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Energy balance of a 2-D model for lubricated oil transportation in a pipe

Balance de energía de un modelo 2-D para transporte de petróleo lubricado en una tubería

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Abstract

We study the equations of motion of two immiscible fluids with comparable densities, but very different viscosities in a two-dimensional horizontal pipe. This is applied to the lubricated transportation of heavy crude oil. First, we write the problem in variational form and next we derive an energy balance for this model.
Key words and phrases:two-phase flow, free surface, surface tension, energy balance.

Resumen

En este trabajo se estudian las ecuaciones de movimiento de dos fluidos no miscibles con densidades comparables pero de viscosidades diferentes en una tubería horizontal. Esto se aplica al transporte lubricado de crudo pesado. Primero, se escribe el problema en forma variacional y después se deriva un balance de energía para este modelo.
Palabras y frases clave: flujo bifásico, superficie libre, tensión superficial, balance de energía.

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1 Introduction

This work is devoted to the equations of motion of the lubricated transportation of heavy crude oil in a horizontal pipeline. In petroleum industry, an efficient way for transporting heavy crude oil in pipelines is by injecting water under pressure along the inner wall of the pipeline. The water acts as a lubricant by coating the wall of the pipeline, thus preventing the oil from adhering to the pipe. This behavior is made possible by the facts that both fluids are immiscible and the oil is much more viscous than the water while both have comparable densities. For more details, the reader can refer to Joseph & Renardy [4].

The full problem is that of a three-dimensional flow in a cylindrical pipe of two immiscible fluids, water and oil, governed by the transient Navier-Stokes equations. On entering the pipe, the fluid with low viscosity (water) is adjacent to the pipe wall and it surrounds the fluid with high viscosity (heavy oil). It is assumed that the flow is sufficiently smooth so that this situation holds until a certain time T, and so that the interface between the two fluids, which is a free surface, can be suitably parametrized and is never adjacent to the pipe wall. The equation of the free surface is given by a transport equation and the transmission conditions on the interface are:

1) the continuity of the velocity;

2) the balance of the normal stress with the surface tension.

Since this is a difficult problem, we consider here the simplified situation of a horizontal pipeline in two dimensions. In this case, we can take advantage of symmetry and consider only one half of the domain, say the upper half, that we denote by Ω.

In this work, we propose to study the energy balance of this problem. Although the flow of two immiscible fluids has been addressed before, to our knowledge, this is the first time that inflow and outflow boundary conditions are considered. Usually, either the pipe has an infinite length, as in the work by Socolowsky [7], or the flow occurs in a closed vessel and the free surface is a smooth closed curve as in the work of Solonnikov [10], [8], [9]. We also refer to our previous work [2] in which we analyze a numerical scheme for solving one time step of a discrete analogue of (5).

This work is organized as follows. In Section 1, we state the fully nonlinear equations. In Section 2, we set the equations in a variational form. The equation for the energy balance is derived in Section 3.

We finish this introduction by recalling the notation that is used in the sequel. We shall use the standard Sobolev space (cf. Adams [1] or Nečas [6]):

$$H^1(\Omega) = \{v \in L^2(\Omega)\,;\, \nabla\, v \in L^2(\Omega)^2\}\,,$$

where $\nabla\, v$ is the gradient of v taken in the sense of distributions:

$$\nabla\, v = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2})^t\,,$$

i.e. in the dual space $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$, the space of indefinitely differentiable functions with compact support in Ω . The space $H^1(\Omega)$ is equipped with the seminorm

$$|v|_{H^1(\Omega)} = \left[\sum_{i=1}^{2}\int_\Omega |\frac{\partial v}{\partial x_i}|^2\, d\mathbf{x}\right]^{1/2}\,,$$

and is a Hilbert space for the norm

$$\|v\|_{H^1(\Omega)} = \left[\|v\|^2_{L^2(\Omega)} + |v|^2_{H^1(\Omega)}\right]^{1/2}\,.$$

The scalar product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. Finally, the definitions of these spaces are extended straightforwardly to vectors, with the same notation. The euclidean vector norm is denoted by $|\cdot|$.

2 The two-phase flow model

Let us consider the $2-D$ flow illustrated by Fig. 1 that depicts the upper half Ω of the domain of interest.

For each time $t \in [0,T]$, the domain Ω is decomposed into two moving subdomains $\Omega^1(t)$ and $\Omega^2(t)$, with boundary

$$\partial\Omega^i(t) = \Gamma^i_{\text{in}} \cup \Gamma^i_0 \cup \Gamma^i_{\text{out}}(t) \cup \Gamma(t), \quad i = 1,2, \tag{1}$$

where $\Gamma_{\text{in}} = \Gamma^1_{\text{in}} \cup \Gamma^2_{\text{in}}$ denotes the inlet boundary that is independent of time, $\Gamma_{\text{out}}(t) = \Gamma^1_{\text{out}}(t) \cup \Gamma^2_{\text{out}}(t)$ denotes the outlet boundary, Γ^2_0 is the upper pipeline boundary, Γ^1_0 is the artificial boundary in the middle of the pipeline, $\Omega^1(t)$ is the region occupied by the high-viscosity fluid (oil) and $\Omega^2(t)$ that occupied by the low-viscosity fluid (water). As stated in the introduction, it is assumed that the interface between the two fluids: $\Gamma(t) = \overline{\Omega^1}(t) \cap \overline{\Omega^2}(t)$, can be parametrized by a function $(x,t) \mapsto \Phi(x,t)$ such that the subdomains can be written

$$\Omega^1(t) = \{(x,y) \in \Omega\ ,\ \ 0 < x < L\ ,\ \ 0 < y < \Phi(x,t)\}, \tag{2}$$

$$\Omega^2(t) = \{(x,y) \in \Omega\ ,\ \ 0 < x < L\ ,\ \ \Phi(x,t) < y < D\}, \tag{3}$$

where D denotes the radius of the pipeline and L its length. Note that whereas $\Gamma_{\rm in}$ is the actual inlet boundary, $\Gamma_{\rm out}$ is an artificial outlet boundary, introduced to cut the domain of interest at a convenient location, in view of numerical computation.

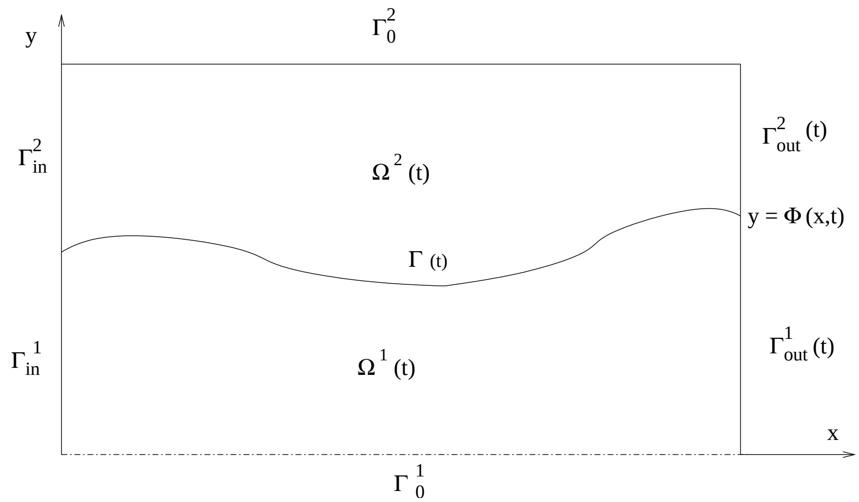

Figure 1: Positioning of the two fluids with water above oil.

To describe the density and viscosity, we introduce the piecewise constant quantities $\rho = \rho(t)$ and $\mu = \mu(t)$ defined by:

$$\rho = \chi^1\,\rho^1 + \chi^2\,\rho^2\ ,\ \ \mu = \chi^1\,\mu^1 + \chi^2\,\mu^2, \tag{4}$$

where $\chi^i = \chi^i(t)$ is the characteristic function of the domain $\Omega^i = \Omega^i(t)$, ρ^i are the constant densities and μ^i the constant viscosities, for $i = 1, 2$. To denote the velocity and pressure, we set:

$$\mathbf{u} = \mathbf{u}^i = (u_x^i, u_y^i)\ ,\ p = p^i \text{ in } \Omega^i\ ,\ \ i = 1, 2\,.$$

Then for almost every $t \in]0, T[$, the fluids must satisfy the following equations (to simplify, we suppress the dependence on t):

$$\begin{cases} \rho^i \left(\dfrac{\partial \mathbf{u}^i}{\partial t} + \mathbf{u}^i \cdot \nabla \mathbf{u}^i \right) - \mu^i \triangle \mathbf{u}^i + \nabla p^i & = & \rho^i \, \mathbf{g} \quad \text{in each } \Omega^i \;, \quad i = 1, 2 \\ \nabla \cdot \mathbf{u}^i & = & 0 \qquad \text{in } \Omega \,, \end{cases} \tag{5}$$

where $\mathbf{g}$ is the gravity and

$$\mathbf{u} \cdot \nabla \mathbf{u} = \sum_{i=1}^{2} u_i \frac{\partial \mathbf{u}}{\partial x_i} \,.$$

The equation for the motion of the free surface Γ, stating the immiscibility of the fluids, is

$$\frac{\partial \Phi}{\partial t} + u_x \frac{\partial \Phi}{\partial x} = u_y \,. \tag{6}$$

The equations (5) are complemented by an adequate initial condition, appropriate inflow and outflow conditions on the vertical boundaries of Ω, a no-slip boundary condition on the top horizontal boundary of Ω, and an artificial symmetry condition on the bottom horizontal boundary of Ω:

$$\begin{cases} \mathbf{u} & = & \mathbf{U} & \text{on } \Gamma_{\text{in}} \\ \mathbf{u}^2 & = & \mathbf{0} & \text{on } \Gamma_0^2 \\ \mathbf{u}^1 \cdot \mathbf{n} & = & 0 & \text{on } \Gamma_0^1 \\ \mathbf{t} \cdot \boldsymbol{\sigma}^1 \cdot \mathbf{n} & = & 0 & \text{on } \Gamma_0^1 \\ \boldsymbol{\sigma} \cdot \mathbf{n} & = & -p_{\text{out}} \, \mathbf{n} & \text{on } \Gamma_{\text{out}}, \end{cases} \tag{7}$$

and interface conditions (continuity of the velocity and balance of the normal stress with the surface tension, across the interface)

$$[\mathbf{u}]_\Gamma = \mathbf{0} \;, \; [\boldsymbol{\sigma}]_\Gamma \cdot \mathbf{n}^1 = -\frac{\kappa}{R} \mathbf{n}^1 \,, \tag{8}$$

where $\mathbf{U} = \mathbf{U}^i$ on Γ^i_{in} for $i = 1, 2$ denotes the given inlet velocity independent of time, p_{out} a given exterior pressure on the outlet boundary, $\mathbf{n}$ is the unit exterior normal vector to the boundary of Ω, $\mathbf{t}$ is the unit tangent vector to Γ_0^1, pointing in the direction of increasing x (i.e. in the counterclockwise direction), $\mathbf{n}^1$ is the unit normal to Γ, exterior to Ω^1, $[\cdot]_\Gamma$ denotes the jump on Γ in the direction of $\mathbf{n}^1$:

$$[f]_\Gamma = f|_{\Omega^1} - f|_{\Omega^2} \,,$$

$\kappa > 0$ is a given constant related to the surface tension, R is the radius of curvature with the appropriate sign, i.e. with the convention that $R > 0$ if the center of curvature of Γ is located in Ω^1, and the stress tensor $\boldsymbol{\sigma}$ satisfies the constitutive equation of a Newtonian fluid:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u}, p) = \mu\, \boldsymbol{A}_1(\mathbf{u}) - p\, \boldsymbol{I} = \mu\big(\nabla\, \mathbf{u} + (\nabla\, \mathbf{u})^t\big) - p\, \boldsymbol{I}\,.$$

We assume that the inlet velocity $\mathbf{U}$ has the form:

$$\mathbf{U} = -U(y)\mathbf{n} = (U(y), 0)^t\,,\ U(y) \geq 0\,, \tag{9}$$

i.e. the inlet velocity is parallel to the normal vector $\mathbf{n}$ and is directed *inside* Ω. Moreover, we assume that $U(D) = 0$; thus $\mathbf{U}$ satisfies the compatibility conditions:

$$\mathbf{U}^2(\Gamma_0^2 \cap \Gamma_{\text{in}}^2) = \mathbf{0}\ ,\ \mathbf{U}^1 \cdot \mathbf{t}^1(\Gamma_{\text{in}}^1 \cap \Gamma_0^1) = 0\,, \tag{10}$$

where $\mathbf{t}^1$ is the unit tangent vector to Γ_{in}^1 (i.e. in the direction of the normal to Γ_0^1).

Remark 2.1. It follows from the second and third boundary conditions in (7) and the fact that $\operatorname{div} \mathbf{u} = 0$ that necessarily,

$$\int_{\Gamma_{\text{out}}} \mathbf{u} \cdot \mathbf{n}\, dy = \int_{\Gamma_{\text{out}}} U(y)\, dy = \int_{\Gamma_{\text{in}}} U(y)\, dy\,. \tag{11}$$

Finally, (6) is complemented by the initial and boundary conditions,

$$\begin{aligned} \forall x \in [0, L]\,,\ \Phi(x, 0) = y_0\,,\\ \forall t \in [0, T]\,,\ \Phi(0, t) = y_0\,, \end{aligned} \tag{12}$$

where $y_0 \in]0, D[$ is a given constant. As a consequence, the inlet velocity $\mathbf{U}$ does not depend on time. Furthermore, since the oulet boundary Γ_{out} is in fact artificial, we shall need to introduce an additional condition there. This will appear when performing the energy balance and doing numerical computation. This situation is somewhat similar to that encountered when studying a meniscus.

3 Variational formulation

Let us put problem (5), (7), (8), (9) and (10) into an equivalent variational formulation. For this, we assume that the interface Γ is Lipschitz continuous. This is compatible with the fact that the interface is very smooth at initial time (in fact, its graph is a straight horizontal line); therefore we can reasonably assume that it remains a sufficiently smooth graph for some time T. Thus each subdomain Ω^i is also Lipschitz continuous. The given function U belongs to $H^1(0,D)$, the outlet pressure p_{out} belongs to $L^2(\Gamma_{\text{out}})$ and $\mathbf{g}$ being the force of gravity is very smooth. Then we assume that the solution $(\mathbf{u},p)$ is also sufficiently smooth during the above-mentioned time T.

First we consider the problem where the first equation in (7) is replaced by the homogeneous boundary condition with $\mathbf{U}=\mathbf{0}$:

$$\mathbf{u}=\mathbf{0} \quad \text{on } \Gamma_{\text{in}}\,.$$

Afterward, we shall introduce an adequate lifting of $\mathbf{U}$ in the variational formulation. In view of the boundary conditions, we choose the following space for the velocity:

$$X=\{\mathbf{v}\in H^1(\Omega)^2\,;\,\mathbf{v}|_{\Gamma_{\text{in}}}=\mathbf{0}\,,\,\mathbf{v}|_{\Gamma_0^2}=\mathbf{0}\,,\,\mathbf{v}\cdot\mathbf{n}|_{\Gamma_0^1}=0\}\,. \tag{13}$$

Both the transmission condition on the interface and the outflow condition involve the stress tensor; thus the pressure has no indeterminate constant and hence the space for the pressure is

$$M=L^2(\Omega)\,, \tag{14}$$

and as usual, we define the space of the velocities with zero divergence:

$$V=\{\mathbf{v}\in X\,;\,\nabla\cdot\mathbf{v}=0\}\,. \tag{15}$$

Now, for the variational formulation, since $\nabla\cdot\mathbf{v}=0$, we have the identity in each Ω^i:

$$\Delta\,\mathbf{u}=\nabla\cdot\boldsymbol{A}_1(\mathbf{u})\,.$$

Therefore, taking the scalar product of the first equation of (5) in $L^2(\Omega^i)^2$ with a test function $\mathbf{v}\in X$, applying Green's formula in each Ω^i (that is valid for a sufficiently smooth solution) and summing over i, we obtain:

$$\begin{aligned}\int_\Omega \rho\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{v}\,d\mathbf{x}+\sum_{i=1}^{2}\int_{\Omega^i}(\mu\,\boldsymbol{A}_1(\mathbf{u}^i)-p^i\,\boldsymbol{I}):\nabla\,\mathbf{v}^i\,d\mathbf{x}+\int_\Omega\rho(\mathbf{u}\cdot\nabla\,\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x}&\\ +\sum_{i=1}^{2}\int_{\partial\Omega^i}(-\mu\,\boldsymbol{A}_1(\mathbf{u}^i)\mathbf{n}^i+p^i\mathbf{n}^i)\cdot\mathbf{v}^i\,ds=\int_\Omega\rho\,\mathbf{g}\cdot\mathbf{v}\,d\mathbf{x}\,.&\end{aligned} \tag{16}$$

The symmetry of the operator $\boldsymbol{A}_1(\mathbf{u})$ gives $\boldsymbol{A}_1(\mathbf{u}) : \nabla\mathbf{v} = \boldsymbol{A}_1(\mathbf{u}) : (\nabla\mathbf{v})^t$ and therefore, as both $\mathbf{u}$ and $\mathbf{v}$ belong to $H^1(\Omega)^2$ we have

$$\sum_{i=1}^{2}\int_{\Omega^i}(\mu\,\boldsymbol{A}_1(\mathbf{u}^i) - p^i\,\boldsymbol{I}) : \nabla\,\mathbf{v}^i\,d\mathbf{x} = \frac{1}{2}\int_{\Omega}\mu\boldsymbol{A}_1(\mathbf{u})\,:\,\boldsymbol{A}_1(\mathbf{v})\,d\mathbf{x} - \int_{\Omega}p\,\nabla\cdot\mathbf{v}\,d\mathbf{x}\,.$$

As far as the boundary terms are concerned observe that $\mathbf{v} = \mathbf{0}$ on Γ_{in} and Γ_0^2 and

$$\mathbf{v} = (v_x, 0)^t = v_x\mathbf{t} \quad \text{on} \quad \Gamma_0^1\,.$$

Therefore the boundary term in (16) reduces to

$$\int_{\Gamma}(-\boldsymbol{\sigma}(\mathbf{u}^1,p^1)\mathbf{n}^1,\mathbf{v}^1)\,ds + \int_{\Gamma}(\boldsymbol{\sigma}(\mathbf{u}^2,p^2)\mathbf{n}^1,\mathbf{v}^2)\,ds + \int_{\Gamma_0^1}(-\boldsymbol{\sigma}(\mathbf{u},p)\mathbf{n},\mathbf{t})v_x\,ds$$
$$+ \int_{\Gamma_{\text{out}}}(-\boldsymbol{\sigma}(\mathbf{u},p)\mathbf{n},\mathbf{v})\,ds\,.$$

Substituting these equalities into (16) and using the second equation of (8) and the last line of (7), we obtain a variational formulation of the homogeneous problem: For almost every t in $]0,T[$, find $\mathbf{u}(t) \in X$ and $p(t) \in M$ solution of:

$$\left\{\begin{array}{l} \int_\Omega \rho\,\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{v}\,d\mathbf{x} + \frac{1}{2}\int_\Omega \mu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right) : \left(\nabla\mathbf{v} + (\nabla\mathbf{v})^t\right)\,d\mathbf{x} + \int_\Omega \rho(\mathbf{u}\cdot\nabla\,\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x} \\ +\kappa\int_\Gamma \mathbf{v}\cdot\frac{\mathbf{n}^1}{R}\,ds - \int_\Omega p\,\nabla\cdot\mathbf{v}\,d\mathbf{x} = \int_\Omega \rho\,\mathbf{g}\cdot\mathbf{v}\,d\mathbf{x} - \int_{\Gamma_{\text{out}}} p_{\text{out}}\mathbf{v}\cdot\mathbf{n}\,ds\;,\;\; \forall\mathbf{v}\in X \\ \int_\Omega q\,\nabla\cdot\mathbf{u}\,d\mathbf{x} = 0\;,\;\; \forall q \in M\,. \end{array}\right. \tag{17}$$

Now, to handle the non-homogeneous boundary condition on Γ_{in}, we must construct a lifting, say $\bar{\mathbf{U}}$, of the inlet velocity $\mathbf{U}$. Recall that owing to the geometry of Ω (see Fig.1), the inlet velocity has the form (9)

$$\mathbf{U} = (U(y), 0)^t\,,$$

where $U \in H^1(0,D)$ is a known function of y, that satisfies:

$$U(D) = 0\,.$$

Then $\bar{\mathbf{U}}$ is obtained by replicating these values for all (x,y) in Ω, i.e.

$$\forall(x,y)\in\Omega\,,\;\bar{\mathbf{U}}(x,y) = (U(y),0)^t\,, \tag{18}$$

which has clearly zero divergence, depends continuously on the function U, belongs to $H^1(\Omega)^2$ and satisfies the boundary conditions :

$$\bar{\mathbf{U}}|_{\Gamma_0^2} = \mathbf{0} \text{ and } \bar{\mathbf{U}} \cdot \mathbf{n}|_{\Gamma_0^1} = 0\,.$$

Moreover, as $\mathbf{U}$ does not depend on time, neither does $\bar{\mathbf{U}}$. Therefore, we propose the variational formulation for the non-homogeneous problem: For almost every t in $]0,T[$, find $\mathbf{u}(t) \in X + \bar{\mathbf{U}}$ and $p(t) \in M$ solution of:

$$\left\{\begin{array}{l} \int_\Omega \rho\, \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v}\, d\mathbf{x} + \frac{1}{2}\int_\Omega \mu \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right) : \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^t\right)\, d\mathbf{x} + \int_\Omega \rho(\mathbf{u}\cdot\nabla\,\mathbf{u})\cdot\mathbf{v}\, d\mathbf{x} \\ +\kappa \int_\Gamma \mathbf{v} \cdot \frac{\mathbf{n}^1}{R}\, ds - \int_\Omega p\,\nabla\cdot\mathbf{v}\, d\mathbf{x} = \int_\Omega \rho\,\mathbf{g}\cdot\mathbf{v}\, d\mathbf{x} - \int_{\Gamma_{\mathrm{out}}} p_{\mathrm{out}}\mathbf{v}\cdot\mathbf{n}\, ds\ ,\ \ \forall \mathbf{v} \in X \\ \int_\Omega q\,\nabla\cdot\mathbf{u}\, d\mathbf{x} = 0\ ,\ \ \forall q \in M\,. \end{array}\right. \tag{19}$$

Remark 3.1. Note that Remark 2.1 applies also to $\bar{\mathbf{U}}$, whatever the lifting chosen. Hence, since the function U is nonnegative, it follows from (11) that

$$\int_{\Gamma_{\mathrm{out}}} \bar{\mathbf{U}} \cdot \mathbf{n}\, dy = \int_{\Gamma_{\mathrm{out}}} U(y)\, dy = \int_{\Gamma_{\mathrm{in}}} U(y)\, dy > 0\,.$$

As a consequence, if p_{out} is a nonnegative constant, which is the case if it is the atmospheric pressure, then

$$\int_{\Gamma_{\mathrm{out}}} p_{\mathrm{out}}\bar{\mathbf{U}} \cdot \mathbf{n}\, dy = \int_{\Gamma_{\mathrm{in}}} p_{\mathrm{out}} U(y)\, dy > 0\,.$$

Although the variational formulation (19) is not used for the energy balance in the next section, the first steps for obtaining both variational formulation and energy balance are the same and it will be worth noting further on the points by which they differ.

4 Energy balance

In this section, we suppose that the solution has sufficient smoothness. Now, observe that at the entrance of the pipe, i.e. when $x = 0$, $\frac{\partial \Phi}{\partial t}$ vanishes since

$$\Phi(0,t) = y_0\,,$$

a fixed number that does not depend on time. In addition, according to (9), $u_y(0,y,t) = 0$. Therefore, at $x = 0$, equation (6) reduces to:

$$U(y_0)\frac{\partial \Phi}{\partial x}(0,t) = 0\,.$$

As $U(y_0) \neq 0$, this implies that:

For almost every $t \in]0,T[$, the interface is horizontal at the intersection with Γ_{in}:

$$\forall t \leq T\,,\ \frac{\partial \Phi}{\partial x}(0,t) = 0\,. \tag{20}$$

The energy balance we present here is based directly on (16). For almost every $t \in]0,T[$, let us choose $\mathbf{v} = \mathbf{u}(t)$ in (16). Then the only difference with (17) is that $\operatorname{div} \mathbf{v} = 0$ and that $\mathbf{v}$ does not vanish on Γ_{in}; therefore, (17) is replaced here by:

$$\begin{aligned}\int_\Omega \rho(t)\,\frac{\partial \mathbf{u}}{\partial t}(t)\cdot\mathbf{u}(t)\,d\mathbf{x} &+ \frac{1}{2}\int_\Omega \mu(t)\ |\boldsymbol{A}_1(\mathbf{u}(t))|^2\ d\mathbf{x} + \int_\Omega \rho(t)\,(\mathbf{u}(t)\cdot\nabla\,\mathbf{u}(t))\cdot\mathbf{u}(t)\,d\mathbf{x}\\ &+ \kappa\int_{\Gamma(t)} \mathbf{u}(t)\cdot\frac{\mathbf{n}^1}{R}(t)\,ds - \int_{\Gamma_{\text{in}}} (\boldsymbol{\sigma}(\mathbf{u}(t),p(t))\mathbf{n})\cdot\mathbf{u}(t)\,ds\\ &= \int_\Omega \rho(t)\,\mathbf{g}\cdot\mathbf{u}(t)\,d\mathbf{x} - \int_{\Gamma_{\text{out}}} p_{\text{out}}\mathbf{u}(t)\cdot\mathbf{n}\,ds\,.\end{aligned} \tag{21}$$

Note that if the solution $\mathbf{u}(t)$ vanishes on Γ_{in}, then (21) simplifies and follows immediately from (19).

Let us examine the terms in (21). First, in view of (9), the integral on Γ_{in} has the expression

$$\int_{\Gamma_{\text{in}}} (\boldsymbol{\sigma}(\mathbf{u}(t),p(t))\mathbf{n})\cdot\mathbf{u}(t)\,ds = \int_{\Gamma_{\text{in}}} \left(-2\,\mu\frac{\partial u_x}{\partial x} + p\right)(0,y,t)U(y)\,dy\,, \tag{22}$$

and in view of the direction of the normal vector to Γ_{out}, the integral on Γ_{out} has the expression:

$$\int_{\Gamma_{\text{out}}} p_{\text{out}}\mathbf{u}(t)\cdot\mathbf{n}\,ds = \int_{\Gamma_{\text{out}}} p_{\text{out}}(y,t)u_x(L,y,t)\,dy\,. \tag{23}$$

Next, the following proposition studies the time derivative.

Proposition 4.1. *If ρ, $\mathbf{u}$ and the function Φ are sufficiently smooth, we have*

$$\begin{aligned}\int_\Omega \rho(t)\,\frac{\partial \mathbf{u}(t)}{\partial t}\cdot\mathbf{u}(t)\,d\mathbf{x} &= \frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho(t)\,|\mathbf{u}(t)|^2\,d\mathbf{x}\right)\\ &\quad - \frac{1}{2}(\rho^1-\rho^2)\int_{\Gamma(t)} \mathbf{u}(s,t)\cdot\mathbf{n}^1(s,t)|\mathbf{u}(s,t)|^2\,ds\,.\end{aligned} \tag{24}$$

Proof. Splitting Ω into Ω^1 and Ω^2, we can write

$$\frac{d}{dt}\left(\int_{\Omega} \rho(t)\,|\mathbf{u}(t)|^2\,d\mathbf{x}\right) = \rho^1 \frac{d}{dt}\left(\int_{\Omega^1(t)} |\mathbf{u}^1(t)|^2\,d\mathbf{x}\right) + \rho^2 \frac{d}{dt}\left(\int_{\Omega^2(t)} |\mathbf{u}^2(t)|^2\,d\mathbf{x}\right).$$

But in view of (2),

$$\int_{\Omega^1(t)} |\mathbf{u}^1(t)|^2\,d\mathbf{x} = \int_0^L \int_0^{\Phi(x,t)} |\mathbf{u}^1(x,y,t)|^2\,dy\,dx\,.$$

Then by definition of the time derivative,

$$\begin{aligned} &\frac{d}{dt}\left(\int_{\Omega^1(t)} |\mathbf{u}^1(t)|^2\,d\mathbf{x}\right) \\ &= \int_0^L \lim_{h\to 0}\frac{1}{h}\left[\int_0^{\Phi(x,t+h)} |\mathbf{u}^1(x,y,t+h)|^2\,dy - \int_0^{\Phi(x,t)} |\mathbf{u}^1(x,y,t)|^2\,dy\right] dx \\ &= \int_0^L \lim_{h\to 0}\frac{1}{h}\int_0^{\Phi(x,t)} \left\{|\mathbf{u}^1(x,y,t+h)|^2 - |\mathbf{u}^1(x,y,t)|^2\right\}\,dy\,dx \\ &\quad + \int_0^L \lim_{h\to 0}\frac{1}{h}\int_{\Phi(x,t)}^{\Phi(x,t+h)} |\mathbf{u}^1(x,y,t+h)|^2\,dy\,dx\,. \end{aligned}$$

As expected, assuming sufficient smoothness, the first term in the right-hand side converges to

$$\int_0^L \int_0^{\Phi(x,t)} \frac{\partial}{\partial t}\left(|\mathbf{u}^1(x,y,t)|^2\right)\,dy\,dx\,.$$

For the second term, assuming again sufficient smoothness, we apply to Φ the mean-value theorem: there exists $\tau \in]t, t+h[$ such that

$$\Phi(x,t+h) = \Phi(x,t) + h\frac{\partial \Phi}{\partial t}(x,\tau)\,,$$

and the first law of the mean for integrals: there exists $\zeta \in]\Phi(x,t), \Phi(x,t+h)[$ such that

$$\int_{\Phi(x,t)}^{\Phi(x,t+h)} |\mathbf{u}^1(x,y,t+h)|^2\,dy = h\frac{\partial \Phi}{\partial t}(x,\tau)|\mathbf{u}^1(x,\zeta,t+h)|^2\,.$$

Therefore, considering that $\mathbf{u}$ belongs to $H^1(\Omega)^2$, the second term converges to

$$\int_0^L \frac{\partial \Phi}{\partial t}(x,t)|\mathbf{u}(x,\Phi(x,t),t)|^2\,dx\,.$$

Hence

$$\begin{aligned}\frac{d}{dt}\left(\int_{\Omega^1(t)} |\mathbf{u}^1(t)|^2\,d\mathbf{x}\right) &= \int_0^L \int_0^{\Phi(x,t)} \frac{\partial}{\partial t}\left(|\mathbf{u}^1(x,y,t)|^2\right)\,dy\,dx \\ &+ \int_0^L \frac{\partial \Phi}{\partial t}(x,t)|\mathbf{u}(x,\Phi(x,t),t)|^2\,dx\,,\end{aligned} \tag{25}$$

with a similar formula in $\Omega^2(t)$:

$$\begin{aligned}\frac{d}{dt}\left(\int_{\Omega^2(t)} |\mathbf{u}^2(t)|^2\,d\mathbf{x}\right) &= \int_0^L \int_{\Phi(x,t)}^{D} \frac{\partial}{\partial t}\left(|\mathbf{u}^2(x,y,t)|^2\right)\,dy\,dx \\ &- \int_0^L \frac{\partial \Phi}{\partial t}(x,t)|\mathbf{u}(x,\Phi(x,t),t)|^2\,dx\,.\end{aligned} \tag{26}$$

Now, let us apply (6):

$$\frac{\partial \Phi}{\partial t} = u_y - u_x \frac{\partial \Phi}{\partial x}\,.$$

Considering that the unit normal vector to $\Gamma(t)$, exterior to $\Omega^1(t)$, is

$$\mathbf{n}^1(x,t) = \frac{1}{\sqrt{1+\left(\frac{\partial \Phi}{\partial x}(x,t)\right)^2}}(-\frac{\partial \Phi}{\partial x}(x,t),1)^t\,, \tag{27}$$

we have

$$\frac{\partial \Phi}{\partial t}(x,t) = \sqrt{1+\left(\frac{\partial \Phi}{\partial x}(x,t)\right)^2}\,\mathbf{u}(x,\Phi(x,t),t)\cdot \mathbf{n}^1(x,t)\,. \tag{28}$$

Substituting into (25), this yields:

$$\begin{aligned}\frac{d}{dt}\left(\int_{\Omega^1(t)} |\mathbf{u}^1(\mathbf{x},t)|^2\,d\mathbf{x}\right) &= \int_{\Omega^1(t)} \frac{\partial}{\partial t}\left(|\mathbf{u}^1(\mathbf{x},t)|^2\right)\,d\mathbf{x} \\ &+ \int_{\Gamma(t)} \mathbf{u}(s,t)\cdot \mathbf{n}^1(s,t)|\mathbf{u}(s,t)|^2\,ds\,,\end{aligned}$$

with a similar equation when substituting into (26). Then (4) gives

$$\begin{aligned} \frac{d}{dt}\left(\int_\Omega \rho(t)\,|\mathbf{u}(t)|^2\,d\mathbf{x}\right) &= \int_\Omega \rho(t)\,\frac{\partial}{\partial t}\left(|\mathbf{u}(t)|^2\right)\,d\mathbf{x} \\ &\quad + (\rho^1-\rho^2)\int_{\Gamma(t)} \mathbf{u}(s,t)\cdot\mathbf{n}^1(s,t)|\mathbf{u}(s,t)|^2\,ds\,, \end{aligned} \tag{29}$$

and (24) follows from (29). □

The next proposition studies the convection term.

Proposition 4.2. *If ρ, $\mathbf{u}$ and the function Φ are sufficiently smooth, we have*

$$\begin{aligned} \int_\Omega \rho(t)\,(\mathbf{u}(t)\cdot\nabla\,\mathbf{u}(t))\cdot\mathbf{u}(t)\,d\mathbf{x} &= \frac{1}{2}(\rho^1-\rho^2)\int_{\Gamma(t)} \mathbf{u}(s,t)\cdot\mathbf{n}^1(s,t)|\mathbf{u}(s,t)|^2\,ds \\ -\frac{1}{2}\int_{\Gamma_{\mathrm{in}}} \rho(0,y,t)\,U(y)^3\,dy &+ \frac{1}{2}\int_{\Gamma_{\mathrm{out}}} \rho(L,y,t)\,u_x(L,y,t)|\mathbf{u}(L,y,t)|^2\,dy\,. \end{aligned} \tag{30}$$

Proof. As usual, we write:

$$\int_\Omega \rho(t)\,(\mathbf{u}(t)\cdot\nabla\,\mathbf{u}(t))\cdot\mathbf{u}(t)\,d\mathbf{x} = \int_\Omega \rho(t)\,\mathbf{u}(t)\cdot\nabla\left(\frac{1}{2}|\mathbf{u}(t)|^2\right)\,d\mathbf{x}\,,$$

and we split the integral in the right-hand side as in the previous proof:

$$\begin{aligned} \int_\Omega \rho(t)\,\mathbf{u}(t)\cdot\nabla\left(|\mathbf{u}(t)|^2\right)\,d\mathbf{x} &= \rho^1\int_{\Omega^1(t)} \mathbf{u}(t)\cdot\nabla\left(|\mathbf{u}(t)|^2\right)\,d\mathbf{x} \\ &\quad + \rho^2\int_{\Omega^2(t)} \mathbf{u}(t)\cdot\nabla\left(|\mathbf{u}(t)|^2\right)\,d\mathbf{x}\,. \end{aligned}$$

Then we apply Green's formula, use the incompressibility condition, the continuity of $\mathbf{u}$ across the interface Γ and the boundary conditions. This yields:

$$\begin{aligned} \int_\Omega \rho(t)\,\mathbf{u}(t)\cdot\nabla\left(\frac{1}{2}|\mathbf{u}(t)|^2\right)\,d\mathbf{x} &= \frac{1}{2}(\rho^1-\rho^2)\int_{\Gamma(t)} \mathbf{u}(s,t)\cdot\mathbf{n}^1(s,t)|\mathbf{u}(s,t)|^2\,ds \\ &\quad + \frac{1}{2}\int_{\Gamma_{\mathrm{in}}} \rho(0,y,t)\,\mathbf{U}(y)\cdot\mathbf{n}|U(y)|^2\,dy \\ &\quad + \frac{1}{2}\int_{\Gamma_{\mathrm{out}}} \rho(L,y,t)\mathbf{u}(L,y,t)\cdot\mathbf{n}|\mathbf{u}(L,y,t)|^2\,dy\,, \end{aligned}$$

and (30) follows immediately from this equation and (9). □

It remains to express suitably the term involving the surface tension. We shall prove that it is directly related to the time derivative of the measure of the interface Γ, a behavior similar to that derived by Murat and Simon in [5] for a closed interface. The next proposition gives an expression for this time derivative.

Proposition 4.3. *If the function Φ is sufficiently smooth, we have:*

$$\begin{aligned}\frac{d}{dt}\left(|\Gamma(t)|\right) = &-\int_0^L (\mathbf{u}\cdot\mathbf{n}^1)(x,\Phi(x,t),t)\frac{\partial^2\Phi}{\partial x^2}(x,t)\frac{1}{1+(\frac{\partial\Phi}{\partial x}(x,t))^2}dx\\ &+\frac{\partial\Phi}{\partial x}(L,t)\left(\mathbf{u}\cdot\mathbf{n}^1\right)\left(L,\Phi(L,t),t\right).\end{aligned}\tag{31}$$

Proof. Let us prove that

$$\begin{aligned}\frac{d}{dt}\left(|\Gamma(t)|\right) = &-\int_0^L (\mathbf{u}\cdot\mathbf{n}^1)(x,\Phi(x,t),t)\frac{\partial^2\Phi}{\partial x^2}(x,t)\frac{1}{1+(\frac{\partial\Phi}{\partial x}(x,t))^2}dx\\ &+\left[\frac{\partial\Phi}{\partial x}(x,t)(\mathbf{u}\cdot\mathbf{n}^1)(x,\Phi(x,t),t)\right]_0^L;\end{aligned}\tag{32}$$

in view of (20), $\frac{\partial\Phi}{\partial x}(0,t)=0$ and this yields (31). Considering the expression (27) for the normal vector $\mathbf{n}^1$, we have:

$$|\Gamma(t)| = \int_0^L\sqrt{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}\,dx\,,$$

where $|\Gamma|$ denotes the measure of Γ. Therefore

$$\frac{d}{dt}\left(|\Gamma(t)|\right) = \int_0^L\frac{1}{\sqrt{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}}\left(\frac{\partial\Phi}{\partial x}(x,t)\right)\left(\frac{\partial^2\Phi}{\partial t\partial x}(x,t)\right)dx\,.\tag{33}$$

Now, it follows from (6) and (27) that

$$\frac{\partial^2\Phi}{\partial t\partial x}=\frac{\partial^2\Phi}{\partial x\partial t}=\frac{\partial}{\partial x}(u_y-u_x\frac{\partial\Phi}{\partial x})=\frac{\partial}{\partial x}\left[(\mathbf{u}\cdot\mathbf{n}^1)\sqrt{1+\left(\frac{\partial\Phi}{\partial x}\right)^2}\right].$$

Hence, substituting into (33) and integrating by parts, we obtain

$$\frac{d}{dt}\left(|\Gamma(t)|\right) = -\int_0^L (\mathbf{u}\cdot\mathbf{n}^1)\sqrt{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}\,\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}}\frac{\partial\Phi}{\partial x}(x,t)\right]dx$$
$$+\left[(\mathbf{u}\cdot\mathbf{n}^1)(x,\Phi(x,t),t)\frac{\partial\Phi}{\partial x}(x,t)\right]_0^L . \tag{34}$$

A straightforward computation gives

$$\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}}\frac{\partial\Phi}{\partial x}(x,t)\right] = \left(\frac{\partial^2\Phi}{\partial x^2}(x,t)\right)\frac{1}{(1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2)^{3/2}} .$$

Therefore, substituting into (34), we readily derive (32). □

In order to compare (31) with the surface tension, we use the fact that, with the convention of sign used for R, we have:

$$\frac{\mathbf{n}^1}{R} = -\frac{\mathbf{n}}{\bar{R}} \text{ and } \frac{\mathbf{n}}{\bar{R}} = \frac{d\mathbf{t}}{ds}, \tag{35}$$

where $\mathbf{t}$ is the tangent to Γ in the direction of increasing s, that is the same as that of increasing x, $\mathbf{n}$ is the principal normal to Γ, i.e. parallel to $\mathbf{n}^1$ and directed toward the center of curvature of Γ, and $\bar{R}$ is the positive radius of curvature, i.e. $\bar{R} = R$ if the center of curvature is located inside Ω^1 and $\bar{R} = -R$ otherwise. Then we have the following result.

Proposition 4.4. *If the function Φ is sufficiently smooth, we have:*

$$\int_{\Gamma(t)} \mathbf{u}(s,t)\cdot\frac{\mathbf{n}^1}{R}(s,t)\,ds = -\int_0^L (\mathbf{u}\cdot\mathbf{n}^1)(x,t)\frac{1}{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}\frac{\partial^2\Phi}{\partial x^2}(x,t)\,dx . \tag{36}$$

Proof. Considering that

$$\frac{dx}{ds} = \frac{1}{\sqrt{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}},$$

we can write

$$\frac{d\mathbf{t}}{ds} = \frac{1}{\sqrt{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}}\left(\frac{d\mathbf{t}}{dx}\right),$$

and (35) implies

$$\int_{\Gamma(t)} \mathbf{u}(s,t)\cdot\frac{\mathbf{n}^1}{R}(s,t)\,ds = -\int_0^L \mathbf{u}(x,\Phi(x,t),t)\cdot\left(\frac{d\mathbf{t}}{dx}(x,t)\right)\,dx\,. \tag{37}$$

It remains to find the expression of $d\mathbf{t}/dx$. In view of (27), $\mathbf{t}$ is given by

$$\mathbf{t}(x,t) = \frac{1}{\sqrt{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}}(1,\frac{\partial\Phi}{\partial x}(x,t))^t\,.$$

A straightforward derivation gives

$$\frac{d\mathbf{t}}{dx}(x,t) = \left[\frac{1}{1+\left(\frac{\partial\Phi}{\partial x}(x,t)\right)^2}\frac{\partial^2\Phi}{\partial x^2}(x,t)\right]\mathbf{n}^1\,, \tag{38}$$

whence (36). □

These two propositions imply immediately the following theorem.

Theorem 4.5. *If the function Φ is sufficiently smooth, we have:*

$$\kappa\int_{\Gamma(t)} \mathbf{u}(s,t)\cdot\frac{\mathbf{n}^1}{R}(s,t)\,ds = \kappa\frac{d}{dt}\left(|\Gamma(t)|\right) - \kappa\frac{\partial\Phi}{\partial x}(L,t)\left(\mathbf{u}\cdot\mathbf{n}^1\right)(L,\Phi(L,t),t)\,. \tag{39}$$

Finally, substituting (24), (30), (39), (22) and (23) into (21), we derive our equation of energy balance.

Theorem 4.6. *If ρ, $\mathbf{u}$ and the function Φ are sufficiently smooth, we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho(t)\,|\mathbf{u}(t)|^2\,d\mathbf{x}\right) + \frac{1}{2}\int_\Omega \mu(t)\,\left|\boldsymbol{A}_1(\mathbf{u}(t))\right|^2\,d\mathbf{x} + \kappa\frac{d}{dt}\left(|\Gamma(t)|\right)\\
&\quad= -\int_{\Gamma_{\text{out}}} p_{\text{out}}(y,t)u_x(L,y,t)\,dy + \int_{\Gamma_{\text{in}}}\left(-2\,\mu\frac{\partial u_x}{\partial x_1}+p\right)(0,y,t)U(y)\,dy\\
&\quad-\frac{1}{2}\int_{\Gamma_{\text{out}}} \rho(L,y,t)\,u_x(L,y,t)|\mathbf{u}(L,y,t)|^2\,dy + \frac{1}{2}\int_{\Gamma_{\text{in}}}\rho(0,y,t)U(y)^3\,dy\\
&\quad+\int_\Omega \rho(t)\,\mathbf{g}\cdot\mathbf{u}(t)\,d\mathbf{x} + \kappa\frac{\partial\Phi}{\partial x}(L,t)\left(\mathbf{u}\cdot\mathbf{n}^1\right)(L,\Phi(L,t),t)\,.
\end{aligned} \tag{40}$$

The relation (40) expresses the transfers between the different forms of energy of the system and the outside world. In the left-hand side, the first term involves the time derivative of the kinetic energy, the third one involves the time derivative of the superficial energy and the second one is the power of the viscous forces. Note that this last term is non-negative, as well as the two energies: kinetic and superficial. In particular, this is important in deriving a stability estimate for this system.

In the right-hand side, the terms in the first line correspond both to the powers of the stress tensor on the inlet and outlet boundaries. The terms in the second line are fluxes of the kinetic energy. The last term in the second line, that involves $\mathbf{g}$, stands for the power of gravitational forces.

Now, let us assume that for almost every $t \in]0,T[$, the horizontal component of the velocity u_x remains non-negative on the outlet boundary Γ_{out}:

$$\forall y \in]0,D[\,,\ \forall t \leq T\,,\ u_x(L,y,t) \geq 0\,. \tag{41}$$

Since this is the case on entering the pipe, it is reasonable to assume that this situation prevails for a certain time T and for a certain distance L.

Then, with this assumption, the first term in the second line of the right-hand side in non-positive, thus expressing the fact that kinetic energy is lost at the outlet boundary, whereas kinetic energy is injected at the entrance of the pipe (whence a positive term).

The term in the last line is problematic. It expresses the fact that some superficial energy is transferred (gained or lost) to the outside world at the point where the pipe is cut. It is possible to control this term (thus stabilizing the system) by prescribing a zero vertical velocity at the outlet boundary, i.e.:

For almost every $t \in]0,T[$, the vertical component of the velocity u_y vanishes on Γ_{out}:

$$\forall y \in [0,D]\,,\ \forall t \leq T\,,\ u_y(L,y,t) = 0\,. \tag{42}$$

This condition is also satisfied on entering the pipe, but it is not necessarily satisfied at all points inside Ω. It has the following consequence:

$$\frac{\partial \Phi}{\partial x}(L,t)\left(\mathbf{u}\cdot\mathbf{n}^1\right)(L,\Phi(L,t),t) = -\left[u_x \frac{(\frac{\partial \Phi}{\partial x})^2}{\sqrt{1+(\frac{\partial \Phi}{\partial x})^2}}\right](L,\Phi(L,t),t)\,,$$

that is a non-positive term, if (41) holds. Of course, if we prescribe (42), then we must relax the last condition in (7) and replace it by

$$\forall y \in [0,D]\,,\ \forall t \leq T\,,\ \mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}(L,y,t) = p_{\text{out}}(y,t)\,. \tag{43}$$

With (42) and (43), Theorem 4.6 has the following corollary.

Corollary 4.7. *If ρ, $\mathbf{u}$ and the function Φ are sufficiently smooth, if (42) is prescribed and the last condition in (7) is replaced by (43), we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho(t)\,|\mathbf{u}(t)|^2\,d\mathbf{x}\right) + \frac{1}{2}\int_\Omega \mu(t)\,\left|\boldsymbol{A}_1(\mathbf{u}(t))\right|^2\,d\mathbf{x} + \kappa\frac{d}{dt}\left(|\Gamma(t)|\right)\\
&\quad = -\int_{\Gamma_{\text{out}}} p_{\text{out}}(y,t)u_x(L,y,t)\,dy + \int_{\Gamma_{\text{in}}}\left(-2\,\mu\frac{\partial u_x}{\partial x_1} + p\right)(0,y,t)U(y)\,dy\\
&\quad - \frac{1}{2}\int_{\Gamma_{\text{out}}} \rho(L,y,t)\,|u_x(L,y,t)|^3\,dy + \frac{1}{2}\int_{\Gamma_{\text{in}}} \rho(0,y,t)U(y)^3\,dy\\
&\quad + \int_\Omega \rho(t)\,\mathbf{g}\cdot\mathbf{u}(t)\,d\mathbf{x} - \kappa\left[u_x\frac{(\frac{\partial\Phi}{\partial x})^2}{\sqrt{1+(\frac{\partial\Phi}{\partial x})^2}}\right](L,\Phi(L,t),t)\,.
\end{aligned} \tag{44}$$

If (41) holds, the last term in the right-hand side is non-positive, expressing the fact that energy is lost at the point where the interface intersects the outflow boundary.

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