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Hamiltonian Virus-Free Digraphs

Digrafos Libres de Virus Hamiltonianos

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Abstract

A hamiltonian virus is a local configuration that, if present in a digraph, forbids this digraph to have a hamiltonian circuit. Unfortunately, there are non-hamiltonian digraphs that are hamiltonian virus-free. Some families of these digraphs will be described here. Moreover, problems and conjectures related to hamiltonian virus-free digraphs are given.
Keywords and phrases: digraph, hamiltonian digraph, hamiltonian virus.

Resumen

Un virus hamiltoniano es una estructura local que, estando presente en un digrafo, impide que éste tenga un circuito hamiltoniano. Desafortunadamente, existen digrafos no hamiltonianos sin virus hamiltonianos. Algunas familias de estos digrafos son descritas aquí. Más aún, se plantean problemas y conjeturas relativas a digrafos sin virus hamiltonianos.
Palabras y frases clave: digrafos, digrafos hamiltonianos, virus hamiltonianos.

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1 Introduction and terminology

The importance of hamiltonian viruses [1, 8] is their relation with the "certification" of non-hamiltonian digraph families, i.e. they are non-hamiltonian if and only if they have a hamiltonian virus. For example, balanced bipartite digraphs are hamiltonian if and only if they are hamiltonian virus-free. This paper has multiple goals:

- To identify non-hamiltonian digraph families which are hamiltonian virus-free, and digraph families that are hamiltonian if and only if they are hamiltonian virus-free.
- To characterize digraphs that do not contain hamiltonian viruses of a given order.
- To present and discuss problems and conjectures related to expected properties of hamiltonian virus-free digraph families.

1.1 Terminology

The terminology described in what follows is taken textually from [2] and it will be used throughout the whole paper.

Invariants are integer or boolean values that are preserved under isomorphism. We will be using the following invariants, relations between invariants, theorems and digraph examples. Let $D = (V(D), E(D))$ be a digraph.

Integer invariants

nodes: number of nodes of a digraph.
arcs: number of arcs of a digraph.
alpha2: maximum size of a set of nodes which induces no circuit of length 2.
alpha0: maximum size of a set of nodes inducing no arc.
woodall: $\min\{d^+(x) + d^-(y) : (x, y) \notin E(D), x \neq y\}$ (if *alpha2* $= 1$, then *woodall* $= 2$*nodes* by convention.)
minimum: $\min\{$*mindegpositive*, *mindegnegative*$\}$.
mindegpositive $\min\{d^+(x) : x \in V(D)\}$.
mindegnegative $\min\{d^-(x) : x \in V(D)\}$.

Boolean invariants

hamiltonian: the digraph contains a hamiltonian circuit.
traceable: the digraph contains a hamiltonian path.
k-connected: the deletion of fewer than k vertices always results in a connected digraph.

bipartite: its vertex set is partitioned into two subsets X and Y such that each arc has one vertex in X and another in Y.
antisymmetric: it does not contain a circuit of length two.
(1,1)-*factor*: it contains a spanning subdigraph H such that $d_H^+(x) = d_H^-(x) = 1$ for all vertices.

Relations between invariants

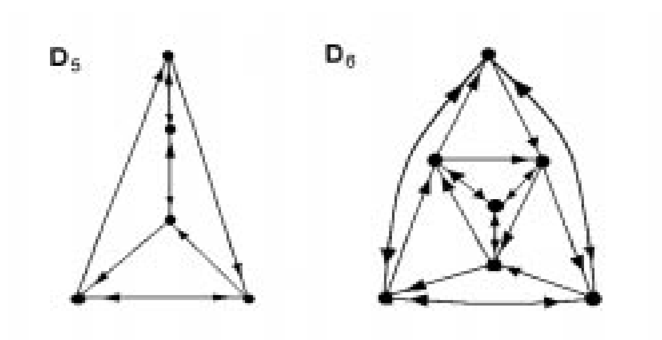
R_{11}: *minimum* $\geq 2 \ \wedge$ *nodes* $\leq 4 \Longrightarrow$ *hamiltonian.*

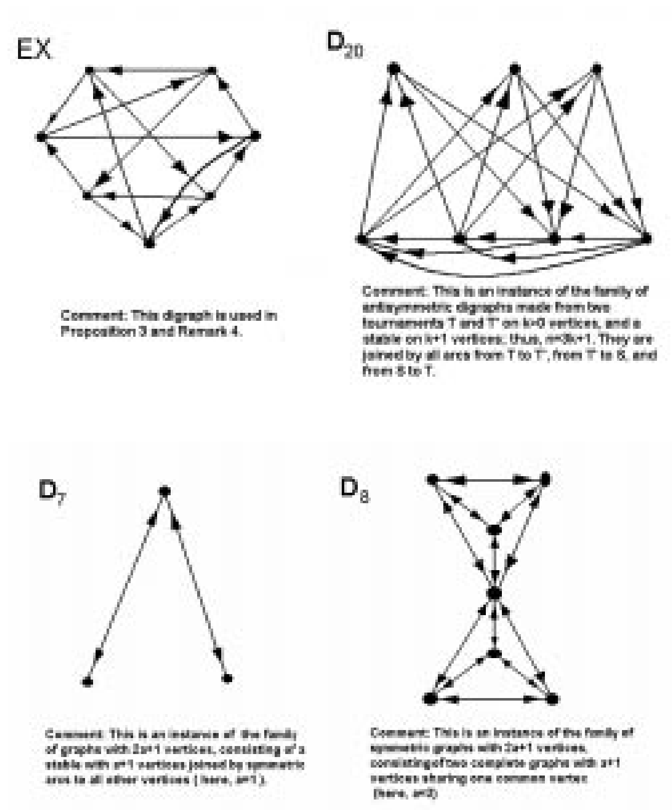
R_{31}: *antisymmetric* $\Longrightarrow$ *arcs* $\leq$ *nodes*$($*nodes* $-1)/2$.

Theorems and conjectures

Theorem 51: k-connected $\wedge$ *(alpha0* $\leq k$*)* $\Longrightarrow$ *(1,1)-factor. Best result: see* D_{20}.
Theorem 64: antisymmetric $\wedge$ *(nodes* $\leq 2h+2$*)* $\wedge$ *(*$h \geq 2$*)* $\wedge$ *(minimum* $\geq h$*)* $\Longrightarrow$ *hamiltonian.*
Theorem 65: antisymmetric $\wedge$ *(nodes* ≥ 6*)* $\wedge$ *(woodall* $\geq$ *nodes* $-\ 2$*)* $\Longrightarrow$ *hamiltonian.*
Theorem 66: antisymmetric $\wedge$ *(*$h \geq 5$*)* $\wedge$ *(minimum* $\geq h$*)* $\wedge$ *(nodes* $\leq 2h+5$*)* $\Longrightarrow$ *hamiltonian.*
Theorem 67: antisymmetric $\wedge$ *2-connected* $\wedge$ *[arcs* $\geq$ *nodes(nodes* $-\ 1$*)/2* $-\ 2$*]* $\Longrightarrow$ *hamiltonian. Best result: see* D_{20}.
Theorem 77: (nodes $= 2a+1$*)* $\wedge$ *(minimum* $\geq a$*)* $\Longrightarrow$ *hamiltonian* $\vee D_5 \vee D_6 \vee D_7 \vee D_8$.
Theorem 78: r-diregular $\wedge$ *(nodes*$= 2r+1$*)* $\Longrightarrow$ *hamiltonian* $\vee D_5 \vee D_6$.

Digraphs examples





2 Hamiltonian viruses

A *hamiltonian virus* is a local configuration that, if present in a digraph, forbids this digraph to have a hamiltonian circuit.

Theorem 1 ([1]). *Let* $H = (V(H), E(H))$ *be a proper induced subdigraph of a given digraph* $D = (V(D), E(D))$. *A 3-uple* (H, T^+, T^-), *where* $T^+ = \{x \in V(H) : d^+_H(x) = d^+_D(x)\}$ *and* $T^- = \{x \in V(H) : d^-_H(x) = d^-_D(x)\}$, *is a hamiltonian virus if and only if for every set of disjoint directed paths* $P_1, \ldots, P_r$ *covering* $V(H)$ *there exists a path* $P_j = x^1_j \ldots x^{q(j)}_j$, *with* $q(j) \geq 1$ *such that either* $x^1_j \in T^-$ *or* $x^{q(j)}_j \in T^+$. *The order of a hamiltonian virus* (H, T^+, T^-) *is defined as the cardinality of* $V(H)$.

In what follows a 3-uple (H, T^+, T^-) is present in a digraph D if and only if there exists in D a proper induced subdigraph H^1 isomorphic to H (for convenience we identify H^1 with H), such that $T^+ = \{x \in V(H) : d_H^+(x) = d_D^+(x)\}$ and $T^- = \{x \in V(H) : d_H^-(x) = d_D^-(x)\}$. Moreover if (H, T^+, T^-) is a hamiltonian virus of a given digraph D, then we must have $T^+ \neq \emptyset$ or $T^- \neq \emptyset$.

In [1] we show that if a 3-uple (H, T^+, T^-) is not a hamiltonian virus for a given digraph, then there exists a digraph D where (H, T^+, T^-) is present and D is hamiltonian.

Theorem 2. *If a digraph of order n is free of hamiltonian viruses of order h for some h with $2 \leq h < n$, then it has no hamiltonian viruses of order less than h.*

Equivalently, from a hamiltonian virus of order $1 \leq h \leq n-2$ we can build a hamiltonian virus of order $h+1$.

Proof. Let us reason ab absurdo. Let D be a digraph of order n and $2 \leq h \leq n-1$. Assuming D contains a hamiltonian virus (H, T^+, T^-) of order $h-1$. Let $x \in V(D) \setminus V(H)$ and H_1 the subdigraph induced by $V(H \cup \{x\})$ in D. It is clear that the 3-uple (H_1, T_1^+, T_1^-) with $T_1^+ = T^+$ and $T_1^- = T^-$ is present in D. Now, we shall see that (H_1, T_1^+, T_1^-) is a hamiltonian virus. Let $P_1, \ldots, P_r$ be a set of disjoint directed paths covering $V(H_1)$. Without loss of generality we can suppose that $x \in V(P_1)$. We consider three cases:

Case 1 $P_1 = xx_1^1 \ldots x_1^s$. In this case $x_1^1 \notin T^-$. Hence $P_1^1 = x_1^1 \ldots x_1^s$, $P_i^1 = P_i$ $(2 \leq i \leq r)$ are disjoint directed paths covering $V(H)$. Since (H, T^+, T^-) is a hamiltonian virus there exists a path $P_j^1 = x_j^1 \ldots x_j^{q(j)}$ such that $x_j^1 \in T^-$ or $x_j^{q(j)} \in T^+$. Therefore $x_j^1 \in T_1^-$ or $x_j^{q(j)} \in T_1^+$.

Case 2 $P_1 = x_1^1 \ldots x_1^s x$. Hence $x_1^s \notin T^+$. This situation will be treated as Case 1.

Case 3 $P_1 = x_1^1 \ldots x_1^i x x_1^{i+1} \ldots x_1^s$. In this case $P_1^1 = x_1^1 \ldots x_1^i$, $P_2^1 = x_1^{i+1} \ldots x_1^s$, $P_{i+1}^1 = P_i (2 \leq i \leq r)$ are disjoint directed paths covering $V(H)$. Since $x \notin T^+ \cup T^-$ and (H, T^+, T^-) is a hamiltonian virus, $x_1^1 \in T^-$ or $x_1^s \in T^+$. In case $x_1^1 \notin T^-$ and $x_1^1 \notin T^-$ there exists a path P_j^1 $(3 \leq j \leq r+1)$ such that $x_{j-1}^1 \in T^-$ or $x_{j-1}^{q(j-1)} \in T^+$. Therefore (H_1, T_1^+, T_1^-) is a hamiltonian virus of order h. A contradiction. □

Corollary 1. *If a digraph of order n has hamiltonian viruses then it contains a hamiltonian virus of order $n-1$.*

Proof. Directly from Theorem 2. □

From Corollary 1 we have:

Remark 1. Let $D = (V(D), E(D))$ be a digraph. If for each $x \in V(D)$, the 3-uple (H, T^+, T^-) with $H = D - x$, $T^+ = \{x \in V(H) : d_H^+(x) = d_D^+(x)\}$ and $T^- = \{x \in V(H) : d_H^-(x) = d_D^-(x)\}$ is not a hamiltonian virus then D is hamiltonian virus-free. Moreover, if $D - x$ is not a hamiltonian virus for some x, then for each hamiltonian virus (H, T^+, T^-) of D we have $x \in V(H)$.

Digraph D_6 shows that there exist non-hamiltonian digraphs hamiltonian virus-free. In D_6 hamiltonian viruses of order 6 are not present. Then by Corollary 1 D_6 does not have viruses.

From Theorem 1 we have:

Remark 2. A hamiltonian virus-free digraph D has the following structure: for each vertex x the remaining part $D - x$ has a covering by vertex disjoint paths $P_1, \ldots, P_r$ such that each one of them makes a circuit with x.

In the next lemma $D[S]$ denotes the subdigraph induced by S.

Lemma 1. *Let (H, T^+, T^-) be a hamiltonian virus present in D. Let $S \subseteq V(H) \setminus (T^+ \cup T^-) \neq \emptyset$ be such that for all $x \in S$, $d^+_{D[T^-]}(x) = d^-_{D[T^+]}(x) = 0$. Then $(H - S, T^+, T^-)$ is a hamiltonian virus present in D.*

Proof. Let $P_1, \ldots, P_r$ be a set of disjoint directed paths covering $V(H - S)$. Then for any set of disjoint directed paths $P_{r+1}, \ldots, P_s$ covering $V(D[S])$ we have that $P_1, \ldots, P_r, P_{r+1}, \ldots, P_s$ is a set of disjoint directed paths covering $V(H)$. Since (H, T^+, T^-) is a hamiltonian virus, there exists a path $P_j (1 \leq j \leq s)$ with $x_j^1 \in T^-$ or $x_j^{q(j)} \in T^+$. On the other hand since $S \cap (T^+ \cup T^-) = \emptyset$, it must be $1 \leq j \leq r$. Therefore $(H - S, T^+, T^-)$ is a hamiltonian virus. □

Theorem 3. *Let D be an antisymmetric digraph with minimum ≥ 2. Then D is free of hamiltonian viruses of order ≤ 4.*

Proof. Let us reason ab absurdo. Let (H, T^+, T^-) be a hamiltonian virus of order 4 present in D. Let us see that $1 \leq |T^+| \leq 2$ and $1 \leq |T^-| \leq 2$. Suppose $|T^+| \geq 3$ (similarly, $|T^-| \geq 3$). By a simple inspection on the relative positions of the arcs in H, we can deduce that either there exists a symmetric arc or (H, T^+, T^-) is not a hamiltonian virus. A contradiction. Therefore it must be $|T^+| \geq 1$ and $|T^-| \geq 1$. If $T^+ = \emptyset$ then $T^- \neq \emptyset$, hence there are disjoint directed paths covering $V(H)$ that do not verify the conditions for (H, T^+, T^-) to be a hamiltonian virus. A contradiction. Consider now the case $|T^+| = 1$ and $|T^-| = 2$ (the case $|T^+| = 2$ and $|T^-| = 1$ is similarly treated). By enumerating several cases resulting from the relative positions of the arcs in H, sets of disjoint directed paths covering $V(H)$ do not verify the condition in order that (H, T^+, T^-) is a hamiltonian virus. For the case $|T^+| = |T^-| = 1$ (or $|T^+| = |T^-| = 2$), we can see the existence of sets of disjoint directed paths covering $V(H)$ that do not verify the conditions to be a hamiltonian virus. By Theorem 2, D is a hamiltonian virus-free digraph of order ≤ 3. □

In Theorem 4, we need the following definition:

Definition 1. [1] A 1-connected virus is a local configuration that, if present in a digraph, forbids this digraph to be 1-connected. Let $H = (V(H), E(H))$ be a proper induced subdigraph of a given digraph $D = (V(D), E(D))$. A 3-uple (H, T^+, T^-), where $T^+ = \{x \in V(H) : d_H^+(x) = d_D^+(x)\}$ and $T^- = \{x \in V(H) : d_H^-(x) = d_D^-(x)\}$, is a 1-connected virus if and only if $V(H) = T^+$ or $V(H) = T^-$.

Theorem 4. *A hamiltonian virus-free digraph is 2-connected.*

Proof. Let $D = (V(D), E(D))$ be hamiltonian virus-free. Let us reason ab absurdo. Let (H, T^+, T^-) be a 1-connected virus present in $D - x$ for some $x \in V(D)$ with $V(H) = T^-$. The case $V(H) = T^+$ is treated in a similar way. Let $y \in V(D - x) \setminus V(H)$. By Remark 2, there exist vertex disjoint paths $P_1, \ldots, P_r$ covering $D - y$ such that each one of them makes a circuit with y. Since $V(H) = T^-$ then for each $P_j = x_j^1 \ldots x_j^{q(j)}$ we have $x_j^1 \notin V(H)$. Moreover, for each $x_i^t \in V(H) \cap V(P_i)$ we have $x_i^{t-1} \in V(H)$. Hence $x_i^1 \in V(H)$. A contradiction. □

3 Hamiltonian virus-free digraph families

In this section we describe non-hamiltonian and hamiltonian virus-free digraph families. There exist non-hamiltonian digraph families with hamiltonian viruses. This fact has allowed to derive problems and conjectures that are presented and discussed in this section.

Theorem 5. *Balanced bipartite digraphs are hamiltonian if and only if they are hamiltonian virus-free.*

Proof. Let $D = (X \cup Y, E(D))$ be a hamiltonian balanced bipartite digraph. Hence D is hamiltonian virus-free (by Theorem 1). Let us assume that D is hamiltonian virus-free. Let $x \in X$ (similarly discussed for $y \in Y$). By

Remark 2, $D - x$ has a covering by vertex disjoint paths $P_1, \ldots, P_r$ such that each one of them makes a circuit with x. Let C_i $(1 \leq i \leq r)$ be these circuits. Since D is a balanced bipartite digraph we have $|V(C_i)| = 2n_i$ (the circuits have even length), $|X| = n_1 + n_2 - 1 + \cdots + n_r - 1$ and $|Y| = n_1 + n_2 + \cdots + n_r$. Therefore D is not balanced. A contradiction. □

The next remark follows directly from Theorem 77.

Remark 3. There are no non-hamiltonian and hamiltonian virus-free digraphs with *minimum* $= 2$ and *nodes* $= 5$. Notice that digraph D_5 has hamiltonian virus.

The only non-hamiltonian and hamiltonian virus-free digraph with *minimum* $= 3$ and *nodes* $= 7$ is D_6.

A non-hamiltonian digraph with *nodes* $= 2$*minimum* $+ 1 \geq 9$ has hamiltonian viruses. By Theorem 77 the only families of digraphs that are non-hamiltonian and where *nodes* $= 2$*minimum* $+ 1 \geq 9$ holds, are D_7 and D_8. These families have viruses.

Proposition 1. *A hamiltonian virus-free digraph with nodes* ≤ 5 *is hamiltonian.*

Proof. Let D be a hamiltonian virus-free digraph; then *minimum* ≥ 2. If *nodes* ≤ 4 then, by R_{11}, D is hamiltonian. The case *nodes* $= 5$ follows directly from Remark 3. □

Proposition 2. *A hamiltonian virus-free digraph with minimum* $= 2$ *is traceable or hamiltonian.*

Proof. Since *minimum* $= 2$, there exists $x \in V(D)$ such that $d^+(x) = 2$ or $d^-(x) = 2$. Then by Remark 2, there exist at most two vertex disjoint paths covering $D - x$, say $P_i = x_i^1 x_i^2 \ldots x_i^{r(i)} (1 \leq i \leq 2)$, such that each one of them makes a circuit with x. If there is only one path then D is hamiltonian, otherwise the path $x_1^1 x_1^2 \ldots x_1^{r(1)} x x_2^1 x_2^2 \ldots x_2^{r(2)}$ makes D *traceable.* □

Proposition 3. *A hamiltonian virus-free antisymmetric digraph with nodes* $= 6$*, 7 or 8 is hamiltonian or traceable. Moreover, the only hamiltonian non-hamiltonian virus-free antisymmetric digraph with* nodes $= 7$ *is digraph* EX.

Proof. Let D be a hamiltonian virus-free digraph. Then *minimum* ≥ 2 and *woodall* ≥ 4. By Theorem 65, if *nodes* $= 6$ then D is hamiltonian. For *nodes* $= 7$ or 8, if *minimum* $= 2$ then, by Proposition 2, D is *hamiltonian* or *traceable.* If *minimum* ≥ 3 then, by Theorem 65, D is hamiltonian. □

The following conjecture should be true:

Conjecture 1. *A hamiltonian virus-free antisymmetric digraph is hamiltonian or traceable.*

Notice that digraph EX is non-hamiltonian, but it is *traceable*.
The following conjecture should be true:

Conjecture 2. *A hamiltonian virus-free antisymmetric* $r-$*diregular digraph with* $r \geq 3$ *and* $nodes \leq 4r+1$ *is hamiltonian.*

We can formulate the following remarks for Conjecture 2: By Theorem 64, the conjecture for $r=3$ is true when $nodes \leq 8$. For case $9 \leq nodes \leq 13$ the hypothesis hamiltonian virus-free perhaps can be useful. Notice that by Theorem 78, the conjecture is true for $nodes = 2r+1$. The conjecture is true from Theorem 66 for $r=5$ and $nodes \leq 15$.

Problem 1. *Let* D *be an antisymmetric and hamiltonian virus-free digraph. Find the greatest positive integer* x *such that when arcs*$\geq$ *nodes(nodes-1)/2*$-x$ *then* D *is hamiltonian.*

By Theorem 4 and Theorem 67 we have $x \geq 2$. Moreover the digraph D_{20} shows that Theorem 67 is the best possible. Notice that D_{20} has hamiltonian viruses.

Problem 2. *Let* D *be a* k*-connected and hamiltonian virus-free digraph. Find the greatest integer* x *such that when* $alpha0 \leq k+x$ *then* D *contains a (1,1)-factor.*

By Theorem 51 we have that $x \geq 0$. Moreover digraph D_{20} shows that this theorem is the best possible. Notice that D_{20} has hamiltonian viruses.

3.1 Hamiltonian virus-free hypohamiltonian digraphs

This section is devoted to study hypohamiltonian hamiltonian virus-free digraphs and those that have hamiltonian viruses. The methods, for building hypohamiltonian digraphs, established in [9] and [4] are given. Some conjectures related to hamiltonian virus-free and hypohamiltonian digraphs are discussed.

A digraph D is *hypohamiltonian* if it has no hamiltonian circuits but every vertex-deleted subdigraph $D-v$ has such a circuit.

It is natural to formulate, as in [6], the following conjectures:

Conjecture 3. *Every hamiltonian virus-free non-hamiltonian digraph is hypohamiltonian.*

Or the weaker one:

Conjecture 4. *Every non-hamiltonian vertex-transitive hamiltonian virus-free digraph is hypohamiltonian.*

Conjecture 5. *Every hypohamiltonian digraph is hamiltonian virus-free.*

Notice that digraph D_6 is hypohamiltonian and hamiltonian virus-free. In [9] Thomassen gives a method for obtaining hypohamiltonian digraphs by forming the cartesian product of cycles. We give here a short summary of his results, in order to give some remarks on Conjectures 3, 4, 5.

Recall that if D_1 and D_2 are digraphs then its *cartesian product* $D_1 \times D_2$ is the digraph with vertex set $V(D_1) \times V(D_2)$ such that the edge from (v_1, v_2) to (u_1, u_2) is present if and only if $v_1 = u_1$ and $v_2u_2 \in E(D_2)$, or $v_2 = u_2$ and $v_1u_1 \in E(D_1)$. The directed cycle of length k, $2 \leq k$, is denoted C_k. With this notation Thomassen gives the following theorems:

Theorem 6 ([9]). *For each $k \geq 3$, $m \geq 2$, $C_k \times C_{mk-1}$ is a hypohamiltonian antisymmetric digraph. Moreover, $C_3 \times C_{6k+4}$ is hypohamiltonian for each $k \geq 0$.*

Theorem 7 ([9]). *There is no hypohamiltonian digraph with fewer than six vertices, and for each odd $m \geq 3$, $C_2 \times C_m$ is a hypohamiltonian digraph.*

Remark 4. The hypohamiltonian digraphs $C_3 \times C_{6k+4}$ with $k \geq 0$ (Theorem 6) and the hypohamiltonian digraphs given in Theorem 7 verify Conjecture 5. However the digraph $C_4 \times C_{11}$, i.e., $k = 4$ and $m = 3$ in Theorem 6, refutes Conjecture 5. We have proved that the only non-hamiltonian vertex-transitive digraph which is also hamiltonian virus-free of order 6, is the hypohamiltonian digraph $C_2 \times C_3$. Which is in favor of Conjecture 4. Nevertheless the Conjecture 4 is false, the digraph EX is non-hamiltonian, vertex-transitive, hamiltonian virus-free and not hypohamiltonian.

In [4] Fouquet and Jolivet give the following theorem for obtaining hypohamiltonian digraphs.

Theorem 8 ([4]). *For each $n \geq 6$, the digraph $F_n = (V(F_n), E(F_n))$ described below is hypohamiltonian.*

- *For $n = 6$, $F_6 = C_2 \times C_3$.*

- *For* $n = 2p+1$ *and* $p \geq 3$*:* $V(F_n) = \{x_o, x_1, \ldots, x_{2p-1}, y\}$ *and* $E(F_n) = \{x_{2p-1}x_o, x_i x_{i+1}\ (0 \leq i \leq 2p-2)\} \cup \{x_k x_{k-2}$ *and* $x_k x_{k+4}\ , 1 \leq k \leq 2p-1, k$ *odd*$\} \cup \{x_k y$ *and* $y x_k\ , 0 \leq k \leq 2p-1, k$ *even*$\}$*. Each index is taken modulo* $2p$*.*
- *For* $n = 2p$ *and* $p \geq 4$*,* F_n *is obtained from* F_{2p-1} *replacing the arc* $x_{2p-3}x_o$ *by the path* $x_{2p-3}xx_o$ *and adding the following arcs:* $x_{2p-4}x_{2p-3}$, $x_{2p-3}x_{2p-4}$, $x_o x$, $x_{2p-3}x_{2p-5}$, $x_{2p-7}x_{2p-3}$, $x_1 x$, xx_3, $x_o x_{2p-4}, x_{2p-3}x$, xx_{2p-3}*. Each index is taken modulo* $2p-2$*.*

In the next theorem let $C = x_o x_1 \ldots x_{2p-1} x_o$ be a circuit. We denote by $C(x_i, x_j)$ the induced path of C beginning at x_i and ending at x_j.

Theorem 9. *For each* $n \geq 8$*,* F_n *is hamiltonian virus-free.*

Proof. We follow Remark 1 and Remark 2. In each step of the proof, we show the paths P_i $(1 \leq i \leq 2)$ that cover $F_n - w$ and make a circuit with w. For all $w \in V(F_n)$. We consider two cases:

Case 1 $n = 2p+1$ and $p \geq 4$.

- $F_n - x_k$ with $0 \leq k \leq 2p-1$.
  For k even, the paths are $P_1 = y$ and $P_2 = C(x_{k+1}, x_{k-1})$.
  For k odd, the paths are $P_1 = C(x_{k+4}, x_{k-4})$ and $P_2 = C(x_{k+1}, x_{k+3})y$ $C(x_{k-3}, x_{k-1})$.
- $F_n - y$: the paths are $P_1 = C(x_2, x_3)x_1 x_{2p-1} x_o$ and $P_2 = C(x_4, x_{2p-2})$.

Case 2 $n = 2p$ and $p \geq 4$.

- $F_n - x_k$ with $0 \leq k \leq 2p-3$.
  For k odd except 3 *and* $2p-3$ the paths are similar to those of Case 1.
  For $k = 3$ the paths are $P_1 = C(x_7, x)$ and $P_2 = C(x_4, x_6)yC(x_o, x_2)$.
  For $k = 2p-3$ the paths are $P_1 = x$ and $P_2 = x_{2p-5}C(x_1, x_{2p-6})yx_o x_{2p-4}$.
  For k even, the paths are $P_1 = y$ and $P_2 = C(x_{k+1}, x_{k-1})$.
- $F_n - y$: the paths are $P_1 = C(x_2, x_{2p-6})$ and $P_2 = x_o x_1 x x_{2p-3} x_{2p-5} x_{2p-4}$.
- $F_n - x$: the paths are $P_1 = x_{2p-3}$ and $P_2 = x_o x_{2p-4} y C(x_2, x_{2p-5}) x_1$.

□

Remark 5. For $n \geq 8$, the hypohamiltonian digraphs F_n given in Theorem 8 is in favor of Conjecture 5.

4 Conclusion

It is well known that the problem to decide when a digraph is hamiltonian is NP-complete [3]. A "yes" answer to the hamiltonicity problem for a given digraph can be verified by checking in polynomial time that a sequence of vertices given by an oracle is a hamiltonian circuit. In case of non-hamiltonian digraphs, as stated in [7] pages 28, 29, there is no known way of verifying a "yes" answer to the complementary problem of deciding if a digraph is non-hamiltonian. A solution to this problem is to provide a hamiltonian virus, whose presence in the digraph can also be checked in polynomial time. In case of the non-hamiltonian hamiltonian virus-free digraphs, they must hold the particular structure given in Remark 2. The virus notion has been used in random generation of digraphs without certain properties [8].

We have built an interactive support tool called GRAPHVIRUS [5] that allows the graphical edition of hamiltonian viruses and the verification that a given structure is a hamiltonian virus. GRAPHVIRUS can also be used to derive a procedure for deciding whether a given digraph is non-hamiltonian. This procedure is of the same complexity of the problem of deciding if a given digraph is hamiltonian, but the interest of the procedure is the fact of using a local structure.

Finally, the theoretic interest of the results presented here is their relation with the extension of known sufficient conditions with the new hamiltonian virus-free condition for the existence of hamiltonian circuits.

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