Divulgaciones Matemáticas v. 5, No. 1/2 (1997, 39–41

Uniqueness of Initial Value Problems

Unicidad del Problema de Valor Inicial

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Abstract

The uniqueness of an initial value problem is established for second order linear differential equations.
Key words and phrases: differential equations, initial value problems.

Resumen

La unicidad de un problema de valor inicial es establecida para ecuaciones diferenciales lineales de segundo orden.
Palabras y frases clave: ecuaciones diferenciales, problemas de valor inicial.

1 Introduction

The existence and uniqueness of solutions for second order linear differential equations are almost always stated without proof in elementary differential equations textbooks following calculus (see [1, p. 136]). The purpose of this paper is to give an elementary proof of the uniqueness part, wich does not even require an understanding of the Fundamental Theorem of Calculus. It can be understood by high school students with a basic understanding of a one semester course in calculus.

It might also be pointed out that in proving the uniqueness theorem below, we do not require, as is customary, that the coefficients be continuous. *We only require that they be bounded on a closed interval, wich would, of course, follow if they are continuous.*

2 Uniqueness of Initial Value Problems

Theorem: *Consider the differential equation*

$$y''(x) + h_1(x)y'(x) + h_0(x)y(x) = g(x) \tag{1}$$

Let $h_0(x)$, $h_1(x)$ and $g(x)$ be defined on an open interval J. Assume that for any finite closed subinterval $[x_1, x_2]$ of J there exists a constant M such that

$$|h_0(x)| \leq M, \text{ and } |h_1(x)| \leq M \text{ for all } x \in [x_1, x_2]. \tag{2}$$

If $x_0 \in J$, and $u(x)$ and $v(x)$ are two solutions of (1) such that $u(x_0) = v(x_0)$, and $u'(x_0) = v'(x_0)$, then $u(x) \equiv v(x)$ for all $x \in J$.

By a solution of (1) we mean a function defined on J, which has two continuous derivatives, and satisfies the differential equation.

Proof: Let $w(x) = u(x) - v(x)$. Then by substituting into (1), we obtain

$$w''(x) + h_1(x)w'(x) + h_0(x)w(x) = 0. \tag{3}$$

Since $u(x_0) = v(x_0)$, and $u'(x_0) = v'(x_0)$, we have

$$w(x_0) = w'(x_0) = 0. \tag{4}$$

Now, choose $x_1, x_2 \in J$ such that $x_1 < x_0 < x_2$. Let M be as in (2), and let

$$z(x) = [w'(x)]^2 + [w(x)]^2.$$

Then, from (4) we have

$$z(x_0) = 0. \tag{5}$$

Also, for $x \in [x_2, x_2]$,

$$\begin{aligned} z'(x) &= 2w'(x)w''(x) + 2w(x)w'(x) \\ &= 2w'(x)[-h_1(x)w'(x) - h_0(x)w(x)] + 2w(x)w'(x) \\ &= -2h_1(x)[w'(x)]^2 - 2h_0(x)w'(x)w(x) + 2w(x)w'(x) \\ &= -2h_1(x)[w'(x)]^2 + 2w(x)w'(x)(1 - h_0(x)). \end{aligned} \tag{6}$$

Therefore,

$$\begin{aligned} |z'(x)| &\leq |-2h_1(x)|[w'(x)]^2 + |2w(x)w'(x)||1 - h_0(x)| \\ &\leq 2M[w'(x)]^2 + 2|w(x)||w'(x)|(1 + M) \\ &\leq 2M([w'(x)]^2 + [w(x)]^2) + 2|w(x)||w'(x)|(1 + M) \\ &\leq 2M([w'(x)]^2 + [w(x)]^2) + (1 + M)([w'(x)]^2 + [w(x)]^2) \\ &\quad (2|w||w'| \leq (w')^2 + (w)^2 \text{ since } (|w| - |w'|)^2 \geq 0) \\ &= (1 + 3M)([w'(x)]^2 + [w(x)]^2) = (1 + 3M)z(x). \end{aligned}$$

Therefore,

$$-(1+3M)z(x) \leq z'(x) \leq (1+3M)z(x)$$

Now, let $k = 1+3M$. We have shown that

$$z'(x) \leq kz(x), \qquad x \in [x_0, x_2] \tag{7}$$

$$z'(x) \geq -kz(x), \qquad x \in [x_1, x_0]. \tag{8}$$

From (7) we get $z'(x)e^{-kx} - ke^{-kx}z(x) \leq 0$ for $x_0 \leq x \leq x_2$. This implies that $\frac{d}{dz}(z(x)e^{-kx}) \leq 0$ for $x_0 \leq x \leq x_2$. Hence $z(x)e^{-kx}$ is non-increasing, wich, along with (5), implies that

$$z(x)e^{-kx} \leq z(x_0)e^{-kx_0} = 0.$$

This shows that $z(x) \leq 0$ for $x_0 \leq x \leq x_2$. But, $z(x) = [w'(x)]^2 + [w(x)]^2 \geq 0$. Therefore, $z(x) \equiv 0$ for $x_0 \leq x \leq x_2$.

Similarly, from (8) we have $z'(x) + kz(x) \geq 0$ for $x_1 \leq x \leq x_0$.

This implies that

$$\begin{aligned} 0 &\leq z'(x)e^{kx} + ke^{kx}z(x) \\ &= \frac{d}{dx}(z(x)e^{kx}), \qquad x_1 \leq x \leq x_0. \end{aligned}$$

Therefore, for $x_1 \leq x \leq x_0$, we have $z(x)e^{kx} \leq z(x_0)e^{kx_0} = 0$ wich implies $z(x) \leq 0$ for $x_1 \leq x \leq x_0$. But $z(x) = [w'(x)]^2 + [w(x)]^2 \geq 0$. Hence $z(x) \equiv 0$ for $x_1 \leq x \leq x_0$.

We have shown that $z(x) \equiv 0$ for $x_1 \leq x \leq x_2$, wich implies that, $w(x) = u(x) - v(x) \equiv 0$ for $x_1 \leq x \leq x_2$. Since x_1 and x_2 are arbitrary numbers in J, it follows that $u(x) \equiv v(x)$ for all x in J.

References

[1] Nagle, R., Saff E., *Fundamentals of Differential Equations*, Addison-Wesley, New York, 1993.