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The Neighborhood Complex of an Infinite Graph

El Complejo de Entornos de un Grafo Infinito

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Abstract

In this paper we study the relation between the neighborhood complex of a graph, introduced by Lovász, and the local finiteness of the graph by means of the ordinal functions Ord and Ht. On the other hand we describe the neighborhood complex associated to the product graph.
Key words and phrases: locally finite graph, transfinite height, Borst's order.

Resumen

En este trabajo estudiamos la relación entre el complejo simplicial de entornos de un grafo, introducido por Lovász, y la finitud local del grafo, mediante las funciones ordinales Ord y Ht. Por otra parte describimos el complejo de entornos asociado al grafo producto.
Palabras y frases clave: grafo localmente finito, altura transfinita, orden de Borst.

The neighborhood complex of a graph was introduced by Lovász (see [6]) to give a negative answer to Kneser's conjecture. Lovász studied the relation between the chromatic number of a graph and the connectivity of its neighborhood complex. Later several authors (see [8] and [7]) have studied some properties of this simplicial complex, mainly related to its homology groups. In this paper we are going to study the behaviour of the neighborhood complex

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under two ordinal functions, that in the finite case represent the degree of the graph. On the other hand we describe the neighborhood complex associated to the product graph.

By $G = (V, E)$ we mean a (non necessarily finite) graph, that is, V is a set, called vertex set, and E is a set of unordered pairs of distinct points of V, called edge set. If we suppose that for every vertex v, the set of edges incident with v is finite (equivalently, the set of vertices with common neighbor is finite) we say that the graph is locally finite.

We are going to associate to each graph a simplicial complex, called the neighborhood complex, in the following way (see [2], just before theorem 4.2).

Definition 1. For any graph $G = (V, E)$ let $\mathcal{N}(G)$ denote the simplicial complex, called the neighborhood complex of G, whose vertex set is V and whose simplices are those finite sets of vertices which have a common neighbor. That is,

$$\mathcal{N}(G) = \{\sigma \in \operatorname{Fin} V : \exists v \in V : \{a, v\} \in E, \forall a \in \sigma\}.$$

For a finite graph G it is clear that the degree of G is $\max\{|\sigma| : \sigma \in \mathcal{N}(G)\}$.

We will use the ordinal functions Ord and Ht associated to a simplicial complex and a partial ordered set respectively. We quote the definition of Ord from [4].

Definition 2. Let L be a set. For a subset M of $\operatorname{Fin} L$ and a $\sigma \in \{\emptyset\} \cup \operatorname{Fin} L$, let

$$M^\sigma = \{\tau \in \operatorname{Fin} L : \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}.$$

If $a \in L$ and $\sigma = \{a\}$, we write M^a instead of M^σ (M^a is called in some occasions $Link_M(a)$ the link of a in M).

We define $\operatorname{Ord} M$ as follows:

1. $\operatorname{Ord} M = 0$, if $M = \emptyset$.
2. For an ordinal number $\alpha > 0$ we say that $\operatorname{Ord} M \leq \alpha$ if and only if $\operatorname{Ord} M^a < \alpha$ for each $a \in L$.
3. $OrdM = \alpha$ if $OrdM \leq \alpha$ and $Ord(M) \leq \beta$ is false for $\beta < \alpha$.
4. Furthermore, we say that $\operatorname{Ord} M$ exists if and only if there exists some ordinal number α such that $\operatorname{Ord} M \leq \alpha$ holds.

Using the definition above we can prove the following result about the relation of the local finiteness of a graph G and the value that the function Ord gets in the simplicial complex $\mathcal{N}(G)$.

Theorem 3. *Let $G = (V, E)$ be an infinite graph.*

1. $\operatorname{Ord}\mathcal{N}(G) = \max\{|\sigma| : \sigma \in N(G)\}$, *whenever the maximum exists (in particular if G is finite, and thus in this case $Ord(N(G))$ is $dim(N(G))+$ 1 the dimension of the complex).*

2. $\operatorname{Ord}\mathcal{N}(G)$ *exists if and only if G is locally finite.*

3. *If G is locally finite, then* $\operatorname{Ord}\mathcal{N}(G) \leq \omega_0$.

Proof:

1. Obvious.

2. From lemma 2.1.3 of [4], $\operatorname{Ord}\mathcal{N}(G)$ does not exist if and only if there exists a sequence of distinct elements $\{a_i\}_{i=1}^{\infty}$ such that there exists v_n, $n = 1, \ldots, \infty$, in V such that $\{a_i\}_{i=1}^{n}$ are neighborhoods of v_n. So if such a sequence exists there are two possible situations. If there are only a finite number of distinct v_n, then one of them has an infinite set of neighbors. If there are an infinite set of distinct v_n, then a_1 has an infinite set of neighbors. In both cases the graph is not locally finite. Conversely if the graph is not locally finite it is easy to find the desired sequence.

3. We are going to prove that $\mathcal{N}(G)^a \subseteq \mathcal{N}(G^a)$ for every vertex $a \in V$, where $G^a = (V - \{a\}, E^a)$ is the graph with edge set $E^a = \{\{v, w\} \in E : v \neq a, w \neq a$ and $\{a, w\} \in E$ or $\{v, a\} \in E\}$.

   On the one hand $\mathcal{N}(G)^a = \{\sigma \in \operatorname{Fin} V : a \notin \sigma, \sigma \cup \{a\} \in \mathcal{N}(G)\} = \{\sigma \in \operatorname{Fin} V - \{a\} : \exists v \in V - \{a\} : \{b, v\} \in E, \forall b \in \sigma$ and $\{a, v\} \in E\}$. On the other hand $\mathcal{N}(G^a) = \{\sigma \in \operatorname{Fin} V - \{a\} : \exists v \in V : \{b, v\} \in E^a, \forall b \in \sigma\}$. Now let $\sigma \in \mathcal{N}(G)^a$, then $\sigma \in \operatorname{Fin} V - \{a\}$ and there exists $v \in V$ such that $\{b, v\} \in E, \forall b \in \sigma$ and $\{a, v\} \in E$, so $\{b, v\} \in E^a$ $\forall b \in \sigma$ and $\sigma \in \mathcal{N}(G^a)$. (Note that $N(G^a)$ is the deletion $N(G)\backslash a$ of the a in $N(G)$ and for a general simplicial complex Δ we have $Link_{\Delta}(a) \subseteq \Delta\backslash a$.)

   Now the local finiteness of G implies that G^a is a finite graph for each $a \in V$, so $\operatorname{Ord}\mathcal{N}(G^a)$ is finite and $\operatorname{Ord}\mathcal{N}(G)^a \leq \operatorname{Ord}\mathcal{N}(G^a)$ (see lemma 2.1.1 (3) of [4]) is finite for each $a \in V$, and finally $\operatorname{Ord}\mathcal{N}(G) \leq \omega_0$ as desired.

$\square$

Here is the other transfinite invariant, defined in [1]:

Definition 4. Let P be a poset and let P_0 be the set of maximal elements of P. Let define for every ordinal number α

$$P_\alpha = \{x \in P : \forall y \in P : x < y \Rightarrow y \in \bigcup_{\beta<\alpha} P_\beta\}.$$

Clearly $P_\alpha \subset P_\eta$ if $\alpha < \eta$, so there exists an ordinal number α such that $P_\alpha = P_{\alpha+1}$. Hence we say that the transfinite height of the poset P exists if $P_\alpha = P$ (in any other case we say it does not exists) and then we define the transfinite height of the poset P as $\mathrm{Ht}\,(P) = \mathrm{Min}\,\{\alpha \text{ ordinal} : P_\alpha = P\}$.

In order to relate this function with the neighborhood complex of a graph we will use the following canonical way to associate a poset to each simplicial complex (again [2], section 9).

Definition 5. Let K be a simplicial complex. We define the poset of faces of K as K ordered by inclusion, $P(K) = (K, \subset)$.

Using the preceding tools we can prove the following result about the relation of the local finiteness of a graph G and the value that the function Ht gets in the poset $P(\mathcal{N}(G))$.

Theorem 6. *Let $G = (V, E)$ be an infinite graph.*

1. $\mathrm{Ht}\,P(\mathcal{N}(G)) = \max\{|\sigma| \colon \sigma \in \mathcal{N}(G)\}$, *whenever the maximum exists (in particular if G is finite, and thus in this case* $\mathrm{Ht}\,(N(G))$ *is dim*$(N(G))+1$ *the dimension of the complex).*

2. $\mathrm{Ht}\,P(\mathcal{N}(G))$ *exists if and only if G is locally finite.*

3. *If G is locally finite, then* $\mathrm{Ht}\,P(\mathcal{N}(G)) \leq \omega_0$.

Proof:

1. Obvious.

2. Note that the existence of Ht is equivalent to the existence of Ord (see (2) of corollary 3.4, [1]), so the existence of $\mathrm{Ht}\,P(\mathcal{N}(G))$ comes from (1) of 3.

3. For each $\sigma \in \mathcal{N}(G)$ there are only a finite number of $\eta \in \mathcal{N}(G)$ such that $\sigma \subseteq \eta$, so the set $\{\eta \in \mathcal{N}(G) \colon \sigma \leq \eta\}$ is finite. Then if $n = \mathrm{card}\{\eta \in \mathcal{N}(G) \colon \sigma \leq \eta\}$, it is clear that $\sigma \in P(\mathcal{N}(G)_n$. Finally this means that $P(\mathcal{N}(G)) = P(\mathcal{N}(G)_{\omega_0})$ and that $\mathrm{Ht}\,P(\mathcal{N}(G)) \leq \omega_0$.

$\square$

The following definition is the natural extension to the infinite case of that in [5], page 22, chapter 1.

Definition 7. Given two infinite locally finite graphs $G_i = (V_i, E_i)$, $i = 1, 2$, we define their product as the graph $G_1 \times G_2$ whose vertex set is $V_1 \times V_2$ and edge set defined as follows: given two distinct vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$, the unordered pair $\{u, v\}$ is in the edge set of $G_1 \times G_2$ if and only if $u_1 = v_1$ and $\{u_2, v_2\}$ is in E_2 or $u_2 = v_2$ and $\{u_1, v_1\}$ is in E_1.

We are finally going to relate the neighborhood complex of the product graph with the neighborhood complexes of the factors.

Theorem 8. *The neighborhood complex $\mathcal{N}(G_1 \times G_2)$ of the product graph can be described as $\bigoplus_{v_2 \in V_2} \mathcal{N}(G_1)_{v_2} \star \bigoplus_{v_1 \in V_1} \mathcal{N}(G_2)_{v_1}$, where $\mathcal{N}(G_1)_{v_2}$ is a copy of $\mathcal{N}(G_1)$, $\mathcal{N}(G_2)_{v_1}$ is a copy of $\mathcal{N}(G_2)$ and $\star$ is the operation between simplicial complexes defined in [2], 9.5.*

Proof: Firstly $\sigma \in \mathcal{N}(G_1 \times G_2)$ if and only if $\sigma \in \mathrm{Fin}\, V_1 \times V_2$ and $\exists (v_1, v_2) \in V_1 \times V_2$ such that $\{(a_1, a_2), (v_1, v_2)\} \in E_1 \times E_2$, $\forall (a_1, a_2) \in \sigma\}$, where $E_1 \times E_2$ denotes the edge set of $G_1 \times G_2$.

From the definition of $G_1 \times G_2$, the unordered pair $\{(a_1, a_2), (v_1, v_2)\} \in E_1 \times E_2$ if and only if $a_1 = v_1$ and $\{a_2, v_2\}$ is in E_2 or $a_2 = v_2$ and $\{a_1, v_1\}$ is in E_1. So $\sigma \in \mathcal{N}(G_1 \times G_2)$ if and only if $\exists v_1 \in V_1$ and $v_2 \in V_2$ such that $\sigma \subset \{(v_1, a_2) : \{a_2, v_2\} \in E_2\} \cup \{(a_1, v_2) : \{a_1, v_1\} \in E_1\}$.

Let consider the simplicial complexes $H_1 = \{\tau \in \mathrm{Fin}\, V_1 \times V_2 : \exists \sigma_1 \in \mathcal{N}(G_1)$ and $\exists v_2 \in V_2 : \tau = \sigma_1 \times \{v_2\}\}$ and $H_2 = \{\tau \in \mathrm{Fin}\, V_1 \times V_2 : \exists \sigma_2 \in \mathcal{N}(G_2)$ and $\exists v_1 \in V_1 : \tau = \{v_1\} \times \sigma_2\}$.

From the above considerations, $\sigma \in \mathcal{N}(G_1 \times G_2)$ if and only if there exists $\tau_i \in H_i$, $i = 1, 2$, such that $\sigma = \tau_1 \cup \tau_2$, that is $\mathcal{N}(G_1 \times G_2) = H_1 \star H_2$.

Finally the simplicial complex H_1 is the direct sum $\bigoplus_{v_2 \in V_2} \mathcal{N}(G_1)_{v_2}$ where $N(G_1)_{v_2}$ is a copy of $N(G_1)$. Analogously $H_2 = \bigoplus_{v_1 \in V_1} \mathcal{N}(G_2)_{v_1}$ with $N(G_2)_{v_1}$ is a copy of $N(G_2)$. $\square$

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