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A Historical Outline of the Theorem of Implicit Functions

Un Bosquejo Histórico del Teorema de las Funciones Implícitas

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Abstract

In this article a historical outline of the implicit functions theory is presented starting from the wiewpoint of Descartes algebraic geometry (1637) and Leibniz (1676 or 1677), Johann Bernoulli (1695) and Euler (1748) infinitesimal calculus. The critical contribution is highlighted due to the italian mathematician Ulisse Dini who settled the matter in modern form inside the real functions theory. The paper supplies the documented proof of Dini's priority in the so called implicit functions theorem. In the meanwhile the historical lack in attributing the theorem to Dini can be ascribed to the fact that he published his proof only in his *Lezioni* [3], written for supporting his teaching.
Key words and phrases: implicit functions, contemporary history

Resumen

En este artículo se presenta un excursus histórico de la teoría de las funciones implícitas, empezando con el punto de vista de la geometría algebraica de Descartes (1637) y el análisis infinitesimal de Leibniz (1676 o 1677), Johan Bernoulli (1695) y Euler (1748). Se pone en evidencia la contribución decisiva del matemático italiano Ulisse Dini, quien planteó la cuestión en términos modernos, en el ámbito de la teoría de las funciones de variables reales, produciendo la prueba documentada de su paternidad en lo que, hoy en día, es el teorema de las funciones implícitas. También se evidencia que el vacío historiográfico en la atribución del teorema a Dini puede depender del hecho de que él publicó una

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demostración rigurosa solamente en sus *Lezioni* [3], como soporte de su actividad docente.
Palabras y frases clave: funciones implícitas, historia contemporánea.

1 Introduction

This paper goes back to the implicit functions theory and to the relevant analytical questions. All this was matter of research for Descartes and other mathematicians since the first half of the 17^{th} century for almost two hundred years, but only at the end of the 1800's the theory received a satisfactory settlement by the Italian mathematician Ulisse Dini (1845-1918). Furthermore bibliographical support is given by us about Dini's priority often not acknowledged due to a lack in historical culture of much international literature. Let us first present, for the sake of completeness, the theorem in a modern notation.

Theorem (Ulisse Dini, 1878). *Let Ω an open set in $\mathbb{R}^2$ and $f : \Omega \to \mathbb{R}$ a $\mathcal{C}^1$ function. Suppose there exists $(\bar{x}, \bar{y}) \in \Omega$ such that:*

$$f(\bar{x}, \bar{y}) = 0, \qquad \frac{\partial f}{\partial y}(\bar{x}, \bar{y}) > 0,$$

then there must be a real interval B centered around $\overline{x}$, a real interval I centered around $\bar{y}$ and a function $\varphi : I \to \mathbb{R}$ such that:

- $B \times I \subset \Omega$,
- *for any $(x, y) \in B \times I$:*
$$\frac{\partial f}{\partial y}(x, y) \neq 0,$$
- *if $(x, y) \in B \times I$ then:*
$$f(x, y) = 0,$$
if and only if $y = \varphi(x)$,
- $\bar{y} = \varphi(\bar{x})$,
- $\varphi \in \mathcal{C}^1(B)$ *and for any $x \in B$:*
$$\varphi'(x) = -\frac{\dfrac{\partial f}{\partial x}(x, \varphi(x))}{\dfrac{\partial f}{\partial y}(x, \varphi(x))}.$$

Careful analysis of the theorem's proof, reveals that the most important fact is determining the radius of the interval B. Indeed if we choose $a, b \in \mathbb{R}$, $a,\ b > 0$ such that if $B_1 = [\bar{x} - a, \bar{x} + a]$ and $I_1 = [\bar{y} - a, \bar{y} + a]$ then $\Omega_{a,b} = B_1 \times I_1 \subset \Omega$, we define:

$$m = \min_{\Omega_{a,b}} \frac{\partial f}{\partial y},$$

and:

$$M = 1 + \max_{\Omega_{a,b}} \left| \frac{\partial f}{\partial x} \right|.$$

We observe that $m > 0,\ M > 0$. Therefore the interval B in the statement of the theorem of implicit functions is any interval of the form:

$$B = [\bar{x} - \delta, \bar{x} + \delta], \tag{1.1}$$

with $\delta \in]0, a[$ and $\delta \leq \frac{mb}{2M}$.

The typical nonlinearities of mechanics, population biology and economics, are modeled by ordinary differential equations; and even if a closed form integration of them can be performed, one will obtain an implicit solution which seldom can be got explicit. If, for example, we consider the Cauchy problem:

$$\begin{cases} \dot{y}(t) = 1 + y^3(t), \\ y\,(0) = 1, \end{cases} \tag{1.2}$$

one will obtain the following implicit relation:

$$\frac{2\sqrt{3}\,\arctan\left(\dfrac{2\,y(t) - 1}{\sqrt{3}}\right) + \ln\left(\dfrac{\left(1 + y(t)\right)^2}{1 - y(t) + y(t)^2}\right)}{6} = t + \frac{\sqrt{3}\,\pi + \ln 64}{18},$$

which cannot anyway be solved in explicit form.

2 Historical outline of implicit functions theory

In the previous pages we used the word *implicit* to describe a function which does not operate upon a given sequence of values of the independent variable, but is rather expressed as an algebraic or transcendental relationship not solved for the dependent one.

When at long last the problem of finding the tangent to a curve - of a given explicit equation - was solved at the end of the 17^{th} century, the attention was turned towards curves given by implicit relationships.

2.1 René Descartes

Implicit functions of the form $F(x, y) = C$ were among the central themes of R. Descartes (1596-1650) *Géometrie* published in 1637 [2]. For example, he sought to find the locus for which $\overline{CB} \cdot \overline{CF} = \overline{CD} \cdot \overline{CH}$ given four straight lines and four angles, $\overline{CX}$ being the distance from C to X taken from the sides and along the angles. In the Book II (pag. 325 of the original 1637 edition) he finds the above locus (x, y) as given by an implicit relationship of the form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Descartes had also devised an algebraic method of constructing the tangent to an implicitly given curve, called *circle method*, which unfortunately led to prohibitive computations. Swifter algorithms were discovered in the 1650's by J. Hudde (1633-1704) and R.F. de Sluse (1622-1685). The latter made it possible to routinely compute the slope of that tangent. He submitted the method without proof, to the Secretary of the Royal Society, H. Oldenburg (1615-1677), who published the Sluse's letter in the *Philosophical Transactions* of 1672.

2.2 Gottfried Wilhelm Leibniz

G. W. Leibniz (1646 - 1716) in an undated letter (presumably of 1676 or 1677, letter XLII vol. I Leibniz Mathematische Schriften [7]), shows how it is easier to obtain the slope using his (as yet unpublished) calculus. Of course, he proceeds by examples; so, given a plane curve of implicit equation:

$$ay^2 + bxy + cx^2 + fx + gy + h = 0,$$

he substitutes something we could call $x + dx$ in place of x and $y + dy$ in place of y, thus obtaining a long expression, which, due to the preceeding equation, and to the reasonable assumption that:

$$a\,(dy)^2 + b\,(dx)\,(dy) + e\,(dx)^2 = 0,$$

becomes:

$$2ay(dy) + by(dx) + bx(dy) + 2cx(dx) + f(dx) + g(dy) = 0,$$

or, solving for $m = dy/dx$:

$$m = -\frac{by + 2cx + f}{2ay + bx + g},$$

Leibniz then gives the differential quotient dy/dx in terms of (x, y), even if the expression of y to be derived with respect to x is not immediately available. Furthermore he observes that his invention:

> *. . . coincidit cum regula Slusiana ostenditque eam statim occurrere hanc methodum intelligenti, sed methodus ipsa nostra longe est amplior.*
> (... the final result, the same as the Sluse rule, is quite soon achieved by people understanding this method; but our method is by far more wide.)

2.3 Johann Bernoulli

Johann Bernoulli (1667-1748) is one of the founding fathers of the Calculus. In a letter [7] sent from Basel to Leibniz, on June 8^{th} 1695, he tackles a geometrical question which leads him to the following first order nonlinear and nonautonomous differential equation:

$$y'(x) = \sqrt{\frac{a^2 + y^2}{a^2 + x^2}},$$

which he writes

$$\frac{dx}{\sqrt{a^2 + x^2}} = \frac{dy}{\sqrt{a^2 + y^2}}, \qquad (2.1)$$

> *aequatio differentialis constans duobus membris omnino inter se similibus et non integrabilibus, quae tamen aequatio sit pro curva algebraica.*
> (a differential equation consisting of two sides, almost similar each other, whose solution nevertheless will be an algebraic curve.)

Thus, he observes that, even though each differential like those of (2.1) could not be integrated alone (at that time), the combination of both, presented in (2.1) is astonishingly integrable.

G. C. Fagnano (1682-1766) and later Euler, attempted to integrate a similar expression, though with a cubic under the square root.

As a consequence, Euler and A. M. Legendre (1752-1833) came to the elliptic integrals, and to the elliptic functions named after Jacobi (1804-1851), from which most nineteenth century Mathematical Physics stems from.

Bernoulli writes (2.1) as:

$$y\frac{xdx}{\sqrt{a^2+x^2}} = x\frac{ydy}{\sqrt{a^2+y^2}},$$

and partially integrates:

$$y\sqrt{a^2+x^2} - 2 - \int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+y^2} - \int \sqrt{a^2+y^2}\,dx + C.$$

But from (2.1)

$$\int \sqrt{a^2+x^2}dy = \int \sqrt{a^2+y^2}dx,$$

and, as a consequence, *manebit aequatio algebraica* (an algebraic equation will be left):

$$y\sqrt{a^2+x^2} = x\sqrt{a^2+y^2} + C.$$

Furthermore Bernoulli analyzes the similar equation:

$$\frac{dx}{\sqrt{a^2-x^2}} = \frac{dy}{\sqrt{a^2-y^2}},$$

reaching the implicit solution:

$$y\sqrt{a^2-x^2} = x\sqrt{a^2-y^2} + C.$$

Therefore while Descartes managed the implicit functions as an algebraic object inside a geometrical question, on the contrary, in Johann Bernoulli they play a role in order to solve an ordinary differential equation and so the explicitation is equivalent to an integration method.

2.4 Leonhard Euler

What were the concepts of implicitness end explicitness at his time? Several notions we now take for granted, such as the signs of x and y on the four quadrants of the Cartesian plane, were not well understood until the 18^{th} century, with the publication of *Introductio in Analysin infinitorum* [4], namely more than one century after the *Géométrie.* In this beautiful treatise, published in 1748 by L. Euler (1706-1783) as an introduction to the differential and integral calculus, one can read the first definition of implicit function:

> *Hae [functiones] commode distinguuntur in esplicitas et implicitas. Explicitae sunt quae per signa radicalia sunt evolutae, cuiusmodi exempla modo sunt data. Implicitae vero functiones irrationales sunt quae ex resolutione equatione ortum habent. Sic Z erit functio implicita ipsius z si per huiusmodi aequationem* $Z^7 = azZ^2 - bz^5$ *definiatur, quoniam valorem explicitum pro Z admissis etiam signis radicalibus exhibere non licet, propterea quod Algebra communis nondum ad hunc perfectionis gradum est erecta.*
> (These functions can be divided in implicit and explicit. The latter are solved through roots, and some exempla have been provided of them. On the contrary, the implicit functions are irrational ones whose birth depends on the equations. So we will have an implicit function Z of z, if Z is defined, e. g., through the equation $Z^7 = azZ^2 - bz^5$, as a matter of fact, even if one uses the roots, it isn't possible here to display any explicit value for Z, Algebra having this perfection raised not yet.)

Furthermore in §281 of *Institutiones Calculi differentialis* (1755), [5], Euler writes:

> *In hoc capite imprimis est propositum earum functionum ipsius x quae non explicite, sed implicite per aequationem quae relatio functionis istius y ad x continetur, definiuntur, differentiationem explicare.*
> (In this chapter, first of all, the process will be explained for differentiating those functions of x, where y and x aren't given explicitly, but are linked in implicit way.)

Euler seems to be unconcerned—for a given $f(x, y)$—about the existence conditions for $y(x)$ itself. There is no evidence that anyone turned on the issue before the last quarter of the 19$^{\text{th}}$ century, despite the fact that implicit functions troubled most mathematicians. It is enough to recall the J. Kepler (1571-1630) famous implicit E-equation:

$$E - e \sin E = M,$$

where E is the eccentric anomaly (and M the mean) of a body orbiting on elliptical trajectory of (fixed) eccentricity e, as the Moon around the Earth, or the Earth around the Sun. The Kepler equation, which comes from an integration, attracted J. L. Lagrange (1736-1813) and F. W. Bessel (1784-1846), besides many others.

2.5 Ulisse Dini

In the 1870's the mathematicians were urged to seek a proof that would guarantee that $y = y(x)$ is equivalent to $F(x, y) = C$ in the neighborhood of a point (x_0, y_0) that satisfies $F(x_0, y_0) = C$. Consider, for example, the circle $x^2 + y^2 = 1$: for each value of x we have two values of y, but this does not constitute a problem, because having chosen for x only one value of y such as $y_0 = +\sqrt{1 - x_0^2}$, we know which branch of the curve we are dealing with. Alas, if we start from $x = 1$ or $x = -1$, we will meet this unsurmountable ambiguity:

for the variable x from $+1$ or -1, shall go: upper or lower branch?

In other words, the implicit function $x^2 + y^2 - 1 = 0$ ambiguously defines y for $x = 1$. This is not due to the local derivative of y going to infinity for $x = 0$. For example the curve of equation $y = x^{1/3}$ also has a vertical tangent for $x = 0$, too, but yet the implicit function $y - x^{1/3} = 0$ unambiguously defines y for $x = 0$.

Anglo-saxon scientific and historic literature *ignores* the italian mathematician U. Dini: even in the excellent, historically oriented textbook *Analysis and its History*, [8], we find at page 309 that:

> *In the Weierstrass era (see Genocchi-Peano 1884) mathematicians felt a need for more rigorous proof that guarantees that* $f(x, y)$ *is equivalent to* $y = y(x)$ *in some neighborhood of a point* (x_0, y_0) *satisfying* $f(x_0, y_0) = C$.

If one goes to check up the Genocchi-Peano tome [6], where there is a simple résumé of Dini's ideas, at page XXVI, one is referred by a note to the source:

V. Dini, Analisi inf. I, pag. 153.

Dini's ideas span pages 180-195 of his *Analisi infinitesimale* [3], which is to say, a book written six years earlier.

A modern Dini's theorem presentation has been already supplied at this paper's introduction.

3 U. Dini's scientific profile

We deem to finish our historical paper by offering to the reader a very concise Dini's scientific biography.

Dini spent all his academic career at Pisa University. Even if his most important work in Mathematics was on the theory of functions of real variables, he made many contributions to differential geometry, power series, and analytic functions, producing 69 original papers, but none of them dealt with implicit functions. For this it is necessary to look to the quoted *Analisi infinitesimale* and *Lezioni di analisi infinitesimale*, his last treatise published in 1907. Pages 197-241 of Volume I of these final *Lezioni*, deal with the conditions sufficient to ensure the local explicitability of an implicit function $f(x, y) = 0$. It is stated that—we are following his original notations—if the equation $f(x, y) = 0$ is satisfied for a value of $x = x_0$, let $y = y_0$ a value such that the point $P_0 = (x_0, y_0)$ belongs to a domain where $f(x, y)$ is given; there exists a neighborhood of P_0 such that $f(x, y), \partial f/\partial x, \partial f/\partial y$ are defined and continuous, and $(\partial f/\partial y)_0 \neq 0$, then: a neighborhood $\mathcal{N} = (x_0 - h, x_0 + h)$ there exists where $f(x, y) = 0$ completely defines a continuous function $y = y(x)$ with $y(x_0) = y_0$. If y can be differentiated, its first derivative at x_0 will be continuous and given by: $dy/dx = -\partial f/\partial x/\partial f/\partial y$, all to be evaluated at P_0.

Furthermore, more complicated formulas can be written for the successive derivatives y', y'', ... allowing the quantities $y'(x_0)$, $y''(x_0)$, $y'''(x_0)$, ... to be computed. This in turn allows one to construct a Taylor series development in $\mathcal{N}$ around x_0, of the $y(x)$ whose existence has just been proven.

Dini carried out much work at a time when those studying Real Analysis where seeking to determine precisely when the theorems earlier stated unprecisely where valid. To achieve this aim, mathematicians tried to see generalizations needing pathological counterexamples for seeing the boundaries of the generalization itself. Dini was one of the greatest masters of generalization and constructing counterexamples.

Luigi Bianchi (1856-1928), a differential geometer, Dini's pupil and colleague at Pisa University, in a Dini's commemoration held four years after his death [1], remembers that Dini—under the Weierstrass influence—turned his attention to the lack of rigour of the proofs in several fields of Mathematics. In such a way he devoted himself:

> *all'ardua impresa di riedificare, sopra solide fondamenta, tutto l'edificio dell'analisi. (...) Queste ricerche avevano riscosso, fin da principio, il pieno consenso e il plauso del Betti; e non questo piccolo titolo di merito pel maestro, ove si pensi che i nuovi studi*

> *venivano, a sconvolgere, in gran parte, l'edificio che egli, come quasi tutti i matematici del suo tempo, aveva finora ritenuto perfettamente sicuro in tutte le sue parti.*
> (to the hard work of re-building, upon sound pillars, all the estate of the Analysis. (...) All these researchs had gained, since early, the full consent and approval by Betti; and this is not a small merit for the master, specially if one thinks that all the new studies were destroying most part of the building believed by himself—as by all the mathematicians of his time—perfectly safe everywhere.)

We deem to put his words as a closure and synthesis of our profile.

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