Divulgaciones Matemáticas v. 6, No. 2 (1998), 123–131

A Proof of Nehari's Theorem through the Coefficient of a Matricial Measure †

Una Prueba del Teorema de Nehari a través del Coeficiente de una Medida Matricial

Marisela Domínguez
(mdomin@euler.ciens.ucv.ve)
Dpto. de Matemáticas. Facultad de Ciencias.
Universidad Central de Venezuela.
Apartado Postal 47159. Caracas 1041-A, Venezuela

Abstract

An introduction to generalized interpolation problems is given. Nehari's theorem and Sarason's commutation theorem are obtained. The proofs are simple and they are obtained using a generalization of the cosine of an angle.
Key words and phrases: interpolation, commutation, Hankel operator, coefficient of a matricial measure.

Resumen

Se da una introducción a problemas de interpolación generalizada. Se obtienen el teorema de Nehari y el teorema de conmutación de Sarason. Las pruebas son simples y se obtienen usando una generalización del coseno del ángulo.
Palabras y frases clave: interpolación, conmutación, operador de Hankel, coeficiente de una medida matricial.

†Recibido 98/05/09. Revisado 98/12/07. Aceptado 98/12/15.
MSC (1991): 47A57.

1 Introduction

Several classical questions concerning moment problems, Toeplitz and Hankel operators, weighted inequalities for the Hilbert transform and prediction theory are closely related. And the theory of analytic functions in the circle gives the natural environment for these problems.

In this paper the coefficient of a matricial measure is introduced, as a generalization of the cosine of an angle. This coefficient is used to give a characterization of the bounded Hankel operators, and to compute the distance of a function f in L^∞ to H^∞. As a consequence a representation theorem of operators commuting with special contractions is obtained.

Much function theory in the circle $\mathbf{T} \approx [0, 2\pi]$ is considered to depend on group properties of the circle and its dual $\mathbf{Z}$. The results given in this note can be extended: $\mathbf{T}$ can be replaced by the bidimensional torus and the usual order in $\mathbf{Z}$ can be replaced by the lexicographic order (although some changes must be done). Details will be given in [4].

2 The coefficient of a matricial measure

Let dx be the Lebesgue measure in $\mathbf{T}$. For $1 \leq p \leq \infty$ let $L^p(\mathbf{T})$ be the Lebesgue space and let $\| \, . \, \|_p$ be the norm in $L^p(\mathbf{T})$.

For $n \in \mathbf{Z}$ set $e_n(x) = e^{inx}$ and for $f \in L^1(\mathbf{T})$ consider the Fourier coefficients: $\widehat{f}(n)$. The trigonometric polynomials are functions $f : \mathbf{T} \to \mathbf{C}$, such that $\mathrm{supp}\widehat{f}$ is finite and

$$f(x) = \sum_{n \in \mathbf{Z}} \widehat{f}(n) \; e_n(x).$$

Let $\mathcal{P}$ be the space of trigonometric polynomials.

Consider the following sets:

$$\mathcal{P}_1 = \left\{ f \in \mathcal{P} : \mathrm{supp}\widehat{f} \subseteq \{n \in \mathbf{Z} : n \geq 0\} \right\},$$

$$\mathcal{P}_2 = \left\{ f \in \mathcal{P} : \mathrm{supp}\widehat{f} \subseteq \{n \in \mathbf{Z} : n < 0\} \right\}.$$

If $\mu = (\mu_{\alpha_\beta})_{\alpha,\beta=1,2}$ is a 2×2 matrix of complex finite Borel measures defined on $\mathbf{T}$, then it is said that μ is a *complex finite Borel matricial measure* on $\mathbf{T}$.

In this paper all the matricial measures considered are hermitian complex finite Borel on $\mathbf{T}$, and they will simply be called *matricial measures.*

Let μ be a matricial measure on $\mathbf{T}$. We shall say that $\mu \in \mathcal{D}$ if μ is hermitian, $\mu_{11} = \mu_{22}$ and μ_{11} is absolutely continuous with respect to Lebesgue measure.

Definition 1. *The coefficient of a matricial measure $\mu \in \mathcal{D}$ is*

$$\rho(\mu) = \sup \left| \int_{\mathbf{T}} f_1 \overline{f_2} d\mu_{12} \right|,$$

where the supremum is taken over all $f_1 \in \mathcal{P}_1$, $f_2 \in \mathcal{P}_2$ such that $\|f_1\|_{w_{11}} = \|f_2\|_{w_{11}} = 1$ and $d\mu_{11}(x) = d\mu_{22}(x) = w_{11}(x)dx$ ($\| \, . \, \|_{w_{11}}$ is the norm in $L^2(\mathbf{T}, w_{11}dx)$).

Observe that if $\mu_{11} = \mu_{12}$ then $\rho(\mu)$ is the cosine of the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$ in $L^2(\mathbf{T}, \mu_{11}dx)$.

Let $C(\mathbf{T})$ be the Banach space of complex continuous functions on $\mathbf{T}$ with the $\| \, . \, \|_\infty$ norm.

Let $\mu \in \mathcal{D}$ and consider the following form B_μ:

$$B_\mu(f_1, f_2) = \sum_{\alpha,\beta=1}^{2} \int_{\mathbf{T}} f_\alpha \overline{f_\beta} d\mu_{\alpha\beta},$$

for $f_1, \ f_2 \in C(\mathbf{T})$.

Let H^p be the set of functions $f \in L^p(\mathbf{T})$ such that $\widehat{f}(n) = 0$ for all $n < 0$ ($1 \le p \le \infty$). These are the usual Hardy spaces H^p.

An easy calculation establishes the following result:

Proposition 1. *Let μ and ν be two matricial measures on $\mathbf{T}$. Then $B_\mu(f_1, f_2) = B_\nu(f_1, f_2)$ for every $(f_1, f_2) \in \mathcal{P}_1 \times \mathcal{P}_2$ if and only if there is $h \in H^1$ such that $\nu_{11} = \mu_{11}$, $\nu_{22} = \mu_{22}$ and $\nu_{12} = \mu_{12} + hdx$.*

If $r \in (0, 1]$, $h \in H^1$ and μ is a matricial measure on $\mathbf{T}$, define $\mu(r, h)$ as the matricial measure given by $\mu(r, h)_{11} = r\mu_{11}$, $\mu(r, h)_{22} = r\mu_{22}$, $\mu(r, h)_{12} = \mu_{12} + hdx$ and $\mu(r, h)_{21} = \mu_{21} + \overline{h}dx$.

Proposition 2. *Let $r \in (0, 1]$ and $\mu \in \mathcal{D}$. Then:*
$B_{\mu(r,0)}(f_1, f_2) \ge 0$ for every $(f_1, f_2) \in \mathcal{P}_1 \times \mathcal{P}_2$ if and only if $\rho(\mu) \le r$.

The proof follows from a simple calculation.

Definition 2. *A matricial measure μ is said to be* positive *when for every Borel set Δ on $\mathbf{T}$ the numerical matrix $\mu(\Delta) = (\mu_{\alpha\beta}(\Delta))_{\alpha,\beta=1,2}$ is definite positive.*

Proposition 3. *Let μ be a matricial measure on $\mathbf{T}$. Then the following conditions are equivalent:*
(a) μ is a positive matricial measure.
(b) $B_\mu(f_1, f_2) \geq 0$ for every $f_1,\ f_2 \in C(\mathbf{T})$.
(c) $B_\mu(f_1, f_2) \geq 0$ for every $f_1,\ f_2 \in \mathcal{P}$.

Proof. We first show that (a) implies (b). Let μ be a positive matricial measure. Given $f_1,\ f_2 \in C(\mathbf{T})$, there are two sequences of simple functions $\{g_{1n}\}_n$ and $\{g_{1n}\}_n$ such that for $\alpha = 1, 2$:

$$f_\alpha(t) = lim_{n\to\infty} g_{\alpha n}(t).$$

For each n let $\{A_{ni}\}_i$ and $\{B_{nk}\}_k$ be the sets which occur in the canonical representations of $\{g_{1n}\}_n$ and $\{g_{2n}\}_n$. Let A_{n0} and B_{n0} be the sets where g_{1n} and g_{2n} are zero. Then the sets $\{\Delta_{nj}\}_j$ obtained by taking all the intersections $A_{ni} \cap B_{nk}$ form a finite disjoint collection of measurable sets, and for $\alpha = 1, 2$ we may write

$$g_{\alpha n} = \sum_{j=1}^{N_n} f_\alpha(x_{nj}) 1_{\Delta_{nj}}.$$

Thus,

$$\begin{aligned}
B_\mu(f_1, f_2) &= \sum_{\alpha,\beta=1}^{2} \int_{\mathbf{T}} f_\alpha \overline{f_\beta} d\mu_{\alpha\beta} \\
&= \sum_{\alpha,\beta=1}^{2} lim_{n\to\infty} \int_{\mathbf{T}} \sum_{j=1}^{N_n} f_\alpha(x_{n_j}) \overline{f_\beta(x_{n_j})} 1_{\Delta_{n_j}} d\mu_{\alpha\beta} \\
&= lim_{n\to\infty} \sum_{j=1}^{N_n} \sum_{\alpha,\beta=1}^{2} f_\alpha(x_{n_j}) \overline{f_\beta(x_{n_j})} \mu_{\alpha\beta}(\Delta_{n_j}) \geq 0.
\end{aligned}$$

That (b) implies (c) is immediate.

Suppose (c) holds. Given a Borel set $\Delta \subset \mathbf{T}$ and $\lambda_1,\ \lambda_2 \in \mathbf{C}$ let's consider the functions $h_1, h_2 \in L^1(\mathbf{T})$ given by $h_1 = \lambda_1 1_\Delta$, $h_2 = \lambda_2 1_\Delta$. We have that h_1 and h_2 can be approximated by continuous functions and these can be approximated by trigonometric polynomials. Therefore

$$\sum_{\alpha,\beta=1}^{2} \lambda_\alpha \overline{\lambda_\beta} \mu_{\alpha\beta}(\Delta) = B_\mu(h_1, h_2) \geq 0.$$

□

Theorem 1. *Let $r \in (0,1]$ and $\mu \in \mathcal{D}$. Then the following conditions are equivalent:*
(a) $\rho(\mu) \leq r$.
(b) μ_{12} *is absolutely continuous with respect to the Lebesgue measure and there exists* $h \in H^1$ *such that*

$$\mid w_{12}(x) + h(x) \mid \leq r w_{11}(x) \quad a.e.,$$

where $d\mu_{\alpha\beta}(x) = w_{\alpha\beta}(x)dx$, *for* $\alpha, \beta = 1, 2$.

Proof. If $\rho(\mu) \leq r$, then $B_{\mu(r,0)}(f_1, f_2) \geq 0$ for every $(f_1, f_2) \in \mathcal{P}_1 \times \mathcal{P}_2$. From the lifting property (see [3]), there exists $h \in H^1$ such that $B_{\mu(r,h)}(f_1, f_2) \geq 0$ for every $f_1, f_2 \in \mathcal{P}$. Then $\mu(r,h)$ is positive. Therefore

$$\mid \mu_{12}(\Delta) + \int_{\Delta} h(x)dx \mid \leq r\mu_{11}(\Delta).$$

If $\mid \Delta \mid = 0$ then $\mu_{11}(\Delta) = 0$, thus $\mu_{12}(\Delta) = 0$. So if we set $d\mu_{\alpha\beta}(x) = w_{\alpha\beta}(x)dx$ for $\alpha, \beta = 1, 2$ then

$$\mid w_{12}(x) + h(x) \mid \leq r w_{11}(x) \quad a.e.$$

□

3 The theorem of Nehari

The shift is the operator S in $L^2(\mathbf{T})$ given by $(Sf)(x) = e^{ix}f(x)$ for all $f \in L^2(\mathbf{T})$.

Let H^{2-} be the set of functions $f \in L^2(\mathbf{T})$ such that $\widehat{f}(n) = 0$ for all $n \geq 0$ and let P^- be the orthogonal projection in $L^2(\mathbf{T})$ with range H^{2-}.

Definition 3. *A linear operator* $\Gamma : H^2 \to H^{2-}$ *such that* $P^- S\Gamma = \Gamma S \mid_{H^2}$ *is called a* Hankel operator.

Proposition 4. *Let* $\Gamma : H^2 \to H^{2-}$ *be a bounded linear operator. Then the following conditions are equivalent:*

(a) Γ *is a Hankel operator.*

(b) There is a sequence $\{A_n\}_{n \in \mathbf{Z}}$ *such that* $\langle \Gamma e_k, e_{-j} \rangle = A_{k+j}$ *for every* $k \geq 0$, $j > 0$.

Proof. Suppose that (a) holds. Given $k \geq 0$, $j > 0$, consider $n = j - 1$. Then $n \geq 0$ and

$$\begin{aligned}\langle \Gamma e_k, e_{-j}\rangle &= \langle \Gamma e_k, e_{-n-1}\rangle = \langle \Gamma e_k, S^{-n} e_{-1}\rangle = \langle S^n \Gamma e_k, e_{-1}\rangle \\ &= \langle P^- S^n \Gamma e_k, e_{-1}\rangle = \langle \Gamma S^n e_k, e_{-1}\rangle = \langle \Gamma e_{k+n}, e_{-1}\rangle \\ &= \langle \Gamma e_{k+j-1}, e_{-1}\rangle.\end{aligned}$$

Define $A_n = \langle \Gamma e_{n-1}, e_{-1}\rangle$.

Suppose on the other hand that (b) holds. For $k \geq 0$ and $j > 0$ we have

$$\begin{aligned}\langle P^- S\Gamma e_k, e_{-j}\rangle &= \langle S\Gamma e_k, e_{-j}\rangle = \langle \Gamma e_k, S^{-1} e_{-j}\rangle = \langle \Gamma e_k, e_{-j-1}\rangle \\ &= A_{k+j+1} = \langle \Gamma e_{k+1}, e_{-j}\rangle = \langle \Gamma S e_k, e_{-j}\rangle.\end{aligned}$$

Thus $P^- S\Gamma = \Gamma S\,|_{H^2}$. □

If $\gamma \in L^2(\mathbf{T})$ let M_γ be the multiplication operator given by $M_\gamma f = \gamma f$ for all $f \in L^2(\mathbf{T})$.

Proposition 5. *Let $\Gamma : H^2 \to H^{2-}$ be a bounded Hankel operator. Then there exists $\gamma_0 \in L^2(\mathbf{T})$ such that $\Gamma = P^- M_{\gamma_0}$.*

Proof. Let $\gamma_0 = \Gamma e_0$. Then $\gamma_0 \in L^2(\mathbf{T})$ and

$$P^- M_{\gamma_0} e_n = P^- S^n \gamma_0 = P^- S^n \Gamma e_0 = \Gamma S^n e_0 = \Gamma e_n.$$

Therefore $\Gamma = P^- M_{\gamma_0}$. □

Proposition 6. *For $\gamma \in L^2(\mathbf{T})$ let $\Gamma : H^2 \to H^{2-}$ be defined by $\Gamma = P^- M_\gamma$ Then Γ is a Hankel operator. Even more $\langle \Gamma e_k, e_{-j}\rangle = \widehat{\gamma}(-k-j)$, for every $k \geq 0$ and $j > 0$.*

Proof. Let $k \geq 0$, $j > 0$

$$\begin{aligned}\langle P^- S\Gamma e_k, e_{-j}\rangle &= \langle S\Gamma e_k, e_{-j}\rangle = \langle \Gamma e_k, S^{-1} e_{-j}\rangle = \langle \Gamma e_k, e_{-j-1}\rangle \\ &= \langle P^- \gamma e_k, e_{-j-1}\rangle = \langle \gamma e_k, e_{-j-1}\rangle = \langle e_1 \gamma e_k, e_{-j}\rangle \\ &= \langle M_\gamma e_{k+1}, e_{-j}\rangle = \langle P^- M_\gamma e_{k+1}, e_{-j}\rangle = \langle \Gamma e_{k+1}, e_{-j}\rangle \\ &= \langle \Gamma S e_k, e_{-j}\rangle\end{aligned}$$

Thus $P^- S\Gamma = \Gamma S\,|_{H^2}$. And

$$\begin{aligned}\langle \Gamma e_k, e_{-j}\rangle &= \langle P^- M_\gamma e_k, e_{-j}\rangle = \langle M_\gamma e_k, e_{-j}\rangle = \langle S_k \gamma, e_{-j}\rangle \\ &= \langle \gamma, S_{-k} e_{-j}\rangle = \langle \gamma, e_{-k-j}\rangle = \widehat{\gamma}(-k-j).\end{aligned}$$

The proof is complete. □

Remark. For $\gamma \in L^\infty(\mathbf{T})$ we can consider $\Gamma : H^2 \to H^{2-}$ as the operator defined by $\Gamma = P^- M_\gamma$. Then

$$\|\Gamma\| = \|P^- M_\gamma\| \leq \|M_\gamma\| = \|\gamma\|_\infty.$$

Nehari's theorem (see [5], [1]) says that:

Theorem 2. *Let $\Gamma : H^2 \to H^{2-}$ be a Hankel operator. If Γ is bounded then there exists $\gamma \in L^\infty(\mathbf{T})$ such that*
(a) $\widehat{\gamma}(-k-j) = \langle \Gamma e_k, e_{-j}\rangle$ for every $k \geq 0,\ j > 0$
(b) $\Gamma = P^- M_\gamma$
(c) $\|\Gamma\| = inf\{\|\gamma - \xi\|_\infty : \xi \in H^\infty\} = \|\gamma\|_\infty$.

Proof. (a) If $\Gamma = 0$ the result is obvious. By homogeneity we may assume that $\|\Gamma\| = 1$.

Let $\gamma_0 = \Gamma e_0$ then $\Gamma = P^- M_{\gamma_0}$.

Consider the matricial measures $\mu_{11} = \mu_{22} = dx$ and $\mu_{12} = \overline{\mu_{21}} = -\gamma_0 dx$. Then $\rho(\mu) = \|\Gamma\|$.

From Theorem 1 it follows that there exists $h \in H^1$ such that

$$\mid \gamma_0(x) + h(x) \mid \leq 1 \quad a.e.$$

Define $\gamma = \gamma_0 + h$. Thus $\gamma \in L^\infty(\mathbf{T})$ and then $\widehat{\gamma}(-n) = \widehat{\gamma_0}(-n)$ for every $n \geq 0$. Therefore for every $k \geq 0$, $j > 0$:

$$\widehat{\gamma}(-k-j) = \widehat{\gamma_0}(-k-j) = \langle \Gamma e_k, e_{-j}\rangle.$$

It is clear that $\Gamma = P^- M_{\gamma_0} = P^- M_{\gamma_0 + h} = P^- M_\gamma$ and

$$inf\{\|\gamma - \xi\|_\infty : \xi \in H^\infty\} \leq \|\gamma\|_\infty = \|\gamma_0 + h\|_\infty \leq 1 = \|\Gamma\|.$$

On the other hand if $\xi \in H^\infty$ then

$$\|\Gamma\| = \|P^- M_\gamma\| = \|P^- M_{\gamma_\varepsilon - \xi}\| \leq \|M_{\gamma - \xi}\| = \|\gamma - \xi\|_\infty.$$

Therefore

$$\|\Gamma\| \leq inf\{\|\gamma - \xi\|_\infty : \xi \in H^\infty\}.$$

□

4 Generalized interpolation in H^∞

For every closed subspace K of H^2 let P_K be the orthogonal projection in $L^2(\mathbf{T})$ with range K.

Definition 4. *ψ is said to be an* inner function *if $\psi \in H^\infty$ and $|\psi| = 1$ a. e.*

Let $\ominus$ stand for orthogonal difference.

Proposition 7. *Let ψ be a non constant inner function and let K be the closed manifold of H^2 given by $K = H^2 \ominus \psi H^2$. Given $\phi \in H^\infty$ we define the operator $X = P_K M_\phi$. Then X commutes with $P_K S|_K$.*

The converse of the last proposition is given by the celebrated interpolation theorem of Sarason (see [6], [1], [2]), which says that:

Theorem 3. *Given a nonconstant inner function ψ let K be the closed subspace of H^2 given by $K = H^2 \ominus \psi H^2$. If X is a bounded linear operator on K such that X commutes with $P_K S|_K$ then there is a function $\phi \in H^\infty$ such that $X = P_K M_\phi$ and $\|X\| = \|\phi\|_\infty$.*

Proof. Let $f \in L^2(\mathbf{T})$. First we prove that $P_K f = \psi P^- \overline{\psi} f$. In order to do that, let $h \in K$. Since $K = H^2 \ominus \psi H^2 \subset (\psi H^2)^\perp = \psi H^{2-}$, we have that $h \in \psi H^{2-}$. Thus $\overline{\psi} h \in H^{2-}$. Using this we obtain that

$$\langle h, \psi P^- \overline{\psi} f\rangle = \langle \overline{\psi} h, P^- \overline{\psi} f\rangle = \langle \overline{\psi} h, \overline{\psi} f\rangle = \langle h, f\rangle.$$

So $P_K f = \psi P^- \overline{\psi} f$. Let $\Gamma : H^2 \to H^{2-}$ be the operator defined by $\Gamma = \overline{\psi} X P_K$. Then

$$\|\Gamma\| = \|X P_K\| = \|X\|.$$

Let $f \in H^2$; since $\overline{\psi} P_K f = P^- \overline{\psi} f$ and X commutes with $P_K S|_K$ it follows that

$$\Gamma S f = \overline{\psi} X P_K S f = \overline{\psi} P_K S X P_K f = P^- \overline{\psi} S X P_K f = P^- S \overline{\psi} X P_K f = P^- S \Gamma f.$$

Thus $P^- S \Gamma = \Gamma S|_{H^2}$. From Nehari's theorem it follows that there exists $\gamma \in L^\infty(\mathbf{T})$ such that:
(a) $\widehat{\gamma}(-k-j) = \langle \Gamma e_k, e_{-j}\rangle$ for every $k \geq 0, j > 0$
(b) $\Gamma = P^- M_\gamma$
(c) $\|\Gamma\| = inf\{\|\gamma - \xi\|_\infty : \xi \in H^\infty\} = \|\gamma\|_\infty$.
Since

$$P^- \gamma \psi = P^- M_\gamma \psi = \Gamma \psi = \overline{\psi} X P_K \psi = \overline{\psi} X \psi P^- \overline{\psi} \psi = \overline{\psi} X \psi P^- |\psi|^2 = 0$$

we have that $\gamma\psi \in H^\infty$. Set $\phi = \gamma\psi$. From $\|X\| = \|\Gamma\|$ and $\|\gamma\|_\infty = \|\gamma\psi\|_\infty = \|\phi\|_\infty$ it is clear that $\|X\| = \|\phi\|_\infty$. Let $g \in K$. Then

$$\begin{aligned} Xg &= XP_K g = \psi\Gamma g = \psi P^- M_\gamma g = \psi P^- \gamma g = \psi P^- \overline{\psi}\psi\gamma_\varepsilon g \\ &= P_K\psi\gamma g = P_K \phi g = P_K M_\phi g. \end{aligned}$$

$\square$

References

[1] Arocena, R. *Unitary extensions of isometries and interpolation problems: dilation and lifting theorems*, Publ. Matemáticas del Uruguay, **6**(1995), 138–158.

[2] Arocena, R., Cotlar, M. *Generalized Toeplitz kernels, Hankel forms and Sarason's commutation theorem*, Acta Científica Venezolana, **33**(1982), 89–98.

[3] Cotlar, M., Sadosky, C. *On the Helson-Szegö theorem and a related class of modified Toeplitz kernels*, Proc. Symp. Pure Math. AMS., **35-I**(1979), 383–407.

[4] Domínguez, M. *Lexicographic lifting and applications to prediction and interpolation problems.* Preprint.

[5] Nehari, Z. *On bounded bilinear forms*, Annals of Mathematics, **65-1**(1957), 153–162.

[6] Sarason, D. *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc., **127**(1967), 179–203.