Divulgaciones Matemáticas v. 6, No. 2 (1998), 133–138

Roughness of (μ_1, μ_2)-Dichotomies under Small Perturbations in L^∞ †

Persistencia de las Dicotomías (μ_1, μ_2) bajo Perturbaciones Pequeñas en L^∞

Marcos Lizana
(lizana@ciens.ula.ve)
Departamento de Matemáticas, Facultad de Ciencias
Universidad de Los Andes
Mérida 5101, Venezuela
fax: +58 074 401286

Abstract

This paper is devoted to the investigation of the concept of (μ_1, μ_2)-dichotomy. More precisely, we prove that under suitable conditions on (μ_1, μ_2), this type of dichotomy has the roughness property with respect to small perturbations in L^∞.
Key words and phrases: (μ_1, μ_2)-dichotomy, roughness of a dichotomy.

Resumen

En este artículo se investiga el concepto de dicotomía del tipo (μ_1, μ_2). Más precisamente se prueba que, bajo condiciones apropiadas para (μ_1, μ_2), las dicotomías de este tipo persisten bajo pequeñas perturbaciones en L^∞.
Palabras y frases clave: dicotomías (μ_1, μ_2), persistencia de una dicotomía.

† Recibido 98/09/25. Aceptado 98/11/27.
MSC (1991): 34A34; 34A99.

1 Introduction

Suppose $A \in C(J, S^{n\times n})$, where $J = (a, b)$ with $-\infty \le a < b \le \infty$ and S is the set of real or complex numbers. Let Φ be a fundamental matrix solution for the linear homogeneous differential system

$$x' = A(t)x, \quad t \in J. \tag{1.1}$$

If μ_1 and μ_2 are continuous real valued functions on J, the system (1.1) will be said to have a (μ_1, μ_2)-*dichotomy* if there exist supplementary projections P_1 and P_2 on $\mathbb{R}^n$ and a constant $K > 0$ such that

$$\|\Phi(t)P_i\Phi^{-1}(s)\| \le K \exp(\int_s^t \mu_i(\tau)\,d\tau) \quad \text{if } (t-s)(-1)^{i-1} \ge 0,\ i = 1, 2. \tag{1.2}$$

In the case that μ_1 and μ_2 are constants, the system (1.1) is said to have an *exponential dichotomy* if $\mu_1 < 0 < \mu_2$, and an *ordinary dichotomy* if $\mu_1 = \mu_2 = 0$.

The condition (1.2) is readily seen to be equivalent to

$$\|\Phi(t)P_i\xi\| \le K \exp(\int_s^t \mu_i(\tau)\,d\tau)\|\Phi(s)P_i\xi\|, \text{ if } (t-s)(-1)^{i-1} \ge 0,\ i = 1, 2, \tag{1.3}$$

$$\|\Phi(t)P_i\Phi^{-1}(t)\| \le K, \quad i = 1, 2. \tag{1.4}$$

Condition (1.4) says that the angle between $\Phi(t)P_i\mathbb{R}^n$ $(i = 1, 2)$ remains bounded away from zero. The two cases of most interest are where J is the positive half-line $\mathbb{R}_+$ and the whole line $\mathbb{R}$. Hereafter we assume that $J = \mathbb{R}$.

This definition was introduced by J. S. Muldowney in [4]. There, three sets of necessary and sufficient conditions for a (μ_1, μ_2)-dichotomy were given in terms of Liapunov functions. Each result gives practical criteria for a dichotomy, including the extension to unbounded matrices of criteria which depend on a concept of diagonal dominance for $A(t)$. An asymptotic analysis is also given for subspaces of the solution set by means of the associated compound equations.

This paper is devoted to further investigations of this concept of dichotomy. More precisely we prove that, under suitable conditions on (μ_1, μ_2), this type of dichotomy has the roughness property with respect to small perturbations in L^∞.

2 Roughness of a (μ_1, μ_2)-dichotomy

In the proof of the roughness property of a (μ_1, μ_2)-dichotomy, it will be very important that the following "functions" :

$$u^*(t) = \int_{-\infty}^{t} \exp(\int_s^t [\mu_1(\tau) - \mu_2(\tau)]\, d\tau)\, ds, \tag{2.5}$$

$$v^*(t) = \int_t^{\infty} \exp(\int_s^t [\mu_2(\tau) - \mu_1(\tau)]\, d\tau)\, ds, \tag{2.6}$$

be well defined and bounded on $\mathbb{R}$. We sketch a few facts that will help us to clarify this point.

Throughout the following, $\mathcal{B} = \{f : \mathbb{R} \to \mathbf{C}^n, f$ continuous and bounded$\}$ and for any f in $\mathcal{B}$, $|f| = \sup\{|f(t)| : t \in \mathbb{R}\}$. The subset $\mathcal{AP}$ of $\mathcal{B}$ denotes the set of almost-periodic functions. The subset $\mathcal{P}_\omega$ of $\mathcal{AP}$ denotes the set of periodic functions of period ω. An $n \times n$ matrix function on $\mathbb{R}$ is said to belong to one of these spaces if each column belongs to the space.

Let us define the operator $Lx := x' - A(t)x$ with $A \in \mathcal{B}$. The operator L is said to be *regular* if the equation $Lx = f$ has a unique solution in $\mathcal{B}$, for any $f \in \mathcal{B}$. If the equation $Lx = f$ has at least one solution in $\mathcal{B}$, for each $f \in \mathcal{B}$, L is said to be *weakly regular*. In general, weak regularity does not imply regularity. Consider for instance the following equation

$$x'(t) = a(t)x(t) + f(t), \tag{2.7}$$

$$a(t) = \begin{cases} -1, & t \geq 1 \\ -t, & -1 < t < 1 \\ 1, & t \leq -1. \end{cases}$$

Nevertheless, if $A \in \mathcal{AP}$, then regularity and weak regularity are equivalent. See [2, Thm. 2.3, p. 26].

In the scalar case (2.7), being $a \in \mathcal{AP}$, J. L. Massera proved in [3] that L is regular if and only if

$$M(a) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} a(t)\, dt \neq 0.$$

From this result it follows that the set of all regular operators L is dense in the space of almost-periodic operators. From the former discussion the next result is immediate:

Lemma 1. *Suppose that $a \in \mathcal{AP}$ and $M(a) < 0$, where $a(t) = \mu_1(t) - \mu_2(t)$. Then (2.5) and (2.6) are the unique solutions in $\mathcal{B}$ of the equations*

$$\begin{aligned} u'(t) &= (\mu_1(t) - \mu_2(t))u(t) + 1, \\ v'(t) &= (\mu_2(t) - \mu_1(t))v(t) - 1, \end{aligned}$$

respectively.

If $a(t) = \mu_1(t) - \mu_2(t)$ is just a bounded function and

$$\limsup_{(t-s)\to\infty} \frac{1}{t-s} \int_s^t a(\tau)\, d\tau < 0,$$

the same assertion holds.

When $A \in \mathcal{P}_\omega$, L is regular if and only if the only solution of (1.1) which belongs to $\mathcal{B}$ is the solution $x = 0$.

The following result shows when a (μ_1, μ_2)-dichotomy cannot be destroyed by small perturbations in L^∞.

Theorem 2. *Let us assume that system (1.1) has a (μ_1, μ_2)-dichotomy and that (2.5) and (2.6) are bounded functions on $\mathbb{R}$. If $B(t) \in \mathcal{B}$ and $\delta = \sup\{|B(t)| : t \in \mathbb{R}\}$, then the perturbed equation*

$$x' = [A(t) + B(t)]x, \quad t \in \mathbb{R}, \tag{2.8}$$

has a (μ_1', μ_2')-dichotomy for δ small enough, where $\mu_1' = \mu_1 + 6K^3\delta$ and $\mu_2' = \mu_2 - 6K^3\delta$.

Proof. Since (1.1) admits a (μ_1, μ_2)-dichotomy, (1.4) holds. Then, by Lemma 2 in [1, p. 40], there exists a continuously differentiable invertible matrix $T(t)$ with $\|T(t)\| \le \sqrt{2}$, $\|T^{-1}(t)\| \le \sqrt{2}K$, such that the change of variables $x = T(t)z$ transforms (2.8) into the system

$$z' = [C(t) + D(t)]z, \tag{2.9}$$

whose coefficient matrix $C(t) = T^{-1}(t)A(t)T(t) - T^{-1}(t)T'(t)$ is Hermitian, commutes with P_1 and P_2, and satisfies $\|C(t)\| \le \|A(t)\|$, $\forall t \in \mathbb{R}$. Moreover, $\|D(t)\| = \|T^{-1}(t)B(t)T(t)\| \le 2K\delta$, $\forall t \in \mathbb{R}$.

Our next object is to transform the system (2.9) into a kinematically similar system whose coefficient matrix commutes with P_1 and P_2.

Denoting by $\Phi(t)$ the fundamental matrix of (1.1) and observing that $Z(t) = T^{-1}(t)\Phi(t)$ is a fundamental matrix of the system $z' = C(t)z$, we obtain that

$$\|Z(t)P_iZ^{-1}(s)\| \leq 2K^2 \exp(\int_s^t \mu_i),\ (t-s)(-1)^{i-1} \geq 0,\ i = 1,2. \tag{2.10}$$

Let us define an operator $\mathcal{F}$ on the space $\mathbf{B}(\mathbb{R}, \mathbb{R}^{n\times n})$ of the continuous and bounded matrix functions as follows :

$$\begin{aligned} \mathcal{F}H(t) \quad := \quad & \int_{-\infty}^t Z(t)P_1Z^{-1}(s)[I - H(s)]D(s)[I + D(s)]Z(s)P_2Z^{-1}(t)\, ds \\ - \quad & \int_t^\infty Z(t)P_2Z^{-1}(s)[I - H(s)]D(s)[I + D(s)]Z(s)P_1Z^{-1}(t)\, ds. \end{aligned}$$

Using (2.10) and the fact that u^* and v^* are bounded on $\mathbb{R}$, it can be proved that $\mathcal{F}$ is a contraction and maps the ball $\|H\| \leq 1/2$ into itself, if δ is sufficiently small. Let H^* be the unique fixed point of $\mathcal{F}$ in $\|H\| \leq 1/2$.

Making the transformation $z = [I + H^*(t)]v$ and arguing like in [1] pp. 42–44, system (2.9) becomes:

$$v' = [C(t) + D^*(t)]v, \tag{2.11}$$

where the coefficient matrices of (2.11) commute with P_1 and P_2. Hence, (2.11) decomposes into two independent subsystems. The rest of the proof goes in a similar manner to [1] pp. 43–44. □

Remark. In the case that $\mu_1 < 0 < \mu_2$ are constants, i.e., the system (1.1) has an exponential dichotomy, the boundedness of u^* and v^* is automatically satisfied.

We point out that [5] proved roughness with respect to small L^1 perturbations of $A(t)$, which in general does not imply roughness with respect to small L^∞ perturbations.

References

[1] Coppel, W. A., *Dichotomies in Stability Theory*, Lect. Notes in Math. V. 629, Springer Verlag, 1978.

[2] Krasnosielski, M. A., Burd V. Sh., Koliesov Yu. S., *Nonlinear Almost-periodic Oscillations*, Nauka-Moscow, 1970 (in russian).

[3] Massera, J. L., *Un criterio de existencia de soluciones casi-periódicas de ciertos sistemas de ecuaciones diferenciales casi-periódicos*, Boletín Fac. de Ing. y Agr. (Montevideo) V.6, N.11, (1958), 345–349.

[4] Muldowney, J. S., *Dichotomies and asymptotic behavior for linear differential systems*, Trans. AMS **283**, N.2 (1984), 465–484.

[5] Naulin, R., Pinto M., *Roughness of* (h, k)*-dichotomies*, JDE **118**(1995), 20–35.