Lie algebra of $\mathbb{T}(\bar{E})$, then there is a map given by differentiation at the identity element:

$$\hat{T} \to \check{\mathfrak{t}},$$

where $\check{\mathfrak{t}}$ is the dual space of $\mathfrak{t}$. It is worth noting that the image in $\check{\mathfrak{t}}$ of a classical point depends only on the highest weight $\psi_W$, since smooth characters have zero derivative. Emerton also proved, assuming his criterion, that the projection

$E(n, K^{\mathfrak{p}}) \to \check{\mathfrak{t}}$ has discrete fibres. As a result, one knows that the dimension of the eigencurve is at most the absolute rank of $\mathbb{G}$.
The purpose of this paper is to investigate Emerton's criterion for connected, simply connected, simple groups. Specifically, we show that Emerton's criterion holds for all such groups in dimension $n = 1$. Emerton's criterion typically fails in dimension $n = 2$. However we prove a weaker form of the criterion for $n = 2$, and we show that the weaker criterion is sufficient for most purposes.

## 1.2 Emerton's Criterion

Let $p$ be the rational prime below $\mathfrak{p}$. In [18] Emerton introduced the following $p$-adic Banach spaces:

$$\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p) = \left( \varprojlim_s \varinjlim_{K_\mathfrak{p}} H^\bullet(Y(K_\mathfrak{p} K^\mathfrak{p}), \mathbb{Z}/p^s) \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For convenience, we also consider the direct limits of these spaces over all tame levels $K^{\mathfrak{p}}$:

$$\tilde{H}^\bullet(\mathbb{G}, \mathbb{Q}_p) = \varinjlim_{K^\mathfrak{p}} \tilde{H}^\bullet(K^\mathfrak{p}, \mathbb{Q}_p).$$

We have the following actions on these spaces:

- The group $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ acts smoothly on $\tilde{H}^\bullet(\mathbb{G}, \mathbb{Q}_p)$; the subspace $\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p)$ may be recovered as the $K^{\mathfrak{p}}$-invariants:

$$\tilde{H}^\bullet(K^\mathfrak{p}, \mathbb{Q}_p) = \tilde{H}^\bullet(\mathbb{G}, \mathbb{Q}_p)^{K^\mathfrak{p}}.$$

- The Hecke algebra $\mathcal{H}(K^{\mathfrak{p}})$ acts on $\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_p) \otimes E$.

- The group $\mathbb{G}(k_{\mathfrak{p}})$ acts continuously, but not usually smoothly on the Banach space $\tilde{H}^\bullet(K^{\mathfrak{p}}, \mathbb{Q}_{\mathfrak{p}})$. This is an admissible continuous representation of $\mathbb{G}(k_{\mathfrak{p}})$ in the sense of [33] (or [16], Definition 7.2.1).

- Recall that we have fixed a finite extension $E/k_{\mathfrak{p}}$, over which $\mathbb{G}$ splits. We let

$$\tilde{H}^\bullet(K^\mathfrak{p}, E) = \tilde{H}^\bullet(K^\mathfrak{p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} E.$$

  The group $\mathbb{G}(k_{\mathfrak{p}})$ is a $\mathfrak{p}$-adic analytic group. Hence, we may define the subspace of $k_{\mathfrak{p}}$-locally analytic vectors in $\tilde{H}^\bullet(K^{\mathfrak{p}}, E)$ (see [16]):

$$\tilde{H}^\bullet(K^\mathfrak{p}, E)_{\mathrm{loc.an.}}.$$

  This subspace is $\mathbb{G}(k_{\mathfrak{p}})$-invariant, and is an admissible locally analytic representation of $\mathbb{G}(k_{\mathfrak{p}})$ (in the sense of [16], Definition 7.2.7). The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ also acts on the subspace $\tilde{H}^\bullet(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}}$.

For an irreducible algebraic representation $W$ of $\mathbb{G}$ over $E$, we shall write $\check{W}$ be the contragredient representation. Emerton showed (Theorem 2.2.11 of [18]) that there is a spectral sequence:

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathfrak{g}}(\check{W}, \tilde{H}^q(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}}) \implies H^{p+q}_{\mathrm{class.}}(K^{\mathfrak{p}}, W). \tag{2}$$

Taking the direct limit over the tame levels $K^{\mathfrak{p}}$, there is also a spectral sequence (Theorem 0.5 of [18]):

$$\mathrm{Ext}^p_{\mathfrak{g}}(\check{W}, \tilde{H}^q(\mathbb{G}, E)_{\mathrm{loc.an.}}) \implies H^{p+q}_{\mathrm{class.}}(\mathbb{G}, W). \tag{3}$$

In particular, there is an edge map

$$H^n_{\mathrm{class.}}(\mathbb{G}, W) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n(\mathbb{G}, E)_{\mathrm{loc.an.}}). \tag{4}$$

DEFINITION 1. We shall say that $\mathbb{G}$ satisfies *Emerton's criterion in dimension* $n$ if the following holds:

For every $W$, the edge map (4) is an isomorphism.

This is equivalent to the edge maps from $H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W)$ to $\mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}})$ being isomorphisms for every $W$ and every tame level $K^{\mathfrak{p}}$.

THEOREM 1 (Theorem 0.7 of [18]). *Suppose Emerton's criterion holds for $\mathbb{G}$ in dimension $n$. Then we have:*

1. *Projection onto the first factor induces a finite map $E(n, K^{\mathfrak{p}}) \to \hat{T}$.*
2. *The map $E(n, K^{\mathfrak{p}}) \to \check{\mathfrak{t}}$ has discrete fibres.*
3. *If $(\chi, \lambda)$ is a point of the Eigencurve such that $\chi$ is locally algebraic and of non-critical slope (in the sense of [17], Definition 4.4.3), then $(\chi, \lambda)$ is a classical point.*
4. *There is a coherent sheaf $\mathcal{M}$ of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$-modules over $E(n, K^{\mathfrak{p}})$ with the following property. For any classical point $(\theta\psi_W, \lambda) \in E(n, K^{\mathfrak{p}})$ of non-critical slope, the fibre of $\mathcal{M}$ over the point $(\theta\psi_W, \lambda)$ is isomorphic (as a $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{ramified}}$-module) to the dual of the $(\theta\psi_W, \lambda)$-eigenspace of the Jacquet module of $H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, \check{W})$.*

In fact Emerton proved this theorem for all reductive groups $\mathbb{G}/k$. He verified his criterion in the case $\mathbb{G} = \mathrm{GL}_2/\mathbb{Q}$, $n = 1$. He also pointed out that the criterion always holds for $n = 0$, since the edge map at $(0,0)$ for any first quadrant $E_2^{\bullet,\bullet}$ spectral sequence is an isomorphism. Of course the cohomology of $\mathbb{G}$ is usually uninteresting in dimension 0, but his argument can be applied in the case where the derived subgroup of $\mathbb{G}$ has real rank zero. This is the case, for example, when $\mathbb{G}$ is a torus, or the multiplicative group of a definite quaternion algebra.

### 1.3 Our Main Results

For our main results, $\mathbb{G}$ is connected, simple and algebraically simply connected. We shall also assume that $\mathbb{G}(k_\infty)$ is not compact. We do not need to assume that $\mathbb{G}$ is absolutely simple. We shall prove the following.

Theorem 2. *Emerton's criterion holds in dimension* 1.

For cohomological dimensions 2 and higher, Emerton's criterion is quite rare. We shall instead use the following criterion.

Definition 2. We shall say that $\mathbb{G}$ satisfies the *weak Emerton criterion* in dimension $n$ if

(a) for every non-trivial irreducible $W$, the edge map (4) is an isomorphism, and

(b) for the trivial representation $W$, the edge map (4) is injective, and its cokernel is a finite dimensional trivial representation of $\mathbb{G}(\mathbb{A}_f)$.

By simple modifications to Emerton's proof of Theorem 1, we shall prove the following in §4.

Theorem 3. *If the weak Emerton criterion holds for* $\mathbb{G}$ *in dimension* $n$*, then*

1. *Projection onto the first factor induces a finite map* $E(n, K^{\mathfrak{p}}) \to \hat{T}$.
2. *The map* $E(n, K^{\mathfrak{p}}) \to \check{\mathfrak{t}}$ *has discrete fibres.*
3. *If* $(\chi, \lambda)$ *is a point of the Eigencurve such that* $\chi$ *is locally algebraic and of non-critical slope, then either* $(\chi, \lambda)$ *is a classical point or* $(\chi, \lambda)$ *is the trivial representation of* $\mathbb{T}(k_{\mathfrak{p}}) \times \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$.

In order to state our next theorems, we recall the definition of the congruence kernel. As before, $\mathbb{G}/k$ is simple, connected and simply connected and $\mathbb{G}(k_\infty)$ is not compact. By a *congruence subgroup* of $\mathbb{G}(k)$, we shall mean a subgroup of the form

$$\Gamma(K_f) = \mathbb{G}(k) \cap (\mathbb{G}(k_\infty) \times K_f),$$

where $K_f \subset \mathbb{G}(\mathbb{A}_f)$ is compact and open. Any two congruence subgroups are commensurable.

An *arithmetic subgroup* is a subgroup of $\mathbb{G}(k)$, which is commensurable with a congruence subgroup. In particular, every congruence subgroup is arithmetic. The *congruence subgroup problem* (see the survey articles [30, 31]) is the problem of determining the difference between arithmetic subgroups and congruence subgroups. In particular, one could naively ask whether every arithmetic subgroup of $\mathbb{G}$ is a congruence subgroup. In order to study this question more precisely, Serre introduced two completions of $\mathbb{G}(k)$:

$$\hat{\mathbb{G}}(k) = \varprojlim_{K_f} \mathbb{G}(k)/\Gamma(K_f),$$

$$\tilde{\mathbb{G}}(k) = \varprojlim_{\Gamma \text{ arithmetic}} \mathbb{G}(k)/\Gamma.$$

There is a continuous surjective group homomorphism $\tilde{\mathbb{G}}(k) \to \hat{\mathbb{G}}(k)$. The *congruence kernel* $\mathrm{Cong}(\mathbb{G})$ is defined to be the kernel of this map. Recall the following:

Theorem 4 (Strong Approximation Theorem [23, 24, 25, 28, 29]). *Suppose $\mathbb{G}/k$ is connected, simple, and algebraically simply connected. Let $S$ be a set of places of $k$, such that $\mathbb{G}(k_S)$ is not compact. Then $\mathbb{G}(k)\mathbb{G}(k_S)$ is dense in $\mathbb{G}(\mathbb{A})$.*

Under our assumptions on $\mathbb{G}$, the strong approximation theorem implies that $\hat{\mathbb{G}}(k) = \mathbb{G}(\mathbb{A}_f)$, and we have the following extension of topological groups:

$$1 \to \mathrm{Cong}(\mathbb{G}) \to \tilde{\mathbb{G}}(k) \to \mathbb{G}(\mathbb{A}_f) \to 1.$$

By the *real rank* of $\mathbb{G}$, we shall mean the sum

$$m = \sum_{\nu|\infty} \mathrm{rank}_{k_\nu} \mathbb{G}.$$

It follows from the non-compactness of $\mathbb{G}(k_\infty)$, that the real rank of $\mathbb{G}$ is at least 1. Serre [37] has conjectured that for $\mathbb{G}$ simple, simply connected and of real rank at least 2, the congruence kernel is finite; for real rank 1 groups he conjectured that the congruence kernel is infinite. These conjectures have been proved in many cases and there are no proven counterexamples (see the surveys [30, 31]).
Our next result is the following.

Theorem 5. *If the congruence kernel of $\mathbb{G}$ is finite then the weak Emerton criterion holds in dimension 2.*

Theorems 2 and 5 follow from our main auxiliary results:

Theorem 6. *Let $\mathbb{G}$ be as described above. Then $\tilde{H}^0(\mathbb{G}, E) = E$, with the trivial action of $\mathbb{G}(\mathbb{A}_f)$.*

Theorem 7. *Let $\mathbb{G}$ be as described above. Then*

$$\tilde{H}^1(\mathbb{G}, E) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}), E)_{\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})-\mathrm{smooth}},$$

*where $\mathrm{Cong}(\mathbb{G})$ denotes the congruence kernel of $\mathbb{G}$.*

The reduction of Theorem 2 to Theorem 6 is given in §2, and the reduction of Theorem 5 to Theorem 7 is given in §3. Theorem 6 is proved in §6 and Theorem 7 is proved in §8.
Before going on, we point out that in some cases these cohomology spaces are uninteresting. In the case $E = \mathbb{C}$, the cohomology groups are related,

via generalizations of the Eichler–Shimura isomorphism, to certain spaces of automorphic forms. More precisely, Franke [19] has shown that

$$H^{\bullet}_{\text{class.}}(K_f, W) = H^{\bullet}_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, W \otimes \mathcal{A}(K^f)),$$

where $\mathcal{A}(K^f)$ is the space of automorphic forms $\phi : \mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})/K_\infty K_f \to \mathbb{C}$. The right hand side is relative Lie algebra cohomology (see for example [9]). Since the constant functions form a subspace of $\mathcal{A}(K^f)$, we have a $(\mathfrak{g}, K_\infty)$-submodule $W \subset W \otimes \mathcal{A}(K^f)$. This gives us a map:

$$H^n_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, W) \to H^n_{\text{class.}}(\mathbb{G}, W). \tag{5}$$

We shall say that the cohomology of $\mathbb{G}$ is *given by constants in dimension* $n$ if the map (5) is surjective. For example the cohomology of $\mathrm{SL}_2/\mathbb{Q}$ is given by constants in dimensions 0 and 2, although (5) is only bijective in dimension 0. On the other hand, if $\mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})$ is compact then (5) is injective.
It is known that the cohomology of $\mathbb{G}$ is given by constants in dimensions $n < m$ and in dimensions $n > d - m$, where $d$ is the common dimension of the spaces $Y(K_f)$ and $m$ is the real rank of $\mathbb{G}$. One shows this by proving that the relative Lie algebra cohomology of any other irreducible $(\mathfrak{g}, K_\infty)$-subquotient of $W \otimes \mathcal{A}(K^f)$ vanishes in such dimensions (see for example Corollary II.8.4 of [9]).
If the cohomology is given by constants in dimension $n$, then $H^n_{\text{class.}}(\mathbb{G}, W)$ is a finite dimensional vector space, equipped with the trivial action of $\mathbb{G}(\mathbb{A}_f)$. From the point of view of this paper, cohomology groups given by constants are uninteresting. Thus Theorem 2 is interesting only for groups of real rank 1, whereas Theorem 5 is interesting, roughly speaking, for groups of real rank 2.
In fact we can often do a little better than Theorem 3. We shall prove the following in §5:

Theorem 8. *Let $\mathbb{G}/k$ be connected, semi-simple and algebraically simply connected and assume that the weak Emerton criterion holds in dimension $n$. Assume also that at least one of the following two conditions holds:*

- *(a) $H^p_{\text{class.}}(\mathbb{G}, \mathbb{C})$ is given by constants in dimensions $p < n$ and $H^{n+1}_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$; or*
- *(b) $\mathbb{G}(k)$ is cocompact in $\mathbb{G}(\mathbb{A})$.*

*Then all conclusions of Theorem 1 hold for the eigenvariety $E(n, K^{\mathfrak{p}})$.*

The theorem is valid, for example, in the following cases where Emerton's criterion fails:

- $\mathrm{SL}_3/\mathbb{Q}$ in dimension 2;
- $\mathrm{Sp}_4/\mathbb{Q}$ in dimension 2;

- Spin groups of quadratic forms over $\mathbb{Q}$ of signature $(2,l)$ with $l \geq 3$ in dimension 2;
- Special unitary groups $\mathrm{SU}(2,l)$ with $l \geq 3$ in dimension 2;
- $\mathrm{SL}_2/k$, where $k$ is a real quadratic field, in dimension 2.

Our results generalize easily to simply connected, semi-simple groups as follows. Suppose $\mathbb{G}/k$ is a direct sum of simply connected simple groups $\mathbb{G}_i/k$. Assume also that the tame level $K^{\mathfrak{p}}$ decomposes as a direct sum of tame levels $K_i^{\mathfrak{p}}$ in $\mathbb{G}_i(\mathbb{A}_f^{\mathfrak{p}})$. By the Künneth formula, we have a decomposition of the sets of classical points:

$$E(n, K^{\mathfrak{p}})_{\mathrm{class.}} = \bigcup_{n_1+\cdots+n_s=n} \prod_{i=1}^{s} E(n_i, K_i^{\mathfrak{p}})_{\mathrm{class.}}.$$

### 1.4 Some History

Coleman and Mazur constructed the first "eigencurve" in [15]. In our current notation, they constructed the $H^1$-eigencurve for $\mathrm{GL}_2/\mathbb{Q}$. In fact they showed that the points of their eigencurve parametrize overconvergent eigenforms. Their arguments were based on earlier work of Hida [20] and Coleman [14] on families of modular forms. Similar results were subsequently obtained by Buzzard [10] for the groups $\mathrm{GL}_1/k$, and for the multiplicative group of a definite quaternion algebra over $\mathbb{Q}$, and later more generally for totally definite quaternion algebras over totally real fields in [11]. Kassaei [21] treated the case that $\mathbb{G}$ is a form of $\mathrm{GL}_2/k$, where $k$ is totally real and $\mathbb{G}$ is split at exactly one archimedean place. Kissin and Lei in [22] treated the case $\mathbb{G} = \mathrm{GL}_2/k$ for a totally real field $k$, in dimension $n = [k : \mathbb{Q}]$.
Ash and Stevens [2, 3] obtained similar results for $\mathrm{GL}_n/\mathbb{Q}$ by quite different methods. More recently, Chenevier [13] constructed eigenvarieties for any twisted form of $\mathrm{GL}_n/\mathbb{Q}$ which is compact at infinity. Emerton's construction is apparently much more general, as his criterion is formulated for any reductive group over a number field. However, it seems to be quite rare for his criterion to hold. One might expect the weak criterion to hold more generally; in particular one might optimistically ask the following:

*Question.* For $\mathbb{G}/k$ connected, simple, algebraically simply connected and of real rank $m$, does the weak Emerton criterion always hold in dimension $m$?

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## 2 Proof of Theorem 2

Let $\mathbb{G}/k$ be simple, algebraically simply connected, and assume that $\mathbb{G}(k_\infty)$ is not compact. We shall prove in §6 that $\tilde{H}^0(\mathbb{G}, E) = E$, with the trivial action of $\mathbb{G}(\mathbb{A}_f)$. As a consequence of this, the terms $E_2^{p,0}$ in Emerton's spectral sequence (3) are Lie-algebra cohomology groups of finite dimensional representations:

$$E_2^{p,0} = H^p_{\mathrm{Lie}}(\mathfrak{g}, W).$$

Such cohomology groups are completely understood. We recall some relevant results:

Theorem 9 (Theorem 7.8.9 of [39]). *Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field of characteristic zero, and let $W$ be a finite-dimensional representation of $\mathfrak{g}$, which does not contain the trivial representation. Then we have for all $n \geq 0$,*

$$H^n_{\mathrm{Lie}}(\mathfrak{g}, W) = 0.$$

Theorem 10 (Whitehead's first lemma (Corollary 7.8.10 of [39])). *Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field of characteristic zero, and let $W$ be a finite-dimensional representation of $\mathfrak{g}$. Then we have*

$$H^1_{\mathrm{Lie}}(\mathfrak{g}, W) = 0.$$

Theorem 11 (Whitehead's second lemma (Corollary 7.8.12 of [39])). *Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field of characteristic zero, and let $W$ be a finite-dimensional representation of $\mathfrak{g}$. Then we have*

$$H^2_{\mathrm{Lie}}(\mathfrak{g}, W) = 0.$$

We shall use these results to verify Emerton's criterion in dimension 1, thus proving Theorem 2. We must verify that the edge map 4 is an isomorphism for $n = 1$ and for every irreducible algebraic representation $W$ of $\mathbb{G}$. The small terms of the spectral sequence are:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{llll} \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^1(\mathbb{G}, E)) & & & \\ \quad H^0_{\mathrm{Lie}}(\mathfrak{g}, W) & & H^1_{\mathrm{Lie}}(\mathfrak{g}, W) & H^2_{\mathrm{Lie}}(\mathfrak{g}, W) \end{array}$$

We therefore have an exact sequence:

$$0 \to H^1_{\mathrm{Lie}}(\mathfrak{g}, W) \to H^1_{\mathrm{class.}}(\mathbb{G}, W) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^1(\mathbb{G}, E)) \to H^2_{\mathrm{Lie}}(\mathfrak{g}, W).$$

By Theorems 10 and 11 we know that the first and last terms are zero. Therefore the edge map is an isomorphism.

□

## 3 Proof of Theorem 5

Let $\mathbb{G}/k$ be connected, simple and simply connected, and assume that $\mathbb{G}(k_\infty)$ is not compact. In §8 we shall prove the isomorphism

$$\tilde{H}^1(\mathbb{G},\mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})-\mathrm{smooth}}.$$

As a consequence, we have:

Corollary 1. *If the congruence kernel of* $\mathbb{G}$ *is finite then* $\tilde{H}^1(\mathbb{G},\mathbb{Q}_p)=0$.

In this context, it is worth noting that the following may be proved by a similar method.

Theorem 12. *If the congruence kernel of* $\mathbb{G}$ *is finite then* $\tilde{H}^{d-1}(\mathbb{G},\mathbb{Q}_p)=0$, *where* $d$ *is the dimension of the symmetric space* $\mathbb{G}(k_\infty)/K_\infty$.

We shall use the corollary to verify the weak Emerton criterion in dimension 2. Suppose first that $W$ is a non-trivial irreducible algebraic representation of $\mathbb{G}$. We must show that the edge map (4) is an isomorphism. By Theorem 9 we know that the bottom row of the spectral sequence is zero, and by the corollary we know that the first row is zero. The small terms of the spectral sequence are as follows:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \mathrm{Hom}_{\mathfrak{g}}(\check{W},\tilde{H}^2(\mathbb{G},E)_{\mathrm{loc.an.}}) & & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{array}$$

Hence in this case the edge map is an isomorphism.

In the case that $W$ is the trivial representation, we must only verify that the edge map is injective and that its cokernel is a finite dimensional trivial representation of $\mathbb{G}(\mathbb{A}_f)$. We still know in this case that the first row of the spectral sequence is zero. For the bottom row, Theorems 10 and 11 tell us that the spectral sequence is as follows:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \mathrm{Hom}_{\mathfrak{g}}(E,\tilde{H}^2(\mathbb{G},E)_{\mathrm{loc.an.}}) & & & \\ 0 & 0 & 0 & \\ E & 0 & 0 & H^3_{\mathrm{Lie}}(\mathfrak{g},E) \end{array}$$

It follows that we have an exact sequence

$$0 \to H^2_{\mathrm{class.}}(\mathbb{G},E) \to \mathrm{Hom}_{\mathfrak{g}}(E,\tilde{H}^2(\mathbb{G},E)_{\mathrm{loc.an.}}) \to H^3_{\mathrm{Lie}}(\mathfrak{g},E). \qquad (6)$$

The action of $\mathbb{G}(\mathbb{A}_f)$ on $H^3_{\mathrm{Lie}}(\mathfrak{g},E)$ is trivial, since this action is defined by the (trivial) action on $\tilde{H}^0(\mathbb{G},E)=E$.

□

*Remark.* It is interesting to calculate the cokernel of the edge map in (6). In fact it is known that for any simple Lie algebra $\mathfrak{g}$ over a field $E$ of characteristic zero, $H^3_{\mathrm{Lie}}(\mathfrak{g}, E) = E$. We therefore have by the Künneth formula:

$$H^3_{\mathrm{Lie}}(\mathfrak{g}, E) = E^d,$$

where $d$ is the number of simple factors of $\mathbb{G} \times_k \bar{k}$. In particular, this is never zero. The exact sequence (6) can be continued for another term as follows:

$$0 \to H^2_{\mathrm{class.}}(\mathbb{G}, E) \to \tilde{H}^2(\mathbb{G}, E)^{\mathfrak{g}}_{\mathrm{loc.an.}} \to H^3_{\mathrm{Lie}}(\mathfrak{g}, E) \to H^3_{\mathrm{class.}}(\mathbb{G}, E)^{\mathbb{G}(\mathbb{A}_f)}.$$

In order to calculate the last term, we first choose an embedding of $E$ in $\mathbb{C}$, and tensor with $\mathbb{C}$. There is a map

$$H^3_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) \to H^3_{\mathrm{class.}}(\mathbb{G}, \mathbb{C})^{\mathbb{G}(\mathbb{A}_f)}.$$

If the $k$-rank of $\mathbb{G}$ is zero, then this map is an isomorphism. In other cases, it is often surjective, although the author does not know how to prove this statement in general. The groups $H^\bullet_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C})$ are the cohomology groups of compact symmetric spaces (see §I.1.6 of [9]) and are completely understood. In particular, it is often the case that $H^3_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. This implies that the edge map in (6) often has a non-trivial cokernel.

## 4 Proof of Theorem 3

Theorem 3 is a variation on Theorem 1. In order to prove it, we recall some of the intermediate steps in Emerton's proof of Theorem 1.

In [17], Emerton introduced a new kind of Jacquet functor, $\mathrm{Jacq}_{\mathbb{B}}$, from the category of essentially admissible (in the sense of Definition 6.4.9 of [16]) locally analytic representations of $\mathbb{G}(k_{\mathfrak{p}})$ to the category of essentially admissible locally analytic representations of $\mathbb{T}(k_{\mathfrak{p}})$. This functor is left exact, and its restriction to the full subcategory of smooth representations is exact. Indeed, its restriction to smooth representations is the usual Jacquet functor of coinvariants.

Applying the Jacquet functor to the space $\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}}$, one obtains an essentially admissible locally analytic representation of $\mathbb{T}(k_{\mathfrak{p}})$. On the other hand, the category of essentially admissible locally analytic representations of $\mathbb{T}(k_{\mathfrak{p}})$ is anti-equivalent to the category of coherent rigid analytic sheaves on $\hat{T}$ (Proposition 2.3.2 of [18]). We therefore have a coherent sheaf $\mathcal{E}$ on $\hat{T}$. Since the action of $\mathcal{H}(K^{\mathfrak{p}})$ on $\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}}$ commutes with that of $\mathbb{G}(k_{\mathfrak{p}})$, it follows that $\mathcal{H}(K^{\mathfrak{p}})$ acts on $\mathcal{E}$. Let $\mathcal{A}$ be the image of $\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ in the sheaf of endomorphisms of $\mathcal{E}$. Thus $\mathcal{A}$ is a coherent sheaf of commutative rings on $\hat{T}$. Writing $\operatorname{Spec} \mathcal{A}$ for the relative spec of $\mathcal{A}$ over $\hat{T}$, we have a Zariski-closed embedding $\operatorname{Spec} \mathcal{A} \to \hat{T} \times \operatorname{Spec} \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$. Since $\mathcal{A}$ acts as endomorphisms of $\mathcal{E}$, we may localize $\mathcal{E}$ to a coherent sheaf $\mathcal{M}$ on $\operatorname{Spec} \mathcal{A}$.

Theorem 1 may be deduced from the following two results.

THEOREM 13 (2.3.3 of [18]). *(i) The natural projection* $\mathrm{Spec}\,\mathcal{A} \to \hat{T}$ *is a finite morphism.*

*(ii) The map* $\mathrm{Spec}\,\mathcal{A} \to \check{\mathfrak{t}}$ *has discrete fibres.*

*(iii) The fibre of* $\mathcal{M}$ *over a point* $(\chi, \lambda)$ *of* $\hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$ *is dual to the* $(\mathbb{T}(k_{\mathfrak{p}}) = \chi, \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}} = \lambda)$*-eigenspace of* $\mathrm{Jacq}_{\mathbb{B}}(\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}})$*. In particular, the point* $(\chi, \lambda)$ *lies in* $\mathrm{Spec}\,\mathcal{A}$ *if and only if this eigenspace is non-zero.*

For any representation $V$ of $\mathbb{G}(k_{\mathfrak{p}})$ over $E$, we shall write $V_{W-\mathrm{loc.alg.}}$ for the subspace of $W$-locally algebraic vectors in $V$. Note that under Emerton's criterion, we have

$$H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W} = \tilde{H}^n(K^{\mathfrak{p}}, E)_{\check{W}-\mathrm{loc.alg.}}. \tag{7}$$

Hence $H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W}$ is a closed subspace of $\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}}$. By left-exactness of $\mathrm{Jacq}_{\mathbb{B}}$ we have an injective map

$$\mathrm{Jacq}_{\mathbb{B}}(H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W}) \to \mathrm{Jacq}_{\mathbb{B}}(\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}})$$

There are actions of $\mathbb{T}(k_{\mathfrak{p}})$ and $\mathcal{H}(K^{\mathfrak{p}})$ on these spaces, so we may restrict this map to eigenspaces:

$$\mathrm{Jacq}_{\mathbb{B}}(H^n_{\mathrm{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W})^{(\chi,\lambda)} \to \mathrm{Jacq}_{\mathbb{B}}(\tilde{H}^n(K^{\mathfrak{p}}, E)_{\mathrm{loc.an.}})^{(\chi,\lambda)},$$
$$(\chi, \lambda) \in \hat{T} \times \mathrm{Spec}\,\mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}.$$

The next result tells us that this restriction is often an isomorphism.

THEOREM 14 (Theorem 4.4.5 of [17]). *Let* $V$ *be an admissible continuous representation of* $\mathbb{G}(k_{\mathfrak{p}})$ *on a Banach space. If* $\chi := \theta\psi_W \in \hat{T}(\bar{E})$ *is of non-critical slope, then the closed embedding*

$$\mathrm{Jacq}_{\mathbb{B}}(V_{W-\mathrm{loc.alg.}}) \to \mathrm{Jacq}_{\mathbb{B}}(V_{\mathrm{loc.an.}})$$

*induces an isomorphism on* $\chi$*-eigenspaces.*

We recall Theorem 3.

THEOREM. *If the weak Emerton criterion holds for* $\mathbb{G}$ *in dimension* $n$*, then*

1. *Projection onto the first factor induces a finite map* $E(n, K^{\mathfrak{p}}) \to \hat{T}$.
2. *The map* $E(n, K^{\mathfrak{p}}) \to \check{\mathfrak{t}}$ *has discrete fibres.*
3. *If* $(\chi, \lambda)$ *is a point of the Eigencurve such that* $\chi$ *is locally algebraic and of non-critical slope, then either* $(\chi, \lambda)$ *is a classical point or* $(\chi, \lambda)$ *is the trivial representation of* $\mathbb{T}(k_{\mathfrak{p}}) \times \mathcal{H}(K^{\mathfrak{p}})^{\mathrm{sph}}$.

*Proof.* To prove the first two parts of the theorem, it is sufficient to show that $E(K^{\mathfrak{p}}, n)$ is a closed subspace of Spec $\mathcal{A}$. Since $E(n, K^{\mathfrak{p}})$ is defined to be the closure of the set of classical points, it suffices to show that each classical point is in Spec $\mathcal{A}$.
Suppose $\pi$ is a representation appearing in $H^n_{\text{class.}}(K^{\mathfrak{p}}, W)$ and let $(\theta\psi_W, \lambda)$ be the corresponding classical point. This means that the $(\theta, \lambda)$-eigenspace in the $\mathrm{Jacq}_{\mathbb{B}}(\pi)$ is non-zero. By exactness of the Jacquet functor on smooth representations, it follows that the $(\theta, \lambda)$ eigenspace in the Jacquet module of $H^n_{\text{class.}}(K^{\mathfrak{p}}, W)$ is non-zero. Hence by Proposition 4.3.6 of [17], the $(\theta\psi_W, \lambda)$-eigenspace in the Jacquet module of $H^n_{\text{class.}}(K^{\mathfrak{p}}, W) \otimes \check{W}$ is non-zero. By left-exactness of the Jacquet functor, it follows that the $(\theta\psi_W, \lambda)$ eigenspace in the Jacquet module of $\tilde{H}^n(\mathbb{G}, E)_{\text{loc.an.}}$ is non-zero. Hence by Theorem 13 (iii) it follows that the classical point is in Spec $\mathcal{A}$.
If $(\theta\psi, \lambda)$ is of non-critical slope then Theorem 14 shows that the converse also holds. □

## 5 Proof of Theorem 8

We first recall the statement:

Theorem. *Let $\mathbb{G}/k$ be connected, semi-simple and algebraically simply connected and assume that the weak Emerton criterion holds in dimension $n$. Assume also, that at least one of the following two conditions holds:*

*(a) $H^p_{\text{class.}}(\mathbb{G}, \mathbb{C})$ is given by constants in dimensions $p < n$ and $H^{n+1}_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$; or*

*(b) $\mathbb{G}(k)$ is cocompact in $\mathbb{G}(\mathbb{A})$.*

*Then all the conclusions of Theorem 1 hold for the eigenvariety $E(n, K^{\mathfrak{p}})$.*

*Proof.* To prove the theorem, we shall find a continuous admissible Banach space representation $V$, such that for every irreducible algebraic representation $W$, there is an isomorphism of smooth $\mathbb{G}(\mathbb{A}_f)$-modules

$$H^n_{\text{class.}}(\mathbb{G}, W) \cong \mathrm{Hom}_{\mathfrak{g}}(\check{W}, V_{\text{loc.an.}}). \tag{8}$$

Recall that by the weak Emerton criterion, we have an exact sequence of smooth $\mathbb{G}(\mathbb{A}_f)$-modules

$$0 \to H^n_{\text{class.}}(\mathbb{G}, E) \to \tilde{H}^n(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} \to E^r \to 0, \qquad r \geq 0. \tag{9}$$

It follows, either from Lemma 1 or from Lemma 2 below, that all such sequences split. We therefore have a subspace $E^r \subset \tilde{H}^n(\mathbb{G}, E)$, on which $\mathbb{G}(\mathbb{A}_f)$ acts trivially. We define $V$ to be the quotient, so that there is an exact sequence of admissible continuous representations of $\mathbb{G}(\mathbb{A}_f)$ on $E$-Banach spaces.

$$0 \to E^r \to \tilde{H}^n(\mathbb{G}, E) \to V \to 0. \tag{10}$$

Taking $\mathfrak{g}$-invariants of (10) and applying Whitehead's first lemma (Theorem 10), we have an exact sequence:

$$0 \to E^r \to \tilde{H}^n(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} \to V^{\mathfrak{g}}_{\text{loc.an.}} \to 0. \tag{11}$$

On the other hand, $E^r$ is a direct summand of $\tilde{H}^n(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}}$, so this sequence also splits. Comparing (9) and (11), we obtain

$$H^n_{\text{class.}}(\mathbb{G}, E) = V^{\mathfrak{g}}_{\text{loc.an.}} = \mathrm{Hom}_{\mathfrak{g}}(E, V_{\text{loc.an.}}).$$

This verifies (8) in the case that $W$ is the trivial representation.
Now taking $W$ to be a non-trivial irreducible representation, and applying $\mathrm{Hom}_{\mathfrak{g}}(\check{W}, -_{\text{loc.an.}})$ to (10), we obtain a long exact sequence:

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n(\mathbb{G}, E)_{\text{loc.an.}}) \to \mathrm{Hom}_{\mathfrak{g}}(\check{W}, V_{\text{loc.an.}}) \to \mathrm{Ext}^1_{\mathfrak{g}}(\check{W}, E^r).$$

By Whitehead's first lemma, the final term above is zero. Hence, by the weak Emerton criterion, we have:

$$H^2_{\text{class.}}(\mathbb{G}, W) = \mathrm{Hom}_{\mathfrak{g}}(\check{W}, \tilde{H}^n(\mathbb{G}, E)_{\text{loc.an.}}) = \mathrm{Hom}_{\mathfrak{g}}(\check{W}, V_{\text{loc.an.}}).$$

□

LEMMA 1. *Assume that* $H^q_{\text{class.}}(\mathbb{G}, \mathbb{C})$ *is given by constants in dimensions* $q < n$ *and* $H^{n+1}_{\text{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. *Then*

$$\mathrm{Ext}^1_{\mathbb{G}(\mathbb{A}_f)}(E, H^n_{\text{class.}}(\mathbb{G}, E)) = 0,$$

*where the* Ext*-group is calculated from the category of smooth representations of* $\mathbb{G}(\mathbb{A}_f)$ *over* $E$.

*Proof.* Since we are dealing with smooth representations, the topology of $E$ plays no role, so it is sufficient to prove that

$$\mathrm{Ext}^1_{\mathbb{G}(\mathbb{A}_f)}(\mathbb{C}, H^n_{\text{class.}}(\mathbb{G}, \mathbb{C})) = 0,$$

To prove this, it is sufficient to show that for every sufficiently large finite set $S$ of finite primes of $k$, we have

$$\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, H^n_{\text{class.}}(\mathbb{G}, \mathbb{C})) = 0.$$

For this, we shall use the spectral sequence of Borel (§3.9 of [7]; see also §2 of [6]):

$$\mathrm{Ext}^p_{\mathbb{G}(k_S)}(E, H^q_{\text{class.}}(\mathbb{G}, \mathbb{C})) \implies H^{p+q}_{S-\text{class.}}(\mathbb{G}, \mathbb{C}),$$

where $H^\bullet_{S-\text{class.}}(\mathbb{G}, -)$ denotes the direct limit over all $S$-congruence subgroups:

$$H^\bullet_{S-\text{class.}}(\mathbb{G}, -) = \varinjlim_{K^S} H^\bullet_{\text{Group}}(\Gamma_S(K^S), -),$$

$$\Gamma_S(K^S) = \mathbb{G}(k) \cap \left(\mathbb{G}(k_{\infty \cup S}) \times K^S\right).$$

By Proposition X.4.7 of [9], we know that

$$\mathrm{Ext}^p_{\mathbb{G}(k_S)}(\mathbb{C}, \mathbb{C}) = 0, \qquad p \geq 1.$$

Since $H^q_{\mathrm{class.}}(\mathbb{G}, \mathbb{C})$ is a trivial representation of $\mathbb{G}(\mathbb{A}_f)$ in dimensions $q < n$, it follows from Borel's spectral sequence that $\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, H^n_{\mathrm{class.}}(\mathbb{G}, \mathbb{C}))$ injects into $H^{n+1}_{S-\mathrm{class.}}(\mathbb{G}, \mathbb{C})$. On the other hand, it is shown in Theorem 1 of [6], that for $S$ sufficiently large, $H^{n+1}_{S-\mathrm{class.}}(\mathbb{G}, \mathbb{C})$ is isomorphic to $H^{n+1}_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C})$. Under the hypothesis that $H^{n+1}_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$, it follows that for $S$ sufficiently large, $\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, H^n_{\mathrm{class.}}(\mathbb{G}, \mathbb{C})) = 0$. □

Lemma 2. *Assume that $\mathbb{G}(k)$ is cocompact in $\mathbb{G}(\mathbb{A})$. Then*

$$\mathrm{Ext}^1_{\mathbb{G}(\mathbb{A}_f)}(E, H^n_{\mathrm{class.}}(\mathbb{G}, E)) = 0,$$

*where the* Ext*-group is calculated from the category of smooth representations of $\mathbb{G}(\mathbb{A}_f)$ over $E$.*

(The argument in fact shows that $\mathrm{Ext}^p_{\mathbb{G}(\mathbb{A}_f)}(E, H^q_{\mathrm{class.}}(\mathbb{G}, E)) = 0$ for all $p > 0$.)

*Proof.* As in the proof of the previous lemma, we shall show that for $S$ sufficiently large,

$$\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, H^n_{\mathrm{class.}}(\mathbb{G}, \mathbb{C})) = 0.$$

Recall that we have a decomposition:

$$L^2(\mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})) = \widehat{\bigoplus_{\pi}} m(\pi) \cdot \pi,$$

with finite multiplicities $m(\pi)$ and automorphic representations $\pi$. Here the $\hat{\oplus}$ denotes a Hilbert space direct sum. We shall write $\pi = \pi_\infty \otimes \pi_f$, where $\pi_\infty$ is an irreducible unitary representation of $\mathbb{G}(k_\infty)$ and $\pi_f$ is a smooth irreducible unitary representation of $\mathbb{G}(\mathbb{A}_f)$. This decomposition may by used to calculate the classical cohomology (Theorem VII.6.1 of [9]):

$$H^\bullet_{\mathrm{class.}}(\mathbb{G}, \mathbb{C}) = \bigoplus_{\pi} m(\pi) \cdot H^\bullet_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \pi_\infty) \otimes \pi_f.$$

It is therefore sufficient to show that for each automorphic representation $\pi$, we have (for $S$ sufficiently large) $\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C}, \pi_f) = 0$. The smooth representation $\pi_f$ decomposes as a tensor product of representations of $\mathbb{G}(k_\mathfrak{q})$ for $\mathfrak{q} \in S$, together with a representation of $\mathbb{G}(\mathbb{A}_f^S)$:

$$\pi_f = \left(\bigotimes_{\mathfrak{q} \in S} \pi_\mathfrak{q}\right) \otimes \pi_f^S.$$

This gives a decomposition of the cohomology:

$$\mathrm{Ext}^{\bullet}_{\mathbb{G}(k_S)}(\mathbb{C},\pi_f)=\left(\bigotimes_{\mathfrak{q}\in S}\mathrm{Ext}^{\bullet}_{\mathbb{G}(k_{\mathfrak{q}})}(\mathbb{C},\pi_{\mathfrak{q}})\right)\otimes\pi_f^S. \tag{12}$$

There are two cases to consider.
*Case 1.* Suppose $\pi$ is the trivial representation, consisting of the constant functions on $\mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A})$. Then by Proposition X.4.7 of [9], we have

$$\mathrm{Ext}^n_{\mathbb{G}(k_{\mathfrak{q}})}(\mathbb{C},\mathbb{C})=0,\qquad n\geq 1.$$

This implies by (12) that $\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C},\mathbb{C})=0$.
*Case 2.* Suppose $\pi$ is non-trivial, and hence contains no non-zero constant functions. If $\mathfrak{q}$ is a prime for which no factor of $\mathbb{G}(k_{\mathfrak{q}})$ is compact, then it follows from the strong approximation theorem that the local representation $\pi_{\mathfrak{q}}$ is non-trivial. This implies that

$$\mathrm{Ext}^0_{\mathbb{G}(k_{\mathfrak{q}})}(\mathbb{C},\pi_{\mathfrak{q}})=\mathrm{Hom}_{\mathbb{G}(k_{\mathfrak{q}})}(\mathbb{C},\pi_{\mathfrak{q}})=0.$$

If $S$ contains at least two such primes, then we have by (12)

$$\mathrm{Ext}^1_{\mathbb{G}(k_S)}(\mathbb{C},\pi_f)=0.$$

□

*Remark.* At first sight, it might appear that $\mathrm{Ext}^1_{\mathbb{G}(\mathbb{A}_f)}(\mathbb{C},H^n_{\text{class.}}(\mathbb{G},\mathbb{C}))$ should always be zero; however this is not the case. For example, if $\mathbb{G}=\mathrm{SL}_2/\mathbb{Q}$ then

$$\mathrm{Ext}^1_{\mathrm{SL}_2(\mathbb{A}_f)}(\mathbb{C},H^1_{\text{class.}}(\mathrm{SL}_2/\mathbb{Q},\mathbb{C}))=\mathbb{C}.$$

This may be verified using the spectral sequence of Borel cited above, together with the fact that $H^2_{\text{rel.Lie}}(\mathfrak{sl}_2,\mathrm{SO}(2),\mathbb{C})=\mathbb{C}$.

## 6 Proof of Theorem 6

We assume in this section that $\mathbb{G}/k$ is connected, simple and algebraically simply connected, and that $\mathbb{G}(k_\infty)$ is not compact.

Proposition 1. *As topological spaces, we have $Y(K_f)=\Gamma(K_f)\backslash\mathbb{G}(k_\infty)/K_\infty$.*

*Proof.* By the strong approximation theorem (Theorem 4), $\mathbb{G}(k)\mathbb{G}(k_\infty)$ is a dense subgroup of $\mathbb{G}(\mathbb{A})$. Since $\mathbb{G}(k_\infty)K_f$ is open in $\mathbb{G}(\mathbb{A})$, this implies that $\mathbb{G}(k)\mathbb{G}(k_\infty)K_f$ is a dense, open subgroup of $\mathbb{G}(\mathbb{A})$. Since open subgroups are closed it follows that

$$\mathbb{G}(k)\mathbb{G}(k_\infty)K_f=\mathbb{G}(\mathbb{A}).$$

Quotienting out on the left by $\mathbb{G}(k)$, we have (as coset spaces):

$$(\mathbb{G}(k)\cap\mathbb{G}(k_\infty)K_f)\backslash(\mathbb{G}(k_\infty)K_f)=\mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A}).$$

Substituting the definition of $\Gamma(K_f)$, we have:

$$\Gamma(K_f)\backslash\mathbb{G}(k_\infty)K_f = \mathbb{G}(k)\backslash\mathbb{G}(\mathbb{A}).$$

Quotienting out on the right by $K_\infty K_f$, we get:

$$\Gamma(K_f)\backslash\mathbb{G}(k_\infty)/K_\infty = Y(K_f).$$

□

In particular, this implies:

Corollary 2. *$Y(K_f)$ is connected.*

*Proof.* $\mathbb{G}(k_\infty)$ is connected. □

If $K_f$ is sufficiently small then the group $\Gamma(K_f)$ is torsion-free. We shall assume that this is the case. Hence $Y(K_f)$ is a manifold. Its universal cover is $\mathbb{G}(\mathbb{R})/K$, and its fundamental group is $\Gamma(K_f)$.

Corollary 3. *If $\Gamma(K_f)$ is torsion-free then $H^\bullet(Y(K_f),-) = H^\bullet_{\mathrm{Group}}(\Gamma(K_f),-)$.*

*Proof.* This follows because $\Gamma(K_f)$ is the fundamental group of $Y(K_f)$, and the universal cover $\mathbb{G}(k_\infty)/K_\infty$ is contractible. See for example [36]. □

Corollary 4. *Let $\mathbb{G}/k$ be connected, simple, simply connected and assume $\mathbb{G}(k_\infty)$ is not compact. Then as $\mathbb{G}(\mathbb{A}_f)$-modules,*

$$\tilde{H}^0(\mathbb{G},E)_{\mathrm{loc.an.}} = \tilde{H}^0(\mathbb{G},E) = E.$$

*Proof.* Since every $Y(K_f)$ is connected, we have a canonical isomorphism:

$$H^0(Y(K_\mathfrak{p}K^\mathfrak{p}),\mathbb{Z}/p^s) = \mathbb{Z}/p^s.$$

Furthermore, the pull-back maps

$$H^0(Y(K_\mathfrak{p}K^\mathfrak{p}),\mathbb{Z}/p^s) \to H^0(Y(K'_\mathfrak{p}K^\mathfrak{p}),\mathbb{Z}/p^s) \qquad (K'_\mathfrak{p}\subset K_\mathfrak{p})$$

are all the identity on $\mathbb{Z}/p^s$. It follows that

$$\varinjlim_{K_\mathfrak{p}} H^0(Y(K^\mathfrak{p}K_\mathfrak{p}),\mathbb{Z}/p^s) = \mathbb{Z}/p^s.$$

Since the pull-back maps are all the identity, it follows that the action of $\mathbb{G}(k_\mathfrak{p})$ on this group is trivial. Taking the projective limit over $s$ and tensoring with $E$ we find that

$$\tilde{H}^0(K^\mathfrak{p},E) = E.$$

The action of $\mathbb{G}(k_\mathfrak{p})$ is clearly still trivial, and hence every vector is locally analytic. The groups $\tilde{H}^0(K^\mathfrak{p},E)$ for varying tame level $K^\mathfrak{p}$ form a direct system with respect to the pullback maps. These pullback maps are all the identity on $E$. Taking the direct limit over the tame levels, we obtain:

$$\tilde{H}^0(\mathbb{G}, E) = E.$$

Since the pullback maps are all the identity, it follows that the action of $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ on $\tilde{H}^0(\mathbb{G}, E)$ is trivial. □

## 7 Some Cohomology Theories

In this section we introduce some notation and recall some results, which will be needed in the proof of Theorem 7. It is worth mentioning that the theorem is much easier to prove in the case that $\mathbb{G}$ has finite congruence kernel. In that case one quite easily shows that $\tilde{H}^1 = 0$ by truncating the proof given in §8 shortly after the end of "step 1" of the proof. Furthermore, our main application (Theorem 5) requires only this easier case.

### 7.1 Discrete cohomology

Let $G$ be a profinite group acting on an abelian group $A$. We say that the action is *smooth* if every element of $A$ has open stabilizer in $G$. For a smooth $G$-module $A$, we define $H^\bullet_{\mathrm{disc.}}(G, A)$ to be the cohomology of the complex of smooth cochains on $G$ with values in $A$. Due to compactness, cochains take only finitely many values, so we have

$$H^\bullet_{\mathrm{disc.}}(G, A) = \lim_{\overrightarrow{U}} H^\bullet_{\mathrm{Group}}(G/U, A^U).$$

Here the limit is taken over the open normal subgroups $U$ of $G$, and the cohomology groups on the right hand side are those of finite groups.

Theorem 15 (Hochschild–Serre spectral sequence (§2.6b of [35])). *Let $G$ be a profinite group and $A$ a discrete $G$-module on which $G$ acts smoothly. Let $H$ be a closed, normal subgroup. Then there is a spectral sequence:*

$$H^p_{\mathrm{disc.}}(G/H, H^q_{\mathrm{disc.}}(H, A)) \implies H^{p+q}_{\mathrm{disc.}}(G, A).$$

For calculations with adèle groups, we need the following result on countable products of groups.

Proposition 2 (see §2.2 of [35]). *Let*

$$G = \prod_{i \in \mathbb{N}} G_i$$

*be a countable product of profinite groups and let $A$ be a discrete $G$-module. For any finite subset $S \subset \mathbb{N}$ we let*

$$G_S = \prod_{i \in S} G_i.$$

*Then*

$$H^n_{\text{disc.}}(G, A) = \varinjlim_{S} H^n_{\text{disc.}}(G_S, A).$$

*Here the limit is taken over all finite subsets with respect to the inflation maps.*

Corollary 5. *Let $G$ and $A$ be as in the previous proposition, and assume that the action of $G$ on $A$ is trivial. Assume also that for a fixed $n$, we have:*

$$H^r_{\text{disc.}}(G_i, A) = 0, \quad r = 1, \dots, n-1, \ i \in \mathbb{N}.$$

*Then*

$$H^n_{\text{disc.}}(G, A) = \bigoplus_{i \in \mathbb{N}} H^n_{\text{disc.}}(G_i, A).$$

*Proof.* Let $S \subset \mathbb{N}$ be a finite set and let $i \notin S$. We have a direct sum decomposition

$$G_{S \cup \{i\}} = G_S \oplus G_i.$$

Regarding this as a (trivial) group extension, we have a spectral sequence:

$$H^p_{\text{disc.}}(G_S, H^q(G_i, A)) \implies H^{p+q}_{\text{disc.}}(G_{S \cup \{i\}}, A).$$

since the sum is direct, it follows that all the maps in the spectral sequence are zero, and we have

$$H^n_{\text{disc.}}(G_{S \cup \{i\}}, A) = \bigoplus_{r=0}^{n} H^{n-r}_{\text{disc.}}(G_S, H^r_{\text{disc.}}(G_i, A)).$$

By our hypothesis, most of these terms vanish, and we are left with:

$$H^n_{\text{disc.}}(G_{S \cup \{i\}}, A) = H^n_{\text{disc.}}(G_S, A) \oplus H^n_{\text{disc.}}(G_i, A).$$

By induction on the size of $S$, we deduce that

$$H^n_{\text{disc.}}(G_S, A) = \bigoplus_{i \in S} H^n_{\text{disc.}}(G_i, A).$$

The result follows from the previous proposition. □

### 7.2 Continuous cohomology

Again suppose that $G$ is a profinite group, acting on an abelian topological group $A$. We call $A$ a continuous $G$-module if the map $G \times A \to A$ is continuous. For a continuous $G$-module $A$, we define the continuous cohomology groups $H^\bullet_{\text{cts}}(G, A)$ to be the cohomology of the complex of continuous cochains. If the topology on $A$ is actually discrete then continuous cochains are in fact smooth, so we have

$$H^\bullet_{\text{cts}}(G, A) = H^\bullet_{\text{disc.}}(G, A).$$

## 7.3 Derived functors of inverse limit

Let **Ab** be the category of abelian groups. By a projective system in **Ab**, we shall mean a collection of objects $A_s$ ($s \in \mathbb{N}$) and morphisms $\phi : A_{s+1} \to A_s$. We shall write $\mathbf{Ab}^{\mathbb{N}}$ for the category of projective systems in **Ab**. There is a functor

$$\varprojlim_s \,:\, \mathbf{Ab}^{\mathbb{N}} \to \mathbf{Ab}.$$

This functor is left-exact. It has right derived functors

$$\left(\varprojlim_s\right)^{\bullet} : \mathbf{Ab}^{\mathbb{N}} \to \mathbf{Ab}.$$

It turns out that $\left(\varprojlim_s\right)^{n}$ is zero for $n \geq 2$. The first derived functor has the following simple description due to Eilenberg. We define a homomorphism

$$\Delta : \prod_s A_s \to \prod_s A_s, \qquad \big(\Delta(a_\bullet)\big)_s = a_s - \phi(a_{s+1}).$$

With this notation we have

$$\varprojlim_s \; A_s = \ker \Delta.$$

Eilenberg showed that

$$\varprojlim_s{}^1 A_s = \operatorname{coker} \Delta.$$

A projective system $A_s$ is said to satisfy the *Mittag–Leffler condition* if for every $s \in \mathbb{N}$ there is a $t \geq s$ such that for every $u \geq t$ the image of $A_u$ in $A_s$ is equal to the image of $A_t$ in $A_s$.

Proposition 3 (Proposition 3.5.7 of [39]). *If $A_s$ satisfies the Mittag–Leffler condition then $\varprojlim_s{}^1 A_s = 0$.*

This immediately implies:

Corollary 6 (Exercise 3.5.2 of [39]). *If $A_s$ is a projective system of finite abelian groups then $\varprojlim_s{}^1 A_s = 0$.*

We shall use the derived functor $\varprojlim_s{}^1$ to pass between discrete and continuous cohomology:

Theorem 16 (Eilenberg–Moore Sequence (Theorem 2.3.4 of [27])). *Let $G$ be a profinite group and $A$ a projective limit of finite discrete continuous $G$-modules*

$$A = \varprojlim_s \; A_s.$$

*Then there is an exact sequence:*

$$0 \to \varprojlim_s{}^1 H^{n-1}_{\text{disc.}}(G, A_s) \to H^n_{\text{cts}}(G, A) \to \varprojlim_s \; H^n_{\text{disc.}}(G, A_s) \to 0.$$

### 7.4 Stable Cohomology

For a continuous representation $V$ of $\mathbb{G}(k_{\mathfrak{p}})$ over $E$, we shall write $V_{\text{st}}$ for the set of smooth vectors. The functor $V \mapsto V_{\text{st}}$ is left exact from the category of continuous admissible representations of $\mathbb{G}(k_{\mathfrak{p}})$ (in the sense of [33]) to the category of smooth representations. We shall write $H^\bullet_{\text{st}}(\mathbb{G}(k_{\mathfrak{p}}), -)$ for the right-derived functors. This is called "stable cohomology" by Emerton (Definition 1.1.5 of [18]). It turns out that stable cohomology may be expressed in terms of continuous group cohomology as follows (Proposition 1.1.6 of [18]):

$$H^\bullet_{\text{st}}(\mathbb{G}(k_{\mathfrak{p}}), V) = \varinjlim_{K_{\mathfrak{p}}} H^\bullet_{\text{cts}}(K_{\mathfrak{p}}, V).$$

There is an alternative description of these derived functors which we shall also use. Let $V_{\text{loc.an.}}$ denote the subspace of locally analytic vectors in $V$. There is an action of the Lie algebra $\mathfrak{g}$ on $V_{\text{loc.an.}}$. Stable cohomology may be expressed in terms of Lie algebra cohomology as follows (Theorem 1.1.13 of [18]):

$$H^\bullet_{\text{st}}(\mathbb{G}(k_{\mathfrak{p}}), V) = H^\bullet_{\text{Lie}}(\mathfrak{g}, V_{\text{loc.an.}}). \tag{13}$$

## 8 Proof of Theorem 7

In this section, we shall assume that $\mathbb{G}/k$ is connected, simply connected and simple, and that $\mathbb{G}(k_\infty)$ is not connected. We regard the vector space $\text{Hom}_{\text{cts}}(\text{Cong}(\mathbb{G}), \mathbb{Q}_p)$ as a $p$-adic Banach space with the supremum norm:

$$||\phi|| = \sup_{x \in \text{Cong}(\mathbb{G})} |\phi(x)|_p.$$

We regard $\text{Hom}_{\text{cts}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)$ as a discrete abelian group. The group $\mathbb{G}(\mathbb{A}_f)$ acts on these spaces as follows:

$$(g\phi)(x) = \phi(g^{-1}xg), \quad g \in \mathbb{G}(\mathbb{A}_f),\ x \in \text{Cong}(\mathbb{G}).$$

Lemma 3. *The action of $\mathbb{G}(\mathbb{A}_f)$ on $\text{Hom}_{\text{cts}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)$ is smooth.*

*Proof.* One may prove this directly; however it is implicit in the Hochschild–Serre spectral sequence. It is sufficient to show that the action of some open subgroup is smooth. Let $K_f$ be a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$, and write write $\tilde{K}_f$ for the preimage of $K_f$ in $\tilde{\mathbb{G}}(k)$. We therefore have an extension of profinite groups:

$$1 \to \text{Cong}(\mathbb{G}) \to \tilde{K}_f \to K_f \to 1.$$

We shall regard $\mathbb{Z}/p^s$ as a trivial, and hence smooth, $\tilde{K}_f$-module. It follows that each $H^q_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)$ is a smooth $K_f$-module. On the other hand we have

$$\text{Hom}_{\text{cts}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s) = H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s).$$

□

Lemma 4. *The action of* $\mathbb{G}(\mathbb{A}_f)$ *on the* $p$*-adic Banach space* $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)$ *is continuous.*

*Proof.* It is sufficient to prove this for the open submodule $\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)$. We have, as topological $\mathbb{G}(\mathbb{A}_f)$-modules:

$$\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p) = \varprojlim_s \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}/p^s).$$

Continuity follows from the previous Lemma. □

We shall say that a vector $v \in \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)$ is $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$-smooth if its stabilizer in $\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})$ is open. The set of such vectors will be written

$$\mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}_f^{\mathfrak{p}})-\mathrm{smooth}}.$$

Theorem. *Assume* $\mathbb{G}/k$ *is connected, simple and simply connected, and that* $\mathbb{G}(k_\infty)$ *is not compact. Then we have an isomorphism of* $\mathbb{G}(\mathbb{A}_f)$*-modules:*

$$\tilde{H}^1(\mathbb{G},\mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}^{\mathfrak{p}})-\mathrm{smooth}}.$$

*Proof.* Choose a level $K_f$ small enough so that $\Gamma(K_f)$ is torsion-free. By Corollary 3, we have:

$$H^1(Y(K_f),\mathbb{Z}/p^s) = H^1_{\mathrm{Group}}(\Gamma(K_f),\mathbb{Z}/p^s).$$

Elements of $H^1_{\mathrm{Group}}(\Gamma(K_f),\mathbb{Z}/p^s)$ are group homomorphisms $\Gamma(K_f)\to\mathbb{Z}/p^s$. Let $\tilde{K}_f$ be the preimage of $K_f$ in $\tilde{\mathbb{G}}(k)$; this is equal to the profinite completion of $\Gamma(K_f)$. It follows that homomorphisms $\Gamma(K_f)\to\mathbb{Z}/p^s$ correspond bijectively to continuous homomorphisms $\tilde{K}_f\to\mathbb{Z}/p^s$. We therefore have:

$$H^1(Y(K_f),\mathbb{Z}/p^s) = H^1_{\mathrm{disc.}}(\tilde{K}_f,\mathbb{Z}/p^s).$$

We have an extension of profinite groups:

$$1\to\mathrm{Cong}(\mathbb{G})\to\tilde{K}_f\to K_f\to 1.$$

This gives rise to a Hochschild–Serre spectral sequence (Theorem 15):

$$H^p_{\mathrm{disc.}}(K_f,H^q_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}/p^s)) \implies H^{p+q}_{\mathrm{disc.}}(\tilde{K}_f,\mathbb{Z}/p^s).$$

From this we have an inflation-restriction sequence containing the following terms:

$$\begin{aligned} 0\to H^1_{\mathrm{disc.}}(K_f,\mathbb{Z}/p^s)\to H^1(Y(K_f),\mathbb{Z}/p^s)\to \\ \to H^1_{\mathrm{disc.}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}/p^s)^{K_f}\to H^2_{\mathrm{disc.}}(K_f,\mathbb{Z}/p^s) \end{aligned} \tag{14}$$

The proof of the theorem consists of applying the functors $\varinjlim_{K_{\mathfrak{p}}}$, $\varprojlim_s$, $-\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $\varinjlim_{K^{\mathfrak{p}}}$ to the sequence (14).

*Step 1.* We first substitute $K_f = K_\mathfrak{p} K^\mathfrak{p}$, and apply the functor $\varinjlim_{K_\mathfrak{p}}$ to (14). We have by the Künneth formula:

$$\varinjlim_{K_\mathfrak{p}} H^1_{\text{disc.}}(K_\mathfrak{p} K^\mathfrak{p}, \mathbb{Z}/p^s) = H^1_{\text{disc.}}(K^\mathfrak{p}, \mathbb{Z}/p^s),$$

$$\varinjlim_{K_\mathfrak{p}} H^2_{\text{disc.}}(K_\mathfrak{p} K^\mathfrak{p}, \mathbb{Z}/p^s) = H^2_{\text{disc.}}(K^\mathfrak{p}, \mathbb{Z}/p^s).$$

By Lemma 3 we have:

$$\varinjlim_{K_\mathfrak{p}} H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K_\mathfrak{p} K^\mathfrak{p}} = H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^\mathfrak{p}}.$$

Since the functor $\varinjlim_{K_\mathfrak{p}}$ is exact, the sequence remains exact:

$$\begin{aligned} 0 \to H^1_{\text{disc.}}(K^\mathfrak{p}, \mathbb{Z}/p^s) \to \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) \to \\ \to H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^\mathfrak{p}} \to H^2_{\text{disc.}}(K^\mathfrak{p}, \mathbb{Z}/p^s). \end{aligned} \tag{15}$$

*Interlude.* Before going on, we make some restrictions on the tame level $K^\mathfrak{p}$, and investigate the first and last terms in the sequence (15).
We shall assume that the tame level $K^\mathfrak{p}$ is a product of local factors:

$$K^\mathfrak{p} = \prod_{\mathfrak{q} \neq \mathfrak{p}} K_\mathfrak{q},$$

where each $K_\mathfrak{q}$ is a compact open subgroup of $\mathbb{G}(k_\mathfrak{q})$. Consider the following sets of finite primes of $k$:

$$S = \{\mathfrak{q} : \mathfrak{q}|p \text{ and } \mathfrak{q} \neq \mathfrak{p}\},$$

$$T = \{\mathfrak{q} : \mathfrak{q} \not| p \text{ and } K_\mathfrak{q} \neq [K_\mathfrak{q}, K_\mathfrak{q}]\}.$$

Both these sets are finite. We shall also assume from now on that for each prime $\mathfrak{q} \in T$, the group $K_\mathfrak{q}$ is chosen small enough so that it is a pro-$q$ group, where $q$ is the rational prime below $\mathfrak{q}$. In particular, for each $\mathfrak{q} \in T$ we have for $n \geq 1$,

$$H^n_{\text{disc.}}(K_\mathfrak{q}, \mathbb{Z}/p^s) = 0. \tag{16}$$

We have a decomposition of $K^\mathfrak{p}$:

$$K^\mathfrak{p} = K_S \times K_T \times K^{S \cup T \cup \{\mathfrak{p}\}}, \tag{17}$$

where we are using the notation:

$$K_S = \prod_{\mathfrak{q} \in S} K_\mathfrak{q}, \quad K^S = \prod_{\mathfrak{q} \notin S} K_\mathfrak{q}.$$

By the Künneth formula and (16), (17), we have:

$$H^{\bullet}_{\mathrm{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = H^{\bullet}_{\mathrm{disc.}}(K_S K^{S\cup T\cup\{\mathfrak{p}\}}, \mathbb{Z}/p^s). \tag{18}$$

By assumption, the group $K^{S\cup T\cup\{\mathfrak{p}\}}$ is perfect, so we have

$$H^{1}_{\mathrm{disc.}}(K^{S\cup T\cup\{\mathfrak{p}\}}, \mathbb{Z}/p^s) = 0. \tag{19}$$

Again by the Künneth formula together with (18), (19), we have:

$$H^{1}_{\mathrm{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = H^{1}_{\mathrm{disc.}}(K_S, \mathbb{Z}/p^s). \tag{20}$$

$$H^{2}_{\mathrm{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = H^{2}_{\mathrm{disc.}}(K_S, \mathbb{Z}/p^s) \oplus H^{2}_{\mathrm{disc.}}(K^{S\cup T\cup\{\mathfrak{p}\}}, \mathbb{Z}/p^s). \tag{21}$$

For each prime $\mathfrak{q} \notin S \cup T \cup \{\mathfrak{p}\}$, there is an open normal pro-$q$ subgroup $L_{\mathfrak{q}} \subset K_{\mathfrak{q}}$. We shall write $G(\mathfrak{q})$ for the (finite) quotient group. We therefore have a Hochschild–Serre spectral sequence:

$$H^{p}_{\mathrm{Group}}(G(\mathfrak{q}), H^{\mathfrak{q}}_{\mathrm{disc.}}(L_{\mathfrak{q}}, \mathbb{Z}/p^s)) \implies H^{p+q}_{\mathrm{disc.}}(K_{\mathfrak{q}}, \mathbb{Z}/p^s).$$

This spectral sequence degenerates: for $n \geq 1$ we have

$$H^n(L_{\mathfrak{q}}, \mathbb{Z}/p^s) = 0.$$

Hence,

$$H^{\bullet}_{\mathrm{disc.}}(K_{\mathfrak{q}}, \mathbb{Z}/p^s) = H^{\bullet}_{\mathrm{Group}}(G(\mathfrak{q}), \mathbb{Z}/p^s), \qquad \mathfrak{q} \notin S \cup T \cup \{\mathfrak{p}\}. \tag{22}$$

Since $G(\mathfrak{q})$ is a finite perfect group, it has a universal central extension. We shall write $\pi_1(G(\mathfrak{q}))$ for the kernel of this extension, i.e. the Schur multiplier of $G(\mathfrak{q})$. By (22) we have:

$$H^{2}_{\mathrm{disc.}}(K_{\mathfrak{q}}, \mathbb{Z}/p^s) = \mathrm{Hom}_{\mathrm{Group}}(\pi_1(G(\mathfrak{q})), \mathbb{Z}/p^s). \tag{23}$$

By Corollary 5 and (23) we have:

$$H^{2}_{\mathrm{disc.}}(K^{S\cup T\cup\{\mathfrak{p}\}}, \mathbb{Z}/p^s) = \bigoplus_{\mathfrak{q}\notin S\cup T\cup\{\mathfrak{p}\}} \mathrm{Hom}_{\mathrm{Group}}(\pi_1(G(\mathfrak{q})), \mathbb{Z}/p^s). \tag{24}$$

From (21) and (24) we have:

$$H^{2}_{\mathrm{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = H^2(K_S, \mathbb{Z}/p^s) \oplus \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{S\cup T\cup\{\mathfrak{p}\}}, \mathbb{Z}/p^s), \tag{25}$$

where we are using the notation

$$\pi_1^{S\cup T\cup\{\mathfrak{p}\}} = \prod_{\mathfrak{q}\notin S\cup T\cup\{\mathfrak{p}\}} \pi_1(G(\mathfrak{q})).$$

The only property of $\pi_1^{S\cup T\cup\{\mathfrak{p}\}}$ which we require, is that it is a product of finite groups, not depending on $s$.

*Step 2.* We are now ready to apply the functor $\varprojlim_s$ to the sequence (15). To keep track of the exactness, we splice the sequence (15) into two:

$$0 \to H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \to \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) \to A(s) \to 0, \tag{26}$$

$$0 \to A(s) \to H^1_{\text{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^\mathfrak{p}} \to H^2_{\text{disc.}}(K^\mathfrak{p}, \mathbb{Z}/p^s). \tag{27}$$

*Step 2a.* Applying the functor $\varprojlim_s$ to (26), we have a long exact sequence:

$$\begin{aligned} 0 \to \varprojlim_s H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \to \varprojlim_s \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) \to \\ \to \varprojlim_s A(s) \to \varprojlim_s{}^1 H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s). \end{aligned} \tag{28}$$

In order to calculate the individual terms in (28), we shall use the Eilenberg–Moore sequence (see Theorem 16):

$$0 \to \varprojlim_s{}^1 H^{n-1}_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \to H^n_{\text{cts}}(K_S, \mathbb{Z}_p) \to \varprojlim_s H^n_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \to 0. \tag{29}$$

Taking $n = 1$ in (29) we have

$$\varprojlim_s H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s) = H^1_{\text{cts}}(K_S, \mathbb{Z}_p).$$

Since $[K_S, K_S]$ is open in $K_S$, it follows that:

$$\varprojlim_s H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s) = 0. \tag{30}$$

Also, since the groups $H^1_{\text{disc.}}(K_S, \mathbb{Z}/p^s)$ are all finite, it follows by Corollary 6 that

$$\varprojlim_s{}^1 H^1_{\text{cts}}(K_S, \mathbb{Z}/p^s) = 0. \tag{31}$$

Substituting (30) and (31) into (28), we get

$$\varprojlim_s \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) = \varprojlim_s A(s). \tag{32}$$

*Step 2b.* Applying the left-exact functor $\varprojlim_s$ to (27) and substituting (32) we obtain the following exact sequence:

$$\begin{aligned} 0 \to \varprojlim_s \varinjlim_{K_\mathfrak{p}} H^1(Y(K^\mathfrak{p} K_\mathfrak{p}), \mathbb{Z}/p^s) \to \varprojlim_s \left( H^1_{\text{disc.}}(\mathrm{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^\mathfrak{p}} \right) \\ \to \varprojlim_s H^2_{\text{disc.}}(K^\mathfrak{p}, \mathbb{Z}/p^s). \end{aligned} \tag{33}$$

We shall investigate the second and third terms in this sequence further. The functors $\varprojlim_s$ and $-^{K^{\mathfrak{p}}}$ commute, so we have

$$\varprojlim_s \left(H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^{\mathfrak{p}}}\right) = \left(\varprojlim_s H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)\right)^{K^{\mathfrak{p}}}. \quad (34)$$

Again by the Eilenberg–Moore sequence (29) we have by (34):

$$\varprojlim_s \left(H^1_{\text{disc.}}(\text{Cong}(\mathbb{G}), \mathbb{Z}/p^s)^{K^{\mathfrak{p}}}\right) = H^1_{\text{cts}}(\text{Cong}(\mathbb{G}), \mathbb{Z}_p)^{K^{\mathfrak{p}}}. \quad (35)$$

To calculate the third term in (33) we shall use (25). This shows that

$$\varprojlim_s H^2_{\text{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = \varprojlim_s H^2_{\text{disc.}}(K_S, \mathbb{Z}/p^s) \oplus \varprojlim_s \text{Hom}_{\text{cts}}(\pi_1^{S\cup T\cup\{\mathfrak{p}\}}, \mathbb{Z}/p^s). \quad (36)$$

Since $\pi_1^{S\cup T\cup\{\mathfrak{p}\}}$ is a product of finite groups, it follows that

$$\varprojlim_s \text{Hom}_{\text{cts}}(\pi_1^{S\cup T\cup \mathfrak{p}}, \mathbb{Z}/p^s) = 0.$$

Substituting this into (36), we obtain:

$$\varprojlim_s H^2_{\text{disc.}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = \varprojlim_s H^2_{\text{disc.}}(K_S, \mathbb{Z}/p^s). \quad (37)$$

Substituting (31) into the Eilenberg–Moore sequence (29), we have:

$$\varprojlim_s H^2_{\text{cts}}(K_S, \mathbb{Z}/p^s) = H^2_{\text{cts}}(K_S, \mathbb{Z}_p). \quad (38)$$

Substituting (38) into (37) we have:

$$\varprojlim_s H^2_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Z}/p^s) = H^2_{\text{cts}}(K_S, \mathbb{Z}_p).$$

The sequence (33) is therefore

$$0 \to \varprojlim_s \varinjlim_{K_{\mathfrak{p}}} H^1(Y(K^{\mathfrak{p}}K_{\mathfrak{p}}), \mathbb{Z}/p^s) \to H^1_{\text{cts}}(\text{Cong}(\mathbb{G}), \mathbb{Z}_p)^{K^{\mathfrak{p}}} \to H^2_{\text{cts}}(K_S, \mathbb{Z}_p). \quad (39)$$

*Step 3.* We next apply the exact functor $-\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ to (39). Note that since $K^{\mathfrak{p}}$ and $\text{Cong}(\mathbb{G})$ are compact, we have

$$C^\bullet_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = C^\bullet_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Q}_p),$$

$$C^\bullet_{\text{cts}}(\text{Cong}(\mathbb{G}), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = C^\bullet_{\text{cts}}(\text{Cong}(\mathbb{G}), \mathbb{Q}_p).$$

Furthermore, since $\mathbb{Q}_p$ is flat over $\mathbb{Z}_p$, we have

$$H^\bullet_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = H^\bullet_{\text{cts}}(K^{\mathfrak{p}}, \mathbb{Q}_p),$$

$$H^\bullet_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = H^\bullet_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p).$$

Since $H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)$ is torsion-free, we have

$$\left(H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Z}_p)^{K^{\mathfrak{p}}}\right)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p = H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)^{K^{\mathfrak{p}}}.$$

Again, since $\mathbb{Q}_p$ is flat over $\mathbb{Z}_p$, we have an exact sequence:

$$0\to \tilde{H}^1(K^{\mathfrak{p}},\mathbb{Q}_p)\to H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)^{K^{\mathfrak{p}}}\to H^2_{\mathrm{cts}}(K_S,\mathbb{Q}_p). \tag{40}$$

*Step 4.* Applying the exact functor $\varinjlim_{K^{\mathfrak{p}}}$ to (40), we have an exact sequence

$$0\to \tilde{H}^1(\mathbb{G},\mathbb{Q}_p)\to H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}^{\mathfrak{p}})-\mathrm{smooth}}\to H^2_{\mathrm{st}}(\mathbb{G}(k_S),\mathbb{Q}_p). \tag{41}$$

As $\mathbb{G}(k_S)$ is a $\mathbb{Q}_p$-analytic group, the stable cohomology may be expressed in terms of Lie algebra cohomology (using (13)):

$$H^2_{\mathrm{st}}(\mathbb{G}(k_S),\mathbb{Q}_p) = H^2_{\mathrm{Lie}}(\mathfrak{g}\otimes_k k_S,\mathbb{Q}_p),$$

where we are regarding $\mathfrak{g}\otimes_k k_S$ as a Lie algebra over $\mathbb{Q}_p$. By Whitehead's second lemma (Theorem 11) we have

$$H^2_{\mathrm{st}}(\mathbb{G}(k_S),\mathbb{Q}_p)=0.$$

Hence

$$\tilde{H}^1(\mathbb{G},\mathbb{Q}_p) = H^1_{\mathrm{cts}}(\mathrm{Cong}(\mathbb{G}),\mathbb{Q}_p)_{\mathbb{G}(\mathbb{A}^{\mathfrak{p}})-\mathrm{smooth}}.$$

□

# 9 Some Examples

## 9.1 $\mathrm{SL}_2/\mathbb{Q}$

Let $\mathbb{G}=\mathrm{SL}_2/\mathbb{Q}$. Since $\mathfrak{g}$ is 3-dimensional, the spectral sequence has non-zero terms only in columns 0 to 3. Since arithmetic subgroups have virtual cohomological dimension 1, it follows that $\tilde{H}^n=0$ for $n>1$. Taking $W$ to be the trivial representation, the $E_2$ sheet of the spectral sequence is as follows:

$$E_2^{\bullet,\bullet}\quad:\quad \begin{array}{cccc} \tilde{H}^1(\mathbb{G},E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & E & 0 & 0\\ E & 0 & 0 & E \end{array}$$

The connection map $E_2^{1,1}\to E_2^{3,0}$ is an isomorphism, and the spectral sequence stabilizes at $E_3$ as follows:

$$E_3^{\bullet,\bullet}\quad:\quad \begin{array}{cccc} \tilde{H}^1(\mathbb{G},E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & 0 & 0 & 0\\ E & 0 & 0 & 0 \end{array}$$

### 9.2 $\mathrm{SL}_1(D)$ for an indefinite quaternion algebra $D$

Let $k$ be a totally real field and let $D$ be a quaternion algebra over $k$, which is indefinite at exactly one real place of $k$. We shall consider the group $\mathbb{G}(-) = \mathrm{SL}_1(D \otimes_k -)$ over $k$. Arithmetic subgroups of $\mathbb{G}$ have virtual cohomological dimension 2, so we have classical cohomology groups in dimensions 0, 1 and 2. In dimensions 0 and 2 these are given by constants, and are 1-dimensional. On the other hand it is easy to show that $\tilde{H}^2(\mathbb{G}, \mathbb{Q}_p) = 0$. The $E_2$ sheet of the spectral sequence is as follows:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \tilde{H}^1(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} & E^2 & 0 & 0 \\ E & 0 & 0 & E \end{array}$$

The connection map $E_2^{1,1} \to E_2^{3,0}$ is surjective, and the spectral sequence stabilizes at $E_3$ as follows:

$$E_3^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \tilde{H}^1(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} & E & 0 & 0 \\ E & 0 & 0 & 0 \end{array}$$

### 9.3 $\mathrm{SL}_2/k$ for $k$ real quadratic

Let $k$ be a real quadratic field and consider the group $\mathbb{G} = \mathrm{SL}_2/k$. The non-zero classical cohomology groups are the following:

$$\begin{array}{lcl} H^0_{\text{class.}}(\mathbb{G}, W) & = & W^{\mathbb{G}}, \\ H^2_{\text{class.}}(\mathbb{G}, W) & & \text{infinite dimensional.} \end{array}$$

It is known in this case (see [37]) that the congruence kernel of $\mathbb{G}$ is trivial. We therefore have $\tilde{H}^1(\mathbb{G}, E) = 0$, and we can also show that $\tilde{H}^3(\mathbb{G}, E) = 0$. Therefore the weak Emerton criterion holds in dimension 2. We also have $H^3(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. Therefore we may apply Theorem 8 to the eigenvariety $E(2, K^{\mathfrak{p}})$. The $E_2$-sheet of the spectral sequence is as follows:

$$E_2^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} \tilde{H}^2(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & E \end{array}$$

The map $\tilde{H}^2(\mathbb{G}, E)^{\mathfrak{g}}_{\text{loc.an.}} \to E$ in the $E_3$-sheet is surjective, and the spectral sequence stabilizes at the $E_4$-sheet:

$$E_4^{\bullet,\bullet} \quad : \quad \begin{array}{cccc} H^2_{\text{class.}}(\mathbb{G}, E) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 \end{array}$$

### 9.4 $\mathrm{SL}_3/\mathbb{Q}$

Arithmetic subgroups of $\mathrm{SL}_3(\mathbb{Q})$ have virtual cohomological dimension 4, as the symmetric space is 5-dimensional. We have the following non-zero classical cohomology groups:

$$\begin{array}{rcl} H^0_{\mathrm{class.}}(\mathbb{G}, W) & = & W^{\mathbb{G}}, \\ H^2_{\mathrm{class.}}(\mathbb{G}, W) & = & \text{infinite dimensional}, \\ H^3_{\mathrm{class.}}(\mathbb{G}, W) & = & \text{infinite dimensional}. \end{array}$$

It was shown in [4] that the congruence kernel is trivial. Hence the weak Emerton criterion holds in dimension 2, and in fact the only non-zero Banach space representations are:

$$\begin{array}{rcl} \tilde{H}^0(\mathbb{G}, E) & = & E, \\ \tilde{H}^2(\mathbb{G}, E) & = & \text{infinite dimensional}, \\ \tilde{H}^3(\mathbb{G}, E) & = & \text{infinite dimensional}. \end{array}$$

Furthermore, $H^3_{\mathrm{rel.Lie}}(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. We may therefore apply Theorem 8 to the eigenvariety $E(2, K^{\mathfrak{p}})$. One can use Poincaré duality to construct an eigenvariety interpolating $H^3_{\mathrm{class.}}$.
The author has not been able to calculate all of the terms of the spectral sequence. However the $E_2$-sheet is as follows:

$$E_2^{\bullet,\bullet} : \quad \begin{array}{ccccccccc} \tilde{H}^3(\mathbb{G}, E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & ? & ? & ? & ? & ? & ? & 0 & 0 \\ \tilde{H}^2(\mathbb{G}, E)^{\mathfrak{g}}_{\mathrm{loc.an.}} & \mathrm{Ext}^1_{\mathfrak{g}}(E, \tilde{H}^2(\mathbb{G}, E)_{\mathrm{loc.an.}}) & ? & ? & ? & ? & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & E & 0 & E & 0 & 0 & E \end{array}$$

This is stable by the $E_5$-sheet, and most things are known:

$$E_5^{\bullet,\bullet} \quad : \quad \begin{array}{ccccccccc} ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ H^2_{\mathrm{class.}}(\mathbb{G}, E) & ? & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

### 9.5 $\mathrm{Sp}_4/\mathbb{Q}$

Arithmetic subgroups of $\mathrm{Sp}_4(\mathbb{Q})$ have cohomological dimension 5, as the symmetric space is 6-dimensional. It was shown in [5] that the congruence kernel is trivial. Furthermore $H^3(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$. We may therefore apply Theorem 8 to give a construction of the $H^2$-eigencurve. By Poincaré duality, it is also possible to construct a reasonable $H^4$-eigenvariety.

### 9.6 Spin$(2, l)$ $(l \geq 3)$

Let $L$ be a $\mathbb{Z}$-lattice equipped with a quadratic form of signature $(2, l)$ with $l \geq 3$. We let $\mathbb{G}/\mathbb{Q}$ be the corresponding Spin group. This has real rank 2, and the corresponding symmetric space has dimension $2l$. The congruence kernel was shown to be trivial for such groups by Kneser [26]. Hence $\mathbb{G}$ satisfies the weak Emerton criterion in dimension 2. It turns out that $H^3(\mathfrak{g}, K_\infty, \mathbb{C}) = 0$, so we may apply Theorem 8 to $E(2, K^{\mathfrak{p}})$.

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# G-Structures Entières et Modules de Wach

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Abstract. In this paper, we study the tannakian properties of the Fontaine-Laffaille functor $\mathbf{V_{cris}}$ thanks to the theory of Wach's modules. We construct a point of the torsor linking cristalline representations and weakly admissible filtered modules, preserving the lattices in the sens of the Fontaine-Laffaille correspondance.

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## Introduction

Dans tout ce travail, $p$ est un nombre premier impair, $\mathcal{K}$ un corps de caractéristique 0, complet pour une valuation discrète, absolument non ramifié et de corps résiduel $k$ parfait de caractéristique $p$. Nous noterons $W$ l'anneau des vecteurs de Witt à coefficients dans $k$, c'est donc l'anneau des entiers de $\mathcal{K}$. Tous trois sont munis d'une action de Frobenius, notée $\sigma$. Fixons $\overline{\mathcal{K}}$ une cloture algébrique de $\mathcal{K}$, et posons $\Gamma_{\mathcal{K}} = \mathrm{Gal}(\overline{\mathcal{K}}, \mathcal{K})$. Nous noterons $\mathbb{C}$ le complété de $\overline{\mathcal{K}}$ et $\mathcal{X} : \Gamma_{\mathcal{K}} \to \mathbb{Z}_p^*$ désignera le caractère cyclotomique de $\Gamma_{\mathcal{K}}$ (c'est-à-dire que $g(z) = z^{\mathcal{X}(g)}$ pour tout $g \in \Gamma_{\mathcal{K}}$ et pour toute racine de l'unité $z \in \overline{\mathcal{K}}$ d'ordre une puissance de $p$). Nous allons étudier les représentations continues de $\Gamma_{\mathcal{K}}$ dans des $\mathbb{Q}_p$-espaces vectoriels de dimension finie.
Nous nous restreindrons aux représentations cristallines, condition vérifiée dans bien des cas issus de la géométrie (par exemple, pour le module de Tate ou la cohomologie étale à coefficients dans $\mathbb{Q}_p$ d'une variété abélienne ayant bonne réduction). L'avantage de ces représentations est que J.-M. Fontaine et P. Colmez ont montré dans [Fon94b] et [CF00] qu'elles forment une catégorie tannakienne, qui est $\otimes$-équivalente à la catégorie tannakienne des $\varphi$-modules filtrés sur $\mathcal{K}$ faiblement admissibles (c'est à dire ceux qui ont des réseaux fortement divisibles).

Le foncteur qui induit cette équivalence de catégories se décrit de la manière suivante : si $V$ est une représentation $p$-adique cristalline, le $\varphi$-module filtré associé est $\mathbf{D_{cris,p}}(V) = (V \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_{\mathcal{K}}}$ (le quasi-inverse est donné par : pour $D$ un $\varphi$-module filtré faiblement admissible, $\mathbf{V_{cris,p}}(D) = \mathrm{Fil}^0(D \otimes_{\mathcal{K}} B_{cris})^{\varphi}$). De plus, l'application

$$\mathbf{V_{cris,p}}(D) \otimes_{\mathbb{Q}_p} B_{cris} \to D \otimes_{\mathcal{K}} B_{cris}$$

issue de la multiplication de $B_{cris}$ est un isomorphisme (préservant l'action de $\Gamma_{\mathcal{K}}$, la filtration, et le morphisme $\varphi$). Cela peut se traduire de la façon suivante : en notant $w_V$ le foncteur oubli qui à la $\mathbb{Q}_p$-représentation cristalline $V$ associe le $\mathbb{Q}_p$-espace vectoriel sous-jacent à $V$, et $w_D$ celui qui associe le $\mathcal{K}$-espace vectoriel sous-jacent à $\mathbf{D_{cris,p}}(V)$, alors les $\otimes$-isomorphismes du foncteur fibre $w_V \otimes_{\mathbb{Q}_p} \mathcal{K}$ sur le foncteur fibre $w_D$, $\mathbf{Isom}(w_V \otimes_{\mathbb{Q}_p} \mathcal{K}, w_D)$, forment un torseur sous $\mathbf{Aut}^{\otimes}(w_V)_{|\mathcal{K}}$ et sous $\mathbf{Aut}^{\otimes}(w_D)$, qui est non vide sur $B_{cris}$.
Du côté des $\varphi$-modules filtrés sur $\mathcal{K}$, nous disposons de la notion de réseaux fortement divisibles (dont l'existence est une condition nécessaire et suffisante pour que le module soit faiblement admissible), qui sont des $\varphi$-modules filtrés sur $W$ (cf. paragraphe 1.3). J.-M. Fontaine et G. Laffaille ont montré dans [FL82] que, si la longueur de la filtration est strictement plus petite que $p-1$, il existe une équivalence de catégories abéliennes entre réseaux fortement divisibles d'un module filtré faiblement admissible, et les réseaux stables de la représentation cristalline associée.
Plus précisément, à $M$ un $\varphi$-module filtré sur $W$ vérifiant $\mathrm{Fil}^1(M) = \{0\}$ et $\mathrm{Fil}^{2-p}(M) = M$, ils associent le réseau $\mathbf{V_{cris}}(M) = \mathrm{Fil}^0(M \otimes_W A_{cris})^{\varphi^0}$, et cette construction induit un foncteur exact, pleinement fidèle (dont nous noterons $\mathbf{D_{cris}}$ un quasi-inverse). Deux problèmes apparaissent : la condition sur la filtration n'est pas stable par produit tensoriel, et l'application naturelle

$$\mathbf{V_{cris}}(M) \otimes_{\mathbb{Z}_p} A_{cris} \to M \otimes_W A_{cris}$$

n'est pas un isomorphisme (le déterminant est une puissance de $t$, non inversible dans $A_{cris}$). De plus, une question naturelle se pose : est-ce qu'il existe un point $f$ de $\mathbf{Isom}(w_V \otimes_{\mathbb{Q}_p} \mathcal{K}, w_D)$ qui envoie un réseau galoisien sur celui qui lui correpond d'après la correspondance de Fontaine-Laffaille ? Répondre à ces questions revient à étudier les propriétés tannakiennes de $\mathbf{V_{cris}}$.
L'idée va être d'introduire la théorie des modules de Wach de L. Berger (voir [Ber04]), qui à un réseau d'une $\mathbb{Q}_p$-représentation cristalline associe un $(\varphi, \Gamma)$-module dont un quotient redonne le $\varphi$-module filtré sur $W$ correspondant à la théorie de Fontaine-Laffaille. L'intérêt des modules de Wach est leur compatibilité avec le produit tensoriel (le module de Wach d'un produit tensoriel est le produit tensoriel des modules de Wach). Le problème se ramène alors à : pouvons-nous à partir d'un $\varphi$-module filtré sur $W$ reconstruire le module de Wach correspondant ? Pouvons-nous le faire de manière à ce que cette construction soit fonctorielle ?

Le résultat technique principal de cet article est la construction à partir des idées de N. Wach d'un foncteur de la catégorie des modules de Fontaine-Laffaille vers la catégorie des modules de Wach. Plus précisément, notons $\mathbf{MF_W^{-h}}$ la catégorie des $\varphi$-modules filtrés $N$ libres sur $W$ tels que $\mathrm{Fil}^{-h}(N) = N$, $\mathrm{Fil}^1(N) = \{0\}$ (cf. paragraphe 1.3 pour plus de détails) et $\mathbf{MF_W} < -\mathbf{h} >$ la catégorie engendrée par $\mathbf{MF_W^{-h}}$ pour les opérations de sous-objets, objets quotients, produit tensoriel et somme directe, $\mathbf{V_{cris}}$ le foncteur de Fontaine-Laffaille, $\mathbf{\Gamma\Phi M_S^{-h}}$ la catégories des duaux des modules de Wach de hauteur $h$ (ce qui correspond à des modules de Wach d'après la définition de [Ber04]), et $\mathbf{\Gamma\Phi M_S^{-}}$ la réunion sur $h \geq 0$ des $\mathbf{\Gamma\Phi M_S^{-h}}$, $\mathbf{N}$ le foncteur "module de Wach", $\mathbf{V}_{\mathcal{O}_{\mathcal{E}}}$ le foncteur de Fontaine pour les $(\varphi, \Gamma)$-module sur $\mathcal{O}_{\mathcal{E}}$, $\mathbf{V_{cris,p}}$ le foncteur de Fontaine pour les $\varphi$-modules filtrés sur $\mathcal{K}$ admissibles, et $j : S \to \mathcal{O}_{\mathcal{E}}$ qui induit le foncteur extension des scalaires $j^*$ de la catégorie des modules de Wach vers la catégorie des $(\varphi, \Gamma)$-modules sur $\mathcal{O}_{\mathcal{E}}$.

Théorème 1. *Soit $h$ un entier compris entre $0$ et $p-2$, alors il existe un foncteur $\mathrm{F}^-$ exact, préservant le produit tensoriel, fidèle et pleinement fidèle de $\mathbf{MF_W} < -\mathbf{h} >$ vers $\mathbf{\Gamma\Phi M_S^{-}}$. Restreint à $\mathbf{MF_W^{-h}}$, ce foncteur est essentiellement surjectif sur $\mathbf{\Gamma\Phi M_S^{-h}}$. De plus, pour tout objet $M$ de $\mathbf{MF_W^{-h}}$, $\mathrm{F}^-(M)$ est fonctoriellement isomorphe à $\mathbf{N}(\mathbf{V_{cris}}(M))$. Dans le cas général, $\mathrm{F}^-(M)$ s'interprète encore comme le module de Wach du réseau galoisien correspondant au $(\varphi, \Gamma)$-module sur $\mathcal{O}_{\mathcal{E}}$ engendré par $\mathrm{F}^-(M)$. En outre, $\mathbf{V}_{\mathcal{O}_{\mathcal{E}}} \circ j^* \circ \mathrm{F}^-$ est isomorphe (comme foncteur) à $\mathbf{V_{cris,p}}$ une fois $p$ rendu inversible, et à $\mathbf{V_{cris}}$ une fois restreint à la catégorie $\mathbf{MF_W^{-h}}$.*

Remarque 1. *Ce théorème est optimal, dans le sens où nous ne pouvons espérer que $\mathrm{F}^-$ soit essentiellement surjectif sans la restriction sur $h$.*

Pour illuster, le théorème nous dit essentiellement que le diagramme suivant est commutatif (où bien sûr il faut resteindre la catégorie des réseaux des représentations cristallines à ceux à poids de Hodge-Tate dans $[\![0, h]\!]$) :

$$\begin{tikzcd}
\mathbf{Rep}^{\mathbf{cris,h}}_{\mathbb{Z}_\mathbf{p}}(\Gamma_{\mathcal{K}}) \arrow[rr, shift left, "\mathbf{D_{cris}}"] \arrow[d, shift right, "\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}"'] \arrow[drr, "\mathbf{N}"] & & \mathbf{MF_W^{-h}} \arrow[ll, shift left, "\mathbf{V_{cris}}"] \arrow[d, dotted, shift right, "\mathrm{F}^-"'] \\
\mathbf{\Gamma\Phi M}^{\mathbf{et}}_{\mathcal{O}_{\mathcal{E}}} \arrow[u, shift right, "\mathbf{V}_{\mathcal{O}_{\mathcal{E}}}"'] & & \mathbf{\Gamma\Phi M_S^{-h}} \arrow[ll, "j^*"] \arrow[u, shift right, "\mathrm{mod}\,\pi"']
\end{tikzcd}$$

et, une fois $p$ rendu inversible,

$$\begin{tikzcd}
\mathbf{Rep}^{\mathbf{cris}}_{\mathbb{Q}_\mathbf{p}}(\Gamma_{\mathcal{K}}) \arrow[d, shift right, "\mathbf{D}_{\mathcal{E}}"'] \arrow[drr, "\mathbf{N}"] & & \mathbf{MF_W} < -\mathbf{h} > \otimes \mathcal{K} \arrow[ll, "\mathbf{V_{cris,p}}"] \arrow[d, dotted, shift right, "\mathrm{F}^-"'] \\
\mathbf{\Gamma\Phi M}^{\mathbf{et}}_{\mathcal{E}} \arrow[u, shift right, "\mathbf{V}_{\mathcal{E}}"'] & & \mathbf{\Gamma\Phi M}^{-}_{\mathbf{S}[\frac{1}{\mathbf{p}}]} \arrow[ll, "j^*"] \arrow[u, shift right, "\mathrm{mod}\,\pi"']
\end{tikzcd}$$

où $\mathrm{F}^-$ est fidèle, pleinement fidèle, préserve le produit tensoriel, et suivant les cas, peut être essentiellement surjectif (et $\mathbf{MF_W} < -\mathbf{h} > \otimes \mathcal{K}$ représente juste la catégorie formée des objets de $\mathbf{MF_W} < -\mathbf{h} >$ où nous avons rendu $p$ inversible, c'est à dire la catégorie engendrée pour les opérations de produit tensoriel, somme directe, sous-objet et objet quotient, par les modules filtrés sur $\mathcal{K}$ admissibles à pente compris entre 0 et $-h$).

REMARQUE 2. *Dans l'article nous étudierons aussi le cas plus général des $\varphi$-modules filtrés de type fini sur $W$ (donc ayant éventuellement de la $p$-torsion).*

De ce théorème, nous en déduisons le corollaire voulu :

THÉORÈME 2. *Il existe un point du torseur $\mathbf{Isom}(w_V \otimes_{\mathbb{Q}_p} \mathcal{K}, w_D)$ à coefficient dans le corps $\widehat{\mathcal{E}}_{nr}$ qui préserve les réseaux de Fontaine-Laffaille, c'est à dire qui identifie les réseaux stables par Galois des représentations cristallines à poids de Hodge-Tate dans $[\![0, \frac{p-2}{2}]\!]$ au $W$-module filtré correspondant par la théorie de Fontaine-Laffaille.*

Pour obtenir un résultat sur $\mathcal{K}$ plutôt que sur $\widehat{\mathcal{E}}_{nr}$, il faut modifier le problème. Considérons $G$ un groupe algébrique lisse sur $\mathbb{Z}_p$ et une représentation $\rho : \Gamma_{\mathcal{K}} \to G(\mathbb{Z}_p)$. Supposons donnée une immersion fermée $\alpha$ de $G$ dans $GL_U$, pour $U$ un $\mathbb{Z}_p$-module libre de rang fini, telle que la représentation $\alpha \circ \rho$ de $\Gamma_{\mathcal{K}}$ (dans $GL(U \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$) soit cristalline à poids de Hodge-Tate dans $[\![0, h]\!]$ avec $h$ un entier compris entre 0 et $\frac{p-2}{2}$. Notons $V = U \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Par un théorème de Chevalley, il existe un $\mathbb{Q}_p$-espace vectoriel $V_G$ dans $\bigoplus_i \mathrm{End}(V)^{\otimes i}$ (en faisant agir $GL_V$ naturellement sur $V^*$ et trivialement sur $V$, dans $\mathrm{End}(V) = V \otimes V^*$) tel que $G \times_{\mathbb{Z}_p} \mathbb{Q}_p$ soit le groupe algébrique formé de l'ensemble des éléments de $GL_V$ qui laissent stable $V_G$. Alors, par le foncteur de Fontaine-Laffaille, nous pouvons définir naturellement un groupe $G_D$ sur $D = \mathbf{D_{cris,p}}(V)$ comme l'ensemble des éléments de $GL_D$ laissant stable $\mathbf{D_{cris,p}}(V_G)$. Un corollaire de la proposition 6.3.3 de [Fon79] nous donne l'existence d'un élément de $\mathbf{Isom}(w_V \otimes_{\mathbb{Q}_p} \mathcal{K}, w_D)(\mathcal{K})$, donc en particulier d'un isomorphisme de $\mathcal{K}$-modules

$$f : V \otimes_{\mathbb{Q}_p} \mathcal{K} \to D$$

qui identifie $G \times_{\mathbb{Z}_p} \mathcal{K}$ à $G_D$. Le comportement de $f$ vis-à-vis des réseaux est à priori inconnu. Pour l'étudier, nous introduisons un $G$-torseur **Isom** défini sur $W$, qui est heuristiquement le $G$-torseur obtenu à partir de $\mathbf{Isom}(w_V \otimes_{\mathbb{Q}_p} \mathcal{K}, w_D)$ (c'est à dire une forme sur $W$ du $G \times_W \mathcal{K}$ torseur obtenu à partir de $\mathbf{Isom}(w_V \otimes_{\mathbb{Q}_p} \mathcal{K}, w_D)$). Le résultat suivant se montre alors en montrant que **Isom** est un $G$-torseur trivial sur $W$ :

THÉORÈME 3. *Sous les hypothèses précédentes, si $M = \mathbf{D_{cris}}(U)$, il existe un sous-groupe algébrique $G_M$ de $GL_M$ sur $W$, avec $G_M \times_W \mathcal{K} = G_D$, et il existe $f$ un isomorphisme de $W$-modules de $U \otimes_{\mathbb{Z}_p} W$ sur $M$, qui identifie $G$ à $G_M$.*

$$f : U \otimes_{\mathbb{Z}_p} W \to M$$

*De plus, si $U'$ est un réseau de $U \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ laissé stable par l'action de $G$, alors $f[\frac{1}{p}]$ envoie $U' \otimes_{\mathbb{Z}_p} W$ sur $\mathbf{D_{cris}}(U')$.*

Remarque 3. *Ce théorème nous donne en particulier que les réseaux $U$ et $M$ ont la même position vis à vis du groupe $G$.*

Avec des hypothèses plus fortes sur $\alpha$, nous pouvons affaiblir l'hypothèse sur $h$. Une application directe de ce résultat concerne la semi-simplifiée d'une représentation cristalline à poids de Hodge-Tate petits : le groupe algébrique $H$ engendré par l'image de Galois sur $\mathbb{Q}_p$ est alors connexe et réductif, donc en appliquant les résultats cités dans [Tit79] (paragraphe 3.2 et 3.4.1), il existe un groupe algébrique lisse $G$ défini sur $\mathbb{Z}_p$, tel que $G(\mathbb{Z}_p)$ contienne l'image de Galois, et dont la fibre générique est $H$. Le Théorème 3 s'applique alors.

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# 1 Rappels

## 1.1 Rappels sur les $(\varphi, \Gamma)$-modules

### 1.1.1 Définition de $\mathcal{O}_{\mathcal{E}}$

Soit $R$ l'ensemble des suites $x = (x^{(n)})_{n\in\mathbb{N}}$ formées d'éléments de $\mathcal{O}_{\bar{\mathcal{K}}}/p\mathcal{O}_{\bar{\mathcal{K}}}$ vérifiant $(x^{(n+1)})^p = x^{(n)}$ pour tout $n$ (cf. [Fon82], p. 535). C'est un anneau parfait de caractéristique $p$, muni d'une valuation ; son corps résiduel s'identifie à $k$. Son corps des fractions $\mathrm{Fr}\, R$ est un corps algébriquement clos de caractéristique $p$, et $R$ est intégralement clos dans $\mathrm{Fr}\, R$.
Si $A$ est une $k$-algèbre, $W(A)$ désigne l'anneau des vecteurs de Witt à coefficients dans $A$. Notons $\mathbb{Z}_{p^s} = W(\mathbb{F}_{p^s})$, $\mathbb{Z}_p^{nr} = W(\overline{\mathbb{F}_p})$, $W = W(k)$, $W_{\mathcal{K}}(A) = \mathcal{K} \otimes_W W(A) = W(A)[\frac{1}{p}]$ et si $a \in A$, $[a] = (a, 0, \cdots, 0, \cdots)$ le représentant de Teichmüller de $a$ dans $W(A)$. Le Frobenius $x \in A \mapsto x^p \in A$ s'étend à $W(A)$ en $\varphi$ (encore appelé l'endomorphisme de Frobenius) par fonctorialité, ainsi qu'à $W_{\mathcal{K}}(A)$ ; nous noterons $\sigma$ le Frobenius sur $W$ et sur $\mathcal{K}$ (si $\lambda \in W$, $\sigma(\lambda) := \varphi(\lambda)$). En particulier ceci s'applique à $W(R)$, $W(\mathrm{Fr}\, R)$ et $W_{\mathcal{K}}(\mathrm{Fr}\, R)$.
D'autre part, le groupe $\Gamma_{\mathcal{K}}$ opère par fonctorialité sur $R$, $\mathrm{Fr}\, R$ et $W(\mathrm{Fr}(R))$, et les anneaux $W(R)$, $W(\mathrm{Fr}\, R)$ et $W_{\mathcal{K}}(R)$ s'identifient à des sous-anneaux de $W_{\mathcal{K}}(\mathrm{Fr}\, R)$ stables par $\varphi$ et $\Gamma_{\mathcal{K}}$.
Notons $\mathbb{Z}_p(1) = \varprojlim_{n\in\mathbb{N}} \mu_{p^n}(\bar{\mathcal{K}})$ le module de Tate du groupe multiplicatif et pour tout $i \in \mathbb{N}$, $\mathbb{Z}_p(i) = \mathbb{Z}_p(1)^{\otimes i}$ et $\mathbb{Z}_p(-i)$ son dual. Pour tout $\mathbb{Z}_p$-module $T$, et pour tout $i \in \mathbb{Z}$, posons $T(i) = T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$.
Le module de Tate $\mathbb{Z}_p(1) = T_p(\mathbb{G}_m)$ s'identifie au sous $\mathbb{Z}_p$-module du groupe multiplicatif des unités de $R$ congrues à 1 modulo l'idéal maximal, formé des $x$ tels que $x^{(0)} = 1$. Choisisons un générateur de ce module, c'est-à-dire un élément $\varepsilon = (\varepsilon^{(n)})_{n\in\mathbb{N}} \in R$ tel que $\varepsilon^{(0)} = 1$ et $\varepsilon^{(1)} \neq 1$, et considérons l'élément $\pi = [\varepsilon] - 1$ dans $W(R)$. Alors l'adhérence $S$ de la sous $W$-algèbre de $W(R)$ engendrée par $\pi$ s'identifie à l'algèbre $W[\![\pi]\!]$ des séries formelles en $\pi$ à coefficients dans $W$ ; de plus $S$ est stable par $\varphi$ et $\Gamma_{\mathcal{K}}$, et nous avons les relations suivantes :

$$\varphi(\pi) = (1 + \pi)^p - 1$$

$$g(\pi) = (1 + \pi)^{\mathcal{X}(g)} - 1$$

pour $g \in \Gamma_{\mathcal{K}}$.
Soit $\mathcal{K}_n$ le sous corps de $\bar{\mathcal{K}}$ engendré sur $\mathcal{K}$ par les racines $p^n$-ièmes de l'unité, et $\mathcal{K}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$. Notons $\Gamma = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K})$ et $H_{\mathcal{K}}$ le noyau de la projection de $\Gamma_{\mathcal{K}}$ sur $\Gamma$. Le groupe $H_{\mathcal{K}}$ agit trivialement sur $S$. Si $\Gamma_f$ est le sous-groupe de torsion de $\Gamma$, posons $S_0 = S^{\Gamma_f}$ ainsi que $\Gamma_0 = \Gamma/\Gamma_f$ ; J.-M. Fontaine a montré (cf. [Fon90], p. 268-273) que $S_0 = W[\![\pi_0]\!]$, où $\pi_0 = -p + \sum_{a\in\mathbb{F}_p} [\varepsilon]^{[a]}$. Notons $q = p + \pi_0$. $S_0$ est munie d'une action naturelle de $\Gamma_0$.

Notons $\mathcal{O}_{\mathcal{E}}$ le complété pour la topologie $p$-adique de $S[\frac{1}{\pi}]$. C'est l'anneau des entiers d'un corps complet pour une valuation discrète, absolument non ramifié, noté $\mathcal{E}$. Comme $\pi$ est inversible dans $W(\mathrm{Fr}\,R)$, l'inclusion de $S$ dans $W(R)$ se prolonge en un plongement de $S[\frac{1}{\pi}]$ dans $W(\mathrm{Fr}\,R)$, et $\mathcal{O}_{\mathcal{E}}$ s'identifie à l'adhérence de $S[\frac{1}{\pi}]$ dans $W(\mathrm{Fr}\,R)$ pour la topologie $p$-adique, tandis que $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[\frac{1}{p}]$ s'identfie à un sous-corps de $W_{\mathcal{K}}(\mathrm{Fr}\,R)$. Alors si $E = \mathcal{O}_{\mathcal{E}}/p$, $\mathcal{O}_E = S/pS = k[\![\tilde{\pi}]\!]$, où $\tilde{\pi}$ est la réduction modulo $p$ de $\pi$.
De plus, si $\hat{\mathcal{E}}_{nr}$ désigne l'adhérence dans $W_{\mathcal{K}}(\mathrm{Fr}\,R)$ de l'extension maximale non ramifiée $\mathcal{E}_{nr}$ de $\mathcal{E}$ contenue dans $W_{\mathcal{K}}(\mathrm{Fr}\,R)$ et $\mathcal{O}_{\hat{\mathcal{E}}_{nr}}$ son anneau des entiers, $\mathcal{O}_{\hat{\mathcal{E}}_{nr}}/p$ est une clôture séparable $E^{sep}$ de $E$, avec une identification des groupes de Galois

$$H_{\mathcal{K}} = \mathrm{Gal}(E^{sep}/E) = \mathrm{Gal}(\mathcal{E}_{nr}/\mathcal{E}).$$

### 1.1.2 $(\varphi, \Gamma)$-modules et représentations galoisiennes

Nous ne considérerons des $(\varphi, \Gamma)$-modules que sur $S$ ou $\mathcal{O}_{\mathcal{E}}$ (nous considérerons aussi des $(\varphi, \Gamma_0)$-modules définis sur $S_0$). Soit $A$ l'un des anneaux précédent. Un $(\varphi, \Gamma)$-module sur $A$ est un $A$-module muni d'un endomorphisme $\varphi$, semi-linéaire par rapport à $\sigma$ muni en plus d'une action continue de $\Gamma$, semi-linéaire par rapport à l'action de $\Gamma$ sur $A$, cette action commutant avec l'endomorphisme $\varphi$. Nous les supposerons toujours *étale*, c'est à dire de type fini sur $A$ et tels que l'application linéaire $\varphi : M^{\sigma} \to M$, déduite de $\varphi$ en posant $\varphi(\lambda \otimes x) = \lambda\varphi(x)$ pour $\lambda \in A$ et $x \in M$ est bijective. Les $(\varphi, \Gamma)$-modules étales (avec comme morphismes les morphismes $A$-linéaires commutants à $\varphi$ et à $\Gamma$) définissent une $\otimes$-catégorie abélienne notée $\mathbf{\Gamma\Phi M}^{\text{ét}}_{\mathbf{A}}$ (cf. [Fon90] p.273).
Appelons représentation $\mathbb{Z}_p$-adique de $\Gamma_{\mathcal{K}}$ la donnée d'un $\mathbb{Z}_p$-module de type fini muni d'une action linéaire et continue de $\Gamma_{\mathcal{K}}$. Un morphisme sera une application $\mathbb{Z}_p$-linéaire commutant à l'action de $\Gamma_{\mathcal{K}}$. Notons $\mathbf{Rep}_{\mathbb{Z}_p}(\mathbf{\Gamma}_{\mathcal{K}})$ la catégorie des représentations $\mathbb{Z}_p$-adique de $\Gamma_{\mathcal{K}}$. La catégorie $\mathbf{Rep}_{\mathbb{Q}_p}(\mathbf{\Gamma}_{\mathcal{K}})$ est défini de même.
J.-M. Fontaine a montré dans [Fon90] (p. 274) qu'il existait une équivalence de catégories entre $\mathbf{\Gamma\Phi M}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$ et $\mathbf{Rep}_{\mathbb{Z}_p}(\mathbf{\Gamma}_{\mathcal{K}})$ induite par le foncteur $\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(T) = (\mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \otimes_{\mathbb{Z}_p} T)^{H_{\mathcal{K}}}$ pour $T$ une $\mathbb{Z}_p$-représentation de $\Gamma_{\mathcal{K}}$, et son quasi inverse $\mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathcal{N}) = (\mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{N})^{\varphi=1}$. La multiplication dans $\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ induit alors une application naturelle et fonctorielle :

$$\mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathcal{N}) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \xrightarrow{\psi_{\mathcal{N}}} \mathcal{N} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$$

pour $\mathcal{N}$ un objet de la catégorie $\mathbf{\Gamma\Phi M}^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}$.

## 1.2 Représentations cristallines

### 1.2.1 Représentations cristallines

Pour la définition de $A_{cris}$ et de $t := \log([\varepsilon])$, nous renvoyons à [Fon94a] par exemple. Nous noterons $B_{cris} = A_{cris}[\frac{1}{t}]$. Soit $V$ un $\mathbb{Q}_p$-espace vectoriel de dimension finie, et $\rho : \Gamma_{\mathcal{K}} \to GL(V)$ une représentation continue de $\Gamma_{\mathcal{K}}$. Définissons $\mathbf{D_{cris,p}}$ par

$$\mathbf{D_{cris,p}}(V) = (V \otimes_{\mathbb{Q}_p} B_{cris})^{\Gamma_{\mathcal{K}}}$$

Alors $\mathbf{D_{cris,p}}(V)$ est un $\mathcal{K}$-espace vectoriel, et $\dim_{\mathcal{K}} \mathbf{D_{cris,p}}(V) \leq \dim_{\mathbb{Q}_p} V$.

DÉFINITION 4. *La représentation* $(\rho, V)$ *est cristalline si* $\dim_{\mathcal{K}} \mathbf{D_{cris,p}}(V) = \dim_{\mathbb{Q}_p} V$.

Notons $\mathbf{Rep_{\mathbb{Q}_p,cris}(\Gamma_{\mathcal{K}})}$ la sous-catégorie pleine de $\mathbf{Rep_{\mathbb{Q}_p}(\Gamma_{\mathcal{K}})}$ formée par les représentations cristallines. Définissons $\mathbf{MF}_{\mathcal{K}}$ la catégorie des $\varphi$-modules filtrés sur $\mathcal{K}$ : un objet $D$ de $\mathbf{MF}_{\mathcal{K}}$ est un $\mathcal{K}$-espace vectoriel de dimension finie muni d'une filtration $(\mathrm{Fil}^i(D))_{i\in\mathbb{Z}}$ formée de sous-espaces vectoriels, fitration qui est décroissante, exhaustive séparée, et muni d'une application $\sigma$-semi-linéaire bijective $\varphi : D \to D$. $\mathbf{D_{cris,p}}(V)$ est alors naturellement un $\varphi$-module filtré. Un élément de l"image essentiel du foncteur $\mathbf{D_{cris,p}}(V)$ restreint à $\mathbf{Rep_{\mathbb{Q}_p,cris}(\Gamma_{\mathcal{K}})}$ est appelé admissible. Notons $\mathbf{MF^{ad}_{\mathcal{K}}}$ la sous-catégorie pleine de $\mathbf{MF}_{\mathcal{K}}$ formée des modules admissibles.
$\mathbf{Rep_{\mathbb{Q}_p,cris}(\Gamma_{\mathcal{K}})}$ et $\mathbf{MF^{ad}_{\mathcal{K}}}$ sont deux catégories tannakiennes, le foncteur $\mathbf{D_{cris,p}}$ induit une équivalence de $\otimes$-catégories entre ces deux catégories, et un quasi-inverse est donné par le foncteur $\mathbf{V_{cris,p}}(D) = \left(\mathrm{Fil}^0(D \otimes_{\mathcal{K}} B_{cris})\right)^{\varphi=1}$ . L'application naturelle (provenant de la multiplication dans $B_{cris}$)

$$\mathbf{V_{cris,p}}(D) \otimes_{\mathbb{Q}_p} B_{cris} \to D \otimes_{\mathcal{K}} B_{cris} \tag{5}$$

est alors une bijection.

### 1.2.2 POIDS DE HODGE-TATE

Rappelons que pour $\rho : \Gamma_{\mathcal{K}} \to GL_{\mathbb{Q}_p}(V)$ une représentation continue sur un $\mathbb{Q}_p$-espace vectoriel de dimension finie, l'action de $\Gamma_{\mathcal{K}}$ peut s'étendre à $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}_p} \mathbb{C}$ via $g(v \otimes x) = \rho(g)(v) \otimes g(x)$. Notons alors pour $i \in \mathbb{Z}$, $V_{\mathbb{C}}\{i\} = \{v \in V_{\mathbb{C}} | \forall g \in \Gamma_{\mathcal{K}},\ g(v) = \mathcal{X}(g)^i v\}$. $V_{\mathbb{C}}\{i\}$ est un $\mathcal{K}$-sous espace vectoriel de $V_{\mathbb{C}}$ tel que l'injection $V_{\mathbb{C}}\{i\} \to V_{\mathbb{C}}$ s'étend en une injection $\mathbb{C}$-linéaire

$$\bigoplus_{i\in\mathbb{Z}} V_{\mathbb{C}}\{i\} \otimes_{\mathcal{K}} \mathbb{C} \to V_{\mathbb{C}}$$

Alors $V$ est dit de Hodge-Tate si cette injection est une bijection. Les poids de Hodge-Tate sont alors les $i \in \mathbb{Z}$ tels que $\dim_{\mathcal{K}} V_{\mathbb{C}}\{i\} \neq 0$. Si $V$ est cristalline, alors elle est de Hodge-Tate, et ses poids de Hodge-Tate sont les opposées des sauts de la filtration de $\mathbf{D_{cris,p}}(V)$.

## 1.3 RAPPELS SUR LES $\varphi$-MODULES

La catégorie qui va nous interesser est la catégorie $\mathbf{MF_{W,tf}}$ dite des $\varphi$-modules filtrés sur $W$, dont les objets sont les $W$-modules $N$ de type fini, muni

– d'une filtration décroissante exhaustive et séparée formée de sous-modules $(\mathrm{Fil}^i(N))_{i\in\mathbb{Z}}$ ;
– pour tout $i \in \mathbb{Z}$, d'une application $\sigma$-semi-linéaire $\varphi^i : \mathrm{Fil}^i(N) \to N$ telle que $\varphi^i|_{\mathrm{Fil}^{i+1}(N)} = p\varphi^{i+1}$ ;
– il existe $i \in \mathbb{Z}$ avec $\mathrm{Fil}^i(N) = \{0\}$ ;
– les $\mathrm{Fil}^i(N)$ sont des facteurs directs dans $N$ ;
– $\sum\limits_{i\in\mathbb{Z}} \varphi^i(\mathrm{Fil}^i(N)) = N$.

Les morphismes de cette catégorie sont donnés par les applications $W$-linéaires compatibles aux filtrations et commutants aux $\varphi^i$. C'est une $\otimes$-catégorie qui est abélienne, $\mathbb{Z}_p$-linéaire, qui possède des Hom internes (cf. [Win84]).
Soit $X$ (respectivement $X_s$ pour $s \in \mathbb{N}^*$) le groupe additif des applications périodiques (respectivement ayant $s$ pour période) de $\mathbb{Z}$ dans $\mathbb{Z}$. Le Frobenius $\sigma$ agit sur $X$ par $\forall \xi \in X, \forall i \in \mathbb{Z}$, $\sigma(\xi)(i) = \xi(i+1)$, et laisse donc stable les $X_s$.
Pour tout objet $N$ de $\mathbf{MF_{W,tf}}$, si $(N_i)_{i\in\mathbb{Z}}$ est un scindage de $(\mathrm{Fil}^i(N))_{i\in\mathbb{Z}}$, posons pour $x \in N$ tel que $x = \sum\limits_i x_i$ avec $x_i \in N_i$, $f_N(x) = \sum\limits_i \varphi^i_N(x_i)$. Soit pour tout $\xi \in X$, le $W$-module $N\{\xi\} := \{x \in N | f_N^j(x) \in N_{\xi(j)}$ pour tout $j \in \mathbb{Z}\}$. Le $\varphi$-module filtré $N$ est dit *élémentaire* si $N = \oplus_{\xi\in X} N\{\xi\}$.

Lemme 6. *Si $N$ est un module élémentaire, dont le module sous-jacent est libre sur $W$ ou sur $k$, alors il existe une base $(e^i_\xi)_{\xi\in X,\, 1\le i\le \mathrm{rg}(N\{\xi\})}$ de $N$ telle que pour tout $\xi$, $(e^i_\xi)_{1\le i\le \mathrm{rg}(N\{\xi\})}$ soit une base de $N\{\xi\}$ et de plus $\varphi^{\xi(0)}(e^i_\xi) = e^i_{\sigma(\xi)}$.*

J.-P. Wintenberger a montré dans [Win84] :

Théorème 7. *Pour tout objet $N$ de $\mathbf{MF_{W,tf}}$, il existe un et un seul scindage de la filtration de $N$ tel que*
– *il existe un (unique) $u_N \in Aut_W(N)$ tel que le $\varphi$-module filtré $(N, (N_i), u_N^{-1} \circ f_N)$ soit élémentaire ;*
– *$N/pN$ ait une suite de composition dont les quotients successifs sont des modules élémentaires.*

*Ce scindage vérifie les propriétés de fonctorialité attendues.*

Posons enfin $\mathbf{MF^{[a,b]}_{W,tf}}$ (resp. $\mathbf{MF^{[a,b]}_{W}}$) la sous-catégorie pleine de $\mathbf{MF_{W,tf}}$ formée des $W$-modules $M$ (resp. modules libres) tels que $\mathrm{Fil}^a(M) = M$ et $\mathrm{Fil}^{b+1}(M) = \{0\}$. Notons $\mathbf{MF^{-h}_{W,tf}} = \mathbf{MF^{[-h,0]}_{W,tf}}$, $\mathbf{MF^{h}_{W,tf}} = \mathbf{MF^{[0,h]}_{W,tf}}$ et $\mathbf{MF^{\pm h}_{W,tf}} = \mathbf{MF^{[-h,h]}_{W,tf}}$ (de même sans le symbole tf). Pour terminer, nous désignerons par $\mathbf{MF_{W,tf}} < \mathbf{h} >$ la catégorie engendrée par $\mathbf{MF^{h}_{W}}$ dans la catégorie $\mathbf{MF_{W,tf}}$ pour les opérations de sous-objet, objet quotient, somme directe et produit tensoriel.
Soit $D$ un $\varphi$-module filtré sur $\mathcal{K}$ admissible. Alors il possède des sous-réseaux fortement divisible, $M$, c'est-à-dire un réseau $M$ vérifiant $\sum\limits_{i\in\mathbb{Z}} p^{-i}\varphi(\mathrm{Fil}^i(D) \cap M) = M$. En posant $\mathrm{Fil}^i(M) = \mathrm{Fil}^i(D) \cap M$, $\varphi^i = p^{-i}\varphi|_{\mathrm{Fil}^i(M)}$, $M$ devient

un $\varphi$-module filtré sur $W$. Réciproquement, si $M$ est un objet de $\mathbf{MF_{W,tf}}$ libre sur $W$, en posant $D := \mathcal{K} \otimes_W M$, $\mathrm{Fil}^i(D) := \mathcal{K} \otimes_W \mathrm{Fil}^i(M)$, et pour $x_i \in \mathrm{Fil}^i(M)$, $\varphi(x_i) := p^i \varphi^i(x_i)$, l'objet $D$ ainsi construit est un $\varphi$-module filtré sur $\mathcal{K}$ faiblement admissible (et donc en fait admissible) dont $M$ est un réseau fortement divisible. Par contre, différents $M$ peuvent donner le même $D$. Nous noterons $D_M$ ce $\varphi$-module filtré sur $\mathcal{K}$ faiblement admissible construit à partir de $M$.

### 1.4 Le théorème de Fontaine-Laffaille

Définition 8. *Pour tout objet $M$ de $\mathbf{MF_{W,tf}}$ tel que $\mathrm{Fil}^1(M) = \{0\}$, soit $\mathbf{V_{cris}}(M)$ la représentation galoisienne définie par :*

$$\mathbf{V_{cris}}(M) = \left(\mathrm{Fil}^0(M \otimes_W A_{cris})\right)^{\varphi^0}$$

*Si $M$ est libre comme $W$-module, $\mathbf{V_{cris}}(M)$ est un $\mathbb{Z}_p$-module libre (c'est un sous-réseau de $\mathbf{V_{cris,p}}(D_M)$).*

Théorème 9 (Théorème de Fontaine-Laffaille). *Si nous nous restreignons à la sous-catégorie pleine des $M$ vérifiant $\mathrm{Fil}^{2-p}(M) = M$ et $\mathrm{Fil}^1(M) = \{0\}$, alors le foncteur $\mathbf{V_{cris}}$ ainsi défini est exact et pleinement fidèle. De plus si $M$ est libre sur $W$, $\mathbf{V_{cris}}(M)$ est un réseau de la représentation galoisienne associée à $D_M$ (c'est-à-dire que $\mathrm{rg}_{\mathbb{Z}_p}(\mathbf{V_{cris}}(M)) = \mathrm{rg}_W(M)$).*

Nous noterons $\mathbf{D_{cris}}$ un quasi-inverse à $\mathbf{V_{cris}}$ (Il est donc défini sur la catégorie formée par les réseaux des représentations cristallines sur $\mathbb{Q}_p$ de $\Gamma_{\mathcal{K}}$ à poids de Hodge-Tate dans $[\![0, p-2]\!]$, et leurs quotients, à valeurs dans $\mathbf{MF^{2-p}_{W,tf}}$).

## 2 Construction du foncteur

### 2.1 Rappels sur $\mathbf{\Gamma_0\Phi M^h_{S_0}}$

Notons $\mathbf{\Gamma_0\Phi M^h_{S_0}}$ ($\mathbf{\Gamma\Phi M^h_S}$ se définit de la même façon) la sous-catégorie pleine de la catégorie des $(\varphi, \Gamma_0)$-modules sur $S_0$ (cf. paragraphe 1.1) formée des objets $\mathcal{N}$ vérifiant :

– le $S_0$-module sous-jacent est de type fini et sans $p'$-torsion (i.e. pour tout élément irréductible $\lambda$ de $S_0$ non associé à $p$, $\mathcal{N}$ est sans $\lambda$-torsion),
– le $S_0$-module $\mathcal{N}/\varphi(\mathcal{N} \otimes_\sigma S_0)$ est annulé par $q^h$ (où $q = \pi_0 + p$),
– le groupe $\Gamma_0$ agit trivialement sur $\mathcal{N}/\pi_0\mathcal{N}$.

Elle est abélienne si $0 \le h \le p-2$, et l'inclusion $j : S_0 \to \mathcal{O}_{\mathcal{E}}$ induit un foncteur $j^* : \mathbf{\Gamma_0\Phi M^h_{S_0}} \to \mathbf{\Gamma\Phi M^{\text{ét}}_{\mathcal{O}_{\mathcal{E}}}}$ pleinement fidèle qui est une équivalence de catégorie pour $0 \le h \le p-2$ sur son image essentielle (cf. [Fon90], p.301). Si $\mathcal{N}$ est un objet de $\mathbf{\Gamma_0\Phi M^h_{S_0}}$, alors $j^*(\mathcal{N})$ a pour espace sous-jacent $\mathcal{N} \otimes_{S_0} \mathcal{O}_{\mathcal{E}}$. Nous ferons souvent l'abus de notation de n'écrire que l'espace sous-jacent pour désigner $j^*(\mathcal{N})$.

Si $0 \leq h \leq p-2$ et $\mathcal{N}$ un objet de $\mathbf{\Gamma_0\Phi M^h_{S_0}}$, N. Wach a montré qu'il est possible de munir $N = \mathcal{N}/\pi_0 N$ d'une structure de $\varphi$-module filtré sur $W$ en posant

$$\mathrm{Fil}^r N = \{x \in N \text{ tels qu'il existe un relèvement } \widehat{x} \in \mathcal{N} \text{ de } x \text{ avec } \varphi(\widehat{x}) \in q^r \mathcal{N}\}$$

et pour tout $x \in \mathrm{Fil}^r N$, $\varphi^r(x)$ égal à l'image de $\frac{\varphi(\widehat{x})}{q^r}$ dans $N$. Elle a alors démontré le théorème suivant (cf. [Wac97], p.231) :

Théorème 10. *Soit $0 \leq h \leq p-2$. Pour tout objet $\mathcal{N}$ de $\mathbf{\Gamma_0\Phi M^h_{S_0}}$, le $\varphi$-module filtré $i^*(\mathcal{N}) = \mathcal{N}/\pi_0\mathcal{N}$ est un objet de $\mathbf{MF^h_{W,tf}}$ ; le foncteur $i^*$ ainsi défini est exact et fidèle.*

## 2.2 Foncteur entre $\mathbf{MF^h_W}$ et $\mathbf{\Gamma_0\Phi M^h_{S_0}}$

N. Wach a donné les idées pour construire un quasi-inverse à $i^*$ : à partir d'un objet $N$ de $\mathbf{MF^h_W}$ avec $0 \leq h \leq p-2$ et d'une base adaptée à la filtration, elle a construit un objet $\mathcal{N}$ tel que $i^*(\mathcal{N}) = N$. Nous allons montrer qu'en se fixant un scindage fonctoriel de la filtration, nous rendons cette construction fonctorielle.

Proposition 11. *Soit $\mathbf{MF^+_{W,tf}}$ la sous-catégorie pleine formée de la réunion des $\mathbf{MF^h_{W,tf}}$ (définition analogue pour $\mathbf{\Gamma_0\Phi M^+_{S_0}}$). A tout scindage fonctoriel de la filtration des objets de $\mathbf{MF^+_{W,tf}}$ nous pouvons associer un foncteur de $\mathbf{MF^+_{W,tf}}$ vers $\mathbf{\Phi M^{\acute{e}t}_{S_0}}$ (la catégorie des $\varphi$-modules sur $S_0$ dont l'extension à $\mathcal{O}_{\mathcal{E}}$ donne un $\varphi$-module étale), qui soit fidèle, additif, exact, et qui préserve le produit tensoriel.*

*Démonstration.* Si $N$ est un objet de $\mathbf{MF^+_{W,tf}}$, et $N = \oplus N_i$ le scindage de la filtration, il suffit de construire sur $N \otimes_W S_0$ une structure de $\varphi$-module par : l'application $\varphi^i$ étant défini sur $\mathrm{Fil}^i(N)$, elle se restreint à $N_i$, permettant de poser $\varphi_N$ égal à $q^i\varphi^i$ sur $N_i$, c'est-à-dire

$$\forall x \in N_i,\ \varphi_N(x) = q^i\varphi^i(x)$$

Nous prolongeons cette définition à $N \otimes_W S_0$ tout entier en utilisant la semi-linéarité de $\varphi_N$. Les propriétés de fonctorialité découlent alors de celles du scindage de la filtration. Au niveau des flèches, ce foncteur est construit de la manière suivante : si $f : N \to N'$ est un morphisme de $\varphi$-modules filtrés, le foncteur lui associe $f \otimes \mathrm{Id}$. □

Remarque 12. *Le fait qu'il existe un scindage de la filtration fonctoriel (notamment préservant le produit tensoriel) nous est donné par le théorème 7.*

Remarque 13. *Nous pouvons étendre ce foncteur de la même façon en un foncteur de la catégorie des $\varphi$-modules filtrés libres sur $W$ vers $\mathbf{\Phi M^{\acute{e}t}_{\mathcal{O}_{\mathcal{E}}}}$, qui préserve sous-objet, objet quotient, somme directe, produit tensoriel et dual.*

N. Wach a montré la proposition suivante (cf. le lemme 3.1.6 p.233 de [Wac97]) :

PROPOSITION 14. *Supposons* $0 \leq h \leq p-2$. *Alors pour tout objet* $N$ *de* $\mathbf{MF}_{\mathbf{W}}^{\mathbf{h}}$, *il existe une unique action de* $\Gamma_0$ *sur* $N \otimes_W S_0$ *triviale modulo* $\pi_0$ *et commutant au* $\varphi_N$ *construit comme dans la proposition 11. Le module* $N \otimes_W S_0$ *est alors muni d'une structure de* $(\varphi, \Gamma_0)$*-module sur* $S_0$ *et devient un objet de* $\mathbf{\Gamma_0 \Phi M}_{\mathbf{S_0}}^{\mathbf{h}}$.*.*

C'est le point de départ pour montrer le théorème suivant :

THÉORÈME 15. *Supposons* $0 \leq h \leq p-2$. *Il existe un* $\otimes$*-foncteur* $\mathrm{F}$ *additif, exact, fidèle et pleinement fidèle de* $\mathbf{MF}_{\mathbf{W},\mathbf{tf}} < \mathbf{h} >$ *dans* $\mathbf{\Gamma_0 \Phi M}_{\mathbf{S_0}}^{+}$, *qui composé avec le foncteur oubli donne juste le foncteur extension des scalaires de* $W$ *à* $S_0$. *De plus, il induit une équivalence de catégories entre* $\mathbf{MF}_{\mathbf{W},\mathbf{tf}}^{\mathbf{h}}$ *et* $\mathbf{\Gamma_0 \Phi M}_{\mathbf{S_0}}^{\mathbf{h}}$, *dont un quasi-inverse est* $i^*$.

*Démonstration.* La *première étape* consiste à construire $\mathrm{F}$ sur $\mathbf{MF}_{\mathbf{W}}^{\mathbf{h}}$. Soit $N$ un objet de $\mathbf{MF}_{\mathbf{W}}^{\mathbf{h}}$ (donc libre comme $W$-module). Considérons $N \otimes_W S_0$ : comme $0 \leq h \leq p-2$, il existe une unique action de $\Gamma_0$ sur $N \otimes_W S_0$ qui commute à $\varphi$ et est triviale modulo $\pi_0$ (c'est le lemme 3.1.6 p.233 de [Wac97]). Le $(\varphi, \Gamma_0)$ module ainsi défini, noté $\mathrm{F}(N)$, est bien un objet de $\mathbf{\Gamma_0 \Phi M}_{\mathbf{S_0}}^{\mathbf{h}}$. Il faut voir que nous définissons bien ainsi un foncteur. Comme la structure de $\varphi$-module provient d'un scindage de la filtration qui préserve le produit tensoriel, l'unicité de l'action de $\Gamma_0$ nous donnera bien que $\mathrm{F}$ préserve le produit tensoriel (tant que celui-ci reste dans $\mathbf{MF}_{\mathbf{W}}^{\mathbf{h}}$). L'exactitude provient de la même raison. N. Wach a montré (lemme 3.1.1.2 de [Wac97]) qu'il existe un unique générateur topologique $g_0$ de $\Gamma_0$ tel que $\frac{g_0(q)-q}{q\pi_0} = 1$ modulo $qS_0$. Il suffit donc d'étudier l'action de $g_0$. Choisissons une base adaptée à la graduation $(e_i)_{1 \leq i \leq d}$ (c'est-à-dire : si $r_i$ est le plus grand entier tel que $e_i \in \mathrm{Fil}^{r_i}(N)$, alors pour tout $r$, $(e_i)_{r_i = r}$ est une base de $N_r$), et si $(a_{i,j})$ est la matrice des applications $\varphi^r$ dans cette base, l'action de $\varphi$ est donné par :

$$\varphi(e_j) = q^{r_j} \sum_{1 \leq i \leq d} a_{i,j} e_i$$

Avant de montrer que $\mathrm{F}$ préserve les sous objets, nous allons étudier plus en détail l'action de $g_0$.
N. Wach construit l'action de $g_0$ sur $N \otimes_W S_0$ par récurrence modulo $\pi_0^n$. Nous avons besoin de voir cette action d'une autre façon : soit $G = (g_{i,j})$ la matrice dans $GL_{\mathrm{rg}(N)}(S_0)$ définie par $g_0(e_j) = \sum_i g_{i,j} e_i$, et $A = (a_{i,j}) \in GL_{\mathrm{rg}(N)}(W)$ donnant l'action de $\varphi^j$ sur $e_j$. Alors, en écrivant $\varphi \circ g_0(e_j) = \sum_{i,k} \varphi(g_{i,j}) a_{k,i} q^{r_i} e_k$ et $g_0 \circ \varphi(e_j) = \sum_{i,k} g(a_{i,j}) g_{k,i} g(q)^{r_j} e_k$, la commutativité $\varphi \circ g_0 = g_0 \circ \varphi$ nous donne pour $G$ l'équation $AQ\varphi(G) = Gg_0(A)g_0(Q)$ avec $Q$ la matrice correspondant à $Q(e_j) = q^{r_j} e_j$ (et $g_0(A) = A$ puisque $A$ est à coefficients dans $W$). Donc $G$ est un point fixe de l'application $f : H \mapsto AQ\varphi(H)g_0(Q^{-1})g_0(A^{-1})$ (et le lemme 3.1.6 p.233 de [Wac97] affirme juste l'unicité d'un tel point fixe à coefficients

dans $S_0$, qui soit congru à Id modulo $\pi_0$). Notons $I$ la matrice identité dans $GL_{\mathrm{rg}(N)}$ et $G_n = f^{(n)}(I)$ (c'est-à-dire la composée $n$ fois de $f$ appliquée à $I$). Alors, en utilisant que $G - I \in \pi_0 M_{\mathrm{rg}(N)}(S_0)$, nous allons montrer :

Lemme 16. *La matrice $G$ est la limite de la suite $G_n$.*

*Démonstration.* Notons $\varphi^{(n)}$ la composée $n$ fois de $\varphi$ et introduisons alors $B_n = AQ\varphi(A)\varphi(Q)\cdots\varphi^{(n-1)}(A)\varphi^{(n-1)}(Q)$ qui est une matrice à coefficients dans $S_0$. Nous avons $G_n = B_n\varphi^{(n)}(I)g_0(B_n^{-1})$, et comme $G$ est un point fixe de $f$, $G = B_n\varphi^{(n)}(G)g_0(B_n^{-1})$, d'où l'égalité $G_n - G = B_n\varphi^{(n)}(I-G)g_0(B_n^{-1})$. Notons $G = I - \pi_0 H$ avec $H \in M_{\mathrm{rg}(N)}(S_0)$, alors nous avons $G_n - G = \varphi^{(n)}(\pi_0)B_n\varphi^{(n)}(H)g_0(B_n^{-1})$. Or, comme $A$ est inversible (dans $GL_{\mathrm{rg}(N)}(W)$), les seuls dénominateurs possibles sont les puissances de $g_0(q)^{r_i}$, et comme $0 \leq r_i \leq p-2$, nous pouvons écrire $G_n - G = \frac{\varphi^{(n)}(\pi_0)}{g_0\left(q\varphi(q)\cdots\varphi^{n-1}(q)\right)^{p-2}}G'_n$ avec $G'_n = B_n\varphi^{(n)}(H)g_0\big(\varphi^{(n-1)}(q^{p-2}Q^{-1})\varphi^{(n-1)}(A^{-1})\cdots q^{p-2}Q^{-1}A^{-1}\big)$ qui est une matrice à coefficients dans $S_0$.
Donc tout revient à montrer que $\frac{\varphi^{(n)}(\pi_0)}{g_0\left(q\varphi(q)\cdots\varphi^{n-1}(q)\right)^{p-2}}$ tend vers 0. Nous avons $g_0(q) = v_g q$ avec $v_g$ inversible dans $S_0$, par conséquent le fait que $\varphi$ et $g_0$ commutent nous donne l'égalité

$$\frac{\varphi^{(n)}(\pi_0)}{g_0\big(q\varphi(q)\cdots\varphi^{n-1}(q)\big)^{p-2}} = \frac{(v_g\varphi(v_g)\cdots\varphi^{(n-1)}(v_g))^{2-p}}{(q\varphi(q)\cdots\varphi^{n-1}(q))^{p-2}}\varphi^{(n)}(\pi_0)$$

En utilisant que $\varphi(\pi_0) = u\pi_0 q^{p-1}$ pour $u$ un certain inversible dans $S_0$, nous obtenons que $\varphi^{(n)}(\pi_0) = (q\varphi(q)\cdots\varphi^{n-1}(q))^{p-1}\pi_0 u\varphi(u)\cdots\varphi^{(n-1)}(u)$. Donc,

$$\frac{\varphi^{(n)}(\pi_0)}{g_0\big(q\varphi(q)\cdots\varphi^{n-1}(q)\big)^{p-2}} = \pi_0\frac{u\varphi(u)\cdots\varphi^{(n-1)}(u)}{(v_g\varphi(v_g)\cdots\varphi^{(n-1)}(v_g))^{p-2}}q\varphi(q)\cdots\varphi^{(n-1)}(q)$$

et, puisque $q\varphi(q)\cdots\varphi^{(n-1)}(q)$ tend vers 0 dans $S_0$ ($q$ est dans l'idéal maximal de $S_0$, idéal qui est stable par $\varphi$), nous pouvons conclure que $\frac{\varphi^{(n)}(\pi_0)}{g_0\left(q\varphi(q)\cdots\varphi^{n-1}(q)\right)^{p-2}}$ tend vers 0 dans $S_0$, c'est à dire que $G_n$ tend vers $G$. □

Montrons alors la proposition suivante (qui est le point technique clé de cet article) :

Proposition 17. *Soit $N_{i,j}$ des objets de $\mathbf{MF}_{\mathbf{W}}^{\mathbf{h}}$ avec $0 \leq h \leq p-2$, $L$ un sous-objet (dans $\mathbf{MF}_{\mathbf{W}}^{+}$) de $M := \bigoplus_i \otimes_j N_{i,j}$, alors l'action de $\Gamma_0$ sur $\bigoplus_i \otimes_j \mathrm{F}(N_{i,j}) = M \otimes_W S_0$ laisse stable $L \otimes_W S_0$.*

*Démonstration.* Il suffit de le montrer pour l'action du générateur $g_0$ de $\Gamma_0$. Fixons pour chaque $N_{i,j}$ une base $(e_k^{(i,j)})$ adaptée à la graduation. Notons $G^{(i,j)}$ la matrice de l'action de $g_0$ sur cette base et $C^{(i,j)}$ la matrice donnant l'action de $\varphi$ sur $N_{i,j} \otimes_W S_0$ (avec les notations précédentes, $C = AQ$). Alors, par le lemme précédent nous avons $\lim_{n\to+\infty} G_n^{(i,j)} = G^{(i,j)}$ avec $C^{(i,j)}\varphi(G_n^{(i,j)})g_0(C^{(i,j)})^{-1} = G_{n+1}^{(i,j)}$ et $G_0^{(i,j)} = I^{(i,j)}$.
Prenons $(u[l])_l$ une base de $L$, et notons $(u[l]_k^{(i,j)})$ les coordonnées de $u[l]$ dans la base $(e_k^{(i,j)})$. Nous voulons montrer (par récurrence) que $\bigoplus_i \otimes_j G_{n+1}^{(i,j)} g_0(u_k^{(i,j)})$ est une combinaison linéaire (à coefficients dans $S_0$) des $(u[l]_k^{(i,j)})$, pour $u$ élément quelconque de $L \otimes_W S$ (et $(u_k^{(i,j)})$ ses coordonnées). Remarquons que par linéarité, il suffit de le montrer pour $u$ égal aux $u[l]$.
Comme $L$ est un sous-objet de $M$, nous avons $L \otimes_W S_0$ qui est stable par $\varphi$. Or $\varphi$ induit une bijection de $L \otimes_W S_0[\frac{1}{q}]$. Cela se traduit alors en disant $\bigoplus_i \otimes_j C^{(i,j)}\varphi(u[l']_k^{(i,j)})$ est une combinaison linéaire (à coefficients dans $S_0$) des $(u[l]_k^{(i,j)})$, et qu'il existe $N \in \mathbb{N}$ tel que $q^N\big(\otimes_j C^{(i,j)}\big)^{-1}(u[l']_k^{(i,j)})$ est une combinaison linéaire (à coefficients dans $S_0$) des $\varphi(u[l]_k^{(i,j)})$.
Par conséquent, $g_0(q)^N g_0\big(\otimes_j C^{(i,j)}\big)^{-1}\big(g_0(u[l']_k^{(i,j)})\big)$ s'écrit comme une combinaison linéaire (à coefficients dans $S_0$) des $\big(g_0(\varphi(u[l]_k^{(i,j)}))\big)$, ceci pour tout $l'$.
Puis, $\bigoplus_i \otimes_j \varphi(G_n^{(i,j)})g_0\big(\varphi(u[l']_k^{(i,j)})\big) = \varphi\big(\bigoplus_i \otimes_j G_n^{(i,j)} g_0(u[l']_k^{(i,j)})\big)$ est pour tout $l'$ une combinaison linéaire (à coefficients dans $S_0$) des $\big(\varphi(u[l]_k^{(i,j)})\big)$, cela provient de notre hypothèse de récurrence.
En reprenant que $\bigoplus_i \otimes_j C^{(i,j)}\varphi(u[l']_k^{(i,j)})$ est une combinaison linéaire (à coefficients dans $S_0$) des $(u[l]_k^{(i,j)})$ pour tout $l'$, et en mettant bout à bout ces affirmations, nous obtenons que

$$g_0(q)^N \bigoplus_i \otimes_j G_{n+1}^{(i,j)} g_0(u[l']_k^{(i,j)}) = \\ g_0(q)^N \bigoplus_i \otimes C^{(i,j)} \otimes \varphi(G_n^{(i,j)}) g_0\big(\otimes C^{(i,j)}\big)^{-1}(g_0(u[l']_k^{(i,j)}))$$

est pour tout $l'$ une combinaison linéaire (à coefficients dans $S_0$) des $(u[l]_k^{(i,j)})$.
Par conséquent, si $g^{[n]}$ désigne l'application $g_0$-linéaire construite à partir de la matrice $\bigoplus_i \otimes_j G_n^{(i,j)}$ (l'hypothèse de récurrence se traduisant par : $L \otimes_W S_0$ est stable par $g^{[n]}$), alors $g^{[n+1]}(L \otimes_W S_0) \subset \frac{1}{g_0(q)^N} L \otimes_W S_0 = \frac{1}{q^N} L \otimes_W S_0$.
Considérons alors $(f_r)_{1\le r\le \mathrm{rg}_W(M)}$ une base de $M$ telle qu'il existe $n_r \in \mathbb{N} \cup \{+\infty\}$ avec $(p^{n_r} f_r)$ base de $L$. Alors $g^{[n+1]}(p^{n_r} f_r) = \sum_s \frac{\alpha_s}{q^N} p^{n_s} f_s$ avec $\alpha_s \in S_0$ (qui dépend de $r$). Mais, par construction, $g^{[n+1]}(M \otimes_W S_0) \subset M \otimes_W S_0$, alors $g^{[n+1]}(p^{n_r} f_r) = \sum_s p^{n_r} \beta_s f_s$ avec $\beta_s \in S_0$ (qui dépend aussi de $r$). D'où $p^{n_r}\beta_s = \frac{\alpha_s}{q^N} p^{n_s}$, ce qui implique que $q^N$ divise $\alpha_s$ dans $S_0$, donc que $g^{[n+1]}(L \otimes_W S_0) \subset L \otimes_W S_0$, ce qui montre bien la récurrence.

Pour initialiser la récurrence ($n = 0$) nous avons $G_0^{(i,j)} = I^{(i,j)}$ (où $I$ est la matrice identité), donc $\bigoplus_i \otimes_j G_0^{(i,j)} g_0(u_k^{(i,j)}) = u_k^{(i,j)}$ pour tout $u$ dans $L$. D'où par récurrence la propriété est vraie pour tout $n$. En passant à la limite, la propriété est vrai pour $\bigoplus_i \otimes_j G^{(i,j)}$. Donc l'action de $g_0$ sur $M \otimes_W S_0$ laisse stable $L \otimes_W S_0$. □

Cette proposition est le coeur du théorème. Elle nous donne en particulier que si $N'$ est un sous-objet de $N$ dans $\mathbf{MF_W^h}$, alors l'action de $g_0$ sur $N \otimes_W S_0$ laisse stable $N' \otimes_W S_0$. Elle est triviale modulo $\pi_0$ : si $(e_i)$ est une base de $N$, telle qu'il existe $(\alpha_i) \in \mathbb{N} \cup \{+\infty\}$ avec $(p^{\alpha_i} e_i)$ base de $N'$, alors il existe des coefficients $x_{i,j}$ et $y_{i,j}$ dans $S_0$ tels que $g_0(e_i) = e_i + \pi \sum_{j \neq i} x_{i,j} e_j$ et $g_0(p^{\alpha_i} e_i) = \sum_j y_{i,j} p^{\alpha_j} e_j$. En identifiant les coordonnées, nous avons $y_{i,i} = 1$ et $y_{i,j} p^{\alpha_j} = p^{\alpha_i} x_{i,j} \pi$ si $j \neq i$, donc $\pi$ divise bien $y_{i,j}$ dans $S_0$ pour $j \neq i$. Donc l'action de $g_0$ sur $\mathrm{F}(N)$ se restreint en une action triviale modulo $\pi_0$ sur $N' \otimes_W S_0$ qui commute à $\varphi$, donc par unicité cette action est celle de $\mathrm{F}(N')$.
La *deuxième étape* consiste alors à définir $\mathrm{F}$ sur tout $\mathbf{MF_{W,tf}} < \mathbf{h} >$. Le point important est que pour tout objet $M$ de $\mathbf{MF_{W,tf}} < \mathbf{h} >$, il existe des objets $N_{i,j}$ dans $\mathbf{MF_W^h}$ et $L$ un sous-objet de $\bigoplus_i \otimes_j N_{i,j}$ tels que $M$ est isomorphe à un quotient $M'$ de $L$. Considérons alors $N$ un sous-objet de $M$, et supposons que sur $M \otimes_W S_0$ nous ayons une structure de $(\varphi, \Gamma_0)$-module qui le rende isomorphe à $M' \otimes_W S_0$ muni de la structure de $(\varphi, \Gamma_0)$-module obtenu à partir de celle de $L \otimes_W S_0$ donnée par la proposition 17. Il faut voir que $N' \otimes_W S_0$ (où $N'$ est l'image de $N$ dans $M'$) est stable par $\Gamma_0$. En notant $\pi : L \to M'$ la projection naturelle, $\pi^{-1}(N')$ est un sous-objet de $L$ (car c'est le noyau du morphisme $L \to M'/N'$, donc par la remarque 1.4.2 et la proposition 1.4.1 de [Win84], c'est bien un sous-objet de $L$), donc la proposition 17 nous donne bien que $\pi^{-1}(N') \otimes_W S_0$ est stable par l'action de $\Gamma_0$. Par conséquent, $N \otimes_W S_0$ sera bien laissé stable par l'action de $\Gamma_0$ de $M \otimes_W S_0$, donc sera un sous-$(\varphi, \Gamma_0)$-module de $M \otimes_W S_0$.
Puis, $\mathrm{F}$ se construit par itération : notons $\mathbf{MF_n}$ la sous-catégorie pleine de $\mathbf{MF_{W,tf}}$, construite en disant qu'un objet de $\mathbf{MF_{n+1}}$ est soit le sous-objet ou le quotient d'un objet de $\mathbf{MF_n}$, soit la somme directe de deux objets de $\mathbf{MF_n}$, soit le produit tensoriel de deux objets de $\mathbf{MF_n}$, soit un objet de $\mathbf{MF_n}$, et posons (pour initialiser la récurrence) $\mathbf{MF_0} = \mathbf{MF_W^h}$. Alors, $\mathbf{MF_{W,tf}} < \mathbf{h} > = \bigcup \mathbf{MF_n}$, et si $\mathrm{F}$ est construit sur $\mathbf{MF_n}$, alors il s'étend naturellement à la somme directe ou le produit tensoriel de deux objets de $\mathbf{MF_n}$, et l'étude précédente montre qu'il s'étend au cas d'un sous-objet, et donc d'un objet quotient, d'un objet de $\mathbf{MF_n}$. Nous pouvons donc donner naturellement une structure de $(\varphi, \Gamma_0)$-module à tout $M \otimes_W S_0$, pour $M$ un objet de $\mathbf{MF_{W,tf}} < \mathbf{h} >$.
La *troisième étape* s'occupe des morphismes. Pour $M$ et $M'$ deux objets de $\mathbf{MF_{W,tf}} < \mathbf{h} >$, et $f : M \to M'$ un morphisme, nous posons $\mathrm{F}(f) = f \otimes \mathrm{Id}$ (En particulier, le foncteur sera exact (car $S_0$ est plat sur $W$) et fidèle). C'est un morphisme de $(\varphi, \Gamma_0)$ module : par construction, c'est un morphisme de $\varphi$-modules. Puis, $f \oplus \mathrm{Id} : M \oplus M' \to M' \oplus M'$ est un morphisme de $\varphi$-modules filtrés, donc par la proposition 1.4.1 de [Win84], $\mathrm{Ker}(f \oplus \mathrm{Id}) = \{x - y | x \in M,\ y \in$

$M'$, $y = f(x)\}$ est naturellement un $\varphi$-module filtré, sous-objet de $M \oplus M'$, donc par la proposition 17, $\mathrm{Ker}(f \oplus \mathrm{Id}) \otimes_W S_0$ est laissé stable par l'action naturelle de $\Gamma_0$ sur $(M \oplus M') \otimes_W S_0$ (obtenue à partir de celle sur $M \otimes_W S_0$ et $M' \otimes_W S_0$), donc $f$ commute à l'action de $\Gamma_0$, car si $x - y \in \mathrm{Ker}(f \oplus \mathrm{Id}) \otimes_W S_0$, dire que $g(x) - g(y) \in \mathrm{Ker}(f \oplus \mathrm{Id}) \otimes_W S_0$, c'est dire que $f(g(y)) = g(x) = g(f(y))$.
Montrons la pleine fidélité. Remarquons que si nous munissons $M \otimes_W S_0$ de la structure de $\varphi$-module filtré donnée par $\mathrm{Fil}^i(M \otimes_W S_0) = \{x \in M \otimes_W S_0 | \varphi(x) \in q^i M \otimes_W S_0\}$, et $\varphi^i = \frac{1}{q^i}\varphi$, alors la restriction modulo $\pi_0$ est un morphisme de $\varphi$-module filtré. Par conséquent, si $f : M \otimes_W S_0 \to M' \otimes_W S_0$ est un morphisme de $(\varphi, \Gamma_0)$-module, alors la réduction modulo $\pi_0$ induit $\bar{f} : M \to M'$ un morphisme de $\varphi$-modules filtrés. Donc $\bar{f} \otimes \mathrm{Id} : M \otimes_W S_0 \to M' \otimes_W S_0$ est un morphisme de $(\varphi, \Gamma_0)$-module par le résultat précédent, donc $g = f - \bar{f} \otimes \mathrm{Id}$ aussi, et il se réduit modulo $\pi_0$ sur l'application nulle. Notons $\mathcal{M}''$ le noyau de $g$, c'est un sous $(\varphi, \Gamma_0)$-module de $M \otimes_W S_0$. Nous avons $\pi_0 \mathcal{M}'' = \mathcal{M}'' \cap \pi_0 M \otimes_W S_0$ car $M' \otimes_W S_0$ est sans $\pi_0$-torsion. Donc $\mathcal{M}''/\pi_0\mathcal{M}'' \subset M$, et par le lemme du serpent, nous avons égalité, car :

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}'' \arrow[r] \arrow[d, "u_1"] & M \otimes_W S_0 \arrow[r, "g"] \arrow[d, "u_2"] & M \otimes_W S_0 \arrow[r] \arrow[d, "u_3"] & 0 \\
0 \arrow[r] & M \arrow[r] & M \arrow[r, "\bar{g}"] & M \arrow[r] & 0
\end{tikzcd}$$

les lignes horizontales sont exactes, $u_1$ est la réduction modulo $\pi_0$ composée avec l'inclusion $\mathcal{M}''/\pi_0\mathcal{M}'' \subset M$, $u_2$ et $u_3$ sont la réduction modulo $\pi_0$, $u_2$ est surjectif ($u_3$ aussi), et l'application naturelle $\mathrm{Ker}(u_2) = \pi_0 M \otimes_W S_0 \to \mathrm{Ker}(u_3) = \pi_0 M \otimes_W S_0$ est surjective. Donc, nous avons $M \otimes_W S_0 = \mathcal{M}'' + \pi_0 M \otimes_W S_0$, l'idéal engendré par $\pi_0$ est inclus dans le radical de Jacobson de $S_0$, $M \otimes_W S_0$ est de type fini sur $S_0$, donc par le lemme de Nakayama, nous avons $M \otimes_W S_0 = \mathcal{M}''$, donc $f = \bar{f} \otimes \mathrm{Id}$. Par conséquent le foncteur est pleinement fidèle.
La *quatrième étape* est l'étude du foncteur restreint à $\mathbf{MF^h_{W,tf}}$. Par construction, nous avons $i^* \mathrm{F}(N) = N$ pour tout objet $N$ de $\mathbf{MF^h_{W,tf}}$. Montrons :

Lemme 18. *Pour tout objet $\mathcal{N}$ de $\mathbf{\Gamma_0 \Phi M^h_{S_0}}$, libre comme $S_0$-module, si $N = i^*(\mathcal{N})$, il existe un unique isomorphisme de $(\varphi, \Gamma_0)$-module $\mathrm{F}(N) \to \mathcal{N}$ (qui se réduit modulo $\pi_0$ sur l'égalité $N = i^*(\mathcal{N})$).*

*Démonstration.* Présentons ici une démonstration de ce fait due à N. Wach. Pour cela, considérons une base $(e_i)_{1 \leq i \leq d}$ de $N$, adaptée à la graduation, et $(a_{i,j})$ la matrice des applications $\varphi^r$ dans cette base (donc l'action de $\varphi$ est donnée sur $\mathrm{F}(N)$ par $\varphi(e_j) = q^{r_j} \sum_{1 \leq i \leq d} a_{i,j} e_i$). Il faut alors prouver l'existence et l'unicité d'une base $(f_i)$ dans $\mathcal{N}$ vérifiant $\varphi(f_j) = q^{r_j} \sum_{1 \leq i \leq d} a_{i,j} f_i$ avec $e_i = f_i$ modulo $\pi_0$. Ce sera suffisant car en posant $h(e_i) = f_i$, nous aurons un

morphisme de $\varphi$-module, qui fera commuter l'action de $\Gamma_0$ par unicité de celle-ci, et qui modulo $\pi_0$ redonnera l'identité.
Par construction du $\varphi$-module filtré $N$, la base $(e_i)$ se relève en une famille $(\hat{e}_i)$ de $\mathcal{N}$ avec $\varphi(\hat{e}_i) \in q^{r_i}\mathcal{N}$. De plus, $\mathcal{N}$ est complet pour la topologie $\pi_0$-adique (car $S_0$ l'est), et modulo $\pi_0$, $(e_i)$ est une base, donc $\mathcal{N}$ étant sans torsion, $(\hat{e}_i)$ est une base de $\mathcal{N}$ (nous pourrions aussi invoquer le lemme de Nakayama). Donc, il existe $\hat{a}_{i,j} \in S_0$ tels que :

$$\varphi(\hat{e}_j) = q^{r_j} \sum_{1 \leq i \leq d} \hat{a}_{i,j}\hat{e}_i$$

et $\hat{a}_{i,j} = a_{i,j}$ modulo $\pi_0$. Posons $\alpha_{i,j} \in S_0$ l'unique élément tel que $\hat{a}_{i,j} = a_{i,j} + \pi_0\alpha_{i,j}$. Nous cherchons à modifier la base $(\hat{e}_i)$ pour obtenir la base $(f_i)$. Cherchons $f_j$ sous la forme $f_j = \hat{e}_j + \pi_0 c_j$, et posons $b_j = \sum_{1\leq i\leq d} \alpha_{i,j}\hat{e}_i$. Alors, puisque $\varphi(\pi_0) = uq^{p-1}\pi_0$,
$\varphi(\hat{e}_j + \pi_0 c_j) = \varphi(\hat{e}_j) + u\pi_0 q^{p-1}\varphi(c_j)$, et en faisant apparaître $\sum_{i=1}^{d} q^{r_j} a_{i,j}\pi_0 c_i$, nous obtenons
$\varphi(\hat{e}_j + \pi_0 c_j) = \sum_{i=1}^{d} q^{r_j} a_{i,j}(\hat{e}_i + \pi_0 c_i) + \pi_0 q^{r_j} b_j + u\pi_0 q^{p-1}\varphi(c_j) - \sum_{i=1}^{d} q^{r_j} a_{i,j}\pi_0 c_i$
autrement dit, nous cherchons les $c_j \in \mathcal{N}$ tels que

$$b_j + uq^{p-1-r_j}\varphi(c_j) - \sum_{1\leq i\leq d} a_{i,j}c_i = 0$$

Nous résolvons ce système de manière unique par récurrence modulo $\pi_0^n$. A chaque étape, le système se résout en faisant une récurrence modulo $p^k$, en utilisant que $p - 1 - r_j \geq 1$ (par hypothèse), donc que $q^{p-1-r_j} = 0$ modulo $(p, \pi_0)$, et que la matrice $(a_{i,j})$ est inversible modulo $p$. □

Pour terminer la démonstration du théorème (c'est à dire prouver le lemme précédent sans l'hypothèse sur la liberté de $N$), nous aurons besoin de résultats sur les modules de Wach, qui apparaitront plus loin dans l'article. La fin de la démonstration sera faite à la section 4. □

Pour la suite, nous aurons besoin de faire intervenir un foncteur légèrement différent. Si $N$ est un objet de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$, son dual $N^* = \mathrm{Hom}_{\mathbb{Z}_p}(N, \mathbb{Z}_p)$ est un objet de $\mathbf{MF}_{\mathbf{W}}^{\mathbf{h}}$, donc $\mathrm{F}(N^*)$ est bien défini.

Définition 19. *Le foncteur* $\mathrm{F}^-$ *est défini sur* $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$ *pour* $h \leq p-2$ *par :*

$$\mathrm{F}^-(N) = \left(\mathrm{F}(N^*) \otimes_{S_0} \mathcal{O}_{\mathcal{E}}\right)^* \ = N \otimes_W \mathcal{O}_{\mathcal{E}}$$

*pour tout objet* $N$ *de* $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$*. Il donne bien un* $(\varphi, \Gamma)$*-module étale sur* $\mathcal{O}_{\mathcal{E}}$ *(donc est à valeurs dans* $\mathbf{\Gamma\Phi M}_{\mathcal{O}_{\mathcal{E}}}^{\acute{e}t}$*). Il s'étend de même à* $\mathbf{MF}_{\mathbf{W}} < -\mathbf{h} >$.

REMARQUE 20. *Le foncteur* $\mathrm{F}$ *consiste à munir* $N \otimes_W S_0$ *(pour* $N$ *objet de* $\mathbf{MF_W} < \mathbf{h} >$*) d'une structure de* $(\varphi, \Gamma_0)$*-module, et pour avoir un foncteur défini sur* $\mathbf{MF_W^{-h}}$*, nous prenons le dual. Pour retrouver exactement les résultats du théorème 1 cité dans l'introduction, il faudrait définir* $\mathrm{F}^-$ *par* $\mathrm{F}^-(N) = \mathrm{F}(N^*)^*$*. Cet objet est le dual d'un* $(\varphi, \Gamma_0)$*-module sur* $S_0$ *de hauteur* $h$*, ce n'est donc pas un* $(\varphi, \Gamma_0)$*-module, car l'action de* $\varphi$ *construite par dualité ne le laisse pas stable (de manière générale, le dual d'un* $\varphi$*-module sur* $S_0$ *n'est pas un* $\varphi$*-module). Mais nous verrons plus tard que c'est un module de Wach, au sens de [Ber04]. En étendant les scalaires à* $\mathcal{O}_{\mathcal{E}}$*, nous retrouvons le foncteur donné dans la définition précédente. Pour éviter d'avoir à introduire la catégorie des duaux des* $(\varphi, \Gamma_0)$*-module sur* $S_0$ *de hauteur* $h$*, nous n'utilise rons que la définition 19 (le théorème 1 se déduira alors directement du théorème 15, des propositions 31 et 35, de la remarque 39 et des propriétés des modules de Wach montrées par L. Berger).*

Remarquons que $\mathrm{F}^-$ peut être défini sur $\mathbf{MF_{W,tf}^{-h}}$ (puis sur $\mathbf{MF_{W,tf}} < -\mathbf{h} >$) en prenant pour un module de $p$-torsion le dual de Pontriaguine, et en passant à la limite projective pour le cas général.

REMARQUE 21. *Nous pouvons définir* $\widetilde{\mathrm{F}}$ *sur* $\mathbf{MF_W^{\pm h}}$ *pour* $h \leq \frac{p-2}{2}$ *en posant* $\widetilde{\mathrm{F}}(N) = \mathrm{F}(N \otimes_W W[h]) \otimes_{S_0} \mathcal{O}_{\mathcal{E}}[-h]$ *avec* $\mathcal{O}_{\mathcal{E}}[-h] = \mathrm{F}(W[h])^*$ *et* $W[-h]$ *l'objet de* $\mathbf{MF_W^{-h}}$ *dont le* $W$*-module sous-jacent est* $W$*, avec* $\mathrm{Fil}^i(W[-h]) = \begin{cases} W \text{ si } i \leq -h \\ 0 \text{ si } i > -h \end{cases}$ *et* $\varphi^{-h}(x) = \sigma(x)$*. Alors,* $\widetilde{\mathrm{F}}(N^*)$ *est canoniquement isomorphe à* $\widetilde{\mathrm{F}}(N)^*$ *(cela se voit à l'aide de l'unicité de l'action de* $\Gamma_0$ *agissant trivialement modulo* $\pi_0$*, et commutant à* $\varphi$*, d'après le lemme 3.1.6 de [Wac97]), et* $\widetilde{\mathrm{F}}$ *s'étend alors en un foncteur sur* $\mathbf{MF_W} < \pm\mathbf{h} >$ *qui a des propriétés similaires à celles de* $\mathrm{F}$*, et qui préserve le dual.*

## 3 LIEN ENTRE LE FONCTEUR ET LES MODULES DE WACH

### 3.1 FONCTORIALITÉ DE $g_N$

Rappelons le Théorème 1' de N. Wach (cf. [Wac97]) :

THÉORÈME 1'. *Si* $\mathcal{N}$ *est un objet de* $\mathbf{\Gamma_0 \Phi M_{S_0}^h}$ *avec* $0 \leq h \leq p-2$*, alors* $\mathrm{Hom}_{\mathbf{MF_W}}(i^*(\mathcal{N}), A_{cris})$ *est isomorphe (en tant que représentation galoisienne) à* $\mathrm{Hom}_{\mathbf{\Phi M_{S_0}}}(\mathcal{N}, \mathcal{O}_{\widehat{\mathcal{E}}_{nr}})$*.*

Enoncé dans le cadre (et avec les notations) qui nous intéresse, il devient :

THÉORÈME 1'. *Si* $N$ *est un objet de* $\mathbf{MF_{W,tf}^{-h}}$ *avec* $0 \leq h \leq p-2$*, alors il existe un isomorphisme* $g_N : \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(N)) \to \mathbf{V_{cris}}(N)$ *de représentations galoisiennes. Si en plus* $N$ *est libre, en passant au dual, cela donne un isomorphisme de représentations galoisiennes* ${}^t g_N^{-1} : \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}\left(\mathrm{F}(N^*) \otimes_{S_0} \mathcal{O}_{\mathcal{E}}\right) \to \mathbf{V_{cris}}(N)^*$*.*

Nous allons vérifier que cet isomorphisme est fonctoriel :

Théorème 22. *Pour tout objet $N$ de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$ avec $0 \le h \le p-2$, l'application $g_N$ construite par N. Wach vérifie les propriétés de fonctorialité suivante :*

1. *pour tout morphisme $f : N \to N'$ entre deux objets $N$ et $N'$ de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$, nous avons $\mathbf{V_{cris}}(f) \circ g_N = g_{N'} \circ \mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathrm{F}^-(f))$ (cela s'applique en particulier pour l'injection d'un sous-objet, ou pour la projection sur un objet quotient).*
2. *pour tout objet $N$ et $N'$ de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$, $g_{N\oplus N'} = g_N \oplus g_{N'}$ ;*
3. *pour tout objet $N$ et $N'$ de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$, pour tout sous-objet $L$ de $N \otimes N'$ tel que $L$ soit un objet de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$, l'application $g_N \otimes g_{N'}$ restreinte à $\mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathrm{F}^-(L))$ est égale à $g_L$. En particulier, si $N \otimes N'$ est un objet de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$, alors $g_{N\otimes N'} = g_N \otimes g_{N'}$ ;*

Remarque 23. *Le point (3) montre en particulier que $\mathbf{V_{cris}}(N \otimes N')$ est égal à $\mathbf{V_{cris}}(N) \otimes_{\mathbb{Z}_p} \mathbf{V_{cris}}(N')$ dès que $N$, $N'$ et $N \otimes N'$ sont des objets de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$ avec $0 \le h \le p-2$.*

Rappelons la construction de $g_N$ : N. Wach construit l'isomorphisme modulo $p^n$ pour tout $n$ à partir des morphismes d'anneaux (avec $A_S^+ = W(R) \cap \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$) : $A_S^+/p^n \to W_n(R)/\pi_0$ et $A_{cris}/p^n \to W_n(R)/\pi_0$. Notons $\mathcal{N} := \mathrm{F}^-(N)$. Nous avons la bijection $\mathbf{V}_{\mathbf{A}_\mathbf{S}^+}(\mathcal{N}/p^n) := (\mathcal{N} \otimes_{S_0} A_S^+/p^n)^\varphi \to (\mathcal{N} \otimes_{S_0} \mathcal{O}_{\mathcal{E}_{nr}/p^n})^\varphi$ (cf [Fon90], p.296, où c'est exprimé pour le foncteur contravariant). Or, N. Wach a montré que pour $N$ objet de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W}}$ avec $0 \le h \le p-2$, le schéma suivant

$$\begin{tikzcd}
N/p^n \otimes_W A_{cris}/p^n \arrow[r, "k_N"] & N/p^n \otimes_W W_n(R)/\pi_0 & \mathcal{N}/p^n \otimes_{S_0} A_S^+/p^n \arrow[l, "j_\mathcal{N}"'] \\
\mathbf{V_{cris}}(N/p^n) \arrow[u, hook, "k"] & & \mathbf{V}_{\mathbf{A}_\mathbf{S}^+}(\mathcal{N}/p^n) \arrow[u, hook, "j"]
\end{tikzcd}$$

induit un isomorphisme de représentations galoisiennes de $\mathbf{V}_{\mathbf{A}_\mathbf{S}^+}(\mathcal{N}/p^n)$ sur $\mathbf{V_{cris}}(N/p^n)$, c'est-à-dire que $K_N = k_N \circ k$ et $J_\mathcal{N} = j_\mathcal{N} \circ j$ sont toutes les deux injectives, et ont même image dans $N/p^n \otimes_W W_n(R)/\pi_0$.
Tout ceci passe à la limite projective, et nous obtenons l'application $g_N$ bijective :

$$\begin{tikzcd}
N \otimes_W A_{cris} \arrow[r, "k_N"] & N \otimes_W W(R)/\pi_0 & \mathcal{N} \otimes_{S_0} A_S^+ \arrow[l, "j_\mathcal{N}"'] \\
\mathbf{V_{cris}}(N) \arrow[u, hook, "k"] \arrow[ur, "K_N"'] & & \mathbf{V}_{\mathbf{A}_\mathbf{S}^+}(\mathcal{N}) \arrow[u, hook, "j"'] \arrow[ul, "J_\mathcal{N}"] \arrow[ll, "g_N"]
\end{tikzcd}$$

où $\mathbf{V}_{\mathbf{A}_\mathbf{S}^+}(\mathcal{N}) = \varprojlim_{n\in\mathbb{N}} \mathbf{V}_{\mathbf{A}_\mathbf{S}^+}(\mathcal{N}/p^n) = (\mathcal{N} \otimes_{S_0} A_S^+)^{\varphi=1} = \mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathcal{N} \otimes_{S_0} \mathcal{O}_\mathcal{E})$.

*Démonstration du théorème 22.* Pour la fonctorialité au niveau des flèches, il suffit de remarquer que le diagramme suivant est commutatif (car $\mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathrm{F}^-(f))$

est juste $f \otimes \mathrm{Id}$) :

$$\begin{tikzcd}
\mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathrm{F}^-(N)) \arrow[r, "\mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathrm{F}^-(f))"] \arrow[d, hook] & \mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathrm{F}^-(N')) \arrow[d, hook] \\
N \otimes A_S^+ \arrow[r, "f\otimes\mathrm{Id}"] \arrow[d, "j_\mathcal{L}"'] & N' \otimes A_S^+ \arrow[d, "j_{\mathcal{L}'}"] \\
N \otimes W(R)/\pi_0 \arrow[r, "f\otimes\mathrm{Id}"] & N' \otimes W(R)/\pi_0 \\
N \otimes A_{cris} \arrow[r, "f\otimes\mathrm{Id}"] \arrow[u, "k_L"] & N' \otimes A_{cris} \arrow[u, "k_{L'}"'] \\
\mathbf{V_{cris}}(N) \arrow[r, "\mathbf{V_{cris}}(f)"] \arrow[u, hook] & \mathbf{V_{cris}}(N') \arrow[u, hook]
\end{tikzcd}$$

Le fait que $g_{N\oplus N'} = g_N \oplus g_{N'}$ se montre de la même façon. Il reste donc à voir le cas du produit tensoriel : considérons $N$ et $N'$ deux objets de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$ avec $h \leq p-2$. Soit $L$ un sous-objet de $N \otimes N'$ qui est dans $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$, posons $\mathcal{L} = L \otimes_W S_0$. Le diagramme suivant est alors commutatif :

$$\begin{tikzcd}
L \otimes_W A_{cris} \arrow[r, hook] \arrow[d, "k_L"] & (N \otimes_W A_{cris}) \otimes_{A_{cris}} (N' \otimes_W A_{cris}) \arrow[d, "k_N \otimes k_{N'}"] \\
L \otimes_W W(R)/\pi_0 \arrow[r, hook] & \big(N \otimes_W W(R)/\pi_0\big) \otimes_{W(R)/\pi_0} \big(N' \otimes_W W(R)/\pi_0\big) \\
\mathcal{L} \otimes_{S_0} A_S^+ \arrow[r, hook] \arrow[u, "j_\mathcal{L}"] & (\mathcal{N} \otimes_{S_0} A_S^+) \otimes_{A_S^+} (\mathcal{N}' \otimes_{S_0} A_S^+) \arrow[u, "j_\mathcal{N} \otimes j_{\mathcal{N}'}"]
\end{tikzcd}$$

Par conséquent, l'application $K_N \otimes K_{N'}$ restreinte à $\mathbf{V_{cris}}(L)$ est égale à $K_L$, et l'application $J_\mathcal{N} \otimes J_{\mathcal{N}'}$ restreinte à $\mathbf{V}_{\mathbf{A_S^+}}(\mathcal{L})$ est égale à $J_\mathcal{L}$.

Le point important est que $L$ étant un objet de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$ (par hypothèse), ce sont bien des bijections, et ce sont celles qui permettent de construire $g_L$.

Donc $g_N \otimes g_{N'}$ envoie $\mathbf{V}_{\mathbf{A_S^+}}(\mathcal{L})$ sur $\mathbf{V_{cris}}(L)$ si $L$ est un sous-objet de $N \otimes N'$ qui soit dans $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$, et plus exactement, l'application $g_N \otimes g_{N'}$ restreinte à $\mathbf{V}_{\mathbf{A_S^+}}(\mathcal{L})$ est égale à $g_L$.

Si $N \otimes N'$ est un objet de $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$, le résultat précédent avec $L = N \otimes N'$ nous donne $g_{N\otimes N'} = g_N \otimes g_{N'}$. $\square$

Remarque 24. *Nous montrons de même que pour* $(N_{i,j})_{1\leq j\leq n,\,1\leq i\leq n_j}$ *objets de* $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$ *avec* $0 \leq h \leq p-2$*, et pour* $L$ *un sous-objet de* $\bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} N_{i,j}$ *qui soit dans* $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$*, alors* $\bigoplus \otimes g_{\mathrm{F}^-(N_{i,j})}$ *restreinte à* $\mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(L))$ *est égale à* $g_L$*.*

Nous pouvons traduire ces résultats en disant :

Théorème 25. *Soit* $0 \leq h \leq p-2$*, et notons* $\mathcal{G}$ *le foncteur exact de la catégorie* $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$ *vers la catégorie des représentations continues de* $\Gamma_{\mathcal{K}}$ *sur les* $\mathbb{Z}_p$*-modules de rang fini, défini par : si* $N$ *objet de* $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$*,* $\mathcal{G}(N) = \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(N))$*. Alors il existe* $g$ *un isomorphisme de foncteurs entre* $\mathcal{G}$ *et* $\mathbf{V}_{\mathbf{cris}}$*. De plus, nous pouvons supposer que :*

- *pour tous objet* $N$ *et* $N'$ *de* $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$*, tel que* $N \otimes N'$ *soit encore un objet de* $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$*, nous avons* $g_{N\otimes N'} = g_N \otimes g_{N'}$ *;*
- *pour tout uplet d'objets* $(N_{i,j})_{1\leq j\leq n,\,1\leq i\leq n_j}$ *de* $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$*, pour tout sous-objet* $L$ *(dans* $\mathbf{MF}^{-\mathbf{h}}_{\mathbf{W},\mathbf{tf}}$*) de* $\bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} N_{i,j}$*, l'application* $\bigoplus \otimes g_{N_{i,j}}$ *restreinte à* $\mathbf{V}_{\mathcal{O}_{\widehat{\mathcal{E}}}}(\mathrm{F}^-(L))$ *est égale à* $g_L$*.*

## 3.2 Lien entre $\mathbf{\Gamma_0\Phi M^h_{S_0}}$ et $\mathbf{\Gamma\Phi M^h_S}$

Avant de parler de modules de Wach (qui sont des $S$-modules), il faut comprendre l'extension des scalaires $S_0 \to S$.

Lemme 26. $S = \bigoplus_{0\leq i\leq p-2} S_i$*, où si* $x \in S_i$ *et* $g \in \Gamma_f$ *est* $[\alpha]$ *(le relèvement de Teichmuller de* $\alpha \in \mathbb{F}_p^*$*), alors* $g$ *agit sur* $x$ *par* $g(x) = [\alpha]^i x$*.*

*Démonstration.* L'application $p_i = \frac{1}{|\Gamma_f|} \sum_{g\in\Gamma_f} \mathcal{X}(g)^{-i} g$ est un projecteur dont l'image est $S_i$, et les $p_i$ vérifient $\sum_{0\leq i\leq p-2} p_i = \mathrm{Id}$. □

Lemme 27. *$S$ a une base normale sur $S_0$, c'est à dire qu'il existe $e \in S$ tel que $(g(e))_{g\in\Gamma_f}$ soit une base de $S$ sur $S_0$. De plus, $p$ ne divise aucun $p_i(e)$.*

*Démonstration.* En effet, il suffit de le montrer modulo $p$ (et ensuite de relever une base normale de $k[[\pi]]$ sur $k[[\pi_0]]$, puisque $S_0$ est complet pour la topologie $p$-adique). Or, Fontaine a montré dans [Fon90], page 270, que le corps des fractions de $k[[\pi]]$, $k((\pi))$, est une extension galoisienne cyclique de degré $p-1$ (donc modérément ramifiée) de $k((\pi_0))$, dont le groupe de Galois est donné par $\Gamma_f$. Donc, par un théorème de E. Noether, il existe une base normale pour les anneaux d'entiers correspondants. Enfin, si $\overline{e}$ est cette base (modulo $p$), alors $p_i(\overline{e}) = \sum_g \frac{\mathcal{X}(g)^{-i}}{|\Gamma_f|} g(\overline{e})$ est non nul (puisque chaque coordonnée suivant la base $(g(\overline{e}))$ est non nulle (même modulo $p$)), donc $p_i(e)$ sera bien non divisible par $p$ si $e$ relève $\overline{e}$. □

En particulier, nous avons $S_i = p_i(e)S_0$ (car $p_i(e)S_0 \subset S_i$, puis $e \in \oplus_i p_i(e)S_0$, $\oplus_i p_i(e)S_0$ est donc un $S_0$-module contenant $e$ et stable par $\Gamma_f$, donc $S = \oplus_i p_i(e)S_0 \subset \oplus_i S_i = S$). Puis, remarquons que $p_i \circ p_j = 0$ pour $i \neq j$, donc pour $\mathcal{M}$ un objet de $\mathbf{\Gamma\Phi M^h_S}$, nous avons les $p_i(\mathcal{M})$ en somme directe dans $\mathcal{M}$. Enfin, $p_i(e)\mathcal{M}^{\Gamma_f} \subset p_i(\mathcal{M})$, et $p_i(e)\mathcal{M}^{\Gamma_f}$ est isomorphe comme $S_0$-module à $\mathcal{M}^{\Gamma_f}$ car $\mathcal{M}$ est sans $p'$-torsion, et $p$ ne divise pas $p_i(e)$. Donc, nous avons que pour $\mathcal{M}$ un objet de $\mathbf{\Gamma\Phi M^h_S}$, $\mathcal{M}^{\Gamma_f} \otimes_{S_0} S = \oplus_i \mathcal{M}^{\Gamma_f} \otimes_{S_0} S_0 p_i(e)$ s'injecte dans $\mathcal{M}$.

PROPOSITION 28. *Soit $\mathcal{M}$ un objet de $\mathbf{\Gamma\Phi M^h_S}$, alors $\mathcal{M}^{\Gamma_f}$ est un objet de $\mathbf{\Gamma_0\Phi M^h_{S_0}}$, et $\mathcal{M} = \mathcal{M}^{\Gamma_f} \otimes_{S_0} S$. De plus, $\mathcal{M}^{\Gamma_f}/\pi_0 = \mathcal{M}/\pi$. Enfin, si $\mathcal{M}$ est $S$-libre, alors $\mathcal{M}^{\Gamma_f}$ est $S_0$-libre.*

*Démonstration.* Nous avons que $p_0(\mathcal{M}) = \mathcal{M}^{\Gamma_f}$. Or, comme l'action de $\Gamma_f$ est triviale modulo $\pi$, nous avons que pour tout $x \in \mathcal{M}$, $x - p_0(x) \in \pi\mathcal{M}$, donc si $\mathcal{N}$ est le $S$-module engendré par $\mathcal{M}^{\Gamma_f}$ (c'est à dire que $\mathcal{N} = \mathcal{M}^{\Gamma_f} \otimes_{S_0} S$ d'après la remarque précédent la proposition), alors $\mathcal{M} = \mathcal{N} + \pi\mathcal{M}$, donc comme $\mathcal{M}$ est de type fini sur $S$, et que l'idéal engendré par $\pi$ est dans le radical de $S$, le lemme de Nakayama nous donne que $\mathcal{M} = \mathcal{N}$.
Puis, $\mathcal{M}$ est de type fini sur $S$, donc sur $S_0$ (car $S$ est un $S_0$-module libre de rang fini par le lemme 27), donc engendré sur $S_0$ par exemple par la famille finie $(m_i)$. Alors, $p_0(\mathcal{M}) = \mathcal{M}^{\Gamma_f}$ est engendré par la famille $(p_0(m_i))$ (car $p_0$ est un morphisme de $S_0$-modules), donc est de type fini. De plus, $\mathcal{M}^{\Gamma_f}$ étant inclus dans $\mathcal{M}$, il est sans $p'$-torsion.
Ensuite, nous avons que $\pi\mathcal{M} \cap \mathcal{M}^{\Gamma_f} = \pi_0\mathcal{M}^{\Gamma_f}$ : pour $S$, l'égalité $\pi S \cap S_0 = \pi_0 S_0$ provient juste de ce que $\pi_0$ est un multiple de $\pi$, donc $\pi_0 S_0 \subset \pi S \cap S_0$, et pour la réciproque, que $S_0 = W[[\pi_0]]$. Cela se traduit par la suite exacte de $S_0$-modules

$$0 \longrightarrow \pi_0 S_0 \longrightarrow S \longrightarrow S/\pi S \oplus S/S_0 \longrightarrow 0$$

(la surjectivité vient juste de ce que $S/\pi = W$, et que $W \subset S_0$), et en tensorisant par $\mathcal{M}^{\Gamma_f}$ au dessus de $S_0$, nous avons la suite exacte de $S_0$-modules

$$0 \longrightarrow \pi_0\mathcal{M}^{\Gamma_f} \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\pi \oplus \mathcal{M}/\mathcal{M}^{\Gamma_f} \longrightarrow 0$$

ce qui traduit bien $\pi\mathcal{M} \cap \mathcal{M}^{\Gamma_f} = \pi_0\mathcal{M}^{\Gamma_f}$. Par conséquent, $\mathcal{M}^{\Gamma_f}/\pi_0$ s'injecte dans $\mathcal{M}/\pi$, et l'action de $\Gamma_0$ provient de celle sur $\mathcal{M}/\pi$, qui est triviale par définition. De plus, nous avons vu que pour tout $x \in \mathcal{M}$, $x - p_0(x) \in \pi\mathcal{M}$, donc comme $p_0(x) \in \mathcal{M}^{\Gamma_f}$, l'application naturelle $\mathcal{M}^{\Gamma_f}/\pi_0 \to \mathcal{M}/\pi$ (dont nous avons vu l'injectivité) est surjective. Par conséquent, si $\mathcal{M}$ est $S$-libre, $\mathcal{M}^{\Gamma_f}/\pi_0$ est $W$-libre et $\mathcal{M}^{\Gamma_f}$ est sans $\pi_0$-torsion, et donc $S_0$ étant complet pour la topologie $\pi_0$-adique, une $W$-base de $\mathcal{M}^{\Gamma_f}/\pi_0$ se relève en une $S_0$-base de $\mathcal{M}^{\Gamma_f}$.
Enfin, $\varphi$ commute à $\Gamma$, donc laisse stable $\mathcal{M}^{\Gamma_f}$, donc induit un morphisme $\varphi_0 : \mathcal{M}^{\Gamma_f} \otimes_\sigma S_0 \to \mathcal{M}^{\Gamma_f}$. Pour étudier le conoyau, remarquons d'abord que $x \otimes y \in S_0 \otimes_\sigma S_0 \mapsto \varphi(x)y \in S_0$ et $x \otimes y \in S \otimes_\sigma S \mapsto \varphi(x)y \in S$ sont des

isomorphismes (préservant l'action naturelle de $\Gamma_f$), donc $S_0 \otimes_{\sigma(S_0)} S_0[\frac{1}{q}] \simeq S_0[\frac{1}{q}]$ et $S \otimes_{\sigma(S)} S[\frac{1}{q}] \simeq S[\frac{1}{q}]$ (puisque $S[\frac{1}{q}]$ est plat sur $S$ et $S_0[\frac{1}{q}]$ est plat sur $S_0$). Par conséquent, $S_0 \otimes_{\sigma(S_0)} S \simeq S \otimes_{\sigma(S)} S$ et $S_0 \otimes_{\sigma(S_0)} S[\frac{1}{q}] \simeq S \otimes_{\sigma(S)} S[\frac{1}{q}]$ ; plus présicément, si $y_i \in S \otimes_{\sigma(S)} S$ s'envoye dans $S$ sur $p_i(e)$ (nous pouvons supposer que $y_0 = 1$ car $p_0(e)$ est inversible dans $S_0$), alors $S \otimes_{\sigma(S)} S = \oplus_i S_0 \otimes_{\sigma(S_0)} S_0 y_i$ et $S \otimes_{\sigma(S)} S[\frac{1}{q}] = \oplus_i S_0 \otimes_{\sigma(S_0)} S_0[\frac{1}{q}] y_i$ (c'est bien le même $y_i$, car $S \otimes_{\sigma(S)} S$ s'injecte dans $S \otimes_{\sigma(S)} S[\frac{1}{q}]$, puisque $S \otimes_{\sigma(S)} S$ est sans $q$-torsion). Et l'action naturelle de $\Gamma_f$ sur $S \otimes_{\sigma(S)} S[\frac{1}{q}]$ revient à dire que $g(y_i) = \mathcal{X}(g)^i y_i$ pour $g \in \Gamma_f$. Puisque $\mathcal{M} = \mathcal{M}^{\Gamma_f} \otimes_{S_0} S$, nous avons que $\mathcal{M} \otimes_{\sigma(S)} S[\frac{1}{q}] = \mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S[\frac{1}{q}] = \oplus_i \mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0[\frac{1}{q}] y_i$. Donc, $\mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0[\frac{1}{q}]$ s'injecte naturellement dans $\mathcal{M} \otimes_{\sigma(S)} S[\frac{1}{q}]$, et $(\mathcal{M} \otimes_{\sigma(S)} S[\frac{1}{q}])^{\Gamma_f} = \mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0[\frac{1}{q}]$.
Ensuite, $\varphi : \mathcal{M} \otimes_\sigma S \to \mathcal{M}$ est injective, de conoyau tué par $q^h$ (par définition), donc comme $S[\frac{1}{q}]$ est plat sur $S$, $\varphi$ induit une bijection $\varphi : \mathcal{M} \otimes_{\sigma(S)} S[\frac{1}{q}] \to \mathcal{M} \otimes_S S[\frac{1}{q}]$. Puis, $S[\frac{1}{q}] = \oplus_i S_0[\frac{1}{q}] p_i(e)$, donc $\mathcal{M} \otimes_S S[\frac{1}{q}] = \mathcal{M}^{\Gamma_f} \otimes_{S_0} S[\frac{1}{q}] = \oplus_i \mathcal{M}^{\Gamma_f} \otimes_{S_0} S_0[\frac{1}{q}] p_i(e)$, donc $\mathcal{M}^{\Gamma_f} \otimes_{S_0} S_0[\frac{1}{q}]$ s'injecte dans $\mathcal{M} \otimes_S S[\frac{1}{q}]$ et $\mathcal{M}^{\Gamma_f} \otimes_{S_0} S[\frac{1}{q}] = (\mathcal{M} \otimes_S S[\frac{1}{q}])^{\Gamma_f}$. Par conséquent, le diagramme

$$\begin{tikzcd}
\mathcal{M} \otimes_{\sigma(S)} S[\frac{1}{q}] \arrow[r, "\varphi"] & \mathcal{M} \otimes_S S[\frac{1}{q}] \\
\mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0[\frac{1}{q}] \arrow[u, hook, "i"] \arrow[r, "\varphi_0"] & \mathcal{M}^{\Gamma_f} \otimes_{S_0} S_0[\frac{1}{q}] \arrow[u, hook, "j"']
\end{tikzcd}$$

est commutatif, avec $\varphi$ bijective, $i$ et $j$ injective, et $\varphi$ (qui commute à l'action de $\Gamma_f$) qui identifie $(\mathcal{M} \otimes_{\sigma(S)} S[\frac{1}{q}])^{\Gamma_f}$ à $(\mathcal{M} \otimes_S S[\frac{1}{q}])^{\Gamma_f}$, donc $\varphi_0$ est bijective (donc $\mathcal{M}^{\Gamma_f}/\varphi_0(\mathcal{M}^{\Gamma_f} \otimes_\sigma S_0)$ est de $q$-torsion, donc tué par une puissance de $q$ car $\mathcal{M}^{\Gamma_f}$ est de type fini sur $S_0$).
Soit alors $x \in \mathcal{M}^{\Gamma_f}$. Par définition, il existe $y \in \mathcal{M} \otimes_{\sigma(S)} S = \oplus_i \mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0 y_i$ tel que $\varphi(y) = q^h x$. La commutativité du diagramme et la bijectivité de $\varphi_0$ nous donne que $y \in \mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0[\frac{1}{q}]$. Donc nous avons $y \in \big(\mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0[\frac{1}{q}]\big) \cap \big(\oplus_i \mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0 y_i\big) = \big(\mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0[\frac{1}{q}]\big) \cap \big(\mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0\big) = \mathcal{M}^{\Gamma_f} \otimes_{\sigma(S_0)} S_0$.
En définitive, nous avons bien que $\mathcal{M}^{\Gamma_f}/\varphi_0(\mathcal{M}^{\Gamma_f} \otimes_\sigma S_0)$ est tué par $q^h$.
Finalement, nous avons bien que $\mathcal{M}^{\Gamma_f}$ est un objet de $\mathbf{\Gamma_0 \Phi M^h_{S_0}}$. □

Remarque 29. *De la même façon que pour $S_i$, nous montrons pour $\mathcal{M}$ un objet de $\mathbf{\Gamma \Phi M^h_S}$ que $p_i(\mathcal{M}) = \mathcal{M}^{\Gamma_f} \otimes_{S_0} S_i = p_i(e)\mathcal{M}^{\Gamma_f}$.*

Théorème 30. *L'extension des scalaires de $S_0$ à $S$ induit une équivalence de catégories entre $\mathbf{\Gamma_0 \Phi M^h_{S_0}}$ et $\mathbf{\Gamma \Phi M^h_S}$, préservant suites exactes et produit tensoriel (si ce dernier est encore dans la catégorie). Un quasi-inverse est donné par les points fixes par $\Gamma_f$.*

*Démonstration.* L'essentielle surjectivité se prouve en remarquant que si $f : \mathcal{M} \to \mathcal{N}$ est un morphisme de $\mathbf{\Gamma \Phi M^h_S}$, alors comme il commute à l'action de

$\Gamma_f$, $f$ induit bien un morphisme de $(\varphi, \Gamma_0)$-modules entre $\mathcal{M}^{\Gamma_f}$ et $\mathcal{N}^{\Gamma_f}$ (qui redonne $f$ en étendant les scalaires de $S_0$ à $S$). Le reste est immédiat à partir des résultats précédents. $\square$

### 3.3 MODULES DE WACH

L. Berger a défini dans [Ber04] le module de Wach $\mathbf{N}(T)$ d'un réseau $T$ d'une $\mathbb{Q}_p$-représentation cristalline $V$ à poids de Hodge-Tate négatifs comme l'unique $S$-sous-module de $\mathbf{D}^+(T) := (A_S^+ \otimes_{\mathbb{Z}_p} T)^{H_\mathcal{K}}$ (avec $A_S^+ = W(R) \cap \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$) vérifiant :
- $\mathbf{N}(T)$ est un $S$-module libre de rang la dimension de $V$ ;
- l'action de $\Gamma$ préserve $\mathbf{N}(T)$ et est triviale sur $\mathbf{N}(T)/\pi\,\mathbf{N}(T)$ ;
- il existe un entier $r \geq 0$ tel que $\pi^r\,\mathbf{D}^+(T) \subset \mathbf{N}(T)$.

Il définit de même le module de Wach $\mathbf{N}(V)$ d'une représentation cristalline $V$ à poids de Hodge-Tate négatifs. L'unicité donne en particulier que $\mathbf{N}$ va préserver somme directe et produit tensoriel, ce qui nous intéressera tout particulièrement.
Donnons un résultat plus précis que le Théorème 1' de N. Wach :

PROPOSITION 31. *Si $N$ est un objet de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$ avec $0 \leq h \leq p-2$, $\mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_N)$ (qui identifie $\mathrm{F}^-(N) = N \otimes_W \mathcal{O}_\mathcal{E}$ à $\mathbf{D}_{\mathcal{O}_\mathcal{E}}(\mathbf{V_{cris}}(N))$) envoie $N \otimes_W S$ sur $\mathbf{N}(\mathbf{V_{cris}}(N))$ (le module de Wach associé à $\mathbf{V_{cris}}(N)$).*

*Démonstration.* En passant au dual, cela revient à dire que $\mathrm{F}(N^*) \otimes_{S_0} S$ est isomorphe à $\mathbf{N}(\mathbf{V_{cris}}(N)^*)$ par fonctorialité du module de Wach envers le dual. Appelons $T = \mathbf{V_{cris}}(N)^*$ et $r \leq p-2$ l'entier tel que $\mathrm{Fil}^r(N^*) \neq \{0\}$, $\mathrm{Fil}^{r+1}(N^*) = \{0\}$. Remarquons que la structure de $(\varphi, \Gamma_0)$-module de $\mathrm{F}(N^*)$ induit une structure de $(\varphi, \Gamma)$-module sur $N^* \otimes_W S$, et que $N^* \otimes_W \frac{1}{\pi^r} S$ est le dual (au sens généralisé des modules de Wach) d'un $(\varphi, \Gamma)$-module de hauteur finie (puisque égale à $r$) sur $S$, donc par le résultat de J.-M. Fontaine (cf [Fon90], p.296), les périodes de $N^* \otimes_W \frac{1}{\pi^r} S$ sont dans $A_S^+$. Par conséquent, $\mathbf{V}_{\mathcal{O}_\mathcal{E}}\big((N^* \otimes_W \frac{1}{\pi^r} S) \otimes_S \mathcal{O}_\mathcal{E}\big) = \mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathrm{F}(N^*) \otimes_{S_0} \mathcal{O}_\mathcal{E}) = T = ((N^* \otimes_W \frac{1}{\pi^r} S) \otimes_S A_S^+)^\varphi \subset N^* \otimes_W \frac{1}{\pi^r} A_S^+$.
Puis, l'identification de $\mathcal{N}$ avec $\mathbf{D}_{\mathcal{O}_\mathcal{E}}(\mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathcal{N}))$ pour $\mathcal{N}$ un $(\varphi, \Gamma)$-module sur $\mathcal{O}_\mathcal{E}$ est induite par la multiplication dans $\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$. Donc, comme $T \subset N^* \otimes_W \frac{1}{\pi^r} A_S^+$, nous avons $\mathbf{D}^+(T) \subset ((N^* \otimes_W \frac{1}{\pi^r} A_S^+) \otimes A_S^+)^{H_\mathcal{K}}$ qui est identifié à $(N^* \otimes_W \frac{1}{\pi^r} A_S^+)^{H_\mathcal{K}} = N^* \otimes_W \frac{1}{\pi^r} S$. Donc la dernière condition de la définition d'un module de Wach, $\pi^r\,\mathbf{D}^+(T) \subset N^* \otimes_W S$, est vérifiée.

LEMME 32. *Sous les notations précédentes, nous avons l'inclusion $\mathbf{N}(T) \subset N^* \otimes_W S$.*

REMARQUE 33. *La démonstration donnée ci-dessous est exactement l'idée principale de la démonstration de l'unicité du module de Wach (cf. proposition II.1.1 de [Ber04])*

*Démonstration.* Notons $\mathcal{N}_1 = \mathbf{N}(T)$ et $\mathcal{N}_2 = N^* \otimes_W S$. $\mathcal{N}_1 \subset \mathbf{D}^+(T)$ par définition, donc nous avons l'inclusion $\pi^r \mathcal{N}_1 \subset \mathcal{N}_2$. Soit $x \in \mathcal{N}_1$ et $s$ l'entier tel que $\pi^s x \in \mathcal{N}_2$, mais $\pi^s x \notin \pi\mathcal{N}_2$. Choisisons $x \notin \pi\mathcal{N}_1$ tel qu'en plus $s$ soit maximal, ce qui fait que $\pi^s\mathcal{N}_1 \subset \mathcal{N}_2$. Comme $\pi^s x \in \mathcal{N}_2$ et que $\Gamma$ agit trivialement sur $\mathcal{N}_2/\pi\mathcal{N}_2$, nous avons pour tout $g \in \Gamma$ que $(g-1)(\pi^s x) \in \pi\mathcal{N}_2$, et nous pouvons écrire $(g-1)(\pi^s x) = g(\pi^s)(g(x)-x) + (g(\pi^s)-\pi^s)x$. Comme $\Gamma$ agit trivialement sur $\mathcal{N}_1/\pi\mathcal{N}_1$, et que $\pi^s\mathcal{N}_1 \subset \mathcal{N}_2$, nous avons que $g(\pi^s)(g(x)-x) \in \pi\mathcal{N}_2$, et donc que $(g(\pi^s)-\pi^s)x \in \pi\mathcal{N}_2$, ce qui est une contradiction si $s \geq 1$, parce qu'alors $g(\pi^s)-\pi^s = (\mathcal{X}(g)^s-1)\pi^s + \cdots$. Donc nous avons bien $\mathcal{N}_1 \subset \mathcal{N}_2$, autrement dit $\mathbf{N}(T) \subset N^* \otimes_W S$. □

L'étude du paragraphe précédent nous donne que le $S_0$-module $\mathcal{N} = \mathbf{N}(T)^{\Gamma_f}$ est libre et $\mathbf{N}(T) = \mathbf{N}(T)^{\Gamma_f} \otimes_{S_0} S$. Utilisons alors le fait que le foncteur $\mathrm{F}$ est essentiellement surjectif (à cause de l'hypothèse sur $h$) pour dire que $\mathcal{N}$ est isomorphe en tant que $(\varphi,\Gamma_0)$-module à $\mathrm{F}(N^*)$, donc $\mathbf{N}(T)$ est isomorphe au $(\varphi,\Gamma)$-module $N^* \otimes_W S$. Notons $i$ cet isomorphisme.
Remarquons que $\mathbf{N}(T) \otimes_S \mathcal{O}_{\mathcal{E}} = \mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(T) = N^* \otimes_W \mathcal{O}_{\mathcal{E}}$, car une représentation cristalline est de hauteur finie. Par conséquent, $i$ induit un isomorphisme de $\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(T)$ qui envoye $\mathbf{N}(T)$ sur $N^* \otimes_W S$, et comme il préserve $D^+(T)$, nous obtenons bien $N^* \otimes_W S \subset D^+(T)$, donc $N^* \otimes_W S = \mathbf{N}(T)$ car il vérifie toutes les conditions de la définition du module de Wach. □

Nous pouvons alors en déduire la proposition qui nous intéresse :

Proposition 34. *Soit $N_{i,j}$ des objets de $\mathbf{MF}_{\mathbf{W}}^{\mathbf{h}}$ avec $0 \leq h \leq p-2$, $L$ un sous-objet (dans $\mathbf{MF}_{\mathbf{W}}^{+}$) de $M = \bigoplus_i \otimes_j N_{i,j}$. Alors les isomorphismes de modules de Wach*

$$^t\,\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(g_{N^*_{i,j}}) : \mathbf{N}(\mathbf{V_{cris}}(N^*_{i,j})^*) \to N_{i,j} \otimes_W S$$

*identifient $L \otimes_W S$ à un module de Wach.*

*Démonstration.* Les isomorphismes $^t\,\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(g_{N^*_{i,j}})$ induisent un isomorphisme

$$\bigoplus_{i=1}^{n} \otimes_{j=1}^{m_i} {}^t\,\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(g_{N^*_{i,j}}) : \mathbf{N}(\bigoplus_{i=1}^{n} \otimes_{j=1}^{m_i} \mathbf{V_{cris}}(N^*_{i,j})^*) \to M \otimes_W S$$

(puisque le module de Wach préserve le produit tensoriel). Nous utiliserons cet isomorphisme pour identifier ces deux espaces.
Notons $(e_i)$ une base de $M$ telle que $(p^{\alpha_i}e_i)$ soit une base de $L$, avec $\alpha_i \in \mathbb{N} \cup \{+\infty\}$. Notons aussi $n = \mathrm{rg}_W(M)$.
La proposition 17 affirme que $L \otimes_W S$ est stable par l'action de $\Gamma$. Considérons alors la sous-représentation galoisienne $T$ de $U_M := \bigoplus_{i=1}^{n} \otimes_{j=1}^{m_i} \mathbf{V_{cris}}(N^*_{i,j})^*$ définie par $T = \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(L \otimes_W \mathcal{O}_{\mathcal{E}})$. Montrons que $\mathbf{N}(T) = L \otimes_W S$, c'est-à-dire vérifions les conditions qui caractérisent un module de Wach :

– $L \otimes_W S \subset T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \cap (U_M \otimes_{\mathbb{Z}_p} A_S^+)^{H_K} = \mathbf{D}^+(T)$ : l'inclusion provient de ce que $T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} = L \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ et $\mathbf{N}(U_M) = M \otimes_W S$ via l'isomorphisme (et donc $M \otimes_W S \subset D^+(U_M) = (U_M \otimes_{\mathbb{Z}_p} A_S^+)^{H_K}$) ; l'égalité se montre en considérant les coordonnées suivant la base $(e_i)$, car si $x \in T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \cap (U_M \otimes_{\mathbb{Z}_p} A_S^+)^{H_\mathcal{K}}$, alors il existe $(\beta_i) \in (A_S^+)^n$ et $(\delta_i) \in \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}^n$ avec $x = \sum_i \beta_i e_i = \sum_i p^{\alpha_i} \delta_i e_i$, donc $\beta_i = p^{\alpha_i}\delta_i$ pour tout $i$, donc $\beta_i \in p^{\alpha_i} A_S^+$ pour tout $i$, ce qui donne $T \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \cap (U_M \otimes_{\mathbb{Z}_p} A_S^+)^{H_\mathcal{K}} \subset \mathbf{D}^+(T)$ (l'inclusion réciproque étant immédiate) ;

- $L \otimes_W S$ est un $S$-module libre de rang égal à celui de $T$ sur $\mathbb{Z}_p$ (qui est celui de $L \otimes_W \mathcal{O}_\mathcal{E}$ sur $\mathcal{O}_\mathcal{E}$, donc celui de $L$ sur $W$) ;
- l'action de $\Gamma$ laisse stable $L \otimes_W S$ (c'est la proposition 17) et est triviale modulo $\pi$ : l'action de $\Gamma$ sur $M \otimes_W S$ étant triviale modulo $\pi$ par construction, pour $\gamma \in \Gamma$, pour $i$ fixé, il existe $(x_j) \in S^{n-1}$ et $(y_j) \in S^n$ tels que $\gamma(e_i) = e_i + \pi \sum_{j \neq i} x_j e_j$ et $\gamma(p^{\alpha_i} e_i) = \sum_j y_j p^{\alpha_j} e_j$ ; donc $y_i = 1$ et $p^{\alpha_j} y_j = \pi x_j p^{\alpha_i}$ pour $j \neq i$, donc $\pi$ divise $y_j$ dans $S$ pour $j \neq i$.
- il existe $r$ un entier positif tel que $\pi^r\, \mathbf{D}^+(U_M) \subset M \otimes_W S$, donc ce $r$ donne $\pi^r\, \mathbf{D}^+(T) \subset M \otimes_W S \cap L \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} = L \otimes_W S$. En effet, si $x \in M \otimes_W S \cap L \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$, alors il existe $(\beta_i) \in S^n$ et $(\delta_i) \in \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}^n$ avec $x = \sum_i \beta_i e_i = \sum_i p^{\alpha_i}\delta_i e_i$, donc $\beta_i = p^{\alpha_i}\delta_i$ pour tout $i$, donc $\beta_i \in p^{\alpha_i} S$ pour tout $i$, ce qui donne $M \otimes_W S \cap L \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \subset L \otimes_W S$ (l'inclusion réciproque étant immédiate).

D'où, nous avons bien $\mathbf{N}(T) = L \otimes_W S$. □

Ce qui nous intéressera tout particulièrement, c'est le corollaire suivant :

PROPOSITION 35. *Soit $N_{i,j}$ des objets de $\mathbf{MF_W^{-h}}$ avec $0 \leq h \leq p-2$, $L$ un sous-objet (dans $\mathbf{MF_W^-}$) facteur direct (comme $W$-module) de $M = \bigoplus_i \otimes_j N_{i,j}$. Alors les isomorphismes de modules de Wach*

$$\mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_{N_{i,j}}) : N_{i,j} \otimes_W S \to \mathbf{N}(\mathbf{V_{cris}}(N_{i,j}))$$

*induisent un isomorphisme de module de Wach*

$$L \otimes_W S \to \mathbf{N}(\overline{\mathbf{V}}_{\mathbf{cris}}(L))$$

*où* $\overline{\mathbf{V}}_{\mathbf{cris}}(L) := \mathbf{V_{cris,p}}(D_L) \cap \bigoplus_{i=1}^n \otimes_{j=1}^{m_i} \mathbf{V_{cris}}(N_{i,j})$.

*Démonstration.* Les isomorphismes $\mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_{N_{i,j}})$ induisent un isomorphisme

$$\mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_M) := \bigoplus_{i=1}^n \otimes_{j=1}^{m_i} \mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_{N_{i,j}}) : M \otimes_W S \to \mathbf{N}(\overline{\mathbf{V}}_{\mathbf{cris}}(M))$$

Par dualité, il suffit de voir que si $L_0 = M/L$, alors $\mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_M)$ induit un isomorphisme de $\overline{\mathbf{V}}_{\mathbf{cris}}(L_0^*)$ sur $L_0^* \otimes_W S$. Posons $T = \mathbf{V}_{\mathcal{O}_\mathcal{E}}(L_0^* \otimes_W \mathcal{O}_\mathcal{E})$. La proposition précédente nous donne bien que $\mathbf{N}(T) = L_0^* \otimes_W S$ (via $\mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_M)$).
Puis, une propriété du module de Wach nous permet de conclure : $\mathbf{N}(T[\frac{1}{p}])/\pi$ s'identifie à $\mathbf{D_{cris,p}}(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ (par le théorème III.4.4 de [Ber04]), et l'application $g_M$ envoie $\mathbf{N}(T)/\pi$ sur $L_0^*$, donc nous avons bien que $\mathbf{D_{cris,p}}(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = L_0^* \otimes_W \mathcal{K}$, donc que $T = \overline{\mathbf{V}}_{\mathbf{cris}}(L_0^*)$. □

Remarque 36. *Si $M'$ est le quotient du $L$ considéré dans la proposition 35 par le sous-objet $L'$ (facteur direct comme $W$-module), alors $\mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_M) : L \otimes_W S \to \mathbf{N}(\overline{\mathbf{V}}_{\mathbf{cris}}(L))$ (qui induit aussi un isomorphisme $L' \otimes_W S \to \mathbf{N}(\overline{\mathbf{V}}_{\mathbf{cris}}(L'))$) induit par passage au quotient un isomorphisme $M' \otimes_W S \to \mathbf{N}(\overline{\mathbf{V}}_{\mathbf{cris}}(M'))$.*

Corollaire 37. *Soit $0 \leq h \leq p-2$. Soit $M$ et $M'$ deux objets de $\mathbf{MF_W} < \mathbf{h} >$, et $f : M \to M'$ un morphisme de $\varphi$-modules filtrés. Alors $\mathbf{V}_\mathcal{E}(\mathrm{F}(f) \otimes \mathrm{Id}_\mathcal{E}) = \mathbf{V}_{\mathbf{cris},\mathbf{p}}(f)$.*

*Démonstration.* Soient $N_{i,j}$ et $N'_{i,j}$ des objets de $\mathbf{MF^h_W}$, $L$ un sous-objet de $\oplus \otimes N_{i,j}$ et $L_0$ un sous-objet facteur direct de $L$, tel que $M = L/L_0$, $L'$ un sous-objet de $\oplus \otimes N'_{i,j}$ et $L'_0$ un sous-objet facteur direct de $L'$, tel que $M' = L'/L'_0$. Nous allons montrer que les isomorphismes $\oplus \otimes \mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_{N_{i,j}})$ et $\oplus \otimes \mathbf{D}_{\mathcal{O}_\mathcal{E}}(g_{N'_{i,j}})$ identifient $f \otimes \mathrm{Id}$ à $\mathbf{D}_\mathcal{E}(\mathbf{V}_{\mathbf{cris},\mathbf{p}}(f))$.
Pour cela, il suffit d'utiliser la fidélité et la pleine fidélité de $\mathrm{F}$ combiné au théorème 30 (pour pouvoir dire que la réduction modulo $\pi$ est injective sur les morphismes de $(\varphi, \Gamma)$-module entre $M \otimes_W S$ et $M' \otimes_W S$), plus le fait que $\mathbf{D}_\mathcal{E}(\mathbf{V}_{\mathbf{cris},\mathbf{p}}(f))$ modulo $\pi$ redonne $f$ (d'après les résultats de L. Berger dans [Ber04]). □

Corollaire 38. *Soit $M$ et $M'$ deux objets de $\mathbf{MF_W} < \mathbf{h} >$, et $f : M \to M'$ un morphisme $\varphi$-modules filtrés. Alors $\mathbf{V}_{\mathbf{cris},\mathbf{p}}(f)$ envoye $\mathbf{V}_{\mathcal{O}_\mathcal{E}}(M \otimes_W \mathcal{O}_\mathcal{E})$ dans $\mathbf{V}_{\mathcal{O}_\mathcal{E}}(M' \otimes_W \mathcal{O}_\mathcal{E})$. En particulier, en passant au dual, $\overline{\mathbf{V}}_{\mathbf{cris}}$ devient un foncteur en posant $\overline{\mathbf{V}}_{\mathbf{cris}}(f) = \mathbf{V}_{\mathbf{cris},\mathbf{p}}(f)$.*

*Démonstration.* C'est une conséquence immédiate du corollaire précédent, et de ce que si $T$ et $T'$ sont deux $\mathbb{Z}_p$-représentations cristallines, alors un morphisme de $(\varphi, \Gamma)$-modules $g : \mathbf{N}(T) \to \mathbf{N}(T')$ induit $\mathbf{V}_{\mathcal{O}_\mathcal{E}}(g) : T = \mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathbf{N}(T) \otimes_S \mathcal{O}_\mathcal{E}) \to T' = \mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathbf{N}(T') \otimes_S \mathcal{O}_\mathcal{E})$. □

Remarque 39. *D'après ce qui précède, le foncteur $\overline{\mathbf{V}}_{\mathbf{cris}}$ et le foncteur $\mathbf{V}_{\mathcal{O}_\mathcal{E}} \circ \mathrm{F}^-$, tous deux définis sur $\mathbf{MF_W} < -\mathbf{h} >$ et à valeurs dans $\mathbf{Rep}^{\mathbf{cris}}_{\mathbb{Z}_\mathbf{p}}(\Gamma_\mathcal{K})$, sont isomorphes (l'isomorphisme est donné par la transformation naturelle $g$).*

## 4 Fin de la démonstration du théorème 15

Théorème 40. *Pour $0 \leq h \leq p-2$, le foncteur $\mathrm{F}$ de $\mathbf{MF^h_{W,tf}}$ vers $\mathbf{\Gamma_0 \Phi M^h_{S_0}}$ a pour image essentielle $\mathbf{\Gamma_0 \Phi M^h_{S_0}}$.*

Pour montrer ce résultat, nous allons utiliser le théorème 31 qui nous dit que pour $N$ objet de $\mathbf{MF^h_W}$, $\mathrm{F}(N) \otimes_{S_0} S$ est le module de Wach de $\mathbf{V_{cris}}(N^*)^*$. Commençons par montrer :

Proposition 41. *Soit $\mathcal{M}$ un objet de $\mathbf{\Gamma_0 \Phi M^h_{S_0}}$ (avec $0 \leq h \leq p-2$) de $p$-torsion, et $T' = \mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathcal{M} \otimes_{S_0} \mathcal{O}_\mathcal{E})$ la $\mathbb{Z}_p$-représentation galoisienne correspondant au $(\varphi, \Gamma)$-module sur $\mathcal{O}_\mathcal{E}$ obtenu à partir de $\mathcal{M}$. Alors il existe $T'' \subset T$ deux*

*$\mathbb{Z}_p$-représentations galoisiennes cristallines (c'est à dire que le module sous-jacent est libre sur $\mathbb{Z}_p$, et en rendant p inversible nous avons une représentation cristalline) à poids de Hodge-Tate dans $[\![-h,0]\!]$ telles que $T'$ s'identifie au quotient de $T$ par $T''$.*

*Démonstration.* Le Théorème 1' de [Wac97] (et la proposition 31) donne que $T' = \mathrm{Hom}_{\mathbb{Z}_p}\big(\mathbf{V_{cris}}\big(\mathrm{Hom}_W(i^*(\mathcal{M}), \varinjlim W/p^n)\big), \varinjlim \mathbb{Z}_p/p^n\big)$. En notant $X^*$ le dual de Pontriaguine d'un module de torsion $X$, cela s'écrit plus simplement en $T' = \mathbf{V_{cris}}\big((\mathcal{M}/\pi_0)^*\big)^*$. Puis, puisque $(\mathcal{M}/\pi_0)^*$ est un objet de la catégorie $\mathbf{MF^{-h}_{W,tf}}$, la proposition 1.6.3 de [Win84] nous donne qu'il existe $M_1 \in \mathbf{MF^{-h}_{W}}$ et un épimorphisme $M_1 \to (\mathcal{M}/\pi_0)^*$. Le foncteur $\mathbf{V_{cris}}$ étant exact, il existe donc une $\mathbb{Z}_p$-repésentation cristalline $T_1$ ($T_1 = \mathbf{V_{cris}}(M_1)$) dont les poids de Hodge-Tate sont dans $[\![0,h]\!]$ et un épimorphisme $T_1 \to \mathbf{V_{cris}}\big((\mathcal{M}/\pi_0)^*\big)$.
Comme $\mathcal{M}$ est supposé de $p$-torsion, $\mathbf{V_{cris}}\big((\mathcal{M}/\pi_0)^*\big)$ est de $p$-torsion et de type fini, donc il existe un entier $n$ tel que $p^n\,\mathbf{V_{cris}}\big((\mathcal{M}/\pi_0)^*\big) = \{0\}$. Alors $T_1/p^n$ se surjecte toujours sur $\mathbf{V_{cris}}\big((\mathcal{M}/\pi_0)^*\big)$, et en passant au dual de Pontriaguine, $T'$ s'injecte dans $\mathrm{Hom}_{\mathbb{Z}_p}\big(T_1/p^n, \varinjlim \mathbb{Z}_p/p^n\big) = \mathrm{Hom}_{\mathbb{Z}_p}(T_1,\mathbb{Z}_p)/p^n$ (car $T_1$ est un $\mathbb{Z}_p$-module libre). Si $f$ est la projection canonique $\mathrm{Hom}_{\mathbb{Z}_p}\big(T_1,\mathbb{Z}_p\big) \to \mathrm{Hom}_{\mathbb{Z}_p}\big(T_1,\mathbb{Z}_p\big)/p^n$, alors $T = f^{-1}(T')$ convient (et il suffit de prendre $T''$ égal au noyau de la projection $f|_T$). □

PROPOSITION 42. *Soit $\mathcal{M}$ un objet de $\mathbf{\Gamma_0\Phi M^h_{S_0}}$ (avec $0 \le h \le p-2$) de $p$-torsion, $T' = \mathbf{V}_{\mathcal{O}_\mathcal{E}}(\mathcal{M} \otimes_{S_0} \mathcal{O}_\mathcal{E})$ et $T'' \subset T$ les représentations données par la proposition ci-dessus. Alors $\mathcal{M} \otimes_{S_0} S$ s'identifie à $\mathbf{N}(T)/\mathbf{N}(T'')$.*

*Démonstration.* Notons $\mathcal{M}_1 = \mathcal{M} \otimes_{S_0} S$ et $\mathcal{M}_2 = \mathbf{N}(T)/\mathbf{N}(T'')$ (tous les deux vus dans le $(\varphi,\Gamma)$-module $\mathcal{M} \otimes_{S_0} \mathcal{O}_\mathcal{E}$, car $\mathbf{N}(T) \cap \mathbf{D}_{\mathcal{O}_\mathcal{E}}(T") = \mathbf{N}(T'')$ : en effet, notons $\mathcal{N} = \mathbf{N}(T) \cap \mathbf{D}_{\mathcal{O}_\mathcal{E}}(T") = \mathbf{N}(T) \cap T'' \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathcal{E}}_{nr}}$ qui est stable par l'action de $\Gamma$, nous avons que $\mathcal{N} \cap \pi\,\mathbf{N}(T) = \pi\mathcal{N}$ puisque $\pi$ est inversible dans $\mathcal{O}_{\hat{\mathcal{E}}_{nr}}$, donc $\mathcal{N}/\pi$ s'injecte dans $\mathbf{N}(T)/\pi$, donc $\Gamma$ agit bien trivialement sur $\mathcal{N}$. Puis, $T'' \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathcal{E}}_{nr}} \cap (T \otimes_{\mathbb{Z}_p} A_S^+)^{H_\mathcal{K}} = \mathbf{D}^+(T'')$, car si $(e_i)$ est une base de $T$ telle que $(p^{\alpha_i}e_i)$ est une base de $T''$ (avec $\alpha_i \in \mathbb{N} \cup \{+\infty\}$), alors un élément $x$ de l'intersection s'écrit $x = \sum_i x_i e_i = \sum_i p^{\alpha_i} y_i e_i$ avec $x_i \in A_S^+$ et $y_i \in \mathcal{O}_{\hat{\mathcal{E}}_{nr}}$ ; donc $y_i \in p^{-a_i} A_S^+ \cap \mathcal{O}_{\hat{\mathcal{E}}_{nr}} = A_S^+$ si $\alpha_i \neq +\infty$, et $\{0\}$ sinon, donc $x \in T'' \otimes_{\mathbb{Z}_p} A_S^+$ et est fixé par $H_\mathcal{K}$, donc $T'' \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathcal{E}}_{nr}} \cap (T \otimes_{\mathbb{Z}_p} A_S^+)^{H_\mathcal{K}} \subset \mathbf{D}^+(T'')$ (l'inclusion réciproque étant immédiate). Donc nous avons $\mathcal{N} \subset \mathbf{D}^+(T'')$ puisque $\mathbf{N}(T) \subset (T \otimes_{\mathbb{Z}_p} A_S^+)^{H_\mathcal{K}} = \mathbf{D}^+(T)$. Enfin, $\pi^h\,\mathbf{D}^+(T) \subset \mathbf{N}(T)$, donc $\pi^h\,\mathbf{D}^+(T'') \subset \mathbf{N}(T)$, et comme $\pi^h\,\mathbf{D}^+(T'') \subset T'' \otimes_{\mathbb{Z}_p} \mathcal{O}_{\hat{\mathcal{E}}_{nr}}$, nous avons bien que $\pi^h\,\mathbf{D}^+(T'') \subset \mathcal{N}$. Ces conditions caractérisent le module de Wach de $T''$, donc $\mathbf{N}(T) \cap \mathbf{D}_{\mathcal{O}_\mathcal{E}}(T") = \mathbf{N}(T'')$).
D'après les résultats p.296 de [Fon90] (l'égalité entre $\mathbf{D}^*_\mathbf{S}$ et $j_* \circ \mathbf{D}^*_\mathcal{E}$) (ou bien le lemme III.5 de [Col99]), nous avons $\mathcal{M}_1 \subset \mathbf{D}^+(T')$ et $\mathcal{M}_2 \subset \mathbf{D}^+(T')$, puisque tout deux sont des $S$-modules de type fini stables par $\varphi$ et $p$-étales (puisque de $q$-hauteur finie). Puis, l'action de $\Gamma$ est triviale modulo $\pi$ dans les deux cas (puisque c'est le cas par définition sur $\mathbf{N}(T)$, et que l'action de $\Gamma_0$ est triviale modulo $\pi_0$ sur $\mathcal{M}$).

D'après le Théorème III.3.1 de [Ber04], nous avons l'inclusion $\pi^h T \otimes_{\mathbb{Z}_p} A_S^+ \subset \mathbf{N}(T) \otimes_S A_S^+$. Par conséquent, en projetant nous obtenons que $\pi^h T' \otimes_{\mathbb{Z}_p} A_S^+ \subset \mathcal{M}_2 \otimes_S A_S^+$. Par définition, nous avons que $\mathbf{D}^+(T') \subset \mathbf{D}^+(T') \otimes_S A_S^+ \subset T' \otimes_{\mathbb{Z}_p} A_S^+$, donc en prenant les points fixes sous l'action de $H_{\mathcal{K}}$, nous avons $\mathbf{D}^+(T') \subset \big(\mathbf{D}^+(T') \otimes_S A_S^+\big)^{H_{\mathcal{K}}} \subset \big(T' \otimes_{\mathbb{Z}_p} A_S^+\big)^{H_{\mathcal{K}}} = \mathbf{D}^+(T')$. Donc, en prenant les points fixes sous $H_{\mathcal{K}}$ dans l'inclusion $\pi^h T' \otimes_{\mathbb{Z}_p} A_S^+ \subset \mathcal{M}_2 \otimes_S A_S^+$, nous obtenons que $\pi^h\, \mathbf{D}^+(T') \subset \big(\mathcal{M}_2 \otimes_S A_S^+\big)^{H_{\mathcal{K}}}$. Donc nous avons $\pi^h\, \mathbf{D}^+(T') \subset \mathcal{M}_2$ en vertu du lemme :

Lemme 43. *Soit $\mathcal{N}$ un $S$-module de type fini sans $p'$-torsion, alors $\big(\mathcal{N} \otimes_S A_S^+\big)^{H_{\mathcal{K}}} = \mathcal{N}$.*

*Démonstration.* C'est une conséquence de la proposition 1.2.7 de [Fon90], qui nous donne (sous les hypothèses du lemme) une filtration décroissante $\mathcal{N}_i$ de $\mathcal{N}$, telle que $\mathcal{N}_i/\mathcal{N}_{i+1}$ est soit $S/p$-libre, soit $S$-libre. La propriété cherchée est stable par suite exacte, c'est à dire vérifie que si $0 \to \mathcal{N}'' \to \mathcal{N} \to \mathcal{N}' \to 0$ est une suite exacte de $S$-modules, et que $\big(\mathcal{N}'' \otimes_S A_S^+\big)^{H_{\mathcal{K}}} = \mathcal{N}''$, $\big(\mathcal{N}' \otimes_S A_S^+\big)^{H_{\mathcal{K}}} = \mathcal{N}'$, alors $\big(\mathcal{N} \otimes_S A_S^+\big)^{H_{\mathcal{K}}} = \mathcal{N}$. Donc il suffit de montrer le lemme pour $\mathcal{N}$ qui est $S$-libre ou $S/p$-libre, ce qui provient de ce que $(A_S^+)^{H_{\mathcal{K}}} = S$ et $(A_S^+/p)^{H_{\mathcal{K}}} = S/p$. ☐

Puis $\frac{1}{\pi^h}\mathcal{M}_1$ est le dual (de Pontriaguine) d'un $(\varphi, \Gamma)$-module sur $S$ de hauteur inférieure ou égale à $h$, sans $p'$-torsion, donc $T' = \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\frac{1}{\pi^h}\mathcal{M} \otimes_{S_0} \mathcal{O}_{\mathcal{E}})$ vérifie $T' = (\frac{1}{\pi^h}\mathcal{M} \otimes_{S_0} A_S^+)^{\varphi}$ (cf [Fon90], p.296) puisque $0 \leq h$. Donc $T' \otimes_{\mathbb{Z}_p} A_S^+ \subset \frac{1}{\pi^h}\mathcal{M} \otimes_{S_0} A_S^+$, et en prenant les points fixes sous $H_{\mathcal{K}}$ (et par le lemme précédent), nous obtenons $\mathbf{D}^+(T') \subset \frac{1}{\pi^h}\mathcal{M}_1$, donc $\pi^h\, \mathbf{D}^+(T') \subset \mathcal{M}_1$.
Ces conditions impliquent que la démonstration du lemme 32 s'applique ici (car $h \leq p-2$, pour que nous ayons si $0 \leq s \leq h$, $\mathcal{X}(g)^s - 1$ inversible dans $\mathbb{Z}_p$ (c'est à dire $\mathcal{X}(g)^s - 1 \neq 0$ modulo $p$) pour un $g \in \Gamma$), et donc $\mathcal{M}_1 = \mathcal{M}_2$. ☐

Remarque 44. *L'unicité d'un tel module n'est plus vrai en général : dans $S/pS$, $S/pS$ et $\pi^{p-1}S/pS$ sont deux $S$-modules de type fini, avec action de $\Gamma$ triviale modulo $\pi$, et si $T = \mathbf{V}_{\mathbf{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}}/p)$ (c'est à dire $\mathbb{F}_p$ avec l'action triviale), alors $\mathbf{D}^+(T) = S/pS$, donc la dernière condition est aussi vérifiée.*

Il ne reste donc plus qu'à passer d'un module sur $S$ à un module sur $S_0$, ce qui est donné par le lemme suivant (qui est une conséquence immédiate de l'égalité $S = \bigoplus_{0 \leq i \leq p-2} S_i)$ :

Lemme 45. *Soit $\mathcal{M}$ un $S_0$-module, alors $\big(\mathcal{M} \otimes_{S_0} S\big)^{\Gamma_f} = \mathcal{M}$.*

Ces propositions et ces lemmes mis bout à bout nous donnent le théorème dans le cas d'un objet de $\mathbf{\Gamma_0 \Phi M^h_{S_0}}$ de $p$-torsion. C'est à dire que si $\mathcal{M}$ est un objet de $\mathbf{\Gamma_0 \Phi M^h_{S_0}}$ (avec $0 \leq h \leq p-2$) de $p$-torsion, alors il existe $M$ un objet de $\mathbf{MF^h_{W,tf}}$ tel que $\mathcal{M} = \mathrm{F}(M)$. Et plus précisément, nous avons

$M = i^*(\mathcal{M}) = \mathcal{M}/\pi$. Donc, dans le cas où $\mathcal{M}$ n'est pas supposé de $p$-torsion, nous avons que $\mathcal{M}/p^n = \mathrm{F}(i^*(\mathcal{M}/p^n))$ pour tout $n$, donc en passant à la limite projective, nous obtenons bien que $\mathcal{M} = \mathrm{F}(i^*(\mathcal{M}))$, ce qui donne bien l'essentielle surjectivité de $\mathrm{F}$, et donc termine la démonstration du théorème 40 (et donc du théorème 15).

## 5 Le point du torseur

### 5.1 Conséquence des théorèmes précédents

Pour tout objet $N$ de $\mathbf{MF_W^{-h}}$ avec $0 \le h \le p-2$, construisons $\mathsf{f}_N$ comme la composée :

$$\begin{array}{ccccc} \mathbf{V_{cris}}(N) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} & \xrightarrow{g_N^{-1}} & \mathbf{V}_{\mathcal{O}_{\widehat{\mathcal{E}}}}(\mathrm{F}^-(N)) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} & \xrightarrow{\psi_{\mathrm{F}^-(N)}} & F^-(N) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \\ & & & & \| \\ & & & & N \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \end{array}$$

où $\psi$ est l'isomorphisme de Fontaine (cf. paragaraphe 1.1.2), $g_N$ l'isomorphisme de N. Wach (cf. paragraphe 3.1), et $\mathrm{F}^-$ est le foncteur construit à la fin de la partie 2.

De la proposition 35 nous déduisons (toujours à $0 \le h \le p-2$ fixé) :

Proposition 46. *Pour tout uplet d'objets $(N_{i,j})_{1\le j\le n,\,1\le i\le n_j}$ de $\mathbf{MF_W^{-h}}$, pour tout sous-$\varphi$-module filtré $L$ facteur direct (comme $W$-module) de $\bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} N_{i,j}$, l'application $\bigoplus \otimes \mathsf{f}_{N_{i,j}}$ envoie $\overline{\mathbf{V}}_{\mathbf{cris}}(L) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ bijectivement sur $L \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$.*

*Démonstration.* Rappelons que $\overline{\mathbf{V}}_{\mathbf{cris}}(L) = \mathbf{V_{cris,p}}(D_L) \cap \bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} \mathbf{V_{cris}}(N_{i,j})$.

Comme corollaire de la proposition 35, l'inverse de la fonction $\psi^{-1}_{\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(\mathbf{V_{cris}}(N))} \circ \big(\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(g_N) \otimes \mathrm{Id}\big)$ vérifie la propriété recherchée, car $\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(g_N)$ envoie $L \otimes_W S$ sur $\mathbf{N}(\overline{\mathbf{V}}_{\mathbf{cris}}(L))$, donc $L \otimes_W \mathcal{O}_{\mathcal{E}}$ sur $\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(\overline{\mathbf{V}}_{\mathbf{cris}}(L))$. Il suffit alors de remarquer que $\mathsf{f}_N = \psi_{\mathrm{F}^-(N)} \circ \big(g_N^{-1} \otimes \mathrm{Id}\big) = \big(\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(g_N^{-1}) \otimes \mathrm{Id}\big) \circ \psi_{\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(\mathbf{V_{cris}}(N))}$ par commutativité du diagramme suivant :

$$\begin{array}{ccc} \mathbf{V_{cris}}(N) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} & \xrightarrow{g_N^{-1} \otimes \mathrm{Id}} & \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(N)) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \\ \downarrow \psi_{\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(\mathbf{V_{cris}}(N))} & & \downarrow \psi_{\mathrm{F}^-(N)} \\ \mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(\mathbf{V_{cris}}(N)) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} & \xrightarrow[\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}(g_N^{-1}) \otimes \mathrm{Id}]{} & \mathrm{F}^-(N) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \end{array}$$

car $\mathbf{D}_{\mathcal{O}_{\mathcal{E}}}$ et $\mathbf{V}_{\mathcal{O}_{\mathcal{E}}}$ sont des foncteurs quasi-inverses l'un de l'autre. $\square$

En combinant ce résultat et celui de la remarque 24, nous obtenons

Proposition 47. *Si $L$ est un sous-objet dans $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$ de $\bigoplus_i \otimes_j N_{i,j}$ avec $N_{i,j}$ des objets de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$ avec $0 \le h \le p-2$, alors $\mathbf{V_{cris}}(L) \subset \bigoplus_i \otimes_j \mathbf{V_{cris}}(N)_{i,j}$.*

Remarque 48. *Cette propriété peut être montré directement, en utilisant les propriétés des périodes des Lubin-Tate (qui donnent par produit les périodes des modules élémentaires) et le fait qu'un $\varphi$-module filtré simple est élémentaire, donc que (par Jordan-Hölder) tout $\varphi$-module filtré tué par p a une filtration dont le gradué associé est somme directe de modules élémentaires.*

Combiné avec le Théorème 1', et en introduisant $\mathcal{F}_1$ et $\mathcal{F}_2$ les foncteurs exacts de la catégorie $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$ vers la catégorie des $\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$-modules libres de rang fini, définis par : si $M$ objet de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$, $\mathcal{F}_1(M) = \mathbf{V_{cris}}(M) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ et $\mathcal{F}_2(M) = M \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$, nous obtenons :

Théorème 49. *Pour $0 \le h \le p-2$ fixé, il existe $\mathsf{f}$ un isomorphisme de foncteur entre $\mathcal{F}_1$ et $\mathcal{F}_2$. De plus, vis à vis du produit tensoriel, l'isomorphisme peut être choisi de telle sorte que :*

– *pour tous objets $M$ et $N$ de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$ tels que $M \otimes N$ est encore un objet de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$, alors le diagramme suivant est commutatif :*

$$\begin{array}{ccc}
\mathbf{V_{cris}}(N \otimes M) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} & \xrightarrow{\mathsf{f}_{N \otimes M}} & (N \otimes M) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \\
\downarrow & & \downarrow \\
(\mathbf{V_{cris}}(M) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}) \otimes (\mathbf{V_{cris}}(N) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}) & \xrightarrow[\mathsf{f}_M \otimes \mathsf{f}_N]{} & (N \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}) \otimes (M \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}})
\end{array}$$

– *pour tout uplet d'objets $(N_{i,j})_{1 \le j \le n,\, 1 \le i \le n_j}$ de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$, pour tout sous-objet $L$ de $\bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} N_{i,j}$ dans $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$, l'application $\bigoplus \otimes \mathsf{f}_{N_{i,j}}$ restreinte à $\mathbf{V_{cris}}(L)$ est égale à $\mathsf{f}_L$ ;*

– *pour tout uplet d'objets $(N_{i,j})_{1 \le j \le n,\, 1 \le i \le n_j}$ de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$, pour tout sous-$\varphi$-module filtré $L$ facteur direct (comme $W$-module) de $\bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} N_{i,j}$, l'application $\bigoplus \otimes \mathsf{f}_{N_{i,j}}$ envoie $\big(\mathbf{V_{cris,p}}(D_L) \cap \bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} \mathbf{V_{cris}}(N_{i,j})\big) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ bijectivement sur $L \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$.*

Il faut juste regarder le comportement de $\mathsf{f}$ vis à vis du dual pour pouvoir déduire du théorème 49 le théorème 2 de l'introduction.

## 5.2 $\mathsf{f}$ ET LE DUAL

Pour la suite, nous aurons besoin de définir $\mathsf{f}_N$ pour $N$ ayant des poids à la fois positifs et négatifs (par exemple si $N = \mathrm{End}(M)$ pour $M$ un objet de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$). Rappelons que $W[-h]$ est l'objet de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$ dont le $W$-module sous-jacent est $W$, avec $\mathrm{Fil}^i(W[-h]) = \begin{cases} W \text{ si } i \leq -h \\ 0 \text{ si } i > -h \end{cases}$ et $\varphi^{-h}(x) = \sigma(x)$. Rappelons que $\mathbb{Z}_p(h)$ est la représentation galoisienne $\mathbb{Z}_p(1)^{\otimes h}$ pour $h \geq 0$ (et $\mathbb{Z}_p(h) = \mathbb{Z}_p(-h)^*$ si $h \leq 0$), et que $\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}(h) = \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(h)$. Nous pouvons alors définir :

DÉFINITION 50. *Supposons* $0 \leq h \leq \frac{p-2}{2}$. *Posons* $\widetilde{\mathbf{V}}_{\mathbf{cris}}(N) = \mathbf{V}_{\mathbf{cris}}(N \otimes W[-h]) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-h)$ *pour* $N$ *objet de* $\mathbf{MF}_{\mathbf{W}}^{\pm\mathbf{h}}$, *et* $\widetilde{\mathbf{V}}_{\mathbf{cris}}(f) = \mathbf{V}_{\mathbf{cris,p}}(f)$ *restreinte à* $\widetilde{\mathbf{V}}_{\mathbf{cris}}(N)$ *pour* $f : N \to N'$ *flèche de* $\mathbf{MF}_{\mathbf{W}}^{\pm\mathbf{h}}$. *Pour tout objet* $N$ *de* $\mathbf{MF}_{\mathbf{W}}^{\pm\mathbf{h}}$ *nous pouvons définir* $\overline{\mathsf{f}}_N$ *de la façon suivante : remarquons que* $\widetilde{\mathbf{V}}_{\mathbf{cris}}(N) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} = \big(\mathbf{V}_{\mathbf{cris}}(N \otimes W[-h]) \otimes \mathbb{Z}_p(-h)\big) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} = \big(\mathbf{V}_{\mathbf{cris}}(N \otimes W[-h]) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}\big) \otimes_{\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}(-h)$, *et notons* $\mathsf{f}_h = {}^t\mathsf{f}^{-1}_{\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}[-h]}$, *alors* $\overline{\mathsf{f}}_N$ *est l'isomorphisme*

$$\begin{array}{ccc} \widetilde{\mathbf{V}}_{\mathbf{cris}}(N) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} & \xrightarrow{\mathsf{f}_{N \otimes W[-h]} \otimes \mathsf{f}_h} & \big((N \otimes W[-h]) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}\big) \otimes \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}[h] \\ & & \downarrow \\ & & N \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \end{array}$$

Nous avons bien que $\widetilde{\mathbf{V}}_{\mathbf{cris}}(N) \simeq \mathbf{V}_{\mathbf{cris}}(N) \otimes_{\mathbb{Z}_p} \mathbf{V}_{\mathbf{cris}}(W[-h]) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-h) \simeq \mathbf{V}_{\mathbf{cris}}(N)$ de manière naturelle pour $N$ un objet de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{h}}$, car $N \otimes W[-h]$ est un objet de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{2h}}$, et comme $2h \leq p-2$, nous pouvons appliquer la remarque 23.
Un quasi-inverse de $\widetilde{\mathbf{V}}_{\mathbf{cris}}$ est donné par $\widetilde{\mathbf{D}}_{\mathbf{cris}}(V) = \mathbf{D}_{\mathbf{cris}}(V \otimes \mathbb{Z}_p(h)) \otimes_W W[h]$.
Une façon naturelle de voir $\widetilde{\mathbf{V}}_{\mathbf{cris}}(N)$ dans $N \otimes_W A_{cris}$ est de dire que

$$\widetilde{\mathbf{V}}_{\mathbf{cris}}(N) = \mathrm{Fil}^0(N \otimes_W t^{-h} A_{cris})^{\varphi} t^{-h}$$

Avant de continuer de regarder les propriétés de $\widetilde{\mathbf{V}}_{\mathbf{cris}}$, introduisons $\overline{g}_N$ pour $N$ un objet de $\mathbf{MF}_{\mathbf{W}}^{\pm\mathbf{h}}$ si $0 \leq h \leq \frac{p-2}{2}$ de la même façon que précédemment. De la proposition 35 et de la remarque 36, nous déduisons :

COROLLAIRE 51. *Pour tout uplet d'objets* $(N_{i,j})_{1 \leq j \leq n,\, 1 \leq i \leq m}$ *de* $\mathbf{MF}_{\mathbf{W}}^{\pm\mathbf{h}}$ *avec* $0 \leq h \leq \frac{p-2}{2}$, *pour tout sous-objet* $L$ *facteur direct (comme* $W$*-module) de* $\bigoplus_{j=1}^{n} \otimes_{i=1}^{m} N_{i,j}$ *et pour tout quotient* $M$ *de* $L$, *les applications* $\overline{g}_{N_{i,j}}$ *induisent un isomorphisme de représentations de* $\Gamma_{\mathcal{K}}$, *de* $\mathbf{V}_{\mathcal{O}_{\widehat{\mathcal{E}}}}(\mathrm{F}^-(M \otimes W[-mh])) \otimes \mathbb{Z}_p(-mh)$ *sur un réseau de* $\mathbf{V}_{\mathbf{cris,p}}(D_M)$ *(qui est l'image de* $\mathbf{V}_{\mathbf{cris,p}}(D_L) \cap \bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} \widetilde{\mathbf{V}}_{\mathbf{cris}}(N_{i,j})$ *par l'application projection).*

Puis, pour l'étude vis à vis du dual, montrons d'abord le lemme suivant :

Lemme 52. *Pour tout objet $N$ de $\mathbf{MF}_{\mathbf{W}}^{\pm\mathbf{h}}$ avec $0 \leq h \leq \frac{p-2}{2}$, l'application surjective naturelle*

$$N \otimes N^* \xrightarrow{\pi} W$$

*induit un isomorphisme*

$$\widetilde{\mathbf{V}}_{\mathbf{cris}}(N^*) \simeq \widetilde{\mathbf{V}}_{\mathbf{cris}}(N)^*$$

*dont le crochet de dualité correspond à $\mathbf{V}_{\mathbf{cris},\mathbf{p}}(\pi)$.*

*Démonstration.* Soient $l \in \mathbb{N}$ et $l' \in \mathbb{N}$ tels que $N \otimes W[-l]$ et $N^* \otimes W[-l']$ soient des objets de $\mathbf{MF}_{\mathbf{W}}^{-\mathbf{2h}}$. Puis, $W$ désigne l'objet trivial de $\mathbf{MF}_{\mathbf{W}}^{\pm\mathbf{h}}$, et donc $\pi$ induit une application $\mathrm{F}^-(\pi) : \mathrm{F}^-(N \otimes W[-l]) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathrm{F}^-(N^* \otimes W[-l']) \to \mathcal{O}_{\mathcal{E}}[-(l+l')]$, dont l'application linéaire sous-jacente est toujours celle obtenue par le crochet de dualité (c'est juste $\pi \otimes \mathrm{Id}$), donc induit un isomorphisme entre le dual de $\mathrm{F}^-(N \otimes W[-l]) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}} e_l$ et $\mathrm{F}^-(N^* \otimes W[-l']) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}} e_{l'}$ (où $\varphi(e_r) = q^r e_r$ et $g(e_r) = \frac{\mathcal{X}(g)^{-r}\pi^{-r}}{g(\pi^{-r})}$ pour $g \in \Gamma$). Cet isomorphisme de $\mathcal{O}_{\mathcal{E}}$-modules est en fait un isomorphisme de $(\varphi, \Gamma)$-module, car $\mathrm{F}^-(\pi)$ est un morphisme de $(\varphi, \Gamma)$-modules. Comme $\mathbf{V}_{\mathcal{O}_{\mathcal{E}}}$ préserve le dual, en notant $\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(N) = \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(N \otimes W[-l])) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-l) = \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}\big(\mathrm{F}^-(N \otimes W[-l]) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}} e_l\big)$ et $\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(N^*) = \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(N \otimes W[-l'])) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(-l') = \mathbf{V}_{\mathcal{O}_{\mathcal{E}}}\big(\mathrm{F}^-(N^* \otimes W[-l']) \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}} e_{l'}\big)$, nous avons que l'application $\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(\pi))$ identifie le dual de $\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(N)$ (comme représentation de $\Gamma_{\mathcal{K}}$) à $\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(N)^*$.

Or, $\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(N)$ est isomorphe à $\widetilde{\mathbf{V}}_{\mathbf{cris}}(N)$ (via $\overline{g}_N$) et $\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(N^*)$ est isomorphe à $\widetilde{\mathbf{V}}_{\mathbf{cris}}(N^*)$ (via $\overline{g}_{N^*}$). Pour conclure, il suffit d'invoquer la commutativité du diagramme suivant (d'après le corollaire 37) :

$$\begin{tikzcd}
\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(N) \otimes \widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(N^*) \arrow[r, "\overline{g}_N \otimes \overline{g}_{N^*}"'] \arrow[d, "\widetilde{\mathbf{V}}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(\pi))"'] & \widetilde{\mathbf{V}}_{\mathbf{cris}}(N) \otimes \widetilde{\mathbf{V}}_{\mathbf{cris}}(N^*) \arrow[d, "\mathbf{V}_{\mathbf{cris},\mathbf{p}}(\pi)"'] \\
\mathbf{V}_{\mathcal{O}_{\mathcal{E}}}(\mathrm{F}^-(W)) \arrow[r, "\overline{g}_W"'] & \widetilde{\mathbf{V}}_{\mathbf{cris}}(W)
\end{tikzcd}$$

□

Nous en déduisons alors :

Lemme 53. *Sous les conditions du lemme 52, l'application $\overline{\mathfrak{f}}_N$ est fonctorielle vis à vis du dual, c'est-à-dire que le diagramme suivant commute :*

$$\begin{tikzcd}
\widetilde{\mathbf{V}}_{\mathbf{cris}}(N)^* \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \arrow[r, "({}^t\overline{\mathfrak{f}}_N)^{-1}"] \arrow[d] & N^* \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \arrow[d] \\
\widetilde{\mathbf{V}}_{\mathbf{cris}}(N^*) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}} \arrow[r, "\overline{\mathfrak{f}}_{N^*}"] & N^* \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}
\end{tikzcd}$$

D'où, en rassemblant tout ceci, nous obtenons le théorème suivant :

THÉORÈME 54. *Soit* $0 \leq h \leq \frac{p-2}{2}$, *et notons* $\widetilde{\mathcal{F}}_1$ *le foncteur exact de la catégorie* $\mathbf{MF}^{\pm \mathbf{h}}_{\mathbf{W}}$ *vers la catégorie des* $\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$*-modules libres de rang fini, défini par : si* $N$ *objet de* $\mathbf{MF}^{\pm \mathbf{h}}_{\mathbf{W}}$, $\widetilde{\mathcal{F}}_1(N) = \widetilde{\mathbf{V}}_{\mathbf{cris}}(N) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$. *Alors il existe* $\overline{\mathsf{f}}$ *un isomorphisme de foncteurs entre* $\widetilde{\mathcal{F}}_1$ *et* $\mathcal{F}_2$*, préservant le dual. Nous pouvons supposer de plus :*

- *pour tous objet* $N$ *et* $N'$ *de* $\mathbf{MF}^{\pm \mathbf{h}}_{\mathbf{W}}$*, tel que* $N \otimes N'$ *soit encore un objet de* $\mathbf{MF}^{\pm \mathbf{h}}_{\mathbf{W}}$, $\overline{\mathsf{f}}_{N \otimes N'} = \overline{\mathsf{f}}_N \otimes \overline{\mathsf{f}}_{N'}$ *;*
- *pour tout uplet d'objets* $(N_{i,j})_{1 \leq j \leq n,\, 1 \leq i \leq n_j}$ *de* $\mathbf{MF}^{\pm \mathbf{h}}_{\mathbf{W}}$*, pour tout sous-*$\varphi$*-module filtré* $L$ *facteur direct (comme* $W$*-module) de* $\bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} N_{i,j}$*, l'application* $\bigoplus \otimes \overline{\mathsf{f}}_{N_{i,j}}$ *envoie* $(\mathbf{V}_{\mathbf{cris},\mathbf{p}}(D_L) \cap \bigoplus_{j=1}^{n} \otimes_{i=1}^{n_j} \widetilde{\mathbf{V}}_{\mathbf{cris}}(N_{i,j})) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ *bijectivement sur* $L \otimes_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$.

## 6 Position des réseaux

Pour tout ce paragraphe, nous supposerons donnés $\rho : \Gamma_{\mathcal{K}} \to G(\mathbb{Z}_p)$ une représentation cristalline à valeurs dans les point sur $\mathbb{Z}_p$ d'un groupe algébrique lisse sur $\mathbb{Z}_p$, $G$, et $U$ un $\mathbb{Z}_p$-module libre de rang $n$, avec $\alpha : G \to GL_U$ une immersion fermée.

### 6.1 Description des groupes plats sur $\mathbb{Z}_p$

Notons $U_W = U \otimes_{\mathbb{Z}_p} W$. Identifions $G$ avec son image dans $GL_U$. Nous allons donner une définition plus exploitable de $G_W = G \times_{\mathbb{Z}_p} W$ dans un cas particulier :
Prenons une base de $U$ et supposons que l'immersion de $G$ dans $GL_U$ induise une immersion dans $\mathrm{End}_U$ (c'est-à-dire $G = \mathrm{Spec}(\mathbb{Z}_p[X_{i,j}]_{1 \leq i,j \leq n}/I)$).
Notons $\mathbb{Z}_p[X_{i,j}]_{\leq d}$ les polynômes de degré total inférieur ou égal à $d$. Le groupe $GL_U$ agit sur $\mathbb{Z}_p[X_{i,j}]$ par : pour toute $\mathbb{Z}_p$-algèbre $R$, si $f \in R[X_{i,j}]$, et $s \in GL_U(R)$, alors $\eta_s(f)$ est le polynôme défini par $\eta_s(f)(y) = f(s^{-1}y)$. Cette action est linéaire et laisse stable $\mathbb{Z}_p[X_{i,j}]_{\leq d}$.

PROPOSITION 55. *Soit* $G = \mathrm{Spec}(\mathbb{Z}_p[X_{i,j}]_{1 \leq i,j \leq n}/I)$ *un groupe algébrique plat sur* $\mathbb{Z}_p$*, soient* $(f_1, \cdots, f_r)$ *des générateurs de l'idéal* $I$*, et si* $d$ *est le maximum des degrés totaux des* $f_i$*, posons* $E = I \cap \mathbb{Z}_p[X_{i,j}]_{\leq d}$. *Alors* $E$ *est facteur direct, et pour toute* $\mathbb{Z}_p$*-algèbre* $R$*, si* $E_R = E \otimes_{\mathbb{Z}_p} R$, $G(R) = \{g \in GL_U(R) | \eta_g(E_R) = E_R\}$.

REMARQUE 56. *Si* $\alpha : G \to GL_U$ *n'induit pas une immersion fermée de* $G$ *dans* $\mathrm{End}_U$*, il suffit de composer* $\alpha$ *avec une immersion fermée* $\beta : GL_U \to GL_{U'}$ *tel que* $\beta$ *induise une immersion fermée de* $GL_U$ *dans* $\mathrm{End}_{U'}$*. Par exemple,* $U' = U \oplus \mathbb{Z}_p$ *avec* $\beta = \mathrm{Id} \oplus \frac{1}{\det}$*, ou bien* $U' = U \oplus U^*$ *(où* $U^*$ *est le dual de* $U$*)*

*avec $\beta(g) = (g, {}^t g^{-1})$. Par contre, le $E$ donné par la proposition 55 dépendra du morphisme $\beta$ considéré.*

*Démonstration.* Commençons par le lemme suivant :

Lemme 57. *Pour toute $\mathbb{Z}_p$-algèbre $R$, le module $I \otimes_{\mathbb{Z}_p} R$ s'injecte dans $R[X_{i,j}]$.*

*Démonstration.* Pour tout $k$, notons $E_k = I \cap \mathbb{Z}_p[X_{i,j}]_{\leq k}$. Le module $\mathbb{Z}_p[X_{i,j}]_{\leq k}$ est un $\mathbb{Z}_p$-module libre de type fini, et $\mathbb{Z}_p[X_{i,j}]_{\leq k}/E_k$ est sans $p$-torsion car il s'injecte dans $\mathbb{Z}_p[X_{i,j}]_{1\leq i,j\leq n}/I$ qui est plat sur $\mathbb{Z}_p$ (par hypothèse). Donc le module $E_k$ est un $\mathbb{Z}_p$-module libre facteur direct dans $\mathbb{Z}_p[X_{i,j}]_{\leq k}$. Donc $E_k$ est un facteur direct de $\mathbb{Z}_p[X_{i,j}]$, par conséquent $E_k \otimes_{\mathbb{Z}_p} R$ s'injecte dans $R[X_{i,j}]$. Or, $I$ est la réunion des $E_k$, donc $I \otimes_{\mathbb{Z}_p} R$ s'injecte dans $R[X_{i,j}]$. □

Au passage, nous avons démontré une assertion de la proposition, à savoir que $E := E_d$ est facteur direct.
Notons $H$ le groupe algébrique défini par $H(R) = \{g \in GL_U(R) | \eta_g(E_R) = E_R\}$. L'application naturelle $H \to GL_U$ est une immersion fermée et $H(R) = \{g \in GL_U(R) | \eta_g(E_R) \subset E_R\}$ car $E$ est facteur direct. Nous voulons montrer que $H = G$. Si $s \in GL_U(R)$ vérifie $\eta_g(E_R) = E_R$, alors pour tout $i$, $\eta(s)(f_i) \in E_R \subset I \otimes_{\mathbb{Z}_p} R$, donc $\eta(s)(f_i)(Id) = 0$ car $Id \in G(R)$. Donc, par définition de l'action, $f_i(s^{-1}) = 0$ pour tout $i$, donc la famille $(f_i)$ étant une famille de générateurs de l'idéal $I$ sur $\mathbb{Z}_p$, nous obtenons $s^{-1} \in G(R)$, or $G$ est un groupe, donc $s \in G(R)$, ce qui montre l'inclusion $H(R) \subset G(R)$. Donc il existe un monomorphisme $H \to G$.
Montrons que c'est une immersion fermée : le morphisme $H \to GL_U$ est une immersion fermée, et $\alpha$ est par hypothèse une immersion fermée de $G$ dans $GL_U$. Donc, en notant $A[K]$ l'algèbre affine d'un groupe $K$, les flèches $A[GL_U] \to A[G]$ et $A[GL_U] \to A[H]$ sont surjectives, et nous avons le diagramme commutatif suivant :

$$\begin{tikzcd} A[G] \arrow[d] & A[GL_U] \arrow[l, two heads] \arrow[dl, two heads] \\ A[H] & \end{tikzcd}$$

par conséquent, la flèche $A[G] \to A[H]$ est surjective, donc $H \to G$ est bien une immersion fermée.
Nous allons maintenant montrer que $G$ et $H$ ont même fibre générique. Pour cela, donnons une description de $G$ semblable à celle de $H$ :

Lemme 58. *Pour toute $\mathbb{Z}_p$-algèbre $R$, nous avons*

$$G(R) = \{g \in GL_U(R) | \eta_g\big((I \otimes_{\mathbb{Z}_p} R) \cap R[X_{i,j}]_{\leq d}\big) = (I \otimes_{\mathbb{Z}_p} R) \cap R[X_{i,j}]_{\leq d}\}$$

*Démonstration.* Fixons $R$, et posons $M = (I \otimes_{\mathbb{Z}_p} R) \cap R[X_{i,j}]_{\leq d}$. Si $s \in GL_U(R)$ vérifie $\eta_g(M) = M$, alors pour tout $i$, $\eta(s)(f_i) \in M \subset I \otimes_{\mathbb{Z}_p} R$, donc

$\eta(s)(f_i)(Id) = 0$ car $Id \in G(R)$. Donc, par définition de l'action, $f_i(s^{-1}) = 0$ pour tout $i$, donc la famille $(f_i)$ étant une famille de générateurs de l'idéal $I$ sur $\mathbb{Z}_p$, nous obtenons $s^{-1} \in G(R)$, or $G$ est un groupe, donc $s \in G(R)$, ce qui montre une inclusion.
L'inclusion réciproque sera un corollaire du lemme de Yoneda : d'abord, il suffit de montrer que $\eta_s(M) \subset M$, car $\eta$ est une action de groupe. Puis, si $s \in G(R)$ et $f \in M$, notons $P = \eta(s)(f)$. Pour toute $R$-algèbre $B$, pour tout $g \in G(B)$, nous avons $P(g) = f(s^{-1}g) = 0$ car $s^{-1}g \in G(B)$ et $f \in I \otimes_{\mathbb{Z}_p} R$, donc $P$ est dans $I \otimes_{\mathbb{Z}_p} R$ (et de bon degré, donc $P \in M$) par la remarque suivante :
Soit $G = \mathrm{Spec}(A/I)$ un groupe algébrique au dessus d'un anneau $R$, alors si $J = \{f \in A |$ pour toute $R$-algebre $B$, $\forall g \in G(B)$, $f(g) = 0\}$, nous avons $I = J$. En effet, si $K = \mathrm{Spec}(A/J)$, alors $K(B) \subset G(B)$ car $I \subset J$, puis la définition de $J$ nous dit que pour tout $\varphi : A/I \to B$, $J$ est inclus dans $\mathrm{Ker}(\varphi)$, d'où se factorise en $\varphi : A/J \to B$, $J$, donc $G(B) \subset K(B)$. Le lemme de Yoneda donne alors $G = K$, donc $I = J$. □

Lemme 59. *Soit $S/R$ une extension d'anneau, supposons que $S$ est plat sur $R$. Alors, $((I \otimes_{\mathbb{Z}_p} R) \cap R[X_{i,j}]_{\leq d}) \otimes_R S = (I \otimes_{\mathbb{Z}_p} S) \cap S[X_{i,j}]_{\leq d}$.*

*Démonstration.* En effet, notons $M_1 = I \otimes_{\mathbb{Z}_p} R$, $M_2 = R[X_{i,j}]_{\leq d}$ et $M_3 = R[X_{i,j}]$, alors nous voulons voir que $(M_1 \cap M_2) \otimes_R S = (M_1 \otimes_R S) \cap (M_2 \otimes_R S)$. Or, nous avons la suite exacte courte de $R$-modules

$$0 \longrightarrow M_1 \cap M_2 \longrightarrow M_3 \longrightarrow M_3/M_1 \oplus M_3/M_2$$

et nous avons supposé que $S$ est plat sur $R$, donc en tensorisant par $S$ au dessus de $R$, nous obtenons que

$$0 \longrightarrow (M_1 \cap M_2) \otimes S \longrightarrow M_3 \otimes S \longrightarrow (M_3/M_1) \otimes S \oplus (M_3/M_2) \otimes S$$

est une suite exacte, ce qui conclut car $(M_3/M_i) \otimes_R S = M_3 \otimes_R S/M_i \otimes_R S$. □

Puis, $G$ et $H$ ont même fibre générique, par application directe des deux lemmes précédents.
Il ne reste plus qu'à voir que si $H \to G$ est une immersion fermée telle que $G$ et $H$ ont même fibre générique, et $G$ plat, alors $H = G$. Nous voulons montrer que la flèche surjective $A[G] \to A[H]$ est aussi injective. $G$ étant plat, l'application naturelle $i_G : A[G] \to A[G] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ est injective. $H$ et $G$ ayant même fibre générique, la flèche $f : A[G] \to A[H]$ donnée par l'immersion fermée induit une bijection $f_p : A[G] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to A[H] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. De plus le diagramme suivant est commutatif :

$$\begin{tikzcd} A[G] \arrow[r, "f"] \arrow[d, hook, "i_G"] & A[H] \arrow[d, "i_H"] \\ A[G] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \arrow[r, hook, two heads, "f_p"] & A[H] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \end{tikzcd}$$

donc $f$ est bien injective, donc $H = G$. □

En fait, lors de l'identification de $GL_U$ avec $GL_{n,\mathbb{Z}_p}$, nous avons identifié la représentation de $GL_U$ que sont les polynômes homogènes de degré $i$ (noté $\mathbb{Z}_p^i[X_{i,j}]$) avec $\mathrm{Sym}^i(\mathrm{End}_U)$ qui est un sous-objet de $\mathrm{End}_U^{\otimes i}$, ou bien, dit autrement, $\mathbb{Z}_p[X_{i,j}]_{\leq d}$ a été identifié à un sous-objet de $\oplus_{0\leq i\leq d}\mathrm{End}_U^{\otimes i}$. L'action de $GL_U$ sur $\mathbb{Z}_p^i[X_{i,j}]$ est le produit tensoriel de l'action naturelle de $GL_U$ sur $U^*$ par l'action triviale de $GL_U$ sur $U$ dans $\mathrm{End}_U^{\otimes i} = (U \otimes_{\mathbb{Z}_p} U^*)^{\otimes i}$ ; par conséquent $E$ est un sous-module de $\bigoplus_{0\leq i\leq d}(\underbrace{U^*\oplus\cdots\oplus U^*}_{n \text{ fois}})^{\otimes i}$ (où $n = \mathrm{rg}(U)$).

Appliquons ceci non pas au plongement $\alpha$, mais à $\alpha^* : G \to GL_{U^*}$, défini par $\alpha^*(g) = {}^t\alpha(g)^{-1}$. Il existe alors $E^* = \bigoplus_{0\leq i\leq k}(\underbrace{U\oplus\cdots\oplus U}_{n \text{ fois}})^{\otimes i} \cap I^*$ un sous $\mathbb{Z}_p$-module (libre facteur direct) tel que $s \in \alpha(G)(R) \Leftrightarrow s(E_R^*) = E_R^*$ (où $I^*$ est l'idéal définissant $\alpha^*(G)$).

Rassemblons tout ceci dans le lemme suivant :

Proposition 60. *Soit $G$ un groupe plat sur $\mathbb{Z}_p$, $U$ un $\mathbb{Z}_p$-module libre de rang $n$ et $\alpha : G \to GL_U$ une représentation qui induit une immersion fermée dans* $\mathrm{End}_U$. *Identifions $G$ avec son image. Alors, il existe un entier $k$ et un sous $\mathbb{Z}_p$-module $E$ (facteur direct) de $\bigoplus_{0\leq i\leq k}(\underbrace{U\oplus\cdots\oplus U}_{n \text{ fois}})^{\otimes i}$ laissé stable par l'action naturelle de $G(\mathbb{Z}_p)$ (provenant de celle de $GL_U$, notée $\eta$) tels que $G(R) = \{g \in GL(R) | \eta_g(E_R) = E_R\}$ pour toute $\mathbb{Z}_p$-algèbre $R$.*

Nous pouvons alors définir sur $M = \mathbf{D_{cris}}(U)$ (ou sur $\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)$ suivant les cas) un groupe algébrique sur $W$, $G_M$, par : si $\eta$ est l'action naturelle de $GL_M$ sur $\bigoplus_{0\leq i\leq k}(\underbrace{M\oplus\cdots\oplus M}_{n \text{ fois}})^{\otimes i}$, alors $G_M(R) = \{g \in GL_M(R) | \eta_g(\overline{E}_R) = \overline{E}_R\}$ pour toute $W$-algèbre $R$. Il ne reste qu'à bien choisir $\overline{E}$ en liaison avec $E$, ce qui nous conduit au théorème suivant :

Théorème 61. *Supposons $G$ lisse sur $\mathbb{Z}_p$. Si $\alpha : G \to GL_U$ induit une immersion fermée dans* $\mathrm{End}_U$ *et si la représentation de $\Gamma_{\mathcal{K}}$ induite sur $U$ par $\alpha$ (et par $\rho : \Gamma_{\mathcal{K}} \to G(\mathbb{Z}_p)$) vérifie*
*– soit elle est à poids de Hodge-Tate dans $[\![0, h]\!]$ avec $0 \leq h \leq p-2$*
*– soit elle est à poids de Hodge-Tate dans $[\![-h, h]\!]$ avec $0 \leq h \leq \frac{p-2}{2}$*
*alors, en prenant*

$$\overline{E} = \mathbf{D_{cris,p}}(E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cap \bigoplus_{0\leq i\leq k}(\underbrace{\mathbf{D_{cris}}(U)\oplus\cdots\oplus\mathbf{D_{cris}}(U)}_{n\ fois})^{\otimes i}$$

*dans le premier cas, ou*

$$\overline{E} = \mathbf{D_{cris,p}}(E \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \cap \bigoplus_{0\leq i\leq k}(\underbrace{\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)\oplus\cdots\oplus\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)}_{n\ fois})^{\otimes i}$$

*dans le deuxième cas, il existe une bijection $\Psi : U \otimes_{\mathbb{Z}_p} W \to M$ qui identifie $G \times_{\mathbb{Z}_p} W$ et $G_M$.*

REMARQUE 62. *Pour qu'une application $\Psi$ bijective identifie $G \times_{\mathbb{Z}_p} W$ et $G_M$, il suffit de montrer que l'application naturelle induite par $\Psi$ envoie $E \otimes_{\mathbb{Z}_p} W$ bijectivement sur $\overline{E}$.*

### 6.2 Demonstration du théorème 61

Soit $h : U_R = U \otimes_{\mathbb{Z}_p} R \simeq M_R = M \otimes_W R$ un isomorphisme de $R$-modules, alors $h$ induit $s(h)$ de $\bigoplus_{0\leq i\leq k}(\underbrace{U_R \oplus \cdots \oplus U_R}_{n \text{ fois}})^{\otimes i}$ sur $\bigoplus_{0\leq i\leq k}(\underbrace{M_R \oplus \cdots \oplus M_R}_{n \text{ fois}})^{\otimes i}$, qui est aussi un isomorphisme.
Considérons alors

$$\mathbf{Isom}(R) = \{h : U \otimes_{\mathbb{Z}_p} R \simeq M \otimes_W R | s(h)(E_R) = \overline{E}_R\}$$

pour $R$ une $W$-algèbre. C'est un sous-$W$-schéma de $\mathbf{Isom_W}(U \otimes_{\mathbb{Z}_p} W, M)$ (les $W$-isomorphismes de $U \otimes_{\mathbb{Z}_p} W$ sur $M$).
Il est non vide, car $\mathsf{f}_{\mathbf{D_{cris}}(U)}$ (ou $\overline{\mathsf{f}}_{\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)}$ suivant les conditions sur les poids de Hodge-Tate) induit un élément de $\mathbf{Isom}(\mathcal{O}_{\widehat{\mathcal{E}}_{nr}})$. C'est une retraduction du théorème 49 (ou du théorème 54)

LEMME 63. *Le schéma $\mathbf{Isom} \times_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ est un torseur trivial sous $G \times_W \mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$.*

*Démonstration.* $G$ agit naturellement et fidèlement à gauche sur $\mathbf{Isom}$ : si $f \in \mathbf{Isom}(R)$ et $g \in G(R)$,

$$U \otimes_{\mathbb{Z}_p} R \xrightarrow{g} U \otimes_{\mathbb{Z}_p} R \xrightarrow{f} M \otimes_W R$$

est bien un isomorphisme.
Puis, l'application naturelle

$$\begin{array}{ccc} \bigoplus_{0\leq i\leq k}(\underbrace{U \oplus \cdots \oplus U}_{n \text{ fois}})^{\otimes i} \otimes_{\mathbb{Z}_p} R & \xrightarrow{\eta_g} & \bigoplus_{0\leq i\leq k}(\underbrace{U \oplus \cdots \oplus U}_{n \text{ fois}})^{\otimes i} \otimes_{\mathbb{Z}_p} R \\ & & \downarrow{\scriptstyle s(f)} \\ & & \bigoplus_{0\leq i\leq k}(\underbrace{M \oplus \cdots \oplus M}_{n \text{ fois}})^{\otimes i} \otimes_W R \end{array}$$

s'identifie naturellement à $s(f \circ g)$.
Enfin, la définition de $\mathbf{Isom}$, la proposition 60 et le fait que $s(f \circ g) = s(f) \circ \eta_g$ donnent bien $s(f \circ g)(E_R) = E_R$, donc que $f \circ g \in \mathbf{Isom}(R)$. Le groupe $G$ agit donc sur $\mathbf{Isom}$ par $(g, f) \mapsto f \circ g^{-1}$.
La fidélité provient de ce qu'un élément de $\mathbf{Isom}$ est un isomorphisme de modules.
Pour finir, il reste à montrer que pour $f, f' \in \mathbf{Isom}(R)$, il existe $g \in G(R)$ avec $f' = f \circ g^{-1}$. Autrement dit, il faut voir que $g = f'^{-1} \circ f$ est bien un élément de $G(R)$. Cela se montre de la même façon que précédement. C'est la propriété 60 qui est le point essentiel. $\square$

Nous venons donc de montrer que $\mathbf{Isom}\times_W\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ est un $G\times_W\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$-espace homogène ayant un point sur $\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$, or $G\times_W\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ est un groupe lisse, donc $\mathbf{Isom}\times_W\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ est lisse, donc **Isom** aussi (car la flèche $\mathrm{Spec}(\mathcal{O}_{\widehat{\mathcal{E}}_{nr}})\to\mathrm{Spec}(W)$ est fidèlement plate et quasi-compact, donc c'est une application directe du corollaire 17.7.3 de EGA IV).

**Isom** est lisse, donc par le lemme de Hensel (cf. théorème 18.5.11 b de EGA IV), $\mathbf{Isom}(W)$ se surjecte (par la réduction modulo $p$) sur $\mathbf{Isom}(k)$. Si ce dernier est non vide, nous aurons bien montré que $\mathbf{Isom}(W)$ est non vide, ce qui prouvera le théorème.

$\mathbf{Isom}\times_W\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}$ est lisse, donc, toujours par le lemme de Hensel, $\mathbf{Isom}(\mathcal{O}_{\widehat{\mathcal{E}}_{nr}})$ se surjecte sur $\mathbf{Isom}(\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}/p)$, donc le $k$-schéma $\mathbf{Isom}\times\mathrm{Spec}(k)$ est non vide (car $\mathcal{O}_{\widehat{\mathcal{E}}_{nr}}/p$ est une $k$-algèbre), donc par le théorème des zéros de Hilbert, $\mathbf{Isom}\times\mathrm{Spec}(k)(k)$ est non vide (car $k$ est algébriquement clos). $\square$

Remarquons qu'en nous donnant un $\mathbb{Z}_p$-module $N\subset M$ qui engendre $M$ comme $W$-module (c'est à dire $N\otimes_{\mathbb{Z}_p}W=M$), et tel que le $\mathbb{Z}_p$-module $N'=\overline{E}\cap\bigoplus_{0\le i\le k}(\underbrace{N\oplus\cdots\oplus N}_{n\text{ fois}})^{\otimes i}$ engendre $\overline{E}$ comme $W$-module (par exemple, avec les notations du paragraphe 1.3, $N=M^{f_M}$ (et alors $N'=\overline{E}^{f_{\overline{E}}}$) ou $N=M_{\mathbb{Z}_p}^{u_N^{-1}f_M}$ (et alors $N'=\overline{E}_{\mathbb{Z}_p}^{u_{\overline{E}}^{-1}f_{\overline{E}}}$)), nous pouvons définir **Isom** sur $\mathbb{Z}_p$ par

$$\mathbf{Isom}(R)=\{h\,:\,U\otimes_{\mathbb{Z}_p}R\simeq N\otimes_{\mathbb{Z}_p}R|s(h)(E_R)=N'_R\}$$

pour $R$ une $\mathbb{Z}_p$-algèbre. Alors, par un théorème de Lang (tout $H$-torseur défini sur $\mathbb{F}_p$ est trivial si $H$ est un groupe algébrique sur $\mathbb{F}_p$ connexe) , en supposant que $G$ est à fibre spéciale connexe, nous avons $\mathbf{Isom}(\mathbb{F}_p)$ non vide, donc par lissité, $\mathbf{Isom}(\mathbb{Z}_p)$ est non vide, et donc sous cette hypothèse, nous pouvons supposer que $\Psi(U)=N$. De plus, $\Psi$ identifie $G$ à une forme sur $\mathbb{Z}_p$ de $G_M$ (celle qui est définie à l'aide de $N'$).

Corollaire 64. *Sous les hypothèses et notations précédentes, si $U'$ est un réseau de $U\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ laissé stable par l'action de $G$, alors $\Psi[\frac{1}{p}]=\Psi\otimes_W\mathcal{K}$ envoie $U'\otimes_{\mathbb{Z}_p}W$ sur $\mathbf{D_{cris}}(U')$ si les poids de Hodge-Tate sont positifs, ou sur $\widetilde{\mathbf{D}}_{\mathbf{cris}}(U')$ sinon.*

*Démonstration.* Notons $M=\mathbf{D_{cris}}(U)$ (ou $M=\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)$ si les poids de Hodge-Tate ne sont pas tous positifs). Tout d'abord, quitte à multiplier par une certaine puissance de $p$, nous pouvons supposer $U'\subset U$. Puis, $\Psi\in\mathbf{Isom}(W)\subset\mathbf{Isom}(\mathcal{O}_{\widehat{\mathcal{E}}_{nr}})$ et $\mathsf{f}_M\in\mathbf{Isom}(\mathcal{O}_{\widehat{\mathcal{E}}_{nr}})$ (respectivement, $\overline{\mathsf{f}}_M\in\mathbf{Isom}(\mathcal{O}_{\widehat{\mathcal{E}}_{nr}})$), donc, comme **Isom** est un $G\times_{\mathbb{Z}_p}W$-espace homogène, il existe $g\in G(\mathcal{O}_{\widehat{\mathcal{E}}_{nr}})$ tel que $\Psi=\mathsf{f}_M\circ g$ (respectivement $\Psi=\overline{\mathsf{f}}_M\circ g$). Il suffit donc (puisque $U'$ est stable par $G$) de vérifier la propriété pour $\mathsf{f}_M$ (ou $\overline{\mathsf{f}}_M$), or ceci provient juste de la fonctorialité de $\mathsf{f}_M$ (et de $\overline{\mathsf{f}}_M$) pour les sous-objets. $\square$

## 6.3 EXEMPLES

Remarquons que $GL_U$ se plonge naturellement par une immersion fermée $\beta$ dans $GL_{U\oplus U^*}$ où l'action sur $U$ est l'action naturelle, et l'action sur le dual $U^*$ est donnée par la transposée de l'inverse. De plus, $\beta$ provient d'une immersion fermée de $GL_U$ dans $\mathrm{End}_{U\oplus U^*}$, car l'image de $\beta$ est un sous-groupe fermé de $SL_{U\oplus U^*}$. Donc, si la représentation galoisienne $U$ est à poids de Hodge-Tate dans $[\![0,h]\!]$ avec $h \leq \frac{p-2}{2}$, ou dans $[\![-h,h]\!]$ avec $h \leq \frac{p-2}{4}$, alors l'immersion $\alpha' = \beta\circ\alpha$ vérifie en partie les hypothèses du théorème 61. Soit alors $E$ le sous-module définissant $G$ dans $U\oplus U^*$ (de la manière décrite dans le paragraphe 6.1). Nous définissons de même sur $\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)\oplus\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)^*$ un groupe $G_M$ à l'aide de

$$\overline{E} = \mathbf{D}_{\mathbf{cris,p}}(E\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)\cap\bigoplus_{0\leq i\leq k}(\underbrace{\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)\oplus\cdots\oplus\widetilde{\mathbf{D}}_{\mathbf{cris}}(U)}_{n\text{ fois}})^{\otimes i}$$

THÉORÈME 65. *Si la représentation de $\Gamma_{\mathcal{K}}$ induite sur $U$ par $\alpha$ (et par $\rho$) vérifie*
*– soit elle est à poids de Hodge-Tate dans $[\![0,h]\!]$ avec $0\leq h\leq\frac{p-2}{2}$*
*– soit elle est à poids de Hodge-Tate dans $[\![-h,h]\!]$ avec $0\leq h\leq\frac{p-2}{4}$*
*alors, avec les notations précédentes, il existe un isomorphisme de $W$-module $\Psi : U\otimes_{\mathbb{Z}_p}W\to M$ qui induit une bijection $\Psi\oplus{}^t\Psi^{-1} : (U\oplus U^*)\otimes_{\mathbb{Z}_p}W\to M\oplus M^*$ identifiant $G\times_{\mathbb{Z}_p}W$ et $G_M$. En particulier, le groupe $G_M$ (plongé dans $GL_{M\oplus M^*}$) laisse stable $M$ et $M^*$.*

*Démonstration.* Considérons le schéma $\mathbf{Isom}(R) = \{h : U\otimes_{\mathbb{Z}_p}R\simeq M\otimes_W R | s(h)(E_R) = \overline{E}_R\}$ (ce n'est à priori pas le même que celui considéré dans la démonstration du théorème 61, qui considère des morphismes définis sur $U\otimes U^*$). C'est un $G\times_{\mathbb{Z}_p}W$ espace homogène, qui a un point sur $W$ (cela se montre de la même façon que lors de la démonstration du théorème 61). $\square$

## 6.4 DONNÉES INITIALES POUR UN $\varphi$-MODULE FILTRÉ

Formulons ici comment les idées introduites précédemment se traduisent dans le formalisme introduit par Rapoport et Zink (cf. [RZ96]). Soit $G$ un groupe algébrique lisse sur $\mathbb{Z}_p$ et $\rho$ un morphisme de groupe de $\Gamma_{\mathcal{K}}$ dans $G(\mathbb{Z}_p)$, $\mu : \mathbb{G}_m\to G$ un cocaractère défini sur $W$, et $b\in G(W)$. Alors, à toute représentation $\beta : G\to GL_U$ où $U$ est un $\mathbb{Z}_p$-module libre de rang fini, nous pouvons associer un objet $\mathcal{I}(U)$ (qui aura vocation à être un $\varphi$-module filtré) :
- le $W$-module sous-jacent est $U\otimes_{\mathbb{Z}_p}W = M$ ;
- une filtration $\mathrm{Fil}^i(M) = \bigoplus_{j\geq i}M_j$ où $M_j$ est l'espace propre de poids $j$ correspondant à $\mu$ ;
- des applications $\varphi^i : \mathrm{Fil}^i(M)\to M[\frac{1}{p}]$ définies par $\varphi^i = p^{-i}b\circ(Id\otimes\sigma)$ sur $\mathrm{Fil}^i(M)$, c'est-à-dire si $v = \sum_k v_k\otimes x_k\in\mathrm{Fil}^i(M)$ avec $v_k\in U$ et $x_k\in W$, $\varphi^i(v) = p^{-i}\sum_k\sigma(x_k)\beta(b)(v_k)$.

Définition 66. *Le triplet* $(\mu, b, \beta)$ *est dit admissible si* $\mathcal{I}(U)$ *est un objet de* $\mathbf{MF_{W,tf}}$.

Si $G$ est supposé lisse, si $\beta$ induit une immersion fermée de $G$ dans $\mathrm{End}_U$, et $\mathcal{I}(U)$ est un objet de $\mathbf{MF_W^{-h}}$ avec $0 \leq h \leq p-2$, nous pouvons faire de même qu'au paragraphe 6.1 : $G$ est défini par $E$ un $\mathbb{Z}_p$-module bien choisi de $\bigoplus_{0\leq i\leq k} (\underbrace{U \oplus \cdots \oplus U}_{n \text{ fois}})^{\otimes i}$ (alors $E \otimes_{\mathbb{Z}_p} W$ sera un objet de $\mathbf{MF_{W,tf}}$), et sur $\mathbf{V_{cris}}(\mathcal{I}(U))$ nous construisons $G_{\mathbf{V_{cris}}(\mathcal{I}(U))}$ sur $\mathbb{Z}_p$ par son foncteur des points (si $R$ est une $\mathbb{Z}_p$-algèbre, $G_{\mathbf{V_{cris}}(\mathcal{I}(U))}(R)$ est le sous-groupe de $GL_{\mathbf{V_{cris}}(\mathcal{I}(U))}(R)$ formée des éléments qui laissent stable

$$\big(\mathbf{V_{cris,p}}(E \otimes_{\mathbb{Z}_p} \mathcal{K}) \cap \bigoplus_{0\leq i\leq k} (\underbrace{\mathbf{V_{cris}}(M) \oplus \cdots \oplus \mathbf{V_{cris}}(M)}_{n \text{ fois}})^{\otimes i}\big) \otimes_{\mathbb{Z}_p} R$$

Théorème 67. *Sous les conditions précédentes, avec*

– *soit* $M$ *est un objet de* $\mathbf{MF_W^{-h}}$ *avec* $0 \leq h \leq p-2$ *et* $\beta$ *induit une immersion fermée de* $G$ *dans* $\mathrm{End}_U$,
– *soit* $M$ *est un objet de* $\mathbf{MF_W^{-h}}$ *avec* $0 \leq h \leq \frac{p-2}{2}$,
– *soit* $M$ *est un objet de* $\mathbf{MF_W^{\pm h}}$ *avec* $0 \leq h \leq \frac{p-2}{2}$ *et* $\beta$ *induit une immersion fermée de* $G$ *dans* $\mathrm{End}_U$,
– *soit* $M$ *est un objet de* $\mathbf{MF_W^{\pm h}}$ *avec* $0 \leq h \leq \frac{p-2}{4}$,

*alors il existe une bijection* $\Psi : M \to \mathbf{V_{cris}}(\mathcal{I}(U)) \otimes_{\mathbb{Z}_p} W$ *qui identifie* $G \times_{\mathbb{Z}_p} W$ *et* $G_{\mathbf{V_{cris}}(\mathcal{I}(U))} \times_{\mathbb{Z}_p} W$. *De plus, la représentation galoisienne associée à* $\mathbf{V_{cris}}(\mathcal{I}(U))$ *est à valeurs dans* $G_{\mathbf{V_{cris}}(\mathcal{I}(U))}(\mathbb{Z}_p)$.

*Démonstration.* L'existence de la bijection se montre de la même façon que pour le théorème 61 : nous introduisons un $G$-espace homogène **Isom** défini sur $\mathbb{Z}_p$, nous montrons qu'il est lisse sur $\mathbb{Z}_p$, $\mathbf{Isom}(k)$ est non vide par le théorème des zéros de Hilbert, et le lemme de Hensel conclut. La définition même de $G_{\mathbf{V_{cris}}(\mathcal{I}(U))}$ implique que la représentation galoisienne associée à $\mathbf{V_{cris}}(\mathcal{I}(U))$ est à valeurs dans $G_{\mathbf{V_{cris}}(\mathcal{I}(U))}(\mathbb{Z}_p)$. □

Remarque 68. *Le théorème de Lang à propos des torseurs définis sur les corps finis implique que si la fibre spéciale de* $G$ *est connexe, alors le torseur* **Isom** *est trivial sur* $\mathbb{F}_p$. *Autrement dit, il a un point sur* $\mathbb{F}_p$, *que nous pouvons relever à* $\mathbb{Z}_p$ *par le lemme de Hensel. Par conséquent, si la fibre spéciale de* $G$ *est connexe, l'isomorphisme* $\Psi$ *est défini sur* $\mathbb{Z}_p$, *c'est à dire qu'il induit une bijection* $\Psi : U \to \mathbf{V_{cris}}(\mathcal{I}(U))$.

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# Rings of Integers of Type $K(\pi, 1)$

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Abstract. We investigate the Galois group $G_S(p)$ of the maximal $p$-extension unramified outside a finite set $S$ of primes of a number field in the (tame) case, when no prime dividing $p$ is in $S$. We show that the cohomology of $G_S(p)$ is 'often' isomorphic to the étale cohomology of the scheme $Spec(\mathcal{O}_k \smallsetminus S)$, in particular, $G_S(p)$ is of cohomological dimension 2 then.

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## 1 Introduction

We call a connected locally noetherian scheme $Y$ a '$K(\pi, 1)$' for a prime number $p$ if the higher homotopy groups of the $p$-completion $Y_{et}^{(p)}$ of its étale homotopy type $Y_{et}$ vanish. In this paper we consider the case of an arithmetic curve, where the $K(\pi, 1)$-property is linked with open questions in the theory of Galois groups with restricted ramification of number fields:

Let $k$ be a number field, $S$ a finite set of nonarchimedean primes of $k$ and $p$ a prime number. For simplicity, we assume that $p$ is odd or that $k$ is totally imaginary. By a $p$-extension we understand a Galois extension whose Galois group is a (pro-) $p$-group. Let $k_S(p)$ denote the maximal $p$-extension of $k$ unramified outside $S$ and put $G_S(p) = Gal(k_S(p)|k)$. A systematic study of this group had been started by Šafarevič, and was continued by Koch, Kuz'min, Wingberg and many others; see [NSW], VIII, §7 for basic properties of $G_S(p)$. In geometric terms (and omitting the base point), we have

$$G_S(p) \cong \pi_1\big((Spec(\mathcal{O}_k) \smallsetminus S)_{et}^{(p)}\big).$$

As is well known to the experts, if $S$ contains the set $S_p$ of primes dividing $p$, then $Spec(\mathcal{O}_k) \smallsetminus S$ is a $K(\pi, 1)$ for $p$ (see Proposition 2.3 below). In particular, if $S \supset S_p$, then $G_S(p)$ is of cohomological dimension less or equal to 2.

The group $G_S(p)$ is most mysterious in the *tame* case, i.e. when $S \cap S_p = \varnothing$. In this case, examples when $Spec(\mathcal{O}_k) \smallsetminus S$ is *not* a $K(\pi,1)$ are easily constructed. On the contrary, until recently not a single $K(\pi,1)$-example was known. The following properties of the group $G_S(p)$ were known so far:

- $G_S(p)$ is a 'fab-group', i.e. $U^{ab}$ is finite for each open subgroup $U \subset G$.
- $G_S(p)$ can be infinite (Golod-Šafarevič).
- $G_S(p)$ is a finitely presented pro-$p$-group (Koch).

A conjecture of Fontaine and Mazur [FM] asserts that $G_S(p)$ has no infinite $p$-adic analytic quotients.

In 2005, Labute considered the case $k = \mathbb{Q}$ and found finite sets $S$ of prime numbers (called strictly circular sets) with $p \notin S$ such that $G_S(p)$ has cohomological dimension 2. In [S1] the author showed that, in the examples given by Labute, $Spec(\mathbb{Z}) \smallsetminus S$ is a $K(\pi,1)$ for $p$.
The objective of this paper is a systematic study of the $K(\pi,1)$-property. Our focus is on the tame case, where we conjecture that rings of integers of type $K(\pi,1)$ are cofinal in the following sense:

Conjecture 1. *Let $k$ be a number field and let $p$ be a prime number. Assume that $p \neq 2$ or that $k$ is totally imaginary. Let $S$ be a finite set of primes of $k$ with $S \cap S_p = \varnothing$. Let, in addition, a set $T$ of primes of Dirichlet density $\delta(T) = 1$ be given. Then there exists a finite subset $T_1 \subset T$ such that $Spec(\mathcal{O}_k) \smallsetminus (S \cup T_1)$ is a $K(\pi,1)$ for $p$.*

Of course we may assume that $T \cap S_p = \varnothing$ in the conjecture. Our main result is the following

Theorem 1. *Conjecture 1 is true if the number field $k$ does not contain a primitive $p$-th root of unity and the class number of $k$ is prime to $p$.*

Explicit examples of rings of integers of type $K(\pi,1)$ can be found in [La], [S1] (for $k = \mathbb{Q}$) and in [Vo] (for $k$ imaginary quadratic).

The $K(\pi,1)$-property has strong consequences. We write $X = Spec(\mathcal{O}_k)$ and assume in all results that $p \neq 2$ or that $k$ is totally imaginary. Primes $\mathfrak{p} \in S \smallsetminus S_p$ with $\mu_p \not\subset k_\mathfrak{p}$ are redundant in the sense that removing these primes from $S$ does not change $(X \smallsetminus S)_{et}^{(p)}$, see section 4. In the tame case, we may therefore restrict our considerations to sets of primes whose norms are congruent to 1 modulo $p$. These are the results.

Theorem 2. *Let $S$ be a finite non-empty set of primes of $k$ whose norms are congruent to* 1 *modulo $p$. If $X \smallsetminus S$ is a $K(\pi,1)$ for $p$ and $G_S(p) \neq 1$, then the following hold.*

(i) *$cd\, G_S(p) = 2$, $scd\, G_S(p) = 3$.*

(ii) *$G_S(p)$ is a duality group.*

*The dualizing module $D$ of $G_S(p)$ is given by $D = tor_p C_S(k_S(p))$, i.e. it is the subgroup of $p$-torsion elements in the $S$-idèle class group of $k_S(p)$.*

Remarks: 1. If $X \smallsetminus S$ is a $K(\pi,1)$ for $p$ and $G_S(p) = 1$, then $k$ is imaginary quadratic, $\#S = 1$ and $p = 2$ or 3. See Proposition 7.4 for a more precise statement.

2. We have a natural exact sequence

$$0 \to \mu_{p^\infty}(k_S(p)) \to \bigoplus_{\mathfrak{p}\in S} \mathrm{Ind}\,{}^{G_\mathfrak{p}(k_S(p)|k)}_{Gal(k_S(p)|k)}\ \mu_{p^\infty}(k_S(p)_\mathfrak{p}) \to$$

$$\mathrm{tor}_p C_S(k_S(p)) \to \mathcal{O}^\times_{k_S(p),S} \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0\ ,$$

where $\mathcal{O}^\times_{k_S(p),S}$ is the group of $S$-units of $k_S(p)$ and $\mu_{p^\infty}(K)$ denotes the group of all $p$-power roots of unity in a field $K$. Note that $\mu_{p^\infty}(k_S(p))$ is finite, while, by Theorem 3 below, for $\mathfrak{p} \in S$ the field $k_S(p)_\mathfrak{p}$ contains all $p$-power roots of unity.

3. In the wild case $S \supset S_p$, where $X \smallsetminus S$ is always a $K(\pi,1)$ for $p$, $G_S(p)$ is of cohomological dimension 1 or 2. The strict cohomological dimension is conjecturally equal to 2 (=Leopoldt's conjecture for each finite subextension of $k$ in $k_S(p)$). In the wild case, $G_S(p)$ is often, but not always a duality group, cf. [NSW] Prop. 10.7.13.

Allowing ramification at a prime $\mathfrak{p}$ does not mean that the ramification is realized globally. Therefore it is a natural and interesting question how far we get locally at the primes in $S$ when going up to $k_S(p)$. See [NSW] X, §3 for results in the wild case. In the tame case, we have the following

Theorem 3. *Let $S$ be a finite non-empty set of primes of $k$ whose norms are congruent to 1 modulo $p$. If $X \smallsetminus S$ is a $K(\pi,1)$ for $p$ and $G_S(p) \neq 1$, then*

$$k_S(p)_\mathfrak{p} = k_\mathfrak{p}(p)$$

*for all primes $\mathfrak{p} \in S$, i.e. $k_S(p)$ realizes the maximal $p$-extension of the local field $k_\mathfrak{p}$.*

Remark: Under the assumptions of the theorem, let $\mathfrak{q} \notin S$. Then either $\mathfrak{q}$ splits completely in $k_S(p)$, or $k_S(p)$ realizes the maximal unramified $p$-extension $k_\mathfrak{q}^{nr}(p)$. We do not know whether the completely split case actually occurs.

The next result addresses the question of enlarging the set $S$ without destroying the $K(\pi,1)$-property.

Theorem 4. *Let $S'$ be a finite non-empty set of primes of $k$ whose norms are congruent to 1 modulo $p$ and let $S \subset S'$ be a nonempty subset. Assume that $X \smallsetminus S$ is a $K(\pi,1)$ for $p$ and that $G_S(p) \neq 1$. If each $\mathfrak{q} \in S' \smallsetminus S$ does not split completely in $k_S(p)$, then $X \smallsetminus S'$ is a $K(\pi,1)$ for $p$. Furthermore, in this* *case, the arithmetic form of Riemann's existence theorem holds: the natural homomorphism*

$$\mathop{\ast}_{\mathfrak{p}\in S'\setminus S(k_S(p))} T_{\mathfrak{p}}(k_{S'}(p)|k_S(p)) \longrightarrow Gal(k_{S'}(p)|k_S(p))$$

*is an isomorphism, i.e. $Gal(k_{S'}(p)|k_S(p))$ is the free pro-$p$ product of a bundle of inertia groups.*

Finally, we deduce a statement on universal norms of unit groups.

Theorem 5. *Let $S$ be a finite non-empty set of primes of $k$ whose norms are congruent to 1 modulo $p$. Assume that $X \smallsetminus S$ is a $K(\pi,1)$ for $p$ and that $G_S(p) \neq 1$. Then*

$$\varprojlim_{K\subset k_S(p)} \mathcal{O}_K^\times \otimes \mathbb{Z}_p \;=\; 0 \;=\; \varprojlim_{K\subset k_S(p)} \mathcal{O}_{K,S}^\times \otimes \mathbb{Z}_p,$$

*where $K$ runs through all finite subextensions of $k$ in $k_S(p)$, $\mathcal{O}_K^\times$ and $\mathcal{O}_{K,S}^\times$ are the groups of units and $S$-units of $K$, respectively, and the transition maps are the norm maps.*

The structure of this paper is as follows. First we give the necessary definitions and make some calculations of étale cohomology groups for which we couldn't find an appropriate reference. In section 4, we deal with the first obstruction against the $K(\pi,1)$-property, the $h^2$-defect. Then we recall Labute's results on mild pro-$p$-groups, which we use in the proof of Theorem 1 given in section 6. In the last three sections we prove Theorems 2–5.

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## 2 First observations

We tacitly assume all schemes to be connected and omit base points from the notation. Let $Y$ be a locally noetherian scheme and let $p$ be a prime number. We denote by $Y_{et}^{(p)}$ the $p$-completion of the étale homotopy type of $Y$, see [AM], [Fr]. By $\widetilde{Y}(p)$ we denote the universal pro-$p$-covering of $Y$. The projection $\widetilde{Y}(p) \to Y$ is Galois with group $\pi_1^{et}(Y)(p) = \pi_1(Y_{et}^{(p)})$, cf. [AM], (3.7). Any discrete $p$-torsion $\pi_1^{et}(Y)(p)$-module $M$ defines a locally constant sheaf on $Y_{et}$, which we denote by the same letter. The Hochschild-Serre spectral sequence defines natural homomorphisms

$$\phi_{M,i} : H^i(\pi_1^{et}(Y)(p), M) \longrightarrow H^i_{et}(Y, M),\; i \geq 0.$$

Since $H^1_{et}(\widetilde{Y}(p), M) = 0$, the map $\phi_{M,i}$ is an isomorphism for $i = 0$ and 1, and is injective for $i = 2$. For a pro-$p$-group $G$ we denote by $K(G,1)$ the associated Eilenberg-MacLane space ([AM], (2.6)).

Proposition 2.1. *The following conditions are equivalent:*

(i) *The classifying map* $Y_{et}^{(p)} \longrightarrow K(\pi_1^{et}(Y)(p), 1)$ *is a weak equivalence.*

(ii) $\pi_i(Y_{et}^{(p)}) = 0$ *for all* $i \geq 2$.

(iii) $H^i_{et}(\widetilde{Y}(p), \mathbb{Z}/p\mathbb{Z}) = 0$ *for all* $i \geq 1$.

(iv) $\phi_{\mathbb{Z}/p\mathbb{Z},i} : H^i(\pi_1^{et}(Y)(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^i_{et}(Y, \mathbb{Z}/p\mathbb{Z})$ *is an isomorphism for all* $i \geq 0$.

(v) $\phi_{M,i} : H^i(\pi_1^{et}(Y)(p), M) \longrightarrow H^i_{et}(Y, M)$ *is an isomorphism for all* $i \geq 0$ *and any discrete* $p$*-torsion* $\pi_1^{et}(Y)(p)$*-module* $M$.

*Proof.* The equivalences (i)⇔(ii)⇔(v) are the content of [AM], (4.3), (4.4). The equivalence (iii)⇔(iv) follows in a straightforward manner from the Hochschild-Serre spectral sequence. The implication (v)⇒(iv) is trivial.
Assume that (iv) holds. As $\pi_1^{et}(Y)(p)$ is a pro-$p$-group, any finite $p$-primary $\pi_1^{et}(Y)(p)$-module $M$ has a composition series with graded pieces isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with trivial $\pi_1^{et}(Y)(p)$-action ([NSW], Corollary 1.7.4). Hence, if $M$ is finite, the five-lemma implies that $\phi_{M,i}$ is an isomorphism for all $i$. An arbitrary discrete $p$-primary $\pi_1^{et}(Y)(p)$-module is the filtered inductive limit of finite $p$-primary $\pi_1^{et}(Y)(p)$-modules. Since group cohomology ([NSW], Proposition 1.5.1) and étale cohomology ([AGV], VII, 3.3) commute with filtered inductive limits, $\phi_{M,i}$ is an isomorphism for all $i$ and all discrete $p$-torsion $\pi_1^{et}(Y)(p)$-modules $M$. This implies (v) and completes the proof. □

Definition. *We say that* $Y$ *is a '*$K(\pi,1)$ *for* $p$*' if the equivalent conditions of Proposition 2.1 are satisfied.*

Now let $k$ be a number field, $S$ a finite set of nonarchimedean primes of $k$ and $p$ a prime number. We put $X = Spec(\mathcal{O}_k)$. The following observation is straightforward.

Corollary 2.2. *Let* $k'$ *be a finite subextension of* $k$ *in* $k_S(p)$ *and let* $X' = Spec(\mathcal{O}_{k'})$, $S' = S(k')$. *Then the following are equivalent.*

(i) $X \smallsetminus S$ *is a* $K(\pi, 1)$ *for* $p$,

(ii) $X' \smallsetminus S'$ *is a* $K(\pi, 1)$ *for* $p$.

*Proof.* Both schemes have the same universal pro-$p$-covering. □

We denote by $S_p$ and $S_\infty$ the set of primes of $k$ dividing $p$ and the set of archimedean primes of $k$, respectively. For a set $S$ of primes (which may contain archimedean places), let $k_S(p)$ be the maximal $p$-extension of $k$ unramified outside $S$ and $G_S(p) = Gal(k_S(p)|k)$. For a finite set $S$ of nonarchimedean primes of $k$ we have the identification

$$\pi_1^{et}((X \smallsetminus S)_{et}^{(p)}) = G_{S\cup S_\infty}(p).$$

If $p$ is odd or $k$ is totally imaginary, then $G_S(p) = G_{S\cup S_\infty}(p)$. The following proposition is given for sake of completeness. It deals with the 'wild' case $S \supset S_p$, and is well known.

Proposition 2.3. *If $S$ contains $S_p$, then $X \smallsetminus S$ is a $K(\pi,1)$ for $p$.*

*Proof.* We verify condition (v) of Proposition 2.1. Let $k_{S\cup S_\infty}$ be the maximal extension of $k$ unramified outside $S\cup S_\infty$ and put $G_{S\cup S_\infty} = Gal(k_{S\cup S_\infty}|k)$. For any $p$-primary discrete $G_{S\cup S_\infty}(p)$-module $M$ the homomorphism $\phi_{M,i}$ factors as

$$H^i(G_{S\cup S_\infty}(p), M) \to H^i(G_{S\cup S_\infty}, M) \to H^i_{et}(X \smallsetminus S, M).$$

By [NSW], Cor. 10.4.8, the left map is an isomorphism. That also the right map is an isomorphism follows in a straightforward manner by using the Kummer sequence, the Principal Ideal Theorem and known properties of the Brauer group, see for example [Zi], Prop. 3.3.1. or [Mi], II Prop. 2.9. □

Remark: If $p = 2$ and $k$ has real places it is useful to work with the modified étale site defined by T. Zink [Zi], which takes the real archimedean places of $k$ into account. Proposition 2.3 holds true also for the modified étale site, see [S2], Thm. 6.

## 3 Calculation of étale cohomology groups

As a basis of our investigations, we need the calculation of the étale cohomology groups of open subschemes of $Spec(\mathcal{O}_k)$ with values in the constant sheaf $\mathbb{Z}/p\mathbb{Z}$. Let $p$ be a fixed prime number. All cohomology groups are taken with respect to the constant sheaf $\mathbb{Z}/p\mathbb{Z}$, which we omit from the notation. Furthermore, we use the notation

$$h^i(-) = \dim_{\mathbb{F}_p} H^i_{et}(-) \quad (= \dim_{\mathbb{F}_p} H^i_{et}(-, \mathbb{Z}/p\mathbb{Z})\ )$$

for the occurring cohomology groups. We start with some well-known local computations.

Proposition 3.1. *Let $k$ be a nonarchimedean local field of characteristic zero and residue characteristic $\ell$. Let $X = Spec(\mathcal{O}_k)$ and let $x$ be the closed point of $X$. Then the étale local cohomology groups $H^i_x(X)$ vanish for $i \leq 1$ and $i \geq 4$, and*

$$h^2_x(X) = \begin{cases} \delta, & \text{if } \ell \neq p, \\ \delta + [k:\mathbb{Q}_p], & \text{if } \ell = p, \end{cases}$$

*where* $\delta = 1$ *if* $\mu_p \subset k$ *and zero otherwise. Furthermore,* $h^3_x(X) = \delta$. *In particular, we have the Euler-Poincaré characteristic formula*

$$\sum_{i=0}^{3}(-1)^i h^i_x(X) = \begin{cases} 0, & \text{if } \ell \neq p, \\ [k:\mathbb{Q}_p], & \text{if } \ell = p. \end{cases}$$

*Proof.* As $X$ is henselian, we have isomorphisms $H^i_{et}(X) \cong H^i_{et}(x)$ for all $i$, and therefore

$$h^i(X) = \begin{cases} 1 & \text{for } i = 0, 1, \\ 0 & \text{for } i \geq 2. \end{cases}$$

Furthermore, $X \smallsetminus \{x\} = \mathit{Spec}(k)$, hence $H^i_{et}(X \smallsetminus \{x\}) \cong H^i(k)$. The local duality theorem (cf. [NSW], Theorem 7.2.15) shows $h^2(X \smallsetminus \{x\}) = \delta$, and by [NSW], Corollary 7.3.9, we have

$$h^1(X \smallsetminus \{x\}) = \begin{cases} 1+\delta & \text{if } \ell \neq p, \\ 1+\delta+[k:\mathbb{Q}_p] & \text{if } \ell = p. \end{cases}$$

Furthermore, the natural homomorphism $H^1_{et}(X) \to H^1_{et}(X \smallsetminus \{x\})$ is injective. Therefore the result of the proposition follows from the exact excision sequence

$$\cdots \to H^i_x(X) \to H^i_{et}(X) \to H^i_{et}(X \smallsetminus \{x\}) \to H^{i+1}_x(X) \to \cdots.$$

□

Now let $k$ be a number field, $S$ a finite set of nonarchimedean primes of $k$ and $X = \mathit{Spec}(\mathcal{O}_k)$. We assume for simplicity that $p$ is odd or that $k$ is totally imaginary, so that we can ignore the archimedean places of $k$ for cohomological considerations. We introduce the following notation

| | |
|---|---|
| $r_1$ | the number of real places of $k$ |
| $r_2$ | the number of complex places of $k$ |
| $r$ | $= r_1 + r_2$, the number of archimedean places of $k$ |
| $S_p$ | the set of places of $k$ dividing $p$ |
| $\delta$ | equal to 1 if $\mu_p \subset k$ and zero otherwise |
| $\delta_\mathfrak{p}$ | equal to 1 if $\mu_p \subset k_\mathfrak{p}$ and zero otherwise |
| $\mathit{Cl}(k)$ | the ideal class group of $k$ |
| $\mathit{Cl}_S(k)$ | the $S$-ideal class group of $k$ |
| $h_k$ | $= \#\mathit{Cl}(k)$, the class number of $k$ |
| ${}_nA$ | $= \ker(A \xrightarrow{\cdot n} A)$, where $A$ is an abelian group and $n \in \mathbb{N}$ |
| $A/n$ | $= \mathrm{coker}(A \xrightarrow{\cdot n} A)$, where $A$ is an abelian group and $n \in \mathbb{N}$. |

Proposition 3.2. *Assume that* $p \neq 2$ *or that* $k$ *is totally imaginary. Then* $H^i_{et}(X \smallsetminus S) = 0$ *for* $i \geq 4$, *and*

$$\chi(X \smallsetminus S) := \sum_{i=0}^{3}(-1)^i h^i(X \smallsetminus S) = r - \sum_{\mathfrak{p} \in S \cap S_p} [k_\mathfrak{p} : \mathbb{Q}_p]\,.$$

*In particular,*

$$\chi(X \smallsetminus S) = \begin{cases} r, & \text{if } S \cap S_p = \varnothing\,, \\ -r_2, & \text{if } S \supset S_p\,. \end{cases}$$

*Proof.* The assertion for $S = S_p$ is well known, see [Mi], II Theorem 2.13 (a). Consider the exact excision sequence

$$\cdots \to \bigoplus_{\mathfrak{p}\in S} H^i_{\mathfrak{p}}(X_{\mathfrak{p}}) \to H^i_{et}(X) \to H^i_{et}(X \smallsetminus S) \to \bigoplus_{\mathfrak{p}\in S} H^{i+1}_{\mathfrak{p}}(X_{\mathfrak{p}}) \to \cdots\ ,$$

where $X_{\mathfrak{p}} = Spec(\mathcal{O}_{k,\mathfrak{p}})$ is the spectrum of the completion of $\mathcal{O}_k$ at $\mathfrak{p}$. Using this excision sequence for $S = S_p$, Proposition 3.1 implies the result for $S = \varnothing$, noting that $\sum_{\mathfrak{p}\in S_p}[k_{\mathfrak{p}} : \mathbb{Q}_p] - r_2 = [k : \mathbb{Q}] - r_2 = r$. The result for arbitrary $S$ follows from the case $S = \varnothing$, the above excision sequence and from Proposition 3.1. □

In order to give formulae for the individual cohomology groups, we consider the Kummer group (cf. [NSW], VIII, §6)

$$V_S(k) := \{a \in k^\times \mid a \in k_{\mathfrak{p}}^{\times p} \text{ for } \mathfrak{p} \in S \text{ and } a \in U_{\mathfrak{p}} k_{\mathfrak{p}}^{\times p} \text{ for } \mathfrak{p} \notin S\}/k^{\times p},$$

where $U_{\mathfrak{p}}$ denotes the unit group of the local field $k_{\mathfrak{p}}$ (convention: $U_{\mathfrak{p}} = k_{\mathfrak{p}}^\times$ if $\mathfrak{p}$ is archimedean).[1] $V_S(k)$ is a finite group. More precisely, we have the following

Proposition 3.3. *There exists a natural exact sequence*

$$0 \longrightarrow \mathcal{O}_k^\times/p \longrightarrow V_\varnothing(k) \longrightarrow {}_p Cl(k) \longrightarrow 0\,.$$

*In particular,*

$$\dim_{\mathbb{F}_p} V_\varnothing(k) = \dim_{\mathbb{F}_p} {}_p Cl(k) + \dim_{\mathbb{F}_p} \mathcal{O}_k^\times/p = \dim_{\mathbb{F}_p} {}_p Cl(k) + r - 1 + \delta.$$

*If $S$ is arbitrary and $\mathfrak{p} \notin S$ is an additional prime of $k$, then we have a natural exact sequence*

$$0 \longrightarrow V_{S\cup\{\mathfrak{p}\}}(k) \xrightarrow{\phi} V_S(k) \longrightarrow U_{\mathfrak{p}} k_{\mathfrak{p}}^{\times p}/k_{\mathfrak{p}}^{\times p}.$$

*For $\mathfrak{p} \notin S_p$, we have* $\dim_{\mathbb{F}_p} \operatorname{coker}(\phi) \le \delta_{\mathfrak{p}}$.

*Proof.* Sending an $a \in V_\varnothing(k)$ to the class in $Cl(k)$ of the fractional ideal $\mathfrak{a}$ with $(a) = \mathfrak{a}^p$ yields a surjective homomorphism $V_\varnothing(k) \to {}_p Cl(k)$ with kernel $\mathcal{O}_k^\times/p$. This, together with Dirichlet's Unit Theorem, shows the first statement. The second exact sequence follows immediately from the definitions. There are natural isomorphisms

$$U_{\mathfrak{p}} k_{\mathfrak{p}}^{\times p}/k_{\mathfrak{p}}^{\times p} \cong U_{\mathfrak{p}}/U_{\mathfrak{p}} \cap k_{\mathfrak{p}}^{\times p} = U_{\mathfrak{p}}/U_{\mathfrak{p}}^p.$$

For $\mathfrak{p} \notin S_p$ we have $\dim_{\mathbb{F}_p} U_{\mathfrak{p}}/U_{\mathfrak{p}}^p = \delta_{\mathfrak{p}}$, showing the last statement. □

[1] In terms of flat cohomology, we have $V_S(k) = \ker\left(H^1_{fl}(X \smallsetminus S, \mu_p) \to \bigoplus_{\mathfrak{p}\in S} H^1(k_{\mathfrak{p}}, \mu_p)\right)$.

The étale cohomology groups of $X \smallsetminus S$ have the following dimensions.

Theorem 3.4. *Let $S$ be a finite set of nonarchimedean primes of $k$. Assume $p \neq 2$ or that $k$ is totally imaginary. Then $H^i_{et}(X \smallsetminus S) = 0$ for $i \geq 4$ and*

$$\begin{aligned} h^0(X \smallsetminus S) &= 1\,, \\ h^1(X \smallsetminus S) &= 1 + \sum_{\mathfrak{p}\in S} \delta_{\mathfrak{p}} - \delta + \dim_{\mathbb{F}_p} V_S + \sum_{\mathfrak{p}\in S\cap S_p} [k_{\mathfrak{p}} : \mathbb{Q}_p] - r, \\ h^2(X \smallsetminus S) &= \sum_{\mathfrak{p}\in S} \delta_{\mathfrak{p}} - \delta + \dim_{\mathbb{F}_p} V_S + \theta\,, \\ h^3(X \smallsetminus S) &= \theta\,. \end{aligned}$$

*Here $\theta$ is equal to $1$ if $\delta = 1$ and $S = \varnothing$, and zero in all other cases.*

*Proof.* The statement on $h^0$ is trivial and the vanishing of $H^i$ for $i \geq 4$ was already part of Proposition 3.2. Artin-Verdier duality (see [Ma], 2.4 or [Mi], Theorem 3.1) shows

$$H^3_{et}(X)^\vee \cong \operatorname{Hom}_X(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m) = \mu_p(k).$$

Consider the exact excision sequence

$$\bigoplus_{\mathfrak{p}\in S} H^3_{\mathfrak{p}}(X_{\mathfrak{p}}) \xrightarrow{\alpha} H^3_{et}(X) \xrightarrow{\beta} H^3_{et}(X \smallsetminus S) \to \bigoplus_{\mathfrak{p}\in S} H^4_{\mathfrak{p}}(X_{\mathfrak{p}}).$$

By Proposition 3.1, the right hand group is zero, hence $\beta$ is surjective. By the local duality theorem (see [Ma], 2.4, [Mi], II Corollary 1.10), the dual map to $\alpha$ is the natural inclusion

$$\mu_p(k) \to \bigoplus_{\mathfrak{p}\in S} \mu_p(k_{\mathfrak{p}}),$$

which is injective, unless $\delta = 1$ and $S = \varnothing$. Therefore $h^3(X \smallsetminus S) = 1$ if $\delta = 1$ and $S = \varnothing$, and zero otherwise. Using the isomorphism $H^1(G_S(p)) \xrightarrow{\sim} H^1_{et}(X \smallsetminus S)$, the statement on $h^1$ follows from the corresponding formula for the first cohomology of $G_S(p)$, see [NSW], Theorem 8.7.11. Finally, the result for $h^2$ follows by using the Euler-Poincaré characteristic formula in Proposition 3.2. □

Corollary 3.5. *Assume that $\delta = 0$ or $S \neq \varnothing$. Then $X \smallsetminus S$ is a $K(\pi,1)$ for $p$ if and only if the following conditions* (i) *and* (ii) *are satisfied.*

(i) *$\phi_2\colon\ H^2(G_S(p)) \hookrightarrow H^2_{et}(X \smallsetminus S)$ is an isomorphism,*

(ii) *$cd\, G_S(p) \leq 2$.*

*Proof.* The given conditions are obviously necessary. Furthermore, $\phi_0$ and $\phi_1$ are isomorphisms and $H^i_{et}(X \smallsetminus S) = 0$ for $i \geq 3$ by Theorem 3.4. Therefore (i) and (ii) imply that $\phi_i$ is an isomorphism for all $i$. Hence condition (iv) of Proposition 2.1 is satisfied for $X \smallsetminus S$ and $p$. □

Let $F$ be a locally constant sheaf on $(X \smallsetminus S)_{et}$. For each prime $\mathfrak{p}$ the composite map $\mathcal{O}_{k,S} \to k \to k_\mathfrak{p}$ induces natural maps $H^i_{et}(X \smallsetminus S, F) \to H^i(k_\mathfrak{p}, F)$ for all $i \geq 0$.

Definition. *For any locally constant sheaf $F$ on $(X \smallsetminus S)_{et}$ we put*

$$Ш^i(k,S,F) := \ker\Big(H^i_{et}(X \smallsetminus S, F) \longrightarrow \prod_{\mathfrak{p}\in S} H^i(k_\mathfrak{p}, F)\Big).$$

*Assume a prime number $p$ is fixed. Then we write $Ш^i(k,S) := Ш^i(k,S,\mathbb{Z}/p\mathbb{Z})$ and, following historical notation, we put $Б_S(k) := V_S(k)^\vee$, where $\vee$ denotes the Pontryagin dual.*

The next theorem is sharper than [NSW], Thm. 8.7.4, as the group $Ш^2(G_S)$, which was considered there, is a subgroup of $Ш^2(k,S)$. If $p = 2$ and $k$ has real places, then Theorem 3.6 remains true after replacing étale cohomology with its modified version.

Theorem 3.6. *Assume $p \neq 2$ or that $k$ is totally imaginary. Then there exists a natural isomorphism*

$$Ш^2(k,S) \xrightarrow{\sim} Б_S(k).$$

*Proof.* The proof of [NSW], Thm. 8.7.4 can be adapted to show also the stronger statement here. However, we decided to take the short cut by using flat duality. For any prime $\mathfrak{p}$ of $k$ one easily computes the local cohomology groups for the flat topology as $H^1_{fl,\mathfrak{p}}(X,\mu_p) = 0$ and $H^2_{fl,\mathfrak{p}}(X,\mu_p) \cong k_\mathfrak{p}^\times / U_\mathfrak{p} k_\mathfrak{p}^{\times p}$. Therefore excision and Kummer theory imply an exact sequence

$$0 \to H^1_{fl}(X,\mu_p) \to k^\times/k^{\times p} \to \bigoplus_{\mathfrak{p}} k_\mathfrak{p}^\times / U_\mathfrak{p} k_\mathfrak{p}^{\times p}.$$

As $H^1_{fl}(X^h_\mathfrak{p},\mu_p) \cong U_\mathfrak{p}/p$, we obtain the exact sequence

$$(*) \qquad 0 \to V_S(k) \to H^1_{fl}(X,\mu_p) \to \bigoplus_{\mathfrak{p}\in S} H^1_{fl}(X^h_\mathfrak{p},\mu_p).$$

By excision, and noting $H^3_\mathfrak{p}(X,\mathbb{Z}/p\mathbb{Z}) \cong H^2(k_\mathfrak{p},\mathbb{Z}/p\mathbb{Z})$, we have an exact sequence

$$(**) \qquad \bigoplus_{\mathfrak{p}\in S} H^2_\mathfrak{p}(X,\mathbb{Z}/p\mathbb{Z}) \to H^2_{et}(X,\mathbb{Z}/p\mathbb{Z}) \to Ш^2(k,S) \to 0.$$

Comparing sequences $(*)$ and $(**)$ via local and global flat duality, we obtain the asserted isomorphism. □

We provide the following lemma for further use.

Lemma 3.7. *Let $K \subset k_S(p)$ be an extension of $k$ inside $k_S(p)$ and let $(X \smallsetminus S)_K$ be the normalization of $X \smallsetminus S$ in $K$. If $\delta = 0$, or $S \neq \varnothing$ or $K|k$ is infinite, then*

$$H^3_{et}((X \smallsetminus S)_K) = 0.$$

*Proof.* We denote the normalization of $X \smallsetminus S$ in any algebraic extension field $k'$ of $k$ by $(X \smallsetminus S)_{k'}$. Étale cohomology commutes with inverse limits of schemes if the transition maps are affine (see [AGV], VII, 5.8). Therefore

$$H^3((X \smallsetminus S)_K) = \varinjlim_{k' \subset K} H^3((X \smallsetminus S)_{k'}),$$

where $k'$ runs through all finite subextensions of $k$ in $K$. If $\delta = 0$ or $S \neq \varnothing$, then, by Theorem 3.4, $H^3_{et}((X \smallsetminus S)_{k'}) = 0$ for all $k'$ and the limit is obviously zero. Assume $\delta = 1$ and $S = \varnothing$. Then, by Artin-Verdier duality,

$$H^3_{et}(X_{k'}) \cong \mu_p(k')^\vee.$$

For $k' \subset k'' \subset K$, the transition map

$$H^3_{et}(X_{k'}) \to H^3_{et}(X_{k''})$$

is the dual of the norm map $N_{k''|k'}\colon\ \mu_p(k'') \to \mu_p(k')$, hence the zero map if $k' \neq k''$. As $K|k$ is infinite, the limit vanishes. □

## 4 Removing the $h^2$-defect

We start by extending the notions introduced before to infinite sets of primes $S$. Let $k$ be a number field and $S$ a set of nonarchimedean primes of $k$. We set $X = \mathit{Spec}(\mathcal{O}_k)$ and

$$X \smallsetminus S = \mathit{Spec}(\mathcal{O}_{k,S}),$$

which makes sense also if $S$ is infinite. Let $F$ be a sheaf on $X \smallsetminus S$ which comes by restriction from $X \smallsetminus T$ for some finite subset $T \subset S$. As each open subscheme of $X$ is affine, we have

$$H^i_{et}(X \smallsetminus S, F) \cong \varinjlim_{\substack{T \subset S' \subset S \\ S' \text{ finite}}} H^i_{et}(X \smallsetminus S', F)$$

for all $i \geq 0$.

We fix a prime number $p$ and put the running assumption that $k$ is totally imaginary if $p = 2$. Hence we may ignore archimedean primes for cohomological considerations. The notion of being a $K(\pi,1)$ for $p$ extends in an obvious manner to the case when $S$ is infinite. Also the isomorphism

$$Ш^2(k,S) \xrightarrow{\sim} Б_S(k)$$

generalizes to infinite $S$ by passing to the limit over all finite subsets $S' \subset S$. In particular, $Ш^2(k,S)$ is finite.

For the remainder of this paper, we assume that $S \cap S_p = \varnothing$. We also keep the running assumption $p \neq 2$ or $k$ is totally imaginary.

> *For shorter notation, we drop $p$ wherever possible. We write $G_S$ instead of $G_S(p)$, $k_S$ for $k_S(p)$, and so on. Unless mentioned otherwise, all cohomology groups are taken with values in $\mathbb{Z}/p\mathbb{Z}$. We keep this notational convention for the rest of this paper.*

If $\mathfrak{p} \nmid p$ is a prime with $\mu_p \not\subset k_\mathfrak{p}$, then every $p$-extension of the local field $k_\mathfrak{p}$ is unramified (see [NSW], Proposition 7.5.1). Therefore primes $\mathfrak{p} \notin S_p$ with $N(\mathfrak{p}) \not\equiv 1 \bmod p$ cannot ramify in a $p$-extension. Removing all these redundant primes from $S$, we obtain a subset $S_{\min} \subset S$ which has the property that $G_S = G_{S_{\min}}$. Moreover, we have the

LEMMA 4.1. *The natural map*

$$(X \smallsetminus S)^{(p)}_{et} \longrightarrow (X \smallsetminus S_{\min})^{(p)}_{et}$$

*is a weak homotopy equivalence.*

*Proof.* By [AM], (4.3), it suffices to show that for every discrete $p$-primary $G_S$-module $M$ the natural maps $H^i_{et}(X \smallsetminus S_{\min}, M) \to H^i_{et}(X \smallsetminus S, M)$ are isomorphisms for all $i$. By the same argument, as in the proof of Proposition 2.1, (iv)⇒(v), we may suppose that $M = \mathbb{Z}/p\mathbb{Z}$. Using the excision sequence, it therefore suffices to show that the group $H^i_\mathfrak{p}(X \smallsetminus S_{\min}, \mathbb{Z}/p\mathbb{Z})$ vanishes for all $\mathfrak{p} \in S \smallsetminus S_{\min}$. This follows from Proposition 3.1. □

Therefore we can replace $S$ by $S_{\min}$ and make the following notational convention for the rest of this paper.

> *The word 'prime' means nonarchimedean prime with norm $\equiv 1 \bmod p$.*

At this point it is useful to redefine the notion of Dirichlet density.

DEFINITION. *Let $S$ be a set of primes of $k$ (of norm $\equiv 1 \bmod p$). The $p$-density $\Delta^p(S)$ is defined by*

$$\Delta^p(S) = \delta_{k(\mu_p)}\big(S(k(\mu_p))\big),$$

*where $S(k(\mu_p))$ is the set of prolongations of primes in $S$ to $k(\mu_p)$ and $\delta_{k(\mu_p)}$ denotes the Dirichlet density on the level $k(\mu_p)$. In other words,*

$$\Delta^p(S) = d \cdot \delta_k(S), \text{ where } d = [k(\mu_p) : k].$$

The set of all primes (of norm $\equiv 1 \bmod p$) has $p$-density equal to 1.

Proposition 4.2. *Let $S$ be a set of primes of $p$-density $\Delta^p(S) = 1$. Then there exists a finite subset $T \subset S$ with* $Б_T(k) = 0$. *In particular,* $Б_S(k) = 0 = Ш^2(k, S)$.

*Proof.* By the Hasse principle for the module $\mu_p$, see [NSW], Thm. 9.1.3 (ii), and Kummer theory, the natural map

$$k^\times / k^{\times p} \longrightarrow \prod_{\mathfrak{p} \in S} k_\mathfrak{p}^\times / k_\mathfrak{p}^{\times p}$$

is injective, hence $V_S(k) = 0$. Furthermore $V_\varnothing(k)$ is finite. Choosing to each nonzero element $\alpha$ of $V_\varnothing(k)$ a prime $\mathfrak{p} \in S$ with $\alpha \notin k_\mathfrak{p}^{\times p}$, we obtain a finite subset $T \subset S$ with $V_T(k) = 0$. □

Theorem 4.3. *Let $k$ be a number field and let $S$ be a set of primes of $k$ of $p$-density $\Delta^p(S) = 1$. Then $X \smallsetminus S$ is a $K(\pi, 1)$ for $p$.*

*Proof.* Let $T \subset S$ be a finite subset. By [NSW], Thm. 9.2.2 (ii), the natural map

$$H^1_{et}(X \smallsetminus (S \cup S_p)) \longrightarrow \prod_{\mathfrak{p} \in T \cup S_p} H^1(k_\mathfrak{p})$$

is surjective. A class in $H^1_{et}(X \smallsetminus (S \cup S_p))$ which maps to zero in $H^1(k_\mathfrak{p})$ for all $\mathfrak{p} \mid p$ is contained in $H^1_{et}(X \smallsetminus S)$. Therefore, also the map

$$H^1_{et}(X \smallsetminus S) \longrightarrow \prod_{\mathfrak{p} \in T} H^1(k_\mathfrak{p})$$

is surjective. Hence the maximal elementary abelian extension of $k$ in $k_S$ realizes the maximal elementary abelian extension of $k_\mathfrak{p}$ in $k_\mathfrak{p}(p)$ for all $\mathfrak{p} \in S$. Applying the same argument to each finite subextension of $k$ in $k_S$, we conclude that $k_S$ realizes $k_\mathfrak{p}(p)$ for all $\mathfrak{p} \in S$. In particular,

$$\prod_{\mathfrak{p} \in S(k_S)} H^2((k_S)_\mathfrak{p}) = 0.$$

Furthermore, by Proposition 4.2, $Ш^2(K, S(K)) = 0$ for all finite subextensions $K$ of $k$ in $k_S$. We obtain

$$H^2_{et}((X \smallsetminus S)_{k_S}) = 0.$$

As there is no cohomology in dimension greater or equal 3, condition (iii) of Proposition 2.1 is satisfied. □

In order to proceed, we make the following definitions.

Definition. *Let $S$ be a finite set of primes (of norm $\equiv 1 \bmod p$).*

(i) *We say that $S$ is $p$-large if* $Б_S(k, p) = 0$.

(ii) *We put*

$$\delta^2_S(k) = h^2(X \smallsetminus S) - h^2(G_S)$$

*and call this number the $h^2$-defect of $S$ (with respect to $p$).*

(iii) *We denote by $k_S^{el}$ the maximal elementary abelian $p$-extension of $k$ inside $k_S$.*

If $S$ is $p$-large, then $Ш^2(k, S) = 0$, and so, for any set $T \supset S$, the natural maps $H^2(G_S) \to H^2(G_T)$ and $H^2_{et}(X \smallsetminus S) \to H^2_{et}(X \smallsetminus T)$ are injective.

Lemma 4.4. *Let $S$ be $p$-large and let $\mathfrak{p}$ be a prime (of norm $\equiv 1 \bmod p$) which does not split completely in $k_S^{el}|k$. Then*

$$\delta^2_{S\cup\{\mathfrak{p}\}}(k) \le \delta^2_S(k).$$

*Furthermore, the natural map $H^2(G_{S\cup\{\mathfrak{p}\}}) \longrightarrow H^2(k_\mathfrak{p})$ is surjective.*

*Proof.* Put $S' = S \cup \{\mathfrak{p}\}$. By Theorem 3.4, the extension $k_{S'}^{el}|k$ is ramified at $\mathfrak{p}$. Therefore $k_{S'}^{el}$ realizes the maximal elementary abelian $p$-extension $k_\mathfrak{p}^{el}$ of the local field $k_\mathfrak{p}$, i.e. the map

$$H^1(G_{S'}(p)) \longrightarrow H^1(k_\mathfrak{p})$$

is surjective. As the cup-product $H^1(k_\mathfrak{p}) \times H^1(k_\mathfrak{p}) \to H^2(k_\mathfrak{p})$ is surjective, the natural map

$$H^2(G_{S'}) \longrightarrow H^2(k_\mathfrak{p})$$

is also surjective. The statement of the lemma now follows from the commutative and exact diagram

$$\begin{tikzcd}
0 \arrow[r] & H^2(G_S) \arrow[r] \arrow[d, hook] & H^2_{et}(X \smallsetminus S) \arrow[d, hook] \\
0 \arrow[r] & H^2(G_{S'}) \arrow[r] & H^2_{et}(X \smallsetminus S') \arrow[d] \\
 & & H^2(k_\mathfrak{p}).
\end{tikzcd}$$

□

Lemma 4.5. *Let $S$ be $p$-large and let $\mathfrak{p}$ be a prime. Let $T$ be a set of primes of $p$-density $\Delta^p(T) = 1$. Then there exists a prime $\mathfrak{p}' \in T$ such that*

(i) *$\mathfrak{p}'$ does not split completely in $k_S^{el}|k$.*

(ii) *$\mathfrak{p}$ does not split completely in $k_{S\cup\{\mathfrak{p}'\}}^{el}|k$.*

*In particular, $\delta^2_{S\cup\{\mathfrak{p},\mathfrak{p}'\}}(k) \le \delta^2_S(k)$.*

*Proof.* If $\mathfrak{p}$ does not split completely in $k_S^{el}|k$, then condition (ii) is void. By assumption, $\Delta^p(T) = \delta_{k(\mu_p)}(T(k(\mu_p)) = 1$. By Čebotarev's density theorem, we can find a prime $\mathfrak{P}' \in T(k(\mu_p))$ which does not split completely in $k_S^{el}(\mu_p)|k(\mu_p)$. Then $\mathfrak{p}' = \mathfrak{P}'|_k$ satisfies (i). Therefore we may assume that $\mathfrak{p}$ splits completely in $k_S^{el}|k$. By class field theory, there exists an $s \in k^\times$ with

(a) $v_\mathfrak{p}(s) \equiv 1 \bmod p$,
(b) $v_\mathfrak{q}(s) \equiv 0 \bmod p$ for all $\mathfrak{q} \notin S$, $\mathfrak{q} \neq \mathfrak{p}$, and
(c) $s \in k_\mathfrak{q}^{\times p}$ for all $\mathfrak{q} \in S$.

Since $S$ is $p$-large, $s$ is well-defined as an element in $k^\times/k^{\times p}$. Now consider the extensions $k(\mu_p, \sqrt[p]{s})$ and $k^{el}_{S\cup\{\mathfrak{p}\}}(\mu_p)$ of $k(\mu_p)$. The first one might be contained in the second (only if $\zeta_p \in k$) but this does not matter. Using Čebotarev's density theorem, we find $\mathfrak{P}' \in T(k(\mu_p))$ such that $Frob_{\mathfrak{P}'}$ is non-zero in $Gal(k(\mu_p, \sqrt[p]{s})|k(\mu_p))$ and non-zero in $Gal(k^{el}_{S\cup\{\mathfrak{p}\}}(\mu_p)|k(\mu_p))$. We put $\mathfrak{p}' = \mathfrak{P}'|_k$. Then $\mathfrak{p}'$ does not split completely in $k_S^{el}|k$ and $s \notin k_{\mathfrak{p}'}^{\times p} = k(\mu_p)_{\mathfrak{P}'}^{\times p}$. We claim that $\mathfrak{p}$ does not split completely in $k^{el}_{S\cup\{\mathfrak{p}'\}}|k$: Otherwise there would exist a $t \in k^\times$ satisfying conditions (a) – (c) above and with $t \in k_{\mathfrak{p}'}^{\times p}$. Since $s/t \in Б_S(k) = 0$, we obtain $s/t \in k^{\times p}$. Hence $s \in k_{\mathfrak{p}'}^{\times p}$ giving a contradiction. Hence condition (i) and (ii) are satisfied. $\square$

Lemma 4.6. *Let $S$ be a finite set of primes and let $T$ be a set of primes of $p$-density $\Delta^p(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $S \cup T_1$ is $p$-large and such that the natural inclusion*

$$H^2(G_{S\cup T_1}(k)) \hookrightarrow H^2_{et}(X \smallsetminus (S \cup T_1))$$

*is an isomorphism.*

*Proof.* We first move finitely many primes from $T$ to $S$ to achieve that $S$ is $p$-large. If $\delta^2_S(k)$ is zero, we are ready. Otherwise, consider the commutative diagram

$$\begin{array}{ccc} H^2(G_S) & \hookrightarrow & H^2(G_{S\cup T}) \\ \big\downarrow & & \big\downarrow \wr \\ H^2_{et}(X \smallsetminus S) & \hookrightarrow & H^2_{et}(X \smallsetminus (S \cup T)) \end{array}$$

in which the right hand isomorphism follows from Theorem 4.3. Let $x \in H^2_{et}(X \smallsetminus S)$ but $x \notin H^2(G_S)$. Then there exists a finite subset $T_0 \subset T$ such that $x \in H^2(G_{S\cup T_0})$. Let $T_0 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. We choose $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_n \in T$ according to Lemma 4.5 and put $T_1 = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n, \mathfrak{p}'_1, \ldots, \mathfrak{p}'_n\}$. Then the natural map

$$H^2(G_{S\cup T_1}) \xrightarrow{\phi} \prod_{i=1}^n H^2(k_{\mathfrak{p}_i}) \times \prod_{i=1}^n H^2(k_{\mathfrak{p}'_i})$$

is surjective. We have $H^2(G_S) \subset \ker(\phi)$ and also $x \in \ker(\phi)$. Hence $\delta^2_{S\cup T_1}(k) < \delta^2_S(k)$. Iterating this process, we obtain a set with trivial $h^2$-defect. $\square$

## 5 Review of mild pro-$p$-groups

In the following we recall definitions and results from J. Labute's paper [La]. Only interested in our application, we are slightly less general than Labute.

Let $R$ be a principal ideal domain and let $L$ be the free $R$-Lie algebra over $\xi_1,\ldots,\xi_n$, $n\geq 1$. We view $L$ as graded algebra where the degree of $\xi_i$ is 1. Let $\rho_1,\ldots,\rho_m$, $m\geq 1$, be homogeneous elements in $L$ with $\rho_i$ of degree $h_i$ and let $\mathfrak{r}=(\rho_1,\ldots,\rho_m)$ be the ideal of $L$ generated by $\rho_1,\ldots,\rho_m$. Let $\mathfrak{g}=L/\mathfrak{r}$ and $U_\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. Then $M=\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is a $U_\mathfrak{g}$-module via the adjoint representation.

Definition. *The sequence $\rho_1,\ldots,\rho_m$ is called* strongly free *if $U_\mathfrak{g}$ is a free $R$-module and $M=\mathfrak{r}/[\mathfrak{r},\mathfrak{r}]$ is the free $U_\mathfrak{g}$-module on the images of $\rho_1,\ldots,\rho_m$ in $M$.*

Let us consider the special case when $R=k[\pi]$ is the polynomial ring in one variable $\pi$ over a field $k$. Then $\bar L=L/\pi$ is a free $k$-Lie algebra and the next theorem shows that we can detect strong freeness by reduction. We denote the image in $\bar L$ of an element $\rho\in L$ by $\bar\rho$.

Theorem 5.1. ([La], Th. 3.10) *The sequence $\rho_1,\ldots,\rho_m$ in $L$ is strongly free if and only if the sequence $\bar\rho_1,\ldots,\bar\rho_m$ is strongly free in $\bar L$.*

Over fields, we have the following criterion for strong freeness. Let $R=k$ be a field, $X$ a finite set and $S\subset X$ a subset. Let $L(X)$ be the free Lie algebra over $X$ and let $\mathfrak{a}$ be the ideal of $L(X)$ generated by the elements $\xi\in X\smallsetminus S$. Put

$$T=\{\,[\xi,\xi']\mid \xi\in X\smallsetminus S,\ \xi'\in S\}\subset\mathfrak{a}.$$

Theorem 5.2. ([La], Th. 3.3, Cor. 3.5) *If $\rho_1,\ldots,\rho_m$ are homogeneous elements of $\mathfrak{a}$ which lie in the $k$-span of $T$ modulo $[\mathfrak{a},\mathfrak{a}]$ and which are linearly independent over $k$ modulo $[\mathfrak{a},\mathfrak{a}]$ then the sequence $\rho_1,\ldots,\rho_m$ is strongly free in $L$.*

Let $p$ be an odd prime number and let $G$ be a pro-$p$-group. We consider the descending $p$-central series $(G_n)_{n\geq 1}$, which is defined recursively by

$$G_1=G,\ G_{n+1}=G_n^p[G,G_n].$$

The quotients $\mathrm{gr}_n(G)=G_n/G_{n+1}$, denoted additively, are $\mathbb{F}_p$-vector spaces. The graded vector space

$$\mathrm{gr}(G)=\bigoplus_{n\geq 1}\mathrm{gr}_n(G)$$

has a Lie algebra structure over the polynomial ring $\mathbb{F}_p[\pi]$, where multiplication by $\pi$ is induced by $x\mapsto x^p$ and the bracket operation for homogeneous elements is induced by the commutator operation in $G$, see [NSW], III, §8. For $g\in G$, $g\neq 1$, let the natural number $h(g)$ be defined by

$$g\in G_{h(G)},\ g\notin G_{h(G)+1}.$$

This definition makes sense because $\bigcap_n G_n = \{1\}$, see [NSW], Prop. 3.8.2. The image $\omega(g)$ of $g$ in $\mathrm{gr}_{h(g)}(G)$ is called the *initial form* of $g$.

Let $F$ be the free pro-$p$-group over elements $x_1, \ldots, x_n$, $n \geq 1$. Then $h(x_i) = 1$, $i = 1, \ldots, n$, and

$$L = \mathrm{gr}(F)$$

is the free $\mathbb{F}_p[\pi]$-Lie algebra over $\xi_1, \ldots, \xi_n$, where $\xi_i = \omega(x_i)$, $i = 1, \ldots, n$, see [Lz]. Let $r_1, \ldots, r_m$, $m \geq 1$, be a sequence of elements in $F_2 = F^p[F, F] \subset F$ and let $R = (r_1, \ldots, r_m)_F$ be the closed normal subgroup of $F$ generated by $r_1, \ldots, r_m$. Put $\rho_i = \omega(r_i) \in L$.

Definition. *A pro-p-group $G$ is called* mild *if there exists a presentation*

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

*with $F$ a free pro-p-group on generators $x_1, \ldots, x_n$ and $R = (r_1, \ldots, r_m)_F$ such that the associated sequence $\rho_1, \ldots, \rho_m$ is strongly free in $L = gr(F)$.*

Essential for our application is the following property of mild pro-$p$-groups.

Theorem 5.3. ([La], Th. 1.2(c)) *If $G$ is a mild pro-p-group, then $cd\, G = 2$.*

Now let $G$ be a finitely presented pro-$p$-group and let

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

be a minimal presentation, i.e. $F$ is the free pro-$p$-group on generators $x_1, \ldots, x_n$ where $n = \dim_{\mathbb{F}_p} H^1(G)$ and $R = (r_1, \ldots, r_m)_F$ with $m = \dim_{\mathbb{F}_p} H^2(G)$, cf. [NSW], (3.9.5). Then the images $\xi_i = \omega(x_i)$, $i = 1, \ldots, n$, of $x_1, \ldots, x_n$ are a basis of the $\mathbb{F}_p$-vector space $F/F_2 = H_1(F) = H_1(G) = G/G_2$. For $y \in F_n$ and $a \in \mathbb{Z}_p$ the class of $y^a$ modulo $F_{n+1}$ only depends on the residue class $\bar{a} \in \mathbb{F}_p$ of $a$. Every element $r \in R \subset F_2$ has a representation

$$r \equiv \prod_{j=1}^{n} (x_j^p)^{a_j} \cdot \prod_{1 \leq k < l \leq n} [x_k, x_l]^{a_{kl}} \mod F_3,$$

where $a_j, a_{k,l} \in \mathbb{F}_p$. These coefficients are uniquely defined and can be calculated as follows. As $F$ is free, we have an isomorphism $H_2(G) = R_G^{ab}/p$. Let $\bar{r} \in H_2(G)$ be the image of $r$ and let $\chi_1, \ldots, \chi_n \in H^1(G)$ be the dual $\mathbb{F}_p$-basis of $\xi_1, \ldots, \xi_n$.

Theorem 5.4. $a_{kl} = -\bar{r}(\chi_k \cup \chi_l)$ *for $k < l$.*

For a proof see [NSW], Prop. 3.9.13, which also gives a description of the $a_j$ using the Bockstein operator.

Using the results above, we obtain a criterion for mildness.

THEOREM 5.5. *Let $G$ be a finitely presented pro-$p$-group. Assume there exists a basis $\chi_1,\ldots,\chi_n$ of $H^1(G)$, a basis $\bar r_1,\ldots,\bar r_m$ of $H_2(G)$ and a natural number $a$, $1\le a<n$, such that the following conditions are satisfied*

(i) $\bar r_i(\chi_k\cup\chi_l)=0$ *for* $a<k<l\le n$ *and* $i=1,\ldots,m$.

(ii) *The* $m\times a(n-a)$ *matrix*

$$\Big(\bar r_i(\chi_k\cup\chi_l)\Big)_{i,(k,l)},\ 1\le i\le m,\ 1\le k\le a<l\le n$$

*has rank* $m$.

*Then $G$ is a mild pro-$p$-group.*

*Proof.* Let $\xi_1,\ldots,\xi_n\in H_1(G)$ be the dual basis of $\chi_1,\ldots,\chi_n$. We choose a minimal presentation

$$(*)\qquad 1\longrightarrow R\longrightarrow F\longrightarrow G\longrightarrow 1$$

and generators $x_1,\ldots,x_n\in F$ mapping to $\xi_1,\ldots,\xi_n\in H_1(F)=H_1(G)$. Then we choose elements $r_1,\ldots,r_m\in R$ mapping to $\bar r_1,\ldots,\bar r_m\in R_G^{ab}/p=H_2(G)$. Let $X=\{\xi_1,\ldots,\xi_n\}$. Then $L=\mathrm{gr}(F)$ is the free $\mathbb{F}_p[\pi]$-Lie algebra over $X$ and $\bar L=L/\pi$ is the free $\mathbb{F}_p$-Lie algebra over $X$. In order to show that $G$ is mild, we have to show that the initial forms $\rho_1,\ldots,\rho_m$ of $r_1,\ldots,r_m$ are a strongly free sequence in $L$. By Theorem 5.1 it suffices to show that $\bar\rho_1,\ldots,\bar\rho_m\subset\bar L$ are a strongly free sequence. By condition (ii) and Theorem 5.4, we have $\bar\rho_1,\ldots,\bar\rho_m\in\mathrm{gr}_2(\bar L)=F_2/F_3F^p$.
Now we apply Theorem 5.2 with $S=\{\xi_{a+1},\ldots,\xi_n\}\subset X$. In the notation of this theorem, $\mathfrak a$ is the ideal generated by $\xi_1,\ldots,\xi_a$ in $\bar L$ and

$$T=\{[\xi_i,\xi_j]\mid 1\le i\le a,\ a+1\le j\le n\}.$$

By condition (i) and Theorem 5.4, we have $\bar\rho_i$ in the $\mathbb{F}_p$-span $H$ of $T$ modulo $[\mathfrak a,\mathfrak a]$. The elements of $T$ are a basis of $H$ and the coefficient matrix of $\bar\rho_1,\ldots,\bar\rho_m$ is up to sign the transpose of the matrix written in condition (ii). Hence $\bar\rho_1,\ldots,\bar\rho_m$ are linearly independent and, by Theorem 5.2, a strongly free sequence. This concludes the proof. □

## 6 EXISTENCE OF $K(\pi,1)$'S

Let $k$ be a number field and let $p$ be a prime number with $\mu_p\not\subset k$ and assume that $Cl(k)(p)=0$. The exact sequence

$$0\longrightarrow\mathcal{O}_k^\times\longrightarrow k^\times\xrightarrow{(v_{\mathfrak q})_{\mathfrak q}}\bigoplus_{\mathfrak q}\mathbb{Z}\longrightarrow Cl(k)\longrightarrow 0$$

implies the exactness of

$$0\longrightarrow\mathcal{O}_k^\times/p\longrightarrow k^\times/k^{\times p}\longrightarrow\bigoplus_{\mathfrak q}\mathbb{Z}/p\mathbb{Z}\longrightarrow 0.$$

Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ be a finite set of primes of norm $\equiv 1 \bmod p$. We choose for $i = 1, \ldots, m$ elements $s_i \in k^\times/k^{\times p}$ with $v_{\mathfrak{p}_i}(s_i) \equiv 1 \bmod p$ and $v_{\mathfrak{q}}(s_i) \equiv 0 \bmod p$ for all primes $\mathfrak{q} \neq \mathfrak{p}_i$ of $k$. Let furthermore, $e_1, \ldots, e_r$, $r = r_1 + r_2 - 1$, be a basis of $\mathcal{O}_k^\times/p$.

Consider the field

$$K = k(\mu_p, \sqrt[p]{s_1}, \ldots, \sqrt[p]{s_m}, \sqrt[p]{e_1}, \ldots, \sqrt[p]{e_r}).$$

An inspection of the ramification behaviour shows that $Gal(K|k(\mu_p))$ has the Galois group $(\mathbb{Z}/p\mathbb{Z})^{m+r}$: Indeed, $k(\mu_p, \sqrt[p]{e_1}, \ldots, \sqrt[p]{e_r})|k(\mu_p)$ is unramified outside $S_p$ and has Galois group $(\mathbb{Z}/p\mathbb{Z})^r$ by Kummer theory. Adjoining $\sqrt[p]{s_i}$, $i = 1, \ldots, m$, yields a cyclic extension of degree $p$ which is unramified outside $S_p \cup \{\mathfrak{p}_i\}$ and ramified at $\mathfrak{p}_i$.
Since $\mu_p \not\subset k$, the extensions $k_S^{el}(\mu_p)|k(\mu_p)$ and $K|k(\mu_p)$ lie in different eigenspaces for the action of $Gal(k(\mu_p)|k)$. Therefore $Kk_S^{el}|k(\mu_p)$ has Galois group $(\mathbb{Z}/p\mathbb{Z})^{m+r+n}$, with $n = \dim_{\mathbb{F}_p} Gal(k_S^{el}|k) = \dim_{\mathbb{F}_p} H^1(G_S)$.

Assume now that we are given

- a set of primes $T$ of $k$ with $T \cap S = \varnothing$ and with $p$-density $\Delta_k^p(T) = 1$,
- a nonzero element $F \in Gal(k_S^{el}|k) = Gal(k_S^{el}(\mu_p)|k(\mu_p))$,
- to each $\mathfrak{p}_i$, $i = 1, \ldots, m$, a condition $C_i$ which says "split" or "inert".

By Čebotarev's density theorem applied to the extension $Kk_S^{el}(\mu_p)|k(\mu_p)$, we find a prime $\mathfrak{P} \in T(Kk_S^{el}(\mu_p))$ such that

- the image of $Frob_{\mathfrak{P}}$ in $Gal(k(\mu_p, \sqrt[p]{e_1}, \ldots, \sqrt[p]{e_r})|k(\mu_p))$ is trivial,
- the image of $Frob_{\mathfrak{P}}$ in $Gal(k(\mu_p, \sqrt[p]{s_i})|k(\mu_p))$ is trivial if $C_i$ is "split" and nontrivial otherwise, and
- the image of $Frob_{\mathfrak{P}}$ in $Gal(k_S^{el}(\mu_p)|k(\mu_p))$ is equal to $F$.

Let $\mathfrak{p} \in T$ be the restriction of $\mathfrak{P}$ to $k$. Then the natural map $\mathcal{O}_k^\times/p \to k_{\mathfrak{p}}^\times/k_{\mathfrak{p}}^{\times p}$ is the zero map. Since ${}_p Cl(k) = 0$, we obtain $\mathcal{O}_k^\times/p \xrightarrow{\sim} V_\varnothing(k) = V_{\{\mathfrak{p}\}}(k)$. By Theorem 3.4, $k_{\{\mathfrak{p}\}}^{el}|k$ is cyclic of order $p$ and $\mathfrak{p}$ is ramified in this extension. Recall that $H^1_{nr}(G_{\mathfrak{p}})$ is defined as the exact annihilator of the inertia group $T_{\mathfrak{p}}(k_{\mathfrak{p}}^{el}|k_{\mathfrak{p}}) \subset H_1(G_{\mathfrak{p}})$ in the natural pairing

$$H_1(G_{\mathfrak{p}}) \times H^1(G_{\mathfrak{p}}) \longrightarrow \mathbb{F}_p.$$

Dually, $T_{\mathfrak{p}}(k_{\mathfrak{p}}^{el}|k_{\mathfrak{p}})$ is the exact annihilator of $H^1_{nr}(G_{\mathfrak{p}})$. The equation $T_{\mathfrak{p}}(k_{\{\mathfrak{p}\}}^{el}|k) = Gal(k_{\{\mathfrak{p}\}}^{el}|k)$ yields an isomorphism

$$H^1(G_{\{\mathfrak{p}\}}) \xrightarrow{\sim} H^1(G_{\mathfrak{p}})/H^1_{nr}(G_{\mathfrak{p}}).$$

By class field theory, $\mathfrak{p}_i$ splits in $k_{\{\mathfrak{p}\}}^{el}|k$ if and only if there exists an element $s_i' \in k^\times/k^{\times p}$ with $v_{\mathfrak{p}_i}(s_i') \equiv 1 \bmod p$, $v_{\mathfrak{q}}(s_i') \equiv 0 \bmod p$ for all $\mathfrak{q} \neq \mathfrak{p}_i$ and $s_i' \in k_{\mathfrak{p}}^{\times p}$. Then $s_i'/s_i$ lies in $\mathcal{O}_k^\times/p$, and therefore $s_i \in k_{\mathfrak{p}}^{\times p}$. Hence $\mathfrak{p}_i$ splits in $k^{el}_{\{\mathfrak{p}\}}|k$ if and only if $s_i$ is a $p$-th power in $k_\mathfrak{p}$. On the other hand, by our choice of $\mathfrak{P}$, $s_i$ is a $p$-th power in $k_\mathfrak{p}$ if and only if $C_i$ is "split". Therefore the following holds:

- the natural map $\mathcal{O}_k^\times/p \to k_\mathfrak{p}^\times/k_\mathfrak{p}^{\times p}$ is the zero map,
- $Frob_\mathfrak{p} = F \in Gal(k_S^{el}|k)$,
- $k^{el}_{\{\mathfrak{p}\}}|k$ is cyclic of order $p$,
- each $\mathfrak{p}_i$, $i = 1, \ldots, m$, satisfies condition $C_i$ in $k^{el}_{\{\mathfrak{p}\}}|k$.

Now assume that $\text{Б}_{S\smallsetminus\{\mathfrak{q}\}}(k) = 0$ for all $\mathfrak{q} \in S$, in particular, $S$ is $p$-large. Then all $\mathfrak{p}_i \in S$ ramify in $k_S^{el}|k$ and the 1-dimensional subspaces $T_{\mathfrak{p}_i}(k_S^{el}|k)$, $i = 1, \ldots, m$, in $H_1(G_S)$ are pairwise different and generate $H_1(G_S)$. Furthermore assume that $\delta_S^2(k) = 0$. As $\mathfrak{p}$ does not split completely in $k_S^{el}|k$, Lemma 4.4 implies $\delta^2_{S\cup\{\mathfrak{p}\}}(k) = 0$. Since $\mu_p \not\subset k$ and by Theorem 3.4, the natural maps $H^2(G_S) \to \prod_{\mathfrak{q}\in S} H^2(G_\mathfrak{q})$ and $H^2(G_{S\cup\{\mathfrak{p}\}}) \to \prod_{\mathfrak{q}\in S\cup\{\mathfrak{p}\}} H^2(G_\mathfrak{q})$ are isomorphisms. We denote the $\mathfrak{q}$-component of a global cohomology class $\alpha$ by $\alpha_\mathfrak{q}$.

Next we fix a primitive $p$-th root of unity in $k(\mu_p)$ and to each $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ a prolongation to $k(\mu_p)$. After this choice we have identifications $\mu_p(k_\mathfrak{p}) = \mu_p((Kk_S^{el})_\mathfrak{P}) \cong \mathbb{F}_p$ and $\mu_p(k_{\mathfrak{p}_i}) \cong \mathbb{F}_p$, $i = 1, \ldots, m$. In particular, we have an isomorphism $H^2(G_\mathfrak{p}) = H^2(G_\mathfrak{p}, \mu_p) = \mathbb{F}_p$, and similarly for the $\mathfrak{p}_i$. Via these isomorphisms we consider the $\mathfrak{q}$-component $\alpha_\mathfrak{q}$ of a class $\alpha \in H^2(G_{S\cup\{\mathfrak{p}\}})$ as an element in $\mathbb{F}_p$. Let

$$\pi_\mathfrak{p} \in H^1(G_\mathfrak{p})/H^1_{nr}(G_\mathfrak{p}) = H^1(G_\mathfrak{p}, \mu_p)/H^1_{nr}(G_\mathfrak{p}, \mu_p) = k_\mathfrak{p}^\times/U_\mathfrak{p} k_\mathfrak{p}^{\times p}$$

be the image of a uniformizer and let $\chi_\mathfrak{p} \in H^1(G_{\{\mathfrak{p}\}})$ be the unique pre-image. We denote the image of $\chi_\mathfrak{p}$ in $H^1(G_{S\cup\{\mathfrak{p}\}})$ by the same letter. Thus $\chi_\mathfrak{p}$ maps to $\pi_\mathfrak{p}$ under the natural map $H^1(G_{S\cup\{\mathfrak{p}\}}) \to H^1(G_\mathfrak{p})/H^1_{nr}(G_\mathfrak{p})$. Consider the exact pairing

$$H^1_{nr}(G_\mathfrak{p}) \times H^1(G_\mathfrak{p})/H^1_{nr}(G_\mathfrak{p}) \to H^2(G_\mathfrak{p}) = \mathbb{F}_p,$$

which is induced by local Tate duality, see [NSW], Thm. 7.2.15. Let $\delta : k_\mathfrak{p}^\times/k_\mathfrak{p}^{\times p} \xrightarrow{\sim} H^1(G_\mathfrak{p})$ be the boundary isomorphism of the Kummer sequence and let $rec : k_\mathfrak{p}^\times/k_\mathfrak{p}^{\times p} \xrightarrow{\sim} H_1(G_\mathfrak{p})$ be the mod-$p$ reciprocity map. Put $\varphi = rec \circ \delta^{-1}$. Then the image of $\chi_\mathfrak{p}$ under the composition

$$H^1(G_S) \longrightarrow H^1(G_\mathfrak{p}) \xrightarrow{\phi} H_1(G_\mathfrak{p}) \longrightarrow H_1(G_S)$$

is $Frob_\mathfrak{p}$, the Frobenius automorphism of the unramified prime $\mathfrak{p}$ in $k_S^{el}|k$. By [NSW], Prop. 7.2.13 [2]), the diagram

$$\begin{array}{ccccc} H^1(G_\mathfrak{p}) & \times & H^1(G_\mathfrak{p}) & \xrightarrow{\cup} & H^2(G_\mathfrak{p}) \\ \Vert & & \wr\downarrow\varphi & & \wr\downarrow inv \\ H^1(G_\mathfrak{p}) & \times & H_1(G_\mathfrak{p}) & \xrightarrow{can} & \mathbb{F}_p \end{array}$$

[2]This proposition contains a sign error, see the errata file on the author's homepage.

commutes. We obtain for any $\chi \in H^1(G_S) \subset H^1(G_{S\cup\{\mathfrak{p}\}})$ the following formula for the $\mathfrak{p}$-component of $\chi \cup \chi_\mathfrak{p} \in H^2(G_{S\cup\{\mathfrak{p}\}})$:

$$(\chi \cup \chi_\mathfrak{p})_\mathfrak{p} = \chi(Frob_\mathfrak{p}).$$

The image of $\chi_\mathfrak{p}$ in $H^1(G_{\mathfrak{p}_i})$ obviously lies in the subgroup $H^1_{nr}(G_{\mathfrak{p}_i})$. By the same argument, noting that the cup-product is anti-symmetric, we obtain the equality

$$(\chi \cup \chi_\mathfrak{p})_{\mathfrak{p}_i} = -\chi_\mathfrak{p}(Frob_{\mathfrak{p}_i}),$$

for any $\chi \in H^1(G_S)$ mapping to $\pi_{\mathfrak{p}_i} \in H^1(G_{\mathfrak{p}_i})/H^1_{nr}(G_{\mathfrak{p}_i})$, where $Frob_{\mathfrak{p}_i}$ is the Frobenius automorphism of the unramified prime $\mathfrak{p}_i$ in $k^{el}_{\{\mathfrak{p}\}}|k$. As $\chi_\mathfrak{p}$ is unramified at $\mathfrak{p}_i$, the element $(\chi \cup \chi_\mathfrak{p})_{\mathfrak{p}_i}$ depends only on the image of $\chi$ in the one-dimensional $\mathbb{F}_p$-vector space $H^1(G_{\mathfrak{p}_i})/H^1_{nr}(G_{\mathfrak{p}_i})$. Since $\mathfrak{p}_i$ ramifies in $k^{el}_S|k$, the map $H^1(G_S) \to \mathbb{F}_p$, $\chi \mapsto (\chi \cup \chi_\mathfrak{p})_{\mathfrak{p}_i}$ is the linear form associated to an element $t_i \in T_{\mathfrak{p}_i}(k^{el}_S|k) \subset H_1(G_S)$.

Summing up and using the notation and choices above, we obtain the

Lemma 6.1. *Let $k$ be a number field and let $p$ be a prime number with $\mu_p \not\subset k$ and $Cl(k)(p) = 0$. Let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ be a finite $p$-large set of primes and assume $\delta^2_S(k) = 0$. Let for $i = 1, \ldots, m$ elements $\varepsilon_i \in \{0, 1\}$ and for $i = 1, \ldots, n$ elements $d_i \in \mathbb{F}_p$ be given, where not all $d_i$ are zero. Let $\chi_1, \ldots, \chi_n$ be a basis of $H^1(G_S)$. Furthermore, let $T$ be a set of primes of $p$-density $\Delta_p(T) = 1$ and with $T \cap S = \varnothing$.*
*Then there exists a prime $\mathfrak{p} \in T$ such that the following conditions hold with respect to the identifications $H^2(G_{\mathfrak{p}_i}) = \mathbb{F}_p$, $i = 1, \ldots, m$, and $H^2(G_\mathfrak{p}) = \mathbb{F}_p$.*

- *$\mathfrak{p}$ does not split completely in $k^{el}_S|k$,*
- *$k^{el}_{\{\mathfrak{p}\}}|k$ is cyclic of order $p$,*
- *$\chi_1, \ldots, \chi_n, \chi_\mathfrak{p}$ is a basis of $H^1(G_{S\cup\{\mathfrak{p}\}})$,*
- *$(\chi_i \cup \chi_\mathfrak{p})_\mathfrak{p} = d_i$ for $i = 1, \ldots, n$,*
- *For $i = 1, \ldots, m$ we have $c_i = 0$ if and only if $\varepsilon_i = 0$, where $c_i \in T_{\mathfrak{p}_i}(k^{el}_S|k) \subset H_1(G_S)$ represents the map $H^1(G_S) \to \mathbb{F}_p$, $\chi \mapsto (\chi \cup \chi_\mathfrak{p})_{\mathfrak{p}_i}$.*

Now we are able to prove the following result, which is unessentially sharper than Theorem 1 of the introduction.

Theorem 6.2. *Let $k$ be a number field and let $p$ be a prime number such that*

$$\mu_p \not\subset k \quad \textit{and} \quad Cl(k)(p) = 0.$$

*Let $S$ be a finite set of primes of $k$ and let $T$ be a set of primes of $p$-density $\Delta^p(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $Spec(\mathcal{O}_k) \smallsetminus (S \cup T_1)$ is a $K(\pi, 1)$ for $p$.*

*Proof.* We may suppose that $T \cap S = \varnothing$. After moving finitely many primes of $T$ to $S$, we may assume that the following conditions hold:

- $Б_{S\smallsetminus\{\mathfrak{p}\}}(k)=0$ for all $\mathfrak{p}\in S$,
- $\delta_S^2(k)=0$.

Now let $S=\{\mathfrak{p}_1,\dots,\mathfrak{p}_m\}$. Then $m=h^2(G_S)$. Let $n=m-r=h^1(G_S)$. We will achieve the $K(\pi,1)$-situation by adding $m$ further primes to $S$.
We choose any basis $\chi_1,\dots,\chi_n$ of $H^1(G_S)$. Let $t_1,\dots,t_m$ be generators of the inertia groups $T_{\mathfrak{p}_i}(k_S^{el}|k)\subset H_1(G_S)$. Now we add a prime $\mathfrak{p}_{m+1}$ in the following way:
Let $i_1\in\{1,\dots,n\}$ be an index such that $\chi_{i_1}(t_1)\neq 0$, and let $i_1'\in\{1,\dots,n\}$, $i_1'\neq i_1$, be any other index. Now, according to Lemma 6.1, we put the conditions

$$\varepsilon_1=1 \text{ and } \varepsilon_i=0 \text{ for } i\in\{2,\dots,m\},$$

$$d_{i_1'}=1 \text{ and } d_i=0 \text{ for } i\in\{1,\dots,n\},\ i\neq i_1'$$

to choose a prime $\mathfrak{p}_{m+1}\in T$ such that for $i=1,\dots,n$

$$(\chi_i\cup\chi_{\mathfrak{p}_{m+1}})_{\mathfrak{p}_1}=\lambda_1\chi_i(t_1),\lambda_1\in\mathbb{F}_p^\times,\ (\chi_i\cup\chi_{\mathfrak{p}_{m+1}})_{\mathfrak{p}_j}=0,\ j=2,\dots,m$$

$$\text{and } (\chi_i\cup\chi_{\mathfrak{p}_{m+1}})_{\mathfrak{p}_{m+1}}=d_i.$$

Then in the matrix

$$\Big((\chi_i\cup\chi_{\mathfrak{p}_{m+1}})_{\mathfrak{p}_j}\Big)_{\substack{i=1,\dots,n\\ j=1,\dots,m+1}}$$

the $i_1$-line has entry $\neq 0$ at $(i_1,1)$ and all other entries zero, while the $i_1'$-line has some entry at $(i_1',1)$, the entry 1 at $(i_1',m+1)$ and all other entries zero.
In order to proceed, we put $\chi_{n+1}=\chi_{\mathfrak{p}_{m+1}}$ and choose an index $i_2\in\{1,\dots,n\}$ with $\chi_{i_2}(t_2)\neq 0$ and any $i_2'\in\{1,\dots,n\}$ with $i_2'\neq i_2$. We choose conditions as before, completed by $\varepsilon_{m+1}=0$ and $d_{n+1}=0$. Then we choose $\mathfrak{p}_{m+2}$ according to Lemma 6.1 and such that in the matrix

$$\Big((\chi_i\cup\chi_{\mathfrak{p}_{m+2}})_{\mathfrak{p}_j}\Big)_{\substack{i=1,\dots,n\\ j=1,\dots,m+2}}$$

the $i_2$-line has entry $\neq 0$ at $(i_2,2)$ and all other entries zero, while the $i_2'$-line has some entry at $(i_2',2)$, the entry 1 at $(i_2',m+2)$ and all other entries zero. In addition, our choice implies

$$(\chi_{\mathfrak{p}_{m+1}}\cup\chi_{\mathfrak{p}_{m+2}})_{\mathfrak{p}_{m+1}}=0=(\chi_{\mathfrak{p}_{m+1}}\cup\chi_{\mathfrak{p}_{m+2}})_{\mathfrak{p}_{m+2}}$$

As $\chi_{\mathfrak{p}_{m+1}}$ and $\chi_{\mathfrak{p}_{m+2}}$ are unramified at $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ by construction, we have furthermore $(\chi_{\mathfrak{p}_{m+1}}\cup\chi_{\mathfrak{p}_{m+2}})_{\mathfrak{p}_i}=0$ for $i=1,\dots,m$.

Now we proceed to construct $\mathfrak{p}_{m+3},\dots,\mathfrak{p}_{2m}$ in a similar way, and apply Theorem 5.5 with $a=m$. For each $j$, the $j$-th of the $m$-steps in the construction produced the two lines $((i_j,j),-)$ and $((i_j',j),-)$ in the $nm{\times}2m$-matrix

$$\Big((\chi_i\cup\chi_{\mathfrak{p}_j})_{\mathfrak{p}_k}\Big)_{\substack{i=1,\dots,n,j=m+1,\dots,2m\\ k=1,\dots,2m}}$$

According to our choices these $2m$ lines are linearly independent, hence the matrix has rank $2m$. Putting $T_1 = \{\mathfrak{p}_{m+1}, \ldots, \mathfrak{p}_{2m}\}$, we conclude by Theorem 5.5 that $G_{S\cup T_1}$ is a mild pro-$p$-group. Hence $cd\, G_{S\cup T_1} = 2$ by Theorem 5.3. By Lemma 4.4, we didn't produce new $h^2$-defect during our construction, hence $\delta^2_{S\cup T_1}(k) = 0$. As the étale cohomology is trivial in dimension $\geq 3$, we conclude that the homomorphisms

$$\phi_i : H^i(G_{S\cup T_1}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^i_{et}(Spec(\mathcal{O}_k) \smallsetminus (S \cup T_1), \mathbb{Z}/p\mathbb{Z})$$

are isomorphisms for all $i \geq 0$. Hence condition (v) of Proposition 2.1 is satisfied. □

## 7 Consequences of the $K(\pi,1)$-property

In this section we assume that $S$ is finite and we exclude the case $S = \varnothing$ from our considerations. Keeping all conventions made before, we assume

> *$p \neq 2$ or $k$ is totally imaginary and $S$ is a non-empty finite set of nonarchimedean primes $\mathfrak{p}$ with norm $N(\mathfrak{p}) \equiv 1 \bmod p$.*

Lemma 7.1. *$G_S$ is a fab-group, i.e. the abelianization $U^{ab}$ of every open subgroup $U$ of $G_S$ is finite.*

*Proof.* Let $U \subset G_S$ be an open subgroup. The abelianization $U^{ab}$ of $U$ is a finitely generated abelian pro-$p$-group. If $U^{ab}$ were infinite, it would have a quotient isomorphic to $\mathbb{Z}_p$, which by Galois theory corresponds to a $\mathbb{Z}_p$-extension $K_\infty$ of the number field $K = k_S^U$ inside $k_S$. By [NSW], Theorem 10.3.20 (ii), a $\mathbb{Z}_p$-extension of a number field is ramified at at least one prime dividing $p$. This contradicts $K_\infty \subset k_S$ and we conclude that $U^{ab}$ is finite. □

The group theoretical structure of the local Galois groups is well known.

Proposition 7.2. *Let $\mathfrak{p} \in S$. Then $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$ is the pro-$p$-group on two generators $\sigma, \tau$ subject to the relation $\sigma\tau\sigma^{-1} = \tau^q$. The element $\tau$ is a generator of the inertia group, $\sigma$ is a Frobenius lift and $q = N(\mathfrak{p})$.*

*Proof.* This follows from [NSW], Thm. 7.5.2 by passing to the maximal pro-$p$-factor group. □

We obtain the following corollary.

Corollary 7.3. *Assume that $G_S$ is infinite. Then, for each $\mathfrak{p} \in S$, the decomposition group $G_\mathfrak{p}$ of $\mathfrak{p}$ in $G_S$ has infinite index.*

*Proof.* The decomposition group $G_\mathfrak{p}$ is a quotient of the local Galois group $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$. If $G_\mathfrak{p} \subset G_S$ would have finite index, it would be an infinite fab-group by Lemma 7.1. By Proposition 7.2, each infinite quotient of $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$ has a surjection to $\mathbb{Z}_p$ and is therefore not a fab-group. This contradiction shows that $G_\mathfrak{p}$ has infinite index in $G_S$. □

The next proposition classifies the degenerate $K(\pi,1)$-case.

Proposition 7.4. *$X \smallsetminus S$ is a $K(\pi,1)$ and $G_S = 1$ if and only if $S = \{\mathfrak{p}\}$ consists of a single prime and one of the following cases occurs.*

(a) *$p = 2$, $k \neq \mathbb{Q}(\sqrt{-1})$ is imaginary quadratic, $2 \nmid h_k$ and $N(\mathfrak{p}) \not\equiv 1 \bmod 4$,*

(b) *$p = 2$, $k = \mathbb{Q}(\sqrt{-1})$ and $N(\mathfrak{p}) \not\equiv 1 \bmod 8$,*

(c) *$p = 3$, $k = (\mathbb{Q}\sqrt{-3})$ and $N(\mathfrak{p}) \not\equiv 1 \bmod 9$.*

*Proof.* Assume $G_S = 1$ and that $X \smallsetminus S$ is a $K(\pi,1)$. Then $H^i_{et}(X \smallsetminus S) = 0$ for all $i \geq 1$. In particular, $p \nmid h_k$. By Theorem 3.4, $h^2(X \smallsetminus S) = 0$ implies $\delta = 1$, $\#S = 1$ and $V_S = 0$. Then, using $h^1(X \smallsetminus S) = 0$, we obtain $r = 1$. As $\delta = 1$, the following possibilities remain

(a) $p = 2$, $k \neq \mathbb{Q}(\sqrt{-1})$ is imaginary quadratic and $2 \nmid h_k$,

(b) $p = 2$, $k = \mathbb{Q}(\sqrt{-1})$,

(c) $p = 3$, $k = (\mathbb{Q}\sqrt{-3})$.

In all cases, Proposition 3.3 yields an isomorphism $\mathcal{O}_k^\times/p \xrightarrow{\sim} V_\varnothing$. The second exact sequence of Proposition 3.3 and the isomorphism $U_\mathfrak{p}/p \cong U_\mathfrak{p} k_\mathfrak{p}^{\times p}/k_\mathfrak{p}^{\times p}$ imply

$$0 = V_S = \ker\big(\mathcal{O}_k^\times/p \to U_\mathfrak{p}/p\big).$$

Note that $\mathcal{O}_k^\times/p$ is one-dimensional. In case (a), the unit $-1$ is a generator of $\mathcal{O}_k^\times/2$ which must not be a square in $U_\mathfrak{p}$, implying $N(\mathfrak{p}) \not\equiv 1 \bmod 4$. In case (b), $\sqrt{-1}$ is a generator, and in case (c), a generator is given by $\zeta_3 = \frac{1}{2}(-1+\sqrt{-3})$. The assertions in the cases (b) and (c) follow similarly. Conversely, assume we are in case (a), (b) or (c). Then we can reverse the given arguments and obtain $h^i(X \smallsetminus S) = 0$ for all $i \geq 1$. □

Theorem 7.5. *Assume $G_S \neq 1$ and that $X \smallsetminus S$ is a $K(\pi,1)$. Then the following hold.*

(i) *$cd\, G_S = 2$, $scd\, G_S = 3$.*

(ii) *$G_S$ is a duality group (of dimension 2).*

*Proof.* By Lemma 7.1 and Corollary 3.5, $G_S$ is a fab-group and $cd\ G_S \leq 2$. Now the assertions follow in a purely group-theoretical way:
As $G_S \neq 1$ and $G_S^{ab}$ is finite, $G_S$ is not free, and we obtain $cd\, G_S = 2$. By [NSW], Proposition 3.3.3, it follows that $scd\, G_S \in \{2,3\}$. Assume $scd\, G = 2$. We consider the $G_S$-module

$$D_2(\mathbb{Z}) := \varinjlim_{U} U^{ab},$$

where the limit runs over all open normal subgroups $U \lhd G_S$ and for $V \subset U$ the transition map is the transfer $\mathrm{Ver}\colon U^{ab} \to V^{ab}$, i.e. the dual of the corestriction map $\mathrm{cor}\colon H^2(V,\mathbb{Z}) \to H^2(U,\mathbb{Z})$ (see [NSW], I, §5). By [NSW], Theorem 3.6.4 (iv), we obtain $G_S^{ab} = D_2(\mathbb{Z})^{G_S}$. On the other hand, $U^{ab}$ is finite for all $U$ and the group theoretical version of the Principal Ideal Theorem (see [Ne], VI, Theorem 7.6) implies $D_2(\mathbb{Z}) = 0$. Hence $G_S^{ab} = 0$ which implies $G_S = 1$ producing a contradiction. Hence $scd\, G_S = 3$.
It remains to show that $G_S$ is a duality group. By [NSW], Theorem 3.4.6, it suffices to show that the terms

$$D_i(G_S,\mathbb{Z}/p\mathbb{Z}) := \varinjlim_U H^i(U,\mathbb{Z}/p\mathbb{Z})^\vee$$

are zero for $i = 0, 1$. Here $U$ runs through the open subgroups of $G_S$, $\vee$ denotes the Pontryagin dual and the transition maps are the duals of the corestriction maps. For $i = 0$, and $V \subsetneqq U$, the transition map

$$\mathrm{cor}^\vee : \mathbb{Z}/p\mathbb{Z} = H^0(V,\mathbb{Z}/p\mathbb{Z})^\vee \to H^0(U,\mathbb{Z}/p\mathbb{Z})^\vee = \mathbb{Z}/p\mathbb{Z}$$

is multiplication by $(U:V)$, hence zero. Therefore $D_0(G_S,\mathbb{Z}/p\mathbb{Z}) = 0$, as $G_S$ is infinite. Furthermore,

$$D_1(G_S,\mathbb{Z}/p\mathbb{Z}) = \varinjlim_U U^{ab}/p = 0$$

by the Principal Ideal Theorem. This finishes the proof. $\square$

In order to proceed, we introduce some notation in order to deal with the case of infinite extensions. For a (possibly infinite) algebraic extension $K$ of $k$ we denote by $S(K)$ the set of prolongations of primes in $S$ to $K$. The set $S(K)$ carries a profinite topology in a natural way. Now assume that $M|K|k$ is a tower of Galois extensions. We denote the inertia group of a prime $\mathfrak{p} \in S(K)$ in the extension $M|K$ by $T_\mathfrak{p}(M|K)$. For $i \geq 0$ we write

$$\sideset{}{'}\bigoplus_{\mathfrak{p}\in S(K)} H^i(T_\mathfrak{p}(M|K),\mathbb{Z}/p\mathbb{Z}) \stackrel{df}{=} \varinjlim_{k'\subset K} \bigoplus_{\mathfrak{p}\in S(k')} H^i(T_\mathfrak{p}(M|k'),\mathbb{Z}/p\mathbb{Z}),$$

where the limit on the right hand side runs through all finite subextensions $k'$ of $k$ in $K$. The $Gal(K|k)$-module $\bigoplus'_{\mathfrak{p}\in S(K)} H^i(T_\mathfrak{p}(M|K),\mathbb{Z}/p\mathbb{Z})$ is the maximal discrete submodule of the product $\prod_{\mathfrak{p}\in S(K)} H^i(T_\mathfrak{p}(M|K),\mathbb{Z}/p\mathbb{Z})$.

Whenever we deal with local terms associated to the elements of $S(K)$ (e.g. étale cohomology groups) we use restricted sums, which are, in the same manner as above, defined as the inductive limit over the similar terms associated to all finite subextensions of $k$ in $K$.

A natural question is how far we get locally at the primes in $S$ when going up to $k_S$.

Proposition 7.6. *Assume that $X \smallsetminus S$ is a $K(\pi,1)$ and that $G_S \neq 1$. Then $k_S$ realizes the maximal unramified $p$-extension of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$, i.e.*

$$k_\mathfrak{p}^{nr}(p) \subset (k_S)_\mathfrak{p} \qquad \textit{for all } \mathfrak{p} \in S.$$

*If $\mathfrak{p} \in S$ ramifies in $k_S$, then $(k_S)_\mathfrak{p} = k_\mathfrak{p}(p)$, i.e. $k_S$ realizes the maximal $p$-extension of $k_\mathfrak{p}$.*

*Proof.* For an integral normal scheme $Y$ we write $Y_L$ for the normalization of $Y$ in an algebraic extension $L$ of its function field. Then $(X \smallsetminus S)_{k_S}$ is the universal pro-$p$ covering of $X \smallsetminus S$. We consider the following part of the excision sequence for the pair $(X_{k_S}, (X \smallsetminus S)_{k_S})$

$$H^2_{et}((X \smallsetminus S)_{k_S}) \to \bigoplus_{\mathfrak{p} \in S(k_S)}{}' H^3_\mathfrak{p}((X_{k_S})_\mathfrak{p}) \to H^3_{et}(X_{k_S}).$$

As $G_S$ is infinite, Lemma 3.7 implies $H^3_{et}(X_{k_S}) = 0$. By condition (iii) of Proposition 2.1 we have $H^2_{et}((X \smallsetminus S)_{k_S}) = 0$. Hence $H^3_\mathfrak{p}((X_{k_S})_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in S(k_S)$. As $H^i_{et}((X_{k_S})_\mathfrak{p}) = 0$ for $i \geq 2$, we obtain

$$H^3_\mathfrak{p}((X_{k_S})_\mathfrak{p}) \cong H^2((k_S)_\mathfrak{p}),$$

where the group on the right hand side is Galois cohomology with values in $\mathbb{Z}/p\mathbb{Z}$. As $\mu_p \subset k_\mathfrak{p}$ by assumption, the vanishing of $H^2((k_S)_\mathfrak{p})$ implies $p^\infty \mid [(k_S)_\mathfrak{p} : k_\mathfrak{p}]$. In other words, the decomposition group $G_\mathfrak{p}(k_S|k)$ of each $\mathfrak{p} \in S$ is infinite. As a subgroup of $G_S$, it has cohomological dimension $\leq 2$. Furthermore, $G_\mathfrak{p}(k_S|k)$ is a factor group of the local Galois group $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$, which, by Proposition 7.2, has only three quotients of cohomological dimension less or equal to 2: itself, the trivial group and the Galois group of the maximal unramified $p$-extension of $k_\mathfrak{p}$. Hence $k_\mathfrak{p}^{nr}(p) \subseteq (k_S)_\mathfrak{p}$ and $(k_S)_\mathfrak{p} = k_\mathfrak{p}(p)$ if $\mathfrak{p}$ ramifies in $k_S$. □

In order to deduce Theorem 3, it remains to show that each $\mathfrak{p} \in S$ ramifies in $k_S$. The following lemma provides a first step.

Lemma 7.7. *Let $\mathfrak{p} \in S$ be a prime and let $S' = S \smallsetminus \{\mathfrak{p}\}$. Assume that the natural injection $V_S \hookrightarrow V_{S'}$ is an isomorphism. Then $\mathfrak{p}$ ramifies in $k_S$.*

*Proof.* Since the map $H^1(G_S) \to H^1_{et}(X \smallsetminus S)$ is an isomorphism, Theorem 3.4 implies

$$\dim_{\mathbb{F}_p} H^1(G_S) = 1 + \#S - \delta + \dim_{\mathbb{F}_p} V_S - r,$$

and the same formula holds with $S$ replaced by $S'$. Hence

$$\dim_{\mathbb{F}_p} H^1(G_S) = \dim_{\mathbb{F}_p} H^1(G_{S'}) + 1\,.$$

In particular, $G_{S'}$ is a proper quotient of $G_S$ and therefore $\mathfrak{p}$ ramifies in $k_S$. □

Corollary 7.8. *Assume that $X \smallsetminus S$ is a $K(\pi,1)$ and that $G_S \neq 1$. Let $\mathfrak{p} \in S$ be a prime and let $S' = S \smallsetminus \{\mathfrak{p}\}$. Assume that $V_{S'} = 0$. Then $(k_S)_\mathfrak{p} = k_\mathfrak{p}(p)$.*

Remark: If $V_\varnothing = 0$, then the given criterion applies to any set $S$ and each $\mathfrak{p} \in S$. This was used in [S1] for $k = \mathbb{Q}$ and in [Vo] for imaginary quadratic number fields. If the unit rank of $k$ is non-zero, then $V_\varnothing \neq 0$ and the criterion applies only to sufficiently large sets $S$.

## 8 Enlarging the set of primes

Next we consider the problem of enlarging the set $S$.

Proposition 8.1. *Let $S \subset S'$ be finite sets of primes of norm congruent to 1 modulo $p$. Assume that $X \smallsetminus S$ is a $K(\pi,1)$ and that $G_S \neq 1$. Further assume that each $\mathfrak{q} \in S' \smallsetminus S$ does not split completely in $k_S$. Then the following holds.*

(i) *$X \smallsetminus S'$ is a $K(\pi,1)$.*

(ii) *$(k_{S'})_\mathfrak{q} = k_\mathfrak{q}(p)$ for all $\mathfrak{q} \in S' \smallsetminus S$.*

*Furthermore, $H^i(Gal(k_{S'}|k_S)) = 0$ for $i \geq 2$. For $i = 1$ we have a natural isomorphism*

$$H^1(Gal(k_{S'}|k_S)) \cong \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S))}{}' H^1(T_\mathfrak{p}(k_{S'}|k_S)),$$

*In particular, $Gal(k_{S'}|k_S)$ is a free pro-$p$-group.*

*Proof.* Let $\mathfrak{q} \in S' \smallsetminus S$. Since $\mathfrak{q}$ does not split completely in $k_S$ and since $cd\, G_S = 2$, the decomposition group of $\mathfrak{q}$ in $k_S|k$ is a non-trivial and torsion-free quotient of $\mathbb{Z}_p \cong G(k_\mathfrak{q}^{nr}(p)|k_\mathfrak{q})$. Therefore $(k_S)_\mathfrak{q}$ is the maximal unramified $p$-extension of $k_\mathfrak{q}$. We denote the normalization of an integral normal scheme $Y$ in an algebraic extension $L$ of its function field by $Y_L$. Then $(X \smallsetminus S)_{k_S}$ is the universal pro-$p$ covering of $X \smallsetminus S$. We consider the étale excision sequence for the pair $((X \smallsetminus S)_{k_S}, (X \smallsetminus S')_{k_S})$. By assumption, $X \smallsetminus S$ is a $K(\pi,1)$, hence $H^i_{et}((X \smallsetminus S)_{k_S}) = 0$ for $i \geq 1$ by condition (iii) of Proposition 2.1. This implies isomorphisms

$$H^i_{et}\big((X \smallsetminus S')_{k_S}\big) \xrightarrow{\sim} \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S)}{}' H^{i+1}_\mathfrak{p}\big(((X \smallsetminus S)_{k_S})_\mathfrak{p}\big)$$

for $i \geq 1$. As $k_S$ realizes the maximal unramified $p$-extension of $k_\mathfrak{q}$ for all $\mathfrak{q} \in S' \smallsetminus S$, the schemes $((X \smallsetminus S)_{k_S})_\mathfrak{p}$, $\mathfrak{p} \in S' \smallsetminus S(k_S)$ have trivial cohomology with values in $\mathbb{Z}/p\mathbb{Z}$ and we obtain isomorphisms

$$H^i((k_S)_\mathfrak{p}) \xrightarrow{\sim} H^{i+1}_\mathfrak{p}\big(((X \smallsetminus S)_{k_S})_\mathfrak{p}\big)$$

for $i \geq 1$. These groups vanish for $i \geq 2$. This implies

$$H^i_{et}((X \smallsetminus S')_{k_S}) = 0$$

for $i \geq 2$. The scheme $(X \smallsetminus S')_{k_{S'}}$ is the universal pro-$p$ covering of $(X \smallsetminus S')_{k_S}$. The Hochschild-Serre spectral sequence yields an inclusion

$$H^2(\mathit{Gal}(k_{S'}|k_S)) \hookrightarrow H^2_{et}((X \smallsetminus S')_{k_S}) = 0.$$

Hence $\mathit{Gal}(k_{S'}|k_S)$ is a free pro-$p$-group and

$$H^1(\mathit{Gal}(k_{S'}|k_S)) \xrightarrow{\sim} H^1_{et}((X \smallsetminus S')_{k_S}) \cong \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S)}{}' H^1((k_S)_\mathfrak{p}).$$

This shows that each $\mathfrak{p} \in S' \smallsetminus S(k_S)$ ramifies in $k_{S'}|k_S$, and since the Galois group is free, $k_{S'}$ realizes the maximal $p$-extension of $(k_S)_\mathfrak{p}$. In particular,

$$H^1(T_\mathfrak{p}(k_{S'}|k_S)) \cong H^1((k_S)_\mathfrak{p})$$

for all $\mathfrak{p} \in S' \smallsetminus S(k_S)$. Using that $\mathit{Gal}(k_{S'}|k_S)$ is free, the Hochschild-Serre spectral sequence induces an isomorphism

$$0 = H^2_{et}((X \smallsetminus S')_{k_S}) \xrightarrow{\sim} H^2_{et}((X \smallsetminus S')_{k_{S'}})^{\mathit{Gal}(k_{S'}|k_S)}.$$

Hence $H^2_{et}((X \smallsetminus S')_{k_{S'}}) = 0$, since $\mathit{Gal}(k_{S'}|k_S)$ is a pro-$p$-group. Condition (iii) of Proposition 2.1 implies that $X \smallsetminus S'$ is a $K(\pi, 1)$. □

Corollary 8.2. *Assume that $X \smallsetminus S$ is a $K(\pi,1)$, and let $S \subset S'$ be a finite set of primes of norm $\equiv 1 \bmod p$. Assume that each $\mathfrak{q} \in S' \smallsetminus S$ does not split completely in $k_S$. Then the arithmetic form of Riemann's existence theorem holds, i.e. the natural homomorphism*

$$\mathop{\ast}_{\mathfrak{p} \in S' \setminus S(k_S)} T_\mathfrak{p}(k_{S'}|k_S) \longrightarrow \mathit{Gal}(k_{S'}|k_S)$$

*is an isomorphism. Here $T_\mathfrak{p}$ is the inertia group and $\ast$ denotes the free pro-$p$-product of a bundle of pro-$p$-groups, cf. [NSW], Ch. IV, §3.*

*Proof.* By Proposition 8.1 and by the calculation of the cohomology of a free product ([NSW], 4.3.10 and 4.1.4), $\phi$ is a homomorphism between free pro-$p$-groups which induces an isomorphism on mod $p$ cohomology. Therefore $\phi$ is an isomorphism. □

## 9 Proof of Theorems 3 and 5

Theorem 9.1. *Assume that $X \smallsetminus S$ is a $K(\pi,1)$ and $G_S \neq 1$. Then $k_S$ realizes the maximal $p$-extension $k_\mathfrak{p}(p)$ of the local field $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$.*

*Proof.* The decomposition groups of primes in $S$ have infinite index by Corollary 7.3. By Corollary 2.2, we may replace $k$ by a finite subextension in $k_S$, and therefore assume that $\#S \geq 2$.
By Proposition 7.6, it suffices to show that each $\mathfrak{p} \in S$ ramifies in $k_S$. Let $\mathfrak{p} \in S$ be a prime which does not ramify in $k_S$ and put $S' = S \smallsetminus \{\mathfrak{p}\}$. By Lemma 7.7, the natural injection $\phi\colon\ V_S \hookrightarrow V_{S'}$ is not an isomorphism. By Proposition 3.3, the cokernel of $\phi$ is one-dimensional. By Theorem 3.4, we obtain

$$h^2(X \smallsetminus S') = h^2(X \smallsetminus S).$$

As $G_S = G_{S'}$, we have $cd\, G_{S'} = 2$ and

$$h^2(G_S) = h^2(G_{S'}) \leq h^2(X \smallsetminus S') = h^2(X \smallsetminus S).$$

As $X \smallsetminus S$ is a $K(\pi,1)$, equality holds. Therefore the injection $H^2(G_{S'}) \hookrightarrow H^2_{et}(X \smallsetminus S')$ is an isomorphism. By Corollary 3.5, $X \smallsetminus S'$ is a $K(\pi,1)$. By Proposition 7.6, $\mathfrak{p}$ does not split completely in $k_{S'} = k_S$. By Proposition 8.1, $k_S$ realizes the maximal $p$-extension of $k_{\mathfrak{p}}$. This yields a contradiction. □

Now we are in the position to show Theorem 5.

*Proof of Theorem 5.* We have $H^2_{et}((X \smallsetminus S)_{k_S}) = 0$ by condition (iii) of Proposition 2.1. By Theorem 9.1, the local cohomology groups $H^2_{\mathfrak{p}}((X_{k_S})_{\mathfrak{p}})$ vanish for all $\mathfrak{p} \in S(k_S)$. Therefore the excision sequence yields $H^2_{et}(X_{k_S}) = 0$. By the flat duality theorem of Artin-Mazur ([Mi], III Corollary 3.2) we have $H^2_{et}(X_K)^\vee \cong H^1_{fl}(X_K, \mu_p)$ for each finite subextension $K$ of $k$ in $k_S$. Hence

$$\varprojlim_{K \subset k_S} H^1_{fl}(X_K, \mu_p) = 0.$$

The flat Kummer sequence $0 \to \mu_p \to \mathbb{G}_m \xrightarrow{\cdot p} \mathbb{G}_m \to 0$ implies compatible exact sequences

$$0 \to \mathcal{O}_K^\times/p \to H^1_{fl}(X_K, \mu_p) \to {}_pH^1_{fl}(X_K, \mathbb{G}_m)$$

for all $K$. We obtain

$$\varprojlim_{K \subset k_S} \mathcal{O}_K^\times/p = 0\,.$$

The topological Nakayama-Lemma (see [NSW], Corollary 5.2.8) for the compact $\mathbb{Z}_p$-module $\varprojlim \mathcal{O}_K^\times \otimes \mathbb{Z}_p$ therefore implies

$$\varprojlim_{K \subset k_S} \mathcal{O}_K^\times \otimes \mathbb{Z}_p = 0\,.$$

Tensoring the exact sequences (cf. [NSW], Lemma 10.3.11)

$$0 \to \mathcal{O}_K^\times \to \mathcal{O}_{K,S}^\times \to \bigoplus_{\mathfrak{p} \in S(K)} (K_{\mathfrak{p}}^\times/U_{\mathfrak{p}}) \to Cl(K) \to Cl_S(K) \to 0$$

by (the flat $\mathbb{Z}$-algebra) $\mathbb{Z}_p$, we obtain exact sequences of finitely generated, hence compact, $\mathbb{Z}_p$-modules. The field $k_S$ admits no unramified $p$-extensions. Therefore class field theory implies $\varprojlim_K Cl(K)(p) = 0$, where $K$ runs through all finite subextensions of $k$ in $k_S$. Thus, passing to the projective limit over $K$, we obtain the exact sequence

$$0 \to \varprojlim_{K \subset k_S} \mathcal{O}_K^\times \otimes \mathbb{Z}_p \to \varprojlim_{K \subset k_S} \mathcal{O}_{K,S}^\times \otimes \mathbb{Z}_p \to \varprojlim_{K \subset k_S} \bigoplus_{\mathfrak{p} \in S(K)} (K_\mathfrak{p}^\times / U_\mathfrak{p}) \otimes \mathbb{Z}_p \to 0.$$

As $k_S$ realizes the maximal unramified $p$-extension of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$, local class field theory implies the vanishing of the right hand limit. Therefore the result for the $S$-units follows from the corresponding result for the units. □

We have proven all assertions but the statement on the dualizing module in Theorem 2. In [S1], Th. 5.2 we showed this statement under the assumption that $k_S$ realizes the maximal $p$-extension $k_\mathfrak{p}(p)$ of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S$. This assumption has been shown above, hence the result follows.

*Added in proof.* As pointed out by K. Wingberg, the proof of Proposition 8.1 does not use that $S$ and $S'$ are disjoint from $S_p$. Hence Theorem 1 also holds if $S \cap S_p \neq \varnothing$, i.e. the result extends to the mixed case.

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# Isotropy of Quadratic Spaces in Finite and Infinite Dimension

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Abstract. In the late 1970s, Herbert Gross asked whether there exist fields admitting anisotropic quadratic spaces of arbitrarily large finite dimensions but none of infinite dimension. We construct examples of such fields and also discuss related problems in the theory of central simple algebras and in Milnor $K$-theory.

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## 1 Introduction

A *quadratic space* over a field $F$ is a pair $(V, q)$ of a vector space $V$ over $F$ together with a map $q : V \longrightarrow F$ such that

- $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in F$, $x \in V$, and
- the map $b_q : V \times V \longrightarrow F$ defined by $b_q(x, y) = q(x + y) - q(x) - q(y)$ $(x, y \in V)$ is $F$-bilinear.

If $\dim V = n < \infty$, one may identify (after fixing a basis of $V$) the quadratic space $(V, q)$ with a form (a homogeneous polynomial) of degree 2 in $n$ variables. Via this identification, a finite-dimensional quadratic space over $F$ will also be referred to as *quadratic form* over $F$. Recall that a quadratic space $(V, q)$ is said to be *isotropic*, if there exists $x \in V \setminus \{0\}$ such that $q(x) = 0$; otherwise, $(V, q)$ is said to be *anisotropic*.

Questions about isotropy are at the core of the algebraic theory of quadratic forms over fields. A natural and much studied field invariant in this context is the so-called $u$-invariant of a field $F$. If $F$ is of characteristic not 2 and nonreal (i.e. $-1$ is a sum of squares in $F$), then $u(F)$ is defined to be the supremum of the dimensions of anisotropic finite-dimensional quadratic forms over $F$. See Section 2 for the general definition of the $u$-invariant. The main purpose of the present article is to give examples of fields having infinite $u$-invariant but not admitting any anisotropic infinite-dimensional quadratic space.

Assume now that the quadratic space $(V, q)$ over $F$ is anisotropic. For any positive integer $n \leq \dim(V)$, let $V_n$ be any $n$-dimensional subspace of $V$ and consider the restriction $q_n = q|_{V_n}$. Clearly, the $n$-dimensional quadratic form $(V_n, q_n)$ is again anisotropic. This simple argument shows that if there is an anisotropic quadratic space over $F$ of infinite dimension, then there exist anisotropic quadratic forms over $F$ of dimension $n$ for all $n \in \mathbb{N}$.

While this observation is rather trivial, it motivates us to examine the converse statement. If we assume that the field $F$ has anisotropic quadratic forms of arbitrarily large finite dimensions, does this imply the existence of some anisotropic quadratic space $(V, q)$ over $F$ of infinite dimension? As already mentioned, this is generally not so.

It appears that originally this question has been formulated by Herbert Gross. He concludes the introduction to his book 'Quadratic forms in infinite-dimensional vector spaces' [12] (published in 1979) by the following sample of 'a number of pretty and unsolved problems' in this area, which we state in his words (cf. [12], p. 3):

1.1 Question (Gross). Is there any commutative field which admits no anisotropic $\aleph_0$-form but which has infinite $u$-invariant, i.e. admits, for each $n \in \mathbb{N}$, some anisotropic form in $n$ variables?

Note that implicitly, Gross is looking for a nonreal field, because anisotropic quadratic spaces of infinite dimension always exist over real fields. (We use the term 'real field' for what is often called 'formally real field'.) Indeed, one observes that the field $F$ is real if and only if the infinite-dimensional quadratic space $(V, q)$ given by $V = F^{(\mathbb{N})}$ and $q : V \longrightarrow F, (x_i) \longmapsto \sum x_i^2$ is anisotropic.

By restricting to those quadratic spaces that are totally indefinite, i.e. indefinite with respect to every field ordering, one can formulate a meaningful analogue of the Gross Question also for real fields, to which we will provide a solution as well.

We also study the Gross Question in characteristic 2 where one has to distinguish between bilinear forms and quadratic forms. For quadratic forms, one furthermore has to distinguish the cases of nonsingular quadratic forms and of arbitrary quadratic forms. The analogue to the Gross Question for nonsingular quadratic forms in characteristic 2 can be treated in more or less the same way as in characteristic not 2, simply by invoking suitable analogues of the results that we use in our proofs in the case of characteristic different from 2. Yet, if translated to bilinear forms or to arbitrary quadratic forms (possibly singular) in characteristic 2, it is not difficult to show that the Gross Question has in fact a negative answer, in other words, the 'bilinear' resp. 'general quadratic' $u$-invariant is infinite if and only if there exist infinite-dimensional anisotropic bilinear resp. quadratic spaces.

The paper is structured as follows. In the next section, we are going to discuss in more detail the $u$-invariant of a field and some related concepts. In Section 3 we will give two different constructions of nonreal fields, each giving a positive answer to the Gross Question.
All our constructions will be based on Merkurjev's method where one starts with an arbitrary field and then uses iterated extensions obtained by composing function fields of quadrics to produce an extension with the desired properties. Our first construction will show the following:

1.2 Theorem I. *Let $F$ be a field of characteristic different from 2. There exists a field extension $K/F$ with the following properties:*

*(i) $K$ has no finite extensions of odd degree.*

*(ii) For any binary quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic quadratic forms over $K$ that contain $\beta$.*

*(iii) For any $k \in \mathbb{N}$, there is an anisotropic $k$-fold Pfister form over $K$.*

*In particular, $K$ is a perfect, nonreal field of infinite $u$-invariant, $I^kK \neq 0$ for all $k \in \mathbb{N}$, and any infinite-dimensional quadratic space over $K$ is isotropic.*

Here and in the sequel, $I^kF$ stands for the $k^{th}$ power of $IF$, the fundamental ideal consisting of classes of even-dimensional forms in the Witt ring $WF$ of $F$. The proof of this theorem only uses some basic properties of Pfister forms and standard techniques from the theory of function fields of quadratic forms. Varying this construction and using this time products of quaternion algebras and Merkurjev's index reduction criterion (see [24] or [38], Théorème 1), we will then show the following:

1.3 Theorem II. *Let $F$ be a field of characteristic different from 2. There exists a field extension $K/F$ with the following properties:*

*(i) $K$ has no finite extensions of odd degree and $I^3K = 0$.*

*(ii) For any binary quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic quadratic forms over $K$ that contain $\beta$.*

*(iii) For any $k \in \mathbb{N}$, there is a central division algebra over $K$ that is decomposable into a tensor product of $k$ quaternion algebras.*

*In particular, $K$ is a nonreal field of infinite u-invariant, and any infinite-dimensional quadratic space over $K$ is isotropic. Furthermore, $K$ is perfect and of cohomological dimension 2.*

In Section 4, we will show two analogous theorems for real fields.

1.4 Theorem III. *Assume that $F$ is real. Then there exists a field extension $K/F$ with the following properties:*

- *(i) $K$ has a unique ordering.*
- *(ii) $K$ has no finite extensions of odd degree and $I^3K$ is torsion free.*
- *(iii) For any totally indefinite quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic quadratic forms over $K$ that contain $\beta$.*
- *(iv) For any $k \in \mathbb{N}$, there is a central division algebra over $K$ that is decomposable into a tensor product of $k$ quaternion algebras.*

*In particular, $K$ is a real field of infinite u-invariant, and any totally indefinite quadratic space of infinite dimension over $K$ is isotropic; moreover, the cohomological dimension of $K(\sqrt{-1})$ is 2.*

While this can be seen as a counterpart to Theorem II for real fields, we can also prove an analogue of Theorem I in this situation.

1.5 Theorem IV. *Assume that $F$ is real. Then there exists a field extension $K/F$ with the following properties:*

- *(i) $K$ has a unique ordering.*
- *(ii) $K$ has no finite extensions of odd degree.*
- *(iii) For any totally indefinite quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic quadratic forms over $K$ that contain $\beta$.*
- *(iv) for any $k \in \mathbb{N}$, there is an element $a \in K^\times$ which is a sum of squares in $K$, but not a sum of $k$ squares.*

*In particular, $K$ is a real field for which the Pythagoras number, the Hasse number, and the u-invariant are all infinite, the torsion part of $I^kK$ is nonzero for all $k \in \mathbb{N}$, and any totally indefinite quadratic space of infinite dimension over $K$ is isotropic.*

In Section 5, we will discuss the Gross Question for quadratic, nonsingular quadratic, and symmetric bilinear forms in characteristic 2. As already mentioned, for nonsingular quadratic forms, we obtain similar results as in characteristic different from 2, whereas for arbitrary quadratic forms and for symmetric bilinear forms the answer turns out to be negative.

In the final Section 6, we discuss an abstract version of the Gross Question, formulated for an arbitrary monoïd together with two subsets satisfying some requirements. We give examples of such monoïds whose elements are well known objects associated to an arbitrary field, such as central simple algebras or symbols in Milnor $K$-theory modulo a prime $p$. In some of the cases that we shall discuss, the answer to (the analogue of) the Gross Question will be positive, in others it will be negative.

For all prerequisites from quadratic form theory in characteristic different from 2 needed in the sequel, we refer to the books of Lam and Scharlau (see [20], [21] and [34]). In general, we use the standard notations introduced there. However, we use a different sign convention for Pfister forms: Given $a_1, \dots, a_r \in F^\times$, we write $\langle\langle a_1, \dots, a_r\rangle\rangle$ for the *r-fold Pfister form* $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_r\rangle$. If $\varphi$ is a quadratic form over $F$ and $n \in \mathbb{N}$, we denote by $n \times \varphi$ the $n$-fold orthogonal sum $\varphi \perp \cdots \perp \varphi$.

A quadratic space $(V, q)$ is said to be *nonsingular* if the radical

$$\mathrm{Rad}(V, q) = \{x \in V \mid b_q(x, y) = 0 \text{ for all } y \in V\}$$

is reduced to 0. Anisotropic quadratic spaces in characteristic different from 2 are obviously always nonsingular, but this need not be so in characteristic 2.

Given two quadratic spaces (resp. forms) $\varphi$ and $\psi$ over $F$. We say that $\psi$ is a *subspace* (resp. *subform*) of $\varphi$ if $\psi$ is isometric to the restriction of $\varphi$ to some subspace of the underlying vector space of $\varphi$. We write $\psi \subset \varphi$ if there exists a quadratic space $\tau$ over $F$ such that $\varphi \cong \psi \perp \tau$. If $\varphi$, $\psi$ are quadratic forms over $F$ with $\psi$ nonsingular, then $\psi \subset \varphi$ if and only if $\psi$ is a subform of $\varphi$.

Unless stated otherwise, the terms 'form' or 'quadratic form' will always stand for 'nonsingular quadratic form'. A *binary form* is a 2-dimensional quadratic form.

We recall the definition of the function field $F(\varphi)$ associated to a nonsingular quadratic form $\varphi$ over $F$ in characteristic different from 2. If $\dim(\varphi) \geq 3$ or if $\dim(\varphi) = 2$ and $\varphi$ is anisotropic, then $F(\varphi)$ is the function field of the projective quadric given by the equation $\varphi = 0$. We put $F(\varphi) = F$ if $\varphi$ is the hyperbolic plane or if $\dim(\varphi) \leq 1$. We refer to [34], Chapter 4, §5, or [21], Chapter X, for the crucial properties of function field extensions. They will play a prominent rôle in all our constructions.

Let $K/F$ be an arbitrary field extension. If $\varphi$ is a quadratic form over $F$, then we denote by $\varphi_K$ the quadratic form over $K$ obtained by scalar extension from $F$ to $K$. Similarly, given an $F$-algebra $A$, we write $A_K$ for the $K$-algebra $A \otimes_F K$. Central simple algebras are by definition finite-dimensional. A central simple algebra without zero-divisors will be called a 'division algebra' for short. For the basics about central simple algebras and the Brauer group of a field, the reader is referred to [34], Chapter 8, or [31], Chapters 12-13.

## 2 The derived $u$-invariant

In this section, all fields are assumed to be of characteristic different from 2.

The question about the existence of an anisotropic infinite-dimensional quadratic space over the field $F$ can be rephrased within the framework of finite-dimensional quadratic form theory, as we shall see now.
We call a sequence of quadratic forms $(\varphi_n)_{n\in\mathbb{N}}$ over $F$ a *chain of quadratic forms over* $F$ if, for any $n \in \mathbb{N}$, we have $\dim(\varphi_n) = n$ and $\varphi_n \subset \varphi_{n+1}$. Given such a chain $(\varphi_n)_{n\in\mathbb{N}}$ over $F$, the direct limit over the quadratic spaces $\varphi_n$ with the appropriate inclusions has itself a natural structure of a nonsingular quadratic space over $F$ of dimension $\aleph_0$ (countably infinite). We denote this quadratic space over $F$ by $\lim_{n\in\mathbb{N}}(\varphi_n)$ and observe that it is anisotropic if and only if $\varphi_n$ is anisotropic for all $n \in \mathbb{N}$. Moreover, any infinite-dimensional nonsingular quadratic space over $F$ contains a subspace isometric to the direct limit $\lim_{n\in\mathbb{N}}(\varphi_n)$ for some chain $(\varphi_n)_{n\in\mathbb{N}}$ and we thus get:

2.1 Proposition. *There exists an anisotropic quadratic space of infinite dimension over $F$ if and only if there exists a chain of anisotropic quadratic forms $(\varphi_n)_{n\in\mathbb{N}}$ over $F$.*

Recall that a form $\varphi$ is *torsion* if its Witt class is a torsion element in the Witt ring $WF$. In [9], Elman and Lam defined the *u-invariant of $F$* as

$$u(F) \;=\; \sup\{\dim(\varphi) \mid \varphi \text{ is an anisotropic torsion form over } F\}\,.$$

It is well known that if $F$ is nonreal, then any form over $F$ is torsion, in which case the above supremum is actually taken over all anisotropic forms over $F$. If $F$ is real, then Pfister's Local-Global Principle says that torsion forms are exactly those forms that have signature zero with respect to each ordering of $F$ (i.e. that are hyperbolic over each real closure of $F$). In the remainder of this section, we are mainly concerned with nonreal fields.

It will be convenient to consider also the following *relative u-invariants*. Given an anisotropic quadratic form $\varphi$ over $F$, we define

$$u(\varphi, F) \;=\; \sup\{\dim(\psi) \mid \psi \text{ anisotropic form over } F \text{ with } \varphi \subset \psi\}\,.$$

Note that, trivially, $\dim(\varphi) \le u(\varphi, F)$. If $F$ is nonreal, we further have that $u(\varphi, F) \le u(F)$ with equality if $\dim\varphi = 1$. Moreover, if $\varphi_1$ and $\varphi_2$ are anisotropic forms over $F$ such that $\varphi_1 \subset \varphi_2$, then $u(\varphi_1, F) \ge u(\varphi_2, F)$.
We introduce now the *derived u-invariant of $F$* as

$$u'(F) \;=\; \sup\{\dim(\varphi) \mid \varphi \text{ anisotropic form over } F \text{ with } u(\varphi, F) = \infty\}.$$

Whenever there exists an anisotropic form $\varphi$ over $F$ with $u(\varphi, F) = \infty$, we have $u'(F) > 0$; if no such forms exist, we put $u'(F) = \sup\emptyset = 0$.

2.2 Proposition. *If there exists an infinite-dimensional quadratic space over $F$, then $u'(F) = \infty$.*

*Proof.* Assume that there exists an anisotropic infinite-dimensional quadratic space over $F$. Then there is also a chain $(\varphi_n)_{n\in\mathbb{N}}$ of anisotropic forms over $F$. Obviously, $u(\varphi_n, F) = \infty$ for any $n \in \mathbb{N}$, and therefore $u'(F) = \infty$. □

In particular, the proposition shows that $u'(F) = \infty$ if $F$ is a real field. Certainly, one could modify the definition of $u'$ to make this invariant more interesting for real fields, but we will not pursue this matter here.

2.3 Proposition. *Assume that $F$ is nonreal. Then $u(F)$ is finite if and only if $u'(F) = 0$.*

*Proof.* If $u(F) = \infty$, then $u(\langle 1\rangle, F) = u(F) = \infty$ and thus $u'(F) \geq 1$. On the other hand, if $u(F) < \infty$, then there is no anisotropic form $\varphi$ over $F$ such that $u(\varphi, F) = \infty$, and therefore $u'(F) = 0$. □

By the previous two propositions, any nonreal field $F$ with $0 < u'(F) < \infty$ will yield an example that answers the Gross Question in the positive. Now Theorem I and Theorem II each say that nonreal fields $K$ with $u'(K) = 1$ do exist.

2.4 Lemma. *For the field $F(\!(t)\!)$ of Laurent series in the variable $t$ over $F$, one has*

$$u'(F(\!(t)\!)) \;=\; 2\,u'(F)\,.$$

The proof of this lemma is straightforward and based on the well known relationship between quadratic forms over $F$ and over $F(\!(t)\!)$ (see [20], Chapter VI, Proposition 1.9). Details are left to the reader.

2.5 Corollary. *Let $m \in \mathbb{N}$. Then there exists a nonreal field $L$ such that $u'(L) = 2^m$. Moreover, $L$ can be constructed such that in addition $I^{m+3}L = 0$, or $I^rL \neq 0$ for all $r \in \mathbb{N}$, respectively.*

*Proof.* Theorem I and Theorem II, respectively, assert the existence of such fields for $m = 0$. The induction step from $m$ to $m+1$ is clear from the preceding lemma. □

This raises the following question.

2.6 Question. Does there exist a nonreal field $F$ with $u'(F) = \infty$ such that every infinite-dimensional quadratic space over $F$ is isotropic?

## 3 Nonreal fields with infinite $u$-invariant

We are going to give a construction, in several variants, which allows us to prove the theorems formulated in the introduction. The proof that the field obtained by this construction has infinite $u$-invariant will be based on known facts about the preservation of properties such as anisotropy of a fixed quadratic form, or absence of zero-divisors in a central simple algebra, under certain types of field extensions.

First, we consider a finite field extension $K/F$ of odd degree. Springer's Theorem (see [20], Chapter VII, Theorem 2.3) says that any anisotropic quadratic form over $F$ stays anisotropic after scalar extension from $F$ to $K$.

Springer's Theorem has an analogue in the theory of central simple algebras. It says that if $D$ is a (central) division algebra over $F$ with exponent equal to a power of 2 and if $K/F$ is a finite field extension of odd degree, then the $K$-algebra $D_K = D \otimes_F K$ is also a division algebra (see [31], Section 13.4, Proposition (vi)).
Both statements also hold in characteristic 2. One can immediately generalise them to 'odd' algebraic extensions that are not necessarily finite.
An algebraic extension $L/F$ is called an *odd closure of* $F$ if $L$ is $F$-isomorphic to $M^G$, where $M$ is an algebraic (resp. separable) closure of $F$ if $\mathrm{char}(F) \neq 2$ (resp. $\mathrm{char}(F) = 2$), and $G$ is a 2-Sylow subgroup of the Galois group of $M/F$. Then $L$ itself has no odd degree extension and all finite subextensions of $F$ inside $L$ are of odd degree. In particular, $L$ is perfect if $\mathrm{char}(F) \neq 2$. We call a field extension $K/F$ an *odd extension* if it can be embedded into an odd closure of $F$. In this case, $K/F$ is algebraic, thus equal to the direct limit of its finite subextensions, which are all of odd degree.
We thus get immediately the following (where we do not make any assumption on the characteristic).

3.1 Lemma. *Let $K/F$ be an odd extension.*

*(i) Any anisotropic quadratic form over $F$ stays anisotropic over $K$.*

*(ii) Any central division algebra of exponent 2 over $F$ remains a division algebra over $K$.*

For the remainder of this section, all fields are assumed to be of characteristic different from 2.
We now consider extensions of the type $F(\varphi)/F$, where $F(\varphi)$ is the function field of a quadratic form $\varphi$ over $F$.

3.2 Lemma. *Let $\pi$ be an anisotropic Pfister form over $F$ and $\varphi$ a form over $F$ with $\dim(\varphi) > \dim(\pi)$. Then $\pi$ stays anisotropic over $F(\varphi)$.*

*Proof.* By the assumption on the dimensions, $\varphi$ is certainly not similar to any subform of $\pi$. Therefore, by [34], Theorem 4.5.4 (ii), $\pi_{F(\varphi)}$ is not hyperbolic. Hence $\pi_{F(\varphi)}$ is anisotropic as it is a Pfister form (see [34], Lemma 2.10.4). $\square$

3.3 Remark. The statement of the last lemma is actually a special case of a more general phenomenon. Let $\varphi$ and $\pi$ be anisotropic forms over $F$ such that, for some $n \in \mathbb{N}$, one has $\dim(\pi) \leq 2^n < \dim(\varphi)$. Then $\pi$ stays anisotropic over $F(\varphi)$ (see [14]). In the particular situation where $\pi$ is an $n$-fold Pfister form, we immediately recover (3.2).

The next statement was the key in Merkurjev's construction of fields of arbitrary even $u$-invariant (see [24]). It is readily derived from [38], Théorème 1.

3.4 Theorem (Merkurjev). *Let $D$ be a division algebra over $F$ of exponent 2 and degree $2^m$, where $m > 0$. Let $\varphi$ be a quadratic form over $F$ such that $\dim(\varphi) > 2m + 2$ or $\varphi \in I^3F$. Then $D_{F(\varphi)}$ is a division algebra.*

3.5 Remark. It is also well known that if $K/F$ is a purely transcendental extension, then anisotropic forms (resp. division algebras) over $F$ stay anisotropic (resp. division) over $K$. We will use this fact repeatedly, especially when $K = F(\varphi)$ is the function field of an *isotropic* quadratic form $\varphi$ over $F$, in which case $F(\varphi)/F$ is purely transcendental of transcendence degree $\dim(\varphi) - 2$ (see [34], 4.5.2 (vi)).

3.6 Proof of Theorem I.

Recall that $F$ is an arbitrary field of characteristic different from 2. We define recursively a tower of fields $(F_n)_{n\in\mathbb{N}}$, starting with $F_0 = F$. Suppose that for a certain $n \geq 1$ the field $F_{n-1}$ has already been defined. Let $F^{\#}_{n-1}$ be an odd closure of $F_{n-1}$ and let

$$F^{(n)}_{n-1} = F^{\#}_{n-1}(X^{(n)}_1, \ldots, X^{(n)}_n)$$

where $X^{(n)}_1, \ldots, X^{(n)}_n$ are indeterminates over $F^{\#}_{n-1}$. We define $F_n$ as the free compositum[1] of all function fields $F^{(n)}_{n-1}(\varphi)$ where $\varphi$ ranges over all anisotropic forms defined over $F_{n-1}$ such that, for some $j < n$, $\dim(\varphi) = 2^j + 1$ and $\varphi$ contains a binary form defined over $F_j$.
Let $K$ be the direct limit of the tower of fields $(F_n)_{n\in\mathbb{N}}$. We are going to show that the field $K$ has the following properties:

(i) $K$ has no finite extensions of odd degree.

(ii) For any binary quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic quadratic forms over $K$ that contain $\beta$.

(iii) For any $k \in \mathbb{N}$, there is an anisotropic $k$-fold Pfister form over $K$.

Once these are established the remaining claims in Theorem I will follow. Indeed, (ii) implies that every infinite-dimensional quadratic space over $K$ is isotropic and that $K$ is nonreal, whereas (iii) implies that $u(K) = \infty$ and that $I^k K \neq 0$ for all $k \in \mathbb{N}$. Finally, since $\mathrm{char}(K) = \mathrm{char}(F) \neq 2$, it follows from (i) that $K$ is perfect.

(i) Consider an irreducible polynomial $f$ over $K$ of odd degree. Then $f$ is defined over $F_n$ for some $n \in \mathbb{N}$. Since $K$ contains $F_{n+1}$ which in turn contains an odd closure of $F_n$, it follows that $f$ has degree one. This shows that $K$ is equal to its odd closure.
(ii) Consider an anisotropic binary form $\beta$ over $K$. There is some $j \in \mathbb{N}$ such that $\beta$ is defined over $F_j$. Let $\varphi$ be a form of dimension $2^j + 1$ over $K$ containing $\beta$. Let $n > j$ be an integer such that $\varphi$ is defined over $F_{n-1}$. Then by construction, $F_n$ contains $F^{(n)}_{n-1}(\varphi)$ and $\varphi$ is therefore isotropic over $F_n$ and

[1] See [21], p. 333, for a precise description of the notion of 'free compositum' of a family of function fields of quadratic forms.

thus over $K$. This shows that $u(\beta, K) \leq 2^j$. Here, $j$ depends on the binary form $\beta$, but in any case we have that $u(\beta, K)$ is finite, proving (ii).
(iii) Given positive integers $n$ and $j$, we write $F_{n,j}$ for the compositum of $F_n$ with the algebraic closure of $F_j$ inside a fixed algebraic closure of $K$. Similarly, we write $F^{\#}_{n-1,j}$ and $F^{(n)}_{n-1,j}$, for the compositum of $F^{\#}_{n-1}$, $F^{(n)}_{n-1}$, respectively, with the algebraic closure of $F_j$.

$$\begin{tikzcd}
 & F_{n,j} \arrow[d, dash] \\
F_n \arrow[ur, dash] \arrow[d, dash, "\textit{compositum}"'] & F^{(n)}_{n-1,j} \arrow[d, dash] \\
F^{(n)}_{n-1} \arrow[ur, dash] \arrow[d, dash, "\textit{purely transc.}"'] & F^{\#}_{n-1,j} \arrow[d, dash] \\
F^{\#}_{n-1} \arrow[ur, dash] \arrow[d, dash, "\textit{odd}"'] & F_{n-1,j} \\
F_{n-1} \arrow[ur, dash] &
\end{tikzcd}$$

From now on, let $n > j$. Note that $F^{(n)}_{n-1,j} = F^{\#}_{n-1,j}(X^{(n)}_1, \ldots, X^{(n)}_n)$ is a purely transcendental extension of $F^{\#}_{n-1,j}$. Further, $F^{\#}_{n-1,j}$ is an odd extension of $F_{n-1,j}$. Using (3.1), it follows that every anisotropic form over $F_{n-1,j}$ stays anisotropic over $F^{\#}_{n-1,j}$, hence also over $F^{(n)}_{n-1,j}$. Moreover, $F_{n,j}$ is obtained from $F^{(n)}_{n-1,j}$ as a free compositum of certain function fields $F^{(n)}_{n-1,j}(\varphi)$ where $\varphi$ is a form defined over the subfield $F_{n-1}$ of $F^{(n)}_{n-1,j}$, either $\dim(\varphi) \geq 2^{j+1} + 1$, or $\dim(\varphi) = 2^\ell + 1$ with $\ell \leq j$ in which case $\varphi$ contains a binary subform defined over $F_\ell \subset F_j$. But in this latter case, $\varphi$ is isotropic over $F^{(n)}_{n-1,j}$ and thus $F^{(n)}_{n-1,j}(\varphi)/F^{(n)}_{n-1,j}$ is purely transcendental.
Consider now an anisotropic $m$-fold Pfister form $\pi$ defined over $F^{(n)}_{n-1,j}$, where $m \leq j+1$. Since, by the above, $F_{n,j}$ is obtained from $F^{(n)}_{n-1,j}$ as a compositum of function fields of forms of dimension at least $2^{j+1}+1$ and of purely transcendental extensions, (3.2) and (3.5) imply that $\pi$ stays anisotropic over $F_{n,j}$. But then $\pi$ stays anisotropic over $F^{(n+1)}_{n,j}$ as well. Repeating this, we see that $\pi$ stays anisotropic over all the fields $F_{m,j}$ for all $m \geq n$.
Let now $k$ be any positive integer. Let $\pi$ denote the $k$-fold Pfister form $\langle\langle X^{(k)}_1, \ldots, X^{(k)}_k \rangle\rangle$. This form is defined over $F^{(k)}_{k-1}$. Since $X^{(k)}_1, \ldots, X^{(k)}_k$ are algebraically independent over $F_{k-1}$, hence also over its algebraic closure

$F_{k-1,k-1} = F^{\#}_{k-1,k-1}$, we know that $\pi$ is still anisotropic when considered as a form over the field $F^{(k)}_{k-1,k-1} = F^{\#}_{k-1,k-1}(X_1^{(k)}, \dots, X_n^{(k)})$. Now the above argument shows that, for any $n \geq k$, the form $\pi$ is anisotropic over $F_{n,k-1}$ and, thus, over $F_n$. This implies that $\pi$ is anisotropic over $K$, the direct limit of the fields $F_n$.
Hence we showed that for any $k \in \mathbb{N}$, there exists an anisotropic $k$-fold Pfister form over $K$.

### 3.7 Proof of Theorem II.

Again, we define recursively a tower of fields $(F_n)_{n\in\mathbb{N}}$, starting with $F_0 = F$. Suppose that for a certain $n \geq 1$, the field $F_{n-1}$ is defined. As before, let $F^{\#}_{n-1}$ denote an odd closure of $F_{n-1}$. This time we define

$$F^{(n)}_{n-1} = F^{\#}_{n-1}(X_1^{(n)}, Y_1^{(n)}, \dots, X_n^{(n)}, Y_n^{(n)})$$

where $X_1^{(n)}, Y_1^{(n)}, \dots, X_n^{(n)}, Y_n^{(n)}$ are indeterminates over $F^{\#}_{n-1}$. Let $F_n$ denote the free compositum of the function fields $F^{(n)}_{n-1}(\varphi)$ where $\varphi$ is an anisotropic form over $F_{n-1}$ such that

- $\varphi$ is a 3-fold Pfister form, or
- $\dim(\varphi) = 2j + 3$ for some $j < n$ and $\varphi$ contains a binary subform defined over $F_j$.

Let $K$ be the direct limit of the tower of fields $(F_n)_{n\in\mathbb{N}}$. We want to show that $K$ has the following properties:

(i) $K$ has no finite extensions of odd degree and $I^3K = 0$.

(ii) For any binary quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic quadratic forms over $K$ which contain $\beta$.

(iii) For any $k \in \mathbb{N}$, there is a central division algebra over $K$ that is decomposable into a tensor product of $k$ quaternion algebras.

Note that (iii) implies that $u(K) = \infty$ (see [24] or [28], Lemma 1.1(d)), while (ii) prohibits the existence of infinite-dimensional anisotropic quadratic spaces over $K$. Now the field $K$ is perfect and nonreal by (i). Furthermore, (i) and (iii) together imply that the cohomological dimension of $K$ is exactly 2 (see [24]).

(i) As in the proof of Theorem I, we see that $K$ has no finite extensions of odd degree.
Let $\pi$ be an arbitrary 3-fold Pfister form over $K$. It is defined as a 3-fold Pfister form over $F_{n-1}$ for some $n \geq 1$. By the construction of the field $F_n$, $\pi$ becomes isotropic over $F_n$ and thus over $K$. Hence, every 3-fold Pfister form over $K$ is

isotropic and therefore hyperbolic. Since $I^3K$ is additively generated by the 3-fold Pfister forms over $K$ (see [34], p. 156), we conclude that $I^3K = 0$.
(ii) Let $\beta$ be an anisotropic binary form over $K$. There is an integer $j \in \mathbb{N}$ such that $\beta$ is defined over $F_j$. Let $\varphi$ be any form of dimension $2j+3$ over $K$ containing $\beta$. There is some integer $n > j$ such that $\varphi$ is defined over $F_{n-1}$. Since $F^{\#}_{n-1}(\varphi)$ is part of the compositum $F_n$, $\varphi$ becomes isotropic over $F_n$ and thus over $K$. Therefore $u(\beta, K) \leq 2j + 2$, establishing (ii).
(iii) For positive integers $n$ and $j$, we denote by $F_{n,j}$, $F^{\#}_{n-1,j}$, $F^{(n)}_{n-1,j}$ the composita of the fields $F_n$, $F^{\#}_{n-1}$, $F^{(n)}_{n-1}$, respectively, with the algebraic closure of $F_j$ inside a fixed algebraic closure of $K$.
Assume from now on that $n > j$. Similarly as in the proof of Theorem I, we have that $F^{(n)}_{n-1,j}$ is equal to $F^{\#}_{n-1,j}(X^{(n)}_1, Y^{(n)}_1, \ldots, X^{(n)}_n, Y^{(n)}_n)$, a purely transcendental extension of $F^{\#}_{n-1,j}$, which in turn is an odd extension of $F_{n-1,j}$. Using (3.1) and (3.5), it follows that every division algebra of exponent 2 over $F_{n-1,j}$ remains a division algebra after scalar extension to $F^{(n)}_{n-1,j}$.
Moreover, $F_{n,j}$ is obtained from $F^{(n)}_{n-1,j}$ as a free compositum of certain function fields $F^{(n)}_{n-1,j}(\varphi)$ where $\varphi$ is a form defined over $F^{(n)}_{n-1,j}$ which is either a 3-fold Pfister form, or which has dimension at least $2j+3$, or which contains a binary form defined over $F_j$ and thus is isotropic over $F^{(n)}_{n-1,j}$. Hence, by (3.4) and (3.5), any division algebra over $F^{(n)}_{n-1,j}$ of exponent 2 and of degree at most $2^j$ remains a division algebra after scalar extension to the field $F_{n,j}$.
Consider now a central simple algebra $D$ of exponent 2 and degree $2^j$ over $F^{(j)}_{j-1}$ for some $j \in \mathbb{N}$. Assume that for some $n > j$, the algebra $D$ will stay a division algebra after extending scalars to $F^{(n)}_{n-1,j}$. Combining the observations above, we see that $D$ also remains a division algebra when we extend scalars to $F_{n,j}$, or even to $F^{(n+1)}_{n,j}$. Repeating this argument shows that $D$ will stay a division algebra after scalar extension to $F^{(N)}_{N-1,j}$ for any $N \geq n$.
Let now $k$ be a positive integer and let $D$ denote the tensor product of quaternion algebras $(X^{(k)}_1, Y^{(k)}_1) \otimes \cdots \otimes (X^{(k)}_k, Y^{(k)}_k)$ over the field $F^{(k)}_{k-1}$. This is a division algebra over $F^{(k)}_{k-1}$ of degree $2^k$ and of exponent 2. Since $X^{(k)}_1, Y^{(k)}_1, \ldots, X^{(k)}_k, Y^{(k)}_k$ are algebraically independent over the field $F_{k-1}$, hence also over its algebraic closure $F_{k-1,k-1} = F^{\#}_{k-1,k-1}$, it follows that $D_{F^{(k)}_{k-1,k-1}}$ is a division algebra over the field $F^{(k)}_{k-1,k-1}$. Now the argument above applies, showing that $D_{F_{n,k-1}}$ is a division algebra over $F_{n,k-1}$ for any $n \geq k$. But then $D_{F_n}$ is a division algebra for any $n \geq k$, implying that the tensor product of $k$ quaternion algebras $D_K$ is a division algebra over $K$.

3.8 Remark. At first glance, it may seem that the fields $K$ constructed in the proofs of the theorems are horrendously big. However, a closer inspection of the proofs reveals that if the field $F$ we start with is infinite, the field $K$ obtained by the construction will have the same cardinality as $F$. For example, if we start with $F = \mathbb{Q}$, then the field $K$ will be countable and thus can be embedded into $\mathbb{C}$.

## 4 Real fields and totally indefinite spaces

In our answer to the Gross Question, we had to construct a field $F$ which in particular has the property that all infinite-dimensional quadratic spaces over $F$ are isotropic. A real such field cannot exist as mentioned previously. In fact, for a quadratic space $\varphi$ (of finite or infinite dimension) over a real field $F$ to be isotropic, a necessary condition is that $\varphi$ be *totally indefinite*, i.e. indefinite with respect to each ordering. To get a meaningful analogue to the Gross Question in the case of real fields, it is therefore reasonable to restrict our attention to quadratic spaces that are totally indefinite. We start this section with the definition of this notion and some general observations before proving the 'real' analogues to the constructions that answer the Gross Question.

Let $F$ be real and let $P$ be an ordering on $F$ with corresponding order relation $<_P$. A quadratic space $(V, q)$ over $F$ is said to be *indefinite at* $P$, if there exist elements $v_1, v_2 \in V$ such that $q(v_1) <_P 0 <_P q(v_2)$. If $(V, q)$ is indefinite at every ordering of $F$, then we say that $(V, q)$ is *totally indefinite*. Note that this definition of (total) indefiniteness extends the common one for quadratic forms. (By definition, if $F$ is nonreal, every form over $F$ is totally indefinite.)
The *Hasse number* $\tilde{u}$ *of* $F$ is defined by

$$\tilde{u}(F) \ = \ \sup\{\dim(\varphi) \mid \varphi \text{ anisotropic, totally indefinite form over } F\}\,.$$

Since any nontrivial torsion form is obviously totally indefinite, one has $u(F) \leq \tilde{u}(F)$. On the other hand, there are examples of real fields $F$ where $u(F) < \infty$ while $\tilde{u}(F) = \infty$. For a survey on the possible pairs of values $(u(F), \tilde{u}(F))$, we refer to [15].
Recall that the *Pythagoras number* $p(F)$ of $F$ is the least integer $m \geq 1$ such that every sum of squares is a sum of $m$ squares in $F$ if such an $m$ exists, otherwise $p(F) = \infty$. It is well known and not difficult to see that if $p(F) = \infty$, then also $u(F) = \tilde{u}(F) = \infty$, and if $u(F) > 0$ then $p(F) \leq u(F)$.
The following observation is useful when dealing with infinite-dimensional totally indefinite quadratic spaces.

4.1 Proposition. *Every totally indefinite quadratic space over a real field $F$ contains a finite-dimensional, nonsingular, totally indefinite quadratic subspace.*

*Proof.* Let $(V, q)$ be a totally indefinite quadratic space over $F$. We may assume $(V, q)$ nonsingular. If $(V, q)$ is isotropic then it contains a hyperbolic plane which yields the desired subspace. Hence, we may assume that $(V, q)$ is anisotropic. In particular, any subspace of $(V, q)$ is nonsingular. After scaling we may furthermore assume that there exists a vector $v_0 \in V$ with $q(v_0) = 1$. Since

$(V,q)$ is totally indefinite, for each ordering $P$ there exists a vector $v_P \in V$ such that $q(v_P) <_P 0$.
Recall that the set of all orderings of $F$, denoted by $X_F$, is a compact topological space that has as a subbasis the clopen sets

$$H(a) = \{ P \in X_F \mid a \in P \}$$

(see [32], Theorem 6.5). We put $a_P = q(v_P)$ for every $P \in X_F$. Clearly, $P \in H(-a_P)$ and hence $X_F = \bigcup_{P\in X_F} H(-a_P)$. The compactness of $X_F$ thus yields that there are finitely many orderings $P_1, \dots, P_n \in X_F$ such that

$$X_F = H(-a_{P_1}) \cup \cdots \cup H(-a_{P_n}).$$

We put $v_i = v_{P_i}$ for $1 \le i \le n$. By the last equality, for each ordering $P$ of $F$ we have $q(v_i) <_P 0$ for at least one $i \in \{1, \dots, n\}$.
Let $W$ be the subspace of $V$ generated by the vectors $v_0, v_1, \dots, v_n$. Then it follows that $(W,q)$ is an anisotropic, finite-dimensional, totally indefinite subspace of $(V,q)$. $\square$

Recall that any ordering $P$ of $F$ can be extended to the odd closure of $F$ as well as to any purely transcendental extension of $F$. From [10], Theorem 3.5, Remark 3.6, we cite the following simple criterion for when an ordering can be extended to the function field of a given quadratic form.

4.2 Lemma. *Let $P$ be an ordering of $F$ and let $\{\varphi_i\}$ be any family of quadratic forms over $F$ of dimension at least 2. Then $P$ can be extended to the free compositum of the $F(\varphi_i)$ if and only if each $\varphi_i$ is indefinite at $P$.*

We are now going to modify the constructions presented in the last section and prove the remaining two theorems formulated in the introduction.

4.3 Proof of Theorem III.

This time, starting with the real field $F = F_0$ and any ordering $P_0$ on it, we construct a tower of fields with orderings $(F_n, P_n)_{n\in\mathbb{N}}$, where the ordering $P_{n+1}$ on $F_{n+1}$ extends the ordering $P_n$ on $F_n$ for all $n$. Suppose now that the pair $(F_{n-1}, P_{n-1})$ has been defined for a certain $n \ge 1$. Let $F^{\#}_{n-1}$ denote an odd closure of $F_{n-1}$ and let $P^{\#}_{n-1}$ be any ordering on $F^{\#}_{n-1}$ extending $P_{n-1}$. Let

$$F^{(n)}_{n-1} = F^{\#}_{n-1}(X^{(n)}_1, Y^{(n)}_1, \dots, X^{(n)}_n, Y^{(n)}_n)$$

where $X^{(n)}_1, Y^{(n)}_1, \dots, X^{(n)}_n, Y^{(n)}_n$ are indeterminates over $F^{\#}_{n-1}$. Let $P^{(n)}_{n-1}$ be any ordering on $F^{(n)}_{n-1}$ extending $P^{\#}_{n-1}$. Let now $F_n$ be the free compositum of the function fields $F^{(n)}_{n-1}(\varphi)$ where $\varphi$ is an anisotropic form over $F_{n-1}$ such that

- $\varphi$ is a 3-fold Pfister form and indefinite at $P_{n-1}$, or
- $\dim(\varphi) = 2j + 3$ for some $j < n$, and $\varphi$ contains a binary form defined over $F_j$ and indefinite at $P_j$.

Note that considered as forms over $F_{n-1}^{(n)}$ and by the construction of our orderings, all the above forms are in fact totally indefinite at $P_{n-1}^{(n)}$. By (4.2), the ordering $P_{n-1}^{(n)}$ extends to an ordering $P_n$ on $F_n$. In particular, $F_n$ is a real field.
Note that, for any 2-fold Pfister form $\rho$ over $F_{n-1}$ and any $a \in F_{n-1}$, at least one of the 3-fold Pfister forms $\rho \otimes \langle\langle a \rangle\rangle$ and $\rho \otimes \langle\langle -a \rangle\rangle$ is indefinite at $P_{n-1}$ and thus becomes hyperbolic over $F_n$ by the construction of this field.
Let $K$ be the direct limit of the tower of fields $(F_n)_{n\in\mathbb{N}}$. We will show that $K$ has the following properties:

(i) $K$ has a unique ordering which is given by $P = \bigcup_{n\in\mathbb{N}} P_n$.

(ii) $K$ has no finite extensions of odd degree and $I^3K$ is torsion free.

(iii) For any totally indefinite quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic quadratic forms over $K$ that contain $\beta$.

(iv) For any $k \in \mathbb{N}$, there is a central division algebra over $K$ that is decomposable into a tensor product of $k$ quaternion algebras.

Once these properties of $K$ are established, the remaining claims in Theorem III are immediate consequences:

- $K$ is a real field and by (iii) and (4.1), every infinite-dimensional anisotropic quadratic space over $K$ is definite with respect to the unique ordering.
- (i) implies that $K$ is *SAP* (see, e.g., [32], § 9, for the definition of and some facts about *SAP*), $I^3K$ is torsion free, and (iv) implies that the symbol length $\lambda(K)$ of $K$ is infinite. (Recall that the symbol length $\lambda(K)$ is the smallest $m \in \mathbb{N}$ such that each central simple algebra of exponent 2 over $K$ is Brauer equivalent to a tensor product of at most $m$ quaternion algebras provided such an integer exists, otherwise $\lambda(K) = \infty$.) It follows from [15], Theorem 1.5, that $u(K) = \infty$.
- (i) and (ii) yield that the cohomological dimension of $K(\sqrt{-1})$ is at most 2. (iv) then implies that it is exactly 2.

We now proceed to the proof of (i)–(iv).

(i) Since all the fields $F_n$ ($n \in \mathbb{N}$) are real, the same holds for $K$. It follows from what we observed during the construction above that, for any $a \in K^\times$, one of the forms $\langle\langle -1, -1, a \rangle\rangle$ and $\langle\langle -1, -1, -a \rangle\rangle$ is hyperbolic over $K$, which means that either $a$ or $-a$ is a sum of four squares in $K$. This shows that $K$ is uniquely ordered. It is clear that the unique ordering on $K$ is given by $\bigcup_{n\in\mathbb{N}} P_n$.
(ii) There is no change — compared to the previous constructions — in the argument that $K$ has no finite extensions of odd degree.

The torsion subgroup of $I^3K$ is generated by those 3-fold Pfister forms over $K$ that are torsion. Indeed, this is a general fact (see [2], Corollary 2.7) which, however, could be proven very easily in our particular situation where $K$ is uniquely ordered.
Let $\pi$ be any torsion 3-fold Pfister form over $K$. Then $\pi$ is defined as a 3-fold Pfister form over $F_{n-1}$ for some $n \geq 1$. Since the unique ordering on $K$ extends the ordering $P_{n-1}$ on $F_{n-1}$, it follows that $\pi$ (considered as 3-fold Pfister form over $F_{n-1}$) is indefinite at $P_{n-1}$. The construction of $F_n$ then yields that $\pi$ becomes isotropic and hence hyperbolic over $F_n$. Therefore, $\pi$ is hyperbolic over $K$. This shows that $I^3K$ is torsion free.
(iii) Since $K$ has a unique ordering, every (totally) indefinite form over $K$ contains an indefinite binary subform. Hence, (iii) needs only to be proven for binary indefinite forms $\beta$. The proof goes along the same lines as that of (ii) in Theorem II.
(iv) This part is identical to the corresponding part (iii) in the proof of Theorem II.

## 4.4 Proof of Theorem IV.

Again, starting with the real field $F = F_0$ and any ordering $P_0$ on it, we define a tower of ordered fields $(F_n, P_n)_{n\in\mathbb{N}}$ where the ordering $P_{n+1}$ on $F_{n+1}$ extends the ordering $P_n$ on $F_n$ for all $n$.
Suppose that for a certain $n \geq 1$ the pair $(F_{n-1}, P_{n-1})$ is already defined. Let $F^{\#}_{n-1}$ be an odd closure of $F_{n-1}$ and let $F^{(n)}_{n-1}$ be the rational function field $F^{\#}_{n-1}(X^{(n)})$. As before, $P_{n-1}$ extends to some ordering $P^{\#}_{n-1}$ of $F^{\#}_{n-1}$ which in turn extends to an ordering $P^{(n)}_{n-1}$ on $F^{(n)}_{n-1} = F^{\#}_{n-1}(X^{(n)})$ at which $X^{(n)}$ is positive.
We define $F_n$ to be the free compositum of all function fields $F^{(n)}_{n-1}(\varphi)$ where $\varphi$ is an anisotropic form defined over $F_{n-1}$ such that, for some $j < n$, we have $\dim(\varphi) = 2^j + 1$ and $\varphi$ contains an binary form which is defined over $F_j$ and indefinite at $P_j$. By (4.2), $P^{(n)}_{n-1}$ extends to an ordering $P_n$ of $F_n$.
Let $K$ be the direct limit of the tower $(F_n)_{n\in\mathbb{N}}$. We are going to establish the following properties:

(i) $K$ has a unique ordering which is given by $P = \bigcup_{n\in\mathbb{N}} P_n$.

(ii) $K$ has no finite extensions of odd degree.

(iii) For any totally indefinite quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic quadratic forms over $K$ which contain $\beta$.

(iv) for any $k \in \mathbb{N}$, there is an element $a \in K^\times$ which is a sum of squares in $K$, but not a sum of $k$ squares.

Note that (iv) implies that the Pythagoras number of $K$ is infinite, which in turn forces the Hasse number and the $u$-invariant of $K$ to be infinite as well. As before, (iii) implies that every infinite-dimensional anisotropic quadratic space over $K$ is definite with respect to the unique ordering of $K$.

(i) Since each $F_n$ is real, so is the direct limit $K$. Consider an arbitrary element $a \in K^\times$. Then $a \in F_n$ for some $n \in \mathbb{N}$. Now either $\langle 1, -a\rangle$ or $\langle 1, a\rangle$ is indefinite at $P_n$. Therefore, by construction, either $2^n \times \langle 1\rangle \perp \langle -a\rangle$ or $2^n \times \langle 1\rangle \perp \langle a\rangle$ becomes isotropic over the field $F_{n+1}$. Hence, $a$ or $-a$ is a sum of $2^n$ squares in $K$. It readily follows that $K$ has a unique ordering given by $\bigcup_{n\in\mathbb{N}} P_n$.
(ii) $K$ is equal to its odd closure, by the same arguments as before.
(iii) The argument here is the same as for (iii) in the last proof.
(iv) We denote by $F_{n-1,j}$, $F^{\#}_{n-1,j}$, and $F^{(n)}_{n-1,j}$, the composita of $F_{n-1}$, $F^{\#}_{n-1}$, and $F^{(n)}_{n-1}$, respectively, with the real closure of $F_j$ at the ordering $P_j$. Assume now that $n > j$. Then we observe as before that every anisotropic quadratic form defined over $F_{n-1,j}$ stays anisotropic over $F^{(n)}_{n-1,j}$. Note that $F_{n,j}$ is obtained from $F^{(n)}_{n-1,j}$ as a compositum of function fields $F^{(n)}_{n-1,j}(\varphi)$ where $\varphi$ is a form defined over $F^{(n)}_{n-1,j}$ which is either of dimension at least $2^{j+1} + 1$, or which contains a binary form defined over $F_j$ and indefinite at $P_j$ and which is therefore isotropic over $F^{(n)}_{n-1,j}$. As in part (iii) of the proof of Theorem I, we conclude that if $\pi$ is an anisotropic $m$-fold Pfister form over $F^{(n)}_{n-1,j}$ with $m \le j + 1$, then $\pi$ stays anisotropic over $F_{n,j}$.
Let now $k \in \mathbb{N}$. Then the $(k + 1)$-fold Pfister form $2^k \times \langle\langle X^{(k)}\rangle\rangle$ is defined over $F^{(k)}_{k-1}$ and is still anisotropic over $F^{(k)}_{k-1,k-1}$. It follows now from the above arguments that this form stays anisotropic over $F_{n,k-1}$, for all $n > k$. In particular, $2^k \times \langle\langle X^{(k)}\rangle\rangle$ is anisotropic over all fields $F_n$ for $n \ge k$, thus also over $K$. This shows that the element $X^{(k)}$ is not a sum of $2^k$ squares in $K$. On the other hand, by the construction we have $X^{(k)} \in P$, so that $X^{(k)}$ is a sum of squares in $K$, by (i).

## 5 Fields of characteristic 2

Throughout this section, all fields considered will be of characteristic 2. To translate the Gross Question into this setting, we have to take into account the different types of objects for which analogous problems might be formulated: quadratic, nonsingular quadratic, and symmetric bilinear spaces. We maintain the convention to use the term 'form(s)' for finite-dimensional spaces. For nonsingular quadratic forms we shall obtain analogues to Theorems I and II stated in the introduction, thus obtaining a positive answer to (the corresponding formulation of) the Gross Question in this case, too. On the other hand, for arbitrary quadratic forms as well as for symmetric bilinear forms, the corresponding answer turns out to be negative. In fact, this is relatively easy to prove, so we treat these types of forms first.

We refer the reader to [3], [30], or [16] for further details on notation, terminology and basic results on quadratic and bilinear forms in characteristic 2. Let $(V, q)$ be a quadratic space over a field $F$ of characteristic 2 and $b_q : V \times V \to F$ its associated bilinear form given by $b_q(x, y) = q(x + y) + q(x) + q(y)$. Recall that the *radical of* $(q, V)$ is the $F$-subspace

$$V^\perp = \mathrm{Rad}(q, V) = \{x \in V \mid b_q(x, y) = 0 \text{ for all } y \in V\}.$$

The quadratic space $(V, q)$ is said to be

- *nonsingular* if $V^\perp = 0$;
- *singular* if $V^\perp \neq 0$;
- *totally singular* if $V^\perp = V$.

If we write $V = V_0 \oplus V^\perp$ and we put $q_0 = q|_{V_0}$ and $q_{ts} = q|_{V^\perp}$, then $q \cong q_0 \perp q_{ts}$ with $q_0$ nonsingular and $q_{ts}$ totally singular. If we also have $q \cong \varphi_0 \perp \varphi_{ts}$ with $\varphi_0$ nonsingular and $\varphi_{ts}$ totally singular, then $q_{ts} \cong \varphi_{ts}$ (any isometry maps radicals bijectively to radicals), but $q_0$ and $\varphi_0$ might not be isometric. Note that $(V, q)$ is totally singular if and only if $q(x+y) = q(x)+q(y)$ for all $x, y \in V$. For $a, b \in F$, the 2-dimensional quadratic form $aX^2 + XY + bY^2$ is nonsingular, and we will denote it by $[a, b]$. The *hyperbolic plane* is then the form $\mathbb{H} = [0, 0] = XY$. For $a_1, \dots, a_s \in F$, the $s$-dimensional quadratic form $\sum_{i=1}^s a_i X_i^2$ is totally singular, and it will be denoted by $\langle a_1, \dots, a_s\rangle$.
Let now $q$ be a quadratic form over $F$ and let $n = \dim(q)$. Then there exist $r, s \in \mathbb{N}$ with $2r + s = n$ and $a_1, b_1, \dots, a_r, b_r, c_1, \dots, c_s \in F$ such that

$$q \cong [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \dots, c_s\rangle,$$

and we clearly have $q_{ts} \cong \langle c_1, \dots, c_s\rangle$. In particular, nonsingular quadratic forms are always of even dimension.

There are two versions of the $u$-invariant in characteristic 2, referring to the different types of quadratic forms, denoted by $u$ and $\widehat{u}$, respectively. They are defined as follows:

$$\begin{aligned} u(F) &= \sup\{\dim(q) \mid q \text{ anisotropic nonsingular quadratic form over } F\} \\ \widehat{u}(F) &= \sup\{\dim(q) \mid q \text{ anisotropic quadratic form over } F\} \end{aligned}$$

Clearly, we have $u(F) \le \widehat{u}(F)$, and $u(F)$ is always even if finite.
One could define corresponding $u$-invariants also for the classes of anisotropic symmetric bilinear forms, and of anisotropic totally singular quadratic forms, respectively, but (5.3) below will show that both suprema thus obtained coincide with $[F : F^2]$, the *degree of inseparability* of $F$.

We will now concentrate for a moment on totally singular quadratic spaces, a case that is very easy to treat.

For a field $F$ of characteristic 2 we fix an algebraic closure $\overline{F}$ and put $\sqrt{F} = \{x \in \overline{F} \mid x^2 \in F\}$. Note that $\sqrt{F}/F$ is a purely inseparable algebraic field extension of degree $[F : F^2]$. Hence the squaring map $\mathrm{sq} : x \mapsto x^2$ yields a quadratic map $\mathrm{sq}_F : \sqrt{F} \to F$ over $F$, and the quadratic space $(\sqrt{F}, \mathrm{sq}_F)$ is clearly of dimension $[F : F^2]$.

5.1 Proposition. *Let $F$ be a field of characteristic 2. The quadratic space $(\sqrt{F}, \mathrm{sq}_F)$ is anisotropic, totally singular, and of dimension $[F : F^2]$. Any anisotropic totally singular quadratic space over $F$ is isometric to a subspace of $(\sqrt{F}, \mathrm{sq}_F)$.*

*Proof.* The first part is obvious. Consider now a totally singular quadratic space $(V, q)$ over $F$ and assume that it is anisotropic. We define

$$\rho : V \longrightarrow \sqrt{F}\,,\ v \longmapsto \sqrt{q(v)}\,.$$

Since $q$ is totally singular, $\rho$ is $F$-linear and we have $sq_F \circ \rho = q$. Since furthermore $q$ is anisotropic, $\rho$ is injective and thus $(V, q)$ is isometric to the subspace $(\rho(V), \mathrm{sq}_F|_{\rho(V)})$ of $(\sqrt{F}, \mathrm{sq_F})$. □

We will now briefly look at symmetric bilinear spaces $(V, b)$ over a field $F$ of characteristic 2. A symmetric bilinear space $(V, b)$ is said to be *isotropic* if there exists $x \in V \setminus \{0\}$ such that $b(x, x) = 0$, *anisotropic* otherwise. In other words, $(V, b)$ is anisotropic if and only if $(V, q_b)$ is so, where $q_b : V \to F$ is the induced quadratic map defined by $q_b(x) = b(x, x)$.

5.2 Lemma. *Let $F$ be a field of characteristic 2 and $V$ an $F$-vector space. There exists an anisotropic symmetric bilinear map $b : V \times V \to F$ if and only if there exists an anisotropic totally singular quadratic map $q : V \to F$.*

*Proof.* By definition, a symmetric bilinear map $b : V \times V \to F$ is anisotropic if and only if the associated totally singular quadratic map $q_b : V \to F$ is so. Now, given an anisotropic totally singular quadratic map $q : V \to F$, it is not difficult to construct a symmetric bilinear map $b : V \times V \to F$ such that $q = q_b$. In fact, picking some $F$-basis $(e_i)_{i \in I}$ of $V$, we can define $b$ by $b(e_i, e_j) = \delta_{ij} q(e_i)$ for $i, j \in I$. All this implies the claim. □

The previous two statements readily imply the following.

5.3 Corollary. *Let $F$ be a field of characteristic 2. Then $[F : F^2] = \infty$ if and only if there exist anisotropic totally singular quadratic spaces and anisotropic symmetric bilinear spaces of infinite dimension over $F$. Moreover, if $[F : F^2] < \infty$, then*

$$\begin{aligned} [F : F^2] &= \sup\{\dim(q) \mid q \textit{ anisotr. tot. singular quadratic form over } F\} \\ &= \sup\{\dim(b) \mid b \textit{ anisotr. symmetric bilinear form over } F\} \end{aligned}$$

We next consider general quadratic forms in characteristic 2 and the corresponding $\widehat{u}$-invariant. The second part of the following statement is [22], Corollary 1.

5.4 Proposition. *Let $F$ be a field of characteristic 2. Then $\widehat{u}(F) < \infty$ if and only if $[F : F^2] < \infty$, in which case*

$$[F : F^2] \leq \widehat{u}(F) \leq 2[F : F^2]\,.$$

*Proof.* If $[F : F^2] = \infty$ then the last corollary readily implies that $\widehat{u}(F) = \infty$. Now suppose $[F : F^2] < \infty$. Then $[F : F^2] \leq \widehat{u}(F)$ also follows from the corollary. To prove the second inequality, consider an anisotropic quadratic form $q$ over $F$, say,

$$q = [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1, \ldots, c_s\rangle\ , \quad a_1, b_1, \ldots, a_r, b_r, c_1, \ldots, c_s \in F.$$

Then the totally singular subform $\langle a_1, \ldots, a_r, c_1, \ldots, c_s\rangle$ is anisotropic as well, hence $r + s \leq [F : F^2]$ and thus $\dim(q) \leq 2\,[F : F^2]$. It follows that $\widehat{u}(F) \leq 2\,[F : F^2]$. □

So far we have shown in this section that the Gross Question (1.1) has actually a negative answer when it is reformulated for general quadratic forms, for totally singular quadratic forms, or for symmetric bilinear forms over a field of characteristic 2.

Let us now return to the case of nonsingular quadratic forms and spaces. To motivate the Gross Question (1.1), we first shall show that the existence of an infinite-dimensional anisotropic nonsingular quadratic space implies the existence of such spaces in every finite even dimension. Again, for quadratic forms $\varphi$ and $\psi$ over $F$ we write $\varphi \subset \psi$ if there exists a quadratic form $\tau$ such that $\psi \cong \varphi \perp \tau$. It is clear that if any two of the quadratic forms $\varphi$, $\psi$, $\tau$ are nonsingular, then so is the third.

We call a sequence of nonsingular quadratic forms $(\varphi_n)_{n\in\mathbb{N}}$ over $F$ a *chain of nonsingular quadratic forms over $F$* if, for any $n \in \mathbb{N}$, we have $\dim(\varphi_n) = 2n$ and $\varphi_n \subset \varphi_{n+1}$. Note that we need even dimension for nonsingularity. Given such a chain $(\varphi_n)_{n\in\mathbb{N}}$ over $F$, the direct limit over the quadratic spaces $\varphi_n$ with the appropriate inclusions is again a nonsingular quadratic space over $F$ of countably infinite dimension. We denote this quadratic space over $F$ by $\lim_{n\in\mathbb{N}}(\varphi_n)$ and observe that it is anisotropic if and only if $\varphi_n$ is anisotropic for all $n \in \mathbb{N}$.

5.5 Lemma. *Any infinite-dimensional nonsingular quadratic space over $F$ has a subspace isometric to $\lim_{n\in\mathbb{N}}(\varphi_n)$ for some chain $(\varphi_n)_{n\in\mathbb{N}}$ of nonsingular quadratic forms.*

*Proof.* Let $(V, q)$ be nonsingular with $\dim(V) = \infty$ and let $b = b_q$.

(i) Let $x \in V \setminus \{0\}$. The nonsingularity implies the existence of $y \in V$ such that $b(x, y) \neq 0$. Clearly, $x$ and $y$ are linearly independent as $b(x, x) = 0$. Let

$U_1 \subset V$ be the subspace spanned by $x$ and $y$. Let $\varphi_1 = q|_{U_1}$. One readily sees that $\varphi_1$ is nonsingular.
(ii) If $U \subset V$ is any *finite-dimensional* subspace with $q|_U$ nonsingular, then the usual argument shows that $V = U \oplus U^\perp$, where $U^\perp = \{v \in V \mid b(v, U) = 0\}$ (see, e.g., [34], Chapter 1, Lemma 3.4). Note that in this situation, the nonsingularity of $(q, V)$ implies that of $q|_{U^\perp}$.
Using (i) and (ii), the lemma follows immediately by induction. $\square$

As a direct consequence, we obtain the following:

5.6 Proposition. *There exists an anisotropic nonsingular quadratic space of infinite dimension over $F$ if and only if there exists a chain of anisotropic nonsingular quadratic forms $(\varphi_n)_{n\in\mathbb{N}}$ over $F$.*

Before we state the analogues of Theorems I and II in characteristic 2, we have to recall a few more definitions and facts.
Let $WF$ denote the Witt ring of nonsingular bilinear forms over $F$, and $W_qF$ the Witt group of nonsingular quadratic forms, which is in fact a $WF$-module. The fundamental ideal of classes of even-dimensional bilinear forms in $WF$ will be denoted by $IF$, and its $n^{th}$ power by $I^nF$. We put $I_q^nF = I^{n-1}F \cdot W_qF$. Then $I_q^nF$ is the submodule of $W_qF$ generated (as a group) by the *$n$-fold quadratic Pfister forms*

$$\langle\langle a_1, \dots, a_n]] = \langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_{n-1}\rangle_b \otimes [1, a_n],$$

with $a_1, \dots, a_{n-1} \in F^\times$ and $a_n \in F$; here, we denote a diagonal bilinear form with $c_1, \dots, c_m$ in the diagonal by $\langle c_1, \dots, c_m\rangle_b$.
Quadratic Pfister forms in characteristic 2 have properties quite analogous to those in characteristic different from 2. For example they are either anisotropic or hyperbolic (i.e. isometric to an orthogonal sum of hyperbolic planes).
Function fields of nonsingular quadratic forms are defined as in characteristic different from 2, again with the convention that $F(\mathbb{H}) = F$. If $q$ is a nonsingular quadratic form of dimension $2m > 0$, then $F(q)/F$ can be realized as a purely transcendental extension of $F$ of transcendence degree $2m - 2$ followed by a separable quadratic extension, and $F(q)/F$ is purely transcendental if and only if $q$ is isotropic.
Recall that (3.1) and (3.5) remain true in characteristic 2: anisotropic quadratic forms (resp. division algebras of exponent a 2-power) over $F$ stay anisotropic (resp. division) over any odd extension of $F$ and equally over any purely transcendental extension of $F$.
Also, (3.2) stays true in characteristic 2 for nonsingular forms: if $\pi$ is an anisotropic $n$-fold quadratic Pfister form and $q$ is any nonsingular form with $\dim(q) > 2^n$, then $\pi_{F(q)}$ is anisotropic. This follows simply by invoking the characteristic 2 analogues of the facts referred to in the proof of (3.2) (see, e.g. [16], Theorem 4.2(i), 4.4).
The characteristic 2 version of Theorem I reads as follows.

5.7 Theorem I(2). *Let $F$ be a field with* $\mathrm{char}(F) = 2$. *There exists a field extension $K/F$ with the following properties:*

*(i) $K$ has no finite extensions of odd degree.*

*(ii) For any binary nonsingular quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic nonsingular quadratic forms over $K$ that contain $\beta$.*

*(iii) For any $k \in \mathbb{N}$, there is an anisotropic $k$-fold quadratic Pfister form over $K$.*

*In particular, $K$ has infinite $u$-invariant, $I_q^k K \neq 0$ for all $k \in \mathbb{N}$, and any infinite-dimensional nonsingular quadratic space over $K$ is isotropic.*

Note that we cannot possibly expect $K$ to be perfect. Indeed, $u(F) = \infty$ implies $\widehat{u}(F) = \infty$ and thus $[K : K^2] = \infty$ by (5.4).
Using the above mentioned facts on nonsingular forms, quadratic Pfister forms and function fields of nonsingular forms, the proof of Theorem I now easily adapts to become a proof of Theorem I(2). Indeed, it suffices to add the adjective 'nonsingular' whenever a quadratic form is mentioned in the proof and to replace 'Pfister form' by 'quadratic Pfister form' (with the appropriate notation). Also, expressions of type $2^j + 1$ referring to the dimension of a form must be replaced by $2^j + 2$ as nonsingularity requires even dimension. We leave the details to the reader.
To treat the characteristic 2 version of Theorem II, we need a few more facts about quaternion algebras and their products over fields of characteristic 2.
A *quaternion algebra* $(a, b]_F$, with $a \in F^\times$ and $b \in F$, is a 4-dimensional central simple $F$-algebra generated by two elements $x, y$ subject to the relations $x^2 = a$, $y^2 + y = b$, $xy = (y + 1)x$.
We now list some relevant facts that allow us to carry over the proofs from characteristic different from 2 to characteristic 2.

5.8 Proposition. *Let $a_1 \dots, a_n \in F^\times$ and $b_1, \dots, b_n \in F$ be such that $A = (a_1, b_1]_F \otimes \cdots \otimes (a_n, b_n]_F$ is a division algebra. Then the following hold:*

*(i) The nonsingular $(2n + 2)$-dimensional quadratic form*

$$\varphi = [1, b_1 + \cdots + b_n] \perp a_1[1, b_1] \perp \cdots \perp a_n[1, b_n]$$

*is anisotropic.*

*(ii) For any field extension $K/F$ of one of the following types, the $K$-algebra $A_K = A \otimes_F K$ is a division algebra and $\varphi_K$ is anisotropic:*

- *$K/F$ is an odd extension;*
- *$K = F(q)$ where $q$ is a nonsingular quadratic form $q$ such that $\dim q \geq 2n + 4$ or $q \in I_q^3 F$;*

- *$K/F$ is purely transcendental.*

*Proof.* (i) This is [23], Proposition 6.
(ii) By Part (i) it suffices to prove in each case that $A_K$ is a division algebra. For a purely transcendental extension $K/F$ this is obvious, and for an odd extension it is also clear as the index of $A$ is a 2-power. In the case $K = F(q)$, this follows from [23], Theorems 3 and 4. $\square$

5.9 Corollary. *Suppose that for every $n \in \mathbb{N}$ there exist $a_1, \dots, a_n \in F^\times$ and $b_1, \dots, b_n \in F$ such that $(a_1, b_1]_F \otimes \cdots \otimes (a_n, b_n]_F$ is a division algebra. Then $u(F) = \infty$.*

The characteristic 2 version of Theorem II reads as follows.

5.10 Theorem II(2). *Let $F$ be a field with $\mathrm{char}(F) = 2$. There exists a field extension $K/F$ with the following properties:*

*(i) $K$ has no finite extensions of odd degree and $I_q^3 K = 0$.*

*(ii) For any binary nonsingular quadratic form $\beta$ over $K$, there is an upper bound on the dimensions of anisotropic nonsingular quadratic forms over $K$ that contain $\beta$.*

*(iii) For any $k \in \mathbb{N}$, there is a central division algebra over $K$ that is decomposable into a tensor product of $k$ quaternion algebras.*

*In particular, $K$ has infinite u-invariant and every infinite-dimensional nonsingular quadratic space over $K$ is isotropic.*

Using (5.8) and (5.9), it is now straightforward to obtain a proof of Theorem II(2) by applying the appropriate changes to the proof of Theorem II, in a similar fashion as was done in the case of Theorem I(2). This time, it is expressions of type $2j + 3$ in the proof of Theorem II which must be replaced by $2j + 4$ because of the nonsingularity of the forms considered. Again, we leave the details to the reader.

5.11 Remark. In Theorem II (where $\mathrm{char}(K) \neq 2$), the facts that $K$ has no odd degree extensions and that $I^3K = 0$ but $I^2K \neq 0$ together imply that $K$ has cohomological dimension $\mathrm{cd}(K) = 2$.
In Theorem II(2) (where $\mathrm{char}(K) = 2$) we have again that $K$ has no odd degree extension. This implies in particular that any finite separable extension $L/K$ also has this property, and therefore $H^1(L, \mu_p) = L^\times / L^{\times p}$ vanishes for every finite separable extension $L/K$ and every odd prime $p$. This implies that $\mathrm{cd}_p(K) = 0$ for the cohomological $p$-dimension of $K$ for any odd prime $p$ (see [37], II.1.2 and II.2.3).
On the other hand, $\mathrm{cd}_2(F) \leq 1$ holds for any field $F$ of characteristic 2 (see [37], II.2.2). In our case, there exist anisotropic nonsingular forms of dimension at least 2 over $K$, thus there certainly are separable quadratic

extensions over $K$. This readily implies that $\mathrm{cd}_2(K) = 1$ and therefore $\mathrm{cd}(K) = \sup\{\mathrm{cd}_p(K) \mid p \text{ prime}\} = 1$.
However, rather than considering $\mathrm{cd}_2(F)$ for a field $F$ with $\mathrm{char}(F) = 2$, it is perhaps more meaningful to ask for the *separable 2-dimension* $\dim_2^{\mathrm{sep}}(F)$ as defined by P. Gille [11]:

$$\dim_2^{\mathrm{sep}}(F) = \sup\{r \geq 0 \mid H_2^r(E) \neq 0 \text{ for some finite separable ext. } E/F\}\,,$$

where the $H_2^n(F)$ $(n \geq 0)$ are Kato's cohomology groups for a field $F$ with $\mathrm{char}(F) = 2$ (see, e.g., [19]).
In the situation of Theorem II(2), we have a field $K$ of characteristic 2 with no odd degree extension and $I_q^3K = 0$. By Kato's proof of the Milnor conjecture in characteristic 2 in [19], we have $H_2^3(K) = 0$. Furthermore, by Galois theory, if $L/K$ is a finite separable extension then $[L : K]$ is a 2-power and $L/K$ can be obtained as a tower of separable quadratic extensions. But for any field $F$ of characteristic 2 and any separable quadratic extension $E/F$, we have that $H_2^n(F) = 0$ implies $H_2^n(E) = 0$ (see, e.g., [4], 6.6). All this together implies that $H_2^3(L) = 0$ for every finite separable extension $L$ of $K$, therefore $\dim_2^{\mathrm{sep}}(K) = 2$ (note that $I_q^2K \neq 0$).

## 6 Analogues of the Gross Question

Let $(\mathcal{M}, *, \varepsilon)$ be a monoïd (associative semi-group) with neutral element $\varepsilon$. Let $\mathcal{A}$ and $\mathcal{S}$ be nonempty subsets of $\mathcal{M}$ with $\varepsilon \notin \mathcal{S} \subset \mathcal{A} \subset \mathcal{M}$. Denoting by $\langle \mathcal{S} \rangle$ the submonoïd of $M$ generated by $\mathcal{S}$, we furthermore assume that for any $a, b \in \langle \mathcal{S} \rangle$, if $a * b \in \mathcal{A}$ then $a, b \in \mathcal{A}$.
We now define a *U-invariant* for this triple $(\mathcal{M}, \mathcal{A}, \mathcal{S})$ by

$$U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}) \;=\; \sup\{m \in \mathbb{N} \mid \exists s_1, \ldots, s_m \in \mathcal{S} \text{ with } s_1 * \cdots * s_m \in \mathcal{A}\}\,.$$

These definitions have of course been motivated by our investigations of quadratic forms. More precisely, let $F$ be a field with $\mathrm{char}(F) \neq 2$. Then we take $\mathcal{M}$ to be the set of nonsingular quadratic forms (up to isometry) over $F$, the operation $*$ the orthogonal sum, $\varepsilon$ the trivial (0-dimensional) quadratic form, $\mathcal{A}$ the set of anisotropic forms over $F$, and $\mathcal{S}$ the set of 1-dimensional (nonzero) quadratic forms over $F$. In this setting, $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S})$ is nothing else but $u(F)$.
The Gross Question has now an obvious reformulation in this more abstract setting.

6.1 Question. Suppose that $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}) = \infty$. Does there exist a sequence $(s_n)_{n \in \mathbb{N}} \subset \mathcal{S}$ such that $s_1 * \cdots * s_n$ belongs to $\mathcal{A}$ for every $n \in \mathbb{N}$?

We proved that this does not always hold for anisotropy of quadratic forms over a field $F$. We will now pass from quadratic forms to other types of algebraic objects defined over a field that also naturally give rise to a triple $(\mathcal{M}, \mathcal{A}, \mathcal{S})$, and we will sketch answers to the above question in these new contexts.

Symbol algebras

Let $F$ be a field and $n \geq 2$ be an integer. We assume that char$(F)$ does not divide $n$, and that $F$ contains a primitive $n^{th}$ root of unity $\zeta$ which we fix. An $F$-algebra generated by two elements $x, y$ subject to the relations $x^n = a$, $y^n = b$, $xy = \zeta yx$, where $a, b \in F^\times$, is denoted by $(a, b)_n$ and called an *$n$-symbol algebra* over $F$. Note that $(a, b)_n$ is a central simple $F$-algebra of degree $n$. For $n = 2$, we recover the case of quaternion algebras. For basic properties of symbol algebras, we refer to [7], §11 (there, such algebras are called 'power norm residue algebras'). In the sequel, we will concentrate on the case where $n = p$ is a prime number.
With $F$ as above, let $\mathcal{M}$ be the set of isomorphism classes of central simple algebras over $F$. The tensor product $\otimes$, taken over $F$, endows $\mathcal{M}$ with a monoïd structure, where the neutral element is given by the class of $F$. Let $\mathcal{A} \subset \mathcal{M}$ be the subset of (finite dimensional) central division algebras over $F$. Further, let $\mathcal{S}_p \subset \mathcal{A}$ be the subset given by the non-split $p$-symbol algebras over $F$.
The Gross Question in this context now becomes the following:

6.2 Question. Suppose that $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}_p) = \infty$, i.e. suppose that to every $n \in \mathbb{N}$ there exist $p$-symbol algebras $Q_1, \ldots, Q_n$ such that $\bigotimes_{i=1}^n Q_i$ is a division algebra. Does there exist a sequence $(A_i)_{i \in \mathbb{N}}$ of $p$-symbol algebras $A_i$ over $F$ such that $\bigotimes_{i=1}^n A_i$ is a division algebra for all $n \in \mathbb{N}$?

Let us first consider the case $p = 2$. If we take $F = K$ to be the field constructed in the proof of Theorem II, then we have in fact shown there that $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}_2) = \infty$, while for any sequence $(A_n)_{n \in \mathbb{N}}$ of quaternion algebras over $K$, the product $A_1 \otimes \cdots \otimes A_n$ fails to be a division algebra for $n \in \mathbb{N}$ sufficiently large. Actually, these two facts do not only follow from the way in which $K$ was constructed, but already from the properties (i)–(iii). We omit the details.
Hence, for $p = 2$, the answer to (6.2) is negative in general. In the sequel, we will sketch how to obtain counterexamples for an arbitrary prime $p$. Our construction is to some extent similar to the one in the proof of Theorem II, but function fields of quadratic forms will now have to be replaced by function fields of generic partial splitting varieties, also called generalized Severi-Brauer (or Brauer-Severi) varieties, and the special case in (3.4) of Merkurjev's index reduction results for function fields of quadratic forms will have to be replaced by an appropriate version concerning index reduction for function fields of generic partial splitting varieties.
Such generic partial splitting varieties have been studied systematically perhaps for the first time by Heuser [13], and then later by Schofield and Van den Bergh [35], [36], and Blanchet [5]. Blanchet derives in particular an index reduction formula for central simple algebras over function fields of generic partial splitting varieties. This formula has been simplified by Wadsworth [39], and it is that simpler formula which we will use. The reader interested in the most general results on index reduction of central simple algebras over function fields of varieties is referred to the two papers by Merkurjev, Panin and Wadsworth [25], [26].

Let $A$ be a central simple algebra over $F$ of degree $n$, and let $s$ be a divisor of $n$. To $A$ we can now associate a generalized Severi-Brauer variety $X = SB(A,n,s)$ such that for any field exension $L/F$, the $L$-points $X(L)$ are the $sn$-dimensional right ideals in $A_L = A \otimes_F L$. If $L$ is a splitting field, so that $A \otimes_F L \cong \mathrm{End}_L(V)$ for an $n$-dimensional $L$-vector space $V$, then $X(L)$ is isomorphic to the Grassmannian $Gr(V,s)$ of $s$-dimensional subspaces of $V$.
The function field $F(X)$ has the property that $\mathrm{ind}(A_{F(X)})$ divides $s$, and it is *generic* for that property in the following sense: If $L$ is any field extension such that $\mathrm{ind}(A_L)$ divides $s$, then there exists an $F$-place $F(X) \longrightarrow L \cup \{\infty\}$ (see [13]). More precisely, we have the following (see [5], Proposition 3):

6.3 Lemma. *Let $A$, $n$, $s$, $X = SB(A,n,s)$ be as above and let $L/F$ be a field extension. Then the following statements are equivalent:*

*(i) $X$ has an $L$-rational point.*

*(ii) $\mathrm{ind}(A_L)$ divides $s$.*

*(iii) The free compositum $L \cdot F(X)$ is a purely transcendental extension of $L$.*

We now have the following index reduction formula for function fields of generic partial splitting varieties, see [39], Theorem 2:

6.4 Theorem. *Let $A$, $n$, $s$, $X = SB(A,n,s)$ be as above, let $K = F(X)$ and let $D$ be a central simple algebra over $F$. Then*

$$\mathrm{ind}(D_K) = \mathrm{gcd}\left\{ \frac{s}{\mathrm{gcd}(i,s)}\,\mathrm{ind}(D \otimes_F A^{-i}) \;\middle|\; 1 \le i \le n \right\}.$$

6.5 Corollary. *Let $p$ be a prime, let $D$ be a central division algebra of index $p^r$ over $F$, and let $A$ be a central simple algebra of degree $p^m$ over $F$ $(m \ge 1)$ and of exponent dividing $p$. Let $X = SB(A, p^m, p^{m-1})$. If $A$ is not a division algebra, or if $m > r$, then $D_{F(X)}$ is a division algebra.*

*Proof.* If $A$ is not a division algebra, then $\mathrm{ind}(A)$ divides $p^{m-1}$ and $F(X)/F$ is purely transcendental by (6.3). This clearly implies that $D$ will stay a division algebra over $F(X)$.
Now assume that $m > r$. We apply the above index reduction formula with $n = p^m$ and $s = p^{m-1}$. Let $i \in \{1,\dots,n\}$.
If $p \mid i$, then $A^{-i}$ is split, because $\exp(A)$ divides $p$, and it follows immediately that $\frac{s}{\mathrm{gcd}(i,s)}\,\mathrm{ind}(D \otimes_F A^{-i})$ is divisible by $\mathrm{ind}(D \otimes_F A^{-i}) = \mathrm{ind}(D)$. Furthermore, for $i = p^m$ we have $\frac{s}{\mathrm{gcd}(i,s)}\,\mathrm{ind}(D \otimes_F A^{-i}) = \mathrm{ind}(D)$.
If $p \nmid i$ then $\mathrm{gcd}(i,s) = 1$. Therefore,

$$\frac{s}{\mathrm{gcd}(i,s)}\,\mathrm{ind}(D \otimes_F A^{-i}) = p^{m-1} \cdot \mathrm{ind}(D \otimes_F A^{-i}),$$

and this number is divisible by $p^{m-1}$ and thus by $\mathrm{ind}(D) = p^r \le p^{m-1}$.

We conclude that

$$\operatorname{ind}(D_{F(X)}) = \gcd\left\{ \frac{s}{\gcd(i,s)} \operatorname{ind}(D \otimes_F A^{-i}) \;\middle|\; 1 \le i \le n \right\} = \operatorname{ind}(D)\ ,$$

in other words, $D$ stays a division algebra over $F(X)$. □

6.6 Theorem. *Let $p$ be a prime and let $F$ be a field with $\operatorname{char}(F) \neq p$. Then there exists a field extension $K/F$ containing a primitive $p^{th}$ root of unity $\zeta$ such that the following holds:*

*(i) Given $a_0 \in K^\times$. Then there exists an $\ell \in \mathbb{N}$ depending on $a_0$ such that for any $a_1, \dots, a_\ell, b_0, \dots, b_\ell \in K^\times$, the product $\bigotimes_{i=0}^{\ell}(a_i, b_i)_p$ is not a division algebra.*

*(ii) For every $n \in \mathbb{N}$ there exist $p$-symbol algebras $A_1, \dots, A_n$ over $K$ such that $\bigotimes_{i=1}^{n} A_i$ is a division algebra.*

*Proof.* Let $F_0 = F(\zeta)$ where $\zeta$ is a primitive $p^{th}$ root of unity in an algebraic closure of $F$. Let $n \ge 1$ and suppose we have have constructed $F_{n-1}$. Let now

$$F_{n-1}^{(n)} = F_{n-1}(\, X_1^{(n)}, Y_1^{(n)}, \dots, X_n^{(n)}, Y_n^{(n)})$$

where $X_1^{(n)}, Y_1^{(n)}, \dots, X_n^{(n)}, Y_n^{(n)}$ are indeterminates over $F_{n-1}$. Let $F_n$ denote the free compositum of function fields $F_{n-1}^{(n)}(SB(A, p^{j+1}, p^j))$ for all central simple algebras $A$ over $F_{n-1}$ of type $A \cong (a_0, b_0)_p \otimes (a_1, b_1)_p \otimes \cdots \otimes (a_j, b_j)_p$ with $j < n$ and $a_0 \in F_j^\times$ and $a_1, \dots, a_j, b_0, \dots, b_j \in F_{n-1}^\times$.
Finally, we define $K = \bigcup_{i=0}^{\infty} F_n$ and claim that $K$ has the desired properties.
(i) Let $a_0 \in K^\times$. Then there exists an integer $\ell > 0$ such that $a_0 \in F_\ell$. Let $a_1, \dots, a_\ell, b_0, \dots, b_\ell \in K^\times$ and consider $B = \bigotimes_{i=0}^{\ell}(a_i, b_i)_p$. It suffices to show that $B$ is not a division algebra over $K$. Now there exists $n > \ell$ such that $a_1, \dots, a_\ell, b_0, \dots, b_\ell \in F_{n-1}^\times$, so $B$ is defined over $F_{n-1}$, and since $F_{n-1}^{(n)}(SB(B, p^{\ell+1}, p^\ell))$ is part of the compositum $F_n$, we have that $\operatorname{ind}(B_{F_n})$ divides $p^\ell$, which implies that $B$ is not a division algebra over $F_n$ and thus also not over $K$.
(ii) For $n \ge 1$, consider over $F_n$ the algebra

$$C_n = (X_1^{(n)}, Y_1^{(n)})_p \otimes \cdots \otimes (X_n^{(n)}, Y_n^{(n)})_p\ .$$

It is well known that $C_n$ is a division algebra over $F_n$ (see, e.g., [25], Corollary 5.2). Part (ii) now follows if we can show that $C_n$ will stay a division algebra over $K$. This can be achieved by mimicking the argument in part (iii) of the proof of Theorem II, this time by invoking (6.3) and (6.5). We omit the details. □

Symbols in Milnor $K$-theory

Recall the definition of the Milnor $K$-groups $K_nF$ of a field $F$ (see [27]). By definition, $K_0F = \mathbb{Z}$, and $K_1F$ is the multiplicative group $F^\times$, written additively with the elements denoted by $\{a\}$, $a \in F^\times$, so that $\{ab\} = \{a\} + \{b\}$ for $a, b \in F^\times$. For $n \geq 2$, $K_nF$ is then defined to be the quotient of the tensor product $(K_1F)^{\otimes n}$ by the subgroup generated by all $\{a_1\} \otimes \cdots \otimes \{a_n\}$ satisfying $a_i + a_{i+1} = 1$ for some $i$. The image of an element $\{a_1\} \otimes \cdots \otimes \{a_n\}$ in the quotient group $K_nF$ is denoted by $\{a_1, \ldots, a_n\}$ and called a *symbol*. We then define the Milnor $K$-ring as the graded $\mathbb{Z}$-algebra $K_*F = \bigoplus_{n=0}^{\infty} K_nF$ with multiplication defined on symbols in the obvious way: $\{a_1, \ldots, a_n\} \cdot \{b_1, \ldots, b_m\} = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$.

We are interested in $K_nF/p$, the Milnor $K$-groups modulo $p$ for some prime $p$. The image of a symbol $\{a_1, \ldots, a_n\}$ in $K_nF/p$ will again be called a symbol and denoted in the same way.

For $p = 2$, these groups are linked to quadratic form theory through the Milnor Conjecture (now a theorem due to Orlov, Vishik, and Voevodsky [29]) which asserts that if $\mathrm{char}(F) \neq 2$ then $K_nF/2$ is isomorphic to $I^nF/I^{n+1}F$, via an isomorphism that maps $\{a_1, \ldots, a_n\}$ to the class of $\langle\langle a_1, \ldots, a_n\rangle\rangle$ modulo $I^{n+1}F$.

We now consider the abstract version of the Gross Question (6.1) in the following setting, where we assume $F^\times \neq F^{\times p}$ because otherwise $K_nF/p = 0$ for all $n \geq 1$. Let $\mathcal{M} = K_*F/p$, $\mathcal{S} = \{\{a\} \mid a \in F^\times \setminus F^{\times p}\}$ (this is nonempty by assumption), $\mathcal{A} = \{\{a_1, \ldots, a_n\} \neq 0 \mid n \in \mathbb{N},\ a_i \in F^\times\}$. It is obvious that for $n \geq 1$ we have $K_nF/p \neq 0$ if and only if there exist $a_1, \ldots, a_n \in F^\times$ with $\{a_1, \ldots, a_n\} \neq 0$. In this setting, Question (6.1) becomes:

6.7 Question. Suppose that $U_{\mathcal{M}}(\mathcal{A}, \mathcal{S}) = \infty$, i.e. $K_nF/p \neq 0$ for all $n \in \mathbb{N}$. Does there exist a sequence $(a_n)_{n\in\mathbb{N}} \subset F^\times$ such that $\{a_1, \ldots, a_n\} \neq 0$ for every $n \in \mathbb{N}$?

Let us first consider the case where $\mathrm{char}(F) = p$. Then the answer to the above question is positive by the following:

6.8 Proposition. *Let $F$ be a field of characteristic $p > 0$. Then the following are equivalent:*

(i) $[F : F^p] = \infty$.

(ii) $K_nF/p \neq 0$ *for all* $n \in \mathbb{N}$.

(iii) *There exists a sequence* $(a_n)_{n\in\mathbb{N}} \subset F^\times$ *such that* $\{a_1, \ldots, a_n\} \neq 0$ *for every* $n \in \mathbb{N}$.

*For $p = 2$, the above statements are further equivalent to any of the following:*

(iv) $\widehat{u}(F) = \infty$.

(v) $\sup\{\dim(b) \mid b$ *anisotropic symmetric bilinear form over* $F\} = \infty$.

(vi) *There exists an infinite-dimensional anisotropic quadratic space over $F$.*

(vii) *There exists an infinite-dimensional anisotropic symmetric bilinear space over $F$.*

*Proof.* Recall that a subset $T \subset F$ is called *p-independent* if, for any finite subset $\{a_1, \ldots, a_n\} \subset T$, one has $[F^p(a_1, \ldots, a_n) : F^p] = p^n$, and that $T \subset F$ is called a *p-basis* of $F$ if $T$ is a minimal generating set of the extension $F/F^p$, i.e. $F = F^p(T)$ and $T$ is $p$-independent.
The key observation here is the fact that for $a_1, \ldots, a_n \in F^\times$ we have that $\{a_1, \ldots, a_n\} \neq 0$ if and only if $a_1, \ldots, a_n$ are $p$-independent, in other words $[F^p(a_1, \ldots, a_n) : F^p] = p^n$. This is an immediate consequence of the Bloch-Kato-Gabber Theorem (see [6], Theorem 2.1, or [17], Appendix A2). The equivalence of the first three statements is now immediate and we leave the details to the reader.
For $p = 2$ it readily follows from (5.3) and (5.4) that (i) is equivalent to any of the statements (iv) to (vii). □

Let us now turn to the case $\mathrm{char}(F) \neq p$. For $p = 2$, the answer to the above question will be negative in general, i.e. there are fields such that $K_nF/2 \neq 0$ for all $n \in \mathbb{N}$, but for any sequence $(a_n)_{n\in\mathbb{N}} \subset F^\times$ one has $\{a_1, \cdots, a_m\} = 0$ for sufficiently large $m$.
Indeed, any field as constructed in Theorem I will do. To see this, it suffices to note that the map from the set of isometry classes of $n$-fold Pfister forms over $F$ into $K_nF/2$ given by $\langle\langle a_1, \ldots, a_n\rangle\rangle \mapsto \{a_1, \ldots, a_n\}$ is well-defined, injective, and sends the hyperbolic Pfister form to zero, see [8], Main Theorem 3.2 (here, we do not need the full thrust of the Milnor Conjecture). We leave the details to the reader.
Now if $p \neq 2$ (and $\mathrm{char}(F) \neq p$), we believe (but have not checked) that in general the answer to the above question should be negative as well. To construct counterexamples, it seems reasonable to try a similar approach as in our other constructions using a tower of iterated function fields. Candidates for these functions fields will naturally be function fields of (generic) splitting varieties of symbols in Milnor $K$-theory modulo $p$. The norm varieties as constructed by Rost (see [33], also [18]) provide examples for such splitting varieties.

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# Generic Observability of Dynamical Systems

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Abstract. We deal with a certain observation mapping defined by means of weighted measurments on a dynamical system and give necessary and sufficient conditions, under which this mapping is generically an injective immersion.

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## 1 Introduction

The observability problem of nonlinear dynamical systems has been an interesting subject and active field of research throughout the last decades. In the present work we consider time invariant systems of the form*

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x). \end{aligned}$$

The first equation describes a real dynamic process. Its state $x(t)$ at time $t$ is assumed to be element of a smooth second countable (hence paracompact) n-dimensional manifold $M$ called state space. The dynamics of the system is given by the vector field $f$ on $M$. The output function $h$ is a mapping from the state space into the reals and stands for a measuring device. The second equation describes the output, which contains partial information of the state. The output $y$ is the only measurable quantity. There is a very broad variety of systems, which can be described in this way. We call the triple $(M, f, h)$ or simply the pair $(f, h)$ a system. The system is called

*We use the customary abbreviations: Time $t$ denotes the natural coordinate on $\mathbb{R}$ and the first equation is identified with its local representative.

$C^r$ iff $M$, $f$ and $h$ are $C^r$. We denote the set of $C^r$ vector fields on $M$ by $\mathbb{X}^r(M)$.

In many applications it is of essential importance to know the state of the system at any time $t$. But measurement of the entire state (e.g. all its coordinates) is often impossible or very difficult. For instance because of high costs or technical reasons. In most cases one has only partial information from measurement of the output $y$, mathematically being described by the second equation. So the issue is to get the state (as well as distinguish different states) by using only output measurement*. If this is possible, then the system is said to be observable. Hence, the issue of the observability problem is the following. Find criteria on the system, such that by means of information from the output trajectory, it is possible to distinguish different states as well as to reconstruct the states. There is no uniform or canonical definition of observability in the literature. The weakest and most natural definition is the following.

Definition 1.1 The system $(f, h)$ is said to be observable (or distinguishable) if for each $(x, x') \in M \times M$ with $x \neq x'$ there exists a time $t_0 = t_0(x, x')$ such that $h(\Phi_{t_0}(x)) \neq h(\Phi_{t_0}(x'))$.

However, the above definition is not well-suited for the treatment of the observability problem. Therefore one seeks to establish a stonger notion of observability as follows. Consider a mapping $\Theta$ (in the sequel called observation mapping), which maps the state space into some finite dimensional Euclidian space, assigning states to data derived from the output trajectories on some observation time interval $J$. Then decide the observability of the system by means of injectivity of $\Theta$. Moreover the following natural question arises. Is observability in this sense (, i.e. with respect to $\Theta$) generic? The latter is the main issue of the present work. In the control theory the noatation of observer is generally standing for another system having the output (and input in the controlled case) of the original system as input and generating an output which is an asymptotic estimate of the original system state.

Beside the task of reconstruction of the state, observability has also application in the theory of chaos and turbulence in the following sense. Suppose an observable system has a global attractor. Then using an observation mapping one can get a homeomorphic picture of the attractor or at least information about some of its characteristic properties. Examples can be found in [RT] and [T]. We consider a certain observation mapping introduced in [KE] and [E]. We derive necessary and sufficient conditions for genericity of observability and local observability with respect to this mapping. Basically, there are two other well-knwon approaches to the observability task: sampling and high-gain approach. In his classical work [T], Takens proved genericity results similar to

*It is worth to mention that often the information by output measurements underly some errors leading to the problem of stabiliy.

ours for these approaches. About the same time Aeyels [Ay] achieved sharper results concerning the sampling approach. Further results on high-gain approach can be found in [GK], [GHK] and [J]. Particularly, for a comrehensive and intensive investigation of deterministic observation theory and applications including many deep results concerning high-gain approach we refer to [GK]. Our approach has the disadvantage that a suitable linear filter has to be constructed. On the other side in contrast to high-gain approach we do not need to restrict ourselves to smooth systems. For more on our approach with applications we refer to [N].

## 2 Main Results

Definition 2.1 Let $0 < \tau < \infty$, $(A, b)$ be a stable and controllable $m$-dimensional linear filter and $(M, f, h)$ a $C^r$-system with flow $\Phi$. We call the mapping $M \ni x \mapsto \Theta_{f,h} := \int_{-\tau}^{0} e^{-At} b h(\Phi_t(x)) dt \in \mathbb{R}^m$ the observation mapping and the number $m$ observation dimension. If $\Theta_{f,h}$ is injective we call the system $(M, f, h)$ $\Theta$-observable or simply observable.
Furthermore we call the system locally observable if $\Theta_{f,h}$ is an immersion.
The indistinguishable subset of $M \times M$ is given by

$$\Omega_{f,h} := \{(x, x') \in M \times M \setminus \Delta_M : \Theta_{f,h}(x) = \Theta_{f,h}(x')\}.$$

The time interval $I := [-\tau, 0]$ has the physical interpretation of observation interval. We treat for simplicity of notation and in view of the physical interpretation of the observation data as history of the output mapping, the case $I \subset \mathbb{R}_{\leq 0}$, $0 \in I$. For unbounded intervals modifications are needed, which we give explicitly for the case $I = \mathbb{R}_{\leq 0}$.

The key point in the proof of the following genericity results is to show that zero is a regular value of the observation mapping. Proving this we then apply transversality density and openness theorems to get locally our statements, which then will be globalized to the whole state space. A Baire argument then yields the final results. Particularly for $\tau < \infty$ in any $C^r$ neighborhood of the system there is a system which is both observable and locally observable with respect to $\Theta$. An appropriate statement is also valid for $\tau = \infty$ in the $C^1$ topology.

If the state space is not compact, it is more suitable to consider the so called strong or Whitney ($C^r$-)topology on $C^r(M, \mathbb{R})$ and $\mathbb{X}^r(M)$. The reason is that in this topology one has more control on the behavior of the functions and vector fields at infinity. Note that density in this topology is a stronger property than in the compact-open (also called weak) topology. For instance, roughly speaking, a sequence of output functions $h_j$ converges in Whitney topology to $h$ iff there exists a compact set $K$ such that $h_j = h$ outside of $K$ except for finitely many $j$ and all the derivatives up to order $k$ converge uniformly on $K$.

The case of smooth vector fields is similar. The well known fact that $C^r(M,\mathbb{R})$ and $\mathbb{X}^r(M)$ are Baire spaces in the Whitney topology (proofs can be found in in [H, 2.4.4] and [P]) is of basic importance for our results. A residual subset of $C^r(M,\mathbb{R})$ or $\mathbb{X}^r(M)$ is dense. *From now on the spaces of $C^r$ functions as well as vector fields on $M$ are equipped with their $C^r$ Whitney topology, unless otherwise indicated.* If the state space $M$ is compact and $r<\infty$, then $C^r(M,\mathbb{R}^k)$ and $\mathbb{X}^r(M)$ endowed with the compact-open topology are Banach spaces (while in general Fréchet spaces), their Whitney and compact-open topology coincide and the flow $\Phi$ of each vector field $f\in\mathbb{X}^r(M)$ is defined globally on $M\times\mathbb{R}$.

For some pairs in $\mathbb{X}^r(M)\times C^r(M,\mathbb{R})$ there is no possibility to distinguish or locally distinguish the states. For genericity results it is an essential fact that the complement of the set of such pairs is residual. Let $Sing(f):=\{x\in M : f(x)=0_x\in T_xM\}$ denote the set of singularities of the vector field $f$ (equilibria of the system), where $0_x$ is the zero of $T_xM$. In the sequel we shall often omit the subscript $x$ and write simply $f(x)=0$ for $x\in Sing(f)$.
Recall that a singularity $x_0\in Sing(f)$ is called simple iff the principial part of the linearization of $f$ at $x_0$, i.e. the linear mapping $d_{x_0}f : T_{x_0}M\to T_{x_0}M$, does not have zero as an eigenvalue.
We denote the set of $C^r$ vector fields, whose singularities are all simple, by $\mathbb{X}^r_0(M)$. It is well known that a simple singularity is isolated and $\mathbb{X}^r_0(M)$ is an open and dense subset of $\mathbb{X}^r(M)$ (for a proof we refer to [PD, 3.3]).

Definition 2.2 We call $x_0\in M$ a $\Theta$-simple singularity iff there exists a cotangent vector $v\in T^*_{x_0}M$ such that the linear system $(T_{x_0}M, d_{x_0}f, v^T)$ is observable, i.e., the linear mapping

$$\int_{-\tau}^{0} e^{-tA}bve^{td_{x_0}f}dt : T_{x_0}M\to\mathbb{R}^m$$

is injective. In this case we say that $v^T$ is a $\Theta$-cocyclic covector of $d_{x_0}f$.
We denote by $\mathbb{X}^r_1(M)$ the set of $C^r$ vector fields on $M$, whose singularities are $\Theta$-simple. Moreover we set $\mathbb{X}^r_{0,1}(M):=\mathbb{X}^r_0(M)\cap\mathbb{X}^r_1(M)$.

In the limiting case $\tau=\infty$ dense orbits as well as nontrivial recurrence cause difficulties and special considerations are necessary. We investigate this case under the assumption that $M$ is compact. Appropriate results in the noncompact case can be similarly deriven if we further restrict (in order to achieve well-definedness of the observation mapping) the systems to be globally Lipschitzian. Furthermore, in order to ensure differentiability of the observation mapping, if $\tau=\infty$ we restrict the vector fields to the open set

$$\mathbb{X}^r(M,a):=\{f\in\mathbb{X}^r(M) : \sup_{x\in M}\|d^j_xf\|<a \text{ for all } j=1,...,r\}$$

Denoting the set of critical elements (equilibria and closed orbits) of a vector field $f$ by $\mathcal{C}(f)$ and the union of the negative limit sets by $\mathcal{L}_-(f)$, we set

$$\mathbb{X}^r_-(M) := \{f \in \mathbb{X}^r(M) : \mathcal{L}_-(f) \subset \mathcal{C}(f)\}.$$

We denote that some ineteresting classes of vector fields like Morse-Smale fields* are contained in $\mathbb{X}^r_-(M)$. Particularly, since the set consisting of Morse-Smale vector fields is open and nonempty in $X^r(M)$, the interior $int(\mathbb{X}^r_-(M))$ of $\mathbb{X}^r_-(M)$ is a Baire space (in the induced topology). Furthermore note that the limit set of a gradient field consists only of the critical points of the potential function. Therefore $\mathbb{X}^r_-(M)$ also contains the set of gradient fields. Moreover we set

$$\mathbb{X}^r_2(M) := \mathbb{X}^r(M, a) \cap int(\mathbb{X}^r_-(M)).$$

It is well known that on a compact manifold $C^1$-generically the nonwandering set of a smooth vector field coincides with the closure of the set of its periodic points. This statement called general density theorem is a consequence of Pugh's closing lemma, which ensures that a nonwandering point can be made periodic by a small $C^1$-petrubation in a neighbourhood of the point. See also [Pu], [AR, 7.3.6] and the references given there*. Particularly $\{f \in \mathbb{X}^1(M) : \mathcal{L}_-(f) \subset \overline{\mathcal{C}(f)}\}$ is a residual subset of $\mathbb{X}^1(M)$.

We set for $y \in E^r_{a,\tau} := \{y \in C^r([-\tau, 0], \mathbb{R}) : \int_{-\tau}^0 e^{at}|y(t)|dt < \infty\}$

$$P_\tau y := \int_{-\tau}^0 e^{-At} b y(t) dt$$

and for fixed $\tau$ simply write $P$ instead of $P_\tau$.

Lemma 2.1 Let $r, \tau \leq \infty$.Furthermore let $q_0 \in \mathbb{R}^m$ and $T, \delta > 0$. Then the followings hold.
a) There exists a function $y \in C^r(\mathbb{R}, \mathbb{R})$ being compactly supported in $]-\delta, 0[$ and satisfying $P_\tau y = q_0$.
b) There exists a $T$-periodic function $y \in C^r(\mathbb{R}, \mathbb{R})$ with $P_\tau y = q_0$ and $supp(y|_{[-(k+1)T,0]}) \subset ]-(k+1)\delta, 0[$ for all $k \in \mathbb{Z}$.

Proof: Ad a) Let $\epsilon := min\{\delta, \tau\}$. The mapping $L^1([-\epsilon, 0]) \ni y \mapsto K(y) := \int_{-\epsilon}^0 e^{-At} b y(t) dt \in \mathbb{R}^m$ is linear, continuous and because of the controllability of $(A, b)$ surjective. $C^r_\epsilon$ is a dense linear subspace of $L^1([-\epsilon, 0])$. Therefore $R := K(C^r_\epsilon)$ is a dense linear subspace of $\mathbb{R}^m$ and consequently $R = \mathbb{R}^m$. Therefore there exists a function $y_0 \in C^r_\epsilon$ having the property $K(y_0) = q_0$. The trivial extension of $y_0$ on $\mathbb{R}$ is obviously the desired function.
Ad b) If $\tau \leq T$, the assertion follows directly from part a). We prove the result for $\tau > T$ using sampling. Assume first $\tau < \infty$ and let $N := max\{k \in \mathbb{N} : NT \leq \tau\}$. Due to the stability of $A$ the series $\sum_{k=0}^\infty e^{kTA}$

*Recall that Morse-Smale vector fields are structurally stable.

*It is still unknown whether the $C^k$-closing lemma with $k \geq 2$ fails in general.

converges to $(I-e^{TA})^{-1}$ and $S_N := \sum_{k=0}^{N} e^{kTA} = (I-e^{TA})^{-1}(I-e^{(N+1)TA})$ is invertible. Let us write $\epsilon := min\{\tau - T, \delta\}$. According to part a) there exists $\tilde{y}_0 \in C^r_\epsilon$ with $P_T\tilde{y}_0 = S_N^{-1}q_0$ (as well as $(I-e^{TA})q_0$ in case $\tau=\infty$). Let $y_0$ denote the trivial extension of $\tilde{y}_0$ on $[-T,0]$. Then we have $P_Ty_0 = S_N^{-1}q_0$. Let $y$ denote the $T$-periodic extension of $y_0$ on $\mathbb{R}$. Consequently

$$\begin{aligned}\int_{-\tau}^{0} e^{-At}by(t)dt &= \sum_{k=0}^{N}\int_{-(1+k)T}^{-kT} e^{-At}by_0(t+kT)dt\\ &= \sum_{k=0}^{N} e^{kTA}\int_{-T}^{0} e^{-At}by_0(t)dt\\ &= q_0,\end{aligned}$$

which yields immediately the desired conclusion. □

Let $\pi$ denote the canonical projection of the tangent bundle $TM$:

$$\pi : TM \to M,\ \pi(v) := x \text{ for } v \in T_xM.$$

Let $M$ be endowed with a Riemannian metric. Denoting the induced norm on the tangent spaces by $|.|$, the unit tangent bundle $T_1M$ is given by

$$T_1M := \underset{x\in M}{\cup}\{v \in T_xM : |v| = 1\}.$$

Recall that $T_1M$ is a $(2n-1)$-dimensional $C^{r-1}$ submanifold of $TM$. It is compact, if $M$ is compact.
Let $K$ be a subset of $M$. We set $T_0K := K\times K\setminus\Delta_K$ and denote the restriction of $T_1M$ to $K$ with $T_1K$, i.e., $T_1K := \{v \in T_1M : \pi(v) \in K\}$. Recall that if $K$ is an $s$-dimensional submanifold, then $T_1K$ has dimension $n+s-1$.

Definition 2.3 We define the $\tau$-history of $K$ by the flow $\Phi$ of the vector field $f$ to be the closure of

$$\Phi(K;\tau) := \{\Phi_t(x) : -\tau < t \le 0, x \in K\}.$$

We denote

$$\Delta\Theta_{f,h}(x,x') := \Theta_{f,h}(x) - \Theta_{f,h}(x') \text{ for } x,x' \in M.$$

Let $V$ be an open subset of $M$ containing the $\tau$-history of $K$ and $L$ be the closure of $V$. Then we denote

$$\mathcal{H}_0(L;K) := \{h \in C^r(L,\mathbb{R}) :\ \text{zero is a regular value of } \Delta\Theta_{f,h}|\Lambda_0K\},$$

and

$$\mathcal{H}_1(L;K) := \{h \in C^{r+1}(L,\mathbb{R}) :\ \text{zero is a regular value of } d\Theta_{f,h}|\Lambda_1K\}.$$

In the sequel we set $i = 0, 1$ as well as $\mathcal{H}_i(K) := \mathcal{H}_i(M; K)$ and $\mathcal{H}_i := \mathcal{H}_i(M)$. If $K' \subset K$ and for an output function $h$, zero is a regular value of the mapping $\Delta\Theta_{f,h}$ on $\Lambda_0 K$, then it is also a regular value of the restricted mapping $\Delta\Theta_{f,h}|_{\Lambda_0 K'}$, that is, $\mathcal{H}_0(K) \subset \mathcal{H}_0(K')$. Similarly $\mathcal{H}_1(K) \subset \mathcal{H}_1(K')$ holds.

In the following, if $f \in \mathbb{X}^r_{0,1}(M)$, then $\mathbb{H}_0(L)$ stands for the set of output functions $h \in C^r(L, \mathbb{R})$ such that

$$h(x_0) \neq h(x_0') \text{ for all } (x_0, x_0') \in \Lambda_0(L \cap Sing(f)),$$

and $\mathbb{H}_1(L)$ for those $h \in C^{r+1}(L, \mathbb{R})$ such that

$$d_{x_0}h \text{ is a } \Theta\text{-cocyclic covector of } d_{x_0}f \text{ for all } x_0 \in L \cap Sing(f).$$

Note that if $L$ is compact, then the number of singularities of $f$ in $L$ is finite and for finite $r$, as an immediate application of the transversality openness and density theorems, it follows that $\mathbb{H}_i(L)$ is an open and dense subset of the Banach space $C^{r+i}(L, \mathbb{R})$.

Lemma 2.2 Assume that $i < r < \infty$ and $f \in \mathbb{X}^r_{0,1}(M)$ is complete. Let $S$ be a $C^r$ submanifold of $M$ such that the $\tau$-history of $S$ is contained in an open subset $V$ of $M$ with compact closure $L := \bar{V}$. Consider the mappings $F^i : \mathbb{H}_i(L) \times \Lambda_i S \to \mathbb{R}^m$ defined by

$$F^0(h, x, x') := \Delta\Theta_{f,h}(x, x')$$

and

$$F^1(h, v) := d_{\pi(v)}\Theta_{f,h}(v).$$

Then the following holds.
a) Zero is a regular value of $F^0$.
b) Zero is a regular value of $F^1$.
Suppose moreover that $f \in X^r_2(M)$. Then the assertions also hold for $\tau = \infty$.

Proof: Ad a) Let $W_0 := \{(h, x, x') \in \mathbb{H}_0(L) \times T_0 S : F^0(h, x, x') = 0\}$. We have to show that the function $F^0$ is submersive on $W_0$. Since $\mathbb{R}^m$ is finite dimensional, it sufficies to prove that the linear mapping $d_{(h,x,x')}F^0 : T_{(h,x,x')}(\mathbb{H}_0(L) \times T_0 S) \to \mathbb{R}^m$ is surjective for all $(h, x, x') \in W_0$. Fix $(h, x, x') \in W_0$ and $q_0 \in \mathbb{R}^m$. According to the condition $h \in \mathbb{H}_0(L)$ we see that $x$ and $x'$ cannot be both equilibrium points. Therefore we assume without loss of generality that $x$ is not an equilibrium point. Since $\mathbb{H}_0(L)$ is open and $\frac{d}{ds}|_{s=0}F^0(h + sg, x, x') = \Delta\Theta_{f,g}(x, x') = F^0(g, x, x')$, it is sufficient to show the existence of an output function $g \in C^r(L, \mathbb{R})$ satisfying $F^0(g, x, x') = q_0$. We use the fact that the flow through a point of the state space $M$ maps each closed finite time interval on a closed subset of $M$ and define a suitable mapping $g$ on an appropriate closed subset of the state space and then extend it to $L$.

Let $\gamma$ and $\gamma'$ denote the $\tau$-histories of the points $x$ and $x'$ respectively and $Z := \gamma \cup \gamma'$. We define $g$ on $Z$. We first treat the case $\tau < \infty$.
Case 1: Both orbits are critical elements. Let $T$ denote the period of $x$. In view of lemma 2.1 there exists a $T$-periodic function $y \in C^k(\mathbb{R}, \mathbb{R})$, which satisfies the condition $P_\tau y = q_0$. We set $g(\Phi_{t+kT}(x)) = y(t)$ for $0 \leq t \leq T, k \in \mathbb{Z}$ and $g = 0$ else.
Case 2: One of the integral curves, say $\Phi(x)$, is injective and the other one is periodic or an equilibrium point. According to Lemma 2.1 there is a function $y \in C^k([-\tau, 0], \mathbb{R})$ with compact support such that $P_\tau y = q_0$. We define $g$ on $\gamma$ by $g(\Phi_t(x)) = y(t)$ for $-\tau \leq t \leq 0$ and $g = 0$ else. If $\Phi(x')$ is injective and $x$ is periodic, we just set $g = 0$ on $\gamma$ and define $g$ on $\gamma'$ such that $P_\tau(g \circ \Phi(x'))) = -q_0$.
Case 3: Both integral curves are injective. In this case we define $g$ on $\gamma$ as in case 2 and on $\gamma'$ by $g = 0$.
If $\tau = \infty$, then because of eventual presence of dense orbits $g$ cannot be simply defined on a part of $Z$ and then trivially extended. Hence the construction of $g$ becomes a little more delicate. Assuming $f \in X_2$ ensures then that the recurrence is trivial and the previous procedure also works. If both orbits are critical elements, i.e. in case 1, then everything remains the same as for finite observation time. Problems could arise in case 2 or 3 if at least one of the integral curves is injective and the past half of one of the orbits, say the one through $x'$, belongs to the negative limit set of the other orbit or itself, i.e., $\{\Phi_t(x') : t \leq 0\} \subset \alpha(x) \cup \alpha(x')$. But this can according to the assumption $f \in \mathbb{X}_2^r(M)$ only occur if $x'$ is periodic or an equilibrium point. We proceed as in case 2 of finite $\tau$ and find again in view of Lemma 2.1 a function $y \in C^r(\mathbb{R}, \mathbb{R})$ compactly supported in an interval $[-\epsilon, 0]$ with $\epsilon > 0$ and set $g(\Phi_t(x)) := y(t)$ for all $t$ and $g = 0$ else.
In all cases we have defined a $C^r$ function on the closed subset $Z$ of the state space $M$ with the property that $Z \subset L$ and $P_\tau(g \circ \Phi(x) - g \circ \Phi(x')) = q_0$. According to the smooth Tietze extension theorem, there exists a $C^r$ extension of the function $g$ to $L$. This function denoted again by $g$ is obviously the desired function, which satisfies $F^0(g, x, x') = q_0$.

Ad b) Denote $W_1 := \{(h, v) \in \mathbb{H}_1(L) \times T_1 S : F^1(h, v) = 0\}$. We fix $(h, v) \in W_1$, set $x_0 := \pi(v)$ and show that $F^1$ is submersive at $(h, v)$. Fix $q_0 \in \mathbb{R}^m$. Since $\frac{d}{ds}|_{s=0} F^1(h + sg, v) = F^1(g, v)$ for arbitrary $g \in C^r(M, \mathbb{R})$, it suffices to prove the existence of a function $g \in C^r(L, \mathbb{R})$ with $F^1(g, v) = q_0$ locally and extend it $L$. If $x_0$ would be an equilibrium point, then in view of the assumption $h \in \mathbb{H}_1(L)$, the linear system $(T_{x_0}M, d_{x_0}f, d_{x_0}h)$ would be $\Theta$-observable and consequently $F^1(h, v) \neq 0$ in contradiction to the assumption $(h, v) \in W_1$. Therefore we may assume that $x_0$ is not an equilibrium point. Hence, in view of the straightening-out theorem there is a local chart $(U, \psi)$ at $x_0$ such that $\psi(U) = U' \times ] - \epsilon, -\epsilon[$ with $\epsilon > 0$, $U'$ an open subset of $\mathbb{R}^{n-1}$, $\psi(x_0) = 0$ and the vector field $f$ has the local representative $(z, t) \mapsto e_n$. Here $e_n$ denotes the $n$th standard base vector in $\mathbb{R}^n$. Denote the induced coordinate function on $T_1M$ at $v$ by $\widehat{\psi}$. Since $v \neq 0$ we can and do assume that $v$ has the local representative $\eta = (\eta_1, \eta_2)^T$ with $\eta_1 \in \mathbb{R}^{n-1}$, $\eta_2 \in \mathbb{R}$, $\eta \neq 0$, i.e. $v = \frac{\partial}{\partial z}(x_0)\eta_1 + \frac{\partial}{\partial t}(x_0)\eta_2$. Furthermore for $t \in ]-\epsilon, -\epsilon[$, the local representative of $d_{x_0}\Phi_t$ reads as $\begin{bmatrix} Id & 0 \\ 0 & 1 \end{bmatrix}$ and subsequently that of $d_{x_0}h \circ \Phi_t v$ as $\nabla h(0,t)\eta$.

By shrinking $U$ if necessary, we can (in the case $\tau = \infty$ on account of the assumption that $f \in \mathbb{X}_2^r(M)$ if the point $x_0$ is recurrent, then it is periodic) and do assume that the intersection of $U$ and the $\tau$-history of $x_0$ is connected. According to lemma 2.1 there exists a function $\hat{y} \in C^{r+1}(\mathbb{R}, \mathbb{R})$ with derivative $y$ being supported (on each period, if $x_0$ has period $> \tau$) in $[-\epsilon, 0]$ such that $P_\epsilon y(t)dt = q_0$. Obviously there exists a function $g \in C^r(U' \times ]-\epsilon, \epsilon[, \mathbb{R})$ being compactly supported, with $\nabla g(0,t)\eta = y(t)$. For instance define $\tilde{g}(z,t) = |\eta|^{-2}(z^T\eta_1 y(t) + \eta_2\hat{y}(t))$. The trivial extension of $\tilde{g} \circ \psi$ to $L$ is the desired function $g$. □

Sometimes in applications one is interested in or limited to observation restricted to a subset of the state space. It can be for instance because of technical or physical reasons, or if it happens that all the information needed can be evaluated from measurements on a certain subset. The latter case being perhaps the most important one, occurs if the subset under observation is an attractor. Other subsets invariant under the flow can also be of interest. Therefore we state our genericity results for observations of subsets of the state space as well.

Lemma 2.3 Suppose that $K$ is a subset of an $s$-dimensional $C^r$ submanifold of $M$ denoted by $S$, $\tau < \infty$ and $f \in \mathbb{X}_{0,1}^r(M)$ is complete. Then the following holds.
a) Assume that $m \geq n + s - r$ and $r \geq 2$. Then $\mathcal{H}_1(K)$ is residual. If $K$ is closed, then $\mathcal{H}_1(K)$ is also open.
b) Assume that $m \geq n + s + 1 - r$. Then $\mathcal{H}_0(K)$ is residual. If $K$ is closed, then $\mathcal{H}_0(K)$ contains an open set.
Suppose moreover that $M$ is compact and $f \in \mathbb{X}_2^r(M)$. Then the assertions hold also in the case $\tau = \infty$.

Proof: Assume first $r < \infty$, $K$ is compact, $U$ is a chart domain of $S$, which contains $K$ and has compact closure. By compactness of $\overline{U}$ and finiteness of $\tau$ in case of finite observation time and because of compactness of $M$ in case $\tau = \infty$, the $\tau$-history of $U$ is compact. By local compactness there is an open set $V \subset M$ with compact closure $L := \overline{V}$ such that $V$ contains the $\tau$-history of $U$.

Local density: We prove residuality with respect to $\mathbb{H}_i(L)$. Since the latter is open and dense in $C^r(L, \mathbb{R})$, density of $\mathcal{H}_i(L; U)$ is also then shown with respect to $C^r(L, \mathbb{R})$.
According to the previous lemma zero is a regular value of the evaluation map-

ping $F^i : \mathbb{H}_i(L) \times T_i U \to \mathbb{R}^m$ defined by

$$F^0(h, x, x') := \Delta\Theta_{f,h}(x, x')$$

and

$$F^1(h, v) := d_{\pi(v)}\Theta_{f,h}(v).$$

Therefore according to the transversality density theorem (see for instance [AR, 19.1]) $\mathcal{H}_i(L; U)$ is a residual subset of $\mathbb{H}_i(L)$, and hence dense in $C^r(L, \mathbb{R})$.
Local Openness: Compactness of $K$ implies that $T_1K$ is also compact in $T_1U$. According to the transversality openness theorem $\mathcal{H}_1(L; K)$ is open (with respect to the compact-open and by compactness of $L$ also with respect to the Whitney topology) in $C^r(L, \mathbb{R})$. These conclusions do not work for $\mathcal{H}_0(L; K)$. Instead let $\Lambda$ be a compact subset of $T_0U$. Then in view of transversality openness theorem

$$H'_0(L; \Lambda) := \{h \in C^r(L, \mathbb{R}) : \text{ zero is a regular value of } \Delta\Theta_{f,h} \text{ on } \Lambda\}$$

is open in $C^r(L, \mathbb{R})$.
Since the assertions are proved for $r$ finite and sufficiently large, they also hold for $r = \infty$. Hence we assume from now on that $r \leq \infty$.
Globalization: This part of the proof is basicly standard. Therefore we give an outline and refer to [H, 2.2] for details. Since $K \subset U$, we have $\mathcal{H}_1(L; U) \subset \mathcal{H}_1(L; K)$. Hence $\mathcal{H}_1(L; K)$ is also dense in $C^r(L, \mathbb{R})$. Likewise one gets density of $\mathcal{H}'_0(L; \Lambda)$ in $C^r(L, \mathbb{R})$ from density of $\mathcal{H}_0(L; U)$, since $\Lambda \subset T_0U$ and subsequently $\mathcal{H}_0(L; U) \subset \mathcal{H}'_0(L; \Lambda)$. Using a bump function we now prove that $\mathcal{H}_1(K)$ is dense in $C^r(M, \mathbb{R})$. Fix $g_0 \in C^r(M, \mathbb{R})$. Since $\mathcal{H}_1(L; K)$ is dense in $C^r(L, \mathbb{R})$, there is a sequence $\{h_j\}$ in $\mathcal{H}_1(L; K)$ converging in the compact-open topology to $g_0|_L$. Since $K \subset U$, there is a $C^r$-function $\rho : M \to [0, 1]$ with compact support in $L$, such that $\rho = 1$ on an open neighborhood of $K$. The sequence $\rho h_j + (1 - \rho)g_0$ converges to $g_0$ with respect to the Whitney topology. Therefore $\mathcal{H}_1(K)$ is a dense subset of $C^r(M, \mathbb{R})$. A similar argument shows that $\mathcal{H}'_0(\Lambda)$ is a dense subset of $C^r(M, \mathbb{R})$.
We now drop the assumption that $K \subset U$. Let $J$ be a countable indexing set, $\{U_j\}$ with $j \in J$ be a covering of $S$ with chart domains $U_j$. Furthermore let $\{K_j\}$ be a subordinate family of compact sets such that $K = \bigcup_J K_j$ and $K_j \subset U_j$. We can and do assume that there is a compact covering of $M$ denoted by $\{L_j\}$ such that the interior of $L_j$ contains the $\tau$-history of $U_j$.
Openness statements: Suppose that $K$ is closed. Then it is also paracompact and the covering can be assumed to be locally finite. Since $\{L_j\}$ covers $M$, it holds that $\mathcal{H}_1(K) = \{h \in C^r(M, \mathbb{R}) : h|_{L_j} \in \mathcal{H}_1(L_j; K_j) \text{ for all} j \in J\}$. Hence local finiteness of the covering implies that $\mathcal{H}_1(K)$ is open. Simliarily it follows that the set $\{h \in C^r(M, \mathbb{R}) : \text{ zero is a regular value of } \Delta\Theta_{f,h} \text{ on } K \times K\}$ is open. The latter is contained in $\mathcal{H}_0(K)$.
Residuality statements: We now drop the assumption that $K$ is closed. By the preceding arguments $\mathcal{H}_1(K_j)$ is open and dense. Therefore $\mathcal{H}_1(K) = \bigcap_j \mathcal{H}_1(K_j)$

is residual (and in view of the Baire property of $C^r(M,\mathbb{R})$ also dense). Taking a compact covering $\{\Lambda_j\}$ of $T_0K$ a similar argument yields the results on $H_0(K)$. □

The openness results in the preceding lemma can also be proved (without applying the transversality openness theorem) as follows. For instance, by compactness of $L$ the mapping $\mathbb{H}_0(U) \ni h \mapsto F^0(h,.)) \in C^r(T_0U,\mathbb{R}^m)$ is continuous in the Whitney topology. This fact and openness of $\{F \in C^r(T_0U,\mathbb{R}^m) :$ zero is a regular value of $F\}$ (this fact follows, for instance, from the lower semicontinuity of the mapping $C^r(L,\mathbb{R}) \times T_0U \ni (h,x,x') \mapsto rank(d_{(h,x,x')}F^0)$ and compactness of $L$) in the Whitney topology implies that $\mathcal{H}_0(L;U)$ is open in the Whitney topology and by compactness of $L$ also open in the compact-open topology of $C^r(L,\mathbb{R})$. Other parts follow likewise.

In light of the preceding lemmas we can now prove the genericity results on output functions just by comparing dimensions.

Theorem 2.1 Suppose that $K$ is contained in an $s$-dimensional $C^r$ submanifold of $M$ and $f \in \mathbb{X}^r_{0,1}(M)$ is complete. Then the following assertions hold.
a) Assume that $r \geq 2$ and $m \geq n+s$. Then functions $h$ belonging to $C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is immersive at each point of $K$, constitute a residual set. This set is also open, if $K$ is closed.
b) Assume that $m \geq 2s+1$. Then the set of functions $h$ belonging to $C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is injective (respectively an injective immersion) on $K$, is residual (respectively, if $r \geq 2$). It contains an open set, if $K$ is closed.
Supposing $m \leq 2s$ and $r > 2s-m$ the same results hold for the set of functions $h$ belonging to $C^r(M,\mathbb{R})$ such that the $\Theta$-unobservable points of $K \times K$ belong to a submanifold of dimension $2s-m$.
Suppose moreover that $M$ is compact and $f \in \mathbb{X}^r_2(M)$. Then the assertions also hold for $\tau = \infty$.

Proof: Ad a) Since $m \geq n+s$, the set of functions $h$ belonging to $C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is an immersion at each point of $K$, coincides with $\mathcal{H}_1(K)$, i.e., $d\Theta_{f,h}|_{T_1K}$ is transversal to $\{0\} \in \mathbb{R}^m$. Therefore the assertion follows immediately from the previous lemma.
Ad b) According to the previous lemma $\mathcal{H}_0(K)$ is residual and contains an open set, if $K$ is closed. If $m \geq 2s+1$, then the set of functions $h$ belonging to $C^r(M,\mathbb{R})$ such that the restriction of $\Theta_{f,h}$ to $K$ is injective (respectively an injective immersion), is precisely $\mathcal{H}_0(K)$ (respectively $\mathcal{H}_0(K) \cap \mathcal{H}_1(K)$). The statement on the indistinguishable set follows from the preimage theorem. □

Remark 2.2 As it can easily be seen from the proof of lemma 2.2, the dimension condition $m \geq n+s$ in part a) of the preceding theorem can be weakend to $m \geq 2s$, if immersivity is replaced by immersivity on $TS$.

If the state space is compact, we get sharper results. In particular the following theorem is important, since embedding of the state space gives information on limit behavior of the system.

Theorem 2.2 Suppose that $m \geq 2n+1$, $f \in \mathbb{X}^r_{0,1}(M)$ and $M$ is compact. Then output functions $h \in C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is an embedding constitute an open and dense set.
If we further assume that $f \in \mathbb{X}^r_2(M)$, then the assertion remains true in the case $\tau = \infty$.

Proof: Recall that by compactness of $M$ an injective immersion is also an embedding (case $2 \leq r \leq \infty$ in Theorem 2.1) and an injective mapping is also a topological embedding (case $r = 1$). Hence density follows from Theorem 2.1. Openness is a consequence of the fact that by compactness of $M$ the mapping $C^r(M,\mathbb{R}) \times T_0M \ni h \mapsto \Delta\Theta_{f,h} \in C^r(M,\mathbb{R}^m)$ is continuous and the set of embeddings $Emb^r(M,\mathbb{R}^m) := \{F \in C^r(M,\mathbb{R}^m) : F \text{ is embedding}\}$ is open. □

Next we shall prove residuality of $\mathbb{X}^r_{0,1}(M)$ by using the characterization of simplicity and $\Theta$-simplicity of a singularity, in terms of transversal nonintersection.

Lemma 2.4 Assume that $m \geq n$. Then $\mathbb{X}^r_{0,1}(M)$ is open and dense in $\mathbb{X}^r(M)$. Moreover, the assertion holds also for $\tau = \infty$, if we restrict the vector fields to $X^r(M,a)$.

Proof: We give a proof for $\tau < \infty$. The arguments for $\tau = \infty$ are similar. It sufficies to show that $\mathbb{X}^r_1(M)$ is open and dense. Let $\mathcal{O}$ resp. $\mathcal{U}$ denote the set of $n$-dimensional $\Theta$-observable resp. unobservable linear systems. Let $f \in \mathbb{X}^r_1(M)$, $x_0 \in Sing(f)$ and $d_{x_0}f$ denote the principial part of the linearization of $f$.
Note that $f \in \mathbb{X}^r_1(M)$ if and only if there exists a $v \in T_{x_0}M$ such that the system $(d_{x_0}f, v)$ avoids the set of $\Theta$-unobservable linear systems on $T_{x_0}M$.
We now give a local characterization. Let $\psi : U \to \mathbb{R}^n$ be a local chart such that $U$ has compact closure and $\psi(x_0) = 0$. The tangent mapping $d\psi : TU \to \mathbb{R}^n \times \mathbb{R}^n$ defined by $d\psi(v) = (\psi(x), d_x\psi v)$ with $x = \pi(v)$ is a local chart for $TM$. Let $Pr_2$ denote the projection $\mathbb{R}^n \times \mathbb{R}^n \ni (x,w) \mapsto w$. Consider the mapping $\xi_f := Pr_2 d\psi f \circ \psi^{-1}$ on $\psi(U)$. Hence $d_0\xi_f = d_{x_0}\psi d_{x_0} f d_0\psi^{-1}$. The singularity $x_0$ is $\Theta$-simple if and only if $(d_0\xi_f, d_{x_0}v) \notin \mathcal{U}$ for some $v \in T_{x_0}M$. Obviously $\mathcal{U}$ is closed, analytic and $\neq End(\mathbb{R}^n) \times \mathbb{R}^n$, hence finite union of closed positive codimensional real analytic submanifold of $End(\mathbb{R}^n) \times \mathbb{R}^n$. Given a pair $(G,w) \in \mathcal{O}$, obviously there exists a vector field $g \in \mathbb{X}^r(M)$ such that $g$ and $f$ coincide on $M \setminus U$, $x_0 \in Sing(g)$ and $d_{x_0}\xi_g = G$. An immediate application (details are similar to those in the proof of lemma 2.3) of the transversality density and openness theorems completes the proof. □

We can now prove the main result on generic observability with respect to the mapping $\Theta$.

Theorem 2.3 Suppose that $m \geq 2n+1$ and $M$ is compact. Then pairs $(f,h)$ in $\mathbb{X}^r(M) \times C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is an embedding constitute an open and dense subset of $\mathbb{X}^r(M) \times C^r(M,\mathbb{R})$. Restricing the vector fields to $\mathbb{X}^r_2(M)$, the assertion remains valid for $\tau = \infty$ as well.

Proof: Recall that $\mathbb{X}^r_{0,1}(M)$ is open and dense. Furthermore $\mathbb{X}^r_2(M)$ is open in $\mathbb{X}^r(M,a)$. The assertions follow immediately from this facts and theorem 2.2. $\square$

Similarily, genericity results for noncompact state spaces can be deriven, if we restrict ourself to the set of complete vector fields, replace open and dense by residual and embedding by injective immersion. We remark also that considering the observation mapping

$$\Theta_{g,h}(x) := \sum_{k=-N}^{0} e^{-kA} b h(g^k(x))$$

for $x \in M$ and $(g,h) \in Diff^r(M) \times C^r(M,\mathbb{R})$ with $g^k := g \circ g^{k-1}$, corresponding results for discrete dynamical systems can be proved likewise.

## 3 Concluding Remarks

Remark 3.1 Since the set of $m$-dimensional controllable linear filters is open and dense in $End(\mathbb{R}^m) \times \mathbb{R}^m$, all genericity results of the last section hold also generically with respect to the linear filters.

The following examples show that the conditions on the observation dimension are also necessary, thus $m \geq 2n$ for generic local observability and $m \geq 2n+1$ for generic observability can not be weakend.

Example 3.1 Let $M = S^1$ and $f(\varphi) = 1$, where $\varphi$ denotes the standard angular coordinate of the circle. Furthermore consider the pair $(\lambda, b)$ with $\lambda < 0$ and $b \neq 0$. Taking $\tau = 2\pi$ and the output function $h(\varphi) = cos\,\varphi$ leads to $\Theta_{f,h}(\varphi) = \frac{1-e^{2\lambda\pi}}{1+\lambda^2}(\lambda cos\,\varphi - sin\,\varphi)$, which is not an immersion. Moreover the zero of $d\Theta_{f,h}$ at $\varphi_0 = -arctan\frac{1}{\lambda}$ is transversal. Hence the nonimmersivity of $\Theta_{f,h}$ is preserved under small perturbations of the output function, the vector field and the linear filter.

Example 3.2 Let $M$, $f$, $\varphi$ and $\tau$ be as in the preceding example. Furthermore let $A = diag(-1,-2)$, $b = (1-e^{-2\pi})^{-1}(1,1)^T$ and $h(\varphi) = 2cos\,\varphi + 5cos\,2\varphi$.

Then a straightforward computation yields

$$\Theta_{f,h}(\varphi) = \begin{bmatrix} cos\,\varphi + sin\,\varphi + cos\,2\varphi + 2sin\,2\varphi \\ (e^{-2\pi}+1)(\frac{2}{5}(2cos\,\varphi + sin\,\varphi) + \frac{5}{4}(cos\,2\varphi + sin\,2\varphi)) \end{bmatrix}.$$

The following figure shows the image of $S^1$ by $\Theta_{f,h}$.

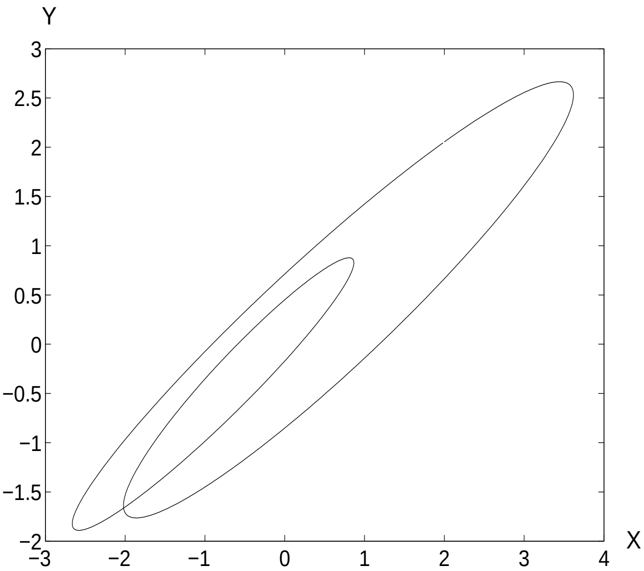


Image of $S^1$ by the continuous linear filter mapping given by
$\varphi \mapsto (cos\varphi + sin\varphi + cos2\varphi + 2sin2\varphi, (e^{-2\pi}+1)(\frac{2}{5}(2cos\varphi + sin\varphi) + \frac{5}{4}(cos2\varphi + sin2\varphi))^T$
in the $XY$-plane

The selfintersection of the image is transversal. Hence the noninjectiveness of $\Theta_{f,h}$ is persistent under small perturbations of the output function, the vector field and the linear filter.
For instance, small perturbations of $h$ do not result in injectivity, i.e., there is an $\epsilon > 0$ such that for each output function $\tilde{h}$, which is $C^r$ near to $h$ within $\epsilon$, the mapping $\Theta_{f,\tilde{h}}$ is not injective.

Note that the considered system is also unobservable with respect to the high-gain mapping given by $\varphi \mapsto (2cos\,\varphi + 5cos\,2\varphi, 2sin\,\varphi - 10sin\,2\varphi)^T$ as well as sampling mapping $\varphi \mapsto (2cos(\varphi + t_1) + 5cos(2\varphi + 2t_1, 2cos(\varphi + t_2) + 5cos(2\varphi + 2t_2))^T$ with sampling times $t_1, t_2$. The following figure shows the image of the state space by the sampling mapping with sampling times 0 and $\frac{\pi}{2}$.

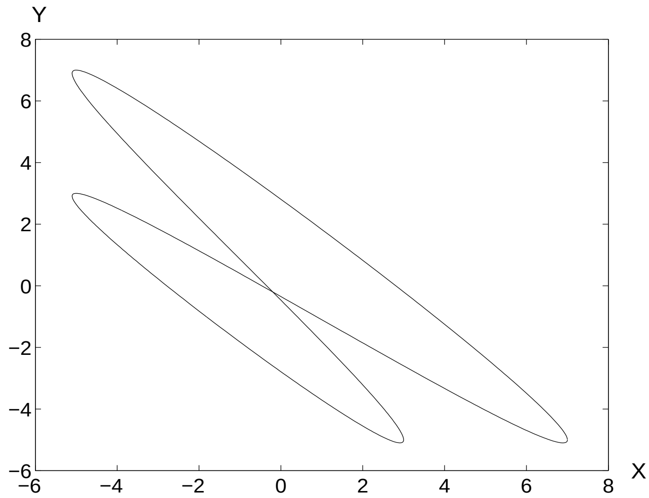


Image of $S^1$ by the sampling mapping given by
$\varphi \mapsto (2cos\,\varphi + 5cos\,2\varphi, -2sin\,\varphi - 5cos\,2\varphi)^T$
in the $XY$-plane

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# Geometric Methods for Cohomological Invariants

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Abstract. We explain how to exploit Rost's theory of Chow groups with coefficients to carry some computations of cohomological invariants. In particular, we use the idea of the "stratification method" introduced by Vezzosi.

We recover a number of known results, with very different proofs. We obtain some new information on spin groups.

Keywords and Phrases: cohomological invariants, algebraic cycles.

## §1. Introduction

*In what follows, $k$ is a base field, and all other fields considered in this paper will be assumed to contain $k$. We fix a prime number $p$ which is different from $char(k)$, and for any field $K/k$, we write $H^i(K)$ for $H^i(Gal(K_s/K), \mathbf{Z}/p(i))$. Here $K_s$ is a separable closure of $K$, and $\mathbf{Z}/p(i)$ is the $i$-th Tate twist of $\mathbf{Z}/p$, that is $\mathbf{Z}/p \otimes \boldsymbol{\mu}_p^{\otimes i}$. (See the end of this introduction for more information on our choice of coefficients.)*

### 1.1. Cohomological invariants

Given a functor $A(-)$ from fields over $k$ to the category of pointed sets, the (mod $p$) *cohomological invariants* of $A$ are all transformations of functors

$$a : A(-) \to H^*(-).$$

Typical examples for $A$ include

$$\begin{array}{lcl} A(K) & = & \text{isomorphism classes of nondegenerate quadratic forms over } K, \\ A(K) & = & \text{isomorphism classes of octonions algebras over } K, \\ \text{etc...} & & \end{array}$$

For more examples and a very good introduction to the subject, see [5].
We are mostly interested in the situation where $A(K)$ is the set of isomorphism classes over K of some type of "algebraic object", especially when all such objects become isomorphic over algebraically closed fields. In this case, if we write $X_K$ for the base point in $A(K)$, then we obtain a group scheme $G$ over $k$ by setting $G(K) = Aut_K(X_K)$, assuming $Aut_K$ is appropriately defined. Moreover, by associating to each object $Y_K \in A(K)$ the variety of isomorphisms (in a suitable sense) from $Y_K$ to the base point $X_K$, we obtain a $1:1$ correspondence between $A(K)$ and $H^1(K, G)$. Recall that this is the (pointed) set of all $G$-torsors over $K$, i.e. varieties acted on by $G$ which become isomorphic to $G$ with its translation action on itself when the scalars are extended to the algebraic closure $\bar{K}$.
For example, in the case of quadratic forms, resp. of octonion algebras, we have $A(K) = H^1(K, \mathbf{O}_n)$, resp. $A(K) = H^1(K, G_2)$.
In more geometric terms, a $G$-torsor over $K$ is a principal $G$-bundle over $\mathrm{Spec}(K)$. In this way we see that cohomological invariants

$$H^1(-, G) \to H^*(-)$$

are analogous to characteristic classes in topology. We shall call them the mod $p$ cohomological invariants of $G$. They form a graded $H^*(k)$-algebra which we denote by $Inv(G)$ (the prime $p$ being implicit).

### 1.2. Versal torsors and classifying spaces

A simple way of constructing a $G$-torsor is to start with a $G$-principal bundle $E \to X$ and take the fibre over a point $\mathrm{Spec}(K) \to X$. As it turns out, there always exist bundles $\pi : E \to X$ with the particularly nice feature that *any* $G$-torsor is obtained, up to isomorphism, as a fibre of $\pi$. In this case, the generic torsor $T_\kappa$ over $\kappa = k(X)$ is called *versal*. We shall also say that $\pi$ is a versal bundle.
A strong result of Rost (presented in [5]) asserts that an invariant $a \in Inv(G)$ is entirely determined by its value $a(T_\kappa)$ on a versal torsor. It follows that $Inv(G) \subset H^*(\kappa)$, and more precisely one can show that

$$Inv(G) \subset A^0(X, H^*).$$

The right hand side refers to the cohomology classes in $H^*(\kappa)$ which are *unramified* at all divisors of $X$ (see *loc. cit.*). (The notation comes from Rost's theory of Chow groups with coefficients, see below and section 2.)
There are two well-known constructions of versal bundles. One can embed $G$ in a "special" group $S$ (eg $S = \mathbf{GL}_n$, $S = \mathbf{SL}_n$ or $\mathbf{Sp}_n$) and take $E = S$, $X = S/G$. Alternatively, one can pick a representation $V$ of $G$ such that the action is free on a nonempty open subset $U \subset V$, and take $E = U$, $X = U/G$. In either situation, there are favorable cases when we actually have

$$Inv(G) = A^0(X, H^*).$$

For versal bundles of the first type, this happens when $S$ is simply connected (see Merkurjev [8]). For versal bundles of the second type, this happens when the complement of $U$ has codimension $\geq 2$ ([5], letter by Totaro).
In this paper we shall restrict our attention exclusively to the second type of versal torsors, for reasons which we shall explain in a second. We shall view $U/G$ as an approximation to the *classifying space* of $G$ (see [13]). It will not cause any confusion, hopefully, to call this variety $BG$ (see §2.3).

### 1.3. Rost's Chow groups with coefficients

The group $A^0(X, H^*)$, for any variety $X$, is the first in a sequence of groups $A^n(X, H^*)$, Rost's *Chow groups with coefficients.* They even form a (bigraded) ring when $X$ is smooth. The term "Chow groups" is used since the usual mod $p$ Chow groups of $X$ may be recovered as $CH^n X \otimes_{\mathbf{Z}} \mathbf{Z}/p = A^n(X, H^0)$.
Rost's groups have extremely good properties: homotopy invariance, long exact sequence associated to an open subset, and so on. There is even a spectral sequence for fibrations.
Our aim in this paper, very briefly, is to use these geometric properties in order to get at $A^0(BG, H^*) = Inv(G)$.
To achieve this, we shall be led to introduce the equivariant Chow groups $A^n_G(X, H^*)$, defined when $X$ is acted on by the algebraic group $G$. The definition of these is in perfect analogy with the case of equivariant, ordinary Chow groups as in [3], [13], where "Borel constructions" are used. When $X = \mathrm{Spec}(k)$, we have $A^n_G(\mathrm{Spec}(k), H^*) = A^n(BG, H^*)$ where $BG$ is a classifying space "of the second type" as above.
The key observation for us will be that $A^0_G(V, H^*) = Inv(G)$ when $V$ is a representation of $G$ (thus a variety which is equivariantly homotopic to $\mathrm{Spec}(k)$, so to speak). This allows us to cut out $V$ into smaller pieces, do some geometry, and eventually implement the "stratification method", which was first introduced by Vezzosi in [14] in the context of ordinary Chow groups.

### 1.4. Results

A large number of the results that we shall obtain in this paper are already known, although we provide a completely different approach for these. Occasionally we refine the results and sometimes we even get something new.
In this introduction we may quote the following theorem (3.2.1 in the text):

Theorem – *Let $p = 2$ and $n \geq 2$. There are exact sequences:*

$$0 \longrightarrow Inv(\mathbf{O}_n) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{O}_{n-1}) \xrightarrow{r} Inv(\mathbf{O}_{n-2})$$
$$0 \longrightarrow Inv(\mathbf{SO}_{2n}) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{SO}_{2n-1}) \xrightarrow{r} Inv(\mathbf{SO}_{2n-2})$$
$$0 \longrightarrow Inv(\mathbf{Spin}_{2n}) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{Spin}_{2n-1}) \xrightarrow{r} Inv(\mathbf{Spin}_{2n-2})$$

*Moreover, the image of the map $r$ contains the image of the restriction map $Inv(\mathbf{O}_n) \to Inv(\mathbf{O}_{n-2})$, resp. $Inv(\mathbf{SO}_{2n}) \to Inv(\mathbf{SO}_{2n-2})$, resp. $Inv(\mathbf{Spin}_{2n}) \to Inv(\mathbf{Spin}_{2n-2})$.*

In each of the three cases the existence of the map $r$ was not known, as far as I am aware, while the other half of the exact sequence is described by Garibaldi in [4]. Of course the computation of $Inv(\mathbf{O}_n)$ and $Inv(\mathbf{SO}_n)$ has been completed (see [5] again), and indeed we shall derive it from the theorem. This affords a new construction of an invariant first defined by Serre (and written originally $b_1$). On the other hand the invariants of $\mathbf{Spin}_n$ are not known in general: they have been computed for $n \leq 12$ only ([4]), so our theorem might be of some help.

In any case, our emphasis in this paper is with methods rather than specific results, and the point that we are trying to make is that the stratification method is a powerful one. It provides us with a place to start when trying to tackle the computation of $Inv(G)$, whatever $G$ may be. It is much more mechanical than any other approach that I am aware of.

Still, the reader will find in what follows a number of examples of computations of $Inv(G)$ for various $G$'s: products of copies of $\boldsymbol{\mu}_p$ in 2.2.3, the group $\mathbf{Spin}_7$ in 4.2.2, the wreath product $\mathbf{G}_m \wr \mathbf{Z}/2$ in 4.2.1, the dihedral group $D_8$ in 5.2.3, etc.

### 1.5. Organization of the paper

We start by presenting Rost's definition of Chow groups with coefficients in section 2. We also indicate how to construct the equivariant Chow groups, and we mention a number of basic tools such as the Künneth formula.

In section 3 we introduce the stratification method, and prove the above results on orthogonal groups. We compute $Inv(\mathbf{O}_n)$ and $Inv(\mathbf{SO}_n)$ completely.

In section 4 we explain how one could use the stratification method with projective representations rather than ordinary ones. We recover a result (corollary 4.1.3) which was proved and exploited fruitfully in [4].

We proceed to introduce the Bloch & Ogus spectral sequence in section 5. As we have mentioned already, the stratification method has been used by many authors (including, Vezzosi, Vistoli and myself) to compute $CH^*BG$ and $H^*BG$, and to study the cycle map $CH^*BG \to H^*BG$. It turns out that the spectral sequence shows, roughly speaking, that cohomological invariants measure the failure of this cycle map to be an isomorphism (in small degrees this is strictly true). This was the motivation to try and extend the stratification method to cohomological invariants.

We conclude in section 6 with some easy remarks on invariants with values in other cycle modules, particularly algebraic $K$-theory.

*Notations & Conventions.* We insist on the assumptions that we have made at the beginning of this introduction: $p$ is a fixed prime, $k$ is a fixed base field, $p \neq char(k)$, and $H^*(K)$ means mod $p$ Galois cohomology as defined more precisely above. Our particular choice of twisting for the coefficients has been dictated by the desire to obtain a "cycle module" in the sense described in section 2, and at the same time keep things as simple as possible. There are more general cycle modules, including Galois cohomology simply with some more general coefficients (see [11]). In most applications, we have $p = 2$ anyway, in which case the twisting does not interfere.

We will say that $G$ is an algebraic group to indicate that $G$ is a smooth, affine group scheme over $k$ [in alternative terminology: $G$ is a linear algebraic group over $\bar{k}$ defined over $k$]. Subgroups will always be assumed to be closed and smooth. We shall encounter many times the stabilizer of an element under an action of $G$: in each case, it will be trivially true that the scheme-theoretic stablizer is closed and smooth. [Alternatively: subgroups are always defined over $k$. So will all stabilizers we encounter.]

We write $Inv(G)$ for the mod $p$ cohomological invariants of $G$, even though the letter $p$ does not appear in the notation. We caution that in [4], these would be called the invariants of $G$ with values in $\boldsymbol{\mu}_p$ (for $p = 2$ there is no difference).

*Acknowledgements.* I would like to thank Burt Totaro and Skip Garibaldi for their many suggestions.

## §2. Rost's Chow groups with coefficients

### 2.1. Definitions

In [11], Rost defines a *cycle module* to be a functor $M : F \mapsto M(F)$ from fields containing $k$ to graded abelian groups, equipped with structural data. Quoting Rost, these divide into "the even ones", namely restriction and corestriction maps, and "the odd ones": it is required that $M(F)$ should have the structure of a module over $K_*(F)$ (Milnor's $K$-theory of $F$), and there should exist residue maps for discrete valuations. The fundamental example for us is $M(F) = H^*(F)$ (mod $p$ Galois cohomology as defined in the introduction). Another example is Milnor's $K$-theory itself, $M(F) = K_*(F)$.

Given a cycle module $M$, Rost defines for any variety $X$ over $k$ the Chow groups with coefficients in $M$, written $A_i(X, M)$ ($i \geq 0$). These are bigraded; for the cycle modules above, we have the summands $A_i(X, H^j)$ and $A_i(X, K_j)$. Moreover, the classical and mod $p$ Chow groups can be recovered as $CH_iX = A_i(X, K_0)$ and $CH_iX \otimes_{\mathbf{Z}} \mathbf{Z}/p = A_i(X, H^0)$.

When $X$ is of pure dimension $n$, we shall reindex the Chow groups by putting $A^i(X, H^j) = A_{n-i}(X, H^j)$. Our interest in the theory of Chow groups with coefficients stems from the concrete description of $A^0(X, H^j)$ when $X$ is irreducible: it is the subgroup of $H^j(k(X))$ comprised of those cohomology classes in degree $j$ which are unramified at all divisors of $X$. Hence, when $X$ is a classifying space for an algebraic group $G$ (as in 1.2), we have $A^0(X, H^j) = Inv^j(G)$.

Rost's Chow groups have all the properties of ordinary Chow groups, and more: in fact they were designed to be more flexible than ordinary Chow groups, particularly in fibred situations. For the time being, we shall be content to list the following list of properties; we will introduce more as we go along. Here $X$ will denote an equidimensional variety.

1. When $X$ is smooth, $A^*(X, H^*)$ is a graded-commutative ring.

2. A map $f : X \to Y$ induces $f^* : A^*(Y, H^*) \to A^*(X, H^*)$ whenever $Y$ is smooth or $f$ is flat.

3. A proper map $f : X \to Y$ induces $f_* : A_*(X, H^*) \to A_*(Y, H^*)$.

4. There is a projection formula: $f_*(xf^*(y)) = f_*(x)y$ (here $f : X \to Y$ is proper, and $X$ and $Y$ are smooth).

5. Let $i : C \to X$ be the inclusion of a closed subvariety, and let $j : U \to X$ denote the inclusion of the open complement of $C$. Then there is a long exact sequence:

$$\cdots \longrightarrow A_m(C, H^n) \xrightarrow{i_*} A_m(X, H^n) \xrightarrow{j^*}$$
$$A_m(U, H^n) \xrightarrow{r} A_{m-1}(C, H^{n-1}) \longrightarrow \cdots$$

   Moreover the residue map $r$, or connecting homomorphism, satisfies

$$r(xj^*(y)) = r(x)i^*(y).$$

6. If $\pi : E \to X$ is the projection map of a vector bundle or an affine bundle, then $\pi^*$ is an isomorphism.

There are particular cases when we can say more about the exact sequence in property (5). When $n = 0$ for instance, we recover the usual localisation sequence:

$$CH_mC \otimes_{\mathbf{Z}} \mathbf{Z}/p \longrightarrow CH_mX \otimes_{\mathbf{Z}} \mathbf{Z}/p \longrightarrow CH_mU \otimes_{\mathbf{Z}} \mathbf{Z}/p \longrightarrow 0.$$

Also, when $X$ is equidimensional and $C$ has codimension $\geq 1$ (for example when $X$ is irreducible and $U$ is nonempty), then the following portion of the sequence is exact:

$$0 \longrightarrow A^0(X, H^n) \xrightarrow{j^*} A^0(U, H^n).$$

Example 2.1.1. Let us compute $A^0(\mathbf{G}_m, H^*)$, where $\mathbf{G}_m = \mathbf{GL}_1$ is the punctured affine line. The inclusion of the origin in $\mathbf{A}^1$ yields the exact sequence:

$$0 \longrightarrow A^0(\mathbf{A}^1, H^*) \longrightarrow A^0(\mathbf{G}_m, H^*) \longrightarrow$$
$$A^0(\mathrm{Spec}(k), H^{*-1}) \longrightarrow A^1(\mathbf{A}^1, H^{*-1}) .$$

Now, $\mathbf{A}^1$ can be seen as a (trivial) vector bundle over $\mathrm{Spec}(k)$, so $A^i(\mathbf{A}^1, H^*) = A^i(\mathrm{Spec}(k), H^*)$. From the definitions, we see directly that $A^i(\mathrm{Spec}(k), H^*) = 0$ when $i > 0$, and $A^0(\mathrm{Spec}(k), H^*) = H^*(k)$.
So we have the exact sequence of $H^*(k)$-modules:

$$0 \longrightarrow H^*(k) \longrightarrow A^0(\mathbf{G}_m, H^*) \longrightarrow H^{*-1}(k) \longrightarrow 0.$$

It follows that $A^0(\mathbf{G}_m, H^*)$ is a free $H^*(k)$-module on two generators, one in degree 0, the other in degree 1. In fact, if we write $k(\mathbf{G}_m) = k(t)$, it is not difficult to see that the element $(t) \in H^1(k(t)) = k(t)^\times/(k(t)^\times)^p$ can be taken as the generator in degree 1. For this one only has to unwind the definition of $r$ as a residue map: see [5], chap. II and III. Essentially the reason is that the divisor of the function $t$ on $\mathbf{A}^1$ is the origin.

In any case, we shall use the letter $t$ for this generator. When $k$ is algebraically closed, so that the ring $H^*(k)$ can be seen as the field $\mathbf{F}_p$ concentrated in degree 0, then we can write $A^0(\mathbf{G}_m, H^*) = \mathbf{F}_p[t]/(t^2)$, an exterior algebra.

Remark 2.1.2 – Let $X$ be smooth, and suppose that $X$ has a $k$-rational point $\mathrm{Spec}(k) \to X$. The induced map on Chow groups gives a splitting for the map $H^*(k) \to A^0(X, H^*)$ coming from the projection $X \to \mathrm{Spec}(k)$. It follows that we can write

$$A^0(X, H^*) = H^*(k) \oplus A^0(X, H^*)_{norm}.$$

The elements in the second summand are said to be *normalised*. Note that the splitting may *a priori* depend on the choice of a $k$-rational "base point" for $X$. There is an analogous notion for cohomological invariants. Namely, an invariant $n \in Inv(G)$ is called normalised when it is 0 on the trivial torsor; an invariant $c \in Inv(G)$ is called constant when there is an $\alpha \in H^*(k)$ such that for any torsor $T$ over a field $K$, the value $c(T)$ is the image of $\alpha$ under the natural map $H^*(k) \to H^*(K)$. Any invariant $a \in Inv(G)$ may be written $a = n + c$ with $n$ normalised and $c$ constant, and we may write

$$Inv(G) = H^*(k) \oplus Inv(G)_{norm}$$

where the second summand consists of normalised elements.

Whenever $X$ is a classifying space of the types considered in the introduction, there is a canonical choice of a point $\mathrm{Spec}(k) \to X$ such that the pullback of the versal torsor over $X$ is the trivial torsor. It follows that $A^0(X, H^*)_{norm} = Inv(G)_{norm}$.

Let us give an application. The group $\boldsymbol{\mu}_p$ acts freely on $\mathbf{G}_m$, and the quotient is again $\mathbf{G}_m$. Since $\mathbf{G}_m$ is special, it follows from the introduction that the torsor $\mathbf{G}_m \to \mathbf{G}_m$ is versal, so $Inv(\boldsymbol{\mu}_p) \subset A^0(\mathbf{G}_m, H^*)$. It is not immediate that this inclusion is an equality, as $\mathbf{G}_m$ is not a classifying space: $\mathbf{G}_m$ is not simply-connected, and the origin has only codimension 1 in $\mathbf{A}^1$, so we are in neither of the two favorable cases.

To show that it is indeed an equality, we remark that $A^0(\mathbf{G}_m, H^*)_{norm}$ is a free $H^*(k)$-module on one generator of degree 1, from the previous example. As a result, if we can find a non-zero, normalised invariant of $\boldsymbol{\mu}_p$ in degree 1, then the map of $H^*(k)$-modules $Inv(\boldsymbol{\mu}_p)_{norm} \to A^0(\mathbf{G}_m, H^*)_{norm}$ will certainly be surjective. Such an invariant is easy to find. Indeed, take the identity $H^1(K, \boldsymbol{\mu}_p) \to H^1(K)$. (Should you try to compute the mod $p$ invariants of $\boldsymbol{\mu}_\ell$, where $\ell$ is a prime different from $p$, you would find no normalized invariants at all. It is an easy exercise to prove this using the techniques to be introduced in this paper).

## 2.2. Spectral sequences and Künneth formula

Apart from Milnor K-theory and Galois cohomology, there is one extra type of cycle module which we shall consider in this paper. In the presence of a map $f : X \to Y$, and having picked a primary cycle module $M$, there is for each $n \geq 0$ a new cycle module written $A_n[f; M]$, see [11],§7. These are functors from fields $\kappa$ "over $Y$", that is fields with a map $\mathrm{Spec}(\kappa) \to Y$, to graded abelian groups – which turns out to be enough to define $A_*(Y, A_n[f; M])$. Quite simply, if we write $X_\kappa = \mathrm{Spec}(\kappa) \times_Y X$, the definition is:

$$A_n[f; M](\kappa) = A_n(X_\kappa, M).$$

When $f$ and $M$ are understood, we shall write $\mathcal{A}_n$ for $A_n[f; M]$. Also, when $f$ is flat, so that the fibres all have the same dimension $d$, we may change the grading to follow the codimension, thus defining $\mathcal{A}^n = \mathcal{A}_{d-n}$.

Theorem 2.2.1 – *Let $f : X \to Y$ be a flat map, and suppose that $Y$ is equidimensional. Let $\mathcal{A}_n = A_n[f; M]$ for a cycle module $M$. Then there exists a convergent spectral sequence:*

$$A^r(Y, \mathcal{A}^s) \Rightarrow A^{r+s}(X, M).$$

See [11],§8, for a proof.

Corollary 2.2.2 – *Let $X$ be a scheme (over $k$), and write $X_\kappa$ for the scheme obtained by extending the scalars to the field $\kappa$. Assume that $A^0(X, H^*)$ is a free $H^*(k)$-module of finite rank, and that*

$$A^0(X_\kappa, H^*) = A^0(X, H^*) \otimes_{H^*(k)} H^*(\kappa).$$

*Then there is a Künneth isomorphism:*

$$A^0(X \times Y, H^*) = A^0(X, H^*) \otimes_{H^*(k)} A^0(Y, H^*)$$

*for any equidimensional $Y$.*

The hypotheses of this corollary should be compared with [5], 16.5.

*Proof.* Let $f : X \times Y \to Y$ be the projection. By the theorem, we know that $A^0(X \times Y, H^*) = A^0(Y, \mathcal{A}^0)$.
However, for any field $\mathrm{Spec}(\kappa) \to Y$, the fibre of $f$ above $\mathrm{Spec}(k)$ is $X_\kappa$ as in the statement of the corollary. Our hypothesis is then that $\mathcal{A}^0(\kappa) = A^0(X_\kappa, H^*)$ is a direct sum of copies of $H^*(\kappa)$, and more precisely that $\mathcal{A}^0$ splits up as the direct sum of several copies of the cycle module $H^*$. The result follows. $\square$

As an illustration, we have the following proposition.

Proposition 2.2.3 – *There are invariants $t^{(i)} \in Inv((\boldsymbol{\mu}_p)^n)$ for $1 \leq i \leq n$, each of degree 1, such that $Inv((\boldsymbol{\mu}_p)^n)$ is a free $H^*(k)$-module on the products $t^{(i_1)}t^{(i_2)} \cdots t^{(i_k)}$, for all sequences $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.*
*When $k$ is algebraically closed, $Inv((\mathbf{Z}/p)^n)$ is an exterior algebra $\Lambda(t^{(1)}, \ldots, t^{(n)})$ over $\mathbf{F}_p$.*

*Proof.* The case $n = 1$ has been dealt with in remark 2.1.2. The general case follows from corollary 2.2.2. □

## 2.3. Equivariant Chow groups

The good properties of Chow groups with coefficients will allow us to define the equivariant Chow groups of a variety acted on by an algebraic group. This will be in strict analogy with [3] and [13], to which we will refer for details.
So let $G$ be a linear algebraic group over $k$. There exists a representation $V$ of $G$ such that the action is free outside of a closed subset $S$; moreover, we can make our choices so that the codimension of $S$ is arbitrarily large (*loc. cit.*). Put $U = V - S$.
Let $X$ be any equidimensional scheme over $k$ with an action of $G$. Define

$$X_G = \frac{U \times X}{G}.$$

We shall restrict our attention to those pairs $(X, U)$ for which $X_G$ is a scheme rather than just an algebraic space. Lemma 9 and Proposition 23 in [3] provide simple conditions on $X$ under which an appropriate $U$ may be found. This will be amply sufficient for the examples that we need to study in this paper.[1]
The notation $X_G$ hides the dependence on $U$ because, as it turns out, the Chow groups $A^i(X_G, H^*)$ do not depend on the choice of $V$ or $S$, as long as the codimension of $S$ is large enough (depending on $i$). One may prove this using the "double fibration argument" as in [13]. We write $A^i_G(X, H^*)$ for this group (or the limit taken over all good pairs $(V, S)$, if you want).
A map $f : X \to Y$ induces a map $f_G : X_G \to Y_G$ (the notation $X_G$ will mean that a choice of $V$ and $S$ has been made). It follows that the equivariant Chow groups with coefficients $A^*_G(-, H^*)$ are functorial, and indeed they have all the properties listed in §2.1. The proof of this uses that if $f$ is flat, proper, an open immersion, a vector bundle projection, etc, then $f_G$ is respectively flat, proper, an open immsersion, or a vector bundle projection (see [3]). It may be useful to spell out that a $G$-invariant open subset $U$ in an equidimensional $G$-variety $X$, whose complement $C$ has codimension $c$, gives rise to a long exact sequence:

$$\cdots \to A^m_G(C, H^n) \to A^{m+c}_G(X, H^n) \to A^{m+c}_G(U, H^n) \to A^{m+1}_G(C, H^{n-1}) \to \cdots .$$

We shall refer to it as the equivariant long exact sequence associated to $U$.

Example 2.3.1. We shall write $BG$ for $\mathrm{Spec}(k)_G$. It follows from the results above and from the introduction that

$$A^0_G(\mathrm{Spec}(k), H^*) = A^0(BG, H^*) = Inv(G).$$

[1] I am grateful to the referee for pointing out this technical difficulty. I also agree with him or her that it would be desirable to extend Rost's theory of Chow groups with coefficients to algebraic spaces. This is certainly not the place to do so.

Example 2.3.2. Suppose that the action of $G$ on $X$ is free, and that the quotient $X/G$ exists as a scheme. Then there is a natural map $X_G \to X/G$. Moreover $X_G$ is an open subset in $(V \times X)/G$, which in turn is (the total space of) a vector bundle over $X/G$. The complement of $X_G$ in $(V \times X)/G$ can be taken to have an arbitrarily large codimension (namely, it is that of $S$). In a given degree $i$, we may thus choose $V$ and $S$ appropriately and obtain:

$$A^i_G(X, H^*) = A^i(X_G, H^*) = A^i((V \times X)/G, H^*) = A^i(X/G, H^*).$$

(The second equality coming from the long exact sequence as in §2.1, property (5).)
Thus when the action is free, the equivariant Chow groups are just the Chow groups of the quotient.

Notations 2.3.3 – As we have already done in this section, we shall write $X_G$ to signify that we have chosen $V$ and $S$ with the codimension of $S$ large enough for the computation at hand. Thus we can and will write indifferently $A^*_G(X, H^*)$ or $A^*(X_G, H^*)$.
Moreover, we shall write $EG$ for $U = V - S$, with the same convention. The quotient $EG/G$ is written $BG$, and we write $A^*(BG, H^*)$ much more often than $A^*_G(\mathrm{Spec}(k), H^*)$.

Remark 2.3.4 – Whenever we have a theory at hand which has the properties listed in §2.1, we can define the equivariant analog exactly as above. Apart from Chow groups with coefficients, we shall have to consider at one point the étale cohomology of schemes, namely the groups $H^i_{\mathrm{et}}(X, \mathbf{Z}/p(i))$, which we shall write simply $H^i_{\mathrm{et}}(X)$, the coefficients being understood. In fact we shall only encounter the expression $H^i_{\mathrm{et}}(BG)$.
When $k = \mathbf{C}$, we have $H^*_{\mathrm{et}}(BG) = H^*(BG, \mathbf{F}_p)$, where on the right hand side we use topological cohomology and a model for the classifying space $BG$ in the classical, topological sense of the word. To see this, note first that $H^*_{\mathrm{et}}((V - S)/G) = H^*((V - S)/G, \mathbf{F}_p)$ since we are using finite coefficients. Moreover, we can arrange $V$ and $S$ so that $V - S$ is a Stiefel variety (see [13]), and this provides sufficiently many $V$'s and $S$'s (that is, the codimension of $S$ can be arbitrarily large, with $V - S$ a Stiefel variety). As a result, we can restrict attention to these particular pairs $(V, S)$ when forming the limit $H^*_{\mathrm{et}}(BG) = \lim H^*((V - S)/G, \mathbf{F}_p)$ without changing the result, which is then clearly $H^*(E/G, \mathbf{F}_p)$ where $E$ is the infinite Stiefel variety. This space $E$ is contractible, so $E/G$ is a topological model for $BG$.

## §3. The stratification method

### 3.1. The idea of the method

Let $G$ be a linear algebraic group. The stratification method is a procedure to compute $A^0(BG, H^*) = Inv(G)$. We shall not try to present it as a mechanical algorithm, but rather as a heuristic recipe. This being said, one of the virtues

of the method is that it is closer to a systematic treatment of the question than any other approach that the author is aware of.
The stratification method rests on two very elementary facts.

1. If $K$ is a closed subgroup of $G$, then the equivariant Chow groups of the variety $G/K$ have an easy description: indeed there is an isomorphism of varieties:
$$(G/K)_G = \frac{EG \times G/K}{G} = \frac{EG}{K} = BK.$$
Thus $A^*_G(G/K, H^*) = A^*(BK, H^*)$.

2. Suppose that $V$ is a representation of $G$. Then it is a $G$-equivariant vector bundle over a point, and therefore $V_G \to \mathrm{Spec}(k)_G = BG$ is a vector bundle, too. As a result, $A^*_G(V, H^*) = A^*(BG, H^*)$.

Things are put together in the following way. One starts with a well-chosen representation $V$, and then cuts $V$ into smaller pieces. These smaller pieces would ideally be orbits or families of orbits parametrized in a simple manner. Using (1), and hoping that the stabilizer groups $K$ showing up are easy enough to understand, one computes the Chow groups of the small pieces. Applying then repeatedly the equivariant long exact sequence from §2.3, one gets at $A^0_G(V, H^*)$. By (2), this is $A^0(BG, H^*)$.
Any particular application of the method will require the use of *ad hoc* geometric arguments. Before giving an example however, we propose to add a couple of lemmas to our toolkit.

Lemma 3.1.1 – *Suppose that $G$ and $K$ are algebraic groups acting on the left on $X$, and suppose that the actions commute. Assume further that the action of $K$ alone gives a $K$-principal bundle $X \to X/K = Y$. Then*
$$A^*_G(Y, H^*) = A^*_{G\times K}(X, H^*).$$

*Proof.* This is simply an elaboration on example 2.3.2. □

We shall have very often the occasion of using this lemma in the following guise:

Lemma 3.1.2 – *Suppose that $G$ contains a subgroup $G'$ which is an extension*
$$1 \to N \to G' \to K \to 1.$$
*Let $G \times K$ act on $G/N$ by $(\sigma, \tau)\cdot[g] = [\sigma g \tau^{-1}]$. Assume moreover that $K$ acts on a variety $X$, and that there is a $K$-principal bundle $G/N \times X \to Y$.*
*Then*
$$A^*_G(Y, H^*) = A^*_{G'}(X, H^*).$$
*If the product $N \times K$ is direct, we have $X_N \times K = BN \times X_K$.*

Here the variety $Y$ is an example of what we call "a family of orbits (of $G$) parametrized in a simple manner". Note that when $K$ is the trivial group and $X = \mathrm{Spec}(k)$, we recover property (1) above.

*Proof.* By lemma 3.1.1, we have

$$A^*_G(Y, H^*) = A^*_{G\times K}(G/N \times X, H^*).$$

Now let $E_1$ be an open subset in a representation of $G$, on which the action is free, and suppose that the codimension of the complement of $E_1$ is large enough. Similarly, pick $E_2$ for $K$; then $E_1 \times E_2$ can be chosen for the group $G \times K$. We regard $E_1$ as a trivial $K$-space and $E_2$ as a trivial $G$-space. Finally, $G'$ has natural maps to both $G$ and $K$, and we combine these to see $E_1 \times E_2$ as a $G'$-space, with non-trivial action on each factor.
We write:

$$\begin{array}{rcl} G/N \times X_G \times K & = & (E_1 \times E_2 \times \frac{G}{N} \times X)/G \times K \\ & = & (\frac{E_1\times G/N}{G} \times E_2 \times X)/K \end{array}.$$

By property (1) above, we may identify $\frac{E_1\times G/N}{G}$ with $E_1/N = BN$. Moreover we have arranged things so that, under this identification, the action of $K$ on $\frac{E_1\times G/N}{G}$ translates into its natural action on $E_1/N$ (that is, the action induced from that which $G'$ possesses as a subgroup of $G$ normalising $N$). We proceed:

$$\begin{array}{rcl} G/N \times X_G \times K & = & (E_1/N \times E_2 \times X)/K \\ & = & (E_1 \times E_2 \times X)/G' \end{array}.$$

When the product is direct, the action of $K$ on $E_1/N$ is trivial. This concludes the proof. □

As an example, we may apply this to actions of $G$ on disjoint unions:

LEMMA 3.1.3 – *Suppose that $G$ acts on $Y$ and that $Y$ is the disjoint union of varieties $Y_1 \coprod \ldots \coprod Y_n$. Let $X = Y_1$, and suppose that $X$ contains a point in each orbit, that is, suppose that the natural map*

$$f : G \times X \to Y$$

*is surjective. Let $K$ be the subgroup of $G$ leaving $X$ invariant. Then there is an identification:*

$$A^0_G(Y, H^*) = A^0_K(X, H^*).$$

*Proof.* This is immediate from the previous lemma, with $N$ the trivial group. □

### 3.2. THE EXAMPLE OF $\mathbf{O}_n$, $\mathbf{SO}_n$, AND $\mathbf{Spin}_n$

For this example we take $p = 2$ (so that $char(k) \neq 2$).
Let $q$ be a non-degenerate quadratic form on a $k$-vector space $V$ of dimension $n$. Assume moreover that $q$ has maximal Witt index. Then its automorphism group, which we will denote by $\mathbf{O}_n$, is split. Similarly one has the groups $\mathbf{SO}_n$ and $\mathbf{Spin}_n$, which are also split.

Theorem 3.2.1 – *Let $p = 2$ and $n \geq 2$. There are exact sequences:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & Inv(\mathbf{O}_n) & \longrightarrow & Inv(\mathbf{Z}/2 \times \mathbf{O}_{n-1}) & \xrightarrow{r} & Inv(\mathbf{O}_{n-2}) \\ 0 & \longrightarrow & Inv(\mathbf{SO}_{2n}) & \longrightarrow & Inv(\mathbf{Z}/2 \times \mathbf{SO}_{2n-1}) & \xrightarrow{r} & Inv(\mathbf{SO}_{2n-2}) \\ 0 & \longrightarrow & Inv(\mathbf{Spin}_{2n}) & \longrightarrow & Inv(\mathbf{Z}/2 \times \mathbf{Spin}_{2n-1}) & \xrightarrow{r} & Inv(\mathbf{Spin}_{2n-2}) \end{array}$$

*Moreover, the image of the map $r$ contains the image of the restriction map $Inv(\mathbf{O}_n) \to Inv(\mathbf{O}_{n-2})$, resp. $Inv(\mathbf{SO}_{2n}) \to Inv(\mathbf{SO}_{2n-2})$, resp. $Inv(\mathbf{Spin}_{2n}) \to Inv(\mathbf{Spin}_{2n-2})$.*

*Proof. Step 1.* Let $G_n$ denote either of $\mathbf{O}_n$, $\mathbf{SO}_n$, or $\mathbf{Spin}_n$. Then $G_n$ has the canonical representation $V$. Let $V'$ denote $V - \{0\}$. We write $U$ for the open subset in $V'$ on which $q$ is non-zero, and we write $C$ for its complement.
The codimension of $\{0\}$ in $V$ is $n$. Consider the long exact sequence:

$$0 \to A^0_{G_n}(V, H^*) \to A^0_{G_n}(V', H^*) \to$$
$$0 = A^{1-n}_{G_n}(\{0\}, H^{*-1}) \to A^1_{G_n}(V, H^{*-1}) \to A^1_{G_n}(V', H^{*-1}).$$

It follows that $A^0_{G_n}(V, H^*) = A^0_{G_n}(V', H^*) = Inv(G_n)$. Moreover the last map above is surjective when $* = 1$ (cf §2.1), so that $A^1_{G_n}(V, H^0) = A^1_{G_n}(V', H^0) = CH^1 BG_n \otimes_{\mathbf{Z}} \mathbf{Z}/2$.
Turning to the equivariant long exact sequence associated to the open set $U$ in $V'$, we finally get:

$$0 \longrightarrow Inv(G_n) \longrightarrow A^0_{G_n}(U, H^*) \longrightarrow A^0_{G_n}(C, H^{*-1}). \qquad (\dagger)$$

When $* = 1$ we have:

$$0 \to Inv^1(G_n) \to A^0_{G_n}(U, H^1) \to$$
$$A^0(C, H^0) \to CH^1 BG_n \otimes_{\mathbf{Z}} \mathbf{Z}/2 \to CH^1_{G_n} U \otimes_{\mathbf{Z}} \mathbf{Z}/2 \to 0. \qquad (\dagger\dagger)$$

*Step 2.* Let $Q = q^{-1}(1)$. We shall use the following result.

Lemma 3.2.2 – *The action of $G_n$ on $Q$ is transitive. Moreover the stabilizer of a $k$-rational point is isomorphic to $G_{n-1}$, and we get an isomorphism of $k$-varieties $Q = G_n/G_{n-1}$.*
*The action of $G_n$ on $C$ is transitive. Moreover the stabilizer of a $k$-rational point is isomorphic to a semi-direct product $H \rtimes G_{n-2}$, where $H$ is an algebraic group isomorphic to affine space as a variety, and we get an isomorphism of $k$-varieties $C = G_n/H \rtimes G_{n-2}$. Finally, the map $B(H \rtimes G_{n-2}) \to BG_{n-2}$ is an affine bundle.*

For a proof, see [10].
From this we can at once identify the last term in the exact sequence ($\dagger$). Indeed, if $S = H \rtimes G_{n-2}$ as in the lemma, then $A^0_{G_n}(C, H^*) = A^0(BS, H^*)$ as explained at the beginning of this §. However, since $BS \to BG_{n-2}$ is an affine

bundle, we draw from §2.1, property (6), that $A^0(BS, H^*) = A^0(BG_{n-2}, H^*) = Inv(G_{n-2})$.

*Step 3.* We turn to the term $A^0_{G_n}(U, H^*)$. Extend the action of $G_n$ on $Q$ to an action on $Q \times \mathbf{G}_m$ which is trivial on the second factor. The group $\mathbf{Z}/2 = \{1, \tau\}$ also acts on $Q \times \mathbf{G}_m$ by $\tau(x,t) = (-x, -t)$. The two actions commute. Finally, there is a map $Q \times \mathbf{G}_m \to U$ defined by $(x,t) \mapsto tx$. It is $G_n$-equivariant, and a $\mathbf{Z}/2$-principal bundle.

We wish to apply lemma 3.1.2 with $N = G_{n-1}$, $K = \mathbf{Z}/2$, and $G' = N \times K$. This is of course the time when we need to assume that $n$ is even if $G_n$ is $\mathbf{SO}_n$ or $\mathbf{Spin}_n$. Then the element $-Id$ generates the copy of $\mathbf{Z}/2$ that we need. We conclude:

$$A^0_{G_n}(U, H^*) = A^0(BG_{n-1} \times (\mathbf{G}_m)_{\mathbf{Z}/2}, H^*).$$

We may easily compute $A^0((\mathbf{G}_m)_{\mathbf{Z}/2}, H^*)$. Indeed, using that the action is free, with quotient $\mathbf{G}_m$, this is $A^0(\mathbf{G}_m, H^*)$ from example 2.3.2. As observed in remark 2.1.2, this happens to equal $A^0(B\mathbf{Z}/2, H^*) = Inv(\mathbf{Z}/2)$.

Now, by the Künneth formula (corollary 2.2.2), which we may use as $Inv(\mathbf{Z}/2)$ is a free $H^*(k)$-module, we draw

$$A^0(BG_{n-1} \times (\mathbf{G}_m)_{\mathbf{Z}/2}, H^*) = Inv(G_{n-1}) \otimes_{H^*(k)} Inv(\mathbf{Z}/2) = Inv(\mathbf{Z}/2 \times G_{n-1}).$$

*Step 4.* We need to prove the last sentence in the theorem. We claim that there is an $x \in Inv(\mathbf{Z}/2 \times G_{n-1})$ such that $r(x) = 1 \in Inv(G_{n-2})$. Granted this, the theorem follows from the formula $r(xj^*(y)) = r(x)i^*(y) = i^*(y)$ (see §2.1), applied here with $i^*$ denoting the restriction $Inv(G_n) \to Inv(G_{n-2})$ and $j^*$ denoting the restriction $Inv(G_n) \to Inv(\mathbf{Z}/2 \times G_{n-1})$.

To prove the claim we start by computing $CH^1_{G_n}U$ (in the rest of this proof we shall write simply $CH$ for the mod 2 Chow groups). As above this is $CH^1 BG_{n-1} \times (\mathbf{G}_m)_{\mathbf{Z}/2}$. Arguing as in example 2.3.2, we see that we may replace $(\mathbf{G}_m)_{\mathbf{Z}/2}$ by $\mathbf{G}_m/\mathbf{Z}/2 = \mathbf{G}_m$, as far as computing the Chow groups goes. Now, $\mathbf{G}_m$ being an open set in affine space, it has a trivial Chow ring and there is a Künneth formula, so:

$$CH^1 BG_{n-1} \times (\mathbf{G}_m)_{\mathbf{Z}/2} = CH^1 BG_{n-1}.$$

Therefore the last map in (††) is the restriction map $CH^1 BG_n \to CH^1 BG_{n-1}$. When $G_n = \mathbf{Spin}_n$, which is simply-connected, we have $CH^1 BG_n = 0$. In [10], the reader will find a proof that $CH^1 B\mathbf{SO}_{2n} = 0$ also and that $CH^1 B\mathbf{O}_n \to CH^1 B\mathbf{O}_{n-1}$ is an isomorphism. The claim follows from the exact sequence (††). □

### 3.3. End of the computation for $\mathbf{O}_n$

The main novelty in theorem 3.2.1 is with the spin groups (specifically, with the second half of the exact sequence presented, as the first half was obtained by Garibaldi in [4]). The invariants of $\mathbf{O}_n$ and $\mathbf{SO}_n$ are completely known, see

[5]. In this section and the next, however, we show how to recover these results from theorem 3.2.1. In the case of $\mathbf{O}_n$ this is so close to the proof given in *loc. cit.*that we shall omit most of the details. In the case of $\mathbf{SO}_n$ on the other hand, our method is considerably different, so we have thought it worthwhile to present it. We continue with $p = 2$.
A major role will be played by *Stiefel-Whitney classes*. Recall that $H^1(K, \mathbf{O}_n)$ is the set of isomorphism classes of non-degenerate quadratic forms on $K^n$. Given such a form $q$, the classes $w_i(q) \in H^*(K)$, for $i \geq 0$, have been defined by Milnor [9] (originally they were defined in mod 2 Milnor K-theory). They are 0 for $i > n$ and if $w_t(q) = \sum w_i(q)t^i$, then one has

$$w_t(q \oplus q') = w_t(q)w_t(q').$$

Each $w_i$ can be seen as a cohomological invariant in $Inv(\mathbf{O}_n)$. In fact one has

Proposition 3.3.1 – *The $H^*(k)$-module $Inv(\mathbf{O}_n)$ is free with a basis given by the classes $w_i$, for $0 \leq i \leq n$. Moreover, the following multiplicative formula holds:*

$$\begin{array}{rcll} w_1 w_{i-1} & = & w_i & (i \text{ odd}), \\ & = & (-1)w_{i-1} & (i \text{ even}). \end{array}$$

In the formula, $(-1)$ stands for the image of $-1 \in k$ in $H^1(k) = k^\times/(k^\times)^2$.

*Sketch proof.* From theorem 3.2.1, we have an injection

$$0 \to Inv(\mathbf{O}_n) \to Inv(\mathbf{Z}/2 \times \mathbf{O}_{n-1})$$

where $\mathbf{Z}/2 \times \mathbf{O}_{n-1}$ is seen as a subgroup of $\mathbf{O}_n$, the nonzero element in the copy of $\mathbf{Z}/2$ corresponding to the $-Id$ matrix. An immediate induction then shows that $Inv(\mathbf{O}_n)$ injects into $Inv((\mathbf{Z}/2)^n)$, the invariants of the elementary abelian 2-group comprised of the diagonal matrices in $\mathbf{O}_n$ with $\pm 1$ as entries. (This could be seen as a consequence of the surjection $H^1(K, (\mathbf{Z}/2)^n) \to H^1(K, \mathbf{O}_n)$, which in turns expresses the fact that quadratic forms may be diagonalized).
We have computed $Inv(\mathbf{Z}/2)$ in remark 2.1.2, and we may apply corollary 2.2.2 to obtain $Inv((\mathbf{Z}/2)^n)$. The symmetric group $S_n$ acts on this ring, and the image of $Inv(\mathbf{O}_n)$ must certainly lie in the subring fixed by this action. The latter is easily seen to be a free $H^*(k)$-module on the images of the Stiefel-Whitney classes.
The multiplicative formula is obtained by direct computation. □

### 3.4. End of the computation for $\mathbf{SO}_n$

Since $\mathbf{SO}_n$ is a subgroup of $\mathbf{O}_n$, the restriction map $Inv(\mathbf{O}_n) \to Inv(\mathbf{SO}_n)$ defines Stiefel-Whitney classes for $\mathbf{SO}_n$.
Now, as we have defined it, the group $\mathbf{SO}_n$ has the following interpretation for its torsors: $H^1(K, \mathbf{SO}_n)$ is the set of isomorphism classes of non-degenerate quadratic forms on $K^n$ whose discriminant is 1. Recall that the discriminant

is the determinant of any matrix representing the bilinear form associated to $q$, viewed as an element of $K^\times/(K^\times)^2$. As it turns out, the discriminant of $q$ is precisely $w_1(q)$ when seen as a class in the additive group $H^1(K)$. So we have $w_1 = 0$ in $Inv(\mathbf{SO}_n)$. From the formula $w_1 w_{2i} = w_{2i+1}$, we have in fact $w_{2i+1} = 0$ in $Inv(\mathbf{SO}_n)$.

Proposition 3.4.1 – *When $n = 2m+1$ is odd, $Inv(\mathbf{SO}_n)$ is a free $H^*(k)$-module with basis $\{w_0, w_2, w_4, \dots, w_{2m}\}$.*
*When $n = 2m$ is even, $Inv(\mathbf{SO}_n)$ is a free $H^*(k)$-module with basis*

$$\{w_0, w_2, w_4, \dots, w_{2m-2}, b^{(1)}_{n-1}\},$$

*where $b^{(1)}_{n-1}$ is an invariant of degree $n-1$.*

The invariant $b^{(1)}_{n-1}$ was introduced by Serre [5]. It was originally denoted by $b_1$, but we try to keep lowerscripts for the degree whenever possible.

*Proof.* The odd case is easy, for we have an isomorphism

$$\mathbf{Z}/2 \times \mathbf{SO}_{2m+1} = \mathbf{O}_{2m+1}.$$

From the Künneth formula (corollary 2.2.2), it follows that $Inv(\mathbf{SO}_{2m+1})$ is the quotient of $Inv(\mathbf{O}_{2m+1})$ by the ideal generated by $w_1$. From the formula in proposition 3.3.1, this is the submodule generated by the odd Stiefel-Whitney classes. The result follows.
Turning to the even case, we use the exact sequence from theorem 3.2.1:

$$0 \longrightarrow Inv(\mathbf{SO}_{2m}) \longrightarrow Inv(\mathbf{Z}/2 \times \mathbf{SO}_{2m-1}) \xrightarrow{\ r\ } Inv(\mathbf{SO}_{2m-2})$$

We proceed by induction, assuming the result for $n-2$.
The group $\mathbf{Z}/2 \times \mathbf{SO}_{2m-1}$ is seen as a subgroup of $\mathbf{SO}_{2m}$, were again the copy of $\mathbf{Z}/2$ is identified with $\{\pm Id\}$. However, there is also a canonical isomorphism of $\mathbf{Z}/2 \times \mathbf{SO}_{2m-1}$ with $\mathbf{O}_{2m-1}$, as above. The corresponding injective map $\mathbf{O}_{2m-1} \to \mathbf{SO}_{2m}$ thus obtained induces the map $H^1(K, \mathbf{O}_{2m-1}) \to H^1(K, \mathbf{SO}_{2m})$ which sends a quadratic form $q$ to $\tilde{q} = (q \otimes \det(q)) \oplus \det(q)$. Here $\det(q)$ is the 1-dimensional quadratic form with corresponding $1 \times 1$ matrix given by the discriminant of $q$.
We have $w_{2i+1}(\tilde{q}) = 0$ as explained above. To compute the even Stiefel-Whitney classes, we note that $w_i(\tilde{q}) = w_i((q \oplus \langle 1 \rangle) \otimes \det(q)) = w_i(q \otimes \det(q))$, and we recall that if we factorise formally

$$\sum w_i(q) t^i = \prod (1 + a_i t^i),$$

then

$$\sum w_i(q \otimes \det(q)) t^i = \prod (1 + (a_i + w_1(q)) t^i).$$

It follows that $w_{2i}(\tilde{q}) = w_{2i}(q) + w_1(q) R_i(q)$, and the remainder $w_1(q) R_i(q)$ can thus be written as a linear combination over $H^*(k)$ of the classes $w_j(q)$ with $j < 2i$. Moreover $w_{2m}(\tilde{q}) = 0$.

Now, write $w_i$, resp $w'_i$, resp $w''_i$, for the Stiefel-Whitney classes in $Inv(\mathbf{SO}_{2m})$, resp $Inv(\mathbf{O}_{2m-1})$, resp $Inv(\mathbf{SO}_{2m-2})$, and regard $Inv(\mathbf{SO}_{2m})$ as a subring of $Inv(\mathbf{O}_{2m-1})$. Thus we have $w_{2i+1} = 0$ and $w_{2i} = w'_{2i} + w'_1 R_i$.

Let $M$, resp $N$, denote the $H^*(k)$-submodule of $Inv(\mathbf{O}_{2m-1})$ generated by the classes $w'_{2i+1}$ for $0 \leq i \leq m-2$, resp by the classes $w_{2i}$ for $0 \leq i \leq m-1$ together with $b^{(1)}_{n-1} = w'_{2m+1} + (-1)w'_1 R_m$. Then $M$ and $N$ are free modules, and $Inv(\mathbf{O}_{2m-1}) = M \oplus N$. We claim (i) that the residue map $r$ is 0 on $N$ (ie $r(b^{(1)}_{n-1}) = 0$), and (ii) that $r$ maps $M$ injectively onto a free submodule in $Inv(\mathbf{SO}_{2m-2})$. This will complete the induction step, as we will have $Inv(\mathbf{SO}_{2m}) = \ker(r) = N$.

Both parts of the claim are proved by the same computation. We first note from theorem 3.2.1 that $r(w'_1) = 1$, and we compute:

$$\begin{array}{rcl} r(w'_{2i+1}) & = & r(w'_1 w'_{2i}) \\ & = & r(w'_1 w_{2i}) - r((w'_1)^2 R'_i) \\ & = & w''_{2i} - r((-1)w'_1 R_i) \\ & = & w''_{2i} - (-1)r(w'_1 R_i) \end{array} .$$

Take $i = m-1$ to obtain (i) (since $w''_{2m-2} = 0$). Property (ii) is clear.

It remains to start the induction. The group $\mathbf{SO}_2$ is a torus, so $Inv(\mathbf{SO}_2) = H^*(k)$. One gets the result for $\mathbf{SO}_4$ from this and theorem 3.2.1. This is very similar to the induction step (but easier), and will be left to the reader. □

## §4. Projective variants

### 4.1. Projective bundles

Consider the $n$-th projective space $\mathbf{P}^n$. It has an open subset isomorphic to $\mathbf{A}^n$, and from the long exact sequence of §2.1, (5), we draw at once $A^0(\mathbf{P}^n, H^*) = H^*(k)$.

Now, if $V \to X$ is a vector bundle, we may form the associated projective bundle $\pi : \mathbf{P}(V) \to X$. The fibre of $\pi$ over $\mathrm{Spec}(\kappa) \to X$ is $\mathbf{P}^{n-1}_\kappa$, where $n$ is the rank of $V$. Thus we see that the induced cycle module $\mathcal{A}^0 = A^0[\pi, H^*]$ on $X$ is isomorphic to $H^*$, and from theorem 2.2.1 we have:

Lemma 4.1.1 – *Let $\pi : \mathbf{P}(V) \to X$ denote the projective bundle associated to the vector bundle $V \to X$. Then*

$$A^0(\mathbf{P}(V), H^*) = A^0(X, H^*).$$

Remark 4.1.2 – It should be kept in mind that the above argument, which rests on the spectral sequence of theorem 2.2.1, has been used for the sake of concision only. The result is a consequence of the following general statement:

$$A^*(\mathbf{P}(V), H^*) = A^*(X, H^*)[\zeta]/(\zeta^n)$$

with $\zeta \in A^1(\mathbf{P}(V), H^0)$. This is perfectly analogous to the usual statement for ordinary Chow groups, and is no harder to prove. We shall have no use for the complete statement in the sequel, however, and therefore we omit the lengthy argument.

Corollary 4.1.3 – *Let $V$ be a representation of the linear algebraic group $G$. Form the projective representation $\mathbf{P}(V)$, and assume that there is a $k$-rational point in $\mathbf{P}(V)$ whose orbit is open, and isomorphic to $G/S$. Then there is an injection*

$$0 \to Inv(G) \to Inv(S).$$

*Proof.* We have a vector bundle $V_G \to BG$ and the associated projective bundle is $\mathbf{P}(V)_G \to BG$. From the lemma,

$$Inv(G) = A^0(BG, H^*) = A^0(\mathbf{P}(V)_G, H^*) = A^0_G(\mathbf{P}(V), H^*).$$

We may view $G/S$ as an open subset in $\mathbf{P}(V)$, and from the equivariant long exact sequence we have

$$0 \to Inv(G) \to A^0_G(G/S, H^*).$$

As noted in §3.1, we have $A^0_G(G/S, H^*) = Inv(S)$. □

### 4.2. Applications

Example 4.2.1. As a very simple illustration, we may compute the invariants of a wreath product $\mathbf{G}_m \wr \mathbf{Z}/2$, that is a semi-direct product $\mathbf{G}_m^2 \rtimes \mathbf{Z}/2$ where $\mathbf{Z}/2$ permutes the two copies of the multiplicative group.
Indeed, this group may be seen as the normalizer of a maximal torus in $\mathbf{GL}_2$, and thus it has a canonical representation $W$ of dimension 2. The space $\mathbf{P}(W)$ is just two orbits, and that of $[1,1]$ is open with stabilizer $\mathbf{G}_m \times \mathbf{Z}/2$. From corollary 4.1.3 (and corollary 2.2.2), we know that the restriction map from $Inv(\mathbf{G}_m^2 \rtimes \mathbf{Z}/2)$ to $Inv(\mathbf{Z}/2)$ is injective. It is also certainly surjective, since the inclusion $\mathbf{Z}/2 \to \mathbf{G}_m^2 \rtimes \mathbf{Z}/2$ is a section for the projection $\mathbf{G}_m^2 \rtimes \mathbf{Z}/2 \to \mathbf{Z}/2$. Finally, we conclude that $Inv(\mathbf{G}_m^2 \rtimes \mathbf{Z}/2) = Inv(\mathbf{Z}/2)$ (regardless of the choice of $p$).

Example 4.2.2. In [4], Garibaldi gives many examples of applications of corollary 4.1.3. It is possible to recover a good number of his results using the techniques of this paper. Let us illustrates this with $\mathbf{Spin}_7$ at $p = 2$.
If $\Delta$ denotes the spin representation of $\mathbf{Spin}_7$, one can show that there exists an open orbit $U$ in $\mathbf{P}(\Delta)$ with stabilizer $G_2 \times \mathbf{Z}/2$ ($G_2$ is the split group of that type). Hence an injection

$$0 \to Inv(\mathbf{Spin}_7) \to Inv(G_2 \times \mathbf{Z}/2).$$

Garibaldi computes the image of this, creating invariants of $\mathbf{Spin}_7$ by restricting invariants of $\mathbf{Spin}_8$, and exploiting the fact that further restriction from $Inv(\mathbf{Spin}_7)$ to $Inv(\mathbf{Z}/2)$ is zero, since it factors through the invariants of a maximal torus.

Our version of this computation is to study the complement of $U$ in $\mathbf{P}(\Delta)$. It consists of a single orbit with stabilizer $K$, and there is an affine bundle map $BK \to B(\mathbf{SL}_3 \times \mathbf{Z}/2)$. Therefore we have an exact sequence

$$0 \to Inv^d(\mathbf{Spin}_7) \to Inv^d(G_2 \times \mathbf{Z}/2) \to Inv^{d-1}(\mathbf{SL}_3 \times \mathbf{Z}/2).$$

This seems to be typical of the stratification method when projective representations are used: one can obtain some results very rapidly, but they are not always as accurate as one may wish.
Instead, if we proceed exactly as in the proof of theorem 3.2.1, we obtain:

$$0 \to Inv^d(\mathbf{Spin}_7) \to Inv^d(G_2 \times \mathbf{Z}/2) \to Inv^{d-1}(\mathbf{SL}_3) \to 0.$$

Since $\mathbf{SL}_3$ is special, $Inv(\mathbf{SL}_3) = H^*(k)$. From [5] we have that $Inv(G_2)$ is a free $H^*(k)$-module on two generators of degree 0 and 3. By corollary 2.2.2, $Inv(G_2 \times \mathbf{Z}/2)$ is a free $H^*(k)$-modules with 4 generators of degree 0, 1, 3 and 4. It follows that $Inv(\mathbf{Spin}_7)$ is a free $H^*(k)$-module on 3 generators of degree 0, 3 and 4.

## §5. The Bloch & Ogus spectral sequence

### 5.1. The spectral sequence

In this section, we assume that $k$ is algebraically closed. The prime $p$ being fixed as always, we shall write $H^i_{\text{et}}(X)$ for the étale cohomology group $H^i_{\text{et}}(X, \mathbf{Z}/p)$.

Theorem 5.1.1 (Rost, Bloch, Ogus) – *Let $k = \bar{k}$, and let $X$ be equidimensional. Then there is a spectral sequence*

$$E_2^{r,s} = A^r(X, H^{s-r}) \Rightarrow H^{r+s}_{\text{et}}(X).$$

*In particular the $E_2$ page is zero under the first diagonal, and the resulting map*

$$A^n(X, H^0) = CH^n X \otimes_{\mathbf{Z}} \mathbf{Z}/p \to H^n_{\text{et}}(X)$$

*is the usual cycle map.*

A word of explanation on the authorship of the spectral sequence. To start with, there is the well-known coniveau spectral sequence, which converges to $H^n_{\text{et}}(X)$ and for which there is a description of the $E_1$ term. In [2] Bloch and Ogus prove that the $E_2$ term can be identified with $H^r_{Zar}(X, \mathcal{H}^s)$, where $\mathcal{H}^s$ is the sheafification of $U \mapsto H^s_{\text{et}}(U)$. They deduce that the groups are 0 under the diagonal, and prove the statement about the cycle map. More than 20 years later, in [11], corollary 6.5, Rost proves that the $E_2$ term can also be described using his Chow groups with coefficients as in the theorem. The sequence seems to be usually refered to as the Bloch & Ogus spectral sequence.
Given an algebraic group $G$, we may take a model for $BG$ to play the role of $X$, and obtain:

Corollary 5.1.2 – *There is a map*

$$H^*_{\mathrm{et}}(BG) \to Inv(G)$$

*which vanishes (when* $* > 0$*) on the image of the cycle map*

$$CH^*BG \otimes_{\mathbf{Z}} \mathbf{Z}/p \to H^{2*}_{\mathrm{et}}(BG).$$

Example 5.1.3. The Stiefel-Whitney classes as in 3.3 can be defined as the images of the elements with the same name in the cohomology of $B\mathbf{O}_n$. The corollary explains why they square to zero in $Inv(\mathbf{O}_n)$, since they square to Chern classes in cohomology, and these come from the Chow ring (see [13] for details).

Remark 5.1.4 – When $G$ is a finite group, viewed as a 0-dimensional algebraic group, there is a natural map

$$H^*(G, \mathbf{F}_p) \to H^*_{\mathrm{et}}(BG),$$

where $H^*(G, \mathbf{F}_p)$ is the usual cohomology of $G$ as a discrete group. Indeed, there is a Galois covering $EG \to BG$ with group $G$, so we may see this map as coming from the corresponding Hoschild-Serre spectral sequence. Since we assume in this section that $k$ is algebraically closed, it follows easily that the map is an isomorphism.
Composing this with the map from the previous corollary, we obtain the homomorphism

$$H^*(G, \mathbf{F}_p) \to Inv(G)$$

which was considered in [5]. Of course the direct definition given in *loc. cit.* is much preferable.

Corollary 5.1.5 (to theorem 5.1.1) – *There are natural isomorphisms:*

$$Inv^1(G) = H^1_{\mathrm{et}}(BG)$$

*and*

$$Inv^2(G) = \frac{H^2_{\mathrm{et}}(BG)}{CH^1BG \otimes_{\mathbf{Z}} \mathbf{Z}/p}.$$

The denominator in the second isomorphism is really the image of $CH^1BG \otimes_{\mathbf{Z}} \mathbf{Z}/p$ in étale cohomology via the cycle map. When $k = \mathbf{C}$, the cycle map is injective in degree 1 and in fact $CH^1BG = H^2(BG, \mathbf{Z})$ (topological cohomology), see [13].

Corollary 5.1.6 (to theorem 5.1.1) – *Any class $x$ in the kernel of the cycle map*

$$CH^2BG \otimes_{\mathbf{Z}} \mathbf{Z}/p \to H^4_{\mathrm{et}}(BG)$$

*determines an invariant $r_x \in Inv^3(BG)$, which is well-defined up to the image of $H^3_{\mathrm{et}}(BG) \to Inv^3(BG)$. If $x$ is nonzero, neither is $r_x$.*

We think of $r_x$ as a simplified version of the Rost invariant.

*Proof.* Our assumption implies that the class $x$, viewed as an element of bidegree $(2,2)$ on the $E_2$ page of the spectral sequence under discussion, must be hit by a differential. Let $r_x \in E_2^{0,3} = Inv^3(G)$ be such that $d_2(r_x) = x$. This element is well-defined up to the kernel of $d_2$. Since further differentials $d_r$ for $r > 2$ are zero on $E_r^{0,3}$, we see that $r_x$ is defined up to elements which survive to the $E_\infty$ page. These are, by definition, the elements in the image of $H^3_{\text{et}}(BG) \to Inv^3(BG)$. $\square$

## 5.2. Applications

For simplicity, we shall take $k = \mathbf{C}$, the complex numbers, in the applications. In this case according to remark 2.3.4, the étale cohomology of $BG$ as above coincides with the topological cohomology of a topological classifying space (i.e. a quotient $EG/G$ of a contractible space $EG$ endowed with a free $G$-action). We start with a proposition which should be compared with statements 31.15 and 31.20 in [7], for which there is no proof available as far as I am aware.

Proposition 5.2.1 – *Over the complex numbers, there are isomorphisms*

$$Inv^1(G) = Hom(\pi_0(G), \mathbf{Z}/p)$$

*and*

$$Inv^2(G) = p\text{-torsion in } H^3(BG, \mathbf{Z}).$$

*In particular, if $G$ is connected then $Inv^1(G) = 0$, and if $G$ is 1-connected then $Inv^2(G) = 0$.*

*Proof.* From corollary 5.1.5, we have $Inv^1(G) = H^1(BG, \mathbf{F}_p)$, and of course this is $Hom(\pi_1(BG), \mathbf{Z}/p)$. However $\pi_1(BG) = \pi_0(G)$.

As noted above, the cycle map is injective in degree 1 over the complex numbers, and $CH^1 BG = H^2(BG, \mathbf{Z})$. From corollary 5.1.5 again, we see that

$$Inv^2(G) = \frac{H^2(BG, \mathbf{F}_p)}{H^2(BG, \mathbf{Z}) \otimes \mathbf{Z}/p},$$

and it is elementary to show that this maps injectively onto the $p$-torsion in $H^3(BG, \mathbf{Z})$ via the Bockstein.

The statement about connected groups is trivial. When $G$ is 1-connected, it is automatically 2-connected, since any real Lie group has $\pi_2(G) = 0$. Thus $BG$ is 3-connected and we draw $H^3(BG, \mathbf{Z}) = 0$ from Hurewicz's theorem. $\square$

Example 5.2.2. Consider the case of the exceptional group $G_2$, for $k = \mathbf{C}$ and $p = 2$. It is reductive and 1-connected, so by the proposition we have $Inv^i(G_2) = 0$ for $i = 1, 2$. Moreover, the Chow ring of $G_2$ has been computed over the complex numbers, see [6]. It turns out that the map $CH^2 BG_2 \otimes_{\mathbf{Z}} \mathbf{Z}/2 \to H^4(BG_2, \mathbf{F}_2)$ has exactly one nonzero element. From corollary 5.1.6,

we know that there is a nonzero invariant $e_3 \in Inv^3(G_2)$. It is uniquely defined as $H^3(BG_2, \mathbf{F}_2) = 0$.
It is proved in [5] that for any field $k$ of characteristic $\neq 2$, $Inv(G_2)$ is in fact a free $H^*(k)$-module on the generators 1 and $e_3$.

EXAMPLE 5.2.3. We take $k = \mathbf{C}$, $p = 2$, and $G = D_8 = \mathbf{Z}/4 \rtimes \mathbf{Z}/2$, the dihedral group. We shall completely compute $Inv(G)$ by showing first that things reduce to corollary 5.1.5.
Let $t$ be a generator of $\mathbf{Z}/4$ in $G$, and let $\tau$ be the second generator, so that $\tau(t) = \tau t \tau^{-1} = t^{-1}$. We let $G$ act on $V = \mathbf{A}^2$ via

$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and

$$\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding map $G \to \mathbf{GL}_2$ is an embedding, so $G$ acts freely outside of a finite number of lines. Let $U$ denote the open complement. We have an injection

$$0 \to A^0_G(V, H^*) = Inv(G) \to A^0_G(U, H^*) = A^0(U/G, H^0),$$

by example 2.3.2. Now, $U/G$ is a variety of dimension 2 over an algebraically closed field, so $H^*(k(U)) = 0$ for $* > 2$. It follows that $Inv(G)$ is concentrated in degrees $\leq 2$.
Since we are working over $\mathbf{C}$, we have $H^*_{\mathrm{et}}(BG) = H^*(BG, \mathbf{F}_2) = H^*(G, \mathbf{F}_2)$. The cohomology of $G$ is well-known, see for example [1]:

$$H^*(G, \mathbf{F}_2) = \mathbf{F}_2[x_1, y_1, w_2]/(x_1 y_1 = 0).$$

More precisely, $G$ can be presented as an "extraspecial group", ie as a central extension

$$0 \to \mathbf{Z}/2 \to G \to E \to 0$$

where $E \approx \mathbf{Z}/2 \times \mathbf{Z}/2$. If we choose $x_1$ and $y_1$ so that $H^*(E, \mathbf{F}_2) = \mathbf{F}_2[x_1, y_1]$, then we can pullback these classes to the cohomology of $G$ where they will give the classes with the same name in the description above. (Besides, the cohomology class of the extention is $x_1 y_1$.)
We have immediately, by corollary 5.1.5, that $Inv^1(G) = \mathbf{F}_2 \cdot x_1 \oplus \mathbf{F}_2 \cdot y_1$. In degree 2, we note that $H^2(G, \mathbf{F}_2)$ is generated additively by the classes $x_1^2$, $y_1^2$, and $w_2$. The Chow ring of $BE$ is $CH^*BE \otimes_{\mathbf{Z}} \mathbf{Z}/2 = \mathbf{F}_2[x_1^2, y_1^2]$, from which we know that the classes $x_1^2$ and $y_1^2$ in the cohomology of $BG$ certainly come from the Chow ring of $BG$.
On the other hand, $Sq^1 w_2 = w_2(x_1 + y_1) \neq 0$ (*loc. cit.*), while the Steenrod operation $Sq^1$ is zero on classes coming from the integral cohomology, and *a fortiori* it is zero on the classes coming from the Chow ring.

As a result, we have finally $Inv^2(G) = \mathbf{F}_2 \cdot w_2$.
Over a general field, we could reach a similar conclusion by studying the geometric situation a bit more carefully. The variety $U/G$, for example, can be shown to be the open subset in $\mathbf{A}^2$ obtained by removing the axis $Y = 0$ and the two parabolae $X \pm 2Y^2 = 0$. Alternatively, you might want to use that $D_8$ is a 2-Sylow in $S_4$, and exploit the double coset formula as in [5], chap. V.

## §6. Other cycle modules

We shall conclude with a few simple remarks on other possible cycle modules. Given any cycle module $M$ as in §2, we may define the invariants $Inv(G, M)$ of $G$ with values in $M$ to be the natural transformations of functors

$$H^1(-, G) \to M(-).$$

It is straightforward to establish the inclusion

$$Inv(G, M) \subset A^0(BG, M)$$

for any $BG$ which is the base of a versal $G$-principal bundle. Indeed, the arguments in [5] hold *verbatim* (and in fact in *loc. cit.* the reader will find a similar inclusion even for invariants with values in the Witt ring, even though the Witt ring satisfies weaker properties than cycle modules do.)
We may also define the equivariant Chow groups $A^*_G(X, M)$ for varieties $X$ acted on by $G$, exactly as we have done for $M = H^*$. When this is done, we may take $BG$ to be $\mathrm{Spec}(k)_G$ (as in the rest of this paper), and we have an equality

$$Inv(G, M) = A^0(BG, M) = A^0_G(\mathrm{Spec}(k), M)$$

by Totaro's arguments as in [5].
The techniques we have used for Galois cohomology may be used for any cycle module. Let us illustrate this for $M = K_*$, the algebraic $K$-theory of fields (Milnor or Quillen, it will not affect the sequel). In this case we have $A^*(X, K_0) = CH^*X$, and the parallel with our previous computations becomes even more obvious.
Arguing as in example 2.1.1, we obtain that $A^0(\mathbf{G}_m, K_*)$ is a free $K_*$-module on two generators, one in dimension 0, the other in dimension 1. If $p$ is a prime number, we may use this to compute the invariants of $\boldsymbol{\mu}_p$.
The group $\boldsymbol{\mu}_p$ has a 1-dimensional representation $V$, and $\mathbf{G}_m$ sits in $V$ as a $\boldsymbol{\mu}_p$-invariant open subset whose complement is a point $\mathrm{Spec}(k)$. We obtain the exact sequence (where $G = \boldsymbol{\mu}_p$):

$$0 \to A^0_G(V, K_*) \to A^0_G(\mathbf{G}_m, K_*) \to A^0_G(\mathrm{Spec}(k), K_{*-1}) \to A^1_G(V, K_{*-1}).$$

The action on $\mathbf{G}_m$ is free with quotient $\mathbf{G}_m$, so as in example 2.3.2 we draw $A^0_G(\mathbf{G}_m, K_*) = A^0(\mathbf{G}_m, K_*)$. Thus we may rewrite the exact sequence:

$$0 \longrightarrow Inv(G, K_*) \longrightarrow A^0(\mathbf{G}_m, K_*) \longrightarrow Inv(G, K_{*-1}),$$

and for $* = 1$ it is worth writing the extra term:

$$0 \to Inv(G, K_1) \to A^0(\mathbf{G}_m, K_1) \to Inv(G, K_0) \xrightarrow{s_*} CH^1BG \to 0.$$

Let us explain the notation $s_*$ and what this map looks like. We call $s$ the zero section $s : BG \to V_G$ of the vector bundle $\pi : V_G \to BG$, and $s_*$ is the induced pushforward map. We know that $\pi^*$ is an isomorphism, and that there is a projection formula $s_*(\pi^*(x)y) = xs_*(y)$. If we use $\pi^*$ as an identification, it follows that $s_*(x) = c_1(V)x$ where $c_1(V) = s_*(1)$ is the first *Chern class* of $V$. Finally the map $s_* : Inv(G, K_0) \to CH^1BG$ is simply the surjective map $\mathbf{Z} \to CH^1BG$ sending 1 to $c_1(V)$. Now, $CH^1BG$ is $p$-torsion by a transfer argument (or see [13], example 13.1, which shows that $CH^1BG$ may well be 0 depending on $k$). In any case, $s_*$ has a kernel isomorphic to $\mathbf{Z}$.

We conclude that the 1-dimensional generator for $A^0(\mathbf{G}_m, K_*)$ must map to a generator for this kernel. As a result, $Inv(G, K_*)$ is reduced to $K_*$, i.e. *the group $\boldsymbol{\mu}_p$ has no nonconstant invariants in algebraic $K$-theory at all.*

For $p = 2$ for example, assuming that $char(k) \neq 2$, we may use the surjection $H^1(K, (\mathbf{Z}/2)^n) \to H^1(K, \mathbf{O}_n)$ for fields containing $k$ to deduce that $Inv(\mathbf{O}_n, K_*)$ injects into $Inv((\mathbf{Z}/2)^n, K_*)$. Thus, after an obvious Künneth argument, we see that $\mathbf{O}_n$ has no nonconstant invariants in algebraic $K$-theory, either. In particular, there is no natural way of lifting the Siefel-Whitney classes to integral Milnor $K$-theory.

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# Erratum

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Abstract. This is an erratum concerning the article "Statistics of Lattice Points in Thin Annuli for Generic Lattices", Documenta Math. 11 (2005), 1–23

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1. The correct statement of Lemma 4.1 is: for every $a > 0$,

$$N_\Lambda(t) = \frac{\pi}{d}t^2 - \frac{\sqrt{t}}{d\pi} \sum_{\substack{\vec{k}\in\Lambda^*\setminus\{0\} \\ |\vec{k}|\le\sqrt{N}}} \frac{\cos\left(2\pi t|\vec{k}| + \frac{\pi}{4}\right)}{|\vec{k}|^{\frac{3}{2}}} + O(N^a) + O\left(\frac{t^3}{\delta_\Lambda(t^2)\sqrt{N}}\right),$$

   provided that

$$\delta_\Lambda(y) < 1 \tag{1}$$

   for all $y > 0$.

   The proof of this lemma proceeds exactly as the proof of the correct statement corresponding to lemma 4.1 of [W] (see the erratum to [W]).

2. In the course of "unsmoothing" (that is, the proof of lemma 4.2), we invoke lemma 4.1 with $\delta_\Lambda(y) = \frac{c_1}{y^{K_0}}$ and $\delta_{\Lambda^*}(y) = \frac{c_2}{y^{K_0}}$, where $c_1$, $c_2$ are constants. We choose $N = T^H$, with

$$H := 8 + 4K_0,$$

   and proceed as in the original text.

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# Rational Real Algebraic Models of Topological Surfaces

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Abstract. Comessatti proved that the set of all real points of a rational real algebraic surface is either a nonorientable surface, or diffeomorphic to the sphere or the torus. Conversely, it is well known that each of these surfaces admits at least one rational real algebraic model. We prove that they admit exactly one rational real algebraic model. This was known earlier only for the sphere, the torus, the real projective plane and the Klein bottle.

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Keywords and Phrases: Real algebraic surface, topological surface, rational surface, rational model, birational map, algebraic diffeomorphism, transitivity, geometrically rational surface, geometrically rational model

## 1 Introduction

Let $X$ be a rational nonsingular projective real algebraic surface. Then the set $X(\mathbb{R})$ of real points of $X$ is a compact connected topological surface. Comessatti showed that $X(\mathbb{R})$ cannot be an orientable surface of genus bigger than 1. To put it otherwise, $X(\mathbb{R})$ is either nonorientable, or it is orientable and diffeomorphic to the sphere $S^2$ or the torus $S^1 \times S^1$ [Co2, p. 257].
Conversely, each of these topological surfaces admits a *rational real algebraic model*, or *rational model* for short. In other words, if $S$ is a compact connected topological surface which is either nonorientable, or orientable and diffeomorphic to the sphere or the torus, then there is a nonsingular rational projective real algebraic surface $X$ such that $X(\mathbb{R})$ is diffeomorphic to $S$. Indeed, this is clear for the sphere, the torus and the real projective plane: the real projective surface defined by the affine equation $x^2 + y^2 + z^2 = 1$ is a rational

model of the sphere $S^2$, the real algebraic surface $\mathbb{P}^1 \times \mathbb{P}^1$ is a rational model of the torus $S^1 \times S^1$, and the real projective plane $\mathbb{P}^2$ is a rational model of the topological real projective plane $\mathbb{P}^2(\mathbb{R})$. If $S$ is any of the remaining topological surfaces, then $S$ is diffeomorphic to the $n$-fold connected sum of the real projective plane, where $n \geq 2$. A rational model of such a topological surface is the real surface obtained from $\mathbb{P}^2$ by blowing up $n-1$ real points. Therefore, any compact connected topological surface which is either nonorientable, or orientable and diffeomorphic to the sphere or the torus, admits at least one rational model.

Now, if $S$ is a compact connected topological surface admitting a rational model $X$, then one can construct many other rational models of $S$. To see this, let $P$ and $\overline{P}$ be a pair of complex conjugate complex points on $X$. The blow-up $\widetilde{X}$ of $X$ at $P$ and $\overline{P}$ is again a rational model of $S$. Indeed, since $P$ and $\overline{P}$ are nonreal points of $X$, there are open subsets $U$ of $X$ and $V$ of $\widetilde{X}$ such that

- $X(\mathbb{R}) \subseteq U(\mathbb{R})$, $\widetilde{X}(\mathbb{R}) \subseteq V(\mathbb{R})$, and
- $U$ and $V$ are isomorphic.

In particular, $X(\mathbb{R})$ and $\widetilde{X}(\mathbb{R})$ are diffeomorphic. This means that $\widetilde{X}$ is a rational model of $S$ if $X$ is so. Iterating the process, one can construct many nonisomorphic rational models of $S$. We would like to consider all such models of $S$ to be equivalent. Therefore, we introduce the following equivalence relation on the collection of all rational models of a topological surface $S$.

Definition 1.1. Let $X$ and $Y$ be two rational models of a topological surface $S$. We say that $X$ and $Y$ are isomorphic as rational models of $S$ if there is a sequence

$$\begin{tikzcd} & X_1 \arrow[dl] \arrow[dr] & & X_3 \arrow[dl] \arrow[dr] & & X_{2n-1} \arrow[dl] \arrow[dr] & \\ X = X_0 & & X_2 & & \cdots & & X_{2n} = Y \end{tikzcd}$$

where each morphism is a blowing-up at a pair of nonreal complex conjugate points.

We note that the equivalence relation, in Definition 1.1, on the collection of all rational models of a given surface $S$ is the smallest one for which the rational models $X$ and $\widetilde{X}$ mentioned above are equivalent.

Let $X$ and $Y$ be rational models of a topological surface $S$. If $X$ and $Y$ are isomorphic models of $S$, then the above sequence of blowing-ups defines a rational map

$$f \colon X \dashrightarrow Y$$

having the following property. There are open subsets $U$ of $X$ and $V$ of $Y$ such that

- the restriction of $f$ to $U$ is an isomorphism of real algebraic varieties from $U$ onto $V$, and
- $X(\mathbb{R}) \subseteq U(\mathbb{R})$ and $Y(\mathbb{R}) \subseteq V(\mathbb{R})$.

It follows, in particular, that the restriction of $f$ to $X(\mathbb{R})$ is an *algebraic diffeomorphism* from $X(\mathbb{R})$ onto $Y(\mathbb{R})$, or in other words, it is a *biregular map* from $X(\mathbb{R})$ onto $Y(\mathbb{R})$ in the sense of [BCR].

Let us recall the notion of an algebraic diffeomorphism. Let $X$ and $Y$ be smooth projective real algebraic varieties. Then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are compact manifolds, not necessarily connected or nonempty. Let

$$f : X(\mathbb{R}) \longrightarrow Y(\mathbb{R}) \tag{1}$$

be a map. Choose affine open subsets $U$ of $X$ and $V$ of $Y$ such that $X(\mathbb{R}) \subseteq U(\mathbb{R})$ and $Y(\mathbb{R}) \subseteq V(\mathbb{R})$. Since $U$ and $V$ are affine, we may assume that they are closed subvarieties of $\mathbb{A}^m$ and $\mathbb{A}^n$, respectively. Then $X(\mathbb{R})$ is a closed submanifold of $\mathbb{R}^m$, and $Y(\mathbb{R})$ is a closed submanifold of $\mathbb{R}^n$. The map $f$ in (1) is *algebraic* or *regular* if there are real polynomials $p_1, \ldots, p_n, q_1, \ldots, q_n$ in the variables $x_1, \ldots, x_m$ such that none of the polynomials $q_1, \ldots, q_n$ vanishes on $X(\mathbb{R})$, and

$$f(x) = \left(\frac{p_1(x)}{q_1(x)}, \ldots, \frac{p_n(x)}{q_n(x)}\right)$$

for all $x \in X(\mathbb{R})$.

One can check that the algebraicity of $f$ depends neither on the choice of the affine open subsets $U$ and $V$ nor on the choice of the embeddings of $U$ and $V$ in affine space. Note that the algebraicity of $f$ immediately implies that $f$ is a $C^\infty$-map.

The map $f$ in (1) is an *algebraic diffeomorphism* if $f$ is algebraic, bijective, and $f^{-1}$ is algebraic.

Again let $X$ and $Y$ be rational models of a topological surface $S$. As observed above, if $X$ and $Y$ are isomorphic models of $S$, then there is an algebraic diffeomorphism

$$f \colon X(\mathbb{R}) \longrightarrow Y(\mathbb{R}) .$$

Conversely, if there is an algebraic diffeomorphism $f \colon X(\mathbb{R}) \longrightarrow Y(\mathbb{R})$, then $X$ and $Y$ are isomorphic models of $S$, as it follows from the well known Weak Factorization Theorem for birational maps between real algebraic surfaces(see [BPV, Theorem III.6.3] for the WFT over $\mathbb{C}$, from which the WFT over $\mathbb{R}$ follows).

Here we address the following question. Given a compact connected topological surface $S$, what is the number of nonisomorphic rational models of $S$?

By Comessatti's Theorem, an orientable surface of genus bigger than 1 does not have any rational model. It is known that the topological surfaces $S^2$, $S^1 \times S^1$ and $\mathbb{P}^2(\mathbb{R})$ have exactly one rational model, up to isomorphism (see also Remark 3.2). Mangolte has shown that the same holds for the Klein bottle [Ma, Theorem 1.3] (see again Remark 3.2).

Mangolte asked how large $n$ should be so that the $n$-fold connected sum of the real projective plane admits more than one rational model, up to isomorphism; see the comments following Theorem 1.3 in [Ma]. The following theorem shows that there is no such integer $n$.

THEOREM 1.2. *Let $S$ be a compact connected real two-manifold.*

1. *If $S$ is orientable of genus greater than 1, then $S$ does not admit any rational model.*
2. *If $S$ is either nonorientable, or it is diffeomorphic to one of $S^2$ and $S^1 \times S^1$, then there is exactly one rational model of $S$, up to isomorphism. In other words, any two rational models of $S$ are isomorphic.*

Of course, statement 1 is nothing but Comessatti's Theorem referred to above. Our proof of statement 2 is based on the Minimal Model Program for real algebraic surfaces developed by János Kollár in [Ko1]. Using this Program, we show that a rational model $X$ of a nonorientable topological surface $S$ is obtained from $\mathbb{P}^2$ by blowing it up successively in a finite number of real points (Theorem 3.1). The next step of the proof of Theorem 1.2 involves showing that the model $X$ is isomorphic to a model $X'$ obtained from $\mathbb{P}^2$ by blowing up $\mathbb{P}^2$ at real points $P_1, \ldots, P_n$ of $\mathbb{P}^2$. At that point, the proof of Theorem 1.2 would have been finished if we were able to prove that the group $\mathrm{Diff}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$ of algebraic diffeomorphisms of $\mathbb{P}^2(\mathbb{R})$ acts $n$-transitively on $\mathbb{P}^2(\mathbb{R})$. However, we were unable to prove such a statement. Nevertheless, a statement we were able to prove is the following.

THEOREM 1.3. *Let $n$ be a natural integer. The group $\mathrm{Diff}_{\mathrm{alg}}(S^1 \times S^1)$ acts $n$-transitively on $S^1 \times S^1$.*

We conjecture, however, the following.

CONJECTURE 1.4. Let $X$ be a smooth projective rational surface. Let $n$ be a natural integer. Then the group $\mathrm{Diff}_{\mathrm{alg}}(X(\mathbb{R}))$ acts $n$-transitively on $X(\mathbb{R})$.[1]

The only true evidence we have for the above conjecture is that it holds for $X = \mathbb{P}^1 \times \mathbb{P}^1$ according to Theorem 1.3.

Now, coming back to the idea of the proof of Theorem 1.2, we know that any rational model of $S$ is isomorphic to one obtained from $\mathbb{P}^2$ by blowing up $\mathbb{P}^2$ at real points $P_1, \ldots, P_n$. Since we have established $n$-transitivity of the group of algebraic diffeomorphisms of $S^1 \times S^1$, we need to realize $X'$ as a blowing-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a finite number of real points.

Let $L$ be the real projective line in $\mathbb{P}^2$ containing $P_1$ and $P_2$. Applying a nontrivial algebraic diffeomorphism of $\mathbb{P}^2$ into itself, if necessary, we may assume that $P_i \notin L$ for $i \geq 3$. Then we can do the usual transformation of $\mathbb{P}^2$ into $\mathbb{P}^1 \times \mathbb{P}^1$ by first blowing-up $P_1$ and $P_2$, and then contracting the strict

[1] This conjecture is now proved [HM].

transform of $L$. This realizes $X'$ as a surface obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing-up $\mathbb{P}^1 \times \mathbb{P}^1$ at $n-1$ distinct real points. Theorem 1.2 then follows from the $(n-1)$-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(S^1 \times S^1)$.

We will also address the question of uniqueness of geometrically rational models of a topological surface. By yet another result of Comessatti, a geometrically rational real surface $X$ is rational if $X(\mathbb{R})$ is nonempty and connected. Therefore, Theorem 1.2 also holds when one replaces "rational models" by "geometrically rational models". Since the set of real points of a geometrically rational surface is not neccesarily connected, it is natural to study geometrically rational models of not necessarily connected topological surfaces. We will show that such a surface has an infinite number of geometrically rational models, in general.

The paper is organized as follows. In Section 2 we show that a real Hirzebruch surface is either isomorphic to the standard model $\mathbb{P}^1 \times \mathbb{P}^1$ of the real torus $S^1 \times S^1$, or isomorphic to the standard model of the Klein bottle. The standard model of the Klein bottle is the real algebraic surface $B_P(\mathbb{P}^2)$ obtained from the projective plane $\mathbb{P}^2$ by blowing up one real point $P$. In Section 3, we use the Minimal Model Program for real algebraic surfaces in order to prove that any rational model of any topological surface is obtained by blowing up one of the following three real algebraic surfaces: $\mathbb{P}^2$, $\mathbb{S}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (Theorem 3.1). Here $\mathbb{S}^2$ is the real algebraic surface defined by the equation $x^2 + y^2 + z^2 = 1$. As a consequence, we get new proofs of the known facts that the sphere, the torus, the real projective plane and the Klein bottle admit exactly one rational model, up to isomorphism of course. In Section 4 we prove a lemma that will have two applications. Firstly, it allows us to conclude the uniqueness of a rational model for the "next" topological surface, the 3-fold connected sum of the real projective plane. Secondly, it also allows us to conclude that a rational model of a nonorientable topological surface is isomorphic to a model obtained from $\mathbb{P}^2$ by blowing up a finite number of distinct real points $P_1, \ldots, P_n$ of $\mathbb{P}^2$. In Section 5 we prove $n$-transitivity of the group of algebraic diffeomorphisms of the torus $S^1 \times S^1$. In Section 6 we construct a nontrivial algebraic diffeomorphism $f$ of $\mathbb{P}^2(\mathbb{R})$ such that the real points $f(P_i)$, for $i = 3, \ldots, n$, are not on the real projective line through $f(P_1)$ and $f(P_2)$. In Section 7 we put all the pieces together and complete the proof of Theorem 1.2. In Section 8 we show by an example that the uniqueness does not hold for geometrically rational models of nonconnected topological surfaces.

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## 2 Real Hirzebruch surfaces

The set of real points of the rational real algebraic surface $\mathbb{P}^1 \times \mathbb{P}^1$ is the torus $S^1 \times S^1$. We call this model the *standard model* of the real torus. Fix a real point $O$ of the projective plane $\mathbb{P}^2$. The rational real algebraic surface $B_O(\mathbb{P}^2)$

obtained from $\mathbb{P}^2$ by blowing up the real point $O$ is a model of the Klein bottle $K$. We call this model the *standard model* of the Klein bottle.
Let $d$ be a natural integer. Let $\mathbb{F}_d$ be the *real Hirzebruch surface* of degree $d$. Therefore, $\mathbb{F}_d$ is the compactification $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d)\oplus\mathcal{O}_{\mathbb{P}^1})$ of the line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ over $\mathbb{P}^1$. Recall that the real algebraic surface $\mathbb{F}_d$ is isomorphic to $\mathbb{F}_e$ if and only if $d = e$. The restriction of the line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ to the set of real points $\mathbb{P}^1(\mathbb{R})$ of $\mathbb{P}^1$ is topologically trivial if and only if $d$ is even. Consequently, $\mathbb{F}_d$ is a rational model of the torus $S^1 \times S^1$ if $d$ is even, and it is a rational model of the Klein bottle $K$ if $d$ is odd (see [Si, Proposition VI.1.3] for a different proof).
The following statement is probably well known, and is an easy consequence of known techniques (compare the proof of Theorem 6.1 in [Ma]). We have chosen to include the statement and a proof for two reasons: the statement is used in the proof of Theorem 3.1, and the idea of the proof turns out also to be useful in Lemma 4.1.

Proposition 2.1. *Let $d$ be a natural integer.*

1. *If $d$ is even, then $\mathbb{F}_d$ is isomorphic to the standard model $\mathbb{P}^1 \times \mathbb{P}^1$ of $S^1 \times S^1$.*
2. *If $d$ is odd, then $\mathbb{F}_d$ is isomorphic to the standard model $B_O(\mathbb{P}^2)$ of the Klein bottle $K$.*

*(All isomorphisms are in the sense of Definition 1.1.)*

*Proof.* Observe that

- the real algebraic surface $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathbb{F}_0$, and
- that the real algebraic surface $B_O(\mathbb{P}^2)$ is isomorphic to $\mathbb{F}_1$.

Therefore, the proposition follows from the following lemma. □

Lemma 2.2. *Let $d$ and $e$ be natural integers. Then the two models $\mathbb{F}_d$ and $\mathbb{F}_e$ are isomorphic if and only if $d \equiv e \pmod 2$.*

*Proof.* Since the torus is not diffeomorphic to the Klein bottle, the rational models $\mathbb{F}_d$ and $\mathbb{F}_e$ are not isomorphic if $d \not\equiv e \pmod 2$. Conversely, if $d \equiv e \pmod 2$, then $\mathbb{F}_d$ and $\mathbb{F}_e$ are isomorphic models, as follows from the following lemma using induction. □

Lemma 2.3. *Let $d$ be a natural integer. The two rational models $\mathbb{F}_d$ and $\mathbb{F}_{d+2}$ are isomorphic.*

*Proof.* Let $E$ be the section at infinity of $\mathbb{F}_d$. The self-intersection of $E$ is equal to $-d$. Choose nonreal complex conjugate points $P$ and $\overline{P}$ on $E$. Let $F$ and $\overline{F}$ be the fibers of the fibration of $\mathbb{F}_d$ over $\mathbb{P}^1$ that contain $P$ and $\overline{P}$, respectively. Let $X$ be the real algebraic surface obtained from $\mathbb{F}_d$ by blowing up $P$ and $\overline{P}$.

Denote again by $E$ the strict transform of $E$ in $X$. The self-intersection of $E$ is equal to $-d-2$. The strict transforms of $F$ and $\overline{F}$, again denoted by $F$ and $\overline{F}$ respectively; they are disjoint smooth rational curves of self-intersection $-1$, and they do not intersect $E$. The real algebraic surface $Y$ obtained from $X$ by contracting $F$ and $\overline{F}$ is a smooth $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. The image of $E$ in $Y$ has self-intersection $-d-2$. It follows that $Y$ is isomorphic to $\mathbb{F}_{d+2}$ as a real algebraic surface. Therefore, we conclude that $\mathbb{F}_d$ and $\mathbb{F}_{d+2}$ are isomorphic models. □

## 3 Rational models

Let $Y$ be a real algebraic surface. A real algebraic surface $X$ is said to be *obtained from $Y$ by blowing up* if there is a nonnegative integer $n$, and a sequence of morphisms

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_0 = Y\,,$$

such that for each $i = 1, \ldots, n$, the morphism $f_i$ is either the blow up of $X_{i-1}$ at a real point, or it is the blow up of $X_{i-1}$ at a pair of distinct complex conjugate points.

The surface $X$ is said to be obtained from $Y$ by blowing up *at real points only* if for each $i = 1, \ldots, n$, the morphism $f_i$ is a blow up of $X_{i-1}$ at a real point of $X_{i-1}$.

One defines, similarly, the notion of a real algebraic surface obtained from $Y$ by blowing up *at nonreal points only.*

The real algebraic surface defined by the affine equation

$$x^2 + y^2 + z^2 = 1$$

will be denoted by $\mathbb{S}^2$. Its set of real points is the two-sphere $S^2$. The real Hirzebruch surface $\mathbb{F}_1$ will be simply denoted by $\mathbb{F}$. Its set of real points is the Klein bottle $K$.

Thanks to the Minimal Model Program for real algebraic surfaces due to János Kollár [Ko1, p. 206, Theorem 30], one has the following statement:

Theorem 3.1. *Let $S$ be a compact connected topological surface. Let $X$ be a rational model of $S$.*

1. *If $S$ is not orientable then $X$ is isomorphic to a rational model of $S$ obtained from $\mathbb{P}^2$ by blowing up at real points only.*
2. *If $S$ is orientable then $X$ is isomorphic to $\mathbb{S}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, as a model.*

*Proof.* Apply the Minimal Model Program to $X$ in order to obtain a sequence of blowing-ups as above, where $Y$ is one of the following:

1. a minimal surface,

2. a conic bundle over a smooth real algebraic curve,

3. a Del Pezzo surface of degree 1 or 2, and

4. $\mathbb{P}^2$ or $\mathbb{S}^2$.

(See [Ko1, p. 206, Theorem 30].) The surface $X$ being rational, we know that $X$ is not birational to a minimal surface. This rules out the case of $Y$ being a minimal surface. Since $X(\mathbb{R})$ is connected, it can be shown that $X$ is not birational to a Del Pezzo surface of degree 1 or 2. Indeed, such Del Pezzo surfaces have disconnected sets of real points [Ko1, p. 207, Theorem 33(D)(c–d)]. This rules out the case of $Y$ being a Del Pezzo surface of degree 1 or 2. It follows that

- either $Y$ is a conic bundle, or
- $Y$ is isomorphic to $\mathbb{P}^2$, or
- $Y$ is isomorphic to $\mathbb{S}^2$.

We will show that the statement of the theorem holds in all these three cases. If $Y$ is isomorphic to $\mathbb{P}^2$, then $Y(\mathbb{R})$ is not orientable. Since $X$ is obtained from $Y$ by blowing up, it follows that $X(\mathbb{R})$ is not orientable either. Therefore, the surface $S$ is not orientable, and also $X$ is isomorphic to a rational model of $S$ obtained from $\mathbb{P}^2$ by blowing up. Moreover, it is easy to see that $X$ is then isomorphic to a rational model of $S$ obtained from $\mathbb{P}^2$ by blowing up at real points only. This settles the case when $Y$ is isomorphic to $\mathbb{P}^2$.
If $Y$ is isomorphic to $\mathbb{S}^2$, then there are two cases to consider: (1) the case of $S$ being orientable, (2) and the case of $S$ being nonorientable. If $S$ is orientable, then $X(\mathbb{R})$ is orientable too, and $X$ is obtained from $Y$ by blowing up at nonreal points only. It follows that $X$ is isomorphic to $\mathbb{S}^2$ as a model.
If $S$ is nonorientable, then $X(\mathbb{R})$ is nonorientable too, and $X$ is obtained from $\mathbb{S}^2$ by blowing up a nonempty set of real points. Therefore, the map $X \longrightarrow Y$ factors through a blow up $\widetilde{\mathbb{S}}^2$ of $\mathbb{S}^2$ at a real point. Now, $\widetilde{\mathbb{S}}^2$ contains two smooth disjoint complex conjugated rational curves of self-intersection $-1$. When we contract them, we obtain a real algebraic surface isomorphic to $\mathbb{P}^2$. Therefore, $X$ is obtained from $\mathbb{P}^2$ by blowing up. It follows again that $X$ is isomorphic to a rational model of $S$ obtained from $\mathbb{P}^2$ by blowing up at real points only. This settles the case when $Y$ is isomorphic to $\mathbb{S}^2$.
The final case to consider is the one where $Y$ is a conic bundle over a smooth real algebraic curve $B$. Since $X$ is rational, $B$ is rational. Moreover, $B$ has real points because $X$ has real points. Hence, the curve $B$ is isomorphic to $\mathbb{P}^1$.
The singular fibers of the the conic bundle $Y$ over $B$ are real, and moreover, the number of singular fibers is even. Since $X(\mathbb{R})$ is connected, we conclude that $Y(\mathbb{R})$ is connected too. it follows that the conic bundle $Y$ over $B$ has either no singular fibers or exactly 2 singular fibers. If it has exactly 2 singular fibers, then $Y$ is isomorphic to $\mathbb{S}^2$ [Ko2, Lemma 3.2.4], a case we have already dealt with.

Therefore, we may assume that $Y$ is a smooth $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Therefore, $Y$ is a real Hirzebruch surface. By Proposition 2.1, we may suppose that $Y = \mathbb{P}^1 \times \mathbb{P}^1$, or that $Y = \mathbb{F}$. Since $\mathbb{F}$ is obtained from $\mathbb{P}^2$ by blowing up one real point, the case $Y = \mathbb{F}$ follows from the case of $Y = \mathbb{P}^2$ which we have already dealt with above.
Therefore, we may assume that $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Again, two cases are to be considered: (1) the case of $S$ being orientable, and (2) the case of $S$ being nonorientable. If $S$ is orientable, $X(\mathbb{R})$ is orientable, and $X$ is obtained from $Y$ by blowing up at non real points only. It follows that $X$ is isomorphic as a model to $\mathbb{P}^1 \times \mathbb{P}^1$. If $S$ is not orientable, $X$ is obtained from $Y$ by blowing up, at least, one real point. Since $Y = \mathbb{P}^1 \times \mathbb{P}^1$, a blow-up of $Y$ at one real point is isomorphic to a blow-up of $\mathbb{P}^2$ at two real points. We conclude again by the case of $Y = \mathbb{P}^2$ dealt with above. □

Note that Theorem 3.1 implies Comessatti's Theorem referred to in the introduction, i.e., the statement to the effect that any orientable compact connected topological surface of genus greater than 1 does not admit a rational model (Theorem 1.2.1).

*Remark* 3.2. For sake of completeness let us show how Theorem 3.1 implies that the surfaces $S^2, S^1 \times S^1, \mathbb{P}^2(\mathbb{R})$ and the Klein bottle $K$ admit exactly one rational model. First, this is clear for the orientable surfaces $S^2$ and $S^1 \times S^1$. Let $X$ be a rational model of $\mathbb{P}^2(\mathbb{R})$. From Theorem 3.1, we know that $X$ is isomorphic to a rational model of $\mathbb{P}^2(\mathbb{R})$ obtained from $\mathbb{P}^2$ by blowing up at real points only. Therefore, we may assume that $X$ itself is obtained from $\mathbb{P}^2$ by blowing up at real points only. Since $X(\mathbb{R})$ is diffeomorphic to $\mathbb{P}^2(\mathbb{R})$, it follows that $X$ is isomorphic to $\mathbb{P}^2$. Thus any rational model of $\mathbb{P}^2(\mathbb{R})$ is isomorphic to $\mathbb{P}^2$ as a model.
Let $X$ be a rational model of the Klein bottle $K$. Using Theorem 3.1 one may assume that $X$ is a blowing up of $\mathbb{P}^2$ at real points only. Since $X(\mathbb{R})$ is diffeomorphic to the 2-fold connected sum of $\mathbb{P}^2(\mathbb{R})$, the surface $X$ is a blowing up of $\mathbb{P}^2$ at exactly one real point. It follows that $X$ is isomorphic to $\mathbb{F}$. Therefore, any rational model of the Klein bottle $K$ is isomorphic to $\mathbb{F}$, as a model; compare with [Ma, Theorem 1.3].

One can wonder whether the case where $S$ is a 3-fold connected sum of real projective planes can be treated similarly. The first difficulty is as follows. It is, a priori, not clear why the following two rational models of $\#^3\mathbb{P}^2(\mathbb{R})$ are isomorphic. The first one is obtained from $\mathbb{P}^2$ by blowing up two real points of $\mathbb{P}^2$. The second one is obtained by a successive blow-up of $\mathbb{P}^2$: first blow up $\mathbb{P}^2$ at a real point, and then blow up a real point of the exceptional divisor. In the next section we prove that these two models are isomorphic.

## 4 The 3-fold connected sum of the real projective plane

We start with a lemma.

Lemma 4.1. *Let $P$ be a real point of $\mathbb{P}^2$, and let $B_P(\mathbb{P}^2)$ be the surface obtained from $\mathbb{P}^2$ by blowing up $P$. Let $E$ be the exceptional divisor of $B_P(\mathbb{P}^2)$ over $P$. Let $L$ be any real projective line of $\mathbb{P}^2$ not containing $P$. Consider $L$ as a curve in $B_P(\mathbb{P}^2)$. Then there is a birational map*

$$f\colon B_P(\mathbb{P}^2) \dashrightarrow B_P(\mathbb{P}^2)$$

*whose restriction to the set of real points is an algebraic diffeomorphism such that $f(L(\mathbb{R})) = E(\mathbb{R})$.*

*Proof.* The real algebraic surface $B_P(\mathbb{P}^2)$ is isomorphic to the real Hirzebruch surface $\mathbb{F} = \mathbb{F}_1$, and any isomorphism between them takes the exceptional divisor of $B_P(\mathbb{P}^2)$ to the section at infinity of the conic bundle $\mathbb{F}/\mathbb{P}^1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$. The line $L$ in $B_P(\mathbb{P}^2)$ is given by a unique section of $\mathcal{O}_{\mathbb{P}^1}(1)$ over $\mathbb{P}^1$; this section of $\mathcal{O}_{\mathbb{P}^1}(1)$ will also be denoted by $L$. We denote again by $E$ the section at infinity of $\mathbb{F}$.
We have to show that there is a birational self-map $f$ of $\mathbb{F}$ such that the equality $f(L(\mathbb{R})) = E(\mathbb{R})$ holds. Let $R$ be a nonreal point of $L$. Let $F$ be the fiber of the conic bundle $\mathbb{F}$ passing through $R$. The blowing-up of $\mathbb{F}$ at the pair of points $R$ and $\overline{R}$ is a real algebraic surface in which we can contract the strict transforms of $F$ and $\overline{F}$. The real algebraic surface one obtains after these two contractions is again isomorphic to $\mathbb{F}$.
Therefore, we have a birational self-map $f$ of $\mathbb{F}$ whose restriction to the set of real points is an algebraic diffeomorphism. The image, by $f$, of the strict transform of $L$ in $\mathbb{F}$ has self-intersection $-1$. Therefore, the image, by $f$, of the strict transform of $L$ coincides with $E$. In particular, we have $f(L(\mathbb{R})) = E(\mathbb{R})$. ∎

Proposition 4.2. *Let $S$ be the 3-fold connected sum of $\mathbb{P}^2(\mathbb{R})$. Then $S$ admits exactly 1 rational model.*

*Proof.* Fix two real points $O_1, O_2$ of $\mathbb{P}^2$, and let $B_{O_1,O_2}(\mathbb{P}^2)$ be the real algebraic surface obtained from $\mathbb{P}^2$ by blowing up $O_1$ and $O_2$. The surface $B_{O_1,O_2}(\mathbb{P}^2)$ is a rational model of the 3-fold connected sum $S$ of $\mathbb{P}^2(\mathbb{R})$.
Let $X$ be a rational model of $S$. We prove that $X$ is isomorphic to $B_{O_1,O_2}(\mathbb{P}^2)$, as a model. By Theorem 3.1, we may assume that $X$ is obtained from $\mathbb{P}^2$ by blowing up real points only. Since $X(\mathbb{R})$ is diffeomorphic to a 3-fold connected sum of the real projective plane, the surface $X$ is obtained from $\mathbb{P}^2$ by blowing up twice real points. More precisely, there is a real point $P$ of $\mathbb{P}^2$ and a real point $Q$ of the blow-up $B_P(\mathbb{P}^2)$ of $\mathbb{P}^2$ at $P$, such that $X$ is isomorphic to the blow-up $B_Q(B_P(\mathbb{P}^2))$ of $B_P(\mathbb{P}^2)$ at $Q$.
Choose any real projective line $L$ in $\mathbb{P}^2$ not containing $P$. Then, $L$ is also a real curve in $B_P(\mathbb{P}^2)$. We may assume that $Q \notin L$. By Lemma 4.1, there is a birational map $f$ from $B_P(\mathbb{P}^2)$ into itself whose restriction to the set of real points is an algebraic diffeomorphism, and such that

$$f(L(\mathbb{R})) = E(\mathbb{R}),$$

where $E$ is the exceptional divisor on $B_P(\mathbb{P}^2)$. Let $R = f(Q)$. Then $R \notin E$, and $f$ induces a birational isomorphism

$$\widetilde{f}\colon B_Q(B_P(\mathbb{P}^2)) \longrightarrow B_R(B_P(\mathbb{P}^2))$$

whose restriction to the set of real points is an algebraic diffeomorphism. Since $R \notin E$, the point $R$ is a real point of $\mathbb{P}^2$ distinct from $P$, and the blow-up $B_R(B_P(\mathbb{P}^2))$ is equal to the blow up $B_{P,R}(\mathbb{P}^2)$ of $\mathbb{P}^2$ at the real points $P, R$ of $\mathbb{P}^2$. It is clear that $B_{P,R}(\mathbb{P}^2)$ is isomorphic to $B_{O_1,O_2}(\mathbb{P}^2)$. It follows that $X$ is isomorphic to $B_{O_1,O_2}(\mathbb{P}^2)$ as rational models of the 3-fold connected sum of $\mathbb{P}^2(\mathbb{R})$. □

Lemma 4.3. *Let $S$ be a nonorientable surface and let $X$ be a rational model of $S$. Then, there are distinct real points $P_1, \dots, P_n$ of $\mathbb{P}^2$ such that $X$ is isomorphic to the blowing-up of $\mathbb{P}^2$ at $P_1, \dots, P_n$, as a model.*

*Proof.* By Theorem 3.1, we may assume that $X$ is obtained from $\mathbb{P}^2$ by blowing up at real points only. Let

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_0 = \mathbb{P}^2. \tag{2}$$

be a sequence of blowing ups, where for each $i = 1, \dots, n$, the map $f_i$ is a blowing up of $X_{i-1}$ at a real point $P_i$ of $X_{i-1}$.

To a sequence of blowing-ups as in (2) is associated a forest $F$ of trees. The vertices of $F$ are the centers $P_i$ of the blow-ups $f_i$. For $i > j$, there is an edge between the points $P_i$ and $P_j$ in $F$ if

- the composition $f_{j+1} \circ \cdots \circ f_{i-1}$ is an isomorphism at a neighborhood of $P_i$, and
- maps $P_i$ to a point belonging to the exceptional divisor $f_j^{-1}(P_j)$ of $P_j$ in $X_j$.

Let $\ell$ be the sum of the lengths of the trees belonging to $F$. We will show by induction on $\ell$ that $X$ is isomorphic, as a model, to the blowing-up of $\mathbb{P}^2$ at a finite number of distinct real points of $\mathbb{P}^2$.

This is obvious if $\ell = 0$. If $\ell \neq 0$, let $P_j$ be the root of a tree of nonzero length, and let $P_i$ be the vertex of that tree lying immediately above $P_j$. By changing the order of the blowing-ups $f_i$, we may assume that $j = 1$ and $i = 2$.

Choose a real projective line $L$ in $\mathbb{P}^2$ which does not contain any of the roots of the trees of $F$. By Lemma 4.1, there is a birational map $g_1$ from $X_1 = B_{P_1}(\mathbb{P}^2)$ into itself whose restriction to the set of real points is an algebraic diffeomorphism and satisfies the condition $g_1(L(\mathbb{R})) = E(\mathbb{R})$, where $E$ is the exceptional divisor of $X_1$.

Put $X_0' = \mathbb{P}^2$, $X_1' = X_1$, and $f_1' = f_1$. We consider $g_1$ as a birational map from $X_1$ into $X_1'$. Put $P_2' = g_1(P_2)$. Let $X_2'$ be the blowing-up of $X_1'$ at $P_2'$, and let

$$f_2'\colon X_2' \longrightarrow X_1'$$

be the blowing-up morphism. Then, $g_1$ induces a birational map $g_2$ from $X_2$ into $X_2'$ which is an algebraic diffeomorphism on the set of real points.
By iterating this construction, one gets a sequence of blowing ups

$$f_i'\colon X_i' \longrightarrow X_{i-1}',$$

where $i = 1, \ldots, n$, and birational morphisms $g_i$ from $X_i$ into $X_i'$ whose restrictions to the sets of real points are algebraic diffeomorphisms. In particular, the rational models $X = X_n$ and $X' = X_n'$ of $S$ are isomorphic.
Let $F'$ be the forest of the trees of centers of $X'$. Then the sum of the lengths $\ell'$ of the trees of $F'$ is equal to $\ell - 1$. Indeed, one obtains $F'$ from $F$ by replacing the tree $T$ of $F$ rooted at $P_1$ by the disjoint union of the tree $T \setminus P_1$ and the tree $\{P_1\}$. This follows from the fact that $P_2'$ does not belong to the exceptional divisor of $f_1'$, and that, no root of the other trees of $F$ belongs to the exceptional divisor of $f_1'$ either. □

As observed in the Introduction, if we are able to prove the $n$-transitivity of the action of the group $\mathrm{Diff}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$ on $\mathbb{P}^2(\mathbb{R})$, then the statement of Theorem 1.2 would follow from Lemma 4.3. However, we did not succeed in proving so. Nevertheless, we will prove the $n$-transitivity of $\mathrm{Diff}(S^1 \times S^1)$, which is the subject of the next section.
Now that we know that the topological surfaces $S^1, S^1 \times S^1$ and $\#^n\mathbb{P}^2(\mathbb{R})$, for $n = 1, 2, 3$, admit exactly one rational model, one may also wonder whether Lemma 4.3 allows us to tackle the "next" surface, which is the 4-fold connected sum of $\mathbb{P}^2(\mathbb{R})$. We note that Theorem 1.2 and Lemma 4.3 imply that a rational model of such a surface is isomorphic to a surface obtained from $\mathbb{P}^2$ by blowing up 3 distinct real points. However, it it is not clear why the two surfaces of the following type are isomorphic as models. Take three distinct non-collinear real points $P_1, P_2, P_3$, and three distinct collinear real points $Q_1, Q_2, Q_3$ of $\mathbb{P}^2$. Then the surfaces $X = B_{P_1,P_2,P_3}(\mathbb{P}^2)$ and $Y = B_{Q_1,Q_2,Q_3}(\mathbb{P}^2)$ are rational models of $\#^4\mathbb{P}^2(\mathbb{R})$ (the 4-fold connected sum of $\mathbb{P}^2(\mathbb{R})$), but it is not clear why they should be isomorphic. One really seems to need some nontrivial algebraic diffeomorphism of $\mathbb{P}^2(\mathbb{R})$, that maps $P_i$ to $Q_i$ for $i = 1, 2, 3$, in order to show that $X$ and $Y$ are isomorphic models. We will come back to this in Section 6 (Lemma 6.1).

## 5 Algebraic diffeomorphisms of $S^1 \times S^1$ and $n$-transitivity

The following statement is a variation on classical polynomial interpolation.

LEMMA 5.1. *Let $m$ be a positive integer. Let $x_1, \ldots, x_m$ be distinct real numbers, and let $y_1 \ldots, y_m$ be positive real numbers. Then there is a real polynomial $p$ of degree $2m$ that does not have real zeros, and satisfies the condition $p(x_i) = y_i$ for all $i$.*

*Proof.* Set

$$p(\zeta) := \sum_{j=1}^{m} \prod_{k\neq j} \frac{(\zeta - x_k)^2}{(x_j - x_k)^2} \cdot y_j.$$

Then $p$ is of degree $2m$, and $p$ does not have real zeros. Furthermore, we have $p(x_i) = y_i$ for all $i$. $\square$

Corollary 5.2. *Let $m$ be a positive integer. Let $x_1, \ldots, x_m$ be distinct real numbers, and let $y_1 \ldots, y_m, z_1, \ldots, z_m$ be positive real numbers. Then there are real polynomials $p$ and $q$ without any real zeros such that* $\text{degree}(p) = \text{degree}(q)$*, and*

$$\frac{p(x_i)}{q(x_i)} = \frac{y_i}{z_i}$$

*for all $1 \leq i \leq m$.* $\square$

The interest in the rational functions $p/q$ of the above type lies in the following fact.

Lemma 5.3. *Let $p$ and $q$ be two real polynomials of same degree that do not have any real zeros. Define the rational map $f\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by*

$$f(x, y) = \left(x, \frac{p(x)}{q(x)} \cdot y\right).$$

*Then $f$ is a birational map of $\mathbb{P}^1 \times \mathbb{P}^1$ into itself whose restriction to the set of real points is an algebraic diffeomorphism.* $\square$

Theorem 5.4. *Let $n$ be a natural integer. The group* $\text{Diff}_{\text{alg}}(\mathbb{P}^1 \times \mathbb{P}^1)$ *acts $n$-transitively on $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$.*

*Proof.* Choose $n$ distinct real points $P_1, \ldots, P_n$ and $n$ distinct real points $Q_1, \ldots, Q_n$ of $\mathbb{P}^1 \times \mathbb{P}^1$. We need to show that there is a birational map $f$ from $\mathbb{P}^1 \times \mathbb{P}^1$ into itself, whose restriction to $(\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{R})$ is an algebraic diffeomorphism, such that $f(P_i) = Q_i$, for $i = 1, \ldots, n$.
First of all, we may assume that $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ are contained in the first open quadrant of $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$. In other words, the coordinates of $P_i$ and $Q_i$ are strictly positive real numbers. Moreover, it suffices to prove the statement for the case where $Q_i = (i, i)$ for all $i$.
By the hypothesis above, there are positive real numbers $x_i, y_i$ such that $P_i = (x_i, y_i)$ for all $i$. By Corollary 5.2, there are real polynomials $p$ and $q$ without any real zeros such that $\text{degree}(p) = \text{degree}(q)$, and such that the real numbers

$$\frac{p(x_i)}{q(x_i)} \cdot y_i$$

are positive and distinct for all $i$. Define $f\colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by

$$f(x, y) := \left(x, \frac{p(x)}{q(x)} \cdot y\right).$$

By Lemma 5.3, $f$ is birational, and its restriction to $(\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{R})$ is an algebraic diffeomorphism. By construction, the points $f(P_i)$ have distinct second coordinates. Therefore, replacing $P_i$ by $f(P_i)$ if necessary, we may assume that the points $P_i$ have distinct second coordinates, which implies that $y_1, \ldots, y_m$ are distinct positive real numbers.

By Corollary 5.2, there are real polynomials $p, q$ without any real zeros such that $\mathrm{degree}(p) = \mathrm{degree}(q)$, and

$$\frac{p(y_i)}{q(y_i)} \cdot x_i = i.$$

Define $f \colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by

$$f(x, y) = \left( \frac{p(y)}{q(y)} \cdot x, y \right).$$

By Lemma 5.3, $f$ is birational and its restriction to the set of real points is an algebraic diffeomorphism. By construction, one has $f(P_i) = (i, y_i)$ for all $i$. Therefore, we may assume that $P_i = (i, y_i)$ for all $i$.

Now, again by Corollary 5.2, there are two real polynomials $p$ and $q$ without any real zeros such that $\mathrm{degree}(p) = \mathrm{degree}(q)$, and

$$\frac{p(i)}{q(i)} \cdot y_i = i$$

for all $i$. Define $f \colon \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ by

$$f(x, y) = \left( x, \frac{p(x)}{q(x)} \cdot y \right).$$

By Lemma 5.3, $f$ is birational, and its restriction to the set of real points is an algebraic diffeomorphism. By construction $f(P_i) = Q_i$ for all $i$. $\square$

*Remark* 5.5. One may wonder whether Theorem 5.4 implies that the group $\mathrm{Diff}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$ acts $n$-transitively on $\mathbb{P}^2(\mathbb{R})$. We will explain the implication of Theorem 5.4 in that direction. Let $P_1, \ldots, P_n$ be distinct real points of $\mathbb{P}^2$, and let $Q_1, \ldots, Q_n$ be distinct real points of $\mathbb{P}^2$. Choose a real projective line $L$ in $\mathbb{P}^2$ not containing any of the points $P_1, \ldots, P_n, Q_1, \ldots, Q_n$. Let $O_1$ and $O_2$ be distinct real points of $L$. Identify $\mathbb{P}^1 \times \mathbb{P}^1$ with the surface obtained from $\mathbb{P}^2$ by, first, blowing up $O_1, O_2$ and, then, contracting the strict transform of $L$. Denote by $E_1$ and $E_2$ the images of the exceptional divisors over $O_1$ and $O_2$ in $\mathbb{P}^1 \times \mathbb{P}^1$, respectively. We denote again by $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ the real points of $\mathbb{P}^1 \times \mathbb{P}^1$ that correspond to the real points $P_1, \ldots, P_n, Q_1, \ldots, Q_n$ of $\mathbb{P}^2$.

Now, the construction in the proof of Theorem 5.4 gives rise to a birational map $f$ from $\mathbb{P}^1 \times \mathbb{P}^1$ into itself which is an algebraic diffeomorphism on $(\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{R})$ and which maps $P_i$ onto $Q_i$, for $i = 1, \ldots, n$. Moreover, if one carries out carefully the construction of $f$, one has that $f(E_1(\mathbb{R})) = E_1(\mathbb{R})$ and $f(E_2(\mathbb{R})) = E_2(\mathbb{R})$ and that the real intersection point $O$ of $E_1$ and $E_2$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is a fixed point of $f$.

Note that one obtains back $\mathbb{P}^2$ from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up $O$ and contracting the strict transforms of $E_1$ and $E_2$. Therefore, the birational map $f$ of $\mathbb{P}^1 \times \mathbb{P}^1$ into itself induces a birational map $g$ of $\mathbb{P}^2$ into itself. Moreover, $g(P_i) = Q_i$. One may think that $g$ is an algebraic diffeomorphism on $\mathbb{P}^2(\mathbb{R})$. However, the restriction of $g$ to the set of real points is not necessarily an algebraic diffeomorphism! In fact, $g$ is an algebraic diffeomorphism on $\mathbb{P}^2(\mathbb{R}) \setminus \{O_1, O_2\}$. The restriction of $g$ to $\mathbb{P}^2(\mathbb{R}) \setminus \{O_1, O_2\}$ does admit a continuous extension $\widetilde{g}$ to $\mathbb{P}^2(\mathbb{R})$, and $\widetilde{g}$ is obviously a homeomorphism. One may call $\widetilde{g}$ an *algebraic homeomorphism*, but $\widetilde{g}$ is *not necessarily an algebraic diffeomorphism.* It is not difficult to find explicit examples of such algebraic homeomorphisms that are not diffeomorphisms.

That is the reason why we do not claim to have proven $n$-transitivity of $\mathrm{Diff}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$. The only statement about $\mathbb{P}^2(\mathbb{R})$ the above arguments prove is the $n$-transitivity of the group $\mathrm{Homeo}_{\mathrm{alg}}(\mathbb{P}^2(\mathbb{R}))$ of algebraic homeomorphisms.

## 6 A nontrivial algebraic diffeomorphism of $\mathbb{P}^2(\mathbb{R})$

The nontrivial diffeomorphisms we have in mind are the following. They have been studied in another recent paper as well [RV].

Let $Q_1, \ldots, Q_6$ be six pairwise distinct complex points of $\mathbb{P}^2$ satisfying the following conditions:

1. the subset $\{Q_1, \cdots, Q_6\}$ is closed under complex conjugation,
2. the subset $\{Q_1, \cdots, Q_6\}$ does not lie on a complex conic,
3. the complex conic passing through any 5 of these six points is nonsingular.

Denote by $C_1, \ldots, C_6$ the nonsingular complex conics one thus obtains. These conics are pairwise complex conjugate. Consider the real Cremona transformation $f = f_Q$ of $\mathbb{P}^2$ defined by first blowing-up $\mathbb{P}^2$ at $Q_1, \ldots, Q_6$ and then contracting the strict transforms of $C_1, \ldots, C_6$. Let $R_1, \ldots, R_6$ denote the points of $\mathbb{P}^2$ that correspond to the contractions of the conics $C_1, \ldots, C_6$.

The restriction to $\mathbb{P}^2(\mathbb{R})$ of the birational map $f$ from $\mathbb{P}^2$ into itself is obviously an algebraic diffeomorphism.

The Cremona transformation $f$ maps a real projective line, not containing any of the points $Q_1, \ldots, Q_6$, to a real rational quintic curve having 6 distinct nonreal double points at the points $R_1, \ldots, R_6$. Moreover, it maps a real rational quintic curve in $\mathbb{P}^2$ having double points at $Q_1, \ldots, Q_6$ to a real projective line in $\mathbb{P}^2$ that does not contain any of the points $R_1, \ldots, R_6$.

Observe that the inverse of the Cremona transformation $f_Q$ is the Cremona transformation $f_R$. It follows that $f = f_Q$ induces a bijection from the set of real rational quintics having double points at $Q_1, \ldots, Q_6$ onto the set of real projective lines in $\mathbb{P}^2$ that do not contain any of $R_1, \ldots, R_6$.

This section is devoted to the proof of following lemma.

Lemma 6.1. *Let $n$ be a natural integer bigger than 1. Let $P_1, \ldots, P_n$ be distinct real points of $\mathbb{P}^2$. Then there is a birational map of $\mathbb{P}^2$ into itself, whose restriction to the set of real points is an algebraic diffeomorphism, such that the image points $f(P_3), \ldots, f(P_n)$ are not contained in the real projective line through $f(P_1)$ and $f(P_2)$.*

*Proof.* Choose complex points $Q_1, \ldots, Q_6$ of $\mathbb{P}^2$ as above. As observed before, the Cremona transformation $f = f_Q$ induces a bijection from the set of real rational quintic curves having double points at $Q_1, \ldots, Q_6$ onto the set of real projective lines of $\mathbb{P}^2$ not containing any of the above points $R_1, \ldots, R_n$. In particular, there is a real rational quintic curve $C$ in $\mathbb{P}^2$ having 6 nonreal double points at $Q_1, \ldots, Q_6$.

We show that there is a real projectively linear transformation $\alpha$ of $\mathbb{P}^2$ such that $\alpha(C)$ contains $P_1$ and $P_2$, and does not contain any of the points $P_3, \ldots, P_n$. The Cremona transformation $f_{\alpha(Q)}$ will then be a birational map of $\mathbb{P}^2$ into itself that has the required properties.

First of all, let us prove that there is $\alpha \in \mathrm{PGL}_3(\mathbb{R})$ such that $P_1, P_2 \in \alpha(C)$. This is easy. Since $C$ is a quintic curve, $C(\mathbb{R})$ is infinite. In particular, $C$ contains two distinct real points. It follows that there is $\alpha \in \mathrm{PGL}_3(\mathbb{R})$ such that $P_1, P_2 \in \alpha(C)$. Replacing $C$ by $\alpha(C)$ if necessary, we may suppose that $P_1, P_2 \in C$.

We need to show that there is $\alpha \in \mathrm{PGL}_3(\mathbb{R})$ such that $\alpha(P_1) = P_1$, $\alpha(P_2) = P_2$ and $\alpha(C)$ does not contain any of the points $P_3, \ldots, P_n$.

To prove the existence of $\alpha$ by contradiction, assume that there is no such automorphism of $\mathbb{P}^2$. Therefore, for all $\alpha \in \mathrm{PGL}_3(\mathbb{R})$ having $P_1$ and $P_2$ as fixed points, the image $\alpha(C)$ contains at least one of the points of $P_3, \ldots, P_n$.

Let $G$ be the stabilizer of the pair $(P_1, P_2)$ for the diagonal action of $\mathrm{PGL}_3$ on $\mathbb{P}^2 \times \mathbb{P}^2$. It is easy to see that $G$ is a geometrically irreducible real algebraic group. Let

$$\rho \colon C \times G \longrightarrow \mathbb{P}^2$$

be the morphism defined by $\rho(P, \alpha) = \alpha(P)$. Let

$$X_i := \rho^{-1}(P_i)$$

be the inverse image, where $i = 3, \ldots, n$. Therefore, $X_i$ is a real algebraic subvariety of $C \times G$. By hypothesis, for every $\alpha \in G(\mathbb{R})$, there is an integer $i$ such that $\alpha(C)$ contains $P_i$. Denoting by $p$ the projection on the second factor from $C \times G$ onto $G$, this means that

$$\bigcup_{i=3}^{n} p(X_i(\mathbb{R})) = G(\mathbb{R}).$$

Since $G(\mathbb{R})$ is irreducible, there is an integer $i_0 \in [3, n]$ such that the semi-algebraic subset $p(X_{i_0}(\mathbb{R}))$ is Zariski dense in $G(\mathbb{R})$. Since $G$ is irreducible and $p$

is proper, one has $p(X_{i_0}) = G$. In particular, $P_{i_0} \in \alpha(C)$ for all $\alpha \in G(\mathbb{C})$. To put it otherwise, $\alpha^{-1}(P_{i_0}) \in C$ for all $\alpha \in G(\mathbb{C})$, which means that the orbit of $P_{i_0}$ under the action of $G$ is contained in $C$. In particular, the dimension of the orbit of $P_{i_0}$ is at most one. It follows that $P_1, P_2$ and $P_{i_0}$ are collinear. Let $L$ be the projective line through $P_1, P_2$. Then the orbit of $P_{i_0}$ coincides with $L \setminus \{P_1, P_2\}$. It now follows that $L \subseteq C$. This is in contradiction with the fact that $C$ is irreducible. □

## 7 Proof of Theorem 1.2.2

Let $S$ be a topological surface, either nonorientable or of genus less than 2. We need to show that any two rational models of $S$ are isomorphic. By Remark 3.2, we may assume that $S$ is the $n$-fold connected sum of $\mathbb{P}^2(\mathbb{R})$, where $n \geq 3$.
Let $O_1, \ldots, O_{n-2}$ be fixed pairwise distinct real points of $\mathbb{P}^1 \times \mathbb{P}^1$, and let $B_{n-2}(\mathbb{P}^1 \times \mathbb{P}^1)$ be the surface obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up the points $O_1, \ldots, O_{n-2}$. It is clear that $B_{n-2}(\mathbb{P}^1 \times \mathbb{P}^1)$ is a rational model of $S$.
Now, it suffices to show that any rational model of $S$ is isomorphic to $B_{n-2}(\mathbb{P}^1 \times \mathbb{P}^1)$, as a model. Let $X$ be any rational model of $S$. By Lemma 4.3, we may assume that there are distinct real points $P_1, \ldots, P_m$ of $\mathbb{P}^2$ such that $X$ is the surface obtained from $\mathbb{P}^2$ by blowing up $P_1, \ldots, P_m$. Since $X$ is a rational model of an $n$-fold connected sum of $\mathbb{P}^2(\mathbb{R})$, one has $m = n - 1$. In particular, $m \geq 2$. By Lemma 6.1, we may assume that the points $P_3, \ldots, P_m$ are not contained in the real projective line $L$ through $P_1$ and $P_2$.
The blow-up morphism $X \longrightarrow \mathbb{P}^2$ factors through the blow up $\widetilde{\mathbb{P}}^2 = B_{P_1,P_2}(\mathbb{P}^2)$. The strict transform $\widetilde{L}$ of $L$ has self-intersection $-1$ in $\widetilde{\mathbb{P}}^2$. If we contract $\widetilde{L}$, then we obtain a surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore, $X$ is isomorphic to a model obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up $m - 1 = n - 2$ distinct real points of $\mathbb{P}^1 \times \mathbb{P}^1$. It follows from Theorem 5.4 that $X$ is isomorphic to $B_{n-2}(\mathbb{P}^1 \times \mathbb{P}^1)$. □

## 8 Geometrically rational models

Recall that a nonsingular projective real algebraic surface $X$ is *geometrically rational* if the complex surface $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ is rational. Comessatti showed that, if $X$ is a geometrically rational real algebraic surface with $X(\mathbb{R})$ connected, then $X$ is rational; see Theorem IV of [Co1] and the remarks thereafter (see also [Si, Corollary VI.6.5]). Therefore, the main result, namely Theorem 1.2, also applies to geometrically rational models. More precisely, we have the following consequence.

Corollary 8.1. *Let $S$ be a compact connected real two-manifold.*

1. *If $S$ is orientable and the genus of $S$ is greater than* 1*, then $S$ does not admit a geometrically rational real algebraic model.*

*2. If $S$ is either nonorientable, or it is diffeomorphic to one of $S^2$ and $S^1 \times S^1$, then there is exactly one geometrically rational model of $S$, up to isomorphism. In other words, any two geometrically rational models of $S$ are isomorphic.* □

Now, the interesting aspect about geometrically rational real surfaces is that their set of real points can have an arbitrary finite number of connected components. More precisely, Comessati proved the following statement [Co2, p. 263 and further] (see also [Si, Proposition VI.6.1]).

Theorem 8.2. *Let $X$ be a geometrically rational real algebraic surface such that $X(\mathbb{R})$ is not connected. Then each connected component of $X(\mathbb{R})$ is either nonorientable or diffeomorphic to $S^2$. Conversely, if $S$ is a nonconnected compact topological surface each of whose connected components is either nonorientable or diffeomorphic to $S^2$, then there is a geometrically rational real algebraic surface $X$ such that $X(\mathbb{R})$ is diffeomorphic to $S$.* □

Let $S$ be a nonconnected topological surface. One may wonder whether the geometrically rational model of $S$ whose existence is claimed above, is unique up to isomorphism of models. The answer is negative, as shown by the following example.

Example 8.3. Let $S$ be the disjoint union of a real projective plane and 4 copies of $S^2$. Then, any minimal real Del Pezzo surface of degree 1 is a geometrically rational model of $S$ [Ko2, Theorem 2.2(D)]. Minimal real Del Pezzo surfaces of degree 1 are rigid; this means that any birational map between two minimal real Del Pezzo surfaces of degree 1 is an isomorphism of real algebraic surfaces [Is, Theorem 1.6]. Now, the set of isomorphism classes of minimal real Del Pezzo surfaces of degree 1 is in one-to-one correspondence with an open dense subset of the quotient $\mathbb{P}^2(\mathbb{R})^8/\mathrm{PGL}_3(\mathbb{R})$ for the diagonal action of the group $\mathrm{PGL}_3(\mathbb{R})$. It follows that the topological surface $S$ admits a 8-dimensional continuous family of nonisomorphic geometrically rational models. In particular, the number of nonisomorphic geometrically rational models of $S$ is infinite.

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# Eigenvalue Clusters of the Landau Hamiltonian in the Exterior of a Compact Domain

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Abstract. We consider the Schrödinger operator with a constant magnetic field in the exterior of a compact domain on the plane. The spectrum of this operator consists of clusters of eigenvalues around the Landau levels. We discuss the rate of accumulation of eigenvalues in a fixed cluster.

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## 1 Introduction

### 1.1 Preliminaries

The Landau Hamiltonian describes a charged particle confined to a plane in a constant magnetic field. The Landau Hamiltonian is one of the earliest explicitly solvable quantum mechanical models. Its spectrum consists of the Landau levels,[1] infinitely degenerate eigenvalues placed at the points of an arithmetic progression.

In [7], the Landau Hamiltonian was considered in the exterior of a compact obstacle. Introducing the obstacle produces clusters of eigenvalues of finite multiplicity around the Landau levels. Various asymptotics (high energy, semi-classical) of these eigenvalue clusters were studied in [7]. In this paper we focus

[1] It is a little known fact that this was worked out by Fock two years before Landau; see [4, 9].

on a different aspect of the spectral analysis of this model: for a fixed eigenvalue cluster, we consider the rate of accumulation of eigenvalues in this cluster to the Landau level. We describe this rate of accumulation rather precisely in terms of the logarithmic capacity of the obstacle.

Our construction is motivated by the recent progress in the study of the Landau Hamiltonian on the whole plane perturbed by a compactly supported or fast decaying electric or magnetic field, see [15, 13, 3, 16]. In particular, we use some operator theoretic constructions from [15] and [13] and some concrete analysis (related to logarithmic capacity) from [3].

### 1.2 The Landau Hamiltonian

We will write $x = (x_1, x_2) \in \mathbb{R}^2$ and identify $\mathbb{R}^2$ with $\mathbb{C}$ in the standard way, setting $z = x_1 + ix_2 \in \mathbb{C}$. The Lebesgue measure in $\mathbb{R}^2$ will be denoted by $dx$ and in $\mathbb{C}$ by $dm(z)$. The derivatives with respect to $x_1$, $x_2$ are denoted by $\partial_k = \partial_{x_k}$; we set, as usual, $\bar\partial = (\partial_1 + i\partial_2)/2$, $\partial = (\partial_1 - i\partial_2)/2$.

We denote by $B > 0$ the magnitude of the constant magnetic field in $\mathbb{R}^2$. We choose the gauge $A(x) = (A_1(x), A_2(x)) = (-\frac{1}{2}Bx_2, \frac{1}{2}Bx_1)$ for the magnetic vector potential associated with this field. The magnetic Hamiltonian on the whole plane is defined as

$$X_0 = -(\nabla - iA)^2 \quad \text{in } L^2(\mathbb{R}^2). \tag{1}$$

More precisely, for $u \in C_0^\infty(\mathbb{R}^2)$ we set

$$\|u\|^2_{H^1_A} = \int_{\mathbb{R}^2} \left|i\nabla u(x) + A(x)u(x)\right|^2 dx \tag{2}$$

and define $X_0$ as the selfadjoint operator which corresponds to the closure of the quadratic form $\|u\|^2_{H^1_A}$, $u \in C_0^\infty(\mathbb{R}^2)$.

It is well known (see [4, 9] or [10]) that the spectrum of $X_0$ consists of the eigenvalues $\Lambda_q = (2q+1)B$, $q = 0, 1, \dots$, of infinite multiplicity. In particular, we have

$$\|u\|^2_{H^1_A} \geq B\|u\|^2_{L^2}, \quad u \in C_0^\infty(\mathbb{R}^2). \tag{3}$$

We will denote by $\mathcal{L}_q$ the eigenspace of $X_0$ corresponding to $\Lambda_q$ and by $P_q$ the operator of orthogonal projection onto $\mathcal{L}_q$ in $L^2(\mathbb{R}^2)$. Later on, we will need an explicit description of $\mathcal{L}_q$; this will be discussed in section 4.2.

Let $\Omega \subset \mathbb{R}^2$ be an open set. In order to define the magnetic Hamiltonian in $\Omega$, it is convenient to consider the associated quadratic form. Following [12], we denote by $H^1_A(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{H^1_A}$. The quadratic form $\|u\|^2_{H^1_A}$ is closed in $L^2(\Omega)$ and (by (3)) positively definite. This form defines a self-adjoint operator in $\mathbb{R}^2$ which we denote by $X(\Omega)$. If $\Omega$ is bounded by a smooth curve, then the usual computations show that this definition of $X(\Omega)$ corresponds to setting the Dirichlet boundary condition on $\partial\Omega$. The operator $X_0$ corresponds to taking $\Omega = \mathbb{R}^2$ in the above definitions.

Note that for a bounded $\Omega$, the norm in $H^1_A(\Omega)$ is equivalent to the standard Sobolev norm $H^1(\Omega)$; in particular, in this case the embedding $H^1_A(\Omega) \subset L^2(\Omega)$ is compact.

### 1.3 Main results

Let $K \subset \mathbb{R}^2$ be a compact set and $K^c$ its complement. Our main results concern the spectrum of the operator $X(K^c)$. First we state a preliminary result which gives a general description of the spectrum of $X(K^c)$. This result is already known (see [7]) but as part of our construction, we provide a simple proof in Section 1.4.

Proposition 1.1. *Let $K \subset \mathbb{R}^2$ be a compact set. Then*

$$\sigma_{ess}(X(K^c)) = \sigma_{ess}(X_0) = \cup_{q=0}^{\infty}\{\Lambda_q\}, \quad \Lambda_q = (2q+1)B.$$

*Moreover, for all $q$ and all $\lambda \in (\Lambda_{q-1}, \Lambda_q)$ the number of eigenvalues of $X(K^c)$ in $(\lambda, \Lambda_q)$ is finite.*

In other words, the last statement means that the eigenvalues of $X(K^c)$ can accumulate to the Landau levels only from above.
For all $q \geq 0$, we enumerate the eigenvalues of $X(K^c)$ in $(\Lambda_q, \Lambda_{q+1})$:

$$\lambda_1^q \geq \lambda_2^q \geq \dots$$

Proposition 1.1 ensures that $\lambda_n^q \to \Lambda_q$ as $n \to \infty$. Below we describe the rate of this convergence. Roughly speaking, we will see that for large $n$,

$$\frac{a^n}{n!} \leq \lambda_n^q - \Lambda_q \leq \frac{b^n}{n!} \tag{4}$$

with some $a, b$ depending on $K$. In order to discuss the dependence of $a, b$ on the domain $K$, let us introduce the following notation:

$$\begin{aligned} \Delta_q(K) &= \limsup_{n\to\infty} [n!(\lambda_n^q - \Lambda_q)]^{1/n}, \\ \delta_q(K) &= \liminf_{n\to\infty} [n!(\lambda_n^q - \Lambda_q)]^{1/n}. \end{aligned} \tag{5}$$

The estimates for these spectral characteristics will be given in terms of the logarithmic capacity of $K$, which is denoted by $\mathrm{Cap}\,(K)$. Below we recall the definition and basic properties of logarithmic capacity; for the details, see e.g. [11] or [18]. For a measure $\mu \geq 0$ in $\mathbb{R}^2$, its logarithmic energy is defined by

$$I(\mu) = \int\int \log\frac{1}{|z-t|} d\mu(z) d\mu(t).$$

For a compact set $K \subset \mathbb{R}^2$, its logarithmic capacity is defined by

$$\mathrm{Cap}\,(K) = e^{-V(K)},$$
$$V(K) = \inf\{I(\mu) \mid \mu \geq 0 \text{ is a measure in } \mathbb{R}^2, \quad \mathrm{supp}\,\mu \subset K, \quad \mu(K) = 1\}.$$

The logarithmic capacity of compact sets in $\mathbb{R}^2$ has the following properties:
(i) if $K_1 \subset K_2$ then Cap $K_1 \leq$ Cap $K_2$;
(ii) the logarithmic capacity of a disc of radius $r$ is $r$;
(iii) if $K_2 = \{\alpha x \mid x \in K_1\}$, $\alpha > 0$, then Cap $K_2 = \alpha$ Cap $K_1$;
(iv) Cap $K$ coincides with the logarithmic capacity of the polynomial convex hull $\mathrm{Pc}\,(K)$ of $K$ (=the complement of the unbounded connected component of $K^c$).
(v) Continuity of capacity: if $K_\varepsilon = \{x \in \mathbb{R}^2 \mid \mathrm{dist}\,(x, K) \leq \varepsilon\}$, then Cap $K_\varepsilon \to$ Cap $K$ as $\varepsilon \to +0$.
To extend the notion of capacity to arbitrary Borel sets $E$, one defines the inner and outer capacities

$$\mathrm{Cap}_i(E) = \sup\{\mathrm{Cap}\,(K) \mid K \subset E, \quad K \text{ compact}\}$$
$$\mathrm{Cap}_o(E) = \inf\{\mathrm{Cap}_i(U) \mid E \subset U, \quad U \text{ open}\}.$$

Then every Borel set $E$ is capacitable in the sense that $\mathrm{Cap}_i(E) = \mathrm{Cap}_o(E)$ and one can simply write $\mathrm{Cap}\,(E)$ for this common value. We will also need another version of inner capacity, which we denote by $\mathrm{Cap}_-(K)$ and define by

$$\sup\{\mathrm{Cap}\; S \mid \; S \subset K \text{ is a compact set with a Lipschitz boundary}\}.$$

THEOREM 1.2. *Let $K \subset \mathbb{R}^2$ be a compact set; then for all $q \geq 0$ one has*

$$\Delta_q(K) \leq \frac{B}{2}(\mathrm{Cap}\,(K))^2,$$
$$\delta_q(K) \geq \frac{B}{2}(\mathrm{Cap}_-(\mathrm{Pc}\,(K)))^2.$$

*In particular, if $K$ is a compact set with a Lipschitz boundary, then $\Delta_q(K) = \delta_q(K) = \frac{B}{2}(\mathrm{Cap}\,(K))^2$ for all $q \geq 0$.*

The lower bound in the above theorem is strictly positive if and only if $\mathrm{Pc}\,(K)$ has a non-empty interior. In particular, for such compacts the number of eigenvalues $\lambda_1^q$, $\lambda_2^q$, ... is infinite for each $q$. However, even for some compacts without interior points, lower spectral bounds can be obtained. In particular, this can be done for the compact $K$ being a smooth (not necessarily closed) curve.

THEOREM 1.3. *Let $K \subset \mathbb{R}^2$ be a $C^\infty$ smooth simple curve of a finite length. Then for all $q \geq 0$, one has*

$$\Delta_q(K) = \delta_q(K) = \frac{B}{2}(\mathrm{Cap}\,(K))^2.$$

REMARK. 1. One can prove that

$$\text{if } \mathrm{Cap}\,(K) = 0, \text{ then } C_0^\infty(K^c) \text{ is dense in } H_A^1(\mathbb{R}^2). \tag{6}$$

It follows that for $K$ of zero capacity, $H^1_A(K^c) = H^1_A(\mathbb{R}^2)$ and therefore $X(K^c) = X_0$. Thus, for such $K$ the spectrum of $X(K^c)$ consists of Landau levels $\Lambda_q$.

The statement (6) seems to be well known to the experts in the field although it is difficult to pinpoint the exact reference. One can use the argument of [1], Theorem 9.9.1; this argument applies to the usual $H^1$ Sobolev norm, but it is very easy to modify it for the norm $H^1_A$. In this theorem the Bessel capacity rather than the logarithmic capacity is used; however, the Bessel capacity of a compact set vanishes if and only if its logarithmic capacity vanishes. In order to prove the latter fact (again, well known to experts) one has to combine Theorem 2.2.7 in [1] and Sect.II.4 in [11].

2. We do not know whether it possible for $\Lambda_q$ to remain eigenvalues of $X(K^c)$ of infinite multiplicity if $\mathrm{Cap}\,(K) > 0$.

3. Following the proof of Theorem 1.2 and using the results of [3], it is easy to show that for $q = 0$, the lower bound in this theorem can be replaced by the following one:

$$\delta_0(K) \geq \frac{B}{2}(\mathrm{Cap}\,(K_-))^2,$$
$$K_- = \{z \in \mathbb{C} \mid \limsup_{r\to+0} \frac{\log m(Pc(K) \cap D_r(z))}{\log r} < \infty\},$$

where $D_r(z) = \{\zeta \in \mathbb{C} \mid |\zeta - z| \leq r\}$, and $m(\cdot)$ is the Lebesgue measure.

4. Analysing the proof of Theorem 1.3, it is easy to see that if we are only interested in its statement for finitely many $q$, it suffices to require some finite smoothness of the curve $K$.

### 1.4 Outline of the proof

Let us write $L^2(\mathbb{R}^2) = L^2(K^c) \oplus L^2(K)$. (If the Lebesgue measure of $K$ vanishes then, of course, $L^2(K) = \{0\}$.) With respect to this decomposition, let us define

$$R(K^c) = X(K^c)^{-1} \oplus 0 \text{ in } L^2(\mathbb{R}^2) = L^2(K^c) \oplus L^2(K). \tag{7}$$

Clearly, for any $\lambda \neq 0$ we have

$$\lambda \in \sigma(X(K^c)) \Leftrightarrow \lambda^{-1} \in \sigma(R(K^c)) \tag{8}$$

with the same multiplicity. Thus, it suffices for our purposes to study the spectrum of the operator $R(K^c)$.

First note that in the "free" case $K = \emptyset$ we have $R(\mathbb{R}^2) = X_0^{-1}$ and the spectrum of $X_0^{-1}$ consists of the eigenvalues $\Lambda_q^{-1}$ of infinite multiplicity and their point of accumulation, zero.

Next, it turns out (see section 3) that

$$R(K^c) = X_0^{-1} - W, \quad \text{where } W \geq 0 \text{ is compact.} \tag{9}$$

Thus, the Weyl's theorem on the invariance of the essential spectrum under compact perturbations ensures that $\sigma_{ess}(R(K^c)) = \sigma_{ess}(X_0^{-1})$. Moreover, a simple operator theoretic argument (see e.g. [2, Theorem 9.4.7]) shows that the eigenvalues of $R(K^c)$ do not accumulate to the inverse Landau levels $\Lambda_q^{-1}$ from above. Thus, the spectrum of $R(K^c)$ consists of zero and the eigenvalue clusters $\{(\lambda_1^q)^{-1}, (\lambda_2^q)^{-1}, \dots\}$ with the eigenvalues in the $q$'th cluster accumulating to $\Lambda_q^{-1}$. In section 2.3 we show that the rate of accumulation of $(\lambda_n^q)^{-1}$ to $\Lambda_q^{-1}$ can be described in terms of the spectral asymptotics of the Toeplitz type operator $P_q W P_q$; here $W$ is defined by (9) and $P_q$ is the projection onto $\mathcal{L}_q = \operatorname{Ker}(X_0 - \Lambda_q) = \operatorname{Ker}(X_0^{-1} - \Lambda_q^{-1})$.
The spectrum of $P_q W P_q$ is studied in sections 4 and 5, using the results of [3]. The fact that the logarithmic capacity of the domain appears in this problem probably deserves some explanation. In [3], the spectral asymptotics of $P_q W P_q$ was related to the asymptotics of the singular numbers of the embedding of the Segal-Bargmann space $\mathcal{F}^2$ (see section 4.2 below) into an $L^2$ space with the weight related to $W$. Following the technique of [14], in [3] the analysis of this asymptotics is then reduced to the analysis of the sequence of polynomials of a complex variable, orthogonal with respect to the relevant weight. After this, the results of [18] ensure that the asymptotics of these polynomials is determined by the logarithmic capacity of the support of the weight.

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## 2 Some abstract results

Here we collect some general operator theoretic statements that are used in the proof. The statements themselves, with the exception of the last one, are almost obvious, but spelling them out explicitly helps explain the main ideas of our construction.

### 2.1 Quadratic forms

Our arguments can be stated most succinctly if we are allowed to deal with quadratic forms whose domains are not necessarily dense in the Hilbert space.

Here is the corresponding abstract framework; related constructions appeared before in the literature; see e.g. [17].
Let $a$ be a closed positive definite quadratic form in a Hilbert space $\mathcal{H}$ with the domain $d[a]$. Let the closure of $d[a]$ in $\mathcal{H}$ be $\mathcal{H}_a$. Then the form $a$ defines a self-adjoint operator $A$ in $\mathcal{H}_a$. Let $J_a : \mathcal{H}_a \to \mathcal{H}$ be the natural embedding operator; its adjoint $J_a^* : \mathcal{H} \to \mathcal{H}_a$ acts as the orthogonal projection onto the subspace $\mathcal{H}_a$ of $\mathcal{H}$. The operator $J_a A^{-1} J_a^*$ in $\mathcal{H}$ can be considered as the direct sum

$$J_a A^{-1} J_a^* = A^{-1} \oplus 0 \text{ in the decomposition } \mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_a^{\perp};$$

here we have in mind (7). Now let $b$ be another closed positive definite form in $\mathcal{H}$ and let $B$, $d[b]$, $\mathcal{H}_b$, $J_b$ be the corresponding objects constructed for this form.

Proposition 2.1. *Suppose that $d[b] \subset d[a]$ and $b[x, y] = a[x, y]$ for all $x, y \in d[b]$. Then:*
*(i) $J_b B^{-1} J_b^* \leq J_a A^{-1} J_a^*$ on $\mathcal{H}$;*
*(ii) if $x \in d[b] \cap \mathrm{Dom}\,(A)$, then $x \in \mathrm{Dom}\,(B)$, $Bx = Ax$, and $J_b B^{-1} J_b^* Ax = J_a A^{-1} J_a^* Ax$.*

*Proof.* It suffices to consider the case $\mathcal{H}_a = \mathcal{H}$.
(i) The hypothesis implies

$$b[x, x] = a[J_b x, J_b x] \quad \text{for all } x \in d[b].$$

This can be recast as $\|B^{1/2}x\| = \|A^{1/2}J_b x\|$, $x \in d[b]$. It follows that the operator $A^{1/2}J_b B^{-1/2}$ is an isometry on $\mathcal{H}_b$ and therefore $A^{1/2}J_b B^{-1/2}J_b^*$ is a contraction on $\mathcal{H}$. By conjugation, we get that $\|J_b B^{-1/2} J_b^* A^{1/2} z\| \leq \|z\|$ for all $z \in d[a]$. The last statement is equivalent to $\|J_b B^{-1/2} J_b^* u\| \leq \|A^{-1/2}u\|$ for all $u \in \mathcal{H}$, and so $J_b B^{-1} J_b^* \leq A^{-1}$ as required.
(ii) Let $y \in d[b]$; then

$$b[x, y] = a[x, y] = (Ax, y),$$

and so $x \in \mathrm{Dom}\,(B)$ and $Bx = Ax$. Next, $J_a A^{-1} J_a^* Ax = A^{-1}Ax = x$, and

$$J_b B^{-1} J_b^* Ax = J_b B^{-1} J_b^* Bx = J_b B^{-1} Bx = J_b x = x,$$

which proves the required statement. □

## 2.2 Shift in enumeration

The asymptotics of the type discussed in Theorems 1.2 and 1.3 is independent of a shift in the enumeration of eigenvalues. This is a consequence of the following elementary fact. Let $b_1 \geq b_2 \geq \dots$ be a sequence of positive numbers such that $\limsup_{n\to\infty} [n! b_n]^{1/n} < \infty$. Then for all $\ell \in \mathbb{Z}$,

$$\lim_{n\to\infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} [n! b_{n+\ell}]^{1/n} = \lim_{n\to\infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} [n! b_n]^{1/n}. \tag{10}$$

### 2.3 ACCUMULATION OF EIGENVALUES

Having in mind (9), let us consider the following general situation. Let $T$ be a self-adjoint operator and let $\Lambda$ be an isolated eigenvalue of $T$ of infinite multiplicity with the corresponding eigenprojection $P_\Lambda$. Let $\tau > 0$ be such that

$$((\Lambda - 2\tau, \Lambda + 2\tau) \setminus \{\Lambda\}) \cap \sigma(T) = \emptyset.$$

Next, let $W \geq 0$ be a compact operator; consider the spectrum of $T - W$. The Weyl's theorem on the invariance of the essential spectrum under compact perturbations ensures that

$$((\Lambda - 2\tau, \Lambda + 2\tau) \setminus \{\Lambda\}) \cap \sigma_{ess}(T - W) = \emptyset.$$

Moreover, a simple argument (see e.g. [2, Theorem 9.4.7]) shows that the eigenvalues of $T - W$ do not accumulate to $\Lambda$ from above (i.e. $(\Lambda, \Lambda + \epsilon) \cap \sigma(T - W) = \emptyset$ for some $\epsilon > 0$).
We will need a description of the eigenvalues of $T - W$ below $\Lambda$ in terms of the eigenvalues of the Toeplitz operator $P_\Lambda W P_\Lambda$. Let $\mu_1 \geq \mu_2 \geq \cdots$ be the eigenvalues of $P_\Lambda W P_\Lambda$; in order to exclude degenerate cases, let us assume that this operator has infinite rank. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of $T - W$ in the interval $(\Lambda - \tau, \Lambda)$.

PROPOSITION 2.2. *Under the above assumptions, for any $\epsilon > 0$ there exists $\ell \in \mathbb{Z}$ such that for all sufficiently large $n$, one has*

$$(1 - \epsilon)\mu_{n+\ell} \leq \Lambda - \lambda_n \leq (1 + \epsilon)\mu_{n-\ell}.$$

The proof borrows its key element from [8, Lemma 1.1]. An alternative proof can be found in [15, Proposition 4.1].

*Proof.* 1. We denote $S = T - W$ and $Q_\Lambda = I - P_\Lambda$ and consider the operators

$$R_\pm = \epsilon P_\Lambda W P_\Lambda + \frac{1}{\epsilon} Q_\Lambda W Q_\Lambda \pm (P_\Lambda W Q_\Lambda + Q_\Lambda W P_\Lambda).$$

and

$$S_\pm = P_\Lambda(T - (1 \pm \epsilon)W)P_\Lambda + Q_\Lambda(T - (1 \pm \frac{1}{\epsilon})W)Q_\Lambda.$$

We have

$$S = S_+ + R_- = S_- - R_+.$$

2. Since $W$ is compact, the operators $R_\pm$ are also compact. Since $R_\pm$ can be represented as

$$R_\pm = (\sqrt{\epsilon} P_\Lambda \pm \frac{1}{\sqrt{\epsilon}} Q_\Lambda) W (\sqrt{\epsilon} P_\Lambda \pm \frac{1}{\sqrt{\epsilon}} Q_\Lambda)$$

and $W \geq 0$, we see that $R_\pm \geq 0$.

3. Let us discuss the spectrum of $S_\pm$ in $(\Lambda-\tau,\Lambda)$. Clearly, the spectrum of $P_\Lambda(T-(1\pm\epsilon)W)P_\Lambda=\Lambda P_\Lambda-(1\pm\epsilon)P_\Lambda W P_\Lambda$ consists of the eigenvalues $\Lambda-(1\pm\epsilon)\mu_n$. Next, since by assumption, $T\mid_{\operatorname{Ran}Q_\Lambda}$ has no spectrum in $(\Lambda-2\tau,\Lambda+2\tau)$ and $W$ is compact, we see that $Q_\Lambda(T-(1\pm\frac{1}{\epsilon})W)Q_\Lambda\mid_{\operatorname{Ran}Q_\Lambda}$ has only finitely many eigenvalues in the interval $(\Lambda-\tau,\Lambda+\tau)$. Since the operators $P_\Lambda(T-(1\pm\epsilon)W)P_\Lambda$ and $Q_\Lambda(T-(1\pm\frac{1}{\epsilon})W)Q_\Lambda$ act in orthogonal subspaces of our Hilbert space, the spectrum of $S_\pm$ is the union of the spectra of these operators.

So we arrive at the following conclusion. Let $\nu_1^\pm\le\nu_2^\pm\le\cdots$ denote the eigenvalues of $S_\pm$ in $(\Lambda-\tau,\Lambda)$. Then

$$\nu_n^+=\Lambda-(1+\epsilon)\mu_{n-i},\quad \nu_n^-=\Lambda-(1-\epsilon)\mu_{n-j}, \tag{11}$$

for some integers $i,j$ and all sufficiently large $n$.

4. Let us prove that $\lambda_n\le\nu_{n+k}^-$ for some integer $k$ and all sufficiently large $n$. Denote $\delta=(\lambda_1-\Lambda+\tau)/2$ and let us write $R_+=R_+^{(1)}+R_+^{(2)}$, where $0\le R_+^{(1)}\le\delta I$ and $\operatorname{rank}R_-^{(2)}<\infty$. Denote by $N_S(\alpha,\beta)$ the number of eigenvalues of $S$ in the interval $(\alpha,\beta)$. Writing $S=S_--R_+^{(1)}-R_+^{(2)}$, we get for any $\lambda\in(\lambda_1,\Lambda)$:

$$\begin{aligned}N_S(\Lambda-\tau,\lambda)&=N_S(\lambda_1-2\delta,\lambda)\\&\ge N_{S_--R_+^{(1)}}(\lambda_1-2\delta,\lambda)-\operatorname{rank}R_-^{(2)}\ge N_{S_-}(\lambda_1-\delta,\lambda)-\operatorname{rank}R_-^{(2)}.\end{aligned}$$

The second inequality above follows from $\sigma(R_+^{(1)})\subset[0,\delta]$ (see [2, Lemma 9.4.3]). These inequalities for the eigenvalue counting functions can be rewritten as $\lambda_n\le\nu_{n+k}^-$ with some integer $k$.

In the same way, one proves that $\lambda_n\ge\nu_{n-k}^+$ for large $n$ and some integer $k$. Taken together with (11), this yields the required result. $\square$

## 3 Preliminaries and reduction to Toeplitz operators

Let $K\subset\mathbb{R}^2$ be a compact set; we return to the discussion of the spectrum of $X(K^c)$ and start with some general remarks.

First we would like to point out that the spectral asymptotics that we are interested in is independent of the "holes" in the domain $K$:

$$\delta_q(K)=\delta_q(Pc(K)),\quad \Delta_q(K)=\Delta_q(Pc(K)). \tag{12}$$

Indeed, let us write $K^c=\Omega\cup\Sigma$, where $\Omega$ is the unbounded connected component of $K^c$ and $\Omega$ and $\Sigma$ are disjoint. With respect to the direct sum decomposition $L^2(K^c)=L^2(\Omega)\oplus L^2(\Sigma)$, we have $X(K^c)=X(\Omega)\oplus X(\Sigma)$. By the compactness of the embedding $H_A^1(\Sigma)\subset L^2(\Sigma)$, the operator $X(\Sigma)$ has a compact resolvent. Thus, on any bounded interval of the real line the spectrum of $X(K^c)$ differs from the spectrum of $X(\Omega)$ by at most finitely many eigenvalues. By (10), this yields (12).

Next, we apply the abstract reasoning of section 2.1 to the quadratic form $a[u] = \|u\|^2_{H^1_A(K^c)}$ with domain $d[a] = H^1_A(K^c)$, considering $L^2(\mathbb{R}^2)$ as the main Hilbert space $\mathcal{H}$. We consider the operator $R(K^c)$ (see (7)) and write $R(K^c) = X_0^{-1} - W$. Proposition 2.2 suggests that in order to find the rate of accumulation of the eigenvalues of $R(K^c)$ to $\Lambda_q^{-1}$, one should study the spectrum of the Toeplitz type operators $P_q W P_q$. This is done in the next section. Denote by $\mu_1^q \geq \mu_2^q \geq \dots$ the eigenvalues of $P_q W P_q$. We will prove

PROPOSITION 3.1. *Let $K \subset \mathbb{R}^2$ be a compact set and $q \geq 0$. Then*

$$\limsup_{n\to\infty} (n!\mu_n^q)^{1/n} \leq \frac{B}{2}(\mathrm{Cap}\,(K))^2,$$
$$\liminf_{n\to\infty} (n!\mu_n^q)^{1/n} \geq \frac{B}{2}(\mathrm{Cap}_-(K))^2.$$

*If $K$ is a $C^\infty$ smooth curve, then one has*

$$\lim_{n\to\infty} (n!\mu_n^q)^{1/n} = \frac{B}{2}(\mathrm{Cap}\,(K))^2.$$

Now we can prove our main statements.

*Proof of Theorem 1.1 and Theorem 1.2.* Combining Proposition 3.1, Proposition 2.2 and (12), we get the estimates for the quantities

$$\lim_{n\to\infty} \sup[n!(\Lambda_q^{-1} - (\lambda_n^q)^{-1})]^{1/n} \leq \frac{B}{2}(\mathrm{Cap}\,(K))^2,$$
$$\lim_{n\to\infty} \inf[n!(\Lambda_q^{-1} - (\lambda_n^q)^{-1})]^{1/n} \geq \frac{B}{2}(\mathrm{Cap}_-(Pc(K))^2$$

for any compact $K$. If $K$ is a $C^\infty$ smooth curve, we get

$$\lim_{n\to\infty} [n!(\Lambda_q^{-1} - (\lambda_n^q)^{-1})]^{1/n} = \frac{B}{2}(\mathrm{Cap}\,(K))^2.$$

An elementary argument shows that

$$\lim_{n\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} [n!(\Lambda_q^{-1} - (\lambda_n^q)^{-1})]^{1/n} = \lim_{n\to\infty} \begin{Bmatrix}\sup\\ \inf\end{Bmatrix} [n!(\lambda_n^q - \Lambda_q)]^{1/n}.$$

This yields the required statements. $\square$

*Proof of* (9). Let $D$ be a disc such that $K \subset D$. By Proposition 2.1(i), we get

$$D^c \subset K^c \subset \mathbb{R}^2 \Rightarrow R(D^c) \leq R(K^c) \leq X_0^{-1}$$

and so

$$0 \leq X_0^{-1} - R(K^c) \leq X_0^{-1} - R(D^c). \tag{13}$$

Thus, $W = X_0^{-1} - R(K^c)$ is non-negative; let us address compactness.

It is well known that if $0 \le V_1 \le V_2$ are self-adjoint operators and $V_2$ is compact, then $V_1$ is also compact. Thus, by (13), in order to prove the compactness of $W$, it suffices to check that $X_0^{-1} - R(D^c)$ is compact.
Let $\Gamma = \partial D$. Employing the same arguments as in the proof of (12), we see that $X(\Gamma^c)^{-1} - R(D^c)$ is the inverse of the magnetic operator on the disc and hence a compact operator. Thus, it suffices to prove that the difference

$$X_0^{-1} - X(\Gamma^c)^{-1} = (X_0^{-1} - R(D^c)) - (X(\Gamma^c)^{-1} - R(D^c))$$

is compact.
Let us compute the quadratic form of this difference. Let $f, g \in L^2(\mathbb{R}^2)$, $X_0^{-1} f = u$, $X(\Gamma^c)^{-1} g = v$. We have

$$((X_0^{-1} - X(\Gamma^c)^{-1}) f, g) = (u, X(\Gamma^c) v) - (X_0 u, v).$$

Integrating by parts and noting that $v \in \mathrm{Dom}\,(X(\Gamma^c))$ vanishes on $\Gamma$, we get

$$(u, X(\Gamma^c) v) - (X_0 u, v) = \int_\Gamma (n_A v(s)^+ + n_A v(s)^-) u(s) ds \tag{14}$$

where $n_A v(s) = (\nabla - iA(s)) v \cdot \mathbf{n}(s)$, $\mathbf{n}(s)$ is the exterior normal to $\Gamma$ at the point $s$ and the superscripts $+$ and $-$ indicate that the limits of the functions are taken on the circle $\Gamma$ by approaching it from the outside or inside.
Take a smooth cut-off function $\omega \in C_0^\infty(\mathbb{R}^2)$ such that $\omega(x) = 1$ in the neighborhood of $D$. Then we can replace $u, v$ by $u_1 = \omega u$, $v_1 = \omega v$ in the r.h.s. of (14). By the local elliptic regularity we have $u_1 \in H^2(\mathbb{R}^2)$, $v_1 \in H^2(\Gamma^c)$, and the corresponding Sobolev norms of $u_1$, $v_1$ can be estimated via the $L^2$-norms of $f, g$. Now it remains to notice that the trace mapping $u_1 \mapsto u_1|_\Gamma$ is compact as considered from $H^2(\mathbb{R}^2)$ to $L^2(\Gamma)$, and the mappings $v_1 \mapsto (n_A v_1)^\pm$ are compact as considered from $H^2(\Gamma^c)$ to $L^2(\Gamma)$. It follows that the difference $X_0^{-1} - X(\Gamma^c)^{-1}$ is compact, as required. □

## 4 The spectrum of Toeplitz operators

### 4.1 Restriction operators and the associated Toeplitz operators

Let $\mu$ be a finite measure in $\mathbb{R}^2$ with a compact support. Consider the restriction operator

$$\gamma_0 : C_0^\infty(\mathbb{R}^2) \ni u \mapsto u\,|_{supp(\mu)} \in L^2(\mu).$$

We are interested in two special cases, namely when $\mu$ is the restriction of the Lebesgue measure to a set with Lipschitz boundary and when $\mu$ is the arc length measure on a simple smooth curve. In both cases $\gamma_0$ can be extended by continuity to a bounded and compact operator $\gamma : H_A^1(\mathbb{R}^2) \to L^2(\mu)$.
Next, let $J : H_A^1(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the embedding operator, $J : u \mapsto u$. Then the adjoint $J^* : L^2(\mathbb{R}^2) \to H_A^1(\mathbb{R}^2)$ acts as $J^* : u \mapsto {X_0}^{-1/2} u$.

For $q \geq 0$, consider the operators $T_q(\mu)$ in $L^2(\mathbb{R}^2)$ defined by the quadratic form

$$(T_q(\mu)u, u)_{L^2(\mathbb{R}^2)} = \int_{\operatorname{supp}\mu} |(P_q u)(x)|^2 d\mu(x), \quad u \in L^2(\mathbb{R}^2).$$

This operator can be represented as

$$T_q(\mu) = (\gamma J^* {X_0}^{1/2} P_q)^* (\gamma J^* {X_0}^{1/2} P_q) = \Lambda_q (\gamma J^* P_q)^* (\gamma J^* P_q).$$

Since $\gamma$ is compact by assumption, the operator $T_q(\mu)$ is also compact.
Fix $q \geq 0$; let $s_1^q \geq s_2^q \geq \dots$ be the eigenvalues of $T_q(\mu)$ in $L^2(\mathbb{R}^2)$.

PROPOSITION 4.1. *(i) Let $\mu$ be the restriction of the Lebesgue measure onto a bounded domain $K \subset \mathbb{R}^2$ with a Lipschitz boundary. Then*

$$\lim_{n\to\infty} (n! s_n^q)^{1/n} = \frac{B}{2} (\operatorname{Cap}(K))^2.$$

*(ii) Let $\mu$ be the arc length measure of a $C^\infty$ smooth simple curve of a finite length. Then*

$$\lim_{n\to\infty} (n! s_n^q)^{1/n} = \frac{B}{2} (\operatorname{Cap}(\operatorname{supp}\mu))^2.$$

Before proving this proposition, we need to recall the description of the subspaces $\mathcal{L}_q$.

### 4.2 THE STRUCTURE OF SUBSPACES $\mathcal{L}_q$

Denote $\Psi(z) = \frac{1}{4}B|z|^2$. Let us define the creation and annihilation operators (first introduced in this context by Fock [5])

$$\mathfrak{Q} = -2ie^{-\Psi}\overline{\partial}e^{\Psi} = -2i\overline{\partial} - \frac{B}{2}iz$$
$$\mathfrak{Q}^* = -2ie^{\Psi}\partial e^{-\Psi} = -2i\partial + \frac{B}{2}i\overline{z}.$$

The Landau Hamiltonian can be expressed as

$$X_0 = \mathfrak{Q}^*\mathfrak{Q} + B = \mathfrak{Q}\mathfrak{Q}^* - B. \tag{15}$$

The spectrum and spectral subspaces of $X_0$ can be described in the following way. The equation $(X_0 - B)u = 0$ is equivalent to

$$\mathfrak{Q}u = -2ie^{-\Psi}\bar{\partial}(e^{\Psi}u) = 0.$$

This means that $f = e^{\Psi}u$ is an entire analytic function such that $e^{-\Psi}f \in L^2(\mathbb{C})$. The space of such functions $f$ is called Fock or Segal-Bargmann space $\mathcal{F}^2$ (see [6] for an extensive discussion). So $\mathcal{L}_0 = e^{-\Psi}\mathcal{F}^2$. Further eigenspaces $\mathcal{L}_q$, $q = 1, 2, \dots$, are obtained as $\mathcal{L}_q = (\mathfrak{Q}^*)^q\mathcal{L}_0$. The operators $\mathfrak{Q}^*, \mathfrak{Q}$ act between the subspaces $\mathcal{L}_q$ as

$$\mathfrak{Q}^* : \mathcal{L}_q \mapsto \mathcal{L}_{q+1}, \quad \mathfrak{Q} : \mathcal{L}_q \mapsto \mathcal{L}_{q-1}, \quad \mathfrak{Q} : \mathcal{L}_0 \mapsto \{0\}, \tag{16}$$

and are, up to constant factors, isometries on $\mathcal{L}_q$. In particular, the substitution

$$\mathcal{L}_q \ni u = C_q^{-1}(\mathfrak{Q}^*)^q e^{-\Psi} f, \quad f \in \mathcal{F}^2, \quad C_q = \sqrt{q!(2B)^q} \tag{17}$$

gives a unitary equivalence of spaces $\mathcal{L}_q$ and $\mathcal{F}^2$.

### 4.3 Proof of Proposition 4.1

(i) The proof is given in [3, Lemma 3.1] for $q = 0$ and [3, Lemma 3.2] for $q \geq 0$.
(ii) For $q = 0$ the result again follows from Lemma 3.1 in [3]. Although the reasoning there concerns the operators $T_q(v) = (vP_q)^*(vP_q)$ where the function $v$ is separated from zero on a compact, it goes through for $T_q(\mu)$. Only notational changes are required; one simply has to replace the measure $v(z)dm(z)$ by $d\mu(z)$.
For $q \geq 1$ below we apply the reduction to the lowest Landau level similar to the proof of Lemma 3.2 in [3].
Denote $d\tilde{\mu}(z) = e^{-\Psi(z)}d\mu(z)$. Applying the unitary equivalence (17), we get for $u \in \mathcal{L}_q$

$$(T_q(\mu)u, u)_{L^2(\mathbb{R}^2)} = C_q^{-2}\|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\mu})}. \tag{18}$$

In particular, for $q = 0$

$$(T_0(\mu)u, u)_{L^2(\mathbb{R}^2)} = C_0^{-2}\|f\|^2_{L^2(\tilde{\mu})}. \tag{19}$$

Below we separately prove the upper and lower bound for the quadratic form (18).
1. *Upper bound.* Consider the open $\delta$-neighborhood $U_\delta \subset \mathbb{C}^1$ of the curve $\Gamma$. As it follows from the Cauchy integral formula, for some constant $C_1(q, \delta)$, the inequality

$$\|\partial^k f\|^2_{L^2(\tilde{\mu})} \leq C_1(q, \delta) \int_{U_\delta} |f(z)|^2 dm(z).$$

holds for all functions $f \in \mathcal{F}^2$. Thus, we have the estimate

$$\|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\mu})} \leq C_2(q, \delta) \int_{U_\delta} |f(z)|^2 dm(z).$$

Using (18), (19), we arrive at the estimate

$$T_q(\mu) \leq C T_0(\chi_{U_\delta}(x)dx), \tag{20}$$

where $\chi_{U_\delta}$ is the characteristic function of the set $U_\delta$. Now we can again apply the estimate of [3, Lemma 3.1] to the eigenvalues $s_1(\delta) \geq s_2(\delta) \geq \ldots$ of $T_0(\chi_{U_\delta}(x)dx)$. This estimate together with (20) yields

$$\varlimsup_{n\to\infty} (n!s_n)^{1/n} \leq \varlimsup_{n\to\infty} (n!s_n(\delta))^{1/n} \leq \frac{B}{2}(\mathrm{Cap}\,(U_\delta))^2.$$

Finally, by the continuity of capacity, $\operatorname{Cap}(U_\delta) \to \operatorname{Cap}(\Gamma)$ as $\delta \to 0$, and this proves the upper bound.

2. *Lower bound.* The lower bound for the spectrum of $T_q(\mu)$ requires a little more work. Let $\sigma : [0, s] \to \mathbb{C}$ be the parameterization of $\Gamma$ by the arc length. Since $f$ is analytic, we have

$$(\partial f)(\sigma(t)) = \rho(t)\frac{d}{dt}f(\sigma(t)) \tag{21}$$

with some smooth factor $\rho(t)$, $|\rho(t)| = 1$.

Next, due to the compactness of the embedding $H^1(0, s) \subset L^2(0, s)$, for any $\beta > 0$ there exists a subspace of $H^1(0, s)$ of a finite codimension such that for any $u$ in this subspace,

$$\int_0^s |u(t)|^2 dt \le \beta^2 \int_0^s |u'(t)|^2 dt. \tag{22}$$

It follows from (21) and (22) that for any $\beta > 0$ there exists a subspace of $\mathcal{F}^2$ of a finite codimension such that for any $f$ in this subspace

$$\int_\Gamma |f(z)|^2 d\tilde{\mu}(z) \le \beta^2 \int_\Gamma |\partial f(z)|^2 d\tilde{\mu}(z).$$

Arguing by induction, we obtain that for any $\beta > 0$ there exists a subspace $N = N(\beta, q) \subset \mathcal{F}^2$ of finite codimension such that for all $f \in N(\beta, q)$

$$\int_\Gamma |\partial^k f(z)|^2 d\tilde{\mu}(z) \le \beta^2 \int_\Gamma |\partial^q f(z)|^2 d\tilde{\mu}(z), \quad \forall k = 0, 1, \dots, q-1. \tag{23}$$

Using (23) and choosing $\beta$ sufficiently small, we can estimate the form (18) from below as follows:

$$\begin{aligned}\|(2\partial - B\overline{z})^q f\|^2_{L^2(\tilde{\mu})} &\ge (\|(2\partial)^q f\|_{L^2(\tilde{\mu})} - \sum_{k=0}^{q-1} C_{q,k}\|\partial^k f\|_{L^2(\tilde{\mu})})^2 \\ &\ge \|(2\partial)^q f\|^2_{L^2(\tilde{\mu})}(1 - \sum_{k=0}^{q-1} C_{q,k}\beta)^2 = C_1\|\partial^q f\|^2_{L^2(\tilde{\mu})} \ge C_2\|f\|^2_{L^2(\tilde{\mu})}\end{aligned}$$

for all $f \in N(\beta, q)$. Using (18) and (19), we arrive at the lower bound

$$T_q(\mu) \ge CT_0(\mu) + F$$

where $F$ is a finite rank operator. For the eigenvalues of $T_0(\mu)$ the required lower estimates are already obtained by reference to [3, Lemma 3.1]; this completes the proof of the lower bound. $\square$

## 5 PROOF OF PROPOSITION 3.1

We will prove separately upper and lower bounds.

### 5.1 Proof of the upper bound

1. Let $U \subset \mathbb{R}^2$ be an open bounded set with a Lipschitz boundary, $K \subset U$, and let $\omega \in C_0^\infty(\mathbb{R}^2)$ be such that $\omega|_K = 1$ and $\omega|_{U^c} = 0$. Denote $\tilde{\omega} = 1 - \omega$. Note that for any $\psi \in \mathcal{H}$, the function $\tilde{\omega} P_q \psi$ belongs both to $\mathrm{Dom}\,(X_0)$ and to the form domain of $X(K^c)$. Thus, by Proposition 2.1(ii) (with $A = X_0$ and $B = X(K^c)$), we have $W X_0 \tilde{\omega} P_q \psi = 0$. Thus, we have

$$
\begin{aligned}
(W P_q \psi, P_q \psi) &= \frac{1}{\Lambda_q^2} (W X_0 P_q \psi, X_0 P_q \psi) \\
&= \frac{1}{\Lambda_q^2} (W X_0 (\omega + \tilde{\omega}) P_q \psi, X_0 (\omega + \tilde{\omega}) P_q \psi) = \frac{1}{\Lambda_q^2} (W X_0 \omega P_q \psi, X_0 \omega P_q \psi).
\end{aligned}
$$

Since $W = X_0^{-1} - R(K^c) \le X_0^{-1}$, we have

$$
(W X_0 \omega P_q \psi, X_0 \omega P_q \psi) \le (X_0^{-1} X_0 \omega P_q \psi, X_0 \omega P_q \psi) = \|\omega P_q \psi\|^2_{H^1_A}.
$$

Using (15), we get

$$
\begin{aligned}
\|\omega P_q \psi\|^2_{H^1_A} &= \|\mathfrak{Q}^* \omega P_q \psi\|^2 - B \|\omega P_q \psi\|^2 \le \|\mathfrak{Q}^* \omega P_q \psi\|^2 \\
&= \|\omega \mathfrak{Q}^* P_q \psi - 2i(\partial \omega) P_q \psi\|^2 \le 2 \|\mathfrak{Q}^* P_q \psi\|^2_{L^2(U)} + C_1 \|P_q \psi\|^2_{L^2(U)}.
\end{aligned}
$$

2. Due to the compactness of the embedding $H^1_0(U) \subset L^2(U)$, for any $\beta > 0$ there exists a subspace of $H^1_0(U)$ of a finite codimension such that for all elements $u$ of this subspace,

$$
\int_U |u(x)|^2 dx \le \beta^2 \int_U |\nabla u(x)|^2 dx = \beta^2 \int_U |2 \partial u(x)|^2 dx.
$$

Taking $\beta$ sufficiently small, we obtain

$$
\begin{aligned}
\|\mathfrak{Q}^* u\|_{L^2(U)} &\ge \|2\partial u\|_{L^2(U)} - \frac{B}{2} \|\overline{z} u\|_{L^2(U)} \ge \|2\partial u\|_{L^2(U)} - \frac{B}{2} \sup_U |z| \|u\|_{L^2(U)} \\
&\ge (1 - \frac{B}{2} \beta \sup_U |z|) \|2\partial u\|_{L^2(U)} \ge \frac{1}{2} \|2\partial u\|_{L^2(U)} \ge \frac{1}{2\beta} \|u\|_{L^2(U)}
\end{aligned}
$$

for all $u$ in our subspace. It follows that on a subspace of $\psi \in L^2(\mathbb{R}^2)$ of a finite codimension,

$$
(W P_q \psi, P_q \psi)_{L^2(\mathbb{R}^2)} \le 2 \|\mathfrak{Q}^* P_q \psi\|^2_{L^2(U)} + 4\beta^2 \|\mathfrak{Q}^* P_q \psi\|^2_{L^2(U)} \le C \|P_{q+1} \psi\|^2_{L^2(U)};
$$

the last inequality holds true by (16).
Thus, we have

$$
P_q W P_q \le C_3 P_{q+1} \chi_U P_{q+1} + F,
$$

where $\chi_U$ is the characteristic function of $U$, and $F$ is a finite rank operator.

3. From Proposition 4.1 we get

$$\limsup_{n\to\infty}(n!\mu_n)^{1/n} \le \frac{1}{2}B(\mathrm{Cap}\ U)^2.$$

Since $U$ can be chosen such that Cap $U$ is arbitrarily close to Cap $K$, by the continuity property of capacity, we get the required upper bound.

### 5.2 Proof of the lower bound

1. Let $\gamma$, $J$, $\mu$ be as in section 4.1. Consider the quadratic form in $L^2(\mathbb{R}^2)$

$$\|u\|^2_{H^1_A(\mathbb{R}^2)} + \int_{\mathrm{supp}\,\mu} |u(x)|^2 d\mu(x) = \|X_0^{1/2}u\|^2_{L^2(\mathbb{R}^2)} + \|\gamma J^* X_0^{1/2} u\|^2_{L^2(\mathbb{R}^2)}$$

defined for $u \in H^1_A(\mathbb{R}^2)$. This form is closed and positively defined on $L^2(\mathbb{R}^2)$. Denote by $\tilde{X}$ the corresponding self-adjoint operator in $L^2(\mathbb{R}^2)$. We have

$$\tilde{X} = X_0 + X_0^{1/2} J\gamma^*\gamma J^* X_0^{1/2} = X_0^{1/2}(I + J\gamma^*\gamma J^*)X_0^{1/2}$$

and therefore

$$X_0^{-1} - \tilde{X}^{-1} = X_0^{-1/2}[J\gamma^*\gamma J^*(I + J\gamma^*\gamma J^*)^{-1}]X_0^{-1/2}.$$

Since $\gamma$ is compact by assumption, we have $J\gamma^*\gamma J^* \le I$ on a subspace of a finite codimension. Thus,

$$X_0^{-1} - \tilde{X}^{-1} \ge \frac{1}{2} X_0{}^{-1/2} J\gamma^*\gamma J^* X_0^{-1/2}$$

on a subspace of finite codimension, and so

$$P_q(X_0^{-1} - \tilde{X}^{-1})P_q \ge \frac{1}{2\Lambda_q}(\gamma J^* P_q)^*(\gamma J^* P_q) + F \tag{24}$$

where $F$ is a finite rank operator.
2. Now let $K \subset \mathbb{R}^2$ be a compact with a non-empty interior. Let $K_1 \subset K$ be a set with a Lipschitz boundary. Let $\mu$ be the restriction of the Lebesgue measure on $K_1$. By Proposition 2.1(i), we have $X_0^{-1} \ge \tilde{X}^{-1} \ge R(K^c)$. It follows that

$$P_q(X_0^{-1} - R(K^c))P_q \ge P_q(X_0^{-1} - \tilde{X}^{-1})P_q. \tag{25}$$

From here, using (24) and Proposition 4.1(i), we get the required lower bound in the first part of Proposition 3.1. Finally, consider the case of $K$ being a smooth curve. Let $\mu$ be the arc measure of the curve. Then, again by (24) and (25), and applying Proposition 4.1(ii), we get the second part of Proposition 3.1.

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# Visibility of Mordell-Weil Groups

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Abstract. We introduce a notion of visibility for Mordell-Weil groups, make a conjecture about visibility, and support it with theoretical evidence and data. These results shed new light on relations between Mordell-Weil and Shafarevich-Tate groups.

11G05, 11G10, 11G18, 11Y40
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## 1 Introduction

Consider an exact sequence $0 \to C \to B \to A \to 0$ of abelian varieties over a number field $K$. We say that the covering $B \to A$ is *optimal* since its kernel $C$ is connected. As introduced in [LT58], there is a corresponding long exact sequence of Galois cohomology

$$0 \to C(K) \to B(K) \to A(K) \xrightarrow{\delta} \mathrm{H}^1(K, C) \to \mathrm{H}^1(K, B) \to \mathrm{H}^1(K, A) \to \cdots$$

The study of the Mordell-Weil group $A(K)$ is central in arithmetic geometry. For example, the Birch and Swinnerton-Dyer conjecture (BSD conjecture) of [Bir71, Tat66]), which is one of the Clay Math Problems [Wil00], asserts that the rank $r$ of $A(K)$ equals the ordering vanishing of $L(A, s)$ at $s = 1$, and also gives a conjectural formula for $L^{(r)}(A, 1)$ in terms of the invariants of $A$.

The group $\mathrm{H}^1(K, A)$ is also of interest in connection with the BSD conjecture, because it contains the Shafarevich-Tate group

$$Ш(A/K) = \mathrm{Ker}\left(\mathrm{H}^1(K, A) \to \bigoplus_v \mathrm{H}^1(K_v, A)\right),$$

which is the most mysterious object appearing in the BSD conjecture.

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Definition 1.0.1 (Visibility). The *visible subgroup* of $\mathrm{H}^1(K, C)$ relative to the embedding $C \hookrightarrow B$ is

$$\begin{aligned}\mathrm{Vis}_B \mathrm{H}^1(K, C) &= \mathrm{Ker}(\mathrm{H}^1(K, C) \to \mathrm{H}^1(K, B)) \\ &\cong \mathrm{Coker}(B(K) \to A(K)).\end{aligned}$$

The *visible quotient* of $A(K)$ relative to the optimal covering $B \to A$ is

$$\begin{aligned}\mathrm{Vis}^B(A(K)) &= \mathrm{Coker}(B(K) \to A(K)) \\ &\cong \mathrm{Vis}_B \mathrm{H}^1(K, C).\end{aligned}$$

We say an abelian variety over $\mathbb{Q}$ is *modular* if it is a quotient of the modular Jacobian $J_1(N) = \mathrm{Jac}(X_1(N))$, for some $N$. For example, every elliptic curve over $\mathbb{Q}$ is modular [BCDT01].

This paper gives evidence toward the following conjecture that Mordell-Weil groups should give rise to many visible Shafarevich-Tate groups.

Conjecture 1.0.2. *Let $A$ be an abelian variety over a number field $K$. For every integer $m$, there is an exact sequence $0 \to C \to B \to A \to 0$ such that:*

1. *The image of $B(K)$ in $A(K)$ is contained in $mA(K)$, so $A(K)/mA(K)$ is a quotient of* $\mathrm{Vis}^B(A(K))$.
2. *If $K = \mathbb{Q}$ and $A$ is modular, then $B$ is modular.*
3. *The rank of $C$ is zero.*
4. *We have* $\mathrm{Coker}(B(K) \to A(K)) \subset \text{Ш}(C/K)$, *via the connecting homomorphism.*

In [Ste04] we give the following computational evidence for this conjecture.

Theorem 1.0.3. *Let $E$ be the rank 1 elliptic curve $y^2 + y = x^3 - x$ of conductor 37. Then Conjecture 1.0.2 is true for all* primes $m = p < 25000$ *with $p \neq 2, 37$.*

Let $f = \sum a_n q^n$ be the newform associated to the elliptic curve $E$ of Theorem 1.0.3. Suppose $p$ is one of the primes in the theorem. Then there is an $\ell \equiv 1 \pmod{p}$ and a surjective Dirichlet character $\chi : (\mathbb{Z}/\ell\mathbb{Z})^* \to \mu_p$ such that $L(f \otimes \chi, 1) \neq 0$. The $C$ of the theorem belongs to the isogeny class of abelian varieties associated to $f^\chi$ and $C$ has dimension $p - 1$.

In general, we expect the construction of [Ste04] to work for any elliptic curve and any odd prime $p$ of good reduction. The main obstruction to proving that it does work is proving a nonvanishing result for the special values $L(f^\chi, 1)$. In [Ste04], we verified this hypothesis using modular symbols for $p < 25000$.

A surprising observation that comes out of the construction of [Ste04] is that $\#\text{Ш}(C) = p \cdot n^2$, where $n^2$ is an integer square. We thus obtained the first ever examples of abelian varieties whose Shafarevich-Tate groups have order neither a square nor twice a square.

### 1.1 Contents

In Section 2, we give a brief review of results about visibility of Shafarevich-Tate groups. In Section 3, we give evidence for Conjecture 1.0.2 using results of Kato, Lichtenbaum and Mazur. Section 4 is about bounding the dimension of the abelian varieties in which Mordell-Weil groups are visible. We prove that every Mordell-Weil group is 2-visible relative to an abelian surface. In Section 5, we describe how to construct visible quotients of Mordell-Weil groups, and carry out a computational study of relations between Mordell-Weil groups of elliptic curves and the arithmetic of rank 0 factors of $J_0(N)$.

### 1.2 Acknowledgement

The author had extremely helpful conversations with Barry Mazur and Grigor Grigorov. Proposition 3.0.3 was proved jointly with Ken Ribet. The author was supported by NSF grant DMS-0400386. He used Magma [BCP97] and SAGE [Sage07] for computing the data in Section 5.

## 2 Review of Visibility of Galois Cohomology

In this section, we briefly review visibility of elements of $\mathrm{H}^1(K, A)$, as first introduced by Mazur in [CM00, Maz99], and later developed by Agashe and Stein in [Aga99a, AS05, AS02]. We describe two basic results about visibility, and in Section 2.2 we discuss modularity of elements of $\mathrm{H}^1(K, A)$.

Consider an exact sequence of abelian varieties

$$0 \to A \to B \to C \to 0$$

over a number field $K$. Elements of $\mathrm{H}^0(K, C)$ are points, so they are relatively easy to "visualize", but elements of $\mathrm{H}^1(K, A)$ are mysterious.

There is a geometric way to view elements of $\mathrm{H}^1(K, A)$. The Weil-Chatalet group $\mathrm{WC}(A/K)$ of $A$ over $K$ is the group of isomorphism classes of principal homogeneous spaces for $A$, where a principal homogeneous space is a variety $X$ and a simply-transitive action $A \times X \to X$. Thus $X$ is a twist of $A$ as a variety, but $X(K) = \emptyset$, unless $X$ is isomorphic to $A$. Also, the elements of $Ш(A)$ correspond to the classes of $X$ that have a $K_v$-rational point for all places $v$. By [LT58, Prop. 4], there is an isomorphism between $\mathrm{H}^1(K, A)$ and $\mathrm{WC}(A/K)$.

In [CM00], Mazur introduced the visible subgroup of $\mathrm{H}^1$ as in Definition 1.0.1 in order to help unify diverse constructions of principal homogeneous spaces. Many papers were subsequently written about visibility, including [Aga99b, Maz99, Kle01, AS02, MO03, DWS03, AS05, Dum01].

*Remark* 2.0.1. Note that $\mathrm{Vis}_B \mathrm{H}^1(K, A)$ depends on the embedding of $A$ into $B$. For example, if $B = B_1 \times A$, then there could be nonzero visible elements if $A$ is embedded into the first factor, but there will be no nonzero visible elements if $A$ is embedded into the second factor.

A connection between visibility and $\mathrm{WC}(A/K)$ is as follows. Suppose

$$0 \to A \to B \xrightarrow{\pi} C \to 0$$

is an exact sequence of abelian varieties and that $c \in \mathrm{H}^1(K, A)$ is visible in $B$. Thus there exists $x \in C(K)$ such that $\delta(x) = c$, where $\delta : C(K) \to \mathrm{H}^1(K, A)$ is the connecting homomorphism. Then $X = \pi^{-1}(x) \subset B$ is a translate of $A$ in $B$, so the group law on $B$ gives $X$ the structure of principal homogeneous space for $A$, and this homogeneous space in $\mathrm{WC}(A/K)$ corresponds to $c$.

## 2.1 Basic Facts

Two basic facts about visibility are that the visible subgroup of $\mathrm{H}^1(K, A)$ in $B$ is finite, and that each element of $\mathrm{H}^1(K, A)$ is visible in some $B$.

Lemma 2.1.1. *The group* $\mathrm{Vis}_B \mathrm{H}^1(K, A)$ *is finite.*

*Proof.* Let $C = B/A$. By the Mordell-Weil theorem $C(K)$ is finitely generated. The group $\mathrm{Vis}_B \mathrm{H}^1(K, A)$ is a homomorphic image of $C(K)$ so it is finitely generated. On the other hand, it is a subgroup of $\mathrm{H}^1(K, A)$, so it is a torsion group. But a finitely generated torsion abelian group is finite. $\square$

Proposition 2.1.2. *Let* $c \in \mathrm{H}^1(K, A)$*. Then there exists an abelian variety* $B$ *and an embedding* $A \hookrightarrow B$ *such that* $c$ *is visible in* $B$*. Moreover,* $B$ *can be chosen to be a twist of a power of* $A$*.*

*Proof.* See [AS02, Prop. 1.3] for a cohomological proof or [JS05, §5] for an equivalent geometric proof. Johan de Jong also proved that everything is visible somewhere in the special case $\dim(A) = 1$ using Azumaya algebras, Néron models, and étale cohomology, as explained in [CM00, pg. 17–18], but his proof gives no (obvious) specific information about the structure of $B$. $\square$

## 2.2 Modularity

Usually one focuses on visibility of elements in $Ш(A) \subset \mathrm{H}^1(K, A)$. The papers [CM00, AS02, AS05] contain a number of results about visibility in various special cases, and tables involving elliptic curves and modular abelian varieties.

For example, if $A \subset J_0(389)$ is the 20-dimensional simple newform abelian variety, then we show that

$$\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \cong E(\mathbb{Q})/5E(\mathbb{Q}) \subset Ш(A),$$

where $E$ is the elliptic curve of conductor 389. The divisibility $5^2 \mid \#Ш(A)$ is as predicted by the BSD conjecture. The paper [AS05] contains a few dozen other examples like this; in most cases, explicit computational construction of the Shafarevich-Tate group seems hopeless using any other known techniques.

The author has conjectured that if $A$ is a modular abelian variety, then every element of $Ш(A)$ is modular, i.e., visible in a modular abelian variety. It is a theorem that if $c \in Ш(A)$ has order either 2 or 3 and $A$ is an elliptic curve, then $c$ is modular (see [JS05]).

## 3 Results Toward Conjecture 1.0.2

The main result of this section is a proof of parts 1 and 2 of Conjecture 1.0.2 for elliptic curves over $\mathbb{Q}$. We prove more generally that Mazur's conjecture on finite generatedness of Mordell-Weil groups over cyclotomic $\mathbb{Z}_p$-extensions implies part 1 of Conjecture 1.0.2. Then we observe that for elliptic curves over $\mathbb{Q}$, Mazur's conjecture is known, and prove that the abelian varieties that appear in our visibility construction are modular, so parts 1 and 2 of Conjecture 1.0.2 are true for elliptic curves over $\mathbb{Q}$.

For a prime $p$, the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ is an extension $\mathbb{Q}_{p^\infty}$ of $\mathbb{Q}$ with Galois group $\mathbb{Z}_p$; also $\mathbb{Q}_{p^\infty}$ is contained in the cyclotomic field $\mathbb{Q}(\mu_{p^\infty})$. We let $\mathbb{Q}_{p^n}$ denote the unique subfield of $\mathbb{Q}_{p^\infty}$ of degree $p^n$ over $\mathbb{Q}$. If $K$ is an arbitrary number field, the cyclotomic $\mathbb{Z}_p$-extension of $K$ is $K_{p^\infty} = K\cdot\mathbb{Q}_{p^\infty}$. We denote by $K_{p^n}$ the unique subfield of $K_{p^\infty}$ of degree $p^n$ over $K$. The extension $K_{p^\infty}$ of $K$ decomposes as a tower

$$K = K_{p^0} \subset K_{p^1} \subset \cdots \subset K_{p^n} \subset \cdots \subset K_{p^\infty} = \bigcup_{n=0}^{\infty} K_{p^n}.$$

Mazur hints at the following conjecture in [Maz78] and [RM05, §3]:

Conjecture 3.0.1 (Mazur). *If $A$ is an abelian variety over a number field $K$ and $p$ is a prime, then $A(K_{p^\infty})$ is a finitely generated abelian group.*

Let $L/K$ be a finite extension of number fields and $A$ an abelian variety over $K$. In much of the rest of this paper we will use the *restriction of scalars* $R = \mathrm{Res}_{L/K}(A_L)$ of $A$ viewed as an abelian variety over $L$. Thus $R$ is an abelian variety over $K$ of dimension $[L : K]$, and $R$ represents the following functor on the category of $K$-schemes:

$$S \mapsto E_L(S_L).$$

If $L/K$ is Galois, then we have an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-modules

$$R(\overline{\mathbb{Q}}) = A(\overline{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Gal}(L/K)],$$

where $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts on $\sum P_\sigma \otimes \sigma$ by

$$\tau\left(\sum P_\sigma \otimes \sigma\right) = \sum \tau(P_\sigma) \otimes \tau_{|L} \cdot \sigma,$$

where $\tau_{|L}$ is the image of $\tau$ in $\mathrm{Gal}(L/K)$.

Theorem 3.0.2. *Conjecture 3.0.1 implies part 1 of Conjecture 1.0.2. More precisely, if $A/K$ is an abelian variety, $m$ is a positive integer, and $A(K_{p^\infty})$ is finitely generated for each $p \mid m$, then there is an optimal covering of the form $B = \mathrm{Res}_{L/K}(A_L) \to A$ such that $L$ is abelian over $K$ and the image of $B(K)$ in $A(K)$ is contained in $mA(K)$.*

*Proof.* Fix a prime $p \mid m$. Let $M = K_{p^\infty}$. Because $A(M)$ is finitely generated, some finite set of generators must be in a single sufficiently large $A(K_{p^n})$, and for this $n$ we have $A(M) = A(K_{p^n})$. For any integer $j > 0$ let

$$R_j = \mathrm{Res}_{K_{p^j}/K}(A_{K_{p^j}}).$$

Then, as explained in [Ste04], the trace map induces an exact sequence

$$0 \to B_j \to R_j \xrightarrow{\pi_j} A \to 0,$$

with $B_j$ an abelian variety. Then for any $j \geq n$, $A(K_{p^j}) = A(K_{p^n})$, so

$$\begin{aligned}
\mathrm{Vis}^{B_j}(A(K)) &\cong A(K)/\pi_j(R_j(K)) \\
&= A(K)/\operatorname{Tr}_{K_{p^j}/K}(A(K_{p^j})) \\
&= A(K)/\operatorname{Tr}_{K_{p^n}/K}(\operatorname{Tr}_{K_{p^j}/K_{p^n}}(A(K_{p^j}))) \\
&= A(K)/\operatorname{Tr}_{K_{p^n}/K}(\operatorname{Tr}_{K_{p^j}/K_{p^n}}(A(K_{p^n}))) \\
&= A(K)/\operatorname{Tr}_{K_{p^n}/K}(p^{j-n}A(K_{p^n})) \\
&= A(K)/p^{j-n}\operatorname{Tr}_{K_{p^n}/K}(A(K_{p^n})) \\
&\to A(K)/p^{j-n}A(K),
\end{aligned}$$

where the last map is surjective since

$$\operatorname{Tr}_{K_{p^n}/K}(A(K_{p^n})) \subset A(K).$$

Arguing as above, for each prime $p \mid m$, we find an extension $L_p$ of $K$ of degree a power of $p$ such that $\operatorname{Tr}_{L_p/K}(A(L_p)) \subset p^{\nu_p}A(K)$, where $\nu_p = \mathrm{ord}_p(m)$. Let $L$ be the compositum of the fields $L_p$. Then for each $p \mid m$,

$$\operatorname{Tr}_{L/K}(A(L)) = \operatorname{Tr}_{L_p/K}(\operatorname{Tr}_{L/L_p}(A(L))) \subset \operatorname{Tr}_{L_p/K}(A(L_p)) \subset p^{\nu_p}A(K).$$

Thus

$$\operatorname{Tr}_{L/K}(A(L)) \subset \bigcap_{p \mid m} p^{\nu_p}A(K) = mA(K), \tag{1}$$

where for the last equality we view $A(K)$ as a finite direct sum of cyclic groups.

Let $R = \mathrm{Res}_{L/K}(A_L)$. Then trace induces an optimal cover $R \to A$, and (1) implies that we have the required surjective map

$$\mathrm{Vis}^R(A(K)) = A(K)/\operatorname{Tr}_{L/K}(A(L)) \to A(K)/mA(K).$$

□

We will next prove parts 1 and 2 of Conjecture 1.0.2 for elliptic curves over $\mathbb{Q}$ by observing that Conjecture 3.0.1 is a theorem of Kato in this case. We first prove a modularity property for restriction of scalars. Recall that a modular abelian variety is a quotient of $J_1(N)$.

Proposition 3.0.3. *If $A$ is a modular abelian variety over $\mathbb{Q}$ and $K$ is an abelian extension of $\mathbb{Q}$, then $\mathrm{Res}_{K/\mathbb{Q}}(A_K)$ is also a modular abelian variety.*

*Proof.* Since $A$ is modular, $A$ is isogenous to a product of abelian varieties $A_f$ attached to newforms in $S_2(\Gamma_1(N))$, for various $N$. Since the formation of restriction of scalars commutes with products, it suffices to prove the proposition under the hypothesis that $A = A_f$ for some newform $f$. Let $R = \mathrm{Res}_{K/\mathbb{Q}}(A_f)$. As discussed in [Mil72, pg. 178], for any prime $p$ there is an isomorphism of $\mathbb{Q}_p$-adic Tate modules

$$V_p(R) \cong \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} V_p(A_K).$$

The induced representation on the right is the direct sum of twists of $V_p(A_K)$ by characters of $\mathrm{Gal}(K/\mathbb{Q})$. This is isomorphic to the $\mathbb{Q}_p$-adic Tate module of some abelian variety $P = \prod_\chi A_{g^\chi}$, where $\chi$ runs through certain Dirichlet characters corresponding to the abelian extension $K/\mathbb{Q}$, and $g$ runs through certain $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of $f$, and $g^\chi$ denotes the twist of $g$ by $\chi$. Falting's theorem (see e.g., [Fal86, §5]) then gives us the desired isogeny $R \to P$.

It is not necessary to use the full power of Falting's theorem to prove this proposition, since Ribet [Rib80] gave a more elementary proof of Falting's theorem in the case of modular abelian varieties. However, we must work some to apply Ribet's theorem, since we do not know yet that $R$ is modular.

Let $R$ and $P$ be as above. Over $\overline{\mathbb{Q}}$, the abelian variety $A$ is isogenous to a power of a simple abelian variety $B$, since if more than one non-isogenous simple occurred in the decomposition of $A/\overline{\mathbb{Q}}$, then $\mathrm{End}(A/\overline{\mathbb{Q}})$ would not be a matrix ring over a (possibly skew) field (see [Rib92, §5]). For any character $\chi$, by the (3) $\implies$ (2) assertion of [Rib80, Thm. 4.7], the abelian varieties $A_f$ and $A_{f^\chi}$ are isogenous over $\overline{\mathbb{Q}}$ to powers of the same abelian variety $A'$, hence to powers of the simple $B$. A basic property of restriction of scalars is that $R_K$ is isomorphic to a power of $(A_f)_K$, hence $R_K$ is isogenous over $\overline{\mathbb{Q}}$ to a power of $B$. Thus $R$ and $P$ are both isogenous over $\overline{\mathbb{Q}}$ to a power of $B$, so $R$ is isogenous to $P$ over $\overline{\mathbb{Q}}$, since they have the same dimension, as their Tate modules are isomorphic. Let $L$ be a Galois number field over which such an isogeny is defined. Consider the natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant inclusion

$$\mathrm{Hom}(R_{\mathbb{Q}}, P_{\mathbb{Q}}) \otimes \mathbb{Q}_p \hookrightarrow \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(V_p(R), V_p(P)). \tag{2}$$

By Ribet's proof of the Tate conjecture for modular abelian varieties [Rib80], the inclusion

$$\mathrm{Hom}(R_L, P_L) \otimes \mathbb{Q}_p \hookrightarrow \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/L)}(V_p(R), V_p(P)) \tag{3}$$

is an isomorphism, since there is an isogeny $P_L \to R_L$ and $P$ is modular. But then (2) must also be an isomorphism, since (2) is the result of taking $\mathrm{Gal}(L/\mathbb{Q})$-invariants of both sides of (3).

By construction of $P$, there is an isomorphism $V_p(R) \cong V_p(P)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules, so by (2) there is an isomorphism in $\mathrm{Hom}(R_{\mathbb{Q}}, P_{\mathbb{Q}}) \otimes \mathbb{Q}_p$. Thus there is a $\mathbb{Q}_p$-linear combination of elements of $\mathrm{Hom}(R_{\mathbb{Q}}, P_{\mathbb{Q}})$ that has nonzero determinant. However, if a $\mathbb{Q}_p$-linear combination of matrices has nonzero determinant, then some $\mathbb{Q}$-linear combination does, since the determinant is a polynomial function of the coefficients and $\mathbb{Q}$ is dense in $\mathbb{Q}_p$. Thus there is an isogeny $R \to P$ defined over $\mathbb{Q}$, so $R$ is modular. □

Corollary 3.0.4. *Parts 1 and 2 of Conjecture 1.0.2 are true for every elliptic curve $E$ over $\mathbb{Q}$.*

*Proof.* Suppose $p$ is a prime, and let $\mathbb{Q}_{p^\infty}$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$. By [BCDT01], $E$ is a modular elliptic curve, so work of Rohrlich [Roh84], Kato [Kat04, Sch98], and Serre [Ser72] implies that $E(\mathbb{Q}_{p^\infty})$ is finitely generated (see [Rub98, Cor. 8.2]). Theorem 3.0.2 implies that if $x \in E(\mathbb{Q})$ and $m \mid \mathrm{order}(x)$, then $x$ is $m$-visible relative to an optimal cover of $E$ by a restriction of scalars $B$ from an abelian extension. Then Proposition 3.0.3 implies that $B$ is modular. □

## 4 The Visibility Dimension

The visibility dimension is analogous to the visibility dimension for elements of $\mathrm{H}^1(K, A)$ introduced in [AS02, §2]. We prove below that elements of order 2 in Mordell-Weil groups of elliptic curves over $\mathbb{Q}$ are 2-visible relative to an abelian surface. Along the way, we make a general conjecture about stability of rank and show that it implies a general bound on the visibility dimension.

Definition 4.0.5 (Visibility Dimension). Let $A$ be an abelian variety over a number field $K$ and suppose $m$ is an integer. Then $A$ has *$m$-visibility dimension $n$* if there is an optimal cover $B \to A$ with $n = \dim(B)$ and the image of $B(K)$ in $A(K)$ is contained in $mA(K)$, so $A(K)/mA(K)$ is a quotient of $\mathrm{Vis}^B(A(K))$.

The following rank-stability conjecture is motivated by its usefulness for proving a result about $m$-visibility.

Conjecture 4.0.6. *Suppose $A$ is an abelian variety over a number field $K$, that $L$ is a finite extension of $K$, and $m > 0$ is an integer. Then there is an extension $M$ of $K$ of degree $m$ such that* $\mathrm{rank}(A(K)) = \mathrm{rank}(A(M))$ *and* $M \cap L = K$.

The following proposition describes how Conjecture 4.0.6 can be used to find an extension where the index of $A(K)$ in $A(M)$ is coprime to $m$.

Proposition 4.0.7. *Let $A$ be an abelian variety over a number field $K$ and suppose $m$ is a positive integer. If Conjecture 4.0.6 is true for $A$ and $m$, then there is an extension $M$ of $K$ of degree $m$ such that $A(M)/A(K)$ is of order coprime to $m$.*

*Proof.* Choose a finite set $P_1, \ldots, P_n$ of generators for $A(K)$. Let

$$L = K\left(\frac{1}{m}P_1, \ldots, \frac{1}{m}P_n\right)$$

be the extension of $K$ generated by *all* $m$th roots of each $P_i$. Since the set of $m$th roots of a point is closed under the action of $\mathrm{Gal}(\overline{K}/K)$, the extension $L/K$ is Galois. Note also that the $m$ torsion of $A$ is defined over $L$, since the differences of conjugates of a given $\frac{1}{m}P_i$ are exactly the elements of $A[m]$. Let $S$ be the set of primes of $K$ that ramify in $L$.

By our hypothesis that Conjecture 4.0.6 is true for $A$ and $m$, there is an extension $M$ of $K$ of degree $m$ such that

$$\mathrm{rank}(A(K)) = \mathrm{rank}(A(M))$$

and $M \cap L = K$. In particular, $C = A(M)/A(K)$ is a finite group. Suppose, for the sake of contradiction, that $\gcd(m, \#C) \neq 1$, so there is some prime divisor $p \mid m$ and an element $[Q] \in C$ of exact order $p$. Here $Q \in A(M)$ is such that $pQ \in A(K)$ but $Q \notin A(K)$. Because $P_1, \ldots, P_n$ generate $A(K)$ and $pQ \in A(K)$, there are integers $a_1, \ldots a_n$ such that

$$pQ = \sum_{i=1}^{n} a_i P_i.$$

Then for any fixed choice of the $\frac{1}{p}P_i$, we have

$$Q - \sum_{i=1}^{n} a_i \cdot \frac{1}{p}P_i \in A[p],$$

since

$$p\left(Q - \sum_{i=1}^{n} a_i \cdot \frac{1}{p}P_i\right) = pQ - \sum_{i=1}^{n} a_i \cdot P_i = 0.$$

Thus $Q \in A(L)$. But then since $L \cap M = K$, so we obtain a contradiction from

$$Q \in A(L) \cap A(M) = A(K).$$

□

With Proposition 4.0.7 in hand, we show that Conjecture 4.0.6 bounds the visibility dimension of Mordell-Weil groups. In particular, we see that Conjecture 4.0.6 implies that for any abelian variety $A$ over a number field $K$, and any $m$, there is an embedding $A(K)/mA(K) \hookrightarrow \mathrm{H}^1(K, C)$ coming from a $\delta$ map, where $C$ is an abelian variety over $K$ of rank 0.

Theorem 4.0.8. *Let $A$ be an abelian variety over a number field $K$ and suppose $m$ is a positive integer. If Conjecture 4.0.6 is true for $A$ and $m$, then there is an optimal covering $B \to A$ with $B$ of dimension $m$ such that*

$$\mathrm{Vis}^B(A(K)) \cong A(K)/mA(K).$$

*Proof.* By Proposition 4.0.7, there is an extension $M$ of $K$ of degree $m$ such that the quotient $A(M)/A(K)$ is finite of order coprime to $m$. Then, as in [Ste04], the restriction of scalars $B = \mathrm{Res}_{M/K}(A_M)$ is an optimal cover of $A$ and

$$\mathrm{Vis}^B(A(K)) \cong A(K)/\operatorname{Tr}(A(M)).$$

However, there is also an inclusion $A \hookrightarrow B$ from which one sees that

$$mA(K) \subset \operatorname{Tr}(A(M)),$$

so $\mathrm{Vis}^B(A(K))$ is an $m$-torsion group.

We have

$$[\operatorname{Tr}(A(M)) : \operatorname{Tr}(A(K))] \;\Big|\; [A(M) : A(K)].$$

We showed above that $\gcd([A(M) : A(K)], m) = 1$, so since

$$\operatorname{Tr}(A(M))/\operatorname{Tr}(A(K))$$

is killed by $m$, it follows that $\operatorname{Tr}(A(M)) = \operatorname{Tr}(A(K))$. We conclude that

$$\mathrm{Vis}^B(A(K)) = A(K)/mA(K).$$

□

Proposition 4.0.9. *If $E$ is an elliptic curve over $\mathbb{Q}$ and $m = 2$, then Conjecture 4.0.6 is true for $E$ and $m$.*

*Proof.* Let $L$ be as in Conjecture 4.0.6, so $L$ is an extension of $\mathbb{Q}$ of possibly large degree. Let $D$ be the discriminant of $L$. By [MM97, BFH90] there are infinitely many quadratic imaginary extensions $M$ of $\mathbb{Q}$ such that $L(E^M, 1) \neq 0$, where $E^M$ is the quadratic twist of $E$ by $M$. By [Kol91, Kol88] all these curves have rank 0. Since there are only finitely many quadratic fields ramified only at the primes that divide $D$, there must be some field $M$ that is ramified at a prime $p \nmid D$. If $M$ is contained in $L$, then all the primes that ramify in $M$ divide $D$, so $M$ is not contained in $L$. Since $M$ is quadratic, it follows that $M \cap L = \mathbb{Q}$, as required. Since the image of $E(\mathbb{Q}) + E^M(\mathbb{Q})$ in $E(M)$ has finite index, it follows that $E(M)/E(\mathbb{Q})$ is finite. □

Corollary 4.0.10. *If $E$ is an elliptic curve over $\mathbb{Q}$, then there is an optimal cover $B \to E$, with $B$ a 2-dimension modular abelian variety, such that*

$$\mathrm{Vis}^B(E(\mathbb{Q})) \cong E(\mathbb{Q})/2E(\mathbb{Q}).$$

*Proof.* Combine Proposition 4.0.9 with Theorem 4.0.8. Also $B$ is modular since it is isogenous to $E \times E'$, where $E'$ is a quadratic twist of $E$. □

Note that the $B$ of Corollary 4.0.10 is isomorphic to $(E \times E^D)/\Phi$, where $E^D$ is a rank 0 quadratic imaginary twist of $E$ and $\Phi \cong E[2]$ is embedded antidiagonally in $E \times E^D$. Note that $E^D$ also has analytic rank 0, since it was constructed using the theorems of [Kol91, Kol88] and [MM97, BFH90]. Thus our construction is compatible with the one of Proposition 5.1.1 below.

## 5 Some Data About Visibility and Modularity

This section contains a computational investigation of modularity of Mordell-Weil groups of elliptic curves relative to abelian varieties that are quotients of $J_0(N)$. One reason that we restrict to $J_0(N)$ is so that computations are more tractable. Also, for $m > 2$, the twisting constructions that we have given in previous sections are no longer allowed since they take place in $J_1(N)$. Furthermore, the work of [KL89] suggests that we understand the arithmetic of $J_0(N)$ better than that of $J_1(N)$.

### 5.1 A Visibility Construction for Mordell-Weil Groups

The following proposition is an analogue of [AS02, Thm. 3.1] but for visibility of Mordell-Weil groups (compare also [CM00, pg. 19]).

Proposition 5.1.1. *Let $E$ be an elliptic curve over a number field $K$, and let $\Phi = E[m]$ as a $\mathrm{Gal}(\overline{K}/K)$-module. Suppose $A$ is an abelian variety over $K$ such that $\Phi \subset A$, as $G_{\mathbb{Q}}$-modules. Let $B = (A \times E)/\Phi$, where $\Phi$ is embedded anti-diagonally. Then there is an exact sequence*

$$0 \to B(K)/(A(K) + E(K)) \to E(K)/mE(K) \to \mathrm{Vis}^B(E(K)) \to 0.$$

*Moreover, if $(A/E[m])(K)$ is finite of order coprime to $m$, then the first term of the sequence is $0$, so*

$$\mathrm{Vis}^B(E(K)) \cong E(K)/mE(K).$$

*Proof.* Using the definition of $B$ and multiplication by $m$ on $E$, we obtain the following commutative diagram, whose rows and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & E[m] \arrow[r] \arrow[d] & E \arrow[r, "m"] \arrow[d] & E \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/E[m] \arrow[r, "\cong"] \arrow[d] & B/E \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Taking $K$-rational points we arrive at the following diagram with exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & E(K)/E(K)[m] \arrow[r, "m"] \arrow[d] & E(K) \arrow[r] \arrow[d, "\cong"] & E(K)/mE(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B(K)/A(K) \arrow[r] \arrow[d] & E(K) \arrow[r] \arrow[d] & \mathrm{Vis}^B(E(K)) \arrow[r] & 0 \\
& B(K)/(A(K)+E(K)) & 0 & &
\end{tikzcd}$$

The snake lemma and the fact that the middle vertical map is an isomorphism implies that the right vertical map is a surjection with kernel isomorphic to $B(K)/(A(K)+E(K))$. Thus we obtain an exact sequence

$$0 \to B(K)/(A(K)+E(K)) \to E(K)/mE(K) \to \mathrm{Vis}^B(E(K)) \to 0.$$

This proves the first statement of the proposition. For the second, note that we have an exact sequence $0 \to E \to B \to A/E[m] \to 0$. Taking Galois cohomology yields an exact sequence

$$0 \to E(K) \to B(K) \to (A/E[m])(K) \to \cdots,$$

so $\#(B(K)/E(K)) \mid \#(A/E[m])(K)$. If $(A/E[m])(K)$ is finite of order coprime to $m$, then $B(K)/(A(K)+E(K))$ has order dividing $\#(A/E[m])(K)$, so the quotient $B(K)/(A(K)+E(K))$ is trivial, since it injects into $E(K)/mE(K)$. □

## 5.2 Tables

The data in this section suggests the following conjecture.

Conjecture 5.2.1. *Suppose $E$ is an elliptic curve over $\mathbb{Q}$ and $p$ is a prime such that $E[p]$ is irreducible. Then there exists infinitely many newforms $g \in S_2(\Gamma_0(N))$, for various integers $N$, such that $L(g,1) \neq 0$ and $E[p] \subset A_g$ and $\mathrm{Vis}^B(E(\mathbb{Q})) = E(\mathbb{Q})/pE(\mathbb{Q})$, where $B = (A_g \times E)/E[p]$.*

Let $E$ be the elliptic curve $y^2 + y = x^3 - x$. This curve has conductor 37 and Mordell-Weil group free of rank 1. According to [Cre97], $E$ is isolated in its isogeny class, so each $E[p]$ is irreducible.

Table 1 gives for each $N$ the *odd* primes $p$ such that there is a mod $p$ congruence between $f_E$ and some newform $g$ in $S_2(\Gamma_0(37N))$ such that $A_g$ has rank 0 and the isogeny class of $A_g$ contains no abelian variety with rational $p$ torsion. The first time a $p$ occurs, it is in bold. We bound the torsion in the isogeny class using the algorithm from [AS05, §3.5] with primes up to 17. Thus by Proposition 5.1.1, the Mordell-Weil group of $E$ is $p$-modular of level $37N$. A $-$ means there are no such $p$. Table 2, which was derived directly from Table 1, gives for a prime $p$, all integers $N$ such that $E(\mathbb{Q})$ is $p$-modular of level $37N$.

Table 1: Visibility of Mordell-Weil for $y^2 + y = x^3 - x$

| $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | **5** | 19 | 5 | 36 | – | 53 | **53** | 70 | – | 87 | – | 104 | – |
| 3 | **7** | 20 | – | 37 | – | 54 | – | 71 | $3,7$ | 88 | – | 105 | – |
| 4 | – | 21 | 7 | 38 | 5 | 55 | – | 72 | – | 89 | **43** | 106 | 5 |
| 5 | – | 22 | – | 39 | – | 56 | – | 73 | $3,5$ | 90 | – | 107 | $3,5$ |
| 6 | – | 23 | **11** | 40 | – | 57 | – | 74 | – | 91 | 3 | 108 | – |
| 7 | **3** | 24 | – | 41 | $3,17$ | 58 | – | 75 | – | 92 | – | 109 | $3,7$ |
| 8 | – | 25 | – | 42 | – | 59 | **13** | 76 | – | 93 | 7 | 110 | – |
| 9 | – | 26 | – | 43 | 7 | 60 | – | 77 | – | 94 | – | 111 | – |
| 10 | – | 27 | 3 | 44 | – | 61 | $5,7$ | 78 | – | 95 | – | 112 | – |
| 11 | **17** | 28 | – | 45 | – | 62 | – | 79 | – | 96 | – | 113 | $3,11$ |
| 12 | – | 29 | 3 | 46 | – | 63 | 3 | 80 | – | 97 | **47** | 114 | – |
| 13 | – | 30 | – | 47 | 3 | 64 | – | 81 | 3 | 98 | – | 115 | – |
| 14 | – | 31 | 3 | 48 | – | 65 | – | 82 | – | 99 | – | 116 | – |
| 15 | – | 32 | – | 49 | – | 66 | – | 83 | $3,11$ | 100 | – | 117 | – |
| 16 | – | 33 | 7 | 50 | 5 | 67 | $3,5$ | 84 | – | 101 | $3,11$ | 118 | – |
| 17 | 3 | 34 | – | 51 | – | 68 | – | 85 | – | 102 | – | 119 | 3 |
| 18 | – | 35 | – | 52 | – | 69 | – | 86 | – | 103 | 43 | 120 | – |

| $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 121 | – | 138 | – | 155 | – | 172 | – | 189 | 3 | 206 | – |
| 122 | – | 139 | 17 | 156 | – | 173 | $3,5,11$ | 190 | – | 207 | – |
| 123 | – | 140 | – | 157 | $3,5$ | 174 | – | 191 | 7 | 208 | – |
| 124 | – | 141 | – | 158 | – | 175 | – | 192 | – | 209 | – |
| 125 | 5 | 142 | – | 159 | – | 176 | – | 193 | $5,11$ | | |
| 126 | – | 143 | – | 160 | – | 177 | – | 194 | – | | |
| 127 | **127** | 144 | – | 161 | – | 178 | – | 195 | – | | |
| 128 | – | 145 | – | 162 | – | 179 | 3 | 196 | – | | |
| 129 | – | 146 | – | 163 | $7,13$ | 180 | – | 197 | $3,5,13$ | | |
| 130 | – | 147 | 7 | 164 | – | 181 | 3, **59** | 198 | – | | |
| 131 | 3 | 148 | – | 165 | – | 182 | – | 199 | $3,11$ | | |
| 132 | – | 149 | 5, **31** | 166 | – | 183 | – | 200 | – | | |
| 133 | – | 150 | – | 167 | $3,5$ | 184 | – | 201 | – | | |
| 134 | – | 151 | 17 | 168 | – | 185 | – | 202 | 5 | | |
| 135 | – | 152 | – | 169 | – | 186 | – | 203 | 3 | | |
| 136 | – | 153 | 3 | 170 | – | 187 | – | 204 | – | | |
| 137 | 3 | 154 | – | 171 | – | 188 | – | 205 | – | | |

Table 2: Levels Where Mordell-Weil is $p$-Visible for $y^2 + y = x^3 - x$

| $p$ | $N$ such that $37N$ is a level of $p$-modularity of $E(\mathbb{Q})$ |
|---|---|
| 3 | 7, 17, 27, 29, 31, 41, 47, 63, 67, 71, 73, 81, 83, 91, 101, 107, 109, 113, 119, 131, 137, 153, 157, 167, 173, 179, 181, 189, 197, 199, 203 |
| 5 | 2, 19, 38, 50, 61, 67, 73, 106, 107, 125, 149, 157, 167, 173, 193, 197, 202 |
| 7 | 3, 21, 33, 43, 61, 71, 93, 109, 147, 163, 191 |
| 11 | 23, 83, 101, 113, 173, 193, 199 |
| 13 | 59, 163, 197 |
| 17 | 11, 41, 139, 151 |
| $19 - 29$ | - |
| 31 | 149 |
| $37 - 41$ | - |
| 43 | 89, 103 |
| 47 | 97 |
| 53 | 53 |
| 59 | 181 |
| $61 - 113$ | - |
| 127 | 127 |

Ribet's level raising theorem [Rib90] gives necessary and sufficient conditions on a prime $N$ for there to be a newform $g$ of level $37N$ that is congruent to $f_E$ modulo $p$. Note that the form $g$ is new rather than just $p$-new since 37 is prime and there are no modular forms of level 1 and weight 2. If, moreover, we impose the condition $L(g, 1) \neq 0$, then Ribet's condition requires that $p$ divides $N + 1 + \varepsilon a_N$, where $\varepsilon$ is the root number of $E$. Since $E$ has odd analytic rank, in this case $\varepsilon = -1$. For each primes $p \leq 127$ and each $N \leq 203$, we find the levels of such $g$. If $f$ is a newform, the *torsion multiple* of $f$ is a positive integer that is a multiple of the order of the rational torsion subgroup of any abelian variety attached to $f$, as computed by the algorithm in [AS05]. The *only* cases in which we don't already find a congruence level already listed in Table 2 corresponding to a newform with torsion multiple coprime to $p$ are

$$p = 3, \quad N = 43 \qquad \text{and} \qquad p = 19, \quad N = 47, 79.$$

In all other cases in which Ribet's theorem produces a congruent $g$ with $\mathrm{ord}_{s=1} L(g, s)$ even (hence possibly 0), we actually find a $g$ with $L(g, 1) \neq 0$ and can show that $\#A_g(\mathbb{Q})_{\mathrm{tor}}$ is coprime to $p$.

For $p = 3$ and $N = 43$ we find a unique newform $g \in S_2(\Gamma_0(1591))$ that is congruent to $f_E$ modulo 3. This form is attached to the elliptic curve $y^2 + y = x^3 - 71x + 552$ of conductor 1591, which has Mordell-Weil groups $\mathbb{Z} \oplus \mathbb{Z}$. Thus this is an example of a congruence relating a rank 1 curve to a rank 2 curve. For $p = 19$ and $N = 47$, the newform $g$ has degree 43, so $A_g$ has dimension 43, we have $L(g, 1) \neq 0$, but the torsion multiple is $76 = 19 \cdot 4$, which is divisible by 19. For $p = 19$ and $N = 79$, the $A_g$ has dimension 57, we have $L(g, 1) \neq 0$, but the torsion multiple is 76 again.

Tables 3–4 are the analogues of Tables 1–2 but for the elliptic curve $y^2 + y = x^3 + x^2$ of conductor 43. This elliptic curve also has rank 1 and all mod $p$ representations are irreducible. The primes $p$ and $N$ such that Ribet's theorem produces a congruent $g$ with $\mathrm{ord}_{s=1} L(g, s)$ even, yet we do not find one with $L(g, 1) \neq 0$ and the torsion multiple coprime to $p$ are

$$p = 3, \quad N = 31, 61 \qquad \text{and} \qquad p = 11, \quad N = 19, 31, 47, 79.$$

The situation for $p = 11$ is interesting since in this case all the $g$ with $\mathrm{ord}_{s=1} L(g, s)$ even fail to satisfy our hypothesis. At level $19 \cdot 43$ we find that $g$ has degree 18 and $L(g, 1) \neq 0$, but the torsion multiple is divisible by 11.

Let $E$ be the elliptic curve $y^2 + y = x^3 + x^2 - 2x$ of conductor 389. This curve has Mordell-Weil group free of rank 2. Tables 5–6 are the analogues of Tables 1–2 but for $E$. The primes $p$ and $N$ such that Ribet's theorem produces a congruent $g$ with $\mathrm{ord}_{s=1} L(g, s)$ even, yet we do not find one with $L(g, 1) \neq 0$ and the torsion multiple coprime to $p$ are

$$p = 3, \quad N = 17 \qquad \text{and} \qquad p = 5, \quad N = 19.$$

For $p = 3$, there is a unique $g$ of level $6613 = 37 \cdot 17$ with $\mathrm{ord}_{s=1} L(g, s)$ even and $E[3] \subset A_g$. This form has degree 5 and $L(g, 1) = 0$, so this is another

Table 3: Visibility of Mordell-Weil for $y^2 + y = x^3 + x^2$

| $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | **5** | 17 | 3, **7** | 32 | – | 47 | – | 62 | – | 77 | – | 92 | – |
| 3 | **3** | 18 | – | 33 | 3 | 48 | – | 63 | – | 78 | – | 93 | – |
| 4 | – | 19 | – | 34 | 5 | 49 | – | 64 | – | 79 | – | 94 | – |
| 5 | 5 | 20 | – | 35 | – | 50 | 5 | 65 | – | 80 | – | 95 | – |
| 6 | – | 21 | – | 36 | – | 51 | 3 | 66 | – | 81 | 3 | 96 | – |
| 7 | – | 22 | 5 | 37 | 19 | 52 | – | 67 | **71** | 82 | – | 97 | 7, **13** |
| 8 | – | 23 | 5 | 38 | – | 53 | **59** | 68 | – | 83 | 3, **23** | 98 | – |
| 9 | – | 24 | – | 39 | 3 | 54 | – | 69 | – | 84 | – | 99 | 3 |
| 10 | – | 25 | – | 40 | – | 55 | 5 | 70 | – | 85 | 5 | 100 | – |
| 11 | 3 | 26 | – | 41 | **37** | 56 | – | 71 | 5, 7 | 86 | – | | |
| 12 | – | 27 | 3 | 42 | – | 57 | 3 | 72 | – | 87 | 3 | | |
| 13 | **19** | 28 | – | 43 | – | 58 | – | 73 | 3 | 88 | – | | |
| 14 | – | 29 | 3 | 44 | – | 59 | 3 | 74 | – | 89 | **47** | | |
| 15 | – | 30 | – | 45 | – | 60 | – | 75 | – | 90 | – | | |
| 16 | – | 31 | – | 46 | – | 61 | 5 | 76 | – | 91 | – | | |

Table 4: Levels Where Mordell-Weil is $p$-Visible for $y^2 + y = x^3 + x^2$

| $p$ | $N$ such that $43N$ is a level of $p$-modularity of $E(\mathbb{Q})$ |
|---|---|
| 3 | 3, 11, 17, 27, 29, 33, 39, 51, 57, 59, 73, 81, 83, 87, 99 |
| 5 | 2, 5, 22, 23, 34, 50, 55, 61, 71, 85 |
| 7 | 17, 71, 97 |
| 11 | - |
| 13 | 97 |
| 17 | - |
| 19 | 13, 37 |
| 23 | 83 |
| $29, 31$ | - |
| 37 | 41 |
| $41, 43$ | - |
| 47 | 89 |
| 53 | - |
| 59 | 53 |
| $61, 67$ | - |
| 71 | 67 |

Table 5: Visibility of Mordell-Weil for $y^2 + y = x^3 + x^2 - 2x$

| $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ | $N$ | $p's$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **5** | 7 | 3 | 13 | **11** | 19 | – | 25 | – |
| 2 | – | 8 | – | 14 | – | 20 | – | 26 | – |
| 3 | – | 9 | 3 | 15 | 3 | 21 | – | 27 | 3 |
| 4 | – | 10 | – | 16 | – | 22 | – | 28 | – |
| 5 | **3** | 11 | – | 17 | – | 23 | 5 | 29 | 3 |
| 6 | – | 12 | – | 18 | – | 24 | – | | |

Table 6: Levels Where Mordell-Weil is $p$-Visible for $y^2 + y = x^3 + x^2 - 2x$

| $p$ | $N$ such that $389N$ is a level of $p$-modularity of $E(\mathbb{Q})$ |
|---|---|
| 3 | 5, 7, 9, 15, 27, 29 |
| 5 | 1, 23 |
| 7 | - |
| 11 | 13 |

example where the rank 0 hypothesis of Proposition 5.1.1 is not satisfied. Note that the torsion multiple in this case is 1. For $p = 5$, there is a unique $g$ of level $7391 = 37 \cdot 19$, with $\mathrm{ord}_{s=1} L(g, s)$ even and $E[5] \subset A_g$. This form has degree 4 and $L(g, 1) \neq 0$, but the torsion multiple is divisible by 5.

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# Réalisation $\ell$-Adique des Motifs Triangulés Géométriques I

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Abstract. In this work, we provide an integral $\ell$-adic realization functor for Voevodsky's triangulated category of geometrical motives over a noetherian separated scheme. Our approach to the realization problem is to study finite correspondences from the Nisnevich and étale local point of view. We set the existence of a local decomposition for finite correspondences which implies the existence of local transfers. This result allows us to provide canonical transfers on the Godement resolution of a Nisnevich sheaf with transfers and then to carry out the construction of the $\ell$-adic realization functor. We also give a moderate $\ell$-adic realization functor in some geometrical situations.

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## Introduction

Nous nous intéressons dans cet article à la réalisation des motifs triangulés géométriques de V. Voevodsky sur un schéma noethérien séparé $S$ dans la catégorie des coefficients $\ell$-adiques de T. Ekedahl [11] [1]. M. Levine a construit dans [25] une catégorie triangulée de motifs munie de foncteurs de réalisation très généraux. Ce travail fournit indirectement des foncteurs de réalisations pour les motifs triangulés de Voevodsky dans les cas où les catégories de [29] et [25] sont équivalentes à savoir essentiellement le cas d'un corps parfait [25, 22]. Dans [18, 19] A. Huber fait directement le lien pour un corps de caractéristique

[1] Nous renvoyons à l'appendice A pour un rappel sur la construction de cette catégorie et pour de plus amples précisions sur les notations utilisées dans le cadre $\ell$-adique.

nulle entre les motifs triangulés de Voevodsky et la catégorie des réalisations mixtes de [16] en construisant un foncteur de réalisation mixte.

Nous commençons ce travail en remarquant que la construction de la catégorie triangulée des motifs mixtes géométriques de [29] se généralise au schéma de base $S$, ce qui nous permet d'obtenir une catégorie triangulée tensorielle que nous notons $DM_{gm}(S)$. L'approche faisceautique sur une base régulière fait l'objet d'un travail en cours de D-C. Cisinski et F. Déglise commencé dans [6]. En convenant de désigner par $\mathrm{D}^+(S,\mathbb{Z}_\ell)$ la catégorie des coefficients $\ell$-adiques définie par T. Ekedahl [11], notre résultat principal — le théorème 4.3 — s'énonce alors comme suit :

THÉORÈME. *Le foncteur de réalisation $\ell$-adique des $S$-schémas lisses de type fini*

$$\begin{array}{rcl} R_\ell : \mathrm{Sm}_S^{\mathrm{op}} & \to & \mathrm{D}^+(S,\mathbb{Z}_\ell) \\ X & \mapsto & R\pi_{X*}\pi_X^*\mathbb{Z}_S/\ell^*. \end{array}$$

*se prolonge canoniquement en un foncteur triangulé quasi-tensoriel*[2]

$$DM_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^+(S,\mathbb{Z}_\ell). \tag{1}$$

*(a) Lorsque $S$ est de type fini sur un schéma noethérien régulier de dimension $\leq 1$, le foncteur (1) prend ses valeurs dans $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S,\mathbb{Z}_\ell)$ sous-catégorie triangulée pleine formée des coefficients constructibles.*

*(b) Lorsque $S$ est de type fini sur un corps fini, le foncteur (1) induit un foncteur triangulé tensoriel*

$$DM_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(S,\overline{\mathbb{Q}}_\ell)$$

*où le second membre désigne la catégorie des coefficients $\ell$-adiques mixtes de P. Deligne [8, 5].*

L'approche des réalisations que nous adoptons repose essentiellement sur une étude locale pour les topologies de Nisnevich et étale des correspondances finies. Le résultat fondamental à ce sujet — la proposition 2.1 — consiste en un raffinement de la proposition 3.1.3 de [29] et assure l'existence d'une homotopie canonique pour certains complexes de Čech associés à la décomposition locale d'un schéma pour la topologie de Nisnevich. Dans l'article [9] P. Deligne et A. Goncharov ont utilisé une approche similaire pour construire certains foncteurs de réalisation. Nous renvoyons à [21] pour le lien entre le travail de A. Huber [18, 19] et le présent article.

Dans certaines situations géométriques nous raffinons également la construction précédente en un foncteur de réalisation $\ell$-adique modérée — corollaire 4.17.

Les résultats contenus dans cet article ainsi que dans [21] proviennent de la thèse de doctorat de l'auteur [20].

[2]Nous renvoyons à la définition B.1 pour ce qui concerne la terminologie utilisée dans cet article.

Conventions

Tout au long de ce travail, nous adoptons la convention suivante.

> *$S$ désigne un schéma noethérien séparé et $\ell$ un nombre premier inversible sur $S$. Tous les schémas considérés sont supposés noethériens et séparés.*

Nous sortirons parfois, de manière anodine, de ce cadre en considérant des réunions disjointes, non nécessairement finies, de schémas pris au sens de la convention précédente. Cela sera notamment le cas dans la section 2. Nous notons $\mathrm{Sch}_S$ (resp. $\mathrm{Var}_S$) la catégorie des $S$-schémas (resp. des $S$-schémas de type fini) et nous désignons par $\mathrm{Sm}_S$ la catégorie des $S$-schémas lisses de type fini. Le morphisme structural d'un $S$-schéma $X$ est noté $\pi_X$.

## 1 Motifs mixtes géométriques

Rappelons que sur un corps [3] Voevodsky définit $DM_{gm}^{\mathrm{eff}}$ à partir de la catégorie homotopique de la catégorie des complexes de variétés lisses munies des correspondances finies, modulo les relations :
– Mayer-Vietoris pour la topologie de Zariski
– invariance par homotopie.
La construction que nous donnons ici est exactement la même (en se fondant sur la théorie des correspondances finies sur $S$ de Suslin-Voevodsky), à une différence près : nous remplaçons les relations de Mayer-Vietoris pour la topologie de Zariski par leurs analogues en topologie de Nisnevich. Il y a donc a priori plus de relations, mais il résulte de la proposition 4.1.23 de [20] et du théorème 3.1.12 de [29] que les deux constructions coïncident sur un corps parfait.

Avant d'introduire les correspondances finies sur une base quelconque, nous rappelons pour la commodité du lecteur quelques constructions tirées de [28], référence à laquelle nous renvoyons pour un exposé complet.

### 1.1 Quelques rappels sur les cycles relatifs

Soient $K$ un corps, $\underline{s}$ un $K$-point de $S$. Un épaississement de $\underline{s}$ est la donnée d'un trait $\mathcal{O}$ et d'une factorisation de $\underline{s}$ sous la forme

$$\begin{tikzcd} \operatorname{Spec}\mathcal{O} \arrow[r, "\tau"] & S \\ \operatorname{Spec}K \arrow[u, "\sigma"] \arrow[ur, "\underline{s}"'] & \end{tikzcd}$$

[3] Dans le texte de référence [29], le schéma de base considéré est un corps et pour les besoins de la démonstration de la conjecture de Bloch-Kato, les constructions sont étendues à des schémas simpliciaux lisses sur un corps dans [31].

où $\sigma$ est le point fermé de $\mathcal{O}$ et $\tau$ un morphisme birationnel de $\mathcal{O}$ sur une composante irréductible de $S$ contenant le lieu $s$ de $\underline{s}$. Le point $\underline{s}$ est épais lorsqu'il admet un épaississement. Soit $Z$ un schéma équidimensionnel sur $S$ de dimension $n$ dominant la même composante irréductible que $\mathcal{O}$. Le schéma $\mathcal{O} \times_S Z$ possède une unique composante irréductible $\mathcal{Z}$ dominant $\mathcal{O}$ et cette dernière est plate et équidimensionnelle sur $\mathcal{O}$ de dimension $n$. Supposons donné un cycle $\alpha$ de la forme

$$\alpha = \sum_Z \alpha_Z[Z]$$

la somme étant prise sur les sous-schémas fermés intègres $Z$ de $X$ qui sont équidimensionnels sur $S$ de dimension $n$, les $\alpha_Z$ non nuls étant en nombre fini. On peut associer à $\alpha$ le cycle

$$(\mathcal{O},\tau,\sigma)^{\circledast}\alpha := \sum_Z \alpha_Z[\mathrm{Spec}(K) \times_{\mathcal{O}} \mathcal{Z}] \tag{2}$$

la somme étant restreinte aux sous-schémas fermés qui dominent la même composante irréductible de $S$ que $\mathcal{O}$. On prendra garde néanmoins que pour un $K$-point épais $\underline{s}$ de $S$, le choix de l'épaississement n'est pas unique et qu'en toute généralité rien n'assure que les cycles (2) pour des épaississements distincts soient égaux. Ceci explique l'introduction dans [28, Definition 3.1.3] du groupe abélien $PropCycl_{equi}(X/S,n)$ formé des cycles

$$\alpha = \sum_{i=1}^r \alpha_i[Z_i]$$

vérifiant les deux conditions suivantes :

– Les $Z_i$ sont des sous-schémas fermés intègres de $X$ équidimensionnels et propres sur $S$ de dimension $n$ [4].
– Pour tout $K$-point épais $\underline{s}$ de $S$, le cycle $(\mathcal{O},\tau,\sigma)^{\circledast}\alpha$ est indépendant de l'épaississement $(\mathcal{O},\tau,\sigma)$ choisi.

L'opération essentielle sur les cycles relatifs est l'opération de changement de base

$$\theta^{\circledast} : PropCycl_{equi}(X/S,n)_{\mathbb{Q}} \to PropCycl_{equi}(X_T/T,n)_{\mathbb{Q}}$$

pour un morphisme de schémas $\theta : T \to S$ qui est construite de sorte que l'on ait

$$\underline{s}^{\circledast}\alpha = (\mathcal{O},\tau,\sigma)^{\circledast}\alpha$$

pour un $K$-point épais $\underline{s}$ et un épaississement $(\mathcal{O},\tau,\sigma)$ de ce dernier. En général un cycle obtenu par changement de base n'est pas nécessairement à coefficients entiers, en revanche par construction les dénominateurs pouvant apparaître ne peuvent avoir comme facteurs premiers que les caractéristiques résiduelles de $S$ aux points images des points génériques des composantes irréductibles de $T$.

[4]Dans le cas $n = 0$ cela revient à dire que les $Z_i$ sont finis et dominants sur une composante irréductible de $S$.

Finalement on introduit le sous-groupe $c_{\mathrm{equi}}(X/S,n)$ de $PropCycl_{equi}(X/S,n)$ formé des cycles universellement entiers c'est à dire des cycles $\alpha$ tels que $\theta^{\circledast}\alpha$ soit à coefficients entiers pour tout morphisme de schéma $\theta : T \to S$.

Le changement de base permet alors de définir l'opération $Cor$. Soient $X$ un $S$-schéma et $Y$ un $X$-schéma. Supposons donnés un cycle $\alpha \in c_{\mathrm{equi}}(Y/X,n)$ et un cycle $\beta \in c_{\mathrm{equi}}(X/S,m)$ que l'on écrit

$$\beta = \sum_Z \beta_Z [Z]$$

la somme étant prise sur les sous-schémas fermés intègres de $X$ propres et équidimensionnels sur $S$ de dimension $m$. Pour un tel sous-schéma, on a le carré cartésien

$$\begin{tikzcd} Y \times_X Z \arrow[r, "(\iota_Z)_Y"] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\ Z \arrow[r, "\iota_Z"] & X \end{tikzcd}$$

et on peut considérer le cycle

$$Cor(\alpha, \beta) = \sum_Z \beta_Z (\iota_Z)_{Y*} \iota_Z^{\circledast} \alpha$$

qui appartient à $c_{\mathrm{equi}}(Y/S, n+m)$ d'après le corollaire 3.7.5 de [28]. Nous utilisons dans la suite le lemme suivant.

Lemme 1.1. *Soient $X$ un $S$-schéma et $p : Y \to Y'$ un morphisme de $X$-schémas. Pour tout cycle $\alpha \in c_{\mathrm{equi}}(Y/X,n)$ et $\beta \in c_{\mathrm{equi}}(X/S,m)$, on a*

$$Cor(p_*\alpha, \beta) = p_* Cor(\alpha, \beta)$$

*dans $c_{\mathrm{equi}}(Y'/S, n+m)$.*

*Démonstration.* Par linéarité on peut supposer $\beta = [Z]$ où $Z$ est un sous-schéma fermé intègre de $X$. Notons $\iota$ l'immersion fermée correspondante, en utilisant les notations du diagramme commutatif suivant

$$\begin{tikzcd} & Y' \times_X Z \arrow[rr, "\iota_{Y'}"] \arrow[dd, dotted] & & Y' \arrow[dd] \\ Y \times_X Z \arrow[ur, "p_Z"] \arrow[rr, "\iota_Y"] \arrow[dr] & & Y \arrow[ur, "p"] \arrow[dr] & \\ & Z \arrow[rr, "\iota"] & & X \end{tikzcd}$$

la proposition 3.6.2 de [28] assure que

$$\begin{aligned} Cor(p_*\alpha, \beta) &= \iota_{Y'*}\iota^{\circledast} p_*\alpha = \iota_{Y'*} p_{Z*} \iota^{\circledast}\alpha \\ &= p_* \iota_{Y*} \iota^{\circledast}\alpha = p_* Cor(\alpha, \beta). \end{aligned}$$

□

### 1.2 Correspondances finies

Hormis un survol dans [26, Appendix 1A], cette notion n'est disponible dans la littérature que dans le cas des corps. L'extension aux schémas de base quelconques [5] ne présente aucune difficulté et s'avère une application de la théorie générale des cycles relatifs de [28]. Nous énonçons sans démonstration certaines propriétés élémentaires des correspondances finies : on trouvera plus de détails dans la thèse de l'auteur [20, §2.1].

Définition 1.2. Soient $X$ et $Y$ deux $S$-schémas. Les $S$-correspondances finies de $X$ dans $Y$ sont les éléments du groupe abélien

$$c_S(X,Y) := c_{\mathrm{equi}}(X \times_S Y/X, 0).$$

La composition des correspondances finies est fournie par la relation

$$\beta \circ \alpha := p^{XYZ}_{XZ*} Cor(p^{XY\circledast}_X \beta, \alpha) \tag{3}$$

dans laquelle $\alpha \in c_S(X,Y)$ et $\beta \in c_S(Y,Z)$.

*Remarque* 1.3. Dans la formule (3) donnant la composition $p^{XYZ}_{XZ}$ désigne la projection de $X \times_S Y \times_S Z$ sur $X \times_S Z$ et $p^{XY}_X$ la projection de $X \times_S Y$ sur $X$. Nous utilisons ce type de notation pour les projections dans le reste de ce travail.

Dans la suite nous utilisons la notation suivante.

*Notation* 1.4. Nous désignons par $\Delta_p$ l'immersion fermée $X \hookrightarrow X \times_S Y$ associée à un morphisme de $S$-schémas $p : X \to Y$ et par $\Gamma_p$ le graphe de ce dernier. Celui-ci définit une correspondance finie de $X$ dans $Y$ que nous notons $[p]$.

Lemme 1.5. *Soient $X, Y, Z$ et $W$ des $S$-schémas.*

*(a) Étant données des correspondances finies $\alpha \in c_S(X,Y)$, $\beta \in c_S(Y,Z)$ et $\gamma \in c_S(Z,W)$ on a*
$$\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha.$$

*(b) Pour tout morphisme de $S$-schémas $p : X \to Y$ et toute correspondance finie $\alpha \in c_S(Y,Z)$ on a*
$$\alpha \circ [p] = p^{\circledast}\alpha.$$

*(c) Pour tout morphisme de $S$-schémas $p : Y \to Z$ et toute correspondance finie $\alpha \in c_S(X,Y)$ on a*
$$[p] \circ \alpha = (\mathrm{id}_X \times_S p)_* \alpha.$$

*(d) Pour tout morphisme de $S$-schémas $p : X \to Y$ et $q : Y \to Z$ on a*
$$[q] \circ [p] = [q \circ p].$$

[5] Le cas plus restreint des schémas lisses de type fini sur une base réguliere est traité dans [6] en utilisant les multiplicités d'intersection de Serre. Nous renvoyons à la proposition 1.8 pour le lien entre ces deux approches.

En particulier le lemme 1.5 assure qu'en prenant pour objets les $S$-schémas et pour morphismes les correspondances finies, on obtient une catégorie $\mathrm{SchCor}_S$ munie d'un foncteur pleinement fidèle

$$[-] : \mathrm{Sch}_S \to \mathrm{SchCor}_S. \tag{4}$$

qui à un morphisme de schémas associe son graphe. La définition suivante est donc licite :

Définition 1.6. Nous désignons par $\mathrm{SchCor}_S$ la catégorie des $S$-schémas munis des correspondances finies ayant pour objet les $S$-schémas et dont les morphismes sont donnés par

$$\mathrm{Hom}_{\mathrm{SchCor}_S}([X],[Y]) := c_S(X,Y)$$

pour des $S$-schémas $X$ et $Y$. La notation $\mathrm{VarCor}_S$ (resp. $\mathrm{SmCor}_S$) fait référence à la sous-categorie strictement pleine obtenue en restreignant les objets aux seuls schémas de type fini sur $S$ (resp. lisses de type fini sur $S$).

L'opération « correspondance » de [28] permet de définir un produit associatif et commutatif sur les cycles relatifs via la composition

$$\begin{tikzcd} \begin{matrix} c_{\mathrm{equi}}(X/S,0) \\ \otimes \\ c_{\mathrm{equi}}(Y/S,0) \end{matrix} \arrow[r, "\pi_{Y/S}^{\circledast}\otimes \mathrm{id}"] \arrow[rr, bend left, "\times_S"] & \begin{matrix} c_{\mathrm{equi}}(X\times_S Y/Y,0) \\ \otimes \\ c_{\mathrm{equi}}(Y/S,0) \end{matrix} \arrow[r, "Cor"] & c_{\mathrm{equi}}(X\times_S Y/S,0). \end{tikzcd}$$

Ce produit est compatible aux morphismes de changement de base. En posant pour $\alpha \in c_S(X,Y)$ et $\beta \in c_S(X',Y')$

$$\alpha\otimes\beta := (p_X^{XX'\circledast}\alpha)\times_{X\times_S X'}(p_{X'}^{XX'}\beta)$$

on définit sur la catégorie additive $\mathrm{SchCor}_S$ une structure tensorielle naturelle compatible via le foncteur (4) avec la structure monoïdale symétrique induite par le produit fibré sur $\mathrm{Sch}_S$.

*Remarque* 1.7. Étant donnés des $S$-schémas $X, Y$ et une correspondance finie $\alpha \in c_S(X,Y)$, le carré

$$\begin{tikzcd} X \arrow[r, "{[\Delta_X]}"] \arrow[d, "\alpha"'] & X\times_S X \arrow[d, "\alpha\otimes\alpha"] \\ Y \arrow[r, "{[\Delta_Y]}"] & Y\times_S Y \end{tikzcd}$$

n'est pas nécessairement commutatif lorsque $\alpha$ n'est pas une correspondance finie obtenue à partir d'un morphisme de $S$-schémas.

La proposition suivante montre que la composition définie par les relations (3) coïncident avec la composition des correspondances finies pour les schémas lisses de type fini sur une base régulière obtenue via la théorie de l'intersection et considérée dans [29, 6].

Proposition 1.8. *Supposons que $S$ soit régulier et que $X, Y, Z$ soient des $S$-schémas lisses de type fini. Alors pour toute correspondance finie $\alpha \in c_S(X,Y)$ et $\beta \in c_S(Y,Z)$*

*(a) les cycles $p_{XY}^{XYZ*}\alpha$ et $p_{YZ}^{XYZ*}\beta$ s'intersectent proprement,*

*(b) la composition des correspondances $\alpha$ et $\beta$ est donnée par*

$$\beta \circ \alpha = p_{XZ*}^{XYZ}\left(p_{XY}^{XYZ*}\alpha \frown p_{YZ}^{XYZ*}\beta\right)$$

*où $\frown$ désigne le produit d'intersection.*

Nous utiliserons dans la suite l'extension aux schémas munis des correspondances finies de l'opération classique de changement de base. Étant donnés un morphisme de schémas $\theta : T \to S$ et des $S$-schémas $X, Y$, nous pouvons considérer le morphisme de changement de base induit par le morphisme $\theta_X$

$$\theta_X^{\circledast} : c_{\mathrm{equi}}(X \times_S Y/X, 0) \to c_{\mathrm{equi}}((T \times_S X) \times_S (X \times_S Y)/T \times_S X, 0).$$

Sachant que l'on a les isomorphismes

$$(T \times_S X) \times_S (X \times_S Y) = T \times_S (X \times_S Y) = (T \times_S X) \times_T (T \times_S Y)$$

ce dernier nous donne en fait un morphisme de changement de base

$$T \times_S - : c_S(X,Y) \to c_T(T \times_S X, T \times_S Y).$$

Nous noterons souvent $\alpha_T$ la $T$-correspondance finie obtenue par changement de base à partir d'une $S$-correspondance finie $\alpha$. Les propriétés élémentaires du changement de base sont rassemblées dans le lemme ci-dessous.

Lemme 1.9. *Soient $\theta : T \to S$ un morphisme de schémas et $X, X', Y, Y', Z$ des $S$-schémas.*

*(a) Étant données $\alpha \in c_S(X,Y)$ et $\beta \in c_S(Y,Z)$, on a*

$$\beta_T \circ \alpha_T = (\beta \circ \alpha)_T.$$

*(b) Étant données $\alpha \in c_S(X,Y)$ et $\beta \in c_S(Y,Y')$, on a*

$$\alpha_T \otimes \beta_T = (\alpha \otimes \beta)_T.$$

*(c) Si $p : X \to Y$ est un morphisme de $S$-schémas alors $[p]_T = [p_T]$.*

Le lemme 1.9 assure que le morphisme de changement de base $T \times_S -$ est fonctoriel, tensoriel et qu'en outre le carré suivant est commutatif

$$\begin{tikzcd} \mathrm{Sch}_S \arrow[r, "T\times_S -"] \arrow[d, "{[-]}"'] & \mathrm{Sch}_T \arrow[d, "{[-]}"] \\ \mathrm{SchCor}_S \arrow[r, "T\times_S -"'] & \mathrm{SchCor}_T. \end{tikzcd}$$

Définition 1.10. Soit $\mathcal{S} = \mathrm{Sch}_S, \mathrm{Var}_S, \mathrm{Sm}_S$. Un préfaisceau avec transferts sur $\mathcal{S}$ est un préfaisceau additif de groupes abéliens sur la catégorie $\mathcal{S}\mathrm{Cor}_S$. Un faisceau Nisnevich (étale) avec transferts est un préfaisceau avec transferts dont la restriction à $\mathcal{S}$ est un faisceau Nisnevich (resp. étale). Nous notons $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathcal{S})$ (resp. $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Et}}(\mathcal{S})$) la catégorie des faisceaux Nisnevich (resp. étales) avec transferts sur $\mathcal{S}$.

### 1.3 Motifs mixtes géométriques

La topologie de Nisnevich intervient dans cette sous-section via la définition suivante.

Définition 1.11. On appelle carré distingué élémentaire pour la topologie de Nisnevich, un carré cartésien excisif de $\mathrm{Sm}_S$

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d, "p"] \\ U \arrow[r, "e"] & X \end{tikzcd} \tag{5}$$

dans lequel $e$ est une immersion ouverte et $p$ un morphisme étale.

Définition 1.12. La catégorie $\underline{DM}^{\mathrm{eff}}_{gm}(S)$ est la catégorie triangulée tensorielle

$$\underline{DM}^{\mathrm{eff}}_{gm}(S) = \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_S)/E_{gm}(S)$$

obtenue par passage au quotient par la sous-catégorie triangulée épaisse $E_{gm}(S)$ de $\mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_S)$ engendrée par les complexes :

– *Homotopie* :

$$[\mathbb{A}^1_X] \xrightarrow{[\pi]} [X] \tag{6}$$

où $X$ est un $S$-schéma lisse de type fini et $\pi$ la projection de $\mathbb{A}^1_X$ sur $X$.

– *Mayer-Vietoris pour la topologie de Nisnevich* :

$$[U \times_S X] \longrightarrow [U] \oplus [V] \xrightarrow{[e]\oplus(-[p])} [X] \tag{7}$$

pour tout carré distingué élémentaire pour la topologie de Nisnevich de la forme (5).

*Remarque* 1.13. Sachant que le complexe (6) est le cône dans $\mathrm{C}^{\mathrm{b}}(\mathrm{SmCor}_S)$ du morphisme $[\mathbb{A}^1_X] \to [X]$ induit par la projection et que le complexe (7) est le cône du morphisme canonique entre le cône du morphisme de $[U \times_X V]$ dans $[U] \oplus [V]$ et $[X]$, un foncteur triangulé

$$F : \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_S) \to \mathcal{T}$$

se prolonge à la catégorie $\underline{DM}^{\mathrm{eff}}_{gm}(S)$ si et seulement si les deux conditions suivantes sont vérifiées.

(a) Le morphisme $F(\mathbb{A}^1_X) \to F(X)$ induit par la projection est un isomorphisme.

(b) Le triangle de Mayer-Vietoris

$$F(U \times_X V) \to F(U) \oplus F(V) \to F(X) \to F(U \times_X V)[1]$$

est distingué.

DÉFINITION 1.14. La catégorie des motifs géométriques effectifs que nous noterons

$$DM^{\mathrm{eff}}_{gm}(S)$$

est l'enveloppe pseudo-abélienne de la catégorie triangulée $\underline{DM}^{\mathrm{eff}}_{gm}(S)$.

*Remarque* 1.15. La catégorie introduite à la définition 1.14 possède une structure naturelle de catégorie triangulée d'après le théorème 1.5 de [4].

Nous notons $M$ le foncteur canonique

$$M : \mathrm{Sm}_S \to DM^{\mathrm{eff}}_{gm}(S)$$

qui à un $S$-schéma lisse de type fini $X$ associe son motif géométrique $M(X)$ image de l'objet $[X]$ de $\mathrm{SmCor}_S$ dans la catégorie des motifs mixtes géométriques effectifs.

En posant $DM^{\mathrm{eff}}_{gm}(S)(n) := DM^{\mathrm{eff}}_{gm}(S)$, on dispose d'un 2-système inductif, indexé par l'ensemble ordonné $\mathbb{Z}$, de catégories triangulées tensorielles

$$n \mapsto DM^{\mathrm{eff}}_{gm}(S)(n)$$

avec pour foncteurs de transition si $m \geq n$

$$- \otimes \mathbb{Z}(m-n) : DM^{\mathrm{eff}}_{gm}(S)(n) \to DM^{\mathrm{eff}}_{gm}(S)(m)$$

Par définition les catégories des motifs mixtes non effectifs sont données par les 2-colimites de ces systèmes

$$DM_{gm}(S) := 2\text{-}\operatorname*{colim}_n DM^{\mathrm{eff}}_{gm}(S)(n).$$

Ces catégories $DM_{gm}(S)$ héritent d'une structure tensorielle. En effet il s'agit de voir que la permutation cyclique des facteurs du motif $\mathbb{Z}(1) \otimes \mathbb{Z}(1) \otimes \mathbb{Z}(1)$ est l'identité dans $DM^{\mathrm{eff}}_{gm}(S)$ et cette propriété résulte du lemme 3.13 de [24].

## 2 Localisation des correspondances finies

Dans cette section nous donnons deux résultats de décomposition locale des correspondances finies pour la topologie de Nisnevich. Nous utilisons de manière cruciale les propriétés des anneaux locaux henséliens lors de la construction de ces décompositions, en particulier ces dernières ne possèdent pas d'analogue en topologie de Zariski.

L'existence de ces décompositions locales est aussi valable lorsque l'on remplace la topologie de Nisnevich par la topologie étale. Les démonstrations sont identiques à condition de substituer les anneaux locaux strictement henséliens aux anneaux locaux henséliens et nous avons choisi pour simplifier la présentation de ne donner les détails que pour la topologie de Nisnevich. Les modifications mineures à effectuer lorsque l'on considère la topologie étale sont données dans la sous-section 2.4.

### 2.1 Schémas décomposés

Nous dirons qu'un $S$-schéma est décomposé pour la topologie de Nisnevich lorsqu'il est une réunion disjointe non nécessairement finie de $S$-schémas locaux henséliens. À un $S$-schéma $X$ on peut associer fonctoriellement un $S$-schéma décomposé pour la topologie de Nisnevich

$$X^{\mathfrak{h}} := \coprod_{x \in X} X_x^{\mathfrak{h}}$$

réunion disjointe sur les points de $X$ des schémas locaux henséliens $X_x^{\mathfrak{h}}$ spectre de l'anneau local hensélien $\mathcal{O}_{X,x}^h$ dont on notera le point fermé abusivement par $x$. Pour tout point $x$ de $X$ on dispose du morphisme canonique

$$\begin{tikzcd} X_x^{\mathfrak{h}} \arrow[r] \arrow[rr, bend left, "\mathfrak{l}_{X,x}^{\mathfrak{h}}"] & \operatorname{Spec}(\mathcal{O}_{X,x}) \arrow[r] & X \end{tikzcd}$$

nous donnant un morphisme de schémas $\mathfrak{l}_X^{\mathfrak{h}} : X^{\mathfrak{h}} \to X$.

La propriété universelle des hensélisés se traduit par le fait que la composition par $\mathfrak{l}_X^{\mathfrak{h}}$ induit un isomorphisme

$$\mathrm{Hom}_{\mathrm{Sch}_S}(D, X^{\mathfrak{h}}) = \mathrm{Hom}_{\mathrm{Sch}_S}(D, X) \tag{8}$$

pour tout $S$-schéma $D$ décomposé pour la topologie de Nisnevich. On remarquera que le morphisme

$$\mathfrak{l}_{X^{\mathfrak{h}}}^{\mathfrak{h}} : (X^{\mathfrak{h}})^{\mathfrak{h}} \to X^{\mathfrak{h}}$$

n'est pas un isomorphisme en général, mais qu'il admet cependant une section canonique $\mathfrak{s}_X^{\mathfrak{h}}$ identifiant $X^{\mathfrak{h}}$ à un sous-schéma fermé de $(X^{\mathfrak{h}})^{\mathfrak{h}}$ et provenant du fait que l'anneau local hensélien de $X_x^{\mathfrak{h}}$ en son point fermé est canoniquement isomorphe à $\mathcal{O}_{X,x}^h$ :

$$\begin{tikzcd} X^{\mathfrak{h}} \arrow[r, "\mathfrak{s}_X^{\mathfrak{h}}"] \arrow[rr, bend right, equal, "\mathrm{id}_{X^{\mathfrak{h}}}"'] & (X^{\mathfrak{h}})^{\mathfrak{h}} \arrow[r, "\mathfrak{l}_{X^{\mathfrak{h}}}^{\mathfrak{h}}"] & X^{\mathfrak{h}} \arrow[r, "\mathfrak{l}_X^{\mathfrak{h}}"] & X \end{tikzcd}$$

Autrement dit dans $(X^{\mathfrak{h}})^{\mathfrak{h}}$ apparaissent des facteurs supplémentaires correspondant aux points non fermés des $X_x^{\mathfrak{h}}$.

## 2.2 Une homotopie canonique

Étant donnés un $S$-schéma $X$ et un $X$-schéma $U$, nous notons $\check{C}_X(U)$ le schéma simplicial de Čech dont les $n$-simplexes sont donnés par le produit fibré sur $X$ de $n+1$-copies de $U$

$$\check{C}_X(U)_n = U_X^{n+1} = \underbrace{U \times_X \cdots \times_X U}_{n+1 \text{ termes}}$$

la $i$-ème dégénérescence $\delta_i^n$ étant le morphisme de projection sur chaque facteur sauf le $i$-ème et la $i$-ème face $\sigma_i^n$ le morphisme induit par l'immersion diagonale sur le $i$-ème facteur. Ce schéma simplicial nous définit un complexe de Čech augmenté dans la catégorie des faisceaux Nisnevich avec transferts [6]

$$\check{C}_{U/X} : \cdots \to \mathbb{Z}_{\mathrm{tr}}[\check{C}_X(U)_n] \overset{d_n}{\to} \mathbb{Z}_{\mathrm{tr}}[\check{C}_X(U)_{n-1}] \to \cdots \to \mathbb{Z}_{\mathrm{tr}}[X]. \tag{9}$$

dont la différentielle est donnée par la somme alternée des morphismes induits par les dégénérescences

$$d_n = \sum_{i=0}^{n} (-1)^i (\delta_i^n)_*.$$

Dans la suite nous nous intéressons dans un premier temps aux sections de (9) sur les schémas locaux henséliens puis nous raffinons les résultats obtenus en considérant cette fois les fibres pour la topologie de Nisnevich.

### 2.2.1 Cas des sections sur les schémas locaux henséliens

Étant donné un $S$-schéma $\mathcal{O}$, en prenant les sections sur $\mathcal{O}$ du complexe de Čech précédent on obtient un complexe de groupes abéliens

$$\check{C}_{U/X}(\mathcal{O}) : \cdots \to \mathbb{Z}_{\mathrm{tr}}\left[U_X^2\right](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}[U](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}[X](\mathcal{O}) \to 0 \to \cdots .$$

[6]Nous renvoyons à définition 1.10 pour la notion de faisceau Nisnevich avec transferts. $\mathbb{Z}_{\mathrm{tr}}[X]$ désigne selon les conventions usuelles le faisceau Nisnevich avec transferts représentable associé à $X$.

Le résultat essentiel assurant l'existence d'une décomposition locale canonique des correspondances finies consiste en un raffinement de la proposition 3.1.3 de [29]. Cette dernière assure que lorsque $U$ est un recouvrement Nisnevich de $X$ et $\mathcal{O}$ est un schéma local hensélien, le complexe $\check{C}_{U/X}(\mathcal{O})$ est exact. Lorsque l'on remplace le recouvrement Nisnevich $U$ par le schéma décomposé $X^{\mathfrak{h}}$ le complexe

$$\check{C}_{X^{\mathfrak{h}}/X}(\mathcal{O}) : \cdots \to \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^2_X\right](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}[X^{\mathfrak{h}}](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}[X](\mathcal{O}) \to 0 \to \cdots \tag{10}$$

est non seulement exact mais devient en fait canoniquement homotope à zéro. Plus précisément :

Proposition 2.1. *Soient $X$ un $S$-schéma et $\mathcal{O}$ un $S$-schéma local hensélien. Il existe des morphismes canoniques*

$$\sigma^{\mathfrak{h}}_{\mathcal{O},X,n} : \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right](\mathcal{O}) \qquad n \geq 0$$

*satisfaisant aux deux propriétés suivantes.*

*(a) (Homotopie) On a pour tout $n$ les relations*

$$d_{n+1} \circ \sigma^{\mathfrak{h}}_{\mathcal{O},X,n} + \sigma^{\mathfrak{h}}_{\mathcal{O},X,n-1} \circ d_n = \mathrm{id}. \tag{11}$$

*(b) (Fonctorialité) Étant donnés un $S$-schéma local hensélien $\mathcal{O}'$ et une correspondance finie $\alpha \in c_S(\mathcal{O}', \mathcal{O})$, on a un carré commutatif*

$$\begin{tikzcd}
\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right](\mathcal{O}) \arrow[r, "\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right](\alpha)"] \arrow[d, "\sigma^{\mathfrak{h}}_{\mathcal{O}',X,n}"] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right](\mathcal{O}') \arrow[d, "\sigma^{\mathfrak{h}}_{\mathcal{O},X,n}"] \\
\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right](\mathcal{O}) \arrow[r, "\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right](\alpha)"] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right](\mathcal{O}').
\end{tikzcd} \tag{12}$$

*Remarque* 2.2. En pratique pour les applications — [21, 22] et section 4 du présent article — seuls les cas $n = 0$ des propositions 2.1 et 2.7 nous seront utiles.

*Démonstration.* Étant donné un sous-schéma fermé $Z$ de $\mathcal{O} \times_S X$ fini et équidimensionnel sur $\mathcal{O}$, on note $W$ le sous-schéma fermé de $\mathcal{O} \times_S X^{\mathfrak{h}}$ défini par le carré cartésien

$$\begin{tikzcd}
W \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\
\mathcal{O} \times_S X^{\mathfrak{h}} \arrow[r] & \mathcal{O} \times_S X.
\end{tikzcd}$$

Sachant que dans le carré précédent les morphismes verticaux sont des immersions fermées et que

$$\begin{aligned}\mathcal{O}\times_S (X^{\mathfrak{h}})^n_X &= \mathcal{O}\times_S \left(\underbrace{X^{\mathfrak{h}}\times_X\cdots\times_X X^{\mathfrak{h}}}_{n+1 \text{ termes}}\right)\\ &= \underbrace{(\mathcal{O}\times_S X^{\mathfrak{h}})\times_{(\mathcal{O}\times_S X)}\cdots\times_{(\mathcal{O}\times_S X)}(\mathcal{O}\times_S X^{\mathfrak{h}})}_{n+1 \text{ termes}}\\ &= (\mathcal{O}\times_S X^{\mathfrak{h}})^n_{\mathcal{O}\times_S X}\end{aligned}$$

on voit que $W^n_Z$ est un sous-schéma fermé de $\mathcal{O}\times_S (X^{\mathfrak{h}})^n_X$. En particulier on a des sous-groupes

$$c_{\mathrm{equi}}(W^n_Z/\mathcal{O},0)\subset \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right](\mathcal{O})$$

définissant un sous-complexe de (10)

$$\cdots\to c_{\mathrm{equi}}(W^2_Z/\mathcal{O},0)\to c_{\mathrm{equi}}(W^1_Z/\mathcal{O},0)\to c_{\mathrm{equi}}(Z/\mathcal{O},0)\to 0\cdots. \tag{13}$$

Les complexes (13) sont fonctoriels pour l'inclusion des sous-schémas fermés $Z$ finis et équidimensionnels sur $\mathcal{O}$ et on voit que le complexe (10) est la colimite sur de tels sous-schémas fermés de ces complexes. En effet il s'agit de voir que pour tout $n$

$$\operatorname*{colim}_Z c_{\mathrm{equi}}(W^n_Z,\mathcal{O},0)=\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right](\mathcal{O}) \tag{14}$$

la colimite étant prise sur les sous-schémas fermés de $\mathcal{O}\times_S X$ équidimensionnels et finis sur $\mathcal{O}$. Soit $\mathcal{W}$ un sous-schéma fermé intègre de $\mathcal{O}\times_S (X^{\mathfrak{h}})^n_X$ équidimensionnel et fini sur $\mathcal{O}$. Comme les images de $\mathcal{W}$ par les morphismes

$$\mathcal{O}\times_S (X^{\mathfrak{h}})^n_X \xrightarrow[\substack{\text{la } i\text{-ème}\\ \text{projection}}]{\substack{\text{morphisme}\\ \text{induit par}}} \mathcal{O}\times_S X^{\mathfrak{h}} \xrightarrow{\mathcal{O}\times_S \mathfrak{l}^{\mathfrak{h}}_X} \mathcal{O}\times_S X$$

sont des sous-schémas fermés intègres de $\mathcal{O}\times_S X$ finis et équidimensionnels sur $\mathcal{O}$, il existe un sous-schéma fermé $Z$ fini et équidimensionnel sur $\mathcal{O}$ qui les contient toutes. Pour un tel $Z$ notre $\mathcal{W}$ est un sous-schéma fermé de $W^n_Z$ ce qui prouve la relation (14).

Nous allons construire pour chacun des complexes (13) une homotopie canonique. Comme $Z$ est fini sur le schéma local hensélien $\mathcal{O}$, il est lui même semi-local hensélien [7] donc décomposé pour la topologie de Nisnevich de la forme

$$Z=\coprod_{\substack{z \text{ point fermé}\\ \text{de } Z}}\operatorname{Spec}(\mathcal{O}_{Z,z}).$$

[7]Cette propriété des anneaux henséliens est cruciale, un raisonnement analogue ne peut donc s'appliquer à la topologie de Zariski.

En particulier la propriété universelle (8) nous assure l'existence d'un unique morphisme $\theta_Z$ factorisant la projection $\pi_Z$ de $Z$ sur $X$ sous la forme

$$\begin{tikzcd} Z \arrow[r, dotted, "\theta_Z"'] \arrow[rr, bend left, "\pi_Z"] & X^\natural \arrow[r, "\iota^\natural_X"'] & X. \end{tikzcd}$$

La propriété universelle des produits fibrés nous fournit alors une section canonique $\sigma^\natural_Z$ du morphisme $r$ via le diagramme

$$\begin{tikzcd} Z \arrow[r, dotted, "\sigma^\natural_Z"] \arrow[rr, bend left, "\mathrm{id}_Z"] \arrow[rdd, bend right, "\theta_Z"'] & W \arrow[r, "r"] \arrow[d] \arrow[rd, phantom, "\square"] & Z \arrow[d] \arrow[dd, bend left, "\pi_Z"] \\ & \mathcal{O} \times_S X^\natural \arrow[r] \arrow[d] \arrow[rd, phantom, "\square"] & \mathcal{O} \times_S X \arrow[d] \\ & X^\natural \arrow[r, "\iota^\natural_X"'] & X. \end{tikzcd} \tag{15}$$

On peut alors considérer les morphismes de schémas

$$\sigma^\natural_{Z,n} = \sigma^\natural_Z \times_Z \mathrm{id}_{W^n_Z} : W^n_Z \to W^{n+1}_Z.$$

Ces derniers vérifient les relations pour $i = 0, \ldots, n$

$$\delta^{n+1}_{i+1} \circ \sigma^\natural_{Z,n} = \sigma^\natural_{Z,n-1} \circ \delta^n_i \qquad \delta^{n+1}_0 \circ \sigma^\natural_{Z,n} = \mathrm{id}$$

ce qui assure que les morphismes induits sur les cycles équidimensionnels

$$\sigma^\natural_{\mathcal{O},X,Z,n} := (\sigma^\natural_{Z,n})_* : c_{\mathrm{equi}}(W^n_Z/\mathcal{O}, 0) \to c_{\mathrm{equi}}(W^{n+1}_Z/\mathcal{O}, 0)$$

définissent une homotopie entre l'identité du complexe (13) et le morphisme nul.

Lorsque $Z'$ est un sous-schéma fermé de $\mathcal{O} \times_S X$ fini et équidimensionnel sur $\mathcal{O}$ contenant $Z$, les factorisations précédemment obtenues sont compatibles

$$\begin{tikzcd} Z \arrow[dd] \arrow[rd, "\theta_Z"'] \arrow[rrd, bend left, "\pi_Z"] & & \\ & X^\natural \arrow[r, "\iota^\natural_X"] & X \\ Z' \arrow[ru, "\theta_{Z'}"] \arrow[rru, bend right, "\pi_{Z'}"'] & & \end{tikzcd}$$

et en particulier la construction de la section $\sigma_Z^\natural$ assure que le carré

$$\begin{tikzcd} Z \arrow[r, "\sigma_Z^\natural"] \arrow[d] & W \arrow[d] \\ Z' \arrow[r, "\sigma_{Z'}^\natural"] & W' \end{tikzcd}$$

est commutatif et donc que le diagramme obtenu au niveau des cycles algébriques

$$\begin{tikzcd} c_{\mathrm{equi}}(W_Z^n/\mathcal{O},0) \arrow[r, "\sigma^\natural_{\mathcal{O},X,Z,n}"] \arrow[d] & c_{\mathrm{equi}}(W_Z^{n+1}/\mathcal{O},0) \arrow[d] \arrow[dr] & \\ & & \mathbb{Z}_{\mathrm{tr}}\left[(X^\natural)^{n+1}_X\right](\mathcal{O}) \\ c_{\mathrm{equi}}\left((W')^n_{Z'}/\mathcal{O},0\right) \arrow[r, "\sigma^\natural_{\mathcal{O},X,Z',n}"] & c_{\mathrm{equi}}\left((W')^{n+1}_{Z'}/\mathcal{O},0\right) \arrow[ur] & \end{tikzcd}$$

l'est aussi. En passant à la colimite sur les sous-schémas fermés $Z$ on obtient ainsi un morphisme

$$\sigma^\natural_{\mathcal{O},X,n} : \mathbb{Z}_{\mathrm{tr}}\left[(X^\natural)^n_X\right](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}\left[(X^\natural)^{n+1}_X\right](\mathcal{O})$$

et ces derniers nous donnent une homotopie canonique du complexe (10).

Montrons maintenant que le carré (12) est commutatif autrement dit que l'on a l'égalité

$$\sigma^\natural_{\mathcal{O}',X,n}(\beta\circ\alpha) = \sigma^\natural_{\mathcal{O},X,n}(\beta)\circ\alpha$$

pour toute correspondance finie $\beta \in c_S\left(\mathcal{O},(X^\natural)^n_X\right)$. Fixons pour cela un sous-schéma fermé $Z$ de $\mathcal{O}\times_S X$ fini et équidimensionnel sur $\mathcal{O}$ de sorte qu'en notant $\iota_n$ l'immersion fermée de $W^n_Z$ dans $\mathcal{O}\times_S (X^\natural)^n_X$ on ait

$$\beta = (\iota_n)_*\overline{\beta} \qquad \overline{\beta}\in c_{\mathrm{equi}}(W^n_Z/\mathcal{O},0).$$

De même fixons un sous-schéma fermé $\mathcal{Z}$ de $\mathcal{O}'\times_S\mathcal{O}$ fini et équidimensionnel sur $\mathcal{O}'$ de sorte qu'en notant $\iota$ l'immersion fermée de $\mathcal{Z}$ dans $\mathcal{O}'\times_S\mathcal{O}$ on ait

$$\alpha = \iota_*\overline{\alpha} \qquad \overline{\alpha}\in c_{\mathrm{equi}}(\mathcal{Z}/\mathcal{O}',0).$$

Notons alors $Z''$ le sous-schéma fermé de $\mathcal{O}'\times_S\mathcal{O}\times_S X$ défini par le carré cartésien

$$\begin{tikzcd} Z'' \arrow[r] \arrow[d, "\mathrm{fini}"] \arrow[dd, bend right, "\mathrm{fini}"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\mathrm{fini}"] \\ \mathcal{Z} \arrow[r] \arrow[d, "\mathrm{fini}"] & \mathcal{O} \\ \mathcal{O}'. & \end{tikzcd}$$

Fixons d'autre part un sous-schéma fermé $Z'$ de $\mathcal{O}' \times_S X$ fini équidimensionnel sur $\mathcal{O}'$ tel que l'on ait le carré commutatif

$$\begin{tikzcd} Z'' \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\text{fini}"'] & \mathcal{O}' \times_S \mathcal{O} \times_S X \arrow[d] \\ Z' \arrow[r] \arrow[d, "\text{fini}"'] & \mathcal{O}' \times_S X \arrow[dl] \\ \mathcal{O}' & \end{tikzcd}$$

et convenons de noter $W''$ le schéma défini par le carré cartésien

$$\begin{tikzcd} W'' \arrow[r, "r''"] \arrow[d] \arrow[dr, phantom, "\square"] & Z'' \arrow[d] \\ \mathcal{O}' \times_S \mathcal{O} \times_S X^{\mathfrak{h}} \arrow[r] & \mathcal{O}' \times_S \mathcal{O} \times_S X. \end{tikzcd}$$

Par définition $Z''$ est fini sur le schéma local hensélien $\mathcal{O}'$, autrement dit semi-local hensélien et donc décomposé pour la topologie de Nisnevich de la forme

$$Z'' = \coprod_{\substack{z'' \text{ point fermé} \\ \text{de } Z''}} \operatorname{Spec}\left(\mathcal{O}_{Z'',z''}\right).$$

La propriété universelle (8) nous assure comme précédemment l'existence d'un unique morphisme $\theta_{Z''}$ factorisant la projection $\pi_{Z''}$ de $Z''$ sur $X$ sous la forme

$$\begin{tikzcd} Z'' \arrow[r, dotted, "\theta_{Z''}"'] \arrow[rr, bend left, "\pi_{Z''}"] & X^{\mathfrak{h}} \arrow[r, "\iota_X^{\mathfrak{h}}"'] & X. \end{tikzcd}$$

La propriété universelle des produits fibrés nous fournit une section canonique $\sigma_{Z''}^{\mathfrak{h}}$ du morphisme $r''$ via le diagramme

$$\begin{tikzcd} Z'' \arrow[r, dotted, "\sigma_{Z''}^{\mathfrak{h}}"] \arrow[rr, bend left, "\mathrm{id}_{Z''}"] \arrow[ddr, bend right, "\theta_{Z''}"'] & W \arrow[r, "r''"] \arrow[d] \arrow[dr, phantom, "\square"] & Z'' \arrow[d] \arrow[dd, bend left=60, "\pi_{Z''}"] \\ & \mathcal{O}' \times_S \mathcal{O} \times_S X^{\mathfrak{h}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{O}' \times_S \mathcal{O} \times_S X \arrow[d] \\ & X^{\mathfrak{h}} \arrow[r, "\iota_X^{\mathfrak{h}}"'] & X. \end{tikzcd}$$

On a par ailleurs le diagramme commutatif

$$\begin{tikzcd}
 & Z'' \arrow[dl] \arrow[d, "\theta_{Z''}"] \arrow[dr] \arrow[dd, bend left=90, looseness=2, "\pi_{Z''}"] & \\
Z \arrow[r, "\theta_Z"] \arrow[dr, "\pi_Z"'] & X^{\mathfrak{h}} \arrow[d, "\iota^{\mathfrak{h}}_X"] & Z' \arrow[l, "\theta_{Z'}"'] \arrow[dl, "\pi_{Z'}"] \\
 & X &
\end{tikzcd}$$

ce qui assure que les carrés suivants sont commutatifs

$$\begin{tikzcd}
Z \arrow[d, "\sigma^{\mathfrak{h}}_Z"] & Z'' \arrow[l] \arrow[r] \arrow[d, "\sigma^{\mathfrak{h}}_{Z''}"] & Z' \arrow[d, "\sigma^{\mathfrak{h}}_{Z'}"] \\
W & W'' \arrow[l] \arrow[r] & W'.
\end{tikzcd}$$

On obtient ainsi que le diagramme suivant dans lequel $p_n$ et $q_n$ désignent respectivement les projections induites par les projections de $Z''$ sur $Z$ et $Z'$

$$\begin{tikzcd}
 & (W'')^{n+1}_{Z''} \arrow[dr, "p_{n+1}"] \arrow[ddl, "q_{n+1}"'] & & (W'')^{n}_{Z''} \arrow[ll, "\sigma^{\mathfrak{h}}_{Z'',n}"'] \arrow[dr, "p_n"] \arrow[dd, "q_n"] & \\
 & & (W')^{n+1}_{Z'} & & (W')^{n}_{Z'} \arrow[ll, dotted, "\sigma^{\mathfrak{h}}_{Z',n}"'] \\
W^{n+1}_Z & & & W^n_Z \arrow[lll, "\sigma^{\mathfrak{h}}_{Z,n}"'] &
\end{tikzcd}$$

est commutatif. D'autre part on a un carré cartésien

$$\begin{tikzcd}
(W'')^{n}_{Z''} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W^n_Z \arrow[d] \\
\mathcal{Z} \arrow[r] & \mathcal{O}.
\end{tikzcd}$$

Cela donne ainsi

$$\begin{aligned}
\sigma^{\mathfrak{h}}_{\mathcal{O}',X,n}(\beta \circ \alpha) &= \sigma^{\mathfrak{h}}_{\mathcal{O}',X,n}\left[\left(p^{\mathcal{O}'\times_S\mathcal{O}\times_S X}_{\mathcal{O}'\times_S X}\right)_* Cor_{\mathcal{O}'\times_S\mathcal{O}/\mathcal{O}'}\left(\left(p^{\mathcal{O}'\times_S\mathcal{O}}_{\mathcal{O}}\right)^{\circledast}\beta, \alpha\right)\right] \\
&= \sigma^{\mathfrak{h}}_{\mathcal{O}',X,n}\left[(\iota'_n)_*(p_n)_* Cor_{\mathcal{Z}/\mathcal{O}'}\left(\left(p^{\mathcal{Z}}_{\mathcal{O}}\right)^{\circledast}\overline{\beta}, \overline{\alpha}\right)\right] \\
&= (\iota'_{n+1})_*(\sigma^{\mathfrak{h}}_{Z',n})_*(p_n)_* Cor_{\mathcal{Z}/\mathcal{O}'}\left(\left(p^{\mathcal{Z}}_{\mathcal{O}}\right)^{\circledast}\overline{\beta}, \overline{\alpha}\right) \\
&= (\iota'_{n+1})_*(p_{n+1})_*(\sigma^{\mathfrak{h}}_{Z'',n})_* Cor_{\mathcal{Z}/\mathcal{O}'}\left(\left(p^{\mathcal{Z}}_{\mathcal{O}}\right)^{\circledast}\overline{\beta}, \overline{\alpha}\right)
\end{aligned}$$

D'autre part en utilisant la proposition 3.6.2 de [28] et le lemme 1.1, on voit que

$$\begin{aligned}(\sigma^{\mathfrak{h}}_{Z'',n})_* Cor_{\mathcal{Z}/\mathcal{O}'}\left(\left(p^{\mathcal{Z}}_{\mathcal{O}}\right)^{\circledast}\overline{\beta},\overline{\alpha}\right) &= Cor_{\mathcal{Z}/\mathcal{O}'}\left((\sigma^{\mathfrak{h}}_{Z'',n})_*\left(p^{\mathcal{Z}}_{\mathcal{O}}\right)^{\circledast}\overline{\beta},\overline{\alpha}\right)\\ &= Cor_{\mathcal{Z}/\mathcal{O}'}\left(\left(p^{\mathcal{Z}}_{\mathcal{O}}\right)^{\circledast}(\sigma^{\mathfrak{h}}_{Z,n})_*\overline{\beta},\overline{\alpha}\right)\end{aligned}$$

ce qui assure finalement que

$$\begin{aligned}\sigma^{\mathfrak{h}}_{\mathcal{O}',X,n}(\beta\circ\alpha) &= (\iota'_{n+1})_*(p_{n+1})_* Cor_{\mathcal{Z}/\mathcal{O}'}\left(\left(p^{\mathcal{Z}}_{\mathcal{O}}\right)^{\circledast}(\sigma^{\mathfrak{h}}_{Z,n})_*\overline{\beta},\overline{\alpha}\right)\\ &= \left[p^{\mathcal{O}\times_S\mathcal{O}'\times_S(X^{\mathfrak{h}})^{n+1}_X}_{\mathcal{O}'\times_S(X^{\mathfrak{h}})^{n+1}_X}\right]_* Cor_{\mathcal{O}'\times_S\mathcal{O}/\mathcal{O}'}\left(\left(p^{\mathcal{O}'\times_S\mathcal{O}}_{\mathcal{O}}\right)^{\circledast}\sigma^{\mathfrak{h}}_{\mathcal{O},X,n}(\beta),\alpha\right)\\ &= \sigma^{\mathfrak{h}}_{\mathcal{O},X,n}(\beta)\circ\alpha.\end{aligned}$$

□

### 2.2.2 Cas des fibres pour la topologie de Nisnevich

Considérons maintenant les fibres Nisnevich du complexe (9). Rappelons que pour un point $x \in X$ on dispose d'un isomorphisme canonique

$$X^{\mathfrak{h}}_x = \lim_{U\in(\mathcal{V}^h_{X,x})^{\mathrm{op}}} U$$

où $\mathcal{V}^h_{X,x}$ désigne la catégorie des voisinages Nisnevich affines de $x$. La fibre Nisnevich d'un préfaisceau $F$ au point $x$ est alors donnée par

$$F_x = \operatorname*{colim}_{U\in(\mathcal{V}^h_{X,x})^{\mathrm{op}}} F(U).$$

Plus généralement considérons la définition suivante.

Définition 2.3. Soit $\mathcal{O}$ un $S$-schéma. On appelle présentation de $\mathcal{O}$ la donnée d'un triplet $(\Lambda, U, u)$ vérifiant les conditions suivantes.

- $\Lambda$ est une catégorie cofiltrante.
- $U : \Lambda \to \mathrm{Sch}_S; \lambda \mapsto U_\lambda$ est un système projectif de $S$-schémas indexé par $\Lambda^{\mathrm{op}}$ tel que pour tout morphisme $\lambda \to \mu$ de $\Lambda$ le morphisme induit $U_\lambda \to U_\mu$ soit plat et affine.
- $u$ est un isomorphisme de $S$-schémas $\mathcal{O} \to \lim_{\lambda\in\Lambda^{\mathrm{op}}} U_\lambda$.

*Exemple* 2.4. La catégorie $\mathcal{V}^h_{X,x}$ des voisinages Nisnevich affines de $x$ fournit une présentation naturelle du schéma local hensélien $X^{\mathfrak{h}}_x$. De même pour un point $y \in X^{\mathfrak{h}}_x$ en remarquant que l'on a un isomorphisme canonique

$$(X^{\mathfrak{h}}_x)^{\mathfrak{h}}_y = \lim_{U\in(\mathcal{V}^h_{X,x})^{\mathrm{op}},V\in(\mathcal{V}^h_{U,z})^{\mathrm{op}}} V$$

on obtient une présention naturelle du $S$-schéma local hensélien $(X^{\mathfrak{h}}_x)^{\mathfrak{h}}_y$. On remarquera que lorsque $X$ est (lisse) de type fini sur $S$ les schémas apparaissant dans les systèmes projectifs précédemment décrit sont tous (lisses) de type fini sur $S$.

*Notation* 2.5. Étant donnés un $S$-schéma $X$ et une présentation d'un $S$-schéma $\mathcal{O}'$ il sera commode de disposer des groupes abéliens [8]

$$c_S\{\mathcal{O}, X\} := \operatorname*{colim}_{\lambda\in\Lambda^{\mathrm{op}}} c_S(U_\lambda, X)$$

$$c_S(\mathcal{O}', \mathcal{O}\} := \lim_{\lambda\in\Lambda^{\mathrm{op}}} c_S(\mathcal{O}', U_\lambda) \qquad c_S\{\mathcal{O}', \mathcal{O}\} := \lim_{\lambda\in\Lambda^{\mathrm{op}}} c_S\{\mathcal{O}', U_\lambda)$$

Lorsque $F$ est un préfaisceau (avec transferts) on pose également

$$F\{\mathcal{O}\} := \operatorname*{colim}_{\lambda\in\Lambda^{\mathrm{op}}} F(U_\lambda)$$

de sorte que $\mathbb{Z}_{\mathrm{tr}}[X]\{\mathcal{O}\} = c_S\{\mathcal{O}, X)$ avec les notations précédentes.

On dispose via $u$ d'un morphisme naturel $F\{\mathcal{O}\} \to F(\mathcal{O})$ et les transferts induisent des morphismes naturels s'inscrivant dans un diagramme commutatif

$$\begin{tikzcd}
c_S\{\mathcal{O}', \mathcal{O}) \otimes F\{\mathcal{O}\} \arrow[r] \arrow[d] & c_S\{\mathcal{O}', \mathcal{O}) \otimes F(\mathcal{O}) \arrow[d] \\
c_S\{\mathcal{O}', \mathcal{O}\} \otimes F\{\mathcal{O}\} \arrow[r] \arrow[d] & F\{\mathcal{O}'\} \arrow[d] \\
c_S(\mathcal{O}', \mathcal{O}\} \otimes F\{\mathcal{O}\} \arrow[r] & F(\mathcal{O}') \\
c_S(\mathcal{O}', \mathcal{O}) \otimes F\{\mathcal{O}\} \arrow[r] \arrow[u] & c_S(\mathcal{O}', \mathcal{O}) \otimes F(\mathcal{O}). \arrow[u]
\end{tikzcd} \tag{16}$$

Étant donné un élément $\alpha \in c_S\{\mathcal{O}', \mathcal{O})$ ou plus généralement un élément $\alpha \in c_S\{\mathcal{O}', \mathcal{O}\}$ nous désignerons dans la suite par

$$F\{\alpha\} : F\{\mathcal{O}\} \to F\{\mathcal{O}'\}$$

le morphisme induit. On remarquera que si $\beta \in c_S\{\mathcal{O}'', \mathcal{O}')$ et $\underline{\alpha}$ désigne l'image de $\alpha$ dans $c_S(\mathcal{O}', \mathcal{O})$ on a la relation

$$F\{\underline{\alpha} \circ \beta\} = F\{\beta\} \circ F\{\alpha\}.$$

Nous noterons également $F\{\alpha\} : F\{\mathcal{O}\} \to F(\mathcal{O}')$ le morphisme induit par un élément $\alpha \in c_S(\mathcal{O}', \mathcal{O}\}$. Le lemme suivant nous sera très utile dans la suite :

LEMME 2.6. *Étant donnés un $S$-schéma $X$ et une présentation d'un $S$-schéma $\mathcal{O}$. Le morphisme canonique*

$$\mathbb{Z}_{\mathrm{tr}}[X]\{\mathcal{O}\} \to \mathbb{Z}_{\mathrm{tr}}[X](\mathcal{O})$$

*est injectif.*

[8] Ces notations peuvent sembler ambiguës à priori puisque l'on néglige de préciser la présentation choisie. Néanmoins dans la suite cela n'entrainera aucune confusion, la présentation étant clairement définie par le contexte. Dans le cas des hensélisés nous utiliserons les présentations décrites dans l'exemple 2.4.

*Démonstration.* Pour tout $\lambda \in \Lambda$, le morphisme $\mathcal{O} \to U_\lambda$ est plat et ainsi

$$c_S(U_\lambda, X) = c_{\mathrm{equi}}(U_\lambda \times_S X/U_\lambda, 0) \to c_S(\mathcal{O}, X) = c_{\mathrm{equi}}(\mathcal{O} \times_S X/\mathcal{O}, 0)$$

est donné par un simple changement de base plat. Soit $\alpha \in c_S(U_\lambda, X)$ une correspondance dont l'image dans $c_S(\mathcal{O}, X)$ soit nulle. Étant donné $\mu \in \Lambda/\lambda$, si $Z_\mu$ désigne le support de l'image $\alpha_\mu$ dans $c_S(U_\mu, X)$ de $\alpha$, il s'ensuit que la limite projective du système $\mu \mapsto Z_\mu$ est vide. Le théorème 8.10.5 de [14] assure alors que $Z_\mu$ est vide pour $\mu$ suffisamment grand et donc que $\alpha_\mu = 0$. L'injectivité en résulte. $\square$

Dans la situation précédente il est possible de raffiner la proposition 2.1 sous la forme suivante.

Proposition 2.7. *Étant donnés un $S$-schéma $X$ et une présentation d'un $S$-schéma local hensélien $\mathcal{O}$. Il existe d'uniques morphismes*

$$\{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,n} : \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right]\{\mathcal{O}\} \to \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right]\{\mathcal{O}\} \qquad n \geq 0$$

*satisfaisant aux deux propriétés suivantes.*

(a) *(Homotopie) On a pour tout $n$ les relations*

$$d_{n+1} \circ \{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,n} + \{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,n-1} \circ d_n = \mathrm{id}. \tag{17}$$

(b) *(Compatibilité) Le carré suivant est commutatif*

$$\begin{tikzcd} \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right]\{\mathcal{O}\} \arrow[r, "\{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,n}"] \arrow[d] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right]\{\mathcal{O}\} \arrow[d] \\ \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right](\mathcal{O}) \arrow[r, "\sigma^{\mathfrak{h}}_{\mathcal{O},X,n}"] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right](\mathcal{O}). \end{tikzcd} \tag{18}$$

*Démonstration.* L'unicité de tels morphismes découle immédiatement de la commutativité de (18) et du lemme 2.6. Il suffit donc de donner une construction de ces morphismes en raffinant la preuve de la proposition 2.1. Introduisons la catégorie $\mathcal{C}$ dont les objets sont les couples $(\lambda, Z)$ où $\lambda \in \Lambda$ et $Z$ est un sous-schéma fermé de $U_\lambda \times_S X$ fini et équidimensionnel sur $U_\lambda$. L'ensemble des morphismes de $(\lambda, Z)$ dans $(\lambda', Z')$ étant réduit à un élément si le schéma $Z_\mathcal{O}$ est contenu dans $Z'_\mathcal{O}$ et vide dans le cas contraire. Étant donné $(\lambda, Z) \in \mathcal{C}$, on note $W$ le sous-schéma fermé de $V \times_S X^{\mathfrak{h}}$ défini par le carré cartésien

$$\begin{tikzcd} W \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ U_\lambda \times_S X^{\mathfrak{h}} \arrow[r] & U_\lambda \times_S X \end{tikzcd}$$

ce qui fournit le complexe

$$\cdots \to \operatorname*{colim}_{\mu\in(\Lambda/\lambda)^{\mathrm{op}}} c_{\mathrm{equi}}((W_{U_\mu})^{n+1}_{Z_{U_\mu}}/U_\mu, 0) \to \operatorname*{colim}_{\mu\in(\Lambda/\lambda)^{\mathrm{op}}} c_{\mathrm{equi}}((W_{U_\mu})^{n}_{Z_{U_\mu}}/U_\mu, 0) \to \cdots . \tag{19}$$

Ces complexes sont fonctoriels par rapport aux morphismes dans $\mathcal{C}$ et en remarquant que

$$\operatorname*{colim}_{(\lambda,Z)\in\mathcal{C}} \operatorname*{colim}_{\mu\in(\Lambda/\lambda)^{\mathrm{op}}} c_{\mathrm{equi}}((W_{U_\mu})^{n}_{Z_{U_\mu}}/U_\mu, 0) = \mathbb{Z}_{\mathrm{tr}}(X^{\mathfrak{h}})^n_X\{\mathcal{O}\}$$

on voit que le complexe $\check{C}_{X^{\mathfrak{h}}/X}\{\mathcal{O}\}$ est la colimite sur $\mathcal{C}$ des complexes (19). Nous allons construire comme précédemment une homotopie canonique pour chacun de ces complexes. Il résulte de la preuve de la proposition 2.1 que le morphisme $r_{\mathcal{O}} : W_{\mathcal{O}} \to Z_{\mathcal{O}}$ possède une section canonique

$$\sigma^{\mathfrak{h}}_{Z_{\mathcal{O}}} : Z_{\mathcal{O}} \to W_{\mathcal{O}}.$$

En particulier quitte à remplacer $(\lambda, Z)$ par $(\mu, Z_{U_\mu})$ pour un certain élément $\mu \in (\Lambda/\lambda)^{\mathrm{op}}$ cette section se relève en une section du morphisme $r$

$$\sigma^{\mathfrak{h}}_Z : Z \to W.$$

On peut alors considérer pour tout $\mu \in (\Lambda/\lambda)^{\mathrm{op}}$ les morphismes de schémas

$$\sigma^{\mathfrak{h}}_{Z_{U_\mu},n} = \sigma^{\mathfrak{h}}_{Z_{U_\mu}} \times_{Z_{U_\mu}} \mathrm{id}_{(W_{U_\mu})^n_{Z_{U_\mu}}} : (W_{U_\mu})^n_{Z_{U_\mu}} \to (W_{U_\mu})^{n+1}_{Z_{U_\mu}}.$$

Ces derniers vérifient les relations pour $i = 0, \ldots, n$

$$\delta^{n+1}_{i+1} \circ \sigma^{\mathfrak{h}}_{Z_{U_\mu},n} = \sigma^{\mathfrak{h}}_{Z_{U_\mu},n-1} \circ \delta^n_i \qquad \delta^{n+1}_0 \circ \sigma^{\mathfrak{h}}_{Z_{U_\mu},n} = \mathrm{id}$$

ce qui assure que les morphismes induits sur les cycles équidimensionnels

$$\{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,(\lambda,Z),n} := \begin{array}{c} \operatorname{colim}_{\mu\in(\Lambda/\lambda)^{\mathrm{op}}} c_{\mathrm{equi}}((W_{U_\mu})^{n}_{Z_{U_\mu}}/U_\mu, 0) \\ \Big\downarrow {\scriptstyle \operatorname{colim}_{\mu\in(\Lambda/\lambda)^{\mathrm{op}}}(\sigma^{\mathfrak{h}}_{Z_{U_\mu},n})_*} \\ \operatorname{colim}_{\mu\in(\Lambda/\lambda)^{\mathrm{op}}} c_{\mathrm{equi}}((W_{U_\mu})^{n+1}_{Z_{U_\mu}}/U_\mu, 0) \end{array}$$

définissent une homotopie entre l'identité du complexe (19) et le morphisme nul.

Supposons donné un morphisme $(\lambda, Z) \to (\lambda', Z')$ dans $\mathcal{C}$. Il résulte de la preuve de la proposition 2.1 que le carré

$$\begin{tikzcd} Z_{\mathcal{O}} \arrow[r, "\sigma^{\mathfrak{h}}_{Z_{\mathcal{O}}}"] \arrow[d] & W_{\mathcal{O}} \arrow[d] \\ Z'_{\mathcal{O}} \arrow[r, "\sigma^{\mathfrak{h}}_{Z'_{\mathcal{O}}}"] & W'_{\mathcal{O}} \end{tikzcd}$$

commute. Il existe donc $\mu \in \Lambda$ et des morphismes $\lambda \leftarrow \mu \rightarrow \lambda'$ tels que le carré

$$\begin{tikzcd} Z_{U_\mu} \arrow[r, "\sigma^{\mathfrak{h}}_{Z_{U_\mu}}"] \arrow[d] & W_{U_\mu} \arrow[d] \\ Z'_{U_\mu} \arrow[r, "\sigma^{\mathfrak{h}}_{Z'_{U_\mu}}"] & W'_{U_\mu} \end{tikzcd}$$

soit commutatif. Cela entraîne que le carré

$$\begin{tikzcd} \underset{\mu \in (\Lambda/\lambda)^{\mathrm{op}}}{\mathrm{colim}}\, c_{\mathrm{equi}}((W_{U_\mu})^n_{Z_{U_\mu}}/U_\mu, 0) \arrow[r, "\{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,(\lambda,Z),n}"] \arrow[d] & \underset{\mu \in (\Lambda/\lambda)^{\mathrm{op}}}{\mathrm{colim}}\, c_{\mathrm{equi}}((W_{U_\mu})^{n+1}_{Z_{U_\mu}}/U_\mu, 0) \arrow[d] \\ \underset{\mu \in (\Lambda/\lambda')^{\mathrm{op}}}{\mathrm{colim}}\, c_{\mathrm{equi}}((W'_{U_\mu})^n_{Z'_{U_\mu}}/U_\mu, 0) \arrow[r, "\{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,(\lambda',Z'),n}"] & \underset{\mu \in (\Lambda/\lambda')^{\mathrm{op}}}{\mathrm{colim}}\, c_{\mathrm{equi}}((W'_{U_\mu})^n_{Z'_{U_\mu}}/U_\mu, 0) \end{tikzcd}$$

l'est aussi et en passant à la colimite sur $\mathcal{C}$ on obtient ainsi des morphismes

$$\{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,n} : \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right]\{\mathcal{O}\} \rightarrow \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right]\{\mathcal{O}\}$$

fournissant une homotopie canonique du complexe $\check{C}_{X^{\mathfrak{h}}/X}\{\mathcal{O}\}$. Il résulte immédiatement de la construction que le carré (18) est commutatif. $\square$

Par application du lemme 2.6, la commutativité du carré supérieur de (16) ainsi que celle du carré (12) assurent que les morphismes construits dans la proposition précédente satisfont également le lemme suivant.

Lemme 2.8. *(Fonctorialité) Étant donnés une présentation d'un $S$-schéma local hensélien $\mathcal{O}'$ et un élément $\alpha \in \mathbb{Z}_{\mathrm{tr}}[\mathcal{O}]\{\mathcal{O}'\}$, on a un carré commutatif*

$$\begin{tikzcd} \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right]\{\mathcal{O}\} \arrow[r, "\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right]\{\alpha\}"] \arrow[d, "\{\sigma\}^{\mathfrak{h}}_{\mathcal{O}',X,n}"] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^n_X\right]\{\mathcal{O}'\} \arrow[d, "\{\sigma\}^{\mathfrak{h}}_{\mathcal{O},X,n}"] \\ \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right]\{\mathcal{O}\} \arrow[r, "\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right]\{\alpha\}"] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^{n+1}_X\right]\{\mathcal{O}'\}. \end{tikzcd}$$

Corollaire 2.9. *Soit $X$ un $S$-schéma. Le complexe de faisceaux Nisnevich avec transferts $\check{C}_{X^{\mathfrak{h}}/X}$ est universellement exact au sens de Grayson [12].*

*Démonstration.* Il suffit de vérifier que pour toute présentation d'un $S$-schéma local hensélien $\mathcal{O}$ le complexe

$$\check{C}_{X^{\mathfrak{h}}/X}\{\mathcal{O}\} : \cdots \rightarrow \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^2_X\right]\{\mathcal{O}\} \rightarrow \mathbb{Z}_{\mathrm{tr}}[X^{\mathfrak{h}}]\{\mathcal{O}\} \rightarrow \mathbb{Z}_{\mathrm{tr}}[X]\{\mathcal{O}\} \rightarrow 0 \rightarrow \cdots$$

est universellement exact au sens de Grayson. Ceci découle de la proposition 2.1 puisque cette dernière assure que $\check{C}_{X^{\mathfrak{h}}/X}\{\mathcal{O}\}$ est en fait homotope à zéro donc à fortiori universellement exact au sens de Grayson . $\square$

*Remarque* 2.10. En dépit de la fonctorialité — lemme 2.8 — des homotopies construites dans la proposition 2.7, ces dernières ne peuvent généralement pas se relever en des morphismes de faisceaux avec transferts. En particulier le complexe de faisceaux Nisnevich avec transferts

$$\check{C}_{X^{\mathfrak{h}}/X} : \cdots \to \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{h}})^2_X\right] \to \mathbb{Z}_{\mathrm{tr}}[X^{\mathfrak{h}}] \to \mathbb{Z}_{\mathrm{tr}}[X] \to 0 \to \cdots$$

bien qu'exact d'après le corollaire 2.9 n'est pas homotope à zéro en général.

## 2.3 Décomposition locale

### 2.3.1 Première décomposition locale

Nous abordons maintenant la décomposition locale d'une correspondance finie pour la topologie de Nisnevich. Les morphismes de schémas $\iota^{\mathfrak{h}}_{X,x}$ nous donnent un morphisme naturel

$$\bigoplus_{x\in X} c_S\left(\mathcal{O}, X^{\mathfrak{h}}_x\right) \xrightarrow{\sum_{x\in X}[\iota^{\mathfrak{h}}_{X,x}]\circ -} c_S(\mathcal{O}, X). \tag{20}$$

La remarque suivante résulte immédiatement de la définition des correspondances finies.

*Remarque* 2.11. On a un isomorphisme naturel $c_S(\mathcal{O}, X^{\mathfrak{h}}) = \bigoplus_{x\in X} c_S\left(\mathcal{O}, X^{\mathfrak{h}}_x\right)$ s'insérant dans le triangle commutatif

$$\begin{tikzcd}
\bigoplus_{x\in X} c_S\left(\mathcal{O}, X^{\mathfrak{h}}_x\right) \arrow[r, "\sum_{x\in X}[\iota^{\mathfrak{h}}_{X,x}]\circ -"] \arrow[d, equal] & c_S(\mathcal{O}, X) \\
c_S(\mathcal{O}, X^{\mathfrak{h}}). \arrow[ur, "{[\iota^{\mathfrak{h}}_X]\circ -}"'] &
\end{tikzcd}$$

Lorsque $\mathcal{O}$ est hensélien, la proposition précédente nous permet d'écrire une correspondance finie $\alpha \in c_S(\mathcal{O}, X)$ sous la forme d'une somme de contributions locales

$$\alpha = \sum_x [\iota^{\mathfrak{h}}_{X,x}] \circ \alpha_x \qquad \alpha_x \in c_S\left(\mathcal{O}, X^{\mathfrak{h}}_x\right)$$

la correspondance $\alpha_x$ étant canoniquement déterminée par $\alpha$. Plus précisément on peut déduire de la proposition 2.1 l'énoncé suivant.

COROLLAIRE 2.12. *Soient $X$ un $S$-schéma et $\mathcal{O}$ un $S$-schéma local hensélien. Il existe un morphisme canonique*

$$c_S\left(\mathcal{O}, X\right) \xrightarrow{\sigma^{\mathfrak{h}}_{\mathcal{O},X}} \bigoplus_{x\in X} c_S\left(\mathcal{O}, X^{\mathfrak{h}}_x\right) \tag{21}$$

*satisfaisant les propriétés suivantes.*

*(a)* $\sigma^{\mathfrak{h}}_{\mathcal{O},X}$ *est une section du morphisme (20) telle que le carré*

$$\begin{tikzcd} c_S\left(\mathcal{O},X\right) \arrow[r, "\sigma^{\mathfrak{h}}_{\mathcal{O},X}"] \arrow[d, "c_S(\alpha,X)"] & \bigoplus_{x\in X} c_S\left(\mathcal{O},X^{\mathfrak{h}}_x\right) \arrow[d, "c_S(\alpha,X^{\mathfrak{h}}_x)"] \\ c_S\left(\mathcal{O}',X\right) \arrow[r, "\sigma^{\mathfrak{h}}_{\mathcal{O}',X}"] & \bigoplus_{x\in X} c_S\left(\mathcal{O}',X^{\mathfrak{h}}_x\right) \end{tikzcd} \tag{22}$$

*soit commutatif pour tout schéma local hensélien* $\mathcal{O}'$ *et toute correspondance finie* $\alpha \in c_S(\mathcal{O}',\mathcal{O})$*.*

*(b) Pour un* $S$*-morphisme* $g : \mathcal{O} \to X$ *la composante suivant le point* $x$ *de l'image de* $[g]$ *par (21) est donnée par*

$$\sigma^{\mathfrak{h}}_{\mathcal{O},X}([g])_x = \begin{cases} [\overline{g}] & \textit{si } x = \tau \\ 0 & \textit{sinon} \end{cases}$$

$\tau$ *étant l'image du point fermé de* $\mathcal{O}$ *et* $\overline{g}$ *le morphisme déduit de* $g$

$$\begin{tikzcd} \operatorname{Spec}(\mathcal{O}) \arrow[r, "\overline{g}"] \arrow[rr, bend left, "g"] & X^{\mathfrak{h}}_x \arrow[r, "\iota^{\mathfrak{h}}_{X,x}"] & X. \end{tikzcd}$$

*Démonstration.* Le premier point est une conséquence immédiate de la proposition 2.1. En effet le morphisme

$$\begin{tikzcd} c_S(\mathcal{O},X) \arrow[r, "\sigma^{\mathfrak{h}}_{\mathcal{O},X,0}"] \arrow[rr, bend left, dotted, "\sigma^{\mathfrak{h}}_{\mathcal{O},X}"] & c_S(\mathcal{O},X^{\mathfrak{h}}) \arrow[r, equal] & \bigoplus_{x\in X} c_S\left(\mathcal{O},X^{\mathfrak{h}}_x\right) \end{tikzcd}$$

est une section du morphisme (20) d'après la relation d'homotopie (11), et la commutativité du diagramme (22) n'est autre que celle du diagramme (12).

Considérons donc la seconde assertion. Pour reprendre les notations utilisées dans la démonstration de la proposition 2.1, nous désignons par $Z$ le sous-schéma fermé de $\mathcal{O} \times_S X$ graphe du morphisme $g$ et par $W$ le schéma défini par le carré cartésien

$$\begin{tikzcd} W \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\ \mathcal{O} \times_S X^{\mathfrak{h}} \arrow[r] & \mathcal{O} \times_S X. \end{tikzcd}$$

Le sous-schéma fermé $Z$ s'identifie à l'immersion fermée $\Delta_g : \mathcal{O} \to \mathcal{O} \times_S X$ et l'on voit que le morphisme $\theta_Z$ de la démonstration de la proposition 2.1

s'identifie à la composée de $\overline{g}$ et du morphisme d'inclusion $\iota : X^{\mathfrak{h}}_x \to X^{\mathfrak{h}}$. Dans ce cas le diagramme (15) est donc de la forme

$$\begin{tikzcd}
\mathcal{O} \arrow[r, dotted, "\sigma^{\mathfrak{h}}_Z"] \arrow[rr, bend left, "\mathrm{id}_{\mathcal{O}}"] \arrow[ddr, bend right, "\iota\circ\overline{g}"'] & W \arrow[r, "r"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{O} \arrow[d, "\Delta_g"] \arrow[dd, bend left, "g"] \\
& \mathcal{O}\times_S X^{\mathfrak{h}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{O}\times_S X \arrow[d] \\
& X^{\mathfrak{h}} \arrow[r, "\mathfrak{i}^{\mathfrak{h}}_X"] & X
\end{tikzcd}$$

ce qui prouve que $(\sigma^{\mathfrak{h}}_Z)_*([g]) = [\overline{g}]$. Par construction du morphisme $\sigma^{\mathfrak{h}}_{\mathcal{O},X}$ cela se traduit par

$$\sigma_{\mathcal{O},X}([g])_x = \begin{cases} [\overline{g}] & \text{si } x = \tau \\ 0 & \text{sinon} \end{cases}$$

qui n'est autre que l'égalité souhaitée. $\square$

*Remarque* 2.13. Supposons que $X$ soit lui même un schéma local hensélien de point fermé $s$. On a vu que pour une correspondance finie $\alpha \in c_S(\mathcal{O}, X)$ on avait une décomposition locale

$$\alpha = \underbrace{\sum_{x\in X\setminus\{s\}} [\mathfrak{i}^{\mathfrak{h}}_{X,x}] \circ \alpha_x}_{\substack{\text{contribution des points} \\ \text{non fermés}}} + \alpha_s.$$

En général les correspondances $\alpha$ et $\alpha_s$ ne sont pas égales, autrement dit les points non fermés de $X$ ont une contribution non nulle dans la décomposition locale. Prenons par exemple le cas où $\alpha$ est la correspondance finie associée à un morphisme de schéma. On voit d'après la seconde assertion du corollaire 2.12 que

$$\alpha_s = \begin{cases} \alpha & \text{si le morphisme est local} \\ 0 & \text{sinon.} \end{cases}$$

et que de plus lorsque le morphisme est local non seulement la contribution globale des points non fermés est nulle mais encore chaque correspondance $\alpha_x$ associée à un point non fermé est nulle.

Compte tenu de la remarque 2.11, le morphisme donnant la décomposition locale peut être vu comme un morphisme

$$\sigma^{\mathfrak{h}}_{\mathcal{O},X} : c_S(\mathcal{O}, X) \to c_S(\mathcal{O}, X^{\mathfrak{h}}). \tag{23}$$

On dispose d'un morphisme d'inclusion

$$[\mathfrak{s}^{\mathfrak{h}}_X] \circ - : c_S(\mathcal{O}, X^{\mathfrak{h}}) \to c_S(\mathcal{O}, (X^{\mathfrak{h}})^{\mathfrak{h}})$$

et d'un morphisme de localisation

$$\sigma^{\mathfrak{h}}_{\mathcal{O}, X^{\mathfrak{h}}} : c_S(\mathcal{O}, X^{\mathfrak{h}}) \to c_S(\mathcal{O}, (X^{\mathfrak{h}})^{\mathfrak{h}}).$$

Ces morphismes ne coïncident naturellement pas comme on le voit déjà avec la remarque 2.13. Sachant que le morphisme $\mathfrak{l}^{\mathfrak{h}}_{X^{\mathfrak{h}}}$ n'est pas un isomorphisme en général, il est légitime de se demander si l'on obtient plus d'informations en localisant à nouveau le résultat fourni par le morphisme (23). Le lemme suivant montre qu'il n'en est rien.

Lemme 2.14. *Soient $X$ un $S$-schéma et $\mathcal{O}$ un $S$-schéma local hensélien. Le diagramme suivant est commutatif*

$$\begin{tikzcd}
c_S(\mathcal{O}, X) \arrow[r, "\sigma^{\mathfrak{h}}_{\mathcal{O},X}"] \arrow[d, "\sigma^{\mathfrak{h}}_{\mathcal{O},X}"] & c_S\left(\mathcal{O}, X^{\mathfrak{h}}\right) \arrow[d, "\sigma^{\mathfrak{h}}_{\mathcal{O},X^{\mathfrak{h}}}"] \\
c_S\left(\mathcal{O}, X^{\mathfrak{h}}\right) \arrow[r, "{[\mathfrak{s}^{\mathfrak{h}}_X]\circ -}"] & c_S\left(\mathcal{O}, (X^{\mathfrak{h}})^{\mathfrak{h}}\right).
\end{tikzcd} \tag{24}$$

*Démonstration.* Fixons une correspondance finie $\alpha \in c_S(\mathcal{O}, X)$ et désignons par $Z$ son support qui est un sous-schéma fermé de $\mathcal{O} \times_S X$ fini et équidimensionnel sur $\mathcal{O}$. Notons $\mathcal{Z}$ le support de la correspondance finie

$$\sigma^{\mathfrak{h}}_{\mathcal{O},X}(\alpha) \in c_S\left(\mathcal{O}, X^{\mathfrak{h}}\right)$$

qui est un sous-schéma fermé de $\mathcal{O} \times_S X^{\mathfrak{h}}$ fini et équidimensionnel sur $\mathcal{O}$. On a alors un diagramme commutatif

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{id}_{\mathcal{Z}}"] \arrow[dr, phantom, "\square"] & \mathcal{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[d] & \\
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W \arrow[r] \arrow[d, "\iota"] \arrow[dr, phantom, "\square"] & Z \arrow[d] \\
\mathcal{O} \times_S X^{\mathfrak{h}} \arrow[r] \arrow[rr, bend right, "\mathrm{id}_{\mathcal{O}\times_S X^{\mathfrak{h}}}"'] & \mathcal{O} \times_S (X^{\mathfrak{h}})^{\mathfrak{h}} \arrow[r] & \mathcal{O} \times_S X^{\mathfrak{h}} \arrow[r] & \mathcal{O} \times_S X.
\end{tikzcd}$$

Le schéma $Z$ est semi-local hensélien donc de la forme

$$Z = \coprod_{\substack{z \text{ point fermé} \\ \text{de } Z}} \operatorname{Spec}\left(\mathcal{O}_{Z,z}\right).$$

Notons $z_1, \ldots, z_n$ ses points fermés et désignons par $x_1, \ldots, x_n$ leur projection sur $X$. Par construction de la section $\sigma^{\mathfrak{h}}_Z$, le diagramme (15) est commutatif et

$$\sigma^{\mathfrak{h}}_{\mathcal{O},X}(\alpha) = \iota_*(\sigma^{\mathfrak{h}}_Z)_*(\alpha).$$

En particulier on voit que les points fermés du schéma semi-local $\mathcal{Z}$ se projettent nécessairement sur $x_1, \ldots, x_n$. Cela entraîne que $\sigma^{\mathfrak{h}}_{\mathcal{Z}}$ est en fait le morphisme d'inclusion de $\mathcal{Z}$ dans $\mathcal{W}$ et donc que

$$\sigma^{\mathfrak{h}}_{\mathcal{O},X^{\mathfrak{h}}}\left[\sigma^{\mathfrak{h}}_{\mathcal{O},X}(\alpha)\right] = \sigma^{\mathfrak{h}}_{\mathcal{O},X}(\alpha)$$

dans $c_S\left(\mathcal{O}, (X^{\mathfrak{h}})^{\mathfrak{h}}\right)$. Ce qui prouve la commutativité du diagramme (24). $\square$

En particulier lorsque l'on se donne une correspondance finie $\alpha \in c_S(X, Y)$, on peut associer à un point $x$ de $X$ et un point $y$ de $Y$ une correspondance finie $\alpha_{x,y}$ du schéma $X^{\mathfrak{h}}_x$ dans le schéma $Y^{\mathfrak{h}}_y$ donnée par

$$\alpha_{x,y} := \sigma^{\mathfrak{h}}_{X^{\mathfrak{h}}_x, Y}(\alpha \circ [\iota^{\mathfrak{h}}_{X,x}])_y$$

On obtient ainsi une décomposition locale pour la topologie de Nisnevich de la correspondance $\alpha$ de la forme

$$\alpha \circ [\iota^{\mathfrak{h}}_{X,x}] = \sum_{y \in Y} [\iota^{\mathfrak{h}}_{Y,y}] \circ \alpha_{x,y}.$$

*Notation* 2.15. Pour un morphisme de $S$-schéma $g : X \to Y$ et un point $x$ nous notons $g^{\mathfrak{h}}_x$ le morphisme de schémas locaux henséliens déduit de $g$

$$\begin{tikzcd} X \arrow[r, "g"] & Y \\ X^{\mathfrak{h}}_x \arrow[u, "\iota^{\mathfrak{h}}_{X,x}"] \arrow[r, "g^{\mathfrak{h}}_x"] & Y^{\mathfrak{h}}_{g(x)} \arrow[u, "\iota^{\mathfrak{h}}_{Y,g(x)}"'] \end{tikzcd}.$$

La proposition suivante explicite le comportement par composition et itération de cette décomposition ainsi que la nature de la décomposition obtenue dans le cas d'un morphisme de schémas.

PROPOSITION 2.16. *Soient $X, Y$ et $Z$ des $S$-schémas.*

*(a) Étant données des correspondances finies $\alpha \in c_S(X, Y)$ et $\beta \in c_S(Y, Z)$, on a pour tout point $x \in X$ et $z \in Z$ l'égalité*

$$(\beta \circ \alpha)_{x,z} = \sum_{y \in Y} \beta_{y,z} \circ \alpha_{x,y}. \tag{25}$$

*(b) Étant donné un morphisme $g : X \to Y$ de $S$-schémas on a*

$$[g]_{x,y} = \begin{cases} [g^{\mathfrak{h}}_x] & \textit{si } y = g(x) \\ 0 & \textit{sinon.} \end{cases}$$

*(c) Étant donnée une correspondance finie $\alpha \in c_S(X,Y)$, on a l'égalité dans $c_S\left(X_x^{\mathfrak{h}}, (Y_y^{\mathfrak{h}})_z^{\mathfrak{h}}\right)$*

$$(\alpha_{x,y})_{x,z} = \begin{cases} \alpha_{x,y} & \text{si } z = y \\ 0 & \text{sinon} \end{cases}$$

*pour tout point $x$ de $X$ et tout point $z$ du schéma local hensélien $Y_y^{\mathfrak{h}}$.*

*Démonstration.* Le second point est une conséquence immédiate du corollaire 2.12. La formule de composition (25), se déduit des égalités

$$\begin{aligned}(\beta \circ \alpha)_{x,z} &= \sigma^{\mathfrak{h}}_{X_x^{\mathfrak{h}},Z}\Big(\beta \circ \alpha \circ [\mathfrak{i}^{\mathfrak{h}}_{X,x}]\Big)_z = \sigma^{\mathfrak{h}}_{X_x^{\mathfrak{h}},Z}\left(\sum_{y\in Y} \beta \circ [\mathfrak{i}^{\mathfrak{h}}_{Y,y}] \circ \alpha_{x,y}\right)_z \\ &= \sum_{y\in Y} \sigma^{\mathfrak{h}}_{Y_y^{\mathfrak{h}},Z}\big(\beta \circ [\mathfrak{i}^{\mathfrak{h}}_{Y,y}]\big)_z \circ \alpha_{x,y} \\ &= \sum_{y\in Y} \beta_{y,z} \circ \alpha_{x,y}\end{aligned}$$

dans lesquelles nous avons utilisé la commutativité du diagramme (22). La dernière assertion est quant à elle un corollaire du lemme 2.14. □

Nous donnons maintenant le lien entre la décomposition locale du produit tensoriel de deux correspondances finies et le produit tensoriel des décompositions locales. Étant donnés deux $S$-schémas $X$ et $Y$, la propriété universelle (8) nous donne un morphisme

$$\begin{tikzcd} (X \times_S Y)^{\mathfrak{h}} \arrow[r, dotted, "\mathfrak{m}^{\mathfrak{h}}_{X,Y}"] \arrow[dr, "\mathfrak{i}^{\mathfrak{h}}_{X\times_S Y}"'] & X^{\mathfrak{h}} \times_S Y^{\mathfrak{h}} \arrow[d, "(\mathfrak{i}^{\mathfrak{h}}_X)\times_S(\mathfrak{i}^{\mathfrak{h}}_Y)"] \\ & X \times_S Y. \end{tikzcd}$$

Ces morphismes font du foncteur $(-)^{\mathfrak{h}}$ un foncteur quasi-monoïdal symétrique. Lorsque l'on se fixe un point $e$ du produit $X \times_S Y$ se projetant sur $x$ et $y$ on a une factorisation

$$\begin{tikzcd} (X \times_S Y)^{\mathfrak{h}}_e \arrow[r, "\mathfrak{m}^{\mathfrak{h}}_{X,Y,e}"] \arrow[d] & (X^{\mathfrak{h}}_x) \times_S (Y^{\mathfrak{h}}_y) \arrow[d] \\ (X \times_S Y)^{\mathfrak{h}} \arrow[r, "\mathfrak{m}^{\mathfrak{h}}_{X,Y}"] & X \times_S Y. \end{tikzcd}$$

Le comportement des décompositions locales par rapport au produit tensoriel des correspondances finies est donné par le résultat suivant.

Proposition 2.17. *Soient $X, Y, X', Y'$ des $S$-schémas, $\alpha \in c_S(X, X')$ et $\beta \in c_S(Y, Y')$ des correspondances finies. Pour tout point $e$ de $X \times_S Y$ et tout point $x'$ de $X'$ et $y'$ de $Y'$, on a l'égalité*

$$\Big(\alpha_{x,x'} \otimes \beta_{y,y'}\Big) \circ \Big[\mathfrak{m}^{\mathfrak{h}}_{X,Y,e}\Big] = \sum_{\substack{e' \text{ point de } X' \times_S Y' \\ \text{se projetant sur } x' \text{ et } y'}} \Big[\mathfrak{m}^{\mathfrak{h}}_{X',Y',e'}\Big] \circ \Big(\alpha \otimes \beta\Big)_{e,e'}$$

*dans* $c_S\Big((X \times_S Y)^{\mathfrak{h}}_e, (X')^{\mathfrak{h}}_{x'} \times_S (Y')^{\mathfrak{h}}_{y'}\Big)$.

*Démonstration.* Pour simplifier convenons de noter $E$ le produit $X \times_S Y$, $E'$ le produit $X' \times_S Y'$ et de désigner par $E'_{x'y'}$ l'ensemble des points de $E'$ se projetant sur $x'$ et $y'$. Le lemme sera démontré lorsque nous aurons vérifié la formule

$$\begin{aligned}[][\mathfrak{m}^{\mathfrak{h}}_{X',Y'}] \circ \sigma^{\mathfrak{h}}_{E^{\mathfrak{h}}_e,E'}\Big(\big(\alpha \otimes \beta\big) \circ [\mathfrak{l}^{\mathfrak{h}}_{E,e}]\Big) \\ = \Big[\sigma^{\mathfrak{h}}_{X^{\mathfrak{h}}_x,X'}\big(\alpha \circ [\mathfrak{l}^{\mathfrak{h}}_{X,x}]\big) \otimes \sigma^{\mathfrak{h}}_{Y^{\mathfrak{h}},Y'}\big(\beta \circ [\mathfrak{l}^{\mathfrak{h}}_{Y,y}]\big)\Big] \circ [\mathfrak{m}^{\mathfrak{h}}_{X,Y,e}].\end{aligned} \tag{26}$$

En effet on a d'une part

$$[\mathfrak{m}^{\mathfrak{h}}_{X',Y'}] \circ \sigma^{\mathfrak{h}}_{E^{\mathfrak{h}}_e,E'}\Big(\big(\alpha \otimes \beta\big) \circ [\mathfrak{l}^{\mathfrak{h}}_{E,e}]\Big) = \sum_{\substack{x' \in X' \\ y' \in Y'}} \sum_{e' \in E'_{x'y'}} \Big[\mathfrak{m}^{\mathfrak{h}}_{X',Y',e'}\Big] \circ \Big(\alpha \otimes \beta\Big)_{e,e'}$$

et d'autre part

$$\begin{aligned}\Big[\sigma^{\mathfrak{h}}_{X^{\mathfrak{h}}_x,X'}\big(\alpha \circ [\mathfrak{l}^{\mathfrak{h}}_{X,x}]\big) \otimes \sigma^{\mathfrak{h}}_{Y^{\mathfrak{h}},Y'}\big(\beta \circ [\mathfrak{l}^{\mathfrak{h}}_{Y,y}]\big)\Big] \circ [\mathfrak{m}^{\mathfrak{h}}_{X,Y,e}] \\ = \sum_{\substack{x' \in X' \\ y' \in Y'}} \Big(\alpha_{x,x'} \otimes \beta_{y,y'}\Big) \circ \Big[\mathfrak{m}^{\mathfrak{h}}_{X,Y,e}\Big].\end{aligned}$$

Il suffit alors d'identifier facteur direct par facteur direct pour voir que l'égalité (26) n'est qu'une reformulation du résultat cherché.

Fixons un sous-schéma fermé $Z_\alpha$ de $X^{\mathfrak{h}}_x \times_S X'$ fini et équidimensionnel sur le schéma local hensélien $X^{\mathfrak{h}}_x$ de sorte qu'en notant $\iota_{Z_\alpha}$ l'immersion fermée associée à ce dernier on ait

$$\alpha \circ [\mathfrak{l}^{\mathfrak{h}}_{X,x}] = (\iota_{Z_\alpha})_* \overline{\alpha}$$

pour un unique élément $\overline{\alpha}$ de $c_{\text{equi}}(Z_\alpha / X^{\mathfrak{h}}_x, 0)$.

Choisissons de même un sous-schéma fermé $Z_\beta$ de $Y^{\mathfrak{h}}_y \times_S Y'$ équidimensionnel fini sur $Y^{\mathfrak{h}}_y$ de sorte qu'en notant $\iota_{Z_\beta}$ l'immersion fermée associée à ce dernier on ait

$$\beta \circ [\mathfrak{l}^{\mathfrak{h}}_{Y,y}] = (\iota_{Z_\beta})_* \overline{\beta}$$

pour un unique élément $\overline{\beta}$ de $c_{\text{equi}}(Z_\beta / X^{\mathfrak{h}}_x, 0)$.

Notons comme précédemment $W_\alpha$ et $W_\beta$ les schémas définis par les carrés cartésiens

$$\begin{tikzcd}
W_\alpha \arrow[r] \arrow[d, "\iota_{W_\alpha}"'] \arrow[dr, phantom, "\square"] & Z_\alpha \arrow[d, "\iota_{Z_\alpha}"'] \\
X_x^\natural \times_S (X')^\natural \arrow[r] & X_x^\natural \times_S X'
\end{tikzcd}
\qquad
\begin{tikzcd}
W_\beta \arrow[r] \arrow[d, "\iota_{W_\beta}"'] \arrow[dr, phantom, "\square"] & Z_\beta \arrow[d, "\iota_{Z_\beta}"'] \\
Y_y^\natural \times_S (Y')^\natural \arrow[r] & Y_y^\natural \times_S Y'
\end{tikzcd}$$

et posons $Z = Z_\alpha \times_S Z_\beta$ et $W = W_\alpha \times_S W_\beta$. Nous avons alors un carré cartésien

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "\iota_W"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "\iota_Z"'] \\
X_x^\natural \times_S Y_y^\natural \times_S (X')^\natural \times_S (Y')^\natural \arrow[r] & X_x^\natural \times_S Y_y^\natural \times_S E'.
\end{tikzcd}$$

Considérons par ailleurs les schémas définis par les carrés cartésiens

$$\begin{tikzcd}
 & & & W \arrow[d] \\
\mathcal{W}' \arrow[r, "\theta"] \arrow[d, "\iota_{\mathcal{W}'}"] \arrow[dr, phantom, "\square"] & \mathcal{W} \arrow[urr, bend left] \arrow[r] \arrow[d, "\iota_{\mathcal{W}}"] \arrow[dr, phantom, "\square"] & \mathcal{Z} \arrow[r] \arrow[d, "\iota_{\mathcal{Z}}"] \arrow[dr, phantom, "\square"] \arrow[ur, phantom, "\square"] & Z \arrow[d, "\iota_Z"] \\
E_e^\natural \times_S (E')^\natural \arrow[r] & E_e^\natural \times_S (X')^\natural \times_S (Y')^\natural \arrow[r] & E_e^\natural \times_S E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_x^\natural \times_S Y_y^\natural \times_S E' \arrow[d] \\
 & & E_e^\natural \arrow[r, "\mathfrak{m}^\natural_{X,Y,e}"'] & X_x^\natural \times_S Y_y^\natural .
\end{tikzcd}$$

Notons $p$ et $q$ les projections de $X_x^\natural \times_S Y_y^\natural$ sur le premier et le second facteur. Par définition du produit tensoriel des correspondances, on a

$$\begin{aligned}
\big(\alpha \circ [\mathfrak{l}^\natural_{X,x}]\big) \otimes \big(\beta \circ [\mathfrak{l}^\natural_{Y,y}]\big) &= \Big[p^{\circledast}\big(\alpha \circ [\mathfrak{l}^\natural_{X,x}]\big)\Big] \times_{X_x^\natural \times_S Y_y^\natural} \Big[q^{\circledast}\big(\beta \circ [\mathfrak{l}^\natural_{Y,y}]\big)\Big] \\
&= \Big[p^{\circledast}(\iota_{Z_\alpha})_*\overline{\alpha}\Big] \times_{X_x^\natural \times_S Y_y^\natural} \Big[q^{\circledast}(\iota_{Z_\beta})_*\overline{\beta}\Big] \\
&= (\iota_{\mathcal{Z}})_*\Big[\big(p^{\circledast}\overline{\alpha}\big) \times_{X_x^\natural \times_S Y_y^\natural} \big(q^{\circledast}\overline{\beta}\big)\Big].
\end{aligned}$$

Convenons de noter $\overline{\gamma}$ le cycle algébrique

$$\overline{\gamma} = \big(p^{\circledast}\overline{\alpha}\big) \times_{X_x^\natural \times_S Y_y^\natural} \big(q^{\circledast}\overline{\beta}\big)$$

appartenant à $c_{\mathrm{equi}}(Z/X^{\mathfrak{h}}_x \times_S Y^{\mathfrak{h}}_y, 0)$. Cela nous donne

$$\begin{aligned}[\mathfrak{m}^{\mathfrak{h}}_{X',Y'}]\circ \sigma^{\mathfrak{h}}_{E^{\mathfrak{h}}_e,E'}\Big(\big(\alpha\otimes\beta\big)\circ[\mathfrak{l}^{\mathfrak{h}}_{E,e}]\Big) &= (\mathrm{id}_{E^{\mathfrak{h}}_e}\times_S \mathfrak{m}^{\mathfrak{h}}_{X',Y'})_*\Big[(\iota_{\mathcal{W}'})_*(\sigma^{\mathfrak{h}}_{\mathcal{Z}})_*\big(\mathfrak{m}^{\mathfrak{h}\circledast}_{X,Y,e}\overline{\gamma}\big)\Big]\\ &= (\iota_{\mathcal{W}})_*\theta_*(\sigma^{\mathfrak{h}}_{\mathcal{Z}})_*\Big[\mathfrak{m}^{\mathfrak{h}\circledast}_{X,Y,e}\overline{\gamma}\Big]\\ &= \mathfrak{m}^{\mathfrak{h}\circledast}_{X,Y,e}\Big[(\iota_W)_*(\sigma^{\mathfrak{h}}_{Z_\alpha}\times_S\sigma^{\mathfrak{h}}_{Z_\beta})_*\overline{\gamma}\Big].\end{aligned}$$

On a par ailleurs l'égalité

$$\begin{aligned}(\iota_W)_*(\sigma^{\mathfrak{h}}_{Z_\alpha}\times_S\sigma^{\mathfrak{h}}_{Z_\beta})_*\overline{\gamma} &= \Big[p^{\circledast}\big((\iota_{W_\alpha})_*(\sigma^{\mathfrak{h}}_{Z_\alpha})_*\overline{\alpha}\big)\Big]\times_{X^{\mathfrak{h}}_x\times_S Y^{\mathfrak{h}}_y}\Big[q^{\circledast}\big((\iota_{W_\beta})_*(\sigma^{\mathfrak{h}}_{Z_\beta})_*\overline{\beta}\big)\Big]\\ &= \Big[\sigma^{\mathfrak{h}}_{X^{\mathfrak{h}}_x,X'}\big(\alpha\circ[\mathfrak{l}^{\mathfrak{h}}_{X,x}]\big)\Big]\otimes\Big[\sigma^{\mathfrak{h}}_{Y^{\mathfrak{h}}_y,Y'}\big(\beta\circ[\mathfrak{l}^{\mathfrak{h}}_{Y,y}]\big)\Big]\end{aligned}$$

On obtient ainsi

$$\begin{aligned}&[\mathfrak{m}^{\mathfrak{h}}_{X',Y'}]\circ\sigma^{\mathfrak{h}}_{E^{\mathfrak{h}}_e,E'}\Big(\big(\alpha\otimes\beta\big)\circ[\mathfrak{l}^{\mathfrak{h}}_{E,e}]\Big)\\ &\qquad = \mathfrak{m}^{\mathfrak{h}\circledast}_{X,Y,e}\Big(\sigma^{\mathfrak{h}}_{X^{\mathfrak{h}}_x,X'}\big(\alpha\circ[\mathfrak{l}^{\mathfrak{h}}_{X,x}]\big)\otimes\sigma^{\mathfrak{h}}_{Y^{\mathfrak{h}}_y,Y'}\big(\beta\circ[\mathfrak{l}^{\mathfrak{h}}_{Y,y}]\big)\Big)\\ &\qquad = \Big[\sigma^{\mathfrak{h}}_{X^{\mathfrak{h}}_x,X'}\big(\alpha\circ[\mathfrak{l}^{\mathfrak{h}}_{X,x}]\big)\otimes\sigma^{\mathfrak{h}}_{Y^{\mathfrak{h}}_y,Y'}\big(\beta\circ[\mathfrak{l}^{\mathfrak{h}}_{Y,y}]\big)\Big]\circ[\mathfrak{m}^{\mathfrak{h}}_{X,Y,e}]\end{aligned}$$

ce qui prouve la formule (26). □

### 2.3.2 Décomposition locale raffinée

Supposons donnés une présentation d'un $S$-schéma local hensélien $\mathcal{O}$ ainsi qu'un élément $\alpha$ de $c_S\{\mathcal{O},X)$. On peut alors constater en utilisant le cas $n=0$ de la proposition 2.7 que la correspondance $\alpha_x$ précédemment construite provient en fait d'un unique élément $\{\alpha\}_x$ de $c_S\left\{\mathcal{O},X^{\mathfrak{h}}_x\right)$ vérifiant

$$\alpha = \sum_x [\mathfrak{l}^{\mathfrak{h}}_{X,x}]\circ\{\alpha\}_x$$

dans $c_S\left\{\mathcal{O},X\right)$. Désignons par $\{\mathfrak{l}\}^{\mathfrak{h}}_{X,x}$ l'élément de $c_S\left\{X^{\mathfrak{h}}_x,X\right)$ induit par les morphismes $U\to X$ pour $U$ parcourant $\mathcal{V}^h_{X,x}$. Une correspondance finie $\alpha\in c_S(X,Y)$ possède alors une décomposition locale raffinée de la forme

$$\alpha\circ\{\mathfrak{l}\}^{\mathfrak{h}}_{X,x} = \sum_{y\in Y}[\mathfrak{l}^{\mathfrak{h}}_{Y,y}]\circ\{\alpha\}_{x,y} \qquad \{\alpha\}_{x,y}\in c_S\left\{X^{\mathfrak{h}}_x,Y^{\mathfrak{h}}_y\right)$$

$\{\alpha\}_{x,y}$ ayant pour image $\alpha_{x,y}$ dans $c_S\left(X^{\mathfrak{h}}_x,Y^{\mathfrak{h}}_y\right)$.

*Notation* 2.18. Soient $g: X\to Y$ un morphisme et $x$ un point de $X$. Pour tout $V\in\mathcal{V}^h_{Y,g(x)}$ le morphisme $g$ définit un élément dans $c_S\left\{X^{\mathfrak{h}}_x,V\right)$ et la collection de ces éléments définit un élément

$$\{g\}^{\mathfrak{h}}_x\in c_S\left\{X^{\mathfrak{h}}_x,Y^{\mathfrak{h}}_y\right\}.$$

Les propriétés de ces décompositions locales raffinées se déduisent directement du lemme 2.6 et de la proposition 2.16.

Proposition 2.19. *Soient $X, Y$ et $Z$ des $S$-schémas.*

(a) *Étant données des correspondances finies $\alpha \in c_S(X,Y)$ et $\beta \in c_S(Y,Z)$, on a pour tout point $x \in X$ et $z \in Z$ l'égalité*

$$\{\beta \circ \alpha\}_{x,z} = \sum_{y \in Y} \beta_{y,z} \circ \{\alpha\}_{x,y}. \tag{27}$$

(b) *Étant donné un morphisme $g : X \to Y$ de $S$-schémas on a*

$$\{[g]\}_{x,y} = \begin{cases} \{g\}_x^{\mathfrak{h}} & si\ y = g(x) \\ 0 & sinon \end{cases}$$

*dans $c_S\left\{X_x^{\mathfrak{h}}, Y_y^{\mathfrak{h}}\right\}$.*

(c) *Étant donnée une correspondance finie $\alpha \in c_S(X,Y)$, on a l'égalité dans $c_S\left\{X_x^{\mathfrak{h}}, (Y_y^{\mathfrak{h}})_z^{\mathfrak{h}}\right)$*

$$\{\{\alpha\}_{x,y}\}_{x,z} = \begin{cases} \{\alpha\}_{x,y} & si\ z = y \\ 0 & sinon \end{cases}$$

*pour tout point $x$ de $X$ et tout point $z$ du schéma local hensélien $Y_y^{\mathfrak{h}}$.*

Soient $X, Y$ deux $S$-schémas et $e$ un point de $X \times_S Y$ se projetant sur $x$ et $y$. L'identité de $X \times_S Y$ induit pour tout $U \in \mathcal{V}_{X,x}^h, V \in \mathcal{V}_{Y,y}^h$ un élément dans $c_S\left\{(X \times_S Y)_e^{\mathfrak{h}}, U \times_S V\right)$ et la collection de ces éléments donne un élément de

$$\{\mathfrak{m}\}_{X,Y,e}^{\mathfrak{h}} \in c_S\left\{(X \times_S Y)_e^{\mathfrak{h}}, X_x^{\mathfrak{h}} \times_S Y_y^{\mathfrak{h}}\right\}$$

ayant même image que $[\mathfrak{m}_{X,Y,e}^{\mathfrak{h}}]$ dans $c_S\left((X \times_S Y)_e^{\mathfrak{h}}, X_x^{\mathfrak{h}} \times_S Y_y^{\mathfrak{h}}\right\}$.

Proposition 2.20. *Soient $X, Y, X', Y'$ des $S$-schémas, $\alpha \in c_S(X,X')$ et $\beta \in c_S(Y,Y')$ des correspondances finies. Pour tout point $e$ de $X \times_S Y$ et tout point $x'$ de $X'$ et $y'$ de $Y'$, on a l'égalité*

$$\Big(\{\alpha\}_{x,x'} \otimes \{\beta\}_{y,y'}\Big) \circ \{\mathfrak{m}\}_{X,Y,e}^{\mathfrak{h}} = \sum_{\substack{e' \text{ point de } X' \times_S Y' \\ \text{se projetant sur } x' \text{ et } y'}} \Big[\mathfrak{m}_{X',Y',e'}^{\mathfrak{h}}\Big] \circ \Big\{\alpha \otimes \beta\Big\}_{e,e'}$$

*dans $c_S\Big\{(X \times_S Y)_e^{\mathfrak{h}}, (X')_{x'}^{\mathfrak{h}} \times_S (Y')_{y'}^{\mathfrak{h}}\Big)$.*

## 2.4 Cas de la topologie étale

Les démonstrations données dans la section précédente s'adaptent parfaitement au cas de la topologie étale, il suffit pratiquement pour cela de remplacer les anneaux locaux henséliens par les anneaux locaux strictement henséliens. Dans cette sous-section nous avons rassemblé les énoncés des résultats obtenus pour la topologie étale tout en précisant les modifications mineures à effectuer.

Nous fixons pour tout point $s$ de $S$ une clôture algébrique $\overline{\kappa}(s)$ de $\kappa(s)$ et nous désignons par $\overline{s}$ le point géométrique de $S$ défini par la clôture algébrique $\overline{\kappa}(s)$.

Définition 2.21. Un bon point géométrique $\overline{x}$ est la donnée d'un point $x \in X$ et d'un plongement $\kappa(x) \to \overline{\kappa}(s)$ le point $s$ étant l'image de $x$ dans $S$.

Lemme 2.22. *Les bons points géométriques possèdent les proprétés suivantes.*

*(a) L'image par un morphisme de $S$-schémas d'un bon point géométrique est un bon point géométrique.*

*(b) Les bons points géométriques forment un ensemble conservatif de points* [9] *pour la topologie étale sur* $\mathrm{Var}_S$.

*Démonstration.* La première assertion est immédiate, quant à la seconde d'après la remarque 3.13.b de l'exposé VIII de [2], il suffit de voir que les points d'un $S$-schéma de type fini $X$ fermés dans leur fibre sont le lieu d'un point bon géométrique. Or pour un point $x$ de $X$ fermé dans sa fibre, l'extension $\kappa(x)/\kappa(s)$ est finie et il existe bien un morphisme de corps de $\kappa(x)$ dans la clôture algébrique □

Nous appellerons bon $S$-schéma local strictement hensélien la donnée d'un $S$-schéma strictement hénsélien $\mathcal{O}$ et d'un isomorphisme $\omega : \kappa \to \overline{\kappa}(s)$ de $\kappa(s)$-extensions où $k$ est le corps résiduel de $\mathcal{O}$ et $s$ est l'image dans $S$ du point fermé de $\mathcal{O}$. Les hensélisés stricts d'un $S$-schéma en un bon point géométrique sont canoniquement des bons $S$-schémas strictement henséliens. Étant donné un bon $S$-schéma strictement hensélien $(\mathcal{O}, \omega)$, on remarquera pour l'adaptation des démonstrations précédentes au cas de la topologie étale que les $\mathcal{O}$-schémas finis sont canoniquement des réunions disjointes de bons $S$-schémas locaux strictement henséliens.

A un $S$-schéma $X$, on peut associer fonctoriellement le $S$-schéma $X^{\mathfrak{sh}}$

$$X^{\mathfrak{sh}} := \coprod_{\substack{\overline{x} \text{ bon point} \\ \text{géométrique}}} X^{\mathfrak{sh}}_{\overline{x}}.$$

Ce dernier est la réunion disjointe, sur les bons points géométriques de $X$, des schémas locaux strictement henséliens $X^{\mathfrak{sh}}_{\overline{x}}$ spectre de l'anneau local strictement hensélien $\mathcal{O}^{sh}_{X,\overline{x}}$ dont on notera le point fermé abusivement par $\overline{x}$.

[9] Pour un schéma qui n'est pas de type fini sur $S$ on prendra garde que l'ensemble des bons points géométriques peut être vide.

Pour tout bon point géométrique $\overline{x}$ de $X$ on dispose du morphisme canonique

$$\begin{tikzcd} X^{\mathfrak{sh}}_{\overline{x}} \arrow[r] \arrow[rr, bend left, "\mathfrak{l}^{\mathfrak{sh}}_{X,\overline{x}}"] & \mathrm{Spec}\,(\mathcal{O}_{X,x}) \arrow[r] & X \end{tikzcd}$$

nous donnant un morphisme de schémas $\mathfrak{l}^{\mathfrak{sh}}_X : X^{\mathfrak{sh}} \to X$. Comme dans le cas hensélien, le morphisme

$$\mathfrak{l}^{\mathfrak{sh}}_{X^{\mathfrak{sh}}} : (X^{\mathfrak{sh}})^{\mathfrak{sh}} \to X^{\mathfrak{sh}}$$

n'est pas un isomorphisme en général mais admet cependant une section canonique $\mathfrak{s}^{\mathfrak{sh}}_X$ identifiant $X^{\mathfrak{sh}}$ à un sous-schéma fermé de $(X^{\mathfrak{sh}})^{\mathfrak{sh}}$ et provenant du fait que l'anneau local strictement hensélien de $X^{\mathfrak{sh}}_{\overline{x}}$ en son point fermé $\overline{x}$ est canoniquement isomorphe à $\mathcal{O}^{sh}_{X,\overline{x}}$ :

$$\begin{tikzcd} X^{\mathfrak{sh}} \arrow[r, "\mathfrak{s}^{\mathfrak{sh}}_X"] \arrow[rr, bend right, equal, "\mathrm{id}_{X^{\mathfrak{sh}}}"'] & (X^{\mathfrak{sh}})^{\mathfrak{sh}} \arrow[r, "\mathfrak{l}^{\mathfrak{sh}}_{X^{\mathfrak{sh}}}"] & X^{\mathfrak{sh}} \arrow[r, "\mathfrak{l}^{\mathfrak{sh}}_X"] & X \end{tikzcd}$$

Autrement dit dans $(X^{\mathfrak{sh}})^{\mathfrak{sh}}$ apparaissent des facteurs supplémentaires correspondants aux bons points géométriques distincts des points fermés des $X^{\mathfrak{sh}}_{\overline{x}}$.

Dans le cas étale, le complexe considéré est le complexe de Čech associé au $X$-schéma $X^{\mathfrak{sh}}$

$$\check{C}_{X^{\mathfrak{sh}}/X}(\mathcal{O}) : \cdots \to \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^2_X\right](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}[X^{\mathfrak{sh}}](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}[X](\mathcal{O}) \to 0.$$

On a alors l'analogue de la proposition 2.1.

Proposition 2.23. *Soient $X$ un $S$-schéma et $\mathcal{O}$ un bon $S$-schéma local strictement hensélien. Il existe des morphismes canoniques*

$$\sigma^{\mathfrak{sh}}_{\mathcal{O},X,n} : \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^n_X\right](\mathcal{O}) \to \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^{n+1}_X\right](\mathcal{O}) \qquad n \geq 0$$

*satisfaisant aux deux propriétés suivantes.*

*(a) (Homotopie) On a pour tout $n$ les relations*

$$d_{n+1} \circ \sigma^{\mathfrak{sh}}_{\mathcal{O},X,n} + \sigma^{\mathfrak{sh}}_{\mathcal{O},X,n-1} \circ d_n = \mathrm{id}.$$

*(b) (Fonctorialité) Étant donnés un bon $S$-schéma local strictement hensélien $\mathcal{O}'$ et une correspondance finie $\alpha \in c_S(\mathcal{O}',\mathcal{O})$ on a un carré commutatif*

$$\begin{tikzcd} \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^n_X\right](\mathcal{O}') \arrow[r, "\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^n_X\right](\alpha)"] \arrow[d, "\sigma^{\mathfrak{sh}}_{\mathcal{O}',X,n}"] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^n_X\right](\mathcal{O}) \arrow[d, "\sigma^{\mathfrak{sh}}_{\mathcal{O},X,n}"] \\ \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^{n+1}_X\right](\mathcal{O}') \arrow[r, "\mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^{n+1}_X\right](\alpha)"] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^{n+1}_X\right](\mathcal{O}). \end{tikzcd}$$

De même on dispose également de l'analogue de la proposition 2.7.

PROPOSITION 2.24. *Soient $X$ un $S$-schéma et une présentation $\mathcal{V}$ d'un bon $S$-schéma local strictement hensélien $\mathcal{O}$. Il existe d'uniques morphismes*

$$\{\sigma\}^{\mathfrak{sh}}_{\mathcal{O},X,n} : \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^n_X\right]\{\mathcal{O}\} \to \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^{n+1}_X\right]\{\mathcal{O}\} \qquad n \geq 0$$

*satisfaisant aux deux propriétés suivantes.*

*(a) (Homotopie) On a pour tout $n$ les relations*

$$d_{n+1} \circ \{\sigma\}^{\mathfrak{sh}}_{\mathcal{O},X,n} + \{\sigma\}^{\mathfrak{sh}}_{\mathcal{O},X,n-1} \circ d_n = \mathrm{id}. \tag{28}$$

*(b) (Compatibilité) Le carré suivant est commutatif*

$$\begin{tikzcd} \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^n_X\right]\{\mathcal{O}\} \arrow[r, "\{\sigma\}^{\mathfrak{sh}}_{\mathcal{O},X,n}"] \arrow[d] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^{n+1}_X\right]\{\mathcal{O}\} \arrow[d] \\ \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^n_X\right](\mathcal{O}) \arrow[r, "\sigma^{\mathfrak{sh}}_{\mathcal{O},X,n}"] & \mathbb{Z}_{\mathrm{tr}}\left[(X^{\mathfrak{sh}})^{n+1}_X\right](\mathcal{O}). \end{tikzcd} \tag{29}$$

Sachant d'après le lemme 2.22 que les bons points géométriques fournissent une famille conservative de points pour la topologie étale, on obtient le corollaire suivant.

COROLLAIRE 2.25. *Soit $X$ un $S$-schéma de type fini. La restriction à la catégorie* $\mathrm{VarCor}_S$ *du complexe de faisceaux étales avec transferts $\check{C}_{X^{\mathfrak{sh}}/X}$ est universellement exact au sens de Grayson [12].*

Lorsque l'on se donne une correspondance finie $\alpha \in c_S(X,Y)$ et des bons points géométriques $\overline{x}, \overline{y}$ de $X$ et $Y$, le cas $n = 0$ de la proposition 2.23 fournit une décomposition locale pour la topologie étale de la correspondance $\alpha$ de la forme

$$\alpha \circ [\mathfrak{l}^{\mathfrak{sh}}_{X,\overline{x}}] = \sum_{\substack{\overline{y} \text{ bon point} \\ \text{géométrique}}} [\mathfrak{l}^{\mathfrak{sh}}_{Y,\overline{y}}] \circ \alpha_{\overline{x},\overline{y}}. \tag{30}$$

En utilisant le cas $n = 0$ de la proposition 2.24 on voit finalement que la décomposition locale précédente se raffine en

$$\alpha \circ \{\mathfrak{l}\}^{\mathfrak{sh}}_{X,\overline{x}} = \sum_{\substack{\overline{y} \text{ bon point} \\ \text{géométrique}}} [\mathfrak{l}^{\mathfrak{sh}}_{Y,\overline{y}}] \circ \{\alpha\}_{\overline{x},\overline{y}}$$

où $\{\mathfrak{l}\}^{\mathfrak{sh}}_{X,\overline{x}} \in c_S\{X^{\mathfrak{sh}}_{\overline{x}}, X)$ est induit par les morphismes $U \to X$ pour $U \in \mathcal{V}^{sh}_{X,\overline{x}}$. Soient $g : X \to Y$ un morphisme et $x$ un point de $X$. Pour tout $V \in \mathcal{V}^{sh}_{Y,g\circ\overline{x}}$ le morphisme $g$ définit un élément dans $c_S\{X^{\mathfrak{sh}}_{\overline{x}}, V)$ et la collection de ces éléments définit un élément

$$\{g\}^{\mathfrak{sh}}_{\overline{x}} \in c_S\{X^{\mathfrak{sh}}_{\overline{x}}, Y^{\mathfrak{sh}}_{\overline{y}}\}.$$

Proposition 2.26. *Soient $X, Y$ et $Z$ des $S$-schémas.*

(a) *Étant données des correspondances finies $\alpha \in c_S(X,Y)$ et $\beta \in c_S(Y,Z)$, on a pour tout point bon point géométrique $\overline{x}$ de $X$ et tout bon point géométrique $\overline{z}$ de $Z$ l'égalité*

$$\{\beta \circ \alpha\}_{\overline{x},\overline{z}} = \sum_{\substack{\overline{y} \text{ bon point} \\ \text{géométrique}}} \beta_{\overline{y},\overline{z}} \circ \{\alpha\}_{\overline{x},\overline{y}}.$$

(b) *Étant donné un morphisme $g : X \to Y$ de $S$-schémas on a*

$$\{[g]\}_{\overline{x},\overline{y}} = \begin{cases} \{g\}_{\overline{x}}^{\mathfrak{sh}} & si\ \overline{y} = g \circ \overline{x} \\ 0 & sinon \end{cases}$$

*dans $c_S\{X_{\overline{x}}^{\mathfrak{sh}}, Y_{\overline{y}}^{\mathfrak{sh}}\}$.*

(c) *Étant donnée une correspondance finie $\alpha \in c_S(X,Y)$, on a l'égalité dans $c_S\{X_{\overline{x}}^{\mathfrak{sh}}, (Y_{\overline{y}}^{\mathfrak{sh}})_{\overline{z}}^{\mathfrak{sh}})$*

$$\{\{\alpha\}_{\overline{x},\overline{y}}\}_{\overline{x},\overline{z}} = \begin{cases} \{\alpha\}_{\overline{x},\overline{y}} & si\ \overline{z} = \overline{y} \\ 0 & sinon \end{cases}$$

*pour tout bon point géométrique $\overline{x}$ de $X$ et tout bon point géométrique $\overline{z}$ du schéma local strictement hensélien $Y_{\overline{y}}^{\mathfrak{sh}}$.*

Comme dans le cas de la topologie Nisnevich, ces décompositions locales se comportent bien par produit tensoriel. Étant donnés deux $S$-schémas $X$ et $Y$ ainsi qu'un bon point géométrique $\overline{e}$ de $X \times_S Y$ se projetant sur $\overline{x}$ et $\overline{y}$ nous avons un diagramme naturel

$$\begin{array}{ccc} (X \times_S Y)_{\overline{e}}^{\mathfrak{sh}} & \xrightarrow{\mathfrak{m}_{X,Y,\overline{e}}^{\mathfrak{sh}}} & (X_{\overline{x}}^{\mathfrak{sh}}) \times_S (Y_{\overline{y}}^{\mathfrak{sh}}) \\ \downarrow & & \downarrow \\ (X \times_S Y)^{\mathfrak{sh}} & \xrightarrow{\mathfrak{m}_{X,Y}^{\mathfrak{sh}}} & X \times_S Y. \end{array}$$

L'identité de $X \times_S Y$ induit par ailleurs un élément dans $c_S\{(X \times_S Y)_{\overline{e}}^{\mathfrak{sh}}, U \times_S V)$ pour tout $U \in \mathcal{V}_{X,\overline{x}}^{sh}, V \in \mathcal{V}_{Y,\overline{y}}^{sh}$. La collection de ces éléments donne un élément

$$\{\mathfrak{m}\}_{X,Y,\overline{e}}^{\mathfrak{sh}} \in c_S\{(X \times_S Y)_{\overline{e}}^{\mathfrak{sh}}, X_{\overline{x}}^{\mathfrak{sh}} \times_S Y_{\overline{y}}^{\mathfrak{sh}}\}$$

ayant même image que $[\mathfrak{m}_{X,Y,\overline{e}}^{\mathfrak{sh}}]$ dans $c_S((X \times_S Y)_{\overline{e}}^{\mathfrak{sh}}, X_{\overline{x}}^{\mathfrak{sh}} \times_S Y_{\overline{y}}^{\mathfrak{sh}}\}$.

Proposition 2.27. *Soient $X, Y, X', Y'$ des $S$-schémas, $\alpha \in c_S(X, X')$ et $\beta \in c_S(Y, Y')$ des correspondances finies. Pour tout bon point géométrique $\overline{e}$ de $X \times_S Y$ et tout bon point géométrique $\overline{e}$' de $X' \times_S Y'$, on a l'égalité*

$$\Big(\{\alpha\}_{\overline{x},\overline{x}'} \otimes \{\beta\}_{\overline{y},\overline{y}'}\Big) \circ \{\mathfrak{m}\}_{X,Y,\overline{e}}^{\mathfrak{sh}} = \Big[\mathfrak{m}_{X',Y',\overline{e}'}^{\mathfrak{sh}}\Big] \circ \Big\{\alpha \otimes \beta\Big\}_{\overline{e},\overline{e}'}$$

*dans $c_S\{(X \times_S Y)_{\overline{e}}^{\mathfrak{sh}}, (X')_{\overline{x}'}^{\mathfrak{sh}} \times_S (Y')_{\overline{y}'}^{\mathfrak{sh}})$.*

## 3 Résolution de Godement et transferts

Rappelons qu'une monade dans une catégorie $\mathcal{C}$ est la donnée d'un endofoncteur $M$ de $\mathcal{C}$ et de transformations naturelles $\mu : MM \to M$ et $\eta : \mathrm{id} \to M$ pour lesquelles les diagrammes

$$\begin{tikzcd} M \arrow[r, "\eta M"] \arrow[dr, "\mathrm{id}"'] & MM \arrow[d, "\mu"] & M \arrow[l, "M\eta"'] \arrow[dl, "\mathrm{id}"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} MMM \arrow[r, "M\mu"] \arrow[d, "\mu M"] & MM \arrow[d, "\mu"] \\ MM \arrow[r, "\mu"] & M \end{tikzcd}$$

sont commutatifs. Étant donnée une monade $(M, \mu, \eta)$, l'un des nombreux avatars de la construction bar permet d'associer à un objet $C$ de $\mathcal{C}$ un objet cosimplicial $B^*(M, C)$ de $\mathcal{C}$ muni d'une coaugmentation de $C$ dans ce dernier. Les $n$-cosimplexes sont donnés par l'objet $M^{n+1}C$ de $\mathcal{C}$, les codégénérescences par les morphismes

$$\sigma_i^n := M^i \mu M^{n-1-i} \ : M^{n+1}C \to M^n C \qquad i = 0, \ldots, n-1$$

et les cofaces par les morphismes

$$\delta_i^{n-1} := M^i \eta M^{n-i} \ : M^n C \to M^{n+1}C \qquad i = 0, \ldots, n.$$

Dans la suite de cette section nous fixons une catégorie $\mathcal{S} = \mathrm{Sch}_S, \mathrm{Var}_S$ ou $\mathrm{Sm}_S$, tous les préfaisceaux sont définis sur $\mathcal{S}$ et nous renvoyons à la définition 1.10 pour la notion de faisceaux avec transferts.

### 3.1 Cas de la topologie de Nisnevich

Nous allons maintenant appliquer les résultats concernant la localisation Nisnevich des correspondances finies que nous avons obtenus dans la sous-section 2.3. Rappelons que $\mathcal{S}$ désigne l'une des catégories suivantes $\mathrm{Sch}_S, \mathrm{Var}_S$ ou $\mathrm{Sm}_S$.

Dans la suite, nous désignons par $s_x$ la composante suivant le point $x$ d'un élément $s$ du produit

$$\prod_{x \in X} F\{X_x^{\mathfrak{h}}\}.$$

Par définition $F\{X_x^{\mathfrak{h}}\}$ correspond à la fibre Nisnevich de $F$ au point $x$. Désignons par $\mathcal{G}_{\mathrm{Nis}}$ la monade de la catégorie des faisceaux Nisnevich définissant la résolution cosimpliciale de Godement. Étant donné un faisceau Nisnevich $F$, cette dernière se caractérise de la manière suivante.

(a) Les sections sur $X$ du faisceau Nisnevich $\mathcal{G}_{\mathrm{Nis}}F$ sont données par

$$\mathcal{G}_{\mathrm{Nis}}F(X) = \prod_{x \in X} F\{X_x^{\mathfrak{h}}\}$$

les morphismes structuraux étant déterminés par les égalités

$$\Big[\mathcal{G}_{\mathrm{Nis}}F(p)(s)\Big]_x = F\left\{\{p\}_x^{\mathfrak{h}}\right\}(s_{p(x)})$$

dans lesquelles $x$ est un point de $X$, $p : X \to Y$ un morphisme de $S$-schémas et $s$ un élément de $\mathcal{G}_{\mathrm{Nis}}F(Y)$.

(b) Le morphisme structural $\eta_F$ se trouve donné par

$$\Big[\eta_F(s)\Big]_x = F\big\{\{\mathfrak{l}\}_{X,x}^{\mathfrak{h}}\big\}(s)$$

pour un élément $s$ de $F(X)$.

(c) Le morphisme structural $\mu_F$ coïncide avec la projection sur les composantes associées aux points fermés des $X_x^{\mathfrak{h}}$ via le morphisme naturel

$$(\mathcal{G}_{\mathrm{Nis}}\mathcal{G}_{\mathrm{Nis}}F)(X) \to \prod_{x\in X}\prod_{z\in X_x^{\mathfrak{h}}} F\left\{\left(X_x^{\mathfrak{h}}\right)_z^{\mathfrak{h}}\right\}.$$

Nous notons $s_{x,z}$ la composante suivant $x, z$ de l'image d'une section $s$ par le morphisme précédent de sorte que $\mu_F(s)_x = s_{x,x}$.

La résolution cosimpliciale de Godement d'un faisceau Nisnevich $F$ est par définition le faisceau Nisnevich cosimplicial

$$\mathcal{G}_{\mathrm{Nis}}^*F = B^*(\mathcal{G}_{\mathrm{Nis}}, F).$$

Ce dernier nous fournit un complexe $G_{\mathrm{Nis}}^*F$ de faisceaux Nisnevich de termes $G_{\mathrm{Nis}}^nF = \mathcal{G}_{\mathrm{Nis}}^nF$ et dont les différentielles sont données par la somme alternée des morphismes cofaces.

*Remarque* 3.1. La famille formée de ces foncteurs fibres $F \mapsto F\{X_x^{\mathfrak{h}}\}$ étant conservative, l'augmentation canonique

$$F \to G_{\mathrm{Nis}}^*F$$

est un quasi-isomorphisme [10] de complexes de faisceaux Nisnevich.

Proposition 3.2. *Il existe une monade canonique* $\mathcal{G}_{\mathrm{Nis}}^{\mathrm{tr}}$ *de la catégorie* $\mathrm{Sh}_{\mathrm{Nis}}^{\mathrm{tr}}(\mathcal{S})$ *rendant le carré suivant commutatif*

$$\begin{tikzcd}
\mathrm{Sh}_{\mathrm{Nis}}^{\mathrm{tr}}(\mathcal{S}) \arrow[r] \arrow[d, "\mathcal{G}_{\mathrm{Nis}}^{\mathrm{tr}}"] & \mathrm{Sh}_{\mathrm{Nis}}(\mathcal{S}) \arrow[d, "\mathcal{G}_{\mathrm{Nis}}"] \\
\mathrm{Sh}_{\mathrm{Nis}}^{\mathrm{tr}}(\mathcal{S}) \arrow[r] & \mathrm{Sh}_{\mathrm{Nis}}(\mathcal{S}).
\end{tikzcd}$$

[10] Il s'agit d'une propriété classique de la résolution de Godement prouvée par exemple dans [25, Part II, Ch. IV, lemma 2.2.2].

*Démonstration.* Il suffit de montrer que pour un faisceau Nisnevich avec transferts $F$, le faisceau Nisnevich $\mathcal{G}_{\mathrm{Nis}}F$ est canoniquement muni de transferts et que les morphismes

$$\eta_F : F \to \mathcal{G}_{\mathrm{Nis}}F \qquad \mu_F : \mathcal{G}_{\mathrm{Nis}}\mathcal{G}_{\mathrm{Nis}}F \to \mathcal{G}_{\mathrm{Nis}}F$$

sont des morphismes de faisceaux Nisnevich avec transferts. Étant donnée une correspondance finie $\alpha \in c_S(X,Y)$, on peut associer à une section $s$ de $\mathcal{G}_{\mathrm{Nis}}F(Y)$ la section de $\mathcal{G}_{\mathrm{Nis}}F(X)$ donnée par

$$\mathcal{G}_{\mathrm{Nis}}F(\alpha)(s)_x = \sum_{y\in Y} F\big\{\{\alpha\}_{x,y}\big\}(s_y).$$

On obtient ainsi un morphisme de groupes abéliens $\mathcal{G}_{\mathrm{Nis}}F(\alpha)$. Lorsque l'on se donne une correspondance finie $\beta \in c_S(Y,Z)$, la formule de composition (27) de la proposition 2.19 nous assure que pour un point $x$ de $X$

$$\begin{aligned}\mathcal{G}_{\mathrm{Nis}}F(\beta\circ\alpha)(s)_x &= \sum_{z\in Z} F\{\{\beta\circ\alpha\}_{x,z}\}(s_z) = \sum_{z\in Z}\sum_{y\in Y} F\{\beta_{y,z}\circ\{\alpha\}_{x,y}\}(s_z)\\ &= \sum_{z\in Z}\sum_{y\in Y} F\{\{\alpha\}_{x,y}\}\Big[F\{\{\beta\}_{y,z}\}(s_z)\Big] = \sum_{y\in Y} F\{\{\alpha\}_{x,y}\}\Big[\mathcal{G}_{\mathrm{Nis}}F(\beta)(s)_y\Big]\\ &= \mathcal{G}_{\mathrm{Nis}}F(\alpha)\Big[\mathcal{G}_{\mathrm{Nis}}F(\beta)(s)\Big]_x\end{aligned}$$

et donc que cette définition est fonctorielle :

$$\mathcal{G}_{\mathrm{Nis}}F(\beta\circ\alpha) = \mathcal{G}_{\mathrm{Nis}}F(\alpha)\circ\mathcal{G}_{\mathrm{Nis}}F(\beta).$$

Par ailleurs pour un morphisme $p : X \to Y$ de $S$-schémas, la seconde assertion de la proposition 2.19 nous donne

$$\mathcal{G}_{\mathrm{Nis}}F([p])(s)_x = F\{\{p\}_x^{\mathfrak{h}}\}(s_{p(x)}) = \mathcal{G}_{\mathrm{Nis}}F(p)(s)_x.$$

Les morphismes $\mathcal{G}_{\mathrm{Nis}}F(\alpha)$ étendent donc la structure usuelle de préfaisceau en une structure de préfaisceau avec transferts. La construction des germes de correspondances locaux utilisés assure que

$$\begin{aligned}F(\alpha)(s)_x &= F\{\alpha\circ\{\mathfrak{l}\}^{\mathfrak{h}}_{X,x}\}(s) = \sum_y F\{[\mathfrak{l}^{\mathfrak{h}}_{Y,y}]\circ\{\alpha\}_{x,y}\}(s) = \sum_y F\{\{\alpha\}_{x,y}\}(s_y)\\ &= \mathcal{G}_{\mathrm{Nis}}F(\alpha)(\eta_F(s))_x\end{aligned}$$

autrement dit que $\eta_F$ est un morphisme de faisceaux avec transferts. Il reste à montrer que $\mu_F$ est aussi un morphisme de faisceaux Nisnevich avec transferts. Pour une section $s$ appartenant à $\mathcal{G}_{\mathrm{Nis}}\mathcal{G}_{\mathrm{Nis}}F(X)$ on obtient en utilisant la troisième assertion de la proposition 2.19

$$\begin{aligned}\mu_F(\mathcal{G}_{\mathrm{Nis}}\mathcal{G}_{\mathrm{Nis}}F(\alpha)(s))_x &= \sum_{y\in Y}\mu_F\Big[\mathcal{G}_{\mathrm{Nis}}F\{\{\alpha\}_{x,y}\}(s_y)\Big]_x\\ &= \sum_{y\in Y}\sum_{z\in Y_y^{\mathfrak{h}}}F\big\{\{\{\alpha\}_{x,y}\}_{x,z}\big\}(s_{y,z})\\ &= \sum_{y\in Y}F\{\{\alpha\}_{x,y}\}(s_{y,y}) = \mathcal{G}_{\mathrm{Nis}}F(\alpha)(\mu_F(s)).\end{aligned}$$

Ce qui achève la démonstration. □

La proposition 3.2 assure en particulier que la résolution cosimpliciale de Godement $\mathcal{G}^*_{\mathrm{Nis}}F$ d'un faisceau Nisnevich avec transferts est canoniquement munie de transferts et qu'à fortiori le complexe $G^*_{\mathrm{Nis}}F$ est un complexe de faisceaux Nisnevich avec transferts quasi-isomorphe à $F$ dans la catégorie des complexes de faisceaux Nisnevich avec transferts. Cela entraîne en particulier que les préfaisceaux de cohomologie Nisnevich

$$X \mapsto H^j_{\mathrm{Nis}}(X, F)$$

d'un faisceau Nisnevich avec transferts sont canoniquement munis de transferts, résultat se déduisant par ailleurs de la proposition 3.1.8 de [29, chapter 5].

Nous allons maintenant voir que les tranferts « naturels » précédemment construits sur la résolution de Godement sont en outre compatibles avec la structure tensorielle de cette dernière.

Définition 3.3. Nous appelons faisceau Nisnevich (resp. avec transferts) quasi-monoïdal symétrique la donnée d'un faisceau Nisnevich (resp. d'un faisceau Nisnevich avec transferts) $F$ et pour tout $S$-schéma $X, Y$ d'un morphisme associatif symétrique

$$\boxtimes^F_{X,Y} : F(X) \otimes F(Y) \to F(X \times_S Y)$$

fonctoriel par rapport aux morphismes de schémas (resp. aux correspondances finies).

Ces derniers sont les objets d'une catégorie $\mathrm{Sh}_{\mathrm{Nis},\otimes}(\mathcal{S})$ (resp. une catégorie $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis},\otimes}(\mathcal{S})$) munie d'un foncteur fidèle vers $\mathrm{Sh}_{\mathrm{Nis}}(\mathcal{S})$ (resp. vers $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathcal{S})$).

*Remarque* 3.4. La monade de Godement $\mathcal{G}_{\mathrm{Nis}}$ induit une monade sur la catégorie des faisceaux Nisnevich quasi-monoïdaux symétriques. Étant donné un faisceau Nisnevich quasi-monoïdal symétrique $(F, \boxtimes^F_{X,Y})$, on voit en effet que $\mathcal{G}_{\mathrm{Nis}}F$ est canoniquement un faisceau Nisnevich quasi-monoïdal symétrique pour les morphismes

$$\boxtimes^{\mathcal{G}_{\mathrm{Nis}}F}_{X,Y} : \mathcal{G}_{\mathrm{Nis}}F(X) \otimes \mathcal{G}_{\mathrm{Nis}}F(Y) \to \mathcal{G}_{\mathrm{Nis}}F(X \times_S Y)$$

donnés par les relations

$$\Big[\boxtimes_{X,Y}^{\mathcal{G}_{\mathrm{Nis}}F}(s\otimes t)\Big]_e = F\big\{\{\mathfrak{m}\}^{\mathfrak{h}}_{X,Y,e}\big\}\Big[\boxtimes^F_{X^{\mathfrak{h}}_x,Y^{\mathfrak{h}}_y}(s_x\otimes t_y)\Big]$$

dans lesquelles $s\in\mathcal{G}_{\mathrm{Nis}}F(X), t\in\mathcal{G}_{\mathrm{Nis}}F(Y)$ et $e$ désigne un point du produit $X\times_S Y$ de projection $x$ et $y$. Avec cette définition il est aisé de voir que les morphismes structuraux $\eta_F$ et $\mu_F$ de la monade $\mathcal{G}_{\mathrm{Nis}}$ sont bien des morphismes de faisceaux Nisnevich quasi-monoïdaux symétriques.

Nous sommes maintenant en mesure d'énoncer le résultat suivant.

PROPOSITION 3.5. *La monade $\mathcal{G}^{\mathrm{tr}}_{\mathrm{Nis}}$ induit une monade sur la catégorie $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis},\otimes}(\mathcal{S})$ compatible avec la monade $\mathcal{G}_{\mathrm{Nis}}$ via le foncteur d'oubli. Autrement dit le carré suivant est commutatif*

$$\begin{tikzcd}
\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis},\otimes}(\mathcal{S}) \arrow[r] \arrow[d, "\mathcal{G}^{\mathrm{tr}}_{\mathrm{Nis}}"] & \mathrm{Sh}_{\mathrm{Nis},\otimes}(\mathcal{S}) \arrow[d, "\mathcal{G}_{\mathrm{Nis}}"] \\
\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis},\otimes}(\mathcal{S}) \arrow[r] & \mathrm{Sh}_{\mathrm{Nis},\otimes}(\mathcal{S}).
\end{tikzcd}$$

*Démonstration.* Supposons que $(F,\boxtimes^F_{X,Y})$ soit un faisceau Nisnevich avec transferts quasi-monoïdal symétrique. En utilisant la proposition 3.2, il suffit de montrer que le carré

$$\begin{tikzcd}
\mathcal{G}_{\mathrm{Nis}}F(X')\otimes\mathcal{G}_{\mathrm{Nis}}F(Y') \arrow[r, "\boxtimes^{\mathcal{G}_{\mathrm{Nis}}F}_{X',Y'}"] \arrow[d, "\mathcal{G}_{\mathrm{Nis}}F(\alpha)\otimes\mathcal{G}_{\mathrm{Nis}}F(\beta)"] & \mathcal{G}_{\mathrm{Nis}}F(X'\times_S Y') \arrow[d, "\mathcal{G}_{\mathrm{Nis}}F(\alpha\otimes\beta)"] \\
\mathcal{G}_{\mathrm{Nis}}F(X)\otimes\mathcal{G}_{\mathrm{Nis}}F(Y) \arrow[r, "\boxtimes^{\mathcal{G}_{\mathrm{Nis}}F}_{X,Y}"] & \mathcal{G}_{\mathrm{Nis}}F(X\times_S Y)
\end{tikzcd} \tag{31}$$

est commutatif pour toute correspondance finie $\alpha\in c_S(X,X')$ et $\beta\in c_S(Y,Y')$. Notons $E$ le produit $X\times_S Y$ et $E'$ le produit $X'\times_S Y'$. Considérons un point $e$ de $E$ de projection $x$ et $y$ ainsi que des sections $s\in\mathcal{G}_{\mathrm{Nis}}(X'), t\in\mathcal{G}_{\mathrm{Nis}}(Y')$. En utilisant la remarque 3.4 nous avons

$$\begin{aligned}
\mathcal{G}_{\mathrm{Nis}}F(\alpha\otimes\beta)\Big[\boxtimes^{\mathcal{G}_{\mathrm{Nis}}F}_{X',Y'}(s\otimes t)\Big]_e &= \sum_{e'\in E'} F\big\{\{\alpha\otimes\beta\}_{e,e'}\big\}\Big[\boxtimes^{\mathcal{G}_{\mathrm{Nis}}F}_{X',Y'}(s\otimes t)\Big]_{e'} \\
&= \sum_{e'\in E'} F\big\{\{\alpha\otimes\beta\}_{e,e'}\big\}F\big\{\{\mathfrak{m}\}^{\mathfrak{h}}_{X',Y',e'}\big\}\Big[\boxtimes^F_{(X')^{\mathfrak{h}}_{x'},(Y')^{\mathfrak{h}}_{y'}}(s_{x'}\otimes t_{y'})\Big] \\
&= \sum_{\substack{x'\in X'\\ y'\in Y'}} F\left\{\sum_{e'\in E'_{x'y'}}[\mathfrak{m}^{\mathfrak{h}}_{X',Y',e'}]\circ\{\alpha\otimes\beta\}_{e,e'}\right\}\Big[\boxtimes^F_{(X')^{\mathfrak{h}}_{x'},(Y')^{\mathfrak{h}}_{y'}}(s_{x'}\otimes t_{y'})\Big].
\end{aligned}$$

En utilisant la proposition 2.20, cette dernière égalité peut se réécrire sous la forme

$$\begin{aligned}
&\mathcal{G}_{\mathrm{Nis}}F(\alpha\otimes\beta)\Big[\boxtimes^{\mathcal{G}_{\mathrm{Nis}}F}_{X',Y'}(s\otimes t)\Big]_e\\
&\qquad=\sum_{\substack{x'\in X'\\ y'\in Y'}} F\Big\{\big(\{\alpha\}_{x,x'}\otimes\{\beta\}_{y,y'}\big)\circ\{\mathfrak{m}\}^{\mathfrak{h}}_{X,Y,e}\Big\}\Big[\boxtimes^{F}_{(X')^{\mathfrak{h}}_{x'},(Y')^{\mathfrak{h}}_{y'}}(s_{x'}\otimes t_{y'})\Big]\\
&\qquad=\sum_{\substack{x'\in X'\\ y'\in Y'}} F\big\{\{\mathfrak{m}\}^{\mathfrak{h}}_{X,Y,e}\big\}\Big[\boxtimes^{F}_{X^{\mathfrak{h}}_{x},Y^{\mathfrak{h}}_{y}}\big(F\{\{\alpha\}_{x,x'}\}(s_{x'})\otimes F\{\{\beta\}_{y,y'}\}(t_{y'})\big)\Big]\\
&\qquad=F\big\{\{\mathfrak{m}\}^{\mathfrak{h}}_{X,Y,e}\big\}\Big[\boxtimes_{X^{\mathfrak{h}}_{x},Y^{\mathfrak{h}}_{y}}\big(\mathcal{G}_{\mathrm{Nis}}F(\alpha)(s)_x\otimes\mathcal{G}_{\mathrm{Nis}}F(\beta)(t)_y\big)\Big].
\end{aligned}$$

Ce qui donne finalement

$$\mathcal{G}_{\mathrm{Nis}}F(\alpha\otimes\beta)\Big[\boxtimes^{\mathcal{G}_{\mathrm{Nis}}F}_{X',Y'}(s\otimes t)\Big]_e=\boxtimes^{\mathcal{G}_{\mathrm{Nis}}F}_{X,Y}\Big[\mathcal{G}_{\mathrm{Nis}}F(\alpha)(s)\otimes\mathcal{G}_{\mathrm{Nis}}F(\beta)(t)\Big]_e$$

et prouve la commutativité du carré (31). □

La remarque 3.4 assure que la cohomologie d'un faisceau Nisnevich quasi-monoïdal symétrique est munie d'un produit associatif et commutatif

$$H^p(X,F)\otimes H^q(Y,F)\xrightarrow{\otimes^F_{X,Y}} H^{p+q}(X\times_S Y,F) \tag{32}$$

induit par la structure quasi-monoïdale symétrique sur la résolution de Godement. Ce produit est commutatif au sens gradué, autrement dit tel que

$$a\otimes^F_{X,Y} b=(-1)^{p+q} b\otimes^F_{Y,X} a$$

pour un élément $a$ de $H^p(X,F)$ et un élément $b$ de $H^q(X,F)$ et compatible au produit induit par $F$ sur le $H^0$. En particulier on dispose d'un cup-produit associatif et commutatif

$$\begin{tikzcd}
H^p(X,F)\otimes H^q(X,F) \arrow[r, "\otimes^F_{X,X}"'] \arrow[rr, bend left, "\smile^F_X"] & H^{p+q}(X\times_S X,F) \arrow[r, "\Delta^*_X"'] & H^{p+q}(X,F)
\end{tikzcd} \tag{33}$$

munissant $H^*(X,F)$ d'une structure d'algèbre graduée commutative. Lorsque $F$ est en outre un faisceau Nisnevich avec transferts quasi-monoïdal symétrique, la proposition 3.5 assure que les produits (32) sont compatibles aux transferts *i.e.* que les carrés

$$\begin{tikzcd}
H^p(X',F)\otimes H^q(Y',F) \arrow[r, "\otimes^F_{X',Y'}"] \arrow[d, "H^p(\alpha,F)\otimes H^q(\beta,F)"'] & H^{p+q}(X'\times_S Y',F) \arrow[d, "H^{p+q}(\alpha\otimes\beta,F)"] \\
H^p(X,F)\otimes H^q(Y,F) \arrow[r, "\otimes^F_{X,Y}"] & H^{p+q}(X\times_S Y,F)
\end{tikzcd}$$

sont commutatifs pour toute correspondance $\alpha\in c_S(X,X')$ et $\beta\in c_S(Y,Y')$.

*Remarque* 3.6. On prendra garde que pour une correspondance finie $\alpha \in c_S(X,Y)$, les carrés

$$\begin{tikzcd} H^p(Y,F)\otimes H^q(Y,F) \arrow[r, "\smile_Y^F"] \arrow[d, "H^p(\alpha,F)\otimes H^q(\alpha,F)"'] & H^{p+q}(Y,F) \arrow[d, "H^{p+q}(\alpha,F)"] \\ H^p(X,F)\otimes H^q(X,F) \arrow[r, "\smile_X^F"] & H^{p+q}(X,F) \end{tikzcd}$$

ne sont pas nécessairement commutatifs — nous renvoyons d'ailleurs à la remarque 1.7 à ce propos. Les cup-produits (33) ne sont donc pas à priori compatibles aux transferts.

### 3.2 Cas de la topologie étale

Nous nous contentons dans cette sous-section d'énoncer les résultats dans le cas de la topologie étale : les démonstrations étant identiques à celles données pour la topologie de Nisnevich. Nous supposons ici que $\mathcal{S} = \mathrm{Var}_S$ ou $\mathrm{Sm}_S$.

Dans la suite, nous désignons par $s_{\overline{x}}$ la composante suivant $\overline{x}$ d'un élément $s$ du produit

$$\prod_{\substack{\overline{x} \text{ bon point} \\ \text{géométrique de } X}} F\{X_{\overline{x}}^{\mathfrak{sh}}\}.$$

Par définition $F\{X_{\overline{x}}^{\mathfrak{sh}}\}$ est la fibre étale de $F$ au point géométrique $\overline{x}$. Pour définir la résolution de Godement dans le cadre étale, nous partons de la monade $\mathcal{G}_{\mathrm{Et}}$ de la catégorie des faisceaux étales caractérisée pour un faisceau étale $F$ par les propriétés suivantes.

(a) Les sections sur $X$ du faisceau étale $\mathcal{G}_{\mathrm{Et}}F$ sont données par

$$\mathcal{G}_{\mathrm{Et}}F(X) = \prod_{\substack{\overline{x} \text{ bon point} \\ \text{géométrique de } X}} F\{X_{\overline{x}}^{\mathfrak{sh}}\}$$

les morphismes structuraux étant déterminés par les égalités

$$\Big[\mathcal{G}_{\mathrm{Et}}F(p)(s)\Big]_{\overline{x}} = F\left\{\{p\}_{\overline{x}}^{\mathfrak{sh}}\right\}(s_{p\circ\overline{x}})$$

dans lesquelles $\overline{x}$ est un bon point géométrique de $X$, $p : X \to Y$ un morphisme de $S$-schémas et $s$ un élément de $\mathcal{G}_{\mathrm{Et}}F(Y)$.

(b) Le morphisme structural $\eta_F$ se trouve donné par

$$\Big[\eta_F(s)\Big]_{\overline{x}} = F\{\{\mathfrak{l}\}_{X,\overline{x}}^{\mathfrak{sh}}\}(s)$$

pour un élément $s$ de $F(X)$.

(c) Le morphisme structural $\mu_F$ coïncide avec la projection sur les composantes associées aux points fermés des $X_{\overline{x}}^{\mathfrak{sh}}$ via le morphisme naturel

$$(\mathcal{G}_{\mathrm{Et}}\mathcal{G}_{\mathrm{Et}})F(X) \to \prod_{\substack{\overline{x} \text{ bon point} \\ \text{géométrique de } X}} \prod_{\substack{\overline{z} \text{ bon point} \\ \text{géométrique de } X_{\overline{x}}^{\mathfrak{sh}}}} F\left\{\left(X_{\overline{x}}^{\mathfrak{sh}}\right)_{\overline{z}}^{\mathfrak{sh}}\right\}.$$

La résolution cosimpliciale de Godement d'un faisceau étale $F$ est par définition le faisceau étale cosimplicial

$$\mathcal{G}_{\mathrm{Et}}^{*}F = B^{*}(\mathcal{G}_{\mathrm{Et}}, F).$$

Ce dernier nous fournit un complexe $G_{\mathrm{Et}}^{*}F$ de faisceaux étales de termes $G_{\mathrm{Et}}^{n}F = \mathcal{G}_{\mathrm{Et}}^{n}F$ et dont les différentielles sont données par la somme alternée des morphismes cofaces.

*Remarque* 3.7. La famille formée des foncteurs fibres $F \mapsto F\{X_{\overline{x}}^{\mathfrak{sh}}\}$ étant conservative sur $\mathrm{Var}_S$ d'après le lemme 2.22, l'augmentation canonique

$$F \to G_{\mathrm{Et}}^{*}F$$

est un quasi-isomorphisme [11] de complexes de faisceaux étales sur $\mathrm{Var}_S$.

Proposition 3.8. *Il existe une monade canonique* $\mathcal{G}_{\mathrm{Et}}^{\mathrm{tr}}$ *de la catégorie* $\mathrm{Sh}_{\mathrm{Et}}^{\mathrm{tr}}(\mathcal{S})$ *rendant le carré suivant commutatif*

$$\begin{tikzcd}
\mathrm{Sh}_{\mathrm{Et}}^{\mathrm{tr}}(\mathcal{S}) \arrow[r] \arrow[d, "\mathcal{G}_{\mathrm{Et}}^{\mathrm{tr}}"] & \mathrm{Sh}_{\mathrm{Et}}(\mathcal{S}) \arrow[d, "\mathcal{G}_{\mathrm{Et}}"] \\
\mathrm{Sh}_{\mathrm{Et}}^{\mathrm{tr}}(\mathcal{S}) \arrow[r] & \mathrm{Sh}_{\mathrm{Et}}(\mathcal{S}).
\end{tikzcd}$$

Comme précédemment les tranferts « naturels » sur la résolution de Godement étale sont compatibles avec la structure tensorielle de cette dernière.

Définition 3.9. Nous appelons faisceau étale (resp. avec transferts) quasi-monoïdal symétrique la donnée d'un faisceau étale (resp. d'un faisceau étale avec transferts) $F$ et pour tout $S$-schéma $X, Y$ d'un morphisme associatif symétrique

$$\boxtimes_{X,Y}^{F} : F(X) \otimes F(Y) \to F(X \times_S Y)$$

fonctoriel par rapport aux morphismes de schémas (resp. aux correspondances finies).

Ces derniers sont les objets d'une catégorie $\mathrm{Sh}_{\mathrm{Et},\otimes}(\mathcal{S})$ (resp. une catégorie $\mathrm{Sh}_{\mathrm{Et},\otimes}^{\mathrm{tr}}(\mathcal{S})$) munie d'un foncteur fidèle vers $\mathrm{Sh}_{\mathrm{Et}}(\mathcal{S})$ (resp. vers $\mathrm{Sh}_{\mathrm{Et}}^{\mathrm{tr}}(\mathcal{S})$).

[11] Il s'agit d'une propriété classique de la résolution de Godement prouvée par exemple dans [25, Part II, Ch. IV, lemma 2.2.2].

*Remarque* 3.10. La monade de Godement $\mathcal{G}_{\mathrm{Et}}$ induit une monade sur la catégorie des faisceaux étales quasi-monoïdaux symétriques. Étant donné un faisceau étale quasi-monoïdal symétrique $(F, \boxtimes^F_{X,Y})$, on voit en effet que $\mathcal{G}_{\mathrm{Et}}F$ est canoniquement un faisceau étale quasi-monoïdal symétrique pour les morphismes

$$\boxtimes^{\mathcal{G}_{\mathrm{Et}}F}_{X,Y} : \mathcal{G}_{\mathrm{Et}}F(X) \otimes \mathcal{G}_{\mathrm{Et}}F(Y) \to \mathcal{G}_{\mathrm{Et}}F(X \times_S Y)$$

donnés par les relations

$$\left[\boxtimes^{\mathcal{G}_{\mathrm{Et}}F}_{X,Y}(s \otimes t)\right]_{\overline{e}} = F\big\{\{\mathfrak{m}\}^{\mathfrak{sh}}_{X,Y,\overline{e}}\big\}\left[\boxtimes^F_{X^{\mathfrak{sh}}_{\overline{x}}, Y^{\mathfrak{sh}}_{\overline{y}}}(s_{\overline{x}} \otimes t_{\overline{y}})\right]$$

dans lesquelles $s \in \mathcal{G}_{\mathrm{Et}}F(X), t \in \mathcal{G}_{\mathrm{Et}}F(Y)$ et $\overline{e}$ désigne un bon point géomérique du produit $X \times_S Y$ de projection $\overline{x}$ et $\overline{y}$. Avec cette définition il est aisé de voir que les morphismes structuraux $\eta_F$ et $\mu_F$ de la monade $\mathcal{G}_{\mathrm{Et}}$ sont bien des morphismes de faisceaux Nisnevich quasi-monoïdaux symétriques.

Nous sommes maintenant en mesure d'énoncer le résultat suivant.

Proposition 3.11. *La monade $\mathcal{G}^{\mathrm{tr}}_{\mathrm{Et}}$ induit une monade sur la catégorie $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Et},\otimes}(\mathcal{S})$ compatible avec la monade $\mathcal{G}_{\mathrm{Et}}$ via le foncteur d'oubli. Autrement dit le carré suivant est commutatif*

$$\begin{tikzcd}
\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Et},\otimes}(\mathcal{S}) \arrow[r] \arrow[d, "\mathcal{G}^{\mathrm{tr}}_{\mathrm{Et}}"] & \mathrm{Sh}_{\mathrm{Et},\otimes}(\mathcal{S}) \arrow[d, "\mathcal{G}_{\mathrm{Et}}"] \\
\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Et},\otimes}(\mathcal{S}) \arrow[r] & \mathrm{Sh}_{\mathrm{Et},\otimes}(\mathcal{S}).
\end{tikzcd}$$

## 4 Réalisation $\ell$-adique des motifs mixtes

Nous allons montrer que le foncteur de réalisation $\ell$-adique des $S$-schémas lisses de type fini [12]

$$\begin{array}{rcl} R_\ell : \mathrm{Sm}^{\mathrm{op}}_S & \to & \mathrm{D}^+(S, \mathbb{Z}_\ell) \\ X & \mapsto & R\pi_{X*}\pi^*_X\mathbb{Z}_S/\ell^* \end{array} \tag{34}$$

se prolonge canoniquement en un foncteur triangulé quasi-tensoriel sur la catégorie des motifs mixtes géométriques $DM_{gm}(S)$. Avant d'aborder le résultat principal, nous tenons à donner quelques précisions quant à la catégorie dans laquelle le foncteur de réalisation $\ell$-adique prend ses valeurs.

Lemme 4.1. *Lorsque le foncteur (34) prend ses valeurs dans une sous-catégorie triangulée pleine $\mathcal{C}$ de $\mathrm{D}^+(S, \mathbb{Z}_\ell)$, il est en est de même d'un prolongement de (34) aux motifs mixtes géométriques.*

*Démonstration.* Cela résulte du fait que la catégorie triangulée des motifs mixtes géométriques est engendrée par les motifs des $S$-schémas lisses de type fini. ☐

[12]La notation $\mathrm{D}^+(S, \mathbb{Z}_\ell)$ désigne la catégorie des coefficients $\ell$-adiques construite par T. Ekedahl dans [11]. Nous renvoyons le lecteur à l'appendice A.

Dans la construction qui suit nous utilisons la proposition suivante que nous tirons de [27, 28].

Proposition 4.2. *Les h-faisceaux en groupes abéliens ont une structure canonique de préfaisceau avec transferts. En particulier les faisceaux de groupes abéliens localements constants pour la topologie étale sont canoniquement munis d'une structure de préfaisceau avec transferts et de plus les faisceaux d'anneaux localement constants sont canoniquement munis d'une structure de préfaisceau avec transferts quasi-monoïdal symétrique.*

### 4.1 Construction

Nous consacrons cette sous-section à la construction du foncteur de réalisation $\ell$-adique des motifs mixtes géométriques. L'ingrédient essentiel réside dans les propriétés de la résolution de Godement que nous prouvées dans la section 3. Nous allons donc prouver le

Théorème 4.3. *Le foncteur symétrique monoïdal (34) se prolonge canoniquement en un foncteur triangulé quasi-tensoriel*

$$DM_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^+(S, \mathbb{Z}_\ell). \tag{35}$$

*(a) Lorsque $S$ est de type fini sur un schéma noethérien régulier de dimension $\leq 1$, le foncteur (35) prend ses valeurs dans $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S, \mathbb{Z}_\ell)$ sous-catégorie triangulée pleine formée des coefficients constructibles.*

*(b) Lorsque $S$ est de type fini sur un corps fini, le foncteur (35) induit un foncteur triangulé tensoriel*

$$DM_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(S, \overline{\mathbb{Q}}_\ell)$$

*où le second membre désigne la catégorie des coefficients $\ell$-adiques mixtes de P. Deligne [8, 5].*

Avant de donner la démonstration de ce résultat, il y a lieu de préciser la terminologie que nous employons. On peut considérer le topos $\mathbb{N}^{\mathrm{op}} S_{\mathrm{Et}}$ des systèmes projectifs de grands faisceaux étales [13] sur $S$ ainsi que le faisceau d'anneaux sur ce dernier

$$\mathbb{Z}_{S,\mathrm{Et}}/\ell^* : \cdots \to \mathbb{Z}_{S,\mathrm{Et}}/\ell^{r+1} \to \mathbb{Z}_{S,\mathrm{Et}}/\ell^r \to \cdots \to \mathbb{Z}_{S,\mathrm{Et}}/\ell$$

et la catégorie de modules associée $\mathrm{Mod}(\mathbb{Z}_{S,\mathrm{Et}}/\ell^*)$. Un système projectif de faisceaux étales avec transferts

$$F : r \mapsto F_r$$

dont les composantes $F_r$ sont des $\mathbb{Z}_{S,\mathrm{Et}}/\ell^{r+1}$-modules sera appelé un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module avec transferts. En prenant pour morphismes, les morphismes de

[13] Il s'agit des faisceaux sur $\mathrm{Var}_S$ munie de la topologie étale.

systèmes projectifs de faisceaux étales avec transferts, on obtient une catégorie munie d'un foncteur d'oubli fidèle. Nous dirons qu'un $\mathbb{Z}_{S,\mathrm{Et}}$-module (resp. un $\mathbb{Z}_{S,\mathrm{Et}}$-module avec transferts) $F$ est quasi-monoïdal symétrique lorsque ses composantes $F_r$ sont des $\mathbb{Z}_{S,\mathrm{Et}}/\ell^{r+1}$-modules (resp. $\mathbb{Z}_{S,\mathrm{Et}}/\ell^{r+1}$-modules avec transferts) quasi-monoïdaux symétriques et que les morphismes de transition sont compatibles avec ces structures.

*Preuve du théorème 4.3.* On sait d'après la proposition 4.2 que le grand faisceau étale $\mathbb{Z}_{S,\mathrm{Et}}/\ell^r$ se trouve canoniquement muni de transferts, les morphismes de transition $\mathbb{Z}_{S,\mathrm{Et}}/\ell^r \to \mathbb{Z}_{S,\mathrm{Et}}/\ell^{r+1}$ étant des morphismes de faisceaux étales avec transferts. On peut donc voir que $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$ est canoniquement un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module avec transferts. En prenant la résolution cosimpliciale de Godement de ce dernier, on obtient un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module cosimplicial

$$\mathcal{G}^*_{\mathrm{Et}}\big[\mathbb{Z}_{S,\mathrm{Et}}/\ell^*\big] \in \Delta\mathrm{Mod}(\mathbb{Z}_{S,\mathrm{Et}}/\ell^*)$$

qui d'après le lemme 3.8 est en fait canoniquement un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module avec transferts cosimplicial. Supposons que $X$ soit un $S$-schéma de type fini, on dispose alors de deux morphismes de topos

$$X_{\mathrm{et}} \underset{p_X}{\overset{\iota_X}{\rightleftarrows}} S_{\mathrm{Et}} \qquad\qquad p_X \circ \iota_X = \mathrm{id}$$

la restriction d'un faisceau $F$ au petit site étale de $X$ étant donnée par $\iota_X^* F$. Ces morphismes s'étendent naturellement aux topos $\mathbb{N}^{\mathrm{op}}X_{\mathrm{et}}$ [14] et $\mathbb{N}^{\mathrm{op}}S_{\mathrm{Et}}$ et sachant que $\iota_X^*\mathbb{Z}_{S,\mathrm{Et}}/\ell^* = \mathbb{Z}_X/\ell^*$, ils induisent un foncteur exact

$$\iota_X^* = p_{X*} : \mathrm{Mod}(\mathbb{Z}_{S,\mathrm{Et}}/\ell^*) \to \mathrm{Mod}(\mathbb{Z}_X/\ell^*).$$

En désignant par $\pi_X$ le morphisme structural de $X$ dans $S$, cela permet d'associer au $S$-schéma de type fini $X$ le $\mathbb{Z}_S/\ell^*$-module cosimplicial

$$\mathcal{R}_\ell(X) := \pi_{X*} i_X^* \mathcal{G}^*_{\mathrm{Et}}\big[\mathbb{Z}_{S,\mathrm{Et}}/\ell^*\big]. \tag{36}$$

En prenant le complexe de chaîne associé, on obtient un objet

$$\underline{R}_\ell(X) := \mathcal{C}\mathcal{R}_\ell(X) = \pi_{X*} i_X^* G^*_{\mathrm{Et}}\big[\mathbb{Z}_{S,\mathrm{Et}}/\ell^*\big] \tag{37}$$

appartenant à $\mathrm{C}^+(S, \mathbb{Z}/\ell^*)$.

*Remarque* 4.4. Par construction de la résolution de Godement, on peut voir que

$$R_\ell(X) = \pi_{X*} G^{X,*}_{\mathrm{Et}}\big[\mathbb{Z}_X/\ell^*\big].$$

Le passage par les grands faisceaux étales ne sert en fait que dans la mesure où il permet de mettre des transferts.

[14] Il s'agit des systèmes projectifs de faisceaux sur le schéma $X$ muni de la topologie étale, ainsi $X_{\mathrm{et}}$ désigne le petit topos étale de $X$.

Pour la définition des complexes normalisés apparaissant dans le lemme suivant, nous renvoyons à la remarque A.3.

Lemme 4.5. *Les objets (36) et (37) sont fonctoriels par rapport aux correspondances finies, autrement dit on dispose de foncteurs additifs*

$$\begin{array}{rcl} \mathcal{R}_\ell & : & \mathrm{VarCor}_S^{\mathrm{op}} \to \Delta\mathrm{Mod}(\mathbb{Z}_S/\ell^*) \\ \underline{R}_\ell & : & \mathrm{VarCor}_S^{\mathrm{op}} \to \mathrm{C}^+(S, \mathbb{Z}_S/\ell^*). \end{array}$$

*De plus le foncteur $\underline{R}_\ell$ est à valeurs dans la sous-catégorie formée des systèmes projectifs « normalisés » et on a un isomorphisme de foncteurs canonique $\theta$*

$$\begin{tikzcd} \mathrm{SmCor}_S^{\mathrm{op}} \arrow[r, "\underline{R}_\ell"] \arrow[dr, phantom, "\overset{\theta}{\Longrightarrow}"] & \mathrm{D}^+(S, \mathbb{Z}_\ell) \\ \mathrm{Sm}_S^{\mathrm{op}} \arrow[u] \arrow[ur, "R_\ell"'] & \end{tikzcd}$$

*Démonstration.* Supposons plus généralement que $F$ soit un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module avec transferts cosimplicial. En posant pour un $S$-schéma $X$

$$\widetilde{F}(X) := \pi_{X*}\iota_X^* F$$

on obtient un préfaisceau avec transferts prenant ses valeurs dans la catégorie $\Delta\mathrm{Mod}(\mathbb{Z}_S/\ell^*)$. En effet lorsque l'on se donne une $S$-correspondance finie $\alpha$ de $X$ dans $Y$ et un $S$-schéma étale $U$, on a un morphisme naturel de $\mathbb{Z}/\ell^*$-modules cosimpliciaux

$$\widetilde{F}(Y)(U) := F(Y_U) \xrightarrow{F(\alpha_U)} F(X_U) =: \widetilde{F}(X)(U)$$

induit par les correspondances $\alpha_U$ construites dans la sous-section 1.2. Ces morphismes nous fournissent un morphisme dans la catégorie de $\mathbb{Z}_S/\ell^*$-modules cosimpliciaux

$$\widetilde{F}(\alpha) : \widetilde{F}(Y) \to \widetilde{F}(X).$$

La compatibilité avec la composition démontrée au lemme 1.9 assure que pour des correspondances finies $\alpha \in c_S(X,Y)$ et $\beta \in c_S(Y,Z)$ on a $(\beta\circ\alpha)_U = \beta_U \circ \alpha_U$ et donc que

$$\widetilde{F}(\alpha) \circ \widetilde{F}(\beta) = \widetilde{F}(\beta \circ \alpha).$$

En particulier ceci assure que les objets (36) et (37) sont fonctoriels par rapport aux correspondances finies. Pour la seconde assertion, il suffit de remarquer que pour un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module $F$, la résolution de Godement $F \to G_{\mathrm{Et}}^* F$ est une résolution flasque. En particulier si $X$ est un $S$-schéma de type fini, les morphismes naturels

$$\begin{tikzcd} R\pi_{X*}i_X^*F \arrow[r] & R\pi_{X*}i_X^*G_{\mathrm{Et}}^*F \\ & \pi_{X*}i_X^*G_{\mathrm{Et}}^*F \arrow[u] \end{tikzcd}$$

sont des quasi-isomorphismes. En prenant $F = \mathbb{Z}_{S,\mathrm{Et}}/\ell^*$, on obtient un isomorphisme de foncteurs

$$\begin{tikzcd} \mathrm{SmCor}_S^{\mathrm{op}} \arrow[r, "R_\ell"] & \mathrm{D}^+(S,\mathbb{Z}/\ell^*) \\ \mathrm{Sm}_S^{\mathrm{op}} \arrow[u] \arrow[ur, phantom, "\overset{\theta}{\Longrightarrow}"] \arrow[r, "R_\ell"'] & \arrow[u] \end{tikzcd}$$

Pour un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module plat $F$, les composantes de sa résolution de Godement $G^n_{\mathrm{Et}}F$ sont des $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-modules plats [15] ce qui assure que $G^*_{\mathrm{Et}}F$ est normalisé lorsque $F$ est $\ell$-adique. En particulier ceci entraîne que le foncteur considéré prend bien ses valeurs dans les complexes normalisés — voir remarque A.3. □

On a donc un foncteur

$$\underline{R_\ell} : \mathrm{C^b}(\mathrm{SmCor}_S)^{\mathrm{op}} \xrightarrow{\mathrm{C^b}\underline{R_\ell}} \mathrm{C^b}\left[\mathrm{C}^+(S,\mathbb{Z}/\ell^*)\right] \xrightarrow{\mathrm{Tot}} \mathrm{C}^+(S,\mathbb{Z}/\ell^*)$$

et donc un foncteur triangulé

$$\underline{R_\ell} : \mathrm{K^b}(\mathrm{SmCor}_S)^{\mathrm{op}} \to \mathrm{D}^+(S,\mathbb{Z}/\ell^*)$$

prenant en fait ces valeurs dans $\mathrm{D}^+(S,\mathbb{Z}_\ell)$. En utilisant l'invariance par homotopie dans le cadre $\ell$-adique, la localisation Nisnevich, on obtient un foncteur triangulé

$$R_\ell : \underline{DM}^{\mathrm{eff}}_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^+(S,\mathbb{Z}_\ell).$$

Comme la catégorie triangulée $\mathrm{D}^+(S,\mathbb{Z}_\ell)$ est pseudo-abélienne — il s'agit d'une conséquence du lemme 2.4 de [4] — ce dernier nous donne un foncteur

$$R_\ell : DM^{\mathrm{eff}}_{gm}(S) \to \mathrm{D}^+(S,\mathbb{Z}_\ell). \tag{38}$$

Il reste à vérifier que ce foncteur est bien compatible aux structures tensorielles de part et d'autre.

LEMME 4.6. *Le foncteur triangulé (38) est quasi-tensoriel*[16] *et pour tout $S$-schémas $X, Y$ le morphisme induit*

$$\boxtimes_{X,Y} : R_\ell(X) \otimes R_\ell(Y) \to R_\ell(X \times_S Y)$$

*coïncide avec le morphisme de Künneth.*

[15] On trouvera une preuve de cette propriété classique par exemple dans [25, Part II, Ch. IV, proposition 2.4.3].

[16] Nous renvoyons à la définition B.1 pour ce qui concerne la terminologie utilisée dans cet article.

*Démonstration.* D'après la proposition 4.2, $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$ est un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module avec transferts quasi-monoïdal symétrique. Le lemme 3.11 assure alors que $\mathcal{G}^*_{\mathrm{Et}}\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$ est en fait un $\mathbb{Z}_{S,\mathrm{Et}}/\ell^*$-module avec transferts quasi-monoïdal symétrique cosimplicial. Cela entraîne que le foncteur

$$\mathcal{R}_\ell : \mathrm{SmCor}_S^{\mathrm{op}} \to \Delta\mathrm{Mod}(\mathbb{Z}_S/\ell^*)$$

du lemme 4.5 est quasi-monoïdal symétrique, autrement dit que l'on dispose d'un morphisme canonique de foncteurs sur $\mathrm{SmCor}_S \otimes \mathrm{SmCor}_S$

$$\boxtimes : \mathcal{R}_\ell(-) \otimes \mathcal{R}_\ell(-) \to \mathcal{R}_\ell(- \times_S -)$$

associatif et commutatif. Ce dernier nous fournit des morphismes de foncteurs associatifs et commutatifs

$$\begin{tikzcd}
\mathcal{C}\mathcal{R}_\ell(-) \otimes \mathcal{C}\mathcal{R}_\ell(-) \arrow[d, equal] & \mathcal{C}\big[\mathcal{R}_\ell(-) \otimes \mathcal{R}_\ell(-)\big] \arrow[l, "\boxtimes^{EML}"'] \arrow[r, "\mathcal{C}\boxtimes"] & \mathcal{C}\mathcal{R}_\ell(- \times_S -) \arrow[d, equal] \\
\underline{R}_\ell(-) \otimes \underline{R}_\ell(-) & & \underline{R}_\ell(- \times_S -)
\end{tikzcd}$$

où $\boxtimes^{EML}$ désigne la transformation d'Eilenberg Mac Lane [10]. On a donc des morphismes de bifoncteurs associatifs et commutatifs

$$\begin{tikzcd}
\mathrm{C^b}\Big[\mathcal{C}\mathcal{R}_\ell(-) \otimes \mathcal{C}\mathcal{R}_\ell(-)\Big] \arrow[d, equal] & \mathrm{C^b}\Big[\mathcal{C}\big[\mathcal{R}_\ell(-) \otimes \mathcal{R}_\ell(-)\big]\Big] \arrow[l, "\mathrm{C^b}\boxtimes^{EML}"'] \arrow[r, "\mathrm{C^b}\mathcal{C}\boxtimes"] & \mathrm{C^b}\Big[\mathcal{C}\mathcal{R}_\ell(- \times_S -)\Big] \arrow[d, equal] \\
\underline{R}_\ell(-) \otimes \underline{R}_\ell(-) & & \underline{R}_\ell(- \times_S -)
\end{tikzcd}$$

Par ailleurs l'augmentation

$$\mathbb{Z}_S/\ell^* \to \underline{R}_\ell(S)$$

est un quasi-isomorphisme qui rend le diagramme suivant commutatif

$$\begin{tikzcd}
\underline{R}_\ell(S) \otimes \underline{R}_\ell(-) & \mathrm{C^b}\Big[\mathcal{C}\big[\mathcal{R}_\ell(S) \otimes \mathcal{R}_\ell(-)\big]\Big] \arrow[l, "\mathrm{C^b}\boxtimes^{EML}"'] \arrow[r, "\mathrm{C^b}\mathcal{C}\boxtimes"] & \underline{R}_\ell(S \times_S -) \arrow[dl] \\
\mathbb{Z}_S/\ell^* \otimes \underline{R}_\ell(-) \arrow[u] \arrow[r, equal] & \underline{R}_\ell(-). &
\end{tikzcd}$$

Cela nous assure que le foncteur (38) est bien quasi-tensoriel. □

En remarquant que le motif de $\mathbb{P}^1$ a pour image l'objet $\mathbb{Z}_\ell \oplus \mathbb{Z}_\ell(-1)[-2]$ de $\mathrm{D^b_c}(S, \mathbb{Z}_\ell)$, on déduit immédiatement de la définition du motif de Tate le résultat suivant.

LEMME 4.7. *Il existe un isomorphisme*

$$\vartheta : R_\ell(\mathbb{Z}(1)) \xrightarrow{\sim} \mathbb{Z}_\ell(-1).$$

Comme $\mathbb{Z}_\ell(-1)$ est inversible dans $\mathrm{D}^+(S, \mathbb{Z}_\ell)$, il résulte du lemme 4.7 que le foncteur (38) se prolonge en un foncteur triangulé quasi-tensoriel sur la catégorie des motifs mixtes géométriques $DM_{gm}(S)$.

Les assertions (a) et (b) du théorème 4.3 découlent du lemme 4.1 compte tenu respectivement du théorème de finitude de P. Deligne [1] et des résultats de P. Deligne sur les conjectures de Weil [7, 8]. □

*Remarque* 4.8. Lorsque $S$ est de type fini sur un corps, il résulte de [3, exposé III] que la flèche de Künneth

$$\boxtimes_{X,Y} : R_\ell(X) \otimes R_\ell(Y) \to R_\ell(X \times_S Y)$$

du lemme 4.6 est un isomorphisme pour tout $S$-schéma lisse de type fini $X, Y$. Cela entraîne que le foncteur de réalisation du théorème 4.3 est en fait tensoriel autrement dit que le morphisme

$$\boxtimes_{M,N} : R_\ell(M) \otimes R_\ell(N) \to R_\ell(M \otimes N)$$

est un isomorphisme pour tout $M, N \in DM_{gm}(S)$.

*Remarque* 4.9. Soit $T$ un schéma noethérien séparé et $\theta : T \to S$ un morphisme lisse de schémas. Pour tout $S$-schéma lisse de type fini $X$, le théorème de changement de base lisse appliqué au carré cartésien

$$\begin{tikzcd} X_T \arrow[r, "\theta_X"] \arrow[d, "\pi_{X_T/T}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi_{X/S}"] \\ T \arrow[r, "\theta"] & S \end{tikzcd}$$

nous donne un isomorphisme canonique

$$\begin{aligned} \theta^* R_\ell(X) = \theta^* R\pi_{X/S*}\pi^*_{X/S}\mathbb{Z}_S/\ell^* &= R\pi_{X_T/T*}\theta^*_X\pi^*_{X/S}\mathbb{Z}_S/\ell^* \\ &= R\pi_{X_T/T*}\pi^*_{X_T/T}\mathbb{Z}_T/\ell^* = R_\ell(X_T). \end{aligned}$$

Il résulte alors de la construction donnée dans la preuve du théorème 4.3 que l'on a un isomorphisme canonique de foncteurs $\phi$

$$\begin{tikzcd} DM_{gm}(S) \arrow[r, "R_\ell"] \arrow[d] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathrm{D}^+(S, \mathbb{Z}_\ell/\ell^*) \arrow[d, "\theta^*"] \\ DM_{gm}(T) \arrow[r, "R_\ell"] & \mathrm{D}^+(T, \mathbb{Z}_\ell). \end{tikzcd}$$

### 4.2 Une variante modérée

La généralité correspondant au théorème 4.3 permet de construire dans différentes situations géométriques des variantes modérées du foncteur de réalisation $\ell$-adique.

Le résultat essentiel à cet égard réside dans la proposition 4.16 du paragraphe suivant dans lequel nous détaillons le comportement des catégories de motifs mixtes géométriques par rapport à certaines limites projectives.

#### 4.2.1 Commutation aux limites projectives

Les systèmes projectifs de schémas que nous considérons sont les systèmes projectifs de schémas $\lambda \mapsto S_\lambda$ indexés par un ensemble ordonné filtrant $\Lambda$ et nous notons $u_{\lambda,\mu} : S_\mu \to S_\lambda$ les morphismes de transition. Notre hypothèse est la suivante.

> *Le schéma $S$ est régulier et limite projective d'un système projectif de schémas réguliers $\lambda \mapsto S_\lambda$ dont les morphismes de transition sont plats et affines.*

Comme $S$ et les $S_\lambda$ sont réguliers, leurs composantes irréductibles coïncident avec leurs composantes connexes. Pour un élément $\lambda$ de $\Lambda$, nous notons dans la suite

$$u_\lambda : S \to S_\lambda$$

le morphisme canonique. Il résulte de notre hypothèse que ce morphisme est plat. Si $X_{\lambda_0}$ est un $S_{\lambda_0}$-schéma, nous posons

$$X_\lambda = S_\lambda \times_{S_{\lambda_0}} X_{\lambda_0} \qquad X = S \times_{S_{\lambda_0}} X_{\lambda_0}.$$

Proposition 4.10. *Soit $X_{\lambda_0}$ un $S_{\lambda_0}$-schéma de type fini. Les morphismes de changement de base associés aux morphismes de transition induisent un isomorphisme*

$$\operatorname*{colim}_{\lambda \geq \lambda_0} c_{\mathrm{equi}}(X_\lambda/S_\lambda, 0) = c_{\mathrm{equi}}(X/S, 0).$$

*Démonstration.* Il s'agit de voir que le morphisme

$$\operatorname*{colim}_{\lambda \geq \lambda_0} c_{\mathrm{equi}}(X_\lambda/S_\lambda, 0) \xrightarrow{\operatorname{colim}_{\lambda \geq \lambda_0} u_\lambda^{\circledast}} c_{\mathrm{equi}}(X/S, 0)$$

est un isomorphisme. On remarquera que nos hypothèses entraînent que tous les changements de base considérés sont plats. Montrons tout d'abord que le morphisme est surjectif. On sait d'après le corollaire 3.4.6 de [28] que le membre de droite est le groupe abélien libre engendré par les sous-schémas fermés intègres $Z$ de $X$ finis et surjectifs sur une composante connexe de $S$. Soit $Z$ un tel sous-schéma fermé de $X$. En utilisant les théorèmes 8.8.2 et 8.10.5 de [14], on peut supposer, quitte à prendre $\lambda_0$ un peu plus grand, qu'il existe un sous-schéma fermé $Z_{\lambda_0}$ de $X_{\lambda_0}$ tel que

$$Z = S \times_{S_{\lambda_0}} Z_{\lambda_0}.$$

Une nouvelle application du théorème 8.10.5 de *loc.cit.* nous assure, quitte à prendre $\lambda_0$ un peu plus grand, que $Z_{\lambda_0}$ est fini sur $S_{\lambda_0}$. Soit $C$ la composante connexe de $S$ dominée par $Z$. Comme le morphisme $u_\lambda$ est plat, $C$ domine une composante connexe $C_\lambda$ de $S_\lambda$. Le système projectif $\lambda \mapsto C_\lambda$ étant à morphismes de transition affines dominants, la proposition 8.4.4 de *loc.cit.* assure que la limite projective $C'$ de $C$ est connexe. Par ailleurs, il résulte du théorème 8.10.5 de *loc.cit.* qu'il d'agit d'un ouvert de $S$. On voit donc que $C'$ est un ouvert connexe de $S$ contenant $C$, ce qui permet de conclure que $C$ est la limite projective des $C_\lambda$. Quitte à prendre $\lambda_0$ un peu plus grand, une nouvelle application du théorème 8.10.5 de *loc.cit.* permet de conclure que $Z_{\lambda_0}$ est surjectif sur la composante connexe $C_{\lambda_0}$. Sachant que nous avons supposé les $S_\lambda$ réguliers, le corollaire 3.4.6 de [28] assure alors que $[Z_{\lambda_0}]$ appartient à $c_{\mathrm{equi}}(X_{\lambda_0}/S_{\lambda_0}, 0)$ et la surjectivité résulte de l'égalité

$$u^{\circledast}_{\lambda_0}[Z_{\lambda_0}] = [S \times_{S_{\lambda_0}} Z_{\lambda_0}] = [Z].$$

Montrons maintenant l'injectivité. Soit $\lambda \geq \lambda_0$ un élément de $\Lambda$ et $\alpha_\lambda$ un élément de $c_{\mathrm{equi}}(X_\lambda/S_\lambda, 0)$ dont l'image par $u^{\circledast}_\lambda$ est nulle. En notant $Z_\mu$ le support du cycle $\alpha_\mu = u^{\circledast}_{\lambda,\mu}\alpha$, il s'ensuit que la limite projective du système $\mu \mapsto Z_\mu$ est vide. Le théorème 8.10.5 de [14] assure alors que $Z_\mu$ est vide pour $\mu$ suffisamment grand et donc que $\alpha_\mu = 0$. Cela achève la preuve de la proposition. □

En appliquant le lemme précédent aux correspondances finies entre schémas lisses de type fini, on obtient alors le corollaire suivant.

Corollaire 4.11. *Soient $\lambda_0 \in \Lambda$ et $X_{\lambda_0}, Y_{\lambda_0}$ des $S_{\lambda_0}$-schémas lisses de type fini. Le morphisme canonique*

$$\operatorname*{colim}_{\lambda \geq \lambda_0} c_{S_\lambda}(X_\lambda, Y_\lambda) \to c_S(X, Y)$$

*est un isomorphisme.*

*Démonstration.* Comme les $X_\lambda$ et $Y_\lambda$ sont des schémas lisses sur un schéma régulier, ils sont eux-mêmes réguliers. La proposition 4.10 nous donne alors

$$\begin{aligned}\operatorname*{colim}_{\lambda \geq \lambda_0} c_{S_\lambda}(X_\lambda, Y_\lambda) &= \operatorname*{colim}_{\lambda \geq \lambda_0} c_{\mathrm{equi}}(X_\lambda \times_{S_\lambda} Y_\lambda / X_\lambda, 0) \\ &= c_{\mathrm{equi}}(X \times_S Y/X, 0) = c_S(X, Y)\end{aligned}$$

ce qui justifie le corollaire. □

Ce résultat nous donne la propriété de commutation à certaines limites projectives des catégories de schémas lisses munis des correspondances finies. Son énoncé est la reformulation ci-dessous du corollaire 4.11.

Corollaire 4.12. *Le foncteur canonique*

$$2\text{-}\operatorname*{colim}_{\lambda} \mathrm{SmCor}_{S_\lambda} \to \mathrm{SmCor}_S \tag{39}$$

*est une équivalence de catégories.*

*Démonstration.* Compte tenu du corollaire 4.11, notre assertion est une conséquence des théorèmes 8.8.2 et 8.10.5 de [14] qui assurent que pour tout $S$-schéma de type fini $X$, il existe un $\lambda_0$ tel que $X$ provienne d'un $S_{\lambda_0}$-schéma de type fini $X_{\lambda_0}$, et de la proposition 17.7.8 de [15] qui assure que $X_{\lambda_0}$ peut-être choisi lisse sur $S_{\lambda_0}$ lorsque $X$ est lisse sur $S$. $\square$

Supposons donné un 2-foncteur

$$\mathcal{C}_{-} : \Lambda \to \text{Cat} \; ; \; \lambda \mapsto \mathcal{C}_{\lambda}$$

où Cat désigne la 2-catégorie des catégories essentiellement petites et notons $F_{\lambda,\mu} : \mathcal{C}_{\lambda} \to \mathcal{C}_{\mu}$ le foncteur de transition pour $\mu \geq \lambda$.

*Remarque* 4.13. Dans la suite il est utile de noter que la 2-colimite $\mathcal{C}$ des $\mathcal{C}_{\lambda}$ admet la description élémentaire suivante.

– Un objet de $\mathcal{C}$ est la donnée $(C, \lambda)$ d'un élément $\lambda \in \Lambda$ et d'un objet $C$ de $\mathcal{C}_{\lambda}$.
– Les morphismes entre deux objets $(C, \lambda)$ et $(C', \lambda')$ de $\mathcal{C}$ sont donnés par

$$\mathrm{Hom}_{\mathcal{C}}((C, \lambda), (C', \lambda')) = \operatorname*{colim}_{\mu \geq \lambda, \lambda'} \mathrm{Hom}_{\mathcal{C}_{\mu}}(F_{\lambda,\mu}(C), F_{\lambda',\mu}(C')).$$

Lorsque les $\mathcal{C}_{\lambda}$ sont des catégories additives et que les foncteurs $F_{\lambda,\mu}$ sont additifs, la catégorie $\mathcal{C}$ est naturellement une catégorie additive et il s'agit aussi de la 2-colimite de $\mathcal{C}_{\lambda}$ dans la 2-catégorie Add des catégories additives.

On dispose du lemme suivant. On remarquera la nécessité dans ce dernier de se restreindre aux complexes bornés.

Lemme 4.14. *Supposons donné un* 2*-foncteur*

$$\mathcal{C}_{-} : \Lambda \to \text{Add} \; ; \; \lambda \mapsto \mathcal{C}_{\lambda}$$

*et notons* $\mathcal{C}$ *la* 2*-colimite des* $\mathcal{C}_{\lambda}$*. Le foncteur canonique*

$$2\text{-}\operatorname*{colim}_{\lambda} \mathrm{K}^{\mathrm{b}}(\mathcal{C}_{\lambda}) \to \mathrm{K}^{\mathrm{b}}(\mathcal{C})$$

*est une équivalence de catégories triangulées.*

*Démonstration.* Notons $F_{\lambda}$ le foncteur additif canonique de $\mathcal{C}_{\lambda}$ dans $\mathcal{C}$. De la description précédente des 2-colimites, il résulte que le foncteur

$$2\text{-}\operatorname*{colim}_{\lambda} \mathrm{C}^{\mathrm{b}}(\mathcal{C}_{\lambda}) \to \mathrm{C}^{\mathrm{b}}(\mathcal{C})$$

est une équivalence de catégories. Un morphisme $c : C \to C'$ de complexes d'objets de $\mathcal{C}$ est donc l'image d'un morphisme $c_{\lambda} : C_{\lambda} \to C'_{\lambda}$ de complexes d'objets de $\mathcal{C}_{\lambda}$ par le foncteur $\mathrm{C}^{\mathrm{b}}(F_{\lambda})$. Il suffit alors juste de remarquer que $c$ est une équivalence d'homotopie si et seulement si il existe $\mu \geq \lambda$ tel que $\mathrm{C}^{\mathrm{b}}(F_{\lambda,\mu})(c_{\lambda})$ soit une équivalence d'homotopie dans $\mathrm{C}^{\mathrm{b}}(\mathcal{C}_{\mu})$. $\square$

En utilisant le corollaire 4.12, le lemme précédent nous donne le résultat suivant.

COROLLAIRE 4.15. *Le foncteur canonique*

$$2\text{-}\operatorname*{colim}_{\lambda} \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_{S_\lambda}) \to \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_S) \tag{40}$$

*est une équivalence de catégories triangulées.*

*Démonstration.* Le lemme 4.14 entraîne que le foncteur canonique

$$2\text{-}\operatorname*{colim}_{\lambda} \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_{S_\lambda}) \to \mathrm{K}^{\mathrm{b}}(2\text{-}\operatorname*{colim}_{\lambda} \mathrm{SmCor}_{S_\lambda})$$

est une équivalence de catégories. Le fait que le foncteur (40) soit une équivalence de catégories découle du corollaire 4.12 via le diagramme commutatif

$$\begin{tikzcd} 2\text{-colim}_\lambda \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_{S_\lambda}) \arrow[r] \arrow[rr, bend right, "(40)"'] & \mathrm{K}^{\mathrm{b}}(2\text{-colim}_\lambda \mathrm{SmCor}_{S_\lambda}) \arrow[r, "\mathrm{K}^{\mathrm{b}}(39)"] & \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_S). \end{tikzcd}$$

□

PROPOSITION 4.16. *La sous-catégorie épaisse $E_{gm}(S)$ est la 2-colimite des sous-catégories $E_{gm}(S_\lambda)$ et les foncteurs canoniques*

$$2\text{-}\operatorname*{colim}_{\lambda} \underline{DM}^{\mathrm{eff}}_{gm}(S_\lambda) \to \underline{DM}^{\mathrm{eff}}_{gm}(S) \tag{41}$$

$$2\text{-}\operatorname*{colim}_{\lambda} DM^{\mathrm{eff}}_{gm}(S_\lambda) \to DM^{\mathrm{eff}}_{gm}(S) \tag{42}$$

$$2\text{-}\operatorname*{colim}_{\lambda} DM_{gm}(S_\lambda) \to DM_{gm}(S) \tag{43}$$

*sont des équivalences de catégories.*

*Démonstration.* Remarquons tout d'abord que pour tout carré distingué élémentaire pour la topologie de Nisnevich sur $S_\lambda$

$$\begin{tikzcd} U_\lambda \times_{X_\lambda} V_\lambda \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V_\lambda \arrow[d] \\ U_\lambda \arrow[r] & X_\lambda \end{tikzcd} \tag{44}$$

le carré obtenu par changement de base

$$\begin{tikzcd} (S \times_{S_\lambda} U_\lambda) \times_{(S \times_{S_\lambda} X_\lambda)} (S \times_{S_\lambda} V_\lambda) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \times_{S_\lambda} V_\lambda \arrow[d] \\ S \times_{S_\lambda} U_\lambda \arrow[r] & S \times_{S_\lambda} X_\lambda \end{tikzcd} \tag{45}$$

est distingué élémentaire pour la topologie de Nisnevich sur $S$. La définition des catégories épaisses $E_{gm}(S_\lambda)$ et $E_{gm}(S)$ entraîne donc que le foncteur (40) induit un foncteur

$$\operatorname*{2\text{-}colim}_{\lambda} E_{gm}(S_\lambda) \to E_{gm}(S). \tag{46}$$

D'après le corollaire 4.15 le foncteur (46) est pleinement fidèle, en ce qui concerne la première assertion, il suffit donc de voir que les objets de $\mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_S)$ de la forme

$$[\mathbb{A}^1_X] \to [X] \qquad\qquad [U \times_X V] \to [U] \oplus [V] \to [X]$$

où $X$ est un $S$-schéma lisse de type fini et

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & V \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \tag{47}$$

un carré distingué élémentaire pour la topologie de Nisnevich sur $S$ proviennent à isomorphisme près d'éléments de la 2-colimite des $E_{gm}(S_\lambda)$. Pour le premier complexe, il s'agit d'une conséquence immédiate de l'assertion (ii) du théorème 8.2.2 de [14]. Pour le second complexe, il suffit de remarquer que le théorème 8.10.5 de *loc.cit.* entraîne l'existence d'un $\lambda \in \Lambda$ et d'un carré distingué pour la topologie Nisnevich de $S_\lambda$ de la forme (44) tel que le carré (47) soit isomorphe au carré (45) obtenu par changement de base.

Montrons maintenant que le foncteur (41) est une équivalence de catégories triangulées. On sait déjà grâce au corollaire 4.15 que ce dernier est essentiellement surjectif. Il suffit donc de prouver sa pleine fidélité. Pour simplifier les notations nous posons

$$\mathbb{K}_\lambda = \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_{S_\lambda}) \qquad \mathbb{K} = \mathrm{K}^{\mathrm{b}}(\mathrm{SmCor}_S)$$

$$\mathbb{D}_\lambda := \underline{DM}^{\mathrm{eff}}_{gm}(S_\lambda) \qquad \mathbb{D} := \underline{DM}^{\mathrm{eff}}_{gm}(S)$$

ainsi que

$$\mathbb{E} := \operatorname*{2\text{-}colim}_{\lambda} \underline{DM}^{\mathrm{eff}}_{gm}(\lambda).$$

Convenons en outre de noter par $\mathcal{S}_\lambda$ (resp. $\mathcal{S}$) la sous-catégorie de $\mathbb{K}_\lambda$ (resp. $\mathbb{K}$) ayant les mêmes objets mais dont les morphismes sont les morphismes de $\mathbb{K}_\lambda$ (resp. $\mathbb{K}$) qui deviennent des isomorphismes dans $\mathbb{D}_\lambda$ (resp. $\mathbb{D}$). On peut reformuler l'assertion que nous venons de prouver concernant les catégories $E_{gm}$ en disant que pour tout objet $M \in \mathbb{K}_\lambda$ et tout objet $M' \in \mathbb{K}_{\lambda'}$ on a un isomorphisme naturel

$$\operatorname*{colim}_{\mu \geq \lambda, \lambda'} \mathcal{S}_\mu(F_{\lambda,\mu}(M), F_{\lambda',\mu}(M')) = \mathcal{S}(\Phi(M,\lambda), \Phi(M',\lambda')) \tag{48}$$

où $\Phi$ désigne l'isomorphisme (40).

Notons $\Psi$ le morphisme (41) et fixons des objets $(M,\lambda)$ et $(M',\lambda')$ de $\mathbb{D}$. En utilisant la description de la remarque 4.13 on obtient

$$\begin{aligned}\mathrm{Hom}_{\mathbb{E}}((M,\lambda),(M',\lambda')) &= \operatorname*{colim}_{\mu} \mathrm{Hom}_{\mathbb{K}_\mu}(F_{\lambda,\mu}(M),F_{\lambda',\mu}(M')) \\ &= \operatorname*{colim}_{\mu} \operatorname*{colim}_{\substack{L\in\mathbb{K}_\mu \\ s\in\mathcal{S}_\mu(F_{\lambda',\mu}(M'),L)}} \mathrm{Hom}_{\mathbb{K}_\mu}(F_{\lambda,\mu}(M),L).\end{aligned}$$

D'autre part en utilisant le corollaire 4.15 ainsi que les isomorphismes (48) on voit que

$$\begin{aligned}\mathrm{Hom}_{\mathbb{D}}(\Psi(M,\lambda),\Psi(M',\lambda')) &= \operatorname*{colim}_{\substack{K\in\mathbb{K} \\ \sigma\in\mathcal{S}(\Phi(M',\lambda'),K)}} \mathrm{Hom}_{\mathbb{K}}(\Phi(M,\lambda),K) \\ &= \operatorname*{colim}_{\mu} \operatorname*{colim}_{\substack{K\in\mathbb{K}_\mu \\ \sigma\in(\Phi(M',\lambda'),\Phi(K,\mu))}} \mathrm{Hom}_{\mathbb{K}}(\Phi(M,\lambda),\Phi(K,\mu)) \\ &= \operatorname*{colim}_{\mu} \operatorname*{colim}_{\substack{K\in\mathbb{K}_\mu \\ \sigma\in\mathcal{S}_\mu(F_{\lambda',\mu}(M'),K)}} \mathrm{Hom}_{\mathbb{K}_\mu}(F_{\lambda,\mu}(M),K).\end{aligned}$$

Ce qui prouve la pleine fidélité recherchée.

Désignons par $(-)^\natural$ le 2-foncteur qui à une catégorie associe son enveloppe pseudo-abélienne. Le morphisme canonique

$$2\text{-}\operatorname*{colim}_{\lambda}(\mathcal{C}_\lambda^\natural) \to \mathcal{C}^\natural$$

est une équivalence de catégories comme on le voit à partir de la propriété universelle des 2-colimites et de la pseudo-abélianisation. En particulier le diagramme commutatif

$$\begin{tikzcd} 2\text{-colim}_\lambda\, \underline{DM}^{\mathrm{eff}}_{gm}(S_\lambda)^\natural \arrow[r, "\text{equiv.}"] \arrow[d, equal] & \left[2\text{-colim}_\lambda\, \underline{DM}^{\mathrm{eff}}_{gm}(S_\lambda)\right]^\natural \arrow[r, "(41)^\natural"] & \underline{DM}^{\mathrm{eff}}_{gm}(S)^\natural \arrow[d, equal] \\ 2\text{-colim}_\lambda\, DM^{\mathrm{eff}}_{gm}(S_\lambda) \arrow[rr, "(42)"'] & & DM^{\mathrm{eff}}_{gm}(S) \end{tikzcd}$$

entraîne que (42) est aussi une équivalence de catégories.

Quant à la dernière assertion, il suffit de remarquer que

$$\begin{aligned}2\text{-}\operatorname*{colim}_{\lambda} DM_{gm}(S_\lambda) &= 2\text{-}\operatorname*{colim}_{\lambda} 2\text{-}\operatorname*{colim}_{n} \left[DM^{\mathrm{eff}}_{gm}(S_\lambda)(n)\right] \\ &= 2\text{-}\operatorname*{colim}_{n} 2\text{-}\operatorname*{colim}_{\lambda} \left[DM^{\mathrm{eff}}_{gm}(S_\lambda)(n)\right] \\ &= 2\text{-}\operatorname*{colim}_{n} \left[2\text{-}\operatorname*{colim}_{\lambda} DM^{\mathrm{eff}}_{gm}(S_\lambda)\right](n) \\ &= 2\text{-}\operatorname*{colim}_{n} DM^{\mathrm{eff}}_{gm}(S)(n) = DM_{gm}(S).\end{aligned}$$

Ce qui achève la preuve de la proposition. $\square$

### 4.2.2 Réalisation modérée

Dans ce qui suit, nos hypothèses sont légèrement plus restrictives que dans le paragraphe 4.2.1.

> *Le schéma $S$ est régulier et limite projective d'un système projectif de schémas réguliers $\lambda \mapsto S_\lambda$ dont les morphismes de transition sont lisses et affines.*

Il résulte des théorèmes 8.8.2 et 8.10.5 de [14] et de la proposition 17.7.6 de [15] que la catégorie des $S$-schémas lisses de type fini admet la description suivante

$$2\text{-}\operatorname*{colim}_{\lambda} \mathrm{Sm}_{S_\lambda} = \mathrm{Sm}_S.$$

En passant à la 2-colimite sur $\Lambda$, les foncteurs de réalisation $\ell$-adique

$$\mathrm{Sm}_{S_\lambda}^{\mathrm{op}} \to \mathrm{D}^+(S_\lambda, \mathbb{Z}_\ell) \qquad \lambda \in \Lambda$$

fournissent donc un foncteur

$$\mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{D}^+(S, \mathbb{Z}_\ell)_{md}. \tag{49}$$

La conjonction du théorème 4.3 et de la description de la catégorie des motifs géométriques de la proposition 4.16 nous donne le corollaire suivant.

Corollaire 4.17. *Le foncteur de réalisation $\ell$-adique modérée (49) se prolonge canoniquement en un foncteur triangulé quasi-tensoriel*

$$R_{md,\ell} : DM_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^+(S, \mathbb{Z}_\ell)_{md}. \tag{50}$$

*(a) Lorsque $\lambda \mapsto S_\lambda$ est un système projectif de schémas de type fini sur un schéma nothérien régulier de dimension $\leq 1$, le foncteur (50) prend ses valeurs dans $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S, \mathbb{Z}_\ell)_{md}$.*

*(b) Lorsque $S$ lisse de type fini sur un corps de nombres, le foncteur (50) induit un foncteur triangulé tensoriel*

$$DM_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(S, \overline{\mathbb{Q}}_\ell)_{md}$$

*où le second membre désigne la catégorie des coefficients $\ell$-adiques mixtes modérés de [17, Definition 3.1].*

*Démonstration.* D'après le théorème 4.3 on a des foncteurs triangulés quasi-tensoriels

$$DM_{gm}(\mathcal{S}_\lambda)^{\mathrm{op}} \to \mathrm{D}(\mathcal{S}_\lambda, \mathbb{Z}_\ell) \qquad \lambda \in \Lambda.$$

Compte tenu de la proposition 4.16 et de la remarque 4.9, en passant à la 2-colimite sur $\Lambda$ on obtient un foncteur triangulé quasi-tensoriel

$$\begin{tikzcd}
2\text{-colim}_\lambda\, DM_{gm}(\mathcal{S}_\lambda)^{\mathrm{op}} \arrow[r] \arrow[d, equal] & 2\text{-colim}_\lambda\, \mathrm{D}(\mathcal{S}_\lambda, \mathbb{Z}_\ell) \arrow[d, equal] \\
DM_{gm}(S)^{\mathrm{op}} \arrow[r, dotted, "R_{md,\ell}"] & \mathrm{D}(S, \mathbb{Z}_\ell)_{md}.
\end{tikzcd}$$

Lorsque $\lambda \mapsto S_\lambda$ est un système projectif de schémas de type fini sur un schéma nothérien régulier de dimension $\leq 1$, le théorème 4.3 entraîne également que ce foncteur prend ses valeurs dans $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{Z}_\ell)_{md}$.

On peut appliquer ce résultat au cas où $\Lambda$ est l'ensemble des ouverts affines non vides d'un schéma intègre régulier $S$ de corps des fonctions $F$. Cela permet pour tout $F$-schéma lisse de type fini $X$ d'obtenir une réalisation $\ell$-adique

$$DM_{gm}(X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X, \mathbb{Z}_\ell)_{md}$$

la catégorie de droite étant étudiée dans [17] dans le cas où $F$ est un corps de nombres. En outre les résultats de [7, 8] et le lemme 4.1 assurent que le foncteur précédent induit un foncteur triangulé tensoriel

$$DM_{gm}(S)^{\mathrm{op}} \to \mathrm{D}^{\mathrm{b}}_{\mathrm{m}}(S, \overline{\mathbb{Q}}_\ell)_{md}$$

lorsque $F$ est un corps de nombres. □

## A La catégorie des coefficients $\ell$-adiques de T. Ekedahl

Avec cet appendice nous précisons la nature des coefficients $\ell$-adiques que nous utilisons pour réaliser les motifs mixtes sur $S$. Nous en profitons pour fixer les notations que nous utilisons dans la cadre $\ell$-adique. Dans son article [8], P. Deligne propose de considérer la catégorie

$$2\text{-}\lim \mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(S, \mathbb{Z}/\ell^r)$$

2-limite projective du système projectif $r \mapsto \mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(S, \mathbb{Z}/\ell^r)$ dans lequel

– les morphismes de transition sont fournis par les produits tensoriels

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(S, \mathbb{Z}/\ell^s) \xrightarrow{\mathbb{Z}_S/\ell^r \otimes^L_{\mathbb{Z}_S/\ell^s} -} \mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(S, \mathbb{Z}/\ell^r) \qquad r \geq s \; ;$$

– $\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(S, \mathbb{Z}/\ell^r)$ désigne la sous-catégorie triangulée de la catégorie dérivée $\mathrm{D}^{\mathrm{b}}(\mathbb{Z}_S/\ell^r)$ formée des objets de tor dimension finie et à cohomologie constructible.

Mais cette construction ne fournit pas en toute généralité une catégorie triangulée. En effet la 2-limite projective d'un système projectif de catégories triangulées à morphismes de transition triangulés n'est pas à priori munie d'une structure triangulée. En revanche d'après la proposition 2.2.15 de [5], cette construction devient tout à fait raisonnable lorsque le groupe abélien

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(S, \mathbb{Z}/\ell^r)}(F, G)$$

est fini pour tout objet $F, G$ de $\mathrm{D}^{\mathrm{b}}_{\mathrm{ctf}}(S, \mathbb{Z}/\ell^r)$.

En pratique on sait que cette hypothèse est satisfaite par exemple lorsque $S$ est de type fini sur $\mathbb{Z}$ ou sur le spectre d'un corps $k$ tel que, pour toute extension finie séparable de $E$ de $k$, les groupes de cohomologie Galoisienne $H^i(G_E, \mathbb{Z}/\ell)$

sont finis — ceci inclut notamment le cas où $k$ est un corps fini ou un corps algébriquement clos.

La construction de P. Deligne recouvre donc un grand nombre de situations arithmético-géométriques mais présente un inconvénient majeur relativement à notre approche : il est extrêmement malaisé de travailler « à homotopie près » dans une limite projective de catégories triangulées même lorsque chacune d'entre elle est la catégorie dérivée d'une catégorie abélienne.

## A.1 La catégorie de T. Ekedahl

La construction inconditionnelle des catégories $\mathrm{D}^+(S, \mathbb{Z}_\ell)$ donnée par T. Ekedahl dans [11] ne présente pas cet inconvénient. Cela tient au fait que le passage àla limite projective n'est pas effectuée au niveau des catégories triangulées mais au niveau des complexes. Naturellement lorsque l'hypothèse de finitude n'est plus satisfaite, le foncteur lim n'est plus exact, et la cohomologie $\ell$-adique n'est pas donnée par la simple limite

$$\lim_r H^i(X, \mathbb{Z}/\ell^r)$$

mais par la cohomologie étale continue de U. Jannsen [23] : en d'autres termes il y a lieu de dériver aussi le foncteur limite projective pour obtenir un résultat raisonnable. Le passage à la limite projective au niveau des complexes permet justement de dériver ce foncteur et la catégorie de T. Ekedahl fournit le « formalisme dérivé » correspondant à la cohomologie étale continue de U. Jannsen.

Nous rappelons maintenant la construction de la catégorie de T. Ekedahl. On peut considérer le topos $\mathbb{N}^{\mathrm{op}} S_{\mathrm{et}}$ des systèmes projectifs de faisceaux étales sur $S$ ainsi que le faisceau d'anneaux sur ce dernier

$$\mathbb{Z}_S/\ell^* : \cdots \to \mathbb{Z}_S/\ell^{r+1} \to \mathbb{Z}_S/\ell^r \to \cdots \to \mathbb{Z}_S/\ell.$$

La catégorie des $\mathbb{Z}/\ell^*$-modules est une catégorie abélienne de Grothendieck dont on note $\mathrm{D}(S, \mathbb{Z}/\ell^*)$ la catégorie dérivée. Pour un objet $F \in \mathrm{D}(S, \mathbb{Z}/\ell^*)$, on pose suivant [11]

$$\widehat{F} := L\pi^* R\pi_* F.$$

où $\pi : \mathbb{N}^{\mathrm{op}} S_{\mathrm{et}} \to S_{\mathrm{et}}$ désigne le morphisme naturel de topos. D'après la proposition 2.2 de *loc.cit.* on dispose du lemme suivant.

Lemme A.1. *Soit $F$ un objet de $\mathrm{D}(X, \mathbb{Z}/\ell^*)$. Les conditions suivantes sont équivalentes*

*(a) Le morphisme naturel $\widehat{F} \to F$ est un isomorphisme.*

*(b) Les morphismes induits par les morphismes de transition*

$$\mathbb{Z}_S/\ell^{s+1} \otimes^L_{\mathbb{Z}_S/\ell^{r+1}} F_r \to F_s \qquad r \geq s$$

*sont des isomorphismes dans $\mathrm{D}(S, \mathbb{Z}/\ell^{s+1})$.*

Définition A.2. Soit $\dagger \in \{\mathrm{b}, +, -, \varnothing\}$. La catégorie triangulée $\mathrm{D}^{\dagger}(S, \mathbb{Z}_{\ell})$ des coefficients $\ell$-adiques est la sous-catégorie pleine de $\mathrm{D}^{\dagger}(\mathbb{Z}_S/\ell^*)$ engendrée par les complexes $F$ satisfaisant les conditions équivalentes du lemme A.1. La catégorie des coefficients constructibles $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S, \mathbb{Z}_{\ell})$ est donnée par la sous-catégorie triangulée de $\mathrm{D}^{\mathrm{b}}(S, \mathbb{Z}_{\ell})$ formée des objets à cohomologie constructible.

*Remarque* A.3. Dans [11] les complexes vérifiant les conditions équivalentes du lemme A.1 sont appelés complexes normalisés.

## A.2 Coefficients $\ell$-adiques modérés

Dans cette section nous donnons la définition de la catégorie des coefficients $\ell$-adiques que nous appelons modérés. Nous nous fixons un ensemble ordonné filtrant $\Lambda$ et nous supposons que $S$ est limite projective d'un système projectif de schémas $\lambda \mapsto S_{\lambda}$ dont les morphismes de transition sont des morphismes plats affines.

Définition A.4. Nous appellerons catégorie des coefficients $\ell$-adiques modérés relativement au système projectif des $S_{\lambda}$ la catégorie triangulée

$$\mathrm{D}(S, \mathbb{Z}_{\ell})_{md} := 2\text{-}\operatorname*{colim}_{\lambda} \mathrm{D}(S_{\lambda}, \mathbb{Z}_{\ell}).$$

On pose de même

$$\mathrm{D}^{+}(S, \mathbb{Z}_{\ell})_{md} := 2\text{-}\operatorname*{colim}_{\lambda} \mathrm{D}^{+}(S_{\lambda}, \mathbb{Z}_{\ell}) \quad \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S, \mathbb{Z}_{\ell})_{md} := 2\text{-}\operatorname*{colim}_{\lambda} \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(S_{\lambda}, \mathbb{Z}_{\ell}).$$

# B Foncteurs monoïdaux

La terminologie concernant les catégories symétriques monoïdales n'étant pas entièrement fixée dans la littérature, nous précisons les conventions valables dans cet article avec la définition ci-dessous.

Définition B.1. Soient $\mathcal{A}$ et $\mathcal{B}$ deux catégories monoïdales symétriques. Un foncteur quasi-monoïdal symétrique consiste en les données suivantes.
- Un foncteur $F : \mathcal{A} \to \mathcal{B}$.
- Une transformation naturelle $\boxtimes : F(-) \otimes F(-) \to F(- \otimes -)$ vérifiant les conditions ci-dessous.
  (a) (Associativité) Pour tout objet $A, A', A''$ de $\mathcal{A}$, le diagramme

$$\begin{tikzcd}
F(A) \otimes F(A') \otimes F(A'') \arrow[r, "F(A)\otimes \boxtimes_{A',A''}"] \arrow[d, "\boxtimes_{A,A'}\otimes F(A'')"'] & F(A) \otimes F(A' \otimes A'') \arrow[d, "\boxtimes_{A,A'\otimes A''}"] \\
F(A \otimes A') \otimes F(A'') \arrow[r, "\boxtimes_{A\otimes A',A''}"'] & F(A \otimes A' \otimes A'')
\end{tikzcd}$$

  est commutatif.

(b) (Commutativité) Pour tout objet $A, A'$ de $\mathcal{A}$, le diagramme

$$\begin{tikzcd} F(A) \otimes F(A') \arrow[r] \arrow[d, "\boxtimes_{A,A'}"'] & F(A') \otimes F(A) \arrow[d, "\boxtimes_{A',A}"] \\ F(A \otimes A') \arrow[r] & F(A' \otimes A) \end{tikzcd}$$

est commutatif.

– Une transformation naturelle $1_{\mathcal{B}} \to F(1_{\mathcal{A}})$ telle que l'on ait le diagramme commutatif

$$\begin{tikzcd} 1_{\mathcal{B}} \otimes F(A) \arrow[r] \arrow[d] & F(A) \\ F(1_{\mathcal{A}}) \otimes F(A) \arrow[r, "\boxtimes_{1_{\mathcal{A}},A}"'] & F(1_{\mathcal{A}} \otimes A) \arrow[u] \end{tikzcd}$$

pour tout objet $A$ de $\mathcal{A}$.

Nous dirons que le foncteur est monoïdal symétrique lorsque le morphisme $1_{\mathcal{B}} \to F(1_{\mathcal{A}})$ est un isomorphisme et que pour tout objet $A, A'$ de $\mathcal{A}'$ le morphisme

$$\boxtimes_{A,A'} : F(A) \otimes F(A') \to F(A \otimes A')$$

est un isomorphisme. Lorsque les catégories $\mathcal{A}, \mathcal{B}$ sont additives *i.e.* tensorielles nous parlerons plutôt de foncteur quasi-tensoriel que de foncteur quasi-monoïdal symétrique, et de même de foncteur tensoriel plutôt que de foncteur monoïdal symétrique.

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# Visibility of the Shafarevich–Tate Group at Higher Level

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Abstract. We study visibility of Shafarevich–Tate groups of modular abelian varieties in Jacobians of modular curves of higher level. We prove a theorem about the existence of visible elements at a specific higher level under certain hypothesis which can be verified explicitly. We also provide a table of examples of visible subgroups at higher level and state a conjecture inspired by our data.

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## 1 Introduction

### 1.1 Motivation

Mazur originally suggested that the Shafarevich–Tate group $Ш(A/K)$ of an abelian variety $A$ over a number field $K$ could be studied via a collection of finite subgroups (the *visible subgroups*) corresponding to different embeddings of the variety into other abelian varieties $C$ over $K$ (see [Maz99] and [CM00]). The advantage of this approach is that the isomorphism classes of principal homogeneous spaces, for which one has *a priori* little geometric information, can be given a much more explicit description as $K$-rational points on the quotient abelian variety $C/A$ (the reason why they are called *visible elements*).

Agashe, Cremona, Klenke and the second author built upon the ideas of Mazur and proved many results about visibility of Shafarevich–Tate groups of abelian varieties over number fields (see [Aga99b, AS02, AS05, CM00, Kle01, Ste00]). More precisely, Agashe and Stein provided sufficient conditions for the existence of visible subgroups of certain order in the Shafarevich–Tate group and applied their general theory to the case of newform subvarieties $A_{f/\mathbb{Q}}$ of the Jacobian $J_0(N)_{/\mathbb{Q}}$ of the modular curve $X_0(N)_{/\mathbb{Q}}$ (here, $f$ is a newform of level $N$ and weight 2 which is an eigenform for the Hecke operators acting on the space $S_2(\Gamma_0(N))$ of cuspforms of level $N$ and weight 2). They gave many examples of nontrivial elements of $Ш(A_f/\mathbb{Q})$ that are visible with respect to the embedding $A_f \hookrightarrow J_0(N)$, along with many examples that are not, assuming the Birch and Swinnerton-Dyer conjecture. The results of the present paper allow us in some cases to remove this dependence on the Birch and Swinnerton-Dyer conjecture.

In this paper we consider the case of modular abelian varieties over $\mathbb{Q}$ and make use of the algebraic and arithmetic properties of the corresponding newforms to provide sufficient conditions for the existence of visible elements of $Ш(A_f/\mathbb{Q})$ in Jacobians of modular curves of levels multiples of the base level $N$. More precisely, we consider morphisms of the form $A_f \hookrightarrow J_0(N) \xrightarrow{\phi} J_0(MN)$, where $\phi$ is a suitable linear combination of degeneracy maps whose kernel is 2-torsion. For specific examples, the sufficient conditions can be verified explicitly. We also provide a table of examples where certain elements of $Ш(A_f/\mathbb{Q})$ which are not visible in $J_0(N)$ become visible at a suitably chosen higher level. At the end, we state a conjecture inspired by our results.

### 1.2 Organization of the paper

Section 2 discusses the basic definitions and notation for modular abelian varieties, modular forms, Hecke algebras, the Shimura construction and modular degrees. Section 3 is a brief introduction to visibility theory for Shafarevich–Tate groups. In Section 4 we state and prove a refinement of a theorem of Agashe-Stein (see [AS05, Thm 3.1]) which guarantees existence of visible elements. The result is stronger since it makes use of the Hecke action on the Jacobian $J_0(N)$.

In Section 5 we introduce the notion of *strong visibility* which is relevant for visualizing cohomology classes in Jacobians of modular curves whose level is a multiple of the level of the original abelian variety. Theorem 5.1.3 guarantees existence of strongly visible elements of the Shafarevich–Tate group under a hypothesis on the component groups, a congruence condition between modular forms, and irreducibility of the Galois representation. In Section 5.4 we prove a variant of the same theorem (Theorem 5.4.2) with hypotheses that are easier to verify.

Section 6 discusses in detail two computational examples for which strongly visible elements of certain order exist. These examples provide evidence for the Birch and Swinnerton-Dyer conjecture. We state a general conjecture (Conjecture 7.1.1) in Section 7 according to which every element of the Shafarevich–Tate group of a modular abelian variety becomes visible at higher level. We provide evidence for the conjecture in Section 7.2 and a table of computational data in Section 7.4.

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## 2 Notation

*1. Abelian varieties.* For a number field $K$, $A_{/K}$ denotes an abelian variety over $K$. We denote the dual of $A$ by $A^\vee_{/K}$. If $\varphi : A \to B$ is an isogeny of degree $n$, we denote the *complementary isogeny* by $\varphi'$; this is the isogeny $\phi' : B \to A$, such that $\varphi \circ \varphi' = \varphi' \circ \varphi = [n]$, the multiplication-by-$n$ map on $A$. Unless otherwise specified, Néron models of abelian varieties will be denoted by the corresponding caligraphic letters, e.g., $\mathcal{A}$ denotes the Néron model of $A$ over the ring of integers of $K$.

*2. Galois cohomology.* For a fixed algebraic closure $\overline{K}$ of $K$, $G_K$ will be the Galois group $\mathrm{Gal}(\overline{K}/K)$. If $v$ is any non-archimedean place of $K$, we let $K_v$ and $k_v$ denote the completion and the residue field of $K$ at $v$, respectively. By $K_v^{\mathrm{ur}}$ we denote the maximal unramified extension of the completion $K_v$. Given a $G_K$-module $M$, we let $\mathrm{H}^1(K, M)$ be the Galois cohomology group $\mathrm{H}^1(G_K, M)$.

*3. Component groups.* The *component group* of $A$ at $v$ is the finite group $\Phi_{A,v} = \mathcal{A}_{k_v}/\mathcal{A}^0_{k_v}$ which also has the structure of a finite group scheme over $k_v$. The *Tamagawa number* of $A$ at $v$ is $c_{A,v} = \#\Phi_{A,v}(k_v)$, and the *component group order* of $A$ at $v$ is $\overline{c}_{A,v} = \#\Phi_{A,v}(\overline{k}_v)$.

*4. Modular abelian varieties.* Let $h = 0$ or $1$. A *$J_h$-modular abelian variety* is an abelian variety $A_{/K}$ which is a quotient of $J_h(N)$ for some $N$, i.e. there exists a surjective morphism $J_h(N) \twoheadrightarrow A$ defined over $K$. We define the *level* of a modular abelian variety $A$ to be the minimal $N$, such that $A$ is a quotient of $J_h(N)$. The modularity theorem of Wiles et al. (see [BCDT01]) implies that all elliptic curves over $\mathbb{Q}$ are modular. Serre's modularity theorem (see [KhW07]) implies that the modular abelian varieties over $\mathbb{Q}$ are precisely the abelian varieties over $\mathbb{Q}$ of $\mathrm{GL}_2$-type (see [Rib92, §4]).

*5. Shimura construction.* Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_2(\Gamma_0(N))$ be a newform of level $N$ and weight 2 for $\Gamma_0(N)$ which is an eigenform for all Hecke operators in the Hecke algebra $\mathbb{T}(N)$ generated by *all* Hecke operators $T_n$ for all integers $n$. Shimura (see [Shi94, Thm. 7.14]) associated to $f$ an abelian subvariety $A_{f/\mathbb{Q}}$ of $J_0(N)$, simple over $\mathbb{Q}$, of dimension $d = [K : \mathbb{Q}]$, where $K = \mathbb{Q}(\dots, a_n, \dots)$ is the Hecke eigenvalue field. More precisely, if $I_f = \mathrm{Ann}_{\mathbb{T}(N)}(f)$ then $A_f$ is the connected component containing the identity of the $I_f$-torsion subgroup of $J_0(N)$, i.e. $A_f = J_0(N)[I_f]^0 \subset J_0(N)$. The quotient $\mathbb{T}(N)/I_f$ of the Hecke algebra $\mathbb{T}(N)$ is a subalgebra of the endomorphism ring $\mathrm{End}_{\mathbb{Q}}(A_{/\mathbb{Q}})$. Also

$L(A_f, s) = \prod_{i=1}^{d} L(f_i, s)$, where the $f_i$ are the $G_{\mathbb{Q}}$-conjugates of $f$. We also consider the dual abelian variety $A_f^\vee$ which is a quotient variety of $J_0(N)$.

*6. $I$-torsion submodules.* If $M$ is a module over a commutative ring $R$ and $I$ is an ideal of $R$, let

$$M[I] = \{x \in M : mx = 0 \text{ all } m \in I\}$$

be the $I$-torsion submodule of $M$.

*7. Hecke algebras.* Let $S_2(\Gamma)$ denote the space of cusp forms of weight 2 for any congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$. Let

$$\mathbb{T}(N) = \mathbb{Z}[\ldots, T_n, \ldots] \subseteq \mathrm{End}_{\mathbb{Q}}(J_0(N))$$

be the Hecke algebra, where $T_n$ is the $n$th Hecke operator. $\mathbb{T}(N)$ also acts on $S_2(\Gamma_0(N))$ and the integral homology $H_1(X_0(N), \mathbb{Z})$.

*8. Modular degree.* If $A$ is an abelian subvariety of $J_0(N)$, let

$$\theta : A \hookrightarrow J_0(N) \cong J_0(N)^\vee \to A^\vee$$

be the induced polarization. The *modular degree* of $A$ is

$$m_A = \sqrt{\#\,\mathrm{Ker}(A \xrightarrow{\theta} A^\vee)}.$$

See [AS02] for why $m_A$ is an integer and for an algorithm to compute it.

## 3 Visible Subgroups of Shafarevich–Tate Groups

Let $K$ be a number field and $\iota : A_{/K} \hookrightarrow C_{/K}$ be an embedding of an abelian variety into another abelian variety over $K$.

Definition 3.0.1. The *visible subgroup* of $\mathrm{H}^1(K, A)$ relative to $\iota$ is

$$\mathrm{Vis}_C\,\mathrm{H}^1(K, A) = \mathrm{Ker}\left(\iota_* : \mathrm{H}^1(K, A) \to \mathrm{H}^1(K, C)\right).$$

The visible subgroup of $Ш(A/K)$ relative to the embedding $\iota$ is

$$\begin{aligned}\mathrm{Vis}_C\,Ш(A/K) &= Ш(A/K) \cap \mathrm{Vis}_C\,\mathrm{H}^1(K, A)\\ &= \mathrm{Ker}\left(Ш(A/K) \to Ш(C/K)\right)\end{aligned}$$

Let $Q$ be the abelian variety $C/\iota(A)$, which is defined over $K$. The Galois cohomology exact sequence associated to $0 \to A \to C \to Q \to 0$ gives rise to

$$0 \to A(K) \to C(K) \to Q(K) \to \mathrm{Vis}_C\,\mathrm{H}^1(K, A) \to 0.$$

Surjectivity of the last map implies that the cohomology classes of $\mathrm{Vis}_C\,\mathrm{H}^1(K, A)$ are exactly the images of $K$-rational points on $Q$, which is why Mazur called these classes *visible*. The group $\mathrm{Vis}_C\,\mathrm{H}^1(K, A)$ is finite since it is torsion and since $Q(K)$ is finitely generated.

*Remark* 3.0.2. If $A_{/K}$ is an abelian variety and $c \in \mathrm{H}^1(K, A)$ is any cohomology class, there exists an abelian variety $C_{/K}$ and an embedding $\iota : A \hookrightarrow C$ defined over $K$, such that $c \in \mathrm{Vis}_C \mathrm{H}^1(K, A)$, i.e. $c$ is visible in $C$ (see [AS02, Prop. 1.3]). The $C$ of [AS02, Prop. 1.3] is the restriction of scalars of $A_L = A \times_K L$ down to $K$, where $L$ is any finite extension of $K$ such that $c$ has trivial image in $\mathrm{H}^1(L, A)$.

## 4 Refined Visibility

Let $K$ be a number field, let $A_{/K}$ and $B_{/K}$ be abelian subvarieties of an abelian variety $C_{/K}$, such that $C = A + B$ and $A \cap B$ is finite. Let $Q_{/K}$ denotes the quotient $C/B$. Let $N$ be a positive integer divisible by all primes of bad reduction for $C$.

Let $\ell$ be a prime such that $B[\ell] \subset A$ and $e < \ell - 1$, where $e$ is the largest ramification index of any prime of $K$ lying over $\ell$. Suppose that

$$\ell \nmid N \cdot \#B(K)_{\mathrm{tor}} \cdot \#Q(K)_{\mathrm{tor}} \cdot \prod_{v \mid N} c_{A,v} c_{B,v}.$$

Under these conditions, Agashe and Stein (see [AS02, Thm. 3.1]) constructed a homomorphism $B(K)/\ell B(K) \to Ш(A/K)[\ell]$ whose kernel has $\mathbb{F}_\ell$-dimension bounded by the rank of $A(K)$.

We refine the above theorem by taking into account the algebraic structure coming from the endomorphism ring $\mathrm{End}_K(C)$. In particular, when we apply the theory to modular abelian varieties, we would like to use the additional structure coming from the Hecke algebra. There are examples (see [AS05]) where the theorem of Agashe and Stein does not apply, but nevertheless, we can use our refinement to prove existence of visible elements of $Ш(A_f/\mathbb{Q})$ at higher level (e.g., see Propositions 6.1.3 and 6.2.1 below).

### 4.1 The main theorem

Let $A_{/K}$, $B_{/K}$, $C_{/K}$, $Q_{/K}$, $N$ and $\ell$ be as above. Let $R$ be a commutative subring of $\mathrm{End}_K(C)$ that leaves $A$ and $B$ stable, and let $\mathfrak{m}$ be a maximal ideal of $R$ of residue characteristic $\ell$. By the Néron mapping property, the subgroups $\Phi_{A,v}(k_v)$ and $\Phi_{B,v}(k_v)$ of $k_v$-points of the corresponding component groups can be viewed as $R$-modules.

Theorem 4.1.1 (Refined Visibility Theorem). *Suppose that $A(K)$ has rank zero and that the groups $Q(K)[\mathfrak{m}]$, $B(K)[\mathfrak{m}]$, $\Phi_{A,v}(k_v)[\mathfrak{m}]$ and $\Phi_{B,v}(k_v)[\ell]$ are all trivial for all nonarchimedean places $v$ of $K$. Then there is an injective homomorphism of $R/\mathfrak{m}$-vector spaces*

$$(B(K)/\ell B(K))[\mathfrak{m}] \hookrightarrow \mathrm{Vis}_C(Ш(A/K))[\mathfrak{m}]. \tag{1}$$

*Remark* 4.1.2. Applying the above result for $R = \mathbb{Z}$, we recover the result of Agashe and Stein in the case when $A(K)$ has rank zero. We could relax the hypothesis that $A(K)$ is finite and instead give a bound on the dimension of the kernel of (1) in terms of the rank of $A(K)$ similar to the bound in [AS02, Thm. 3.1]. We will not need this stronger result in our paper.

### 4.2 Some commutative algebra

Before proving Theorem 4.1.1 we recall some well-known lemmas from commutative algebra. Let $M$ be a module over a commutative ring $R$ and let $\mathfrak{m}$ be a finitely generated prime ideal of $R$.

Lemma 4.2.1. *If $M_{\mathfrak{m}}$ is Artinian, then $M_{\mathfrak{m}} \neq 0 \iff M[\mathfrak{m}] \neq 0$.*

*Proof.* We first prove that $M_{\mathfrak{m}} = 0$ implies $M[\mathfrak{m}] = 0$ by a slight modification of the proof of [AM69, Prop. I.3.8]. Suppose $M_{\mathfrak{m}} = 0$, yet there is a nonzero $x \in M[\mathfrak{m}]$. Let $I = \mathrm{Ann}_R(x)$. Then $I \neq (1)$ is an ideal that contains $\mathfrak{m}$, so $I = \mathfrak{m}$. Consider $x/1 \in M_{\mathfrak{m}}$. Since $M_{\mathfrak{m}} = 0$, we have $x/1 = 0$ and hence, $x$ is killed by some element of the set-theoretic difference $R - \mathfrak{m}$. But $\mathrm{Ann}_R(x) = \mathfrak{m}$, a contradiction, so $M[\mathfrak{m}] = 0$.

Conversely, we will show that $M_{\mathfrak{m}} \neq 0$ implies $M[\mathfrak{m}] \neq 0$. Since $M_{\mathfrak{m}}$ is Artinian over $R_{\mathfrak{m}}$, by [AM69, Prop. 6.8], $M_{\mathfrak{m}}$ has a composition series:

$$M_{\mathfrak{m}} = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = 0,$$

where each quotient $M_i/M_{i+1}$ is a simple $R_{\mathfrak{m}}$-module. In particular, $M_{n-1}$ is a simple $R_{\mathfrak{m}}$-module. Suppose $x \in M_{n-1}$ is nonzero, and let $I = \mathrm{Ann}_{R_{\mathfrak{m}}}(x)$. Then

$$R_{\mathfrak{m}}/I \cong R_{\mathfrak{m}} \cdot x \subset M_{n-1},$$

so by simplicity $R_{\mathfrak{m}}/I \cong M_{n-1}$ is simple. Thus $I = \mathfrak{m}$, otherwise $R_{\mathfrak{m}}/I$ would have $\mathfrak{m}/I$ as a proper submodule. Thus $x \in M_{n-1}[\mathfrak{m}]$ is nonzero.

Write $x = y/a$ with $y \in M$ and $a \in R - \mathfrak{m}$. Since $a \in R - \mathfrak{m}$, the element $a$ acts as a unit on $M_{\mathfrak{m}}$, hence $ax = y/1 \in M_{n-1}$ is nonzero and also still annihilated by $\mathfrak{m}$ (by commutativity).

To say that $y/1$ is annihilated by $\mathfrak{m}$ means that for all $\alpha \in \mathfrak{m}$ there exists $t \in R - \mathfrak{m}$ such that $t\alpha y = 0$ in $M$. Since $\mathfrak{m}$ is finitely generated, we can write $\mathfrak{m} = (\alpha_1, \ldots, \alpha_n)$ and for each $\alpha_i$ we get corresponding elements $t_1, \ldots, t_n$ and a product $t = t_1 \cdots t_n$. Also $t \notin \mathfrak{m}$ since $\mathfrak{m}$ is a prime ideal and each $t_i \notin \mathfrak{m}$. Let $z = ty$. Then for all $\alpha \in \mathfrak{m}$ we have $\alpha z = t\alpha y = 0$. Also $z \neq 0$ since $t$ acts as a unit on $M_{n-1}$. Thus $z \in M[\mathfrak{m}]$, and is nonzero, which completes the proof of the lemma. $\square$

Lemma 4.2.2. *Suppose $0 \to M_1 \to N \to M_2 \to 0$ is an exact sequence of $R$-modules, such that $(M_1)_{\mathfrak{m}}, N_{\mathfrak{m}}$ and $(M_2)_{\mathfrak{m}}$ are all Artinian. Then*

$$N[\mathfrak{m}] \neq 0 \iff (M_1 \oplus M_2)[\mathfrak{m}] \neq 0.$$

*Proof.* By Lemma 4.2.1 we have $N[\mathfrak{m}] \neq 0$ if and only if $N_\mathfrak{m} \neq 0$. By Proposition 3.3 on page 39 of [AM69], the localized sequence

$$0 \to (M_1)_\mathfrak{m} \to N_\mathfrak{m} \to (M_2)_\mathfrak{m} \to 0$$

is exact. Thus $N_\mathfrak{m} \neq 0$ if and only if at least one of $(M_1)_\mathfrak{m}$ or $(M_2)_\mathfrak{m}$ is nonzero. Again by Lemma 4.2.1, at least one of $(M_1)_\mathfrak{m}$ or $(M_2)_\mathfrak{m}$ is nonzero if and only if at least one of $M_1[\mathfrak{m}]$ or $M_2[\mathfrak{m}]$ is nonzero. The latter is the case if and only if $(M_1 \oplus M_2)[\mathfrak{m}] \neq 0$. □

*Remark* 4.2.3. One could also prove the lemmas by using that $M[\mathfrak{m}] \cong \mathrm{Hom}_R(R/\mathfrak{m}, M)$ and the exactness properties of Hom, but many of the same details have to be checked.

Lemma 4.2.4. *Let $G$ be a finite cyclic group, $M$ be a finite $G$-module that is also a module over a commutative ring $R$ such that the action of $G$ and $R$ commute (i.e., $M$ is an $R[G]$-module). Suppose $\mathfrak{p}$ is a finitely-generated prime ideal of $R$, and $\mathrm{H}^0(G, M)[\mathfrak{p}] = 0$. Then $\mathrm{H}^1(G, M)[\mathfrak{p}] = 0$.*

*Proof.* The proof is exactly the same as [Se79, Prop. VIII.4.8], but we note that all modules are modules over $R$ and all maps are morphisms of $R$-modules. □

## 4.3 Proof of Theorem 4.1.1

*Proof of Theorem 4.1.1.* The proof is very similar to the proof of [AS02, Thm. 3.1], except that $[\ell]$ is replaced by $[\mathfrak{m}]$ and we apply the above lemmas to verify properties of various maps between $\mathfrak{m}$-torsion modules.

We now give the details of the proof, for the benefit of the reader who is not convinced by the above brief sketch. The construction of [AS02, Lem. 3.6] yields the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & M_0 \arrow[d] & M_1 \arrow[d] & M_2 \arrow[d] & \\
0 \arrow[r] & B(K)/(B(K)[\ell]) \arrow[r, "\ell"] \arrow[d] & B(K) \arrow[r] \arrow[d] \arrow[dr, "\pi"] & B(K)/\ell B(K) \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & C(K)/A(K) \arrow[r] \arrow[d] & Q(K) \arrow[r] & \mathrm{Vis}_C(\mathrm{H}^1(K, A)) \arrow[r] & 0 \\
 & M_3. & & &
\end{tikzcd}$$

Here, $M_0$, $M_1$ and $M_2$ denote the kernels of the corresponding vertical maps and $M_3$ denotes the cokernel of the first map. Since $R$ preserves $A$, $B$, and $B[\ell]$, all objects in the diagram are $R$-module and the morphisms of abelian varieties are also $R$-module homomorphisms.

The snake lemma yields an exact sequence

$$0 \to M_0 \to M_1 \to M_2 \to M_3.$$

By hypothesis, $B(K)[\mathfrak{m}] = 0$, so $N_0 = \mathrm{Ker}(B(K) \to C(K)/A(K))$ has no $\mathfrak{m}$ torsion. Noting that $B(K)[\ell] \subset N_0$, it follows that $M_0 = N_0/(B(K)[\ell])$ has no $\mathfrak{m}$ torsion either, by Lemma 4.2.2. Also, $M_1[\mathfrak{m}] = 0$ again since $B(K)[\mathfrak{m}] = 0$.

By the long exact sequence on Galois cohomology, the quotient $C(K)/B(K)$ is isomorphic to a subgroup of $Q(K)$ and by hypothesis $Q(K)[\mathfrak{m}] = 0$, so $(C(K)/B(K))[\mathfrak{m}] = 0$. Since $Q$ is isogenous to $A$ and $A(K)$ is finite and $C(K)/B(K) \hookrightarrow Q(K)$, we see that $C(K)/B(K)$ is finite. Thus $M_3$ is a quotient of the finite $R$-module $C(K)/B(K)$ that has no $\mathfrak{m}$-torsion, so Lemma 4.2.2 implies that $M_3[\mathfrak{m}] = 0$. The same lemma implies that $M_1/M_0$ has no $\mathfrak{m}$-torsion, since it is a quotient of the finite module $M_1$ which has no $\mathfrak{m}$-torsion. Thus, we have an exact sequence

$$0 \to M_1/M_0 \to M_2 \to M_3 \to 0,$$

and both of $M_1/M_0$ and $M_3$ have trivial $\mathfrak{m}$-torsion. It follows by Lemma 4.2.2, that $M_2[\mathfrak{m}] = 0$. Therefore, we have an injective morphism of $R/\mathfrak{m}$-vector spaces

$$\varphi : (B(K)/\ell B(K))[\mathfrak{m}] \hookrightarrow \mathrm{Vis}_C(\mathrm{H}^1(K, A))[\mathfrak{m}].$$

It remains to show that for any $x \in B(K)$, we have $\varphi(x) \in \mathrm{Vis}_C(\text{Ш}(A/K))$, i.e., that $\varphi(x)$ is locally trivial.

We proceed exactly as in Section 3.5 of [AS05]. In both cases $\mathrm{char}(v) \neq \ell$ and $\mathrm{char}(v) = \ell$ we arrive at the conclusion that the restriction of $\varphi(x)$ to $\mathrm{H}^1(K_v, A)$ is an element $c \in \mathrm{H}^1(K_v^{\mathrm{ur}}/K_v, A(K_v^{\mathrm{ur}}))$. (Note that in the case $\mathrm{char}(v) \neq \ell$ the proof uses that $\ell \nmid \#\Phi_{B,v}(k_v)$.) By [Mil86, Prop I.3.8], there is an isomorphism

$$\mathrm{H}^1(K_v^{\mathrm{ur}}/K_v, A(K_v^{\mathrm{ur}})) \cong \mathrm{H}^1(\overline{k}_v/k_v, \Phi_{A,v}(\overline{k}_v)). \tag{2}$$

We will use our hypothesis that

$$\Phi_{A,v}(k_v)[\mathfrak{m}] = \Phi_{B,v}(k_v)[\ell] = 0$$

for all places $v$ of bad reduction to deduce that the image of $\varphi$ lies in $\mathrm{Vis}_C(\text{Ш}(A/K))[\mathfrak{m}]$. Let $d$ denote the image of $c$ in $\mathrm{H}^1(\overline{k}_v/k_v, \Phi_{A,v}(\overline{k}_v))$. The construction of $d$ is compatible with the action of $R$ on Galois cohomology, since (as is explained in the proof of [Mil86, Prop. I.3.8]) the isomorphism (2) is induced from the exact sequence of $\mathrm{Gal}(K_v^{\mathrm{ur}}/K_v)$-modules

$$0 \to \mathcal{A}^0(K_v^{\mathrm{ur}}) \to \mathcal{A}(K_v^{\mathrm{ur}}) \to \Phi_{A,v}(\overline{k}_v) \to 0,$$

where $\mathcal{A}$ is the Néron model of $A$ and $\mathcal{A}^0$ is the subgroup scheme whose generic fiber is $A$ and whose closed fiber is the identity component of $\mathcal{A}_{k_v}$. Since $\varphi(x) \in \mathrm{H}^1(K, A)[\mathfrak{m}]$, it follows that

$$d \in \mathrm{H}^1(\overline{k}_v/k_v, \Phi_{A,v}(\overline{k}_v))[\mathfrak{m}].$$

Lemma 4.2.4, our hypothesis that $\Phi_{A,v}(k_v)[\mathfrak{m}] = 0$, and that

$$\mathrm{H}^1(\overline{k}_v/k_v, \Phi_{A,v}(\overline{k}_v)) = \varinjlim \mathrm{H}^1(\mathrm{Gal}(k'_v/k_v), \Phi_{A,v}(k'_v))),$$

together imply that $\mathrm{H}^1(\overline{k}_v/k_v, \Phi_{A,v}(\overline{k}_v))[\mathfrak{m}] = 0$, hence $d = 0$. Thus $c = 0$, so $\varphi(x)$ is locally trivial, which completes the proof. □

# 5 Strong Visibility at Higher Level

## 5.1 Strongly visible subgroups

Let $A_{/\mathbb{Q}}$ be an abelian subvariety of $J_0(N)_{/\mathbb{Q}}$ and let $p \nmid N$ be a prime. Let

$$\varphi = \delta_1^* + \delta_p^* : J_0(N) \to J_0(pN), \tag{3}$$

where $\delta_1^*$ and $\delta_p^*$ are the pullback maps on equivalence classes of degree-zero divisors of the degeneracy maps $\delta_1, \delta_p : X_0(pN) \to X_0(N)$. Let $\mathrm{H}^1(\mathbb{Q}, A)^{\mathrm{odd}}$ be the prime-to-2-part of the group $\mathrm{H}^1(\mathbb{Q}, A)$.

Definition 5.1.1 (Strong Visibility). The *strongly visible* subgroup of $\mathrm{H}^1(\mathbb{Q}, A)$ for $J_0(pN)$ is

$$\mathrm{Vis}_{pN}\,\mathrm{H}^1(\mathbb{Q}, A) = \mathrm{Ker}\left(\mathrm{H}^1(\mathbb{Q}, A)^{\mathrm{odd}} \xrightarrow{\varphi_*} \mathrm{H}^1(\mathbb{Q}, J_0(pN))\right) \subset \mathrm{H}^1(\mathbb{Q}, A).$$

Also,

$$\mathrm{Vis}_{pN}\,\text{Ш}(A/\mathbb{Q}) = \text{Ш}(A/\mathbb{Q}) \cap \mathrm{Vis}_{pN}\,\mathrm{H}^1(\mathbb{Q}, A).$$

The reason we replace $\mathrm{H}^1(\mathbb{Q}, A)$ by $\mathrm{H}^1(\mathbb{Q}, A)^{\mathrm{odd}}$ is that the kernel of $\varphi$ is 2-torsion (see [Rib90b]).

*Remark* 5.1.2. We could obtain more visible subgroups by considering the map $\delta_1^* - \delta_p^*$ in Definition 5.1.1. However, the methods of this paper do not apply to this map.

For a positive integer $N$, let

$$\nu(N) = \frac{1}{6} \cdot \prod_{q^r \| N} (q^r + q^{r-1}) \frac{1}{6} \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)].,$$

where each $q^r$ exactly divides $N$.

We refer to $\nu(N)$ as the *Sturm bound* (see [Stu87]).

Theorem 5.1.3. *Let $A = A_f$ be a newform abelian subvariety of $J_0(N)$ for which $L(A_{/\mathbb{Q}}, 1) \neq 0$ and let $p \nmid N$ be a prime. Suppose that there is a maximal ideal $\lambda \subset \mathbb{T}(N)$ and an elliptic curve $E_{/\mathbb{Q}}$ of conductor $pN$ such that the following properties are satisfied:*

1. *[Nondivisibility] The residue characteristic $\ell$ of $\lambda$ satisfies*

$$\ell \nmid 2 \cdot N \cdot p \cdot \prod_{q \mid N} c_{E,q}.$$

2. *[Component Groups] For each prime $q \mid N$,*

$$\Phi_{A,q}(\mathbb{F}_q)[\lambda] = 0.$$

3. *[Fourier Coefficients] Let $a_n(E)$ be the $n$-th Fourier coefficient of the modular form attached to $E$, and $a_n(f)$ the $n$-th Fourier coefficient of $f$. We have $a_p(E) = -1$,*

$$a_p(f) \equiv -(p+1) \pmod{\lambda} \quad \textit{and} \quad a_q(f) \equiv a_q(E) \pmod{\lambda},$$

*for all primes $q \neq p$ with $q \leq \nu(pN)$.*

4. *[Irreducibility] The mod $\ell$ representation $\overline{\rho}_{E,\ell}$ is irreducible.*

*Then there is an injective homomorphism*

$$E(\mathbb{Q})/\ell E(\mathbb{Q}) \hookrightarrow \mathrm{Vis}_{pN}(\text{Ш}(A_f/\mathbb{Q}))[\lambda].$$

*Remark* 5.1.4. In fact, we have

$$E(\mathbb{Q})/\ell E(\mathbb{Q}) \hookrightarrow \mathrm{Ker}(\text{Ш}(A_f/\mathbb{Q}) \to \text{Ш}(C/\mathbb{Q}))[\lambda] \subset \mathrm{Vis}_{pN}(\text{Ш}(A_f/\mathbb{Q}))[\lambda],$$

where $C \subset J_0(pN)$ is isogenous to $A_f \times E$.

## 5.2 Some auxiliary lemmas

We will use the following lemmas in the proof of Theorem 5.1.3. The notation is as in the previous section. In addition, if $f \in S_2(\Gamma_0(N))$, we denote by $a_n(f)$ the $n$-th Fourier coefficient of $f$ and by $K_f$ and $\mathcal{O}_f$ the Hecke eigenvalue field and its ring of integers, respectively.

Lemma 5.2.1. *Suppose $A_f \subset J_0(N)$ and $A_g \subset J_0(pN)$ are attached to newforms $f$ and $g$ of level $N$ and $pN$, respectively, with $p \nmid N$. Suppose that there is a prime ideal $\lambda$ of residue characteristic $\ell \nmid 2pN$ of an integrally closed subgroup $\mathcal{O}$ of $\overline{\mathbb{Q}}$ that contains the ring of integers of the composite field $K = K_f K_g$ such that for $q \leq \nu(pN)$,*

$$a_q(f) \equiv \begin{cases} a_q(g) \pmod{\lambda} & \textit{if } q \neq p, \\ (p+1)a_p(g) \pmod{\lambda} & \textit{if } q = p. \end{cases}$$

*Assume that $a_p(g) = -1$. Let $\lambda_f = \mathcal{O}_f \cap \lambda$ and $\lambda_g = \mathcal{O}_g \cap \lambda$ and assume that $A_f[\lambda_f]$ is an irreducible $G_{\mathbb{Q}}$-module. Then we have an equality*

$$\varphi(A_f[\lambda_f]) = A_g[\lambda_g]$$

*of subgroups of $J_0(pN)$, where $\varphi$ is the morphism of equation* (3) *from Section 5.1.*

*Proof.* Our hypothesis that $a_p(f) \equiv -(p+1) \pmod{\lambda_f}$ implies, by the proofs in [Rib90b], that

$$\varphi(A_f[\lambda_f]) \subset \varphi(A_f) \cap J_0(pN)_{p\text{-new}},$$

where $J_0(pN)_{p\text{-new}}$ is the $p$-new abelian subvariety of $J_0(N)$.

By [Rib90b, Lem. 1], the operator $U_p = T_p$ on $J_0(pN)$ acts as $-1$ on $\varphi(A_f[\lambda_f])$. Consider the action of $U_p$ on the 2-dimensional vector space spanned by $\{f(q), f(q^p)\}$. The matrix of $U_p$ with respect to this basis is

$$U_p = \begin{pmatrix} a_p(f) & p \\ -1 & 0 \end{pmatrix}.$$

In particular, neither of $f(q)$ and $f(q^p)$ is an eigenvector for $U_p$. The characteristic polynomial of $U_p$ acting on the span of $f(q)$ and $f(q^p)$ is $x^2 - a_p(f)x + p$. Using our hypothesis on $a_p(f)$ again, we have

$$x^2 - a_p(f)x + p \equiv x^2 + (p+1)x + p \equiv (x+1)(x+p) \pmod{\lambda}.$$

Thus we can choose an algebraic integer $\alpha$ such that

$$f_1(q) = f(q) + \alpha f(q^p)$$

is an eigenvector of $U_p$ with eigenvalue congruent to $-1$ modulo $\lambda$. (It does not matter whether $x^2 + a_p(f)x + p$ has distinct roots; nonetheless, since $p \nmid N$, [CV92, Thm. 2.1] implies that it does have distinct roots.) The cusp form $f_1$ has the same prime-indexed Fourier coefficients as $f$ at primes other than $p$. If necessary, replace $\mathcal{O}$ by $\mathcal{O}[\alpha]$ so that $\alpha \in \mathcal{O}$. The $p$-th coefficient of $f_1$ is congruent modulo $\lambda$ to $-1$ and $f_1$ is an eigenvector for the full Hecke algebra. It follows from the recurrence relation for coefficients of the eigenforms that

$$a_n(g) \equiv a_n(f_1) \pmod{\lambda}$$

for all integers $n \leq \nu(pN)$.

By [Stu87], we have $g \equiv f_1 \pmod{\lambda}$, so $a_q(g) \equiv a_q(f) \pmod{\lambda}$ for all primes $q \neq p$. Thus by the Brauer-Nesbitt theorem [CR62, page. 215], the 2-dimensional $G_{\mathbb{Q}}$-representations $\varphi(A_f[\lambda_f])$ and $A_g[\lambda_g]$ are isomorphic.

Because $A_g[\lambda_g]$ is irreducible as a Galois module, the annihilator $\mathfrak{m}$ of $A_g[\lambda_g]$ in the Hecke algebra $\mathbb{T}(pN)$ is a maximal ideal. Thus $\mathfrak{m}$ gives rise to an irreducible Galois representation $\overline{\rho}_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{T}(pN)/\mathfrak{m})$ isomorphic to $A_g[\lambda_g]$. Finally, we apply [Wil95, Thm. 2.1(i)] for $H = (\mathbb{Z}/N\mathbb{Z})^\times$ (i.e., $J_H = J_0(N)$) to conclude that $J_0(N)(\overline{\mathbb{Q}})[\mathfrak{m}] \cong (\mathbb{T}(pN)/\mathfrak{m})^2$, i.e., the representation $\overline{\rho}_{\mathfrak{m}}$ occurs with multiplicity one in $J_0(pN)$. Thus

$$A_g[\lambda_g] = \varphi(A_f[\lambda_f]).$$

□

Lemma 5.2.2. *Suppose $\varphi : A \to B$ and $\psi : B \to C$ are homomorphisms of abelian varieties over a number field $K$, with $\varphi$ an isogeny and $\psi$ injective. Suppose $n$ is an integer that is relatively prime to the degree of $\varphi$. If $G = \mathrm{Vis}_C(\text{Ш}(B/\mathbb{Q}))[n^\infty]$, then there is some injective homomorphism*

$$f : G \hookrightarrow \mathrm{Ker}\,\{(\psi \circ \varphi)_* : \text{Ш}(A/\mathbb{Q}) \longrightarrow \text{Ш}(C/\mathbb{Q})\},$$

*such that $\varphi_*(f(G)) = G$.*

*Proof.* Let $m$ be the degree of the isogeny $\varphi : A \to B$. Consider the complementary isogeny $\varphi' : B \to A$, which satisfies $\varphi \circ \varphi' = \varphi' \circ \varphi = [m]$. By hypothesis $m$ is coprime to $n$, so $\gcd(m, \#G) = \gcd(m, n^\infty) = 1$, hence

$$\varphi_*(\varphi'_*(G)) = [m]G = G.$$

Thus $\varphi'_*(G)$ maps, via $\varphi_*$, to $G \subset \text{Ш}(B/\mathbb{Q})$, which in turn maps to 0 in $\text{Ш}(C/\mathbb{Q})$. □

Lemma 5.2.3. *Let $M$ be an odd integer coprime to $N$ and let $R$ be the subring of $\mathbb{T}(N)$ generated by all Hecke operators $T_n$ with $\gcd(n, M) = 1$. Then $R = \mathbb{T}(N)$.*

*Proof.* See the lemma on page 491 of [Wil95]. (The condition that $M$ is odd is necessary, as there is a counterexample when $N = 23$ and $M = 2$.) □

Lemma 5.2.4. *Suppose $\lambda$ is a maximal ideal of $\mathbb{T}(N)$ with generators a prime $\ell \in \mathbb{Z}$ and $T_n - a_n$, with $a_n \in \mathbb{Z}$. For each integer $p \nmid N$, and let $\lambda_p$ be the ideal in $\mathbb{T}(N)$ generated by $\ell$ and all $T_n - a_n$ with with $p \nmid n$. Then $\lambda = \lambda_p$.*

*Proof.* Since $\lambda_p \subset \lambda$ and $\lambda$ is maximal, it suffices to prove that $\lambda_p$ is maximal. Let $R$ be the subring of $\mathbb{T}(N)$ generated by Hecke operators $T_n$ with $p \nmid n$. The quotient $R/\lambda_p$ is a quotient of $\mathbb{Z}$ since each generator $T_n$ is equivalent to an integer. Also, $\ell \in \lambda_p$, so $R/\lambda_p = \mathbb{F}_\ell$. But by Lemma 5.2.3, $R = \mathbb{T}(N)$, so $\mathbb{T}(N)/\lambda_p = \mathbb{F}_\ell$, hence $\lambda_p$ is a maximal ideal. □

Lemma 5.2.5. *Suppose that $A$ is an abelian variety over a field $K$. Let $R$ be a commutative subring of $\mathrm{End}(A)$ and $I$ an ideal of $R$. Then*

$$(A/A[I])[I] \cong A[I^2]/A[I],$$

*where the isomorphism is an isomorphism of $R[G_K]$-modules.*

*Proof.* Let $a + A[I]$ for some $a \in A$ be an $I$-torsion element of $A/A[I]$. Then by definition, $xa \in A[I]$ for each $x \in I$. Therefore, $a \in A[I^2]$. Thus $a + A[I] \mapsto a + A[I]$ determines a well-defined homomorphism of $R[G_K]$-modules

$$\varphi : (A/A[I])[I] \to A[I^2]/A[I].$$

Clearly this homomorphism is injective. It is also surjective as every element $a + A[I] \in A[I^2]/A[I]$ is $I$-torsion as an element of $A/A[I]$, as $Ia \in A[I]$. Therefore, $\varphi$ is an isomorphism of $R[G_K]$-modules. □

Lemma 5.2.6. *Suppose $\ell$ is a prime and $\phi : E \to E'$ is an isogeny of degree coprime to $\ell$ over a number field $K$ between two elliptic curve over $K$. If $v$ is any place of $K$ then $\ell \mid c_{E,v}$ if and only if $\ell \mid c_{E',v}$.*

*Proof.* Consider the complementary isogeny $\phi' : E' \to E$. Both $\phi$ and $\phi'$ induce homomorphisms $\phi : \Phi_{E,v}(k_v) \to \Phi_{E',v}(k_v)$ and $\phi' : \Phi_{E',v}(k_v) \to \Phi_{E,v}(k_v)$ and $\phi \circ \phi'$ and $\phi' \circ \phi$ are multiplication-by-$n$ maps. Since $(n, \ell) = 1$ then $\# \ker \phi$ and $\# \ker \phi'$ must be coprime to $\ell$ which implies the statement. ☐

## 5.3 Proof of Theorem 5.1.3

*Proof of Theorem 5.1.3.* By [BCDT01] $E$ is modular, so there is a rational newform $f \in S_2^{\mathrm{new}}(pN)$ which is an eigenform for the Hecke operators and an isogeny $E \to E_f$ defined over $\mathbb{Q}$, which by Hypothesis 4 can be chosen to have degree coprime to $\ell$. Indeed, every cyclic rational isogeny is a composition of rational isogenies of prime degree, and $E$ admits no rational $\ell$-isogeny since $\overline{\rho}_{E,\ell}$ is irreducible.

By Hypothesis 1 the Tamagawa numbers of $E$ are coprime to $\ell$. Since $E$ and $E_f$ are related by an isogeny of degree coprime to $\ell$, the Tamagawa numbers of $E_f$ are also not divisible by $\ell$ by Lemma 5.2.6. Moreover, note that

$$E(\mathbb{Q}) \otimes \mathbb{F}_\ell \cong E_f(\mathbb{Q}) \otimes \mathbb{F}_\ell.$$

Let $\mathfrak{m}$ be the ideal of $\mathbb{T}(pN)$ generated by $\ell$ and $T_n - a_n(E)$ for all integers $n$ coprime to $p$. Note that $\mathfrak{m}$ is maximal by Lemma 5.2.4.

Let $\varphi$ be as in (3), and let $A = \varphi(A_f)$. Note that if $T_n \in \mathbb{T}(pN)$ then $T_n(E_f) \subset E_f$ since $E_f$ is attached to a newform, and if, moreover $p \nmid n$, then $T_n(A) \subset A$ since the Hecke operators with index coprime to $p$ commute with the degeneracy maps. Lemma 5.2.1 implies that

$$E_f[\ell] = E_f[\mathfrak{m}] = \varphi(A_f[\lambda]) \subset A,$$

so $\Psi = E_f[\ell]$ is a subgroup of $A$ as a $G_{\mathbb{Q}}$-module. Let

$$C = (A \times E_f)/\Psi,$$

where we embed $\Psi$ in $A \times E_f$ anti-diagonally, i.e., by the map $x \mapsto (x, -x)$. The antidiagonal map $\Psi \to A \times E_f$ commutes with the Hecke operators $T_n$ for $p \nmid n$, so $(A \times E_f)/\Psi$ is preserved by the $T_n$ with $p \nmid n$. Let $R$ be the subring of $\mathrm{End}(C)$ generated by the action of all Hecke operators $T_n$, with $p \nmid n$. Also note that $T_p \in \mathrm{End}(J_0(pN))$ acts by Hypothesis 3 as $-1$ on $E_f$, but $T_p$ need *not* preserve $A$.

Suppose for the moment that we have verified that the hypothesis of Theorem 4.1.1 are satisfied with $A$, $B = E_f$, $C$, $Q = C/B$, $R$ as above and $K = \mathbb{Q}$. Then we obtain an injective homomorphism

$$E(\mathbb{Q})/\ell E(\mathbb{Q}) \cong E_f(\mathbb{Q})/\ell E_f(\mathbb{Q}) \hookrightarrow \mathrm{Ker}(Ш(A/\mathbb{Q}) \to Ш(C/\mathbb{Q}))[\mathfrak{m}].$$

We then apply Lemma 5.2.2 with $n = \ell$, $A_f$, $A$, and $C$, respectively, to see that

$$E_f(\mathbb{Q})/\ell E_f(\mathbb{Q}) \subset \mathrm{Ker}(\text{Ш}(A_f/\mathbb{Q}) \to \text{Ш}(C/\mathbb{Q}))[\lambda].$$

That $E_f(\mathbb{Q})/\ell E_f(\mathbb{Q})$ lands in the $\lambda$-torsion is because the subgroup of $\mathrm{Vis}_C(\text{Ш}(E_f/\mathbb{Q}))$ that we constructed is $\mathfrak{m}$-torsion.

Finally, consider $A \times E_f \to J_0(pN)$ given by $(x, y) \mapsto x + y$. Note that $\Psi$ maps to 0, since $(x, -x) \mapsto 0$ and the elements of $\Psi$ are of the form $(x, -x)$. We have a (not-exact!) sequence of maps

$$\text{Ш}(A_f/\mathbb{Q}) \to \text{Ш}(C/\mathbb{Q}) \to \text{Ш}(J_0(pN)/\mathbb{Q}),$$

hence inclusions

$$\begin{aligned} E_f(\mathbb{Q})/\ell E_f(\mathbb{Q}) &\subseteq \mathrm{Ker}(\text{Ш}(A_f/\mathbb{Q}) \to \text{Ш}(C/\mathbb{Q})) \\ &\subseteq \mathrm{Ker}(\text{Ш}(A_f/\mathbb{Q}) \to \text{Ш}(J_0(pN)/\mathbb{Q})), \end{aligned}$$

which gives the conclusion of the theorem.

It remains to verify the hypotheses of Theorem 4.1.1. That $C = A + B$ is clear from the definition of $C$. Also, $A \cap E_f = E_f[\ell]$, which is finite. We explained above when defining $R$ that each of $A$ and $E_f$ is preserved by $R$. Since $K = \mathbb{Q}$ and $\ell$ is odd the condition $1 = e < \ell - 1$ is satisfied. That $A(\mathbb{Q})$ is finite follows from our hypothesis that $L(A_f, 1) \neq 0$ (by [KL89]).

It remains is to verify that the groups

$$Q(\mathbb{Q})[\mathfrak{m}], \quad E_f(\mathbb{Q})[\mathfrak{m}], \quad \Phi_{A,q}(\mathbb{F}_q)[\mathfrak{m}], \quad \text{and } \Phi_{E_f,q}(\mathbb{F}_q)[\ell],$$

are 0 for all primes $q \mid pN$. Since $\ell \in \mathfrak{m}$, we have by Hypothesis 4 that

$$E_f(\mathbb{Q})[\mathfrak{m}] = E_f(\mathbb{Q})[\ell] = 0.$$

We will now verify that $Q(\mathbb{Q})[\mathfrak{m}] = 0$. From the definition of $C$ and $\Psi$ we have $Q \cong A/\Psi$. Let $\lambda_p$ be as in Lemma 5.2.4 with $a_n = a_n(E)$. The map $\varphi$ induces an isogeny of 2-power degree

$$A_f/(A_f[\lambda]) \to A/\Psi.$$

Thus there is $\lambda_p$-torsion in $(A_f/(A_f[\lambda]))(\mathbb{Q})$ if and only if there is $\mathfrak{m}$-torsion in $(A/\Psi)(\mathbb{Q})$. (Note that $\lambda_p$ and $\mathfrak{m}$ are both ideals generated by $\ell$ and $T_n - a_n$ for all $n$ coprime to $p$, but for $\lambda_p$ the $T_n \in \mathbb{T}(N)$, and for $\mathfrak{m}$ they are in $\mathbb{T}(pN)$.) Thus it suffices to prove that $(A_f/A_f[\lambda])(\mathbb{Q})[\lambda_p] = 0$.

By Lemma 5.2.4, we have $\lambda_p = \lambda$, and by Lemma 5.2.5,

$$(A_f/A_f[\lambda])[\lambda] \cong A_f[\lambda^2]/A_f[\lambda].$$

By [Maz77, §II.14], the quotient $A_f[\lambda^2]/A_f[\lambda]$ injects into a direct sum of copies of $A_f[\lambda]$ as Galois modules. But $A_f[\lambda] \cong E[\ell]$ is irreducible, so $(A_f[\lambda^2]/A_f[\lambda])(\mathbb{Q}) = 0$, as required.

By Hypothesis 2, we have $\Phi_{A_f,q}(\mathbb{F}_q)[\lambda] = 0$ for each prime divisor $q \mid N$. Since $A$ is 2-power isogenous to $A_f$ and $\ell$ is odd, this verifies the Tamagawa number hypothesis for $A$. Our hypothesis that $a_p(E) = -1$ implies that $\mathrm{Frob}_p$ on $\Phi_{E_f,p}(\overline{\mathbb{F}}_p)$ acts as $-1$. Thus $\Phi_{E_f,p}(\mathbb{F}_p)[\ell] = 0$ since $\ell$ is odd. This completes the proof. □

*Remark* 5.3.1. An essential ingrediant in the proof of the above theorem is the multiplicity one result used in the paper of Wiles (see [Wil95, Thm. 2.1.]). Since this result holds for Jacobians $J_H$ of the curves $X_H(N)$ that are intermediate covers for the covering $X_1(N) \to X_0(N)$ corresponding to subgroups $H \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$ (i.e., the Galois group of $X_1(N) \to X_H$ is $H$), one should be able to give a generalization of Theorem 5.1.3 which holds for newform subvarieties of $J_H$. This would likely require generalizing some of [Rib90b] to the case of arbitrary $H$.

## 5.4 A Variant of Theorem 5.1.3 with Simpler Hypothesis

PROPOSITION 5.4.1. *Suppose $A = A_f \subset J_0(N)$ is a newform abelian variety and $q$ is a prime that exactly divides $N$. Suppose $\mathfrak{m} \subset \mathbb{T}(N)$ is a non-Eisenstein maximal ideal of residue characteristic $\ell$ and that $\ell \nmid m_A$, where $m_A$ is the modular degree of $A$. Then $\Phi_{A,q}(\overline{\mathbb{F}}_q)[\mathfrak{m}] = 0$.*

*Proof.* The component group of $\Phi_{J_0(N),q}(\overline{\mathbb{F}}_q)$ is Eisenstein by [Rib87], so

$$\Phi_{J_0(N),q}(\overline{\mathbb{F}}_q)[\mathfrak{m}] = 0.$$

By Lemma 4.2.2, the image of $\Phi_{J_0(N),q}(\overline{\mathbb{F}}_q)$ in $\Phi_{A^\vee,q}(\overline{\mathbb{F}}_q)$ has no $\mathfrak{m}$ torsion. By the main theorem of [CS01], the cokernel $\Phi_{J_0(N),q}(\overline{\mathbb{F}}_q)$ in $\Phi_{A^\vee,q}(\overline{\mathbb{F}}_q)$ has order that divides $m_A$. Since $\ell \nmid m_A$, it follows that the cokernel also has no $\mathfrak{m}$ torsion. Thus Lemma 4.2.2 implies that $\Phi_{A^\vee,q}(\overline{\mathbb{F}}_q)[\mathfrak{m}] = 0$. Finally, the modular polarization $A \to A^\vee$ has degree coprime to $\ell$, so the induced map $\Phi_{A,q}(\overline{\mathbb{F}}_q) \to \Phi_{A^\vee,q}(\overline{\mathbb{F}}_q)$ is an isomorphism on $\ell$ primary parts. In particular, that $\Phi_{A^\vee,q}(\overline{\mathbb{F}}_q)[\mathfrak{m}] = 0$ implies that $\Phi_{A,q}(\overline{\mathbb{F}}_q)[\mathfrak{m}] = 0$. □

If $E$ is a semistable elliptic curve over $\mathbb{Q}$ with discriminant $\Delta$, then we see using Tate curves that $\overline{c}_p = \mathrm{ord}_p(\Delta)$.

THEOREM 5.4.2. *Suppose $A = A_f \subset J_0(N)$ is a newform abelian variety with $L(A_{/\mathbb{Q}}, 1) \neq 0$ and $N$ square free, and let $\ell$ be a prime. Suppose that $p \nmid N$ is a prime, and that there is an elliptic curve $E$ of conductor $pN$ such that:*

1. *[Rank] The rank of $E(\mathbb{Q})$ is positive.*
2. *[Divisibility] We have $\ell \mid \overline{c}_{E,p}$ but $\ell \nmid 2 \cdot N \cdot p \cdot c_{E,p} \cdot \prod_{q \mid N} \overline{c}_{E,q}$.*
3. *[Irreducibility] The mod $\ell$ representation $\overline{\rho}_{E,\ell}$ is irreducible.*

4. *[Noncongruence] The representation $\overline{\rho}_{E,\ell}$ is not isomorphic to any representation $\overline{\rho}_{g,\lambda}$ where $g \in S_2(\Gamma_0(N))$ is a newform of level dividing $N$ that is not conjugate to $f$.*

*Then there is an element of order $\ell$ in $\text{Ш}(A_f/\mathbb{Q})$ that is not visible in $J_0(N)$ but is strongly visible in $J_0(pN)$. More precisely, there is an inclusion*

$$E(\mathbb{Q})/\ell E(\mathbb{Q}) \hookrightarrow \mathrm{Ker}(\text{Ш}(A_f/\mathbb{Q}) \to \text{Ш}(C/\mathbb{Q}))[\lambda] \subset \mathrm{Vis}_{pN}(\text{Ш}(A_f/\mathbb{Q}))[\lambda],$$

*where $C \subset J_0(pN)$ is isogenous to $A_f \times E$, the homomorphism $A_f \to C$ has degree a power of 2, and $\lambda$ is the maximal ideal of $\mathbb{T}(N)$ corresponding to $\overline{\rho}_{E,\ell}$.*

*Proof.* The divisibility assumptions of Hypothesis 2 on the $\overline{c}_{E,q}$ imply that the Serre level of $\overline{\rho}_{E,\ell}$ is $N$ and since $\ell \nmid N$, the Serre weight is 2 (see [RS01, Thm. 2.10]). We have $\overline{c}_{E,p} \neq c_{E,p}$ since one is divisible by $\ell$ and the other is not, so $E$ has nonsplit multiplicative reduction, hence $a_p(E) = -1$. Since $\ell$ is odd, Ribet's level lowering theorem [Rib91] implies that there is *some* newform $h = \sum b_n q^n \in S_2(\Gamma_0(N))$ and a maximal ideal $\lambda$ over $\ell$ such that $a_q(E) \equiv b_q \pmod{\lambda}$ for all primes $q \neq p$. By our non-congruence hypothesis, the only possibility is that $h$ is a $G_{\mathbb{Q}}$-conjugate of $f$. Since we can replace $f$ by any Galois conjugate of $f$ without changing $A_f$, we may assume that $f = h$. Also $a_p(f) \equiv -(p+1) \pmod{\lambda}$, as explained in [Rib83, pg. 506].

Hypothesis 3 implies that $\lambda$ is not Eisenstein, and by assumption $\ell \nmid m_A$, so Proposition 5.4.1 implies that $\Phi_{A,q}(\overline{\mathbb{F}}_q)[\lambda] = 0$ for each $q \mid N$.

The theorem now follows from Theorem 5.1.3. □

*Remark* 5.4.3. The non-congruence hypothesis of Theorem 5.4.2 can be verified using modular symbols as follows. Let $W \subset H_1(X_0(N), \mathbb{Z})_{\mathrm{new}}$ be the saturated submodule of $H_1(X_0(N), \mathbb{Z})$ that corresponds to all newforms in $S_2(\Gamma_0(N))$ that are not Galois conjugate to $f$. Let $\overline{W} = W \otimes \mathbb{F}_\ell$. We require that the intersection of the kernels of $T_q|_{\overline{W}} - a_q(E)$, for $q \neq p$, has dimension 0.

## 6 Computational Examples

In this section we give examples that illustrate how to use Theorem 5.4.2 to prove existence of elements of the Shafarevich–Tate group of a newform subvariety of $J_0(N)$ (for 767 and 959) which are invisible at the base level, but become visible in a modular Jacobian of higher level.

*Hypothesis* 6.0.4. The statements in this section all make the hypothesis that certain commands of various computer algebra systems such as Magma [BCP97] produced correct output.

The main point of the examples below is to clearly illustrate how the theoretical quantities elsewhere in this paper behave in practice.

## 6.1 Level 767

Consider the modular Jacobian $J_0(767)$. Using the modular symbols package in Magma, one decomposes $J_0(767)$ (up to isogeny) into a product of six optimal quotients of dimensions 2, 3, 4, 10, 17 and 23. The duals of these quotients are subvarieties $A_2, A_3, A_4, A_{10}, A_{17}$ and $A_{23}$ defined over $\mathbb{Q}$, where $A_d$ has dimension $d$. Consider the subvariety $A_{23}$.

We first show that the Birch and Swinnerton-Dyer conjectural formula predicts that the orders of the groups $Ш(A_{23}/\mathbb{Q})$ and $Ш(A_{23}^\vee/\mathbb{Q})$ are both divisible by 9.

Proposition 6.1.1. *Assume [AS05, Conj. 2.2]. Then*

$$3^2 \mid \#Ш(A_{23}/\mathbb{Q}) \quad \textit{and} \quad 3^2 \mid \#Ш(A_{23}^\vee/\mathbb{Q}).$$

*Proof.* Let $A = A_{23}^\vee$. We use [AS05, §3.5 and §3.6] (see also [Ka81]) to compute a multiple of the order of the torsion subgroup $A(\mathbb{Q})_{\mathrm{tor}}$. This multiple is obtained by injecting the torsion subgroup into the group of $\mathbb{F}_p$-rational points on the reduction of $A$ for odd primes $p$ of good reduction and then computing the order of that group. Hence, the multiple is an isogeny invariant, so one gets the same multiple for $A^\vee(\mathbb{Q})_{\mathrm{tor}}$. For producing a divisor of $\#A(\mathbb{Q})_{\mathrm{tor}}$, we use the injection of the subgroup of rational cuspidal divisor classes of degree 0 into $A(\mathbb{Q})_{\mathrm{tor}}$. Using the implementation in Magma we obtain $120 \mid \#A(\mathbb{Q})_{\mathrm{tor}} \mid 240$. To compute a divisor of $A^\vee(\mathbb{Q})_{\mathrm{tor}}$, we use the algorithm described in [AS05, §3.3] to find that the modular degree $m_A = 2^{34}$, which is not divisible by any odd primes, hence $15 \mid \#A^\vee(\mathbb{Q})_{\mathrm{tor}} \mid 240$.

Next, we use [AS05, §4] to compute the ratio of the special value of the $L$-function of $A_{/\mathbb{Q}}$ at 1 over the real Néron period $\Omega_A$. We obtain $\dfrac{L(A_{/\mathbb{Q}},1)}{\Omega_A} = c_A \cdot \dfrac{2^9 \cdot 3}{5}$, where $c_A \in \mathbb{Z}$ is the Manin constant. Since $c_A \mid 2^{\dim(A)}$ by [ARS06] then

$$\frac{L(A_{/\mathbb{Q}},1)}{\Omega_A} = \frac{2^{n+2} \cdot 3}{5},$$

for some $0 \leq n \leq 23$. In particular, the modular abelian variety $A_{/\mathbb{Q}}$ has rank zero over $\mathbb{Q}$.

Next, using the algorithms from [CS01, KS00] we compute the Tamagawa number $c_{A,13} = 1920 = 2^3 \cdot 3 \cdot 5$. We also find that $c_{A,59}$ is a power of 2 because $W_{23}$ acts as 1 on $A$, and on the component group $\mathrm{Frob} = -W_{23}$, so the fixed subgroup $\Phi_{A,59}(\mathbb{F}_{59})$ of Frobenius is a 2-group (for more details, see [Rib90a, Prop.3.7–8]).

Finally, the Birch and Swinnerton-Dyer conjectural formula for abelian varieties of rank zero (see [AS05, Conj. 2.2]) asserts that

$$\frac{L(A_{/\mathbb{Q}},1)}{\Omega_A} = \frac{\#Ш(A/\mathbb{Q}) \cdot c_{A,13} \cdot c_{A,59}}{\#A(\mathbb{Q})_{\mathrm{tor}} \cdot \#A^\vee(\mathbb{Q})_{\mathrm{tor}}}.$$

By substituting what we computed above, we obtain $3^2 \mid \#Ш(A/\mathbb{Q})$. Since $L(A_{/\mathbb{Q}}, 1) \neq 0$, [KL89] implies that $Ш(A/\mathbb{Q})$ is finite. By the nondegeneracy of the Cassels-Tate pairing, $\#Ш(A/\mathbb{Q}) = \#Ш(A^\vee/\mathbb{Q})$. Thus, if the BSD conjectural formula is true then $3^2 \mid \#Ш(A/\mathbb{Q}) = \#Ш(A^\vee/\mathbb{Q})$. □

We next observe that there are no visible elements of odd order for the embedding $A_{23/\mathbb{Q}} \hookrightarrow J_0(767)_{/\mathbb{Q}}$.

Lemma 6.1.2. *Any element of* $Ш(A_{23}/\mathbb{Q})$ *which is visible in* $J_0(767)$ *has order a power of* 2.

*Proof.* Since $m_{A_{23}} = 2^{34}$, [AS05, Prop. 3.15] implies that any element of $Ш(A_{23}/\mathbb{Q})$ that is visible in $J_0(767)$ has order a power of 2. □

Finally, we use Theorem 5.4.2 to prove the existence of non-trivial elements of order 3 in $Ш(A_{23}/\mathbb{Q})$ which are invisible at level 767, but become visible at higher level. In particular, we prove unconditionally that $3 \mid \#Ш(A_{23}/\mathbb{Q})$ which provides evidence for the Birch and Swinnerton-Dyer conjectural formula.

Proposition 6.1.3. *There is an element of order 3 in* $Ш(A_{23}\mathbb{Q})$ *which is not visible in* $J_0(767)$ *but is strongly visible in* $J_0(2 \cdot 767)$.

*Proof.* Let $A = A_{23}$, and note that $A$ has rank 0, since $L(A_{/\mathbb{Q}}, 1) \neq 0$. Using [Cre] or [Sage] we find that the elliptic curve

$$E: \qquad y^2 + xy = x^3 - x^2 + 5x + 37$$

has conductor $2 \cdot 767$ and $E(\mathbb{Q}) = \mathbb{Z} \oplus \mathbb{Z}$. Also

$$c_2 = 2, c_{13} = 2, c_{59} = 1, \overline{c}_2 = 6, \overline{c}_{13} = 2, \overline{c}_{59} = 1.$$

We apply Theorem 5.4.2 with $\ell = 3$ and $p = 2$. Since $E$ does not admit any rational 3-isogeny (by [Cre]), Hypothesis 3 is satisfied. The level is square free and the modular degree of $A$ is a power of 2, so Hypothesis 2 is satisfied.

We have $a_3(E) = -3$. Using Magma we find

$$\det(T_3|_{\overline{W}} - (-3)) \equiv 1 \pmod 3,$$

where $\overline{W}$ is as in Remark 5.4.3. This verifies the noncongruence hypothesis and completes the proof.

□

## 6.2 Level 959

We do similar computations for a 24-dimensional abelian subvariety of $J_0(959)$. We have $959 = 7 \cdot 137$, which is square free. There are five newform abelian subvarieties of the Jacobian, $A_2, A_7, A_{10}, A_{24}$ and $A_{26}$, whose dimensions are the corresponding subscripts. Let $A_f = A_{24}$ be the 24-dimensional newform abelian subvariety.

Proposition 6.2.1. *There is an element of order 3 in* $Ш(A_f/\mathbb{Q})$ *which is not visible in* $J_0(959)$ *but is strongly visible in* $J_0(2 \cdot 959)$.

*Proof.* Using Magma we find that $m_A = 2^{32} \cdot 583673$, which is coprime to 3. Thus we apply Theorem 5.4.2 with $\ell = 3$ and $p = 2$. Consulting [Cre] we find the curve E=1918C1, with Weierstrass equation

$$y^2 + xy + y = x^3 - 22x - 24,$$

with $E(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$, and

$$c_2 = 2, c_7 = 2, c_{137} = 1, \overline{c}_2 = 6, \overline{c}_7 = 2, \overline{c}_{137} = 1.$$

Using [Cre] we find that $E$ has no rational 3-isogeny. The modular form attached to $E$ is

$$g = q - q^2 - 2q^3 + q^4 - 2q^5 + \cdots,$$

and we have

$$\det(T_2|_{\overline{W}} - (-2)) = 2177734400 \equiv 2 \pmod 3,$$

where $\overline{W}$ is as in Remark 5.4.3. □

## 7 Conjecture, evidence and more computational data

We state a conjecture, provide some evidence and finally, provide a table that we computed using similar techniques to those in Section 6

### 7.1 The conjecture

The two examples computed in Section 6 show that for an abelian subvariety $A$ of $J_0(N)$ an invisible element of $Ш(A/\mathbb{Q})$ at the base level $N$ might become visible at a multiple level $NM$. We state a general conjecture according to which any element of $Ш(A/\mathbb{Q})$ should have such a property.

Conjecture 7.1.1. *Let* $h = 0$ *or* 1. *Suppose* $A$ *is a* $J_h$*-modular abelian variety and* $c \in Ш(A/\mathbb{Q})$. *Then there is a* $J_h$*-modular abelian variety* $C$ *and an inclusion* $\iota : A \to C$ *such that* $\iota_* c = 0$.

*Remark* 7.1.2. For any prime $\ell$, the Jacobian $J_h(N)$ comes equipped with two morphisms $\alpha^*, \beta^* : J_h(N) \to J_h(N\ell)$ induced by the two degeneracy maps $\alpha, \beta : X_h(\ell N) \to X_h(N)$ between the modular curves of levels $\ell N$ and $N$, and it is natural to consider visibility of $Ш(A/\mathbb{Q})$ in $J_h(N\ell)$ via morphisms $\iota$ constructed from these degeneracy maps.

*Remark* 7.1.3. It would be interesting to understand the set of all levels $N$ of $J_h$-modular abelian varieties $C$ that satisfy the conclusion of the conjecture.

## 7.2 Theoretical Evidence for the Conjecture

The first piece of theoretical evidence for Conjecture 7.1.1 is Remark 3.0.2, according to which any cohomology class $c \in \mathrm{H}^1(K, A)$ is visible in some abelian variety $C_{/K}$.

The next proposition gives evidence for elements of $Ш(E/\mathbb{Q})$ for an elliptic curve $E$ and elements of order 2 or 3.

Proposition 7.2.1. *Suppose $E$ is an elliptic curve over $\mathbb{Q}$. Then Conjecture 7.1.1 for $h = 0$ is true for all elements of order 2 and 3 in $Ш(E/\mathbb{Q})$.*

*Proof.* We first show that there is an abelian variety $C$ of dimension 2 and an injective homomorphism $i : E \hookrightarrow C$ such that $c \in \mathrm{Vis}_C(Ш(E/\mathbb{Q}))$. If $c$ has order 2, this follows from [AS02, Prop. 2.4] or [Kle01], and if $c$ has order 3, this follows from [Maz99, Cor. pg. 224]. The quotient $C/E$ is an elliptic curve, so $C$ is isogenous to a product of two elliptic curves. Thus by [BCDT01], $C$ is a quotient of $J_0(N)$, for some $N$. □

We also prove that Conjecture 7.1.1 is true with $h = 1$ for all elements of $Ш(A/\mathbb{Q})$ which split over abelian extensions.

Proposition 7.2.2. *Suppose $A_{/\mathbb{Q}}$ is a $J_1$-modular abelian variety over $\mathbb{Q}$ and $c \in Ш(A/\mathbb{Q})$ splits over an abelian extension of $\mathbb{Q}$. Then Conjecture 7.1.1 is true for $c$ with $h = 1$.*

*Proof.* Suppose $K$ is an abelian extension such that $\mathrm{res}_K(c) = 0$ and let $C = \mathrm{Res}_{K/\mathbb{Q}}(A_K)$. Then $c$ is visible in $C$ (see Section 3.0.2). It remains to verify that $C$ is modular. As discussed in [Mil72, pg. 178], for any abelian variety $B$ over $K$, we have an isomorphism of Tate modules

$$\mathrm{Tate}_\ell(\mathrm{Res}_{K/\mathbb{Q}}(B_K)) \cong \mathrm{Ind}_{G_K}^{G_\mathbb{Q}} \mathrm{Tate}_\ell(B_K),$$

and by Faltings's isogeny theorem [Fal86], the Tate module determines an abelian variety up to isogeny. Thus if $B = A_f$ is an abelian variety attached to a newform, then $\mathrm{Res}_{K/\mathbb{Q}}(B_K)$ is isogenous to a product of abelian varieties $A_{f^\chi}$, where $\chi$ runs through Dirichlet characters attached to the abelian extension $K/\mathbb{Q}$. Since $A$ is isogenous to a product of abelian varieties of the form $A_f$ (for various $f$), it follows that the restriction of scalars $C$ is modular. □

*Remark* 7.2.3. Suppose that $E$ is an elliptic curve and $c \in Ш(E/\mathbb{Q})$. Is there an abelian extension $K/\mathbb{Q}$ such that $\mathrm{res}_K(c) = 0$? The answer is "yes" if and only if there is a $K$-rational point (with $K$-abelian) on the locally trivial principal homogeneous space corresponding to $c$ (this homogeneous space is a genus one curve). Recently, M. Ciperiani and A. Wiles proved that any genus one curve over $\mathbb{Q}$ which has local points everywhere and whose Jacobian is a semistable elliptic curve admits a point over a solvable extension of $\mathbb{Q}$ (see [CW06]). Unfortunately, this paper does not answer our question about the existence of abelian points.

*Remark* 7.2.4. As explained in [Ste04], if $K/\mathbb{Q}$ is an abelian extension of prime degree then there is an exact sequence

$$0 \to A \to \mathrm{Res}_{K/\mathbb{Q}}(E_K) \xrightarrow{\mathrm{Tr}} E \to 0,$$

where $A$ is an abelian variety with $L(A_{/\mathbb{Q}}, s) = \prod L(f_i, s)$ (here, the $f_i$'s are the $G_{\mathbb{Q}}$-conjugates of the twist of the newform $f_E$ attached to $E$ by the Dirichlet character associated to $K/\mathbb{Q}$). Thus one could investigate the question in the previous remark by investigating whether or not $L(f_E, \chi, 1) = 0$ which one could do using modular symbols (see [CFK06]). The authors expect that $L$-functions of twists of degree larger than three are very unlikely to vanish at $s = 1$ (see [CFK06]), which suggests that in general, the question might have a negative answer for cohomology classes of order larger than 3.

## 7.3 Visibility of Kolyvagin cohomology classes

It would also be interesting to study visibility at higher level of Kolyvagin cohomology classes. The following is a first "test question" in this direction.

Question 7.3.1. Suppose $E \subset J_0(N)$ is an elliptic curve with conductor $N$, and fix a prime $\ell$ such that $\overline{\rho}_{E,\ell}$ is surjective. Fix a quadratic imaginary field $K$ that satisfies the Heegner hypothesis for $E$. For any prime $p$ satisfying the conditions of [Rub89, Prop. 5], let $c_p \in \mathrm{H}^1(\mathbb{Q}, E)[\ell]$ be the corresponding Kolyvagin cohomology class. There are two natural homomorphisms $\delta_1^*, \delta_p^*$ : $E \to J_0(Np)$. When is

$$(\delta_1^* \pm \delta_\ell^*)_*(c_\ell) = 0 \in \mathrm{H}^1(\mathbb{Q}, J_0(Np))?$$

When is

$$\mathrm{res}_v((\delta_1^* \pm \delta_\ell^*)_*(c_\ell)) = 0 \in \mathrm{H}^1(\mathbb{Q}_v, J_0(Np))?$$

## 7.4 Table of Strong Visibility at Higher Level

The following is a table that gives the known examples of $A_{f_{/\mathbb{Q}}}$ with square free conductor $N \leq 1339$, such that the Birch and Swinnerton-Dyer conjectural formula predicts an odd prime divisor $\ell$ of $Ш(A_f/\mathbb{Q})$, but $\ell$ does not divide the modular degree of $A_f$. These were taken from [AS05]. If there is an entry in the fourth column, this means we have verified the hypothesis of Theorem 5.4.2, hence there really is a nonzero element in $Ш(A_f/\mathbb{Q})$ that is not visible in $J_0(N)$, but is strongly visible in $J_0(pN)$. The notation in the fourth column is $(p, E, q)$, where $p$ is the prime used in Theorem 5.4.2, $E$ is an elliptic curve, denoted using a Cremona label, and $q \neq p$ is a prime such that

$$\bigcap_{q' \leq q} \mathrm{Ker}(T'_q|_{\overline{W}} - a_{q'}(E)) = 0.$$

| $A_f$ | dim | $\ell \mid Ш(A_f)$? | moddeg | $(p, E, q)$'s |
|---|---|---|---|---|
| 551H | 18 | 3 | $2^? \cdot 13^2$ | (2, 1102A1, -) |
| 767E | 23 | 3 | $2^{34}$ | (2, 1534B1, 3) |
| 959D | 24 | 3 | $2^{32} \cdot 583673$ | (2, 1918C1, 5), (7, 5369A1,2) |
| 1337E | 33 | 3 | $2^{59} \cdot 71$ | (2, 2674A1, 5) |
| 1339G | 30 | 3 | $2^{48} \cdot 5776049$ | (2, 2678B1, 3), (11, 14729A1,2) |

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