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Operator Compactification of Topological Spaces

Compactificación de Espacios Topológicos mediante Operadores

Ennis Rosas (erosas@cumana.sucre.udo.edu.ve)
Universidad de Oriente
Departamento de Matemáticas
Cumaná, Venezuela.
Jorge Vielma (email:vielma@ciens.ula.ve)
Universidad de los Andes
Departamento de Matemáticas
Facultad de Ciencias
Mérida, Venezuela.

Abstract

The concepts of α-locally compact spaces and α-compactification are introduced. It is shown that every α-T_2 and α-locally compact space has an α-compactification.
Key words and phrases: α-locally compact, β-compactification.

Resumen

Se introducen los conceptos de espacio α-localmente compacto y α-compactificación de un espacio y se muestra que todo espacio α-T_2, α localmente compacto, tiene una α-compactificación.
Palabras y frases clave: α-localmente compacto, β-compactificación.

In this paper, we try to show that every α-T_2, α-locally compact space has an α-compactification which generalizes the Alexandroff compactification of locally compact Hausdorff spaces.

We recall the definitions of operator associated to a topology, α-open set, α-T_2 space and α-compact space.

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Definition 1 ([3]). Let (X, Γ) be a topological space, B a subset of X and α an operator from Γ to $P(X)$, i.e., $\alpha : \Gamma \to P(X)$. We say that α is an operator *associated* with Γ if (O) $U \subseteq \alpha(U)$ for all $U \in \Gamma$.

We say that the operator α associated with Γ is *stable* with respect to B if (S) α induces an operator $\alpha_B : \Gamma_B \to P(B)$ such that $\alpha_B(U \cap B) = \alpha(U) \cap B$ for every U in Γ, where Γ_B is the relative topology on B.

Definition 2 ([1]). Let (X, Γ) be a topological space and α an operator associated with Γ. A subset A of X is said to be α-*open* if for each $x \in A$ there exists an open set U containing x such that $\alpha(U) \subset A$. A subset B is said to be α-*closed* if its complement is α-open.

We observe that the collection of α-open sets, in general, is not a topology, but if α is considered to be regular (see [1], [3] for the definition) then this collection is a topology.

Definition 3 ([1]). Let (X, Γ) be a topological space and α an operator associated with Γ. We say that a subset A of X is α-*compact* if for every open covering Π of A there exists a finite subcollection $\{C_1, C_2, \ldots, C_n\}$ of Π such that $A \subset \bigcup_{i=1}^{n} \alpha(C_i)$.

Properties of α-compact spaces has been investigated in [1, 3]. The following theorems were given in [3].

Theorem 1 ([3]). *Let (X, Γ) be a topological space, α an operator associated with Γ, $A \subset X$ and $K \subset A$. If A is α-compact and K is α-closed, then K is α-compact.*

Theorem 2 ([3]). *Let (X, Γ) be a topological space and α a regular operator on Γ. If X is α-T_2 (see [1,3]) and $K \subset X$ is α-compact, then K is α-closed.*

Lemma 1. *The collection of α-compact subsets of X is closed under finite unions. If α is a regular operator and X is an α-T_2 space then it is closed under arbitrary intersections.*

Proof. Trivial. □

Definition 4. Let (X, Γ) be a topological space and α an operator associated with Γ. The operator α is *subadditive* if for every collection of open sets $\{U_\beta\}$, $\alpha(\bigcup U_\beta) \subseteq \bigcup \alpha(U_\beta)$.

Theorem 3. *Let (X, Γ) be a topological space and α a regular subadditive operator associated with Γ. If $Y \subset X$ is α-compact, $x \in X \setminus Y$ and (X, Γ) is α-T_2, then there exist open sets U and V with $x \in U$, $Y \subset \alpha(V)$ and $\alpha(U) \cap \alpha(V) = \varnothing$.*

Proof. For each $y \in Y$, let V_y and V_x^y be open sets such that $\alpha(V_y) \cap \alpha(V_x^y) = \varnothing$, with $y \in V_y$ and $x \in V_x^y$. The collection $\{V_y : y \in Y\}$ is an open cover of Y. Now, since Y is α-compact, there exists a finite subcollection $\{V_{y_1}, \ldots, V_{y_n}\}$ such that $\{\alpha(V_{y_1}), \ldots, \alpha(V_{y_n})\}$ covers Y. Let $U = \bigcap_{i=1}^{n} V_x^{y_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$. Since $U \subset V_x^{y_i}$ for every $i \in \{1, 2, \ldots, n\}$, then $\alpha(U) \cap \alpha(V_{y_i}) = \varnothing$ for every $i \in \{1, 2, \ldots, n\}$. Then $\alpha(U) \cap \alpha(V) = \varnothing$. Also $Y \subset \alpha(V)$. □

Now we give the definition of β-compactification of a topological space.

Definition 5. Let (X, Γ) and (Y, Ψ) be two topological spaces, α and β operators associated with Γ and Ψ, respectively. We say that (Y, Ψ) is a β-*compactification* of X if

1. (Y, Ψ) is β-compact.
2. (X, Γ) is a subspace of (Y, Ψ).
3. The operator $\beta/_X$ is equal to α.
4. The Ψ-closure of X is Y.

Now we give the definition of α-locally compact space.

Definition 6. Let (X, Γ) be a topological space and α an operator associated with Γ. The space (X, Γ) is α-*locally compact* at the point x if there exists an α-compact subset C of X and an α-open neighborhood U of x such that $\alpha(U) \subset C$. The space (X, Γ) is said to be α-*locally compact* if it is α-locally compact at each of its points.

Clearly every α-compact space is α-locally compact. Observe that $\mathbb{R}$ with the usual topology and α defined as $\alpha(U) = \overline{U}$ (the closure of U) is α-locally compact but not compact.

Now we give the main theorem.

Theorem 4. *Let (X, Γ) be a space, where α is regular, monotone (see [4]), subadditive, stable with respect to all α-closed subsets of (X, Γ) and satisfies the additional condition that $\alpha(\varnothing) = \varnothing$. If (X, Γ) is α-locally compact, not α-compact and α-T_2, then there exist a space (Y, Ψ) and an operator β on Ψ such that:*

1. *(Y, Ψ) is a β-compactification of (X, Γ).*
2. *$|Y \setminus X| = 1$.*
3. *(Y, Ψ) is β-T_2.*

Proof. Define $Y = X \cup \{\infty\}$, where ∞ is an object not in X. On Y, we define a topology as follows:

1. If $U \in \Gamma$, then $U \in \Psi$.
2. If C is an α-compact subset of X, then $X \setminus C \cup \{\infty\} \in \Psi$.

Let us show that Ψ is a topology on Y. In fact, the empty set is a set of type (1) and being α-compact, it makes Y to be a set of type (2). If U_1 and U_2 belongs to Ψ, checking that $U_1 \cap U_2$ belongs to Ψ involves three cases:

(a) If U_1 and U_2 belong to Γ, then $U_1 \cap U_2$ belongs to Γ and so to Ψ.

(b) If $U_1 = Y \setminus C_1$ and $U_2 = Y \setminus C_2$, where C_1 and C_2 are α-compact subsets of X, we get that $U_1 \cap U_2 = Y \setminus (C_1 \cup C_2)$, which is of type (2), since $C_1 \cup C_2$ is an α-compact subset of X.

(c) If $U_1 \in \Gamma$ and $U_2 = Y \setminus C_2$, where C_2 is an α-compact subset of X, then $U_1 \cap U_2 = U_1 \cap (Y \setminus C_2) = U_1 \cap (X \setminus C_2)$ which is a set of type (1).

Similarly we check that the union of any collection $\{U_\beta\}$ of elements of Ψ belongs to Ψ. Again we consider three cases:

(a) If each $U_\beta \in \Gamma$ then $\cup U_\beta \in \Gamma$ and so it belongs to Ψ.

(b) If each $U_\beta = Y \setminus C_\beta$, where each C_β is α-compact, we have that $\cup U_\beta = Y \setminus (\cap C_\beta)$ which is of type (2), since $\cap C_\beta$ is an α-compact subset of X.

(c) If some U_β are of type (1) and some are of type (2), the problem reduces to the case in which one is of type (1) and the other is of type (2). So we need to show that every set of the form $U \cup (Y \setminus C)$, where $U \in \Gamma$ and C is α-compact, is an element of Ψ. In fact $U \cup (Y \setminus C) = Y \setminus (C \setminus U)$.

To prove that $C \setminus U$ is α-compact we go as follows: let $\{W_i\}$ be a Γ open covering of $C \setminus U$. Then the collection $\{W_i\} \cup \{U\}$ is an open covering of C and, since C is α-compact, there exist finitely many indices $i_1, i_2, \ldots, i_k$ such that $C \subset \bigcup_{j=1}^{k} \alpha(W_{i_j}) \cup \alpha(U)$. Now,

$$C \setminus U \subset \Big(\bigcup_{j=1}^{k} \alpha(W_{i_j}) \cap (C \setminus U) \Big) \cup (\alpha(U) \cap (C \setminus U)).$$

Since α is stable with respect to all α-closed subsets of X and since $C \setminus U$ is α-closed we have that $\alpha(U) \cap (C \setminus U) = \alpha_{C \setminus U}(U \cap (C \setminus U)) = \alpha_{C \setminus U}(\varnothing) = \varnothing$, by hypothesis. Therefore $C \setminus U \subset \bigcup_{j=1}^{k} \alpha(W_{i_j})$. This implies that $C \setminus U$ is α-compact. So Ψ is a topology on Y.

Define an operator $\beta : \Psi \longrightarrow P(Y)$ as follows:

$$\beta(V) = \begin{cases} \alpha(V) & \text{if V} \in \Gamma \\ \alpha(X \setminus C) \cup \{\infty\} & \text{if } V = X \setminus C \cup \{\infty\}, \text{where } C \text{ is } \alpha\text{-compact.} \end{cases}$$

Let us show that (Y, Ψ) is β-compact.

Let $\{U_i : i \in \Lambda\}$ be a Ψ-open cover of Y. This collection must have at least one element of type (2), say $U_{i_0} = X \setminus C \cup \{\infty\}$, where C is α-compact. Let $V_i = U_i \cap X$ for $i \neq i_0$. Then $\{V_i : i \in \Lambda\}$ is a Γ-open cover of C. Since C is α-compact, there exists a finite subcollection $\{i_1, \ldots, i_n\}$ of Λ such that $C \subseteq \bigcup_{j=1}^{n} \alpha(V_{i_j})$. Now $Y \subset (\bigcup_{j=1}^{n} \beta(V_{i_j}) \cup \beta(Y/C))$. Therefore (Y, Ψ) is β-compact.

Proving that X is a subspace of Y is trivial and, since X is not α-compact, then X is dense in Y. Also, since α is subadditive and stable with respect to all α-closed subsets of X, it is easy to prove that (Y, Ψ) is β-T_2. □

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