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Every Graph is a Self-Similar Set

Todo Grafo es un Conjunto Autosimilar

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Abstract

In this paper we prove that every graph (in particular $\mathbb{S}^1$) is a self-similar space and that $[0, 1]$ is a self-similar set that is not the product of topological spaces, answering two questions posed by C. Ruiz and S. Sabogal in [6].
Key words and phrases: Self-similar, graph.

Resumen

En este artículo probamos que todo grafo (en particular $\mathbb{S}^1$) es un espacio autosimilar y que $[0, 1]$ es un conjunto autosimilar que no es el producto de espacios topológicos, contestando así dos preguntas formuladas por C. Ruiz y S. Sabogal en [6].
Palabras y frases clave: Auto-similar, grafo.

1 Introduction

It is known that $\mathbb{S}^1$ is not a strict self-similar space, since strict self-similar spaces are self-homeomorphic (see section 2 of [1]) and $\mathbb{S}^1$ is not. In [3] and [5] appears a weakening of that notion, called self-similar symbolic spaces. In [6], C. Ruiz and S. Sabogal raised the following question: is $\mathbb{S}^1$ self-similar symbolic? We find in this paper that the answer is “yes”. We are going to prove even more: every graph is (non-strict) self-similar.

First, let us recall the main concepts from [4].

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Consider first a finite set of contractions f_i, each with Lipschitz constant $s < 1$, taking a compact metric space K into itself, $i = 1, 2, \dots n$. Such a setup $K(A, \{f_i\}_{i=1,\dots,n})$ is called an *iterated function system* (IFS). Use this IFS to construct a mapping W from the space $\mathbb{H}$ of nonempty compact subsets of K into itself by defining, in the self-explanatory notation, $W(B) = \bigcup_{i=1}^{n} f_i(B)$ for all $B \in \mathbb{H}$.

Then W is a contraction, with Lipschitz constant $s < 1$, with respect to the Hausdorff metric h on $\mathbb{H}$.

Moreover, $\mathbb{H}$ endowed with h is complete. In this setting W admits a unique fixed point; that is, there is exactly one nonempty compact subset A of K such that $A = W(A)$. A is called the *attractor* of the IFS. A space is called *self-similar* if it is the attractor of some IFS, and *strict* (see [2]) *self-similar* if the mappings f_i are not only contractions but similarities.

We recall that a *graph* is a locally connected continuum with a finite number of end points and ramification points.

Definition 1.1. Let (K, d) be a metric space. We say that (K, d) is a *Lipschitz image* of $[0, 1]$ if there exists a Lipschitz mapping from $[0, 1]$ with the usual metric onto (K, d).

We say that (K, d) is a *non-expansive image* of $[0, 1]$ if there exists a Lipschitz mapping from $[0, 1]$ with the usual metric onto (K, d), with Lipschitz constant not greater than 1.

Remark 1.2. Note that if (K, d) is a Lipschitz image of $[0, 1]$, and $x \in K$, we can take a Lipschitz mapping f from $[0, 1]$ onto (K, d) such that $f(0) = x$.

Proof. Let $g : [0, 1] \to K$ be an onto Lipschitz mapping with Lipschitz constant L. Let $r \in [0, 1]$ be such that $g(r) = x$, $e : [0, 1] \to [0, 1 + r]$ be defined by $e(x) = (1 + r)x$ for every $x \in [0, 1]$, $h : [0, 1 + r] \to K$ be defined by $h(x) = g(d_u(x, r))$ for every $x \in [0, 1 + r]$ (where d_u is the usual metric on $\mathbb{R}$) and $f : [0, 1] \to K$ be defined by $f(x) = h \circ e(x)$ for every $x \in [0, 1]$. Let $x, y \in [0, 1]$, then $d(f(x), f(y)) = d(g(d_u(e(x), r)), g(d_u(e(y), r))) \leq L d_u(d_u(e(x), r), d_u(e(y), r)) \leq L d_u(e(x), e(y)) = L(1 + r) d_u(x, y)$, and hence f is a Lipschitz mapping. Since h and e are onto mapping, it follows that f is an onto mapping. Finally $f(0) = g(d(0, r)) = g(r) = x$. □

2 Main results

Proposition 2.1. *Let (X, d) be a compact metric space, and suppose that $X = \bigcup_{i=1}^{n} K_i$, where each $(K_i, d|_{K_i \times K_i})$ is a Lipschitz image of $[0, 1]$. Then* *there exists* K_{ij} *for* $j = 1, \dots, m$ *and* $i = 1, \dots, n$ *such that* $X = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} K_{ij}$ *and* K_{ij} *is a non-expansive image of* $[0, 1]$ *for any* $j = 1, \dots, m$ *and* $i = 1, \dots, n$.

Proof. Let $h_i : [0, 1] \to K_i$ be a Lipschitz mapping with Lipschitz constant L for each $i = 1, \dots, n$ (we can suppose that L is the same Lipschitz constant for each h_i and that $L \in \mathbb{Z}$). Let $K_{ij} = h_i([\frac{j}{L}, \frac{j+1}{L}])$ for each $j = 0, \dots, L-1$ and each $i = 1, \dots, n$. Given $j \in \{0, \dots, L-1\}$, let $r_j : [0, 1] \to [\frac{j}{L}, \frac{j+1}{L}]$ be defined by $r_j(x) = \frac{x+j}{L}$ for each $x \in [0, 1]$, and let $h_{ij} : [0, 1] \to K_{ij}$ be defined by $h_{ij} = h_i \circ r_j$ for each $j = 0, \dots, L-1$ and $i = 1, \dots, n$. It is straightforward to check that h_{ij} is a non-expansive onto mapping, and hence K_{ij} is a non-expansive image of $[0, 1]$ for any $j = 0, \dots, L-1$ and $i = 1, \dots, n$. □

Definition 2.2. Let (X, d) be a metric space and $x \in X$. We denote by $\delta_x(X) = \sup\{d(x, y) : y \in X\}$.

Note that if (X, d) is a compact metric space then $\delta_x(X) = \max\{d(x, y) : y \in X\}$ for any $x \in X$.

Theorem 2.3. *Let* (X, d) *be a non-degenerate compact metric space, and suppose that* $X = \bigcup_{i=1}^{n} K_i$*, where* $(K_i, d|_{K_i \times K_i})$ *is a Lipschitz image of* $[0, 1]$*. Then there exist onto contractions* $f_i : X \to K_i$ *for* $i = 1, \dots, n$ *such that* $(X, \{f_i : i = 1, \dots, n\})$ *is a self similar set.*

Proof. Let $\delta(X) = \min\{\delta_x(X) : x \in X\}$ (note that $\delta(X) > 0$, since X is non-degenerate). If $\delta(X) \le 1$, we can replace d by sd, where $s = 1 + \frac{1}{\delta(X)}$. So we will suppose that $\delta_x(X) > 1$ for each $x \in X$.

Suppose that $n > 1$ (if $n = 1$ and $h : [0, 1] \to X$ is a Lipschitz onto mapping, then we can consider $K_1 = h([0, \frac{1}{2}])$ and $K_2 = h([\frac{1}{2}, 1])$, and hence we are in the case for which $n = 2$). Let $x_i \in K_i$ and $h_i : [0, 1] \to K_i$ be non-expansive onto mappings with $h_i(0) = x_i$ (apply Remark 1.2 and Proposition 2.1). Let $f_i : X \to K_i$ be defined by $f_i(x) = h_i(\frac{d(x_i, x)}{\delta_{x_i}(X)})$. Then f_i is an onto contraction with Lipschitz constant $\frac{1}{\delta_{x_i}(X)}$, and thus X is the attractor of the system $K(X, \{f_i : i = 1, \dots, n\})$. Indeed, it is clear that f_i is onto and given $x, y \in X$ it follows that $d(f_i(x), f_i(y)) = d(h_i(\frac{d(x_i, x)}{\delta_{x_i}(X)}), h_i(\frac{d(x_i, y)}{\delta_{x_i}(X)})) \le \frac{1}{\delta_{x_i}(X)} d_u(d(x_i, x), d(x_i, y)) \le \frac{1}{\delta_{x_i}(X)} d(x, y)$ (where d_u is the usual metric for $\mathbb{R}$), and since $\frac{1}{\delta_{x_i}(X)} < 1$ we have that f_i is a contraction. □

Since it is clear that every graph is the union of a finite number of Lipschitz images of $[0, 1]$, the following corollary is apparent.

Corollary 2.4. *Every graph is a self similar set.*

Corollary 2.5. *The circle $\mathbb{S}^1$ is a self similar set.*

Question 2.6. Which spaces (in particular graphs) are attractors of an IFS with only two mappings?

Next, we give some examples of graphs with an IFS with only two mappings.

Example 2.7.

1. Intervals: In $[0,1]$, let defined the mappings $h_1 : [0,1] \to [0, \frac{1}{2}]$ and $h_2 : [0,1] \to [\frac{1}{2}, 1]$ by $h_1(x) = \frac{x}{2}$ and $h_2(x) = \frac{1+x}{2}$ for each $x \in [0,1]$.

2. Simple triod: Let $X = A \cup B$, where $A = [0,2] \times \{0\}$ and $B = \{0\} \times [-1,1]$. Let d be defined by $d_u(x,(0,0)) + d_u((0,0),y)$ if $x \in A$ and $y \in B$ or viceversa and by $d_u(x,y)$ if $x,y \in A$ or $x,y \in B$, where d_u is the usual metric on $\mathbb{R}^2$. Let $a = (2,0) \in A$ and $b = (0,1) \in B$. Let $h_1 : X \to A$ be defined by $h_1(x) = (\frac{2}{3}d(a,x), 0)$ for each $x \in X$ and $h_2 : X \to B$ be defined by $h_2(x) = (0, 1 - \frac{2}{3}d(b,x))$ for each $x \in X$. It is straightforward to check that h_1 and h_2 are contractions with Lipschitz constant $\frac{2}{3}$.

3. Triangle with a stick: Let $X = A \cup B$, where $A = [(0,0),(1,0)] \cup [(-\frac{\sqrt{3}}{2}, -\frac{1}{2}),(0,0)]$ and $B = [(-\frac{\sqrt{3}}{2}, -\frac{1}{2}),(-\frac{\sqrt{3}}{2}, \frac{1}{2})] \cup [(-\frac{\sqrt{3}}{2}, \frac{1}{2}),(0,0)]$, where $[,]$ means the segment between both points in $\mathbb{R}^2$. Let d be a metric on X such that the distance of any two neighboring vertices ($(0,0)$, $(1,0)$, $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$) is one, each edge is isometric to $[0,1]$, and the distance between any two points is the length of the shortest arc joining them. Let $a = (1,0) \in A$ and $b = (-\frac{\sqrt{3}}{2}, 0) \in B$. Let $h_1 : X \to A$ be defined by $h_1(x) = i_1(\frac{d(a,x)}{d(a,b)})$ for each $x \in X$, where i_1 is the isometry from $([0,1], d_u)$ onto $(A, \frac{1}{2}d)$, and let $h_2 : X \to B$ be defined by $h_2(x) = i_2(\frac{d(b,x)}{d(a,b)})$ for each $x \in X$, where i_2 is the isometry from $([0,1], d_u)$ onto $(B, \frac{1}{2}d)$ (where d_u is the usual metric on $\mathbb{R}$). It is easy to check that h_1 and h_2 are contractions with Lipschitz constant $\frac{4}{5}$.

4. Let $X = A \cup B$, where $A = [(-1,0),(2,0)]$ and $B = [(0,-1),(0,2)]$. Let d be a metric on X such that the distance of any two neighboring vertices($(0,0)$, $(1,0)$, $(2,0)$, $(-1,0)$, $(0,-1)$, $(0,1)$ and $(0,2)$) is one, each

edge is isometric to $[0,1]$, and the distance between any two points is the length of the shortest arc joining them. Let $a=(2,0)\in A$ and $b=(0,2)\in B$. Let $h_1:X\to A$ be defined by $h_1(x)=(2-\frac{3d(a,x)}{4},0)$ for each $x\in X$, and $h_2:X\to B$ be defined by $h_2(x)=(0,2-\frac{3d(b,x)}{4})$ for each $x\in X$. It is straightforward to check that h_1 and h_2 are contractions with Lipschitz constant $\frac{3}{4}$.

In [6] it is also asked if every self-similar space is the product of topological spaces. The negative answer is in the next result.

Theorem 2.8. $[0,1]$ *is an strict self-similar set, but it cannot be the product of two topological spaces.*

Proof. Take $f_1(x)=\frac{x}{2}$ and $f_2(x)=\frac{x+1}{2}$. Then $[0,1]$ is the attractor of $K([0,1],\{f_1,f_2\})$.

Now suppose that $[0,1]=X\times Y$, then $[0,1]\setminus\{\{p\}\times\{q_1,q_2,q_3\}\}$ must be connected (it is widely known that if X and Y are connected spaces, $X\times Y\setminus A\times B$ is connected if A and B are proper subsets of X and Y). Hence $X=\{p\}$ or $Y=\{q_1,q_2,q_3\}$ and is connected (which is a contradiction since Y is a metric space with 3 points). $\square$

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