

## Weyl's theorems and Kato spectrum

*Los teoremas de Weyl y el espectro de Kato*

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### Abstract

In this paper we study Weyl's theorem,  $a$ -Weyl's theorem, and property  $(w)$  for bounded linear operators on Banach spaces. These theorems are treated in the framework of local spectral theory and in particular we shall relate these theorems to the single-valued extension property at a point. Weyl's theorem is also described by means some special parts of the spectrum originating from Kato theory.

**Key words and phrases:** Local spectral theory, Fredholm theory, Weyl's theorem,  $a$ -Weyl's theorem, property  $(w)$ .

### Resumen

En este artículo estudiamos el teorema de Weyl, el teorema  $a$ -Weyl y la propiedad  $(w)$  para operadores lineales acotados sobre espacios de Banach. Estos teoremas se tratan en el contexto de la teoría espectral local y en particular relacionamos estos teoremas con la propiedad de extensión univaluada en un punto. El teorema de Weyl se describe también mediante algunas partes especiales el espectro derivados de la teoría de Kato.

**Palabras y frases clave:** Teoría espectral local, Teoría de Fredholm, Teorema de Weyl, Teorema  $a$ -Weyl, propiedad  $(w)$ .

## 1 Introduction and definitions

In 1909 H. Weyl [39] studied the spectra of all compact perturbations  $T+K$  of a hermitian operator  $T$  acting on a Hilbert space and showed that  $\lambda \in \mathbb{C}$  belongs to the spectrum  $\sigma(T+K)$  for every compact operator  $K$  precisely when  $\lambda$  is not an isolated point of finite multiplicity in  $\sigma(T)$ . Today this classical result may be stated by saying that the spectral points of a hermitian operator  $T$  which do not belong to the Weyl spectrum are precisely the eigenvalues having finite multiplicity which are isolated point of the spectrum. More recently Weyl's theorem, and some of its variant,  $a$ -Weyl's theorem and property  $(w)$ , has been extended to several classes of operators acting in Banach spaces by several authors. In this expository article Weyl's theorem,  $a$ -Weyl's theorem will be related to an important property which has a leading role on local spectral theory: the single-valued extension theory. Other characterizations of Weyl's theorem and  $a$ -Weyl's theorem are given by using special parts of the spectrum defined in the context of Kato decomposition theory. In the last part we shall characterize property  $(w)$ , recently studied in [12].

We begin with some standard notations on Fredholm theory. Throughout this note by  $L(X)$  we will denote the algebra of all bounded linear operators acting on an infinite dimensional complex Banach space  $X$ . For every  $T \in L(X)$  we shall denote by  $\alpha(T)$  and  $\beta(T)$  the dimension of the kernel  $\ker T$  and the codimension of the range  $T(X)$ , respectively. Let

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$$

denote the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$$

denote the class of all *lower semi-Fredholm* operators. The class of all semi-Fredholm operators is defined by  $\Phi_{\pm}(X) := \Phi_+(X) \cup \Phi_-(X)$ , while the class of all Fredholm operators is defined by  $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ . The *index* of a semi-Fredholm operator is defined by  $\text{ind } T := \alpha(T) - \beta(T)$ . Recall that the *ascent*  $p := p(T)$  of a linear operator  $T$  is the smallest non-negative integer  $p$  such that  $\ker T^p = \ker T^{p+1}$ . If such integer does not exist we put  $p(T) = \infty$ . Analogously, the *descent*  $q := q(T)$  of an operator  $T$  is the smallest non-negative integer  $q$  such that  $T^q(X) = T^{q+1}(X)$ , and if such integer does not exist we put  $q(T) = \infty$ . It is well-known that if  $p(T)$  and  $q(T)$  are both finite then  $p(T) = q(T)$ , see [30, Proposition 38.3]. Other important classes of operators in Fredholm theory are the class of all *upper semi-Browder operators*

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and the class of all *lower semi-Browder operators*

$$\mathcal{B}_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The two classes  $\mathcal{B}_+(X)$  and  $\mathcal{B}_-(X)$  have been introduced in [28] and studied by several other authors, for instance [37]. The class of all *Browder operators* (known in the literature also as *Riesz-Schauder operators*) is defined by  $\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$ . Recall that a bounded operator  $T \in L(X)$  is said to be a *Weyl operator* if  $T \in \Phi(X)$  and  $\text{ind } T = 0$ . Clearly, if  $T$  is Browder then  $T$  is Weyl, since the finiteness of  $p(T)$  and  $q(T)$  implies, for a Fredholm operator, that  $T$  has index 0, see Heuser [30, Proposition 38.5].

The classes of operators defined above motivate the definition of several spectra. The *upper semi-Browder spectrum* of  $T \in L(X)$  is defined by

$$\sigma_{\text{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(X)\},$$

the *lower semi-Browder spectrum* of  $T \in L(X)$  is defined by

$$\sigma_{\text{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(X)\},$$

while the *Browder spectrum* of  $T \in L(X)$  by

$$\sigma_{\text{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}(X)\}.$$

The *Weyl spectrum* of  $T \in L(X)$  is defined by

$$\sigma_{\text{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}.$$

We have that  $\sigma_{\text{w}}(T) = \sigma_{\text{w}}(T^*)$ , while

$$\sigma_{\text{ub}}(T) = \sigma_{\text{lb}}(T^*), \quad \sigma_{\text{lb}}(T) = \sigma_{\text{ub}}(T^*).$$

Evidently,

$$\sigma_{\text{w}}(T) \subseteq \sigma_{\text{b}}(T) = \sigma_{\text{w}}(T) \cup \text{acc } \sigma(T),$$

where we write  $\text{acc } K$  for the accumulation points of  $K \subseteq \mathbb{C}$ , see [1, Chapter 3].

For a bounded operator  $T \in L(X)$  let us denote by

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\text{b}}(T) = \{\lambda \in \sigma(T) : \lambda I - T \text{ is Browder}\}.$$

and, if we write  $\text{iso } K$  for the set of all isolated points of  $K \subseteq \mathbb{C}$ , then

$$\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}$$

will denote the set of isolated eigenvalues of finite multiplicities. Obviously,

$$p_{00}(T) \subseteq \pi_{00}(T) \quad \text{for every } T \in L(X). \quad (1)$$

Following Coburn [17], we say that *Weyl's theorem holds* for  $T \in L(X)$  if

$$\Delta(T) := \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T), \quad (2)$$

while we say that  $T$  satisfies *Browder's theorem* if

$$\sigma(T) \setminus \sigma_w(T) = p_{00}(T),$$

or equivalently,  $\sigma_w(T) = \sigma_b(T)$ . Note that

$$\text{Weyl's theorem} \Rightarrow \text{Browder's theorem},$$

see, for instance [1, p. 166].

The classical result of Weyl shows that for a normal operator  $T$  on a Hilbert space then the equality (2) holds. Weyl's theorem has, successively, extended to several classes of operators, see [3] and the classes of operators (a)-(i) mentioned after Theorem 2.2.

The single valued extension property dates back to the early days of local spectral theory and was introduced by Dunford [23], [24]. This property has a basic role in local spectral theory, see the recent monograph of Laursen and Neumann [31] or Aiena [1]. In this article we shall consider a local version of this property, which has been studied in recent papers by several authors [10], [6], [11], and previously by Finch [25], and Mbekhta [32].

Let  $X$  be a complex Banach space and  $T \in L(X)$ . The operator  $T$  is said to have *the single valued extension property* at  $\lambda_0 \in \mathbb{C}$  (abbreviated SVEP at  $\lambda_0$ ), if for every open disc  $U$  of  $\lambda_0$ , the only analytic function  $f : U \rightarrow X$  which satisfies the equation  $(\lambda I - T)f(\lambda) = 0$  for all  $\lambda \in U$  is the function  $f \equiv 0$ .

An operator  $T \in L(X)$  is said to have the SVEP if  $T$  has the SVEP at every point  $\lambda \in \mathbb{C}$ .

Evidently, an operator  $T \in L(X)$  has the SVEP at every point of the resolvent  $\rho(T) := \mathbb{C} \setminus \sigma(T)$ . The identity theorem for analytic function ensures that every  $T \in L(X)$  has the SVEP at the points of the boundary  $\partial\sigma(T)$  of the spectrum  $\sigma(T)$ . In particular, every operator has the SVEP at every isolated point of the spectrum.

The *quasi-nilpotent part* of  $T$  is defined by

$$H_0(T) := \{x \in X : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is easily seen that  $\ker(T^m) \subseteq H_0(T)$  for every  $m \in \mathbb{N}$  and  $T$  is quasi-nilpotent if and only if  $H_0(T) = X$ , see [38, Theorem 1.5].

The *analytic core* of  $T$  is the set  $K(T)$  of all  $x \in X$  such that there exists a sequence  $(u_n) \subset X$  and  $\delta > 0$  for which  $x = u_0$ , and  $Tu_{n+1} = u_n$  and  $\|u_n\| \leq \delta^n \|x\|$  for every  $n \in \mathbb{N}$ . It easily follows, from the definition, that  $K(T)$  is a linear subspace of  $X$  and that  $T(K(T)) = K(T)$ . Recall that  $T \in L(X)$  is said *bounded below* if  $T$  is injective and has closed range. Let  $\sigma_a(T)$  denote the classical *approximate point spectrum* of  $T$ , i.e. the set

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

and let

$$\sigma_s(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}$$

denote the *surjectivity spectrum* of  $T$ .

**Theorem 1.1.** *For a bounded operator  $T \in L(X)$ ,  $X$  a Banach space, and  $\lambda_0 \in \mathbb{C}$  the following implications hold:*

- (i)  $H_0(\lambda_0 I - T)$  closed  $\Rightarrow T$  has SVEP at  $\lambda_0$  [6].
- (ii) If  $\sigma_a(T)$  does not cluster at  $\lambda_0$  then  $T$  has SVEP at  $\lambda_0$ , [11].
- (iii) If  $\sigma_s(T)$  does not cluster at  $\lambda_0$  then  $T^*$  has SVEP at  $\lambda_0$ , [11].
- (iv) If  $p(\lambda_0 I - T) < \infty$  then  $T$  has SVEP at  $\lambda_0$  [10];
- (v) If  $q(\lambda_0 I - T) < \infty$  then  $T^*$  has SVEP at  $\lambda_0$  [10].

**Definition 1.2.** *An operator  $T \in L(X)$ ,  $X$  a Banach space, is said to be semi-regular if  $T(X)$  is closed and  $\ker T \subseteq T^\infty(X)$ , where*

$$T^\infty(X) := \bigcap_{n \in \mathbb{N}} T^n(X)$$

*denotes the hyper-range of  $T$ . An operator  $T \in L(X)$  is said to admit a generalized Kato decomposition at  $\lambda$ , abbreviated a GKD at  $\lambda$ , if there exists a pair of  $T$ -invariant closed subspaces  $(M, N)$  such that  $X = M \oplus N$ , the restriction  $\lambda I - T|_M$  is semi-regular and  $\lambda I - T|_N$  is quasi-nilpotent.*

A relevant case is obtained if we assume in the definition above that  $\lambda I - T|_N$  is nilpotent. In this case  $T$  is said to be of *Kato type at  $\lambda$* , see for details [1]. Recall that every semi-Fredholm operator is of Kato type at 0, by the classical result of Kato, see [1, Theorem 1.62]. Note that a semi-Fredholm operator need not to be semi-regular. A semi-Fredholm operator  $T$  is semi-regular precisely when its *jump*  $j(T)$  is equal to zero, see [1, Theorem 1.58]. The following characterizations of SVEP for operators of Kato type have been proved in [6] and [11].

**Theorem 1.3.** [1, Chapter 3] If  $T \in L(X)$  is of Kato type at  $\lambda_0$  then the following statements are equivalent:

- (i)  $T$  has SVEP at  $\lambda_0$ ;
- (ii)  $p(\lambda_0 I - T) < \infty$ ;
- (iii)  $\sigma_a(T)$  does not cluster at  $\lambda_0$ ;
- (iv)  $H_0(\lambda_0 I - T)$  is closed.

If  $\lambda_0 I - T \in \Phi_{\pm}(X)$  then the assertions (i)–(iv) are equivalent to the following statement:

- (v)  $H_0(\lambda_0 I - T)$  is finite-dimensional.

If  $\lambda_0 I - T$  is semi-regular then the assertions (i)–(iv) are equivalent to the following statement:

- (vi)  $\lambda_0 I - T$  is injective.

Dually, we have:

**Theorem 1.4.** [1, Chapter 3] If  $T \in L(X)$  is of Kato type at  $\lambda_0$  then the following statements are equivalent:

- (i)  $T^*$  has SVEP at  $\lambda_0$ ;
- (ii)  $q(\lambda_0 I - T) < \infty$ ;
- (iii)  $\sigma_s(T)$  does not cluster at  $\lambda_0$ ;

If  $\lambda_0 I - T \in \Phi_{\pm}(X)$  then the assertions (i)–(iii) are equivalent to the following statement:

- (iv)  $K(\lambda_0 I - T)$  is finite-codimensional.

If  $\lambda_0 I - T$  is semi-regular then the assertions (i)–(iv) are equivalent to the following statement:

- (vi)  $\lambda_0 I - T$  is surjective.

## 2 Weyl's theorem

In this section we shall see that the classes of operators satisfying Weyl's theorem is rather large. First we give a precise description of operators which satisfy Weyl's theorem by means of the localized SVEP.

**Theorem 2.1.** ([2] [4] [22]) If  $T \in L(X)$  then the following assertions are equivalent:

- (i) Weyl's theorem holds for  $T$ ;

- (ii)  $T$  satisfies Browder's theorem and  $\pi_{00}(T) = p_{00}(T)$ ;
- (iii)  $T$  has SVEP at every point  $\lambda \notin \sigma_w(T)$  and  $\pi_{00}(T) = p_{00}(T)$ ;
- (iv)  $T$  satisfies Browder's theorem and is of Kato type at all  $\lambda \in \pi_{00}(T)$ .

The conditions  $p_{00}(T) = \pi_{00}(T)$  is equivalent to several other conditions, see [1, Theorem 3.84]. Let  $\mathcal{P}_0(X)$ ,  $X$  a Banach space, denote the class of all operators  $T \in L(X)$  such that there exists  $p := p(\lambda) \in \mathbb{N}$  for which

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \pi_{00}(T). \quad (3)$$

The condition (3), and in general the properties of the quasi-nilpotent part  $H_0(\lambda I - T)$  as  $\lambda$  ranges in certain subsets of  $\mathbb{C}$ , seems to have a crucial role for Weyl's theorem, see [3]. In fact, we have the following result.

**Theorem 2.2.**  $T \in \mathcal{P}_0(X)$  if and only if  $p_{00}(T) = \pi_{00}(T)$ . In particular, if  $T$  has SVEP then Weyl's theorem holds for  $T$  if and only if  $T \in \mathcal{P}_0(X)$ .

Theorem 2.2 is very useful in order to show whenever Weyl's theorem holds. In fact, as we see now, a large number of the commonly considered operators on Banach spaces and Hilbert spaces have SVEP and belong to the class  $\mathcal{P}_0(X)$ .

(a) A bounded operator  $T \in L(X)$  on a Banach space  $X$  is said to be *paranormal* if  $\|Tx\|^2 \leq \|T^2x\|\|x\|$  for all  $x \in X$ .  $T \in L(X)$  is called *totally paranormal* if  $\lambda I - T$  is paranormal for all  $\lambda \in \mathbb{C}$ . For every totally paranormal operator it is easy to see that

$$H_0(\lambda I - T) = \ker(\lambda I - T) \quad \text{for all } \lambda \in \mathbb{C}. \quad (4)$$

The condition (4) entails SVEP by (i) of Theorem 1.1 and, obviously  $T \in \mathcal{P}_0(X)$ , so Weyl's theorem holds for totally paranormal operators. Weyl's theorem holds also for paranormal operators on Hilbert spaces, since these operators satisfies property (3) and have SVEP, see [2]. Note that the class of totally paranormal operators includes all hyponormal operators on Hilbert spaces  $H$ . In the sequel denote by  $T'$  the Hilbert adjoint of  $T \in L(H)$ . The operator  $T \in L(H)$  is said to be *hyponormal* if

$$\|T'x\| \leq \|Tx\| \quad \text{for all } x \in X.$$

A bounded operator  $T \in L(H)$  is said to be *quasi-hyponormal* if

$$(T'T)^2 \leq T'^2T^2$$

. Quasi-normal operators are paranormal, since these operators are hyponormal, see Conway [18].

An operator  $T \in L(H)$  is said to be *\*-paranormal* if

$$\|T'x\|^2 \leq \|T^2x\|$$

holds for all unit vectors  $x \in H$ .  $T \in L(H)$  is said to be *totally \*-paranormal* if  $\lambda I - T$  is \*-paranormal for all  $\lambda \in \mathbb{C}$ . Every totally \*-paranormal operator  $T$  satisfies property (4), see [29], and hence it satisfies Weyl's theorem.

(b) The condition (4) is also satisfied by every injective  $p$ -hyponormal operator, see [13], where an operator  $T \in L(H)$  on a Hilbert space  $H$  is said to be *p-hyponormal*, with  $0 < p \leq 1$ , if  $(T'T)^p \geq (TT')^p$ , [13].

(c) An operator  $T \in L(H)$  is said to be *log-hyponormal* if  $T$  is invertible and satisfies  $\log(T'T) \geq \log(TT')$ . Every log-hyponormal operator satisfies the condition (4), see [13].

(d) A bounded operator  $T \in L(X)$  is said to be *transaloid* if the spectral radius  $r(\lambda I - T)$  is equal to  $\|\lambda I - T\|$  for every  $\lambda \in \mathbb{C}$ . Every transaloid operator satisfies the condition (4), see [19].

(e) Given a Banach algebra  $A$ , a map  $T : A \rightarrow A$  is said to be a *multiplier* if  $(Tx)y = x(Ty)$  holds for all  $x, y \in A$ . For a commutative semi-simple Banach algebra  $A$ , let  $M(A)$  denote the commutative Banach algebra of all multipliers, [31]. If  $T \in M(A)$ ,  $A$  a commutative semi-simple Banach algebra, then  $T \in L(A)$  and the condition (4) is satisfied, see [6]. In particular, this condition holds for every convolution operator on the group algebra  $L^1(G)$ , where  $G$  is a locally compact Abelian group.

(f) An operator  $T \in L(X)$ ,  $X$  a Banach space, is said to be *generalized scalar* if there exists a continuous algebra homomorphism  $\Psi : \mathcal{C}^\infty(\mathbb{C}) \rightarrow L(X)$  such that

$$\Psi(1) = I \quad \text{and} \quad \Psi(Z) = T,$$

where  $\mathcal{C}^\infty(\mathbb{C})$  denote the Fréchet algebra of all infinitely differentiable complex-valued functions on  $\mathbb{C}$ , and  $Z$  denotes the identity function on  $\mathbb{C}$ . An operator similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces is called *subscalar*. The interested reader can find a well organized study of these operators in [31]. Every subscalar operator satisfies the following property  $H(p)$ :

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \quad (5)$$

for some  $p = p(\lambda) \in \mathbb{N}$ , see [34]. The condition (5) implies SVEP, by Theorem 1.1, and by Theorem 2.2 is obvious that Weyl's theorem holds for  $T$  whenever (5) is satisfied.



(g) An operator  $T \in L(H)$  on a Hilbert space  $H$  is said to be  $M$ -hyponormal if there is  $M > 0$  for which  $TT' \leq MT'T$ .  $M$ -hyponormal operators,  $p$ -hyponormal operators, log-hyponormal operators, and algebraically hyponormal operators are subscalar, so they satisfy the condition (5), see [34]. Also  $w$ -hyponormal operators on Hilbert spaces are subscalar, see for definition and details [16].

(h) An operator  $T \in L(X)$  for which there exists a complex non constant polynomial  $h$  such that  $h(T)$  is paranormal is said to be *algebraically paranormal*. If  $T \in L(H)$  is algebraically paranormal then  $T$  satisfies the condition (3), see [2], but in general the condition (5) is not satisfied by paranormal operators, (for an example see [8, Example 2.3]). Since every paranormal operator satisfies SVEP then, by Theorem 2.40 of [1], every algebraically paranormal operator has SVEP, so that Weyl's theorem holds for all algebraically paranormal operators, see also [27].

(i) An operator  $T \in L(X)$  is said to be *hereditarily normaloid* if every restriction  $T|M$  to a closed subspace of  $X$  is normaloid, i.e. the spectral radius of  $T|M$  coincides with the norm  $\|T|M\|$ . If, additionally, every invertible part of  $T$  is also normaloid then  $T$  is said to be *totally hereditarily normaloid*.

Let  $CHN$  denote the class of operators such that either  $T$  is totally hereditarily normaloid or  $\lambda I - T$  is hereditarily normaloid for every  $\lambda \in \mathbb{C}$ . The class  $CHN$  is very large; it contains  $p$ -hyponormal operators,  $M$ -hyponormal operators, paranormal operators and  $w$ -hyponormal operators, see [26]. Also every *totally \*-paranormal* operator belongs to the class  $CHN$ . Note that every operator  $T \in CHN$  satisfies the condition (3) with  $p(\lambda) = 1$  for all  $\lambda \in \pi_{00}(T)$ , see [21], so that  $T \in \mathcal{P}_0(X)$ . Therefore, if  $T \in CHN$  has SVEP then Weyl's theorem holds for  $T$ .

(l) Tensor products  $Z = T_1 \otimes T_2$  and multiplications  $Z = L_{T_1} R_{T_2}$  do not inherit Weyl's theorem from Weyl's theorem for  $T_1$  and  $T_2$  (here we assume that the tensor product  $X_1 \otimes X_2$  of the Banach spaces  $X_1$  and  $X_2$  is complete with respect to a "reasonable uniform crossnorm"). Also, Weyl's theorem does not transfer from  $Z$  to  $Z^*$ . We prove that if  $T_i$ ,  $i = 1, 2$ , satisfies Browder's theorem, or equivalently  $T_i$  has SVEP at points  $\lambda \notin \sigma_w(T_i)$ , and if the operators  $T_i$  are Kato type at the isolated points of  $\sigma(T_i)$ , then both  $Z$  and  $Z^*$  satisfy Weyl's theorem ([7]).

Weyl's theorem and Browder's theorem may be characterized by means of certain parts of the spectrum originating from Kato decomposition theory. To see this, let us denote by  $\sigma_k(T)$  the *Kato spectrum* of  $T \in L(X)$  defined by

$$\sigma_k(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}.$$

Note that  $\sigma_k(T)$  is a non-empty compact set of  $\mathbb{C}$  containing the topological boundary of  $\sigma(T)$ , see [1, Theorem 1.5]. A bounded operator  $T \in L(X)$  is said to admit a *generalized inverse*  $S \in L(X)$  if  $TST=T$ . It is well known that  $T$  admits a generalized inverse if and only if both the subspaces  $\ker T$  and  $T(X)$  are complemented in  $X$ . Every Fredholm operator admits a generalized inverse, see Theorem 7.3 of [1]. A "complemented" version of semi-regular operators is given by the Saphar operators:  $T \in L(X)$  is said to be *Saphar* if  $T$  is semi-regular and admits a generalized inverse. The *Saphar spectrum* is defined by

$$\sigma_{sa}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Saphar}\}.$$

Clearly,  $\sigma_k(T) \subseteq \sigma_{sa}(T)$ , so  $\sigma_{sa}(T)$  is non-empty compact subset of  $\mathbb{C}$ ; for other properties on Saphar operators we refer to Müller [33, Chapter II, §13].

**Theorem 2.3.** [4] *For a bounded operator  $T \in L(X)$  the following statements are equivalent:*

- (i) *Browder's theorem holds for  $T$ ;*
- (ii)  $\Delta(T) \subseteq \sigma_k(T)$ ;
- (iii)  $\Delta(T) \subseteq iso \sigma_k(T)$ ;
- (iv)  $\Delta(T) \subseteq \sigma_{sa}(T)$ ;
- (v)  $\Delta(T) \subseteq iso \sigma_{sa}(T)$ .

Denote by  $\sigma_0(T)$  the set of all  $\lambda \in \mathbb{C}$  for which  $0 < \alpha(\lambda I - T) < \infty$  and such that there exists a punctured open disc  $\mathbb{D}(\lambda)$  centered at  $\lambda$  such that  $\mu I - T \in W(X)$  and

$$\ker(\mu I - T) \subseteq (\mu I - T)^\infty(X) \quad \text{for all } \mu \in \mathbb{D}(\lambda). \quad (6)$$

Since  $(\mu I - T)(X)$  is closed then the condition (6) is equivalent to saying that  $\mu I - T \in W(X)$  is semi-regular in punctured open disc  $\mathbb{D}(\lambda)$ . Hence

$$\lambda \in \sigma_0(T) \Rightarrow \lambda \notin \text{acc}(\sigma_k(T) \cap \sigma_w(T)).$$

Every invertible operator is semi-regular and Weyl. From this we obtain

$$p_{00}(T) \subseteq \pi_{00}(T) \subseteq \sigma_0(T) \quad \text{for all } T \in L(X). \quad (7)$$

Indeed, if  $\lambda \in \pi_{00}(T)$ , then  $0 < \alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$  and  $\lambda$  is an isolated point of  $\sigma(T)$ , so  $\mu I - T$  is invertible and hence semi-regular near  $\lambda$ , and hence the inclusion (6) is satisfied.

The following result has been proved in [14, Theorem 1.5]. We shall give a different proof by using SVEP.

**Theorem 2.4.** *For a bounded operator  $T \in L(X)$  Weyl's theorem holds if and only if  $\sigma_0(T) = p_{00}(T)$ .*

*Proof.* Assume that  $T$  satisfies Weyl's theorem. From the inclusion (7), in order to show the equality  $\sigma_0(T) = p_{00}(T)$ , it suffices only to prove the inclusion  $\sigma_0(T) \subseteq p_{00}(T)$ . By Theorem 2.1 we know that  $\pi_{00}(T) = p_{00}(T)$ .

Suppose that  $\lambda \notin p_{00}(T)$ . Clearly, if  $\lambda \notin \sigma(T)$  then  $\lambda \notin \sigma_0(T)$ .

Consider the second case  $\lambda \in \sigma(T)$ . Assume that  $\lambda \in \sigma_0(T)$ . By definition of  $\sigma_0(T)$  there exists  $\varepsilon > 0$  such that  $\mu I - T$  is Browder and  $\ker(\mu I - T) \subseteq (\mu I - T)^\infty(X)$  for all  $\mu \in \mathbb{D}(\lambda)$  for all  $0 < |\mu - \lambda| < \varepsilon$ . Since  $\mu I - T$  is semi-regular and the condition  $p(\mu I - T) < \infty$  entails SVEP en  $\mu$  then, by Theorem 1.3,  $\mu I - T$  is injective and therefore, since  $\alpha(\mu I - T) = \beta(\mu I - T)$ , it follows that  $\mu I - T$  is invertible. Hence  $\lambda$  is an isolated point of  $\sigma(T)$ . But  $\lambda \in \sigma_0(T)$ , so that  $0 < \alpha(\lambda I - T) < \infty$ , hence  $\lambda \in \pi_{00}(T) = p_{00}(T)$ , a contradiction. Hence, even in the second case we have  $\lambda \notin \sigma_0(T)$ . Therefore  $\sigma_0(T) \subseteq p_{00}(T)$ .

Conversely, assume that  $\sigma_0(T) = p_{00}(T)$ . From (7) we have

$$\pi_{00}(T) \subseteq \sigma_0(T) = p_{00}(T) \subseteq \pi_{00}(T),$$

so that  $\pi_{00}(T) = p_{00}(T)$ . To show that  $T$  satisfies Weyl's theorem it suffices, by Theorem 2.1 only to prove that  $T$  satisfies Browder's theorem, or equivalently, by Theorem 2.3, that  $\Delta(T) \subseteq \sigma_k(T)$ . Let  $\lambda \in \Delta(T) = \sigma(T) \setminus \sigma_w(T)$  and assume that  $\lambda \notin \sigma_k(T)$ . Then  $\lambda I - T$  is Weyl and

$$0 < \alpha(\lambda I - T) = \beta(\lambda I - T) < \infty.$$

Furthermore, since  $W(X)$  is a open subset of  $L(X)$ , there exists  $\varepsilon > 0$  such that  $\mu I - T \in W(X)$  for all  $|\lambda - \mu| < \varepsilon$ . On the other hand,  $\lambda I - T$  is semi-regular and hence by Theorem 1.31 of [1] we can choice  $\varepsilon$  such that  $\mu I - T$  is semi-regular for all  $|\lambda - \mu| < \varepsilon$ . Therefore,  $\lambda \in \sigma_0(T) = p_{00}(T)$ , so that  $\lambda I - T$  is Browder. The condition  $p(\lambda I - T) < \infty$  entails that  $T$  has SVEP at  $\lambda$  and hence  $\alpha(\lambda I - T) = 0$ ; a contradiction. Therefore,  $\Delta(T) \subseteq \sigma_k(T)$ , as desired. ■

Set

$$\sigma_1(T) := \sigma_w(T) \cup \sigma_k(T),$$

and denote by

$$\pi_{0f}(T) := \{\lambda \in \mathbb{C} : 0 < \alpha(\lambda I - T) < \infty\},$$

the set of all eigenvalues of finite multiplicity.

**Theorem 2.5.** [15] *For a bounded operator  $T \in L(X)$  Weyl's theorem holds if and only if*

$$\pi_{0f}(T) \cap \text{iso } \sigma_1(T) = \sigma(T) \setminus \sigma_w(T). \quad (8)$$

*Proof.* Suppose that Weyl's theorem holds for  $T$ , i.e.  $\Delta(T) = \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$ . From Theorem 2.1 and Theorem 2.3 we then have

$$\sigma(T) = \sigma_w(T) \cup \Delta(T) \subseteq \sigma_w(T) \cup \sigma_k(T) = \sigma_1(T),$$

so that  $\sigma(T) = \sigma_1(T)$ . Therefore,

$$\text{iso } \sigma_1(T) \cap \pi_{0f}(T) = \text{iso } \sigma(T) \cap \pi_{0f}(T) = \pi_{00}(T) = \Delta(T).$$

Conversely, assume that the equality (8) holds. We have

$$\pi_{00}(T) \subseteq \text{iso } \sigma_1(T) \cap \pi_{0f}(T) = \Delta(T).$$

To prove the opposite inclusion, let  $\lambda_0 \in \Delta(T) = \sigma(T) \setminus \sigma_w(T)$ . Then there exists  $\varepsilon > 0$  such that  $\lambda \notin \sigma_1(T)$  for all  $|\lambda - \lambda_1| < \varepsilon$ . We prove now that  $\lambda \notin \sigma(T)$  for all  $0 < |\lambda - \lambda_1| < \varepsilon$ . In fact, if there exists  $\lambda_1$  such that  $0 < |\lambda_1 - \lambda_0| < \varepsilon$  and  $\lambda_1 \in \sigma(T)$ , then

$$\lambda_1 \in \sigma(T) \setminus \sigma_w(T) = \Delta(T) = \text{iso } \sigma_1(T) \cap \pi_{0f}(T).$$

Hence,  $\lambda_1 \in \sigma_1(T)$ ; a contradiction. Therefore,  $\lambda_0 \in \text{iso } \sigma(T)$  and consequently  $\lambda_0 \in \pi_{00}(T)$ . This shows that  $\Delta(T) = \pi_{00}(T)$ , i.e. Weyl's theorem holds for  $T$ . ■

In general the spectral mapping theorem is liable to fail for  $\sigma_1(T)$ . There is only the inclusion  $\sigma_1(f(T)) \subseteq f(\sigma_1(T))$  for every  $f \in \mathcal{H}(\sigma(T))$  [15, Lemma 2.5]. However, the spectral mapping theorem holds for  $\sigma_1(T)$  whenever  $T$  or  $T^*$  has SVEP, see [15, Theorem 2.6]. In fact, if  $T$  has SVEP (respectively,  $T^*$  has SVEP),  $\lambda I - T \in \Phi(X)$  then  $\text{ind}(\lambda I - T) \leq 0$  (respectively,  $\text{ind}(\lambda I - T) \geq 0$ ), see [1, Corollary 3.19], so that the condition of [15, Theorem 2.6] are satisfied.

### 3 $\alpha$ -Weyl's theorem

For a bounded operator  $T \in L(X)$  on a Banach space  $X$  let us denote

$$\pi_{00}^\alpha(T) := \{\lambda \in \text{iso } \sigma_a(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

and

$$p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ub}(T) = \{\lambda \in \text{iso } \sigma_a(T) : \lambda I - T \in \mathcal{B}_+(X)\}.$$

We have

$$p_{00}^a(T) \subseteq \pi_{00}^a(T) \quad \text{for every } T \in L(X).$$

In fact, if  $\lambda \in p_{00}^a(T)$  then  $\lambda I - T \in \Phi_+(X)$  and  $p(\lambda I - T) < \infty$ . By Theorem 1.3 then  $\lambda$  is isolated in  $\sigma_a(T)$ . Furthermore,  $0 < \alpha(\lambda I - T) < \infty$  since  $(\lambda I - T)(X)$  is closed and  $\lambda \in \text{iso } \sigma_a(T)$ .

The *Weyl (or essential) approximate point spectrum*  $\sigma_{wa}(T)$  of a bounded operator  $T \in L(X)$  is the complement of those  $\lambda \in \mathbf{C}$  for which  $\lambda I - T \in \Phi_+(X)$  and  $\text{ind } (\lambda I - T) \leq 0$ . Note that  $\sigma_{wa}(T)$  is the intersection of all approximate point spectra  $\sigma_a(T + K)$  of compact perturbations  $K$  of  $T$ , see [36]. Analogously, *Weyl surjectivity spectrum*  $\sigma_{ws}(T)$  of a bounded operator  $T \in L(X)$  is the complement of those  $\lambda \in \mathbf{C}$  for which  $\lambda I - T \in \Phi_-(X)$  and  $\text{ind } (\lambda I - T) \geq 0$ . Note that  $\sigma_{ws}(T)$  is the intersection of all surjectivity spectra  $\sigma_s(T + K)$  of compact perturbations  $K$  of  $T$ , see [36].

Following Rakočević [36], we shall say that *a-Weyl's theorem holds* for  $T \in L(X)$  if

$$\Delta_a(T) := \sigma_a(T) \setminus \sigma_{wa}(T) = \pi_{00}^a(T),$$

while,  $T$  satisfies *a-Browder's theorem* if

$$\sigma_{wa}(T) = \sigma_{ub}(T).$$

We have, see for instance [1, Chap.3],

$$a\text{-Weyl's theorem} \Rightarrow \text{Weyl's theorem},$$

and

$$a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

The next theorem shows that also *a-Browder's theorem* may be characterized by means of the Kato spectrum.

**Theorem 3.1.** *For a bounded operator  $T \in L(X)$  the following statements are equivalent:*

- (i) *a-Browder's theorem holds for  $T$ ;*
- (ii)  $\Delta_a(T) \subseteq \sigma_k(T)$ ;
- (ii)  $\Delta_a(T) \subseteq \text{iso } \sigma_k(T)$ ;
- (iii)  $\Delta_a(T) \subseteq \sigma_{sa}(T)$ ;
- (v)  $\Delta_a(T) \subseteq \text{iso } \sigma_{sa}(T)$ .

*Proof.* The equivalences (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii) have been proved in [5, Theorem 2.7]. The proof of the equivalences (i)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv) is analogous to the proof of [4, Theorem 2.13].  $\blacksquare$

The following result is analogous to the result stated in Theorem 2.1.

**Theorem 3.2.** ([2], [5]) *Let  $T \in L(X)$ . Then the following statements are equivalent:*

- (i)  $T$  satisfies  $a$ -Weyl's theorem;
- (ii)  $T$  satisfies  $a$ -Browder's theorem and  $p_{00}^a(T) = \pi_{00}^a(T)$ ;
- (iii)  $T$  has SVEP at every point  $\lambda \notin \sigma_{\text{wa}}(T)$  and  $p_{00}^a(T) = \pi_{00}^a(T)$ ;
- (iv)  $T$  satisfies  $a$ -Browder's theorem and is of Kato type at all  $\lambda \in \pi_{00}^a(T)$ .

We give now an example of operator  $T \in L(X)$  which has SVEP, satisfies Weyl's theorem but does not satisfy  $a$ -Weyl's theorem.

**Example 3.3.** Let  $T$  be the hyponormal operator  $T$  given by the direct sum of the 1-dimensional zero operator and the unilateral right shift  $R$  on  $\ell^2(\mathbf{N})$ . Then 0 is an isolated point of  $\sigma_a(T)$  and  $0 \in \pi_{00}^a(T)$ , while  $0 \notin p_{00}^a(T)$ , since  $p(T) = p(R) = \infty$ . Hence,  $T$  does not satisfy  $a$ -Weyl's theorem.

Denote by  $\mathcal{H}(\sigma(T))$  the set of all analytic functions defined on a neighborhood of  $\sigma(T)$ , let  $f(T)$  be defined by means of the classical functional calculus.

**Theorem 3.4.** [2] *If  $T \in L(X)$  has property  $(H_p)$ . Then  $a$ -Weyl's holds for  $f(T^*)$  for every  $f \in \mathcal{H}(\sigma(T))$ . Analogously, if  $T^*$  has property  $(H_p)$  then  $a$ -Weyl's holds for  $f(T)$  for every  $f \in \mathcal{H}(\sigma(T))$ .*

An analogous result holds for paranormal operators on Hilbert spaces. By  $T'$  we shall denote the Hilbert adjoint of  $T \in L(H)$ .

**Theorem 3.5.** [2] *Suppose that  $H$  is a Hilbert space. If  $T \in L(H)$  is algebraically paranormal then  $a$ -Weyl's holds for  $f(T')$  for every  $f \in \mathcal{H}(\sigma(T))$ . Analogously, if  $T'$  has property  $(H_p)$  then  $a$ -Weyl's holds for  $f(T)$  for every  $f \in \mathcal{H}(\sigma(T))$ .*

The results of Theorem 3.4 applies to the classes listed in (a)-(i).

Define by  $\sigma_2(T)$  the set of all  $\lambda \in \mathbb{C}$  for which  $0 < \alpha(\lambda I - T) < \infty$  and there exists a punctured open disc  $\mathbb{D}(\lambda)$  centered at  $\lambda$  such that  $\mu I - T \in W_+(X)$  and

$$\ker(\mu I - T) \subseteq (\mu I - T)^\infty(X) \quad \text{for all } \mu \in \mathbb{D}(\lambda).$$

The following characterization of  $a$ -Weyl's theorem is analogous to that established for Weyl's theorem in Theorem 2.4.

**Theorem 3.6.** [14] *Let  $T \in L(X)$  be a bounded linear operator. Then  $a$ -Weyl's theorem holds for  $T$  if and only if  $\sigma_2(T) = p_{00}^a(T)$ .*

*Proof.* The proof is similar to that of Theorem 2.4. Use the result of Theorem 3.1. ■

Define

$$\sigma_3(T) := \sigma_{\text{wa}}(T) \cup \sigma_{\text{k}}(T).$$

The proof of the following result is similar to that of Theorem 2.5.

**Theorem 3.7.** [15] *For a bounded operator  $T \in L(X)$   $a$ -Weyl's theorem holds if and only if*

$$\pi_{0f}(T) \cap \text{iso } \sigma_3(T) = \Delta_a(T). \quad (9)$$

In general the spectral mapping theorem is liable to fail also for  $\sigma_3(T)$ . There is only the inclusion  $\sigma_3(f(T)) \subseteq f(\sigma_3(T))$  for every  $f \in \mathcal{H}(\sigma(T))$  [15, Lemma 3.4]. Also here, the spectral mapping theorem holds for  $\sigma_3(T)$  whenever  $T$  or  $T^*$  has SVEP, see [15, Theorem 3.6].

## 4 Property ( $w$ )

The following variant of Weyl's theorem has been introduced by Rakočević [35] and studied in [12].

**Definition 4.1.** *A bounded operator  $T \in L(X)$  is said to satisfy property ( $w$ ) if*

$$\Delta_a(T) = \sigma_a(T) \setminus \sigma_{\text{wa}}(T) = \pi_{00}(T).$$

Note that

$$\text{property } (w) \text{ for } T \Rightarrow a\text{-Browder's theorem for } T,$$

and precisely we have.

**Theorem 4.2.** [12] *If  $T \in L(X)$  the following statements are equivalent:*

- (i)  $T$  satisfies property ( $w$ );
- (ii)  $a$ -Browder's theorem holds for  $T$  and  $p_{00}^a(T) = \pi_{00}(T)$ .

Define

$$\Lambda(T) := \{\lambda \in \Delta_a(T) : \text{ind}(\lambda I - T) < 0\}.$$

Clearly,

$$\Delta_a(T) = \Delta(T) \cup \Lambda(T) \quad \text{and} \quad \Lambda(T) \cap \Delta(T) = \emptyset. \quad (10)$$

Property (w), despite of the study of it in literature has been neglected, seems to be of interest. Infact, exactly like  $a$ -Weyl's theorem, property (w) implies Weyl's theorem. The next result relates Weyl's theorem and property (w).

**Theorem 4.3.** [12] *If  $T \in L(X)$  satisfies property (w) then  $\Lambda(T) = \emptyset$ . Moreover, the following statements are equivalent:*

- (i)  $T$  satisfies property (w);
- (ii)  $T$  satisfies Weyl's theorem and  $\Lambda(T) = \emptyset$ ;
- (iii)  $T$  satisfies Weyl's theorem and  $\Delta_a(T) \subseteq \text{iso } \sigma(T)$ ;
- (iv)  $T$  satisfies Weyl's theorem and  $\Delta_a(T) \subseteq \partial\sigma(T)$ ,  $\partial\sigma(T)$  the topological boundary of  $\sigma(T)$ ;

We give now two sufficient conditions for which  $a$ -Weyl's theorem for  $T$  (respectively, for  $T^*$ ) implies property (w) for  $T$  (respectively, for  $T^*$ ). Observe that these conditions are a bit stronger than the assumption that  $T$  satisfies  $a$ -Browder's theorem, see [5].

**Theorem 4.4.** [12] *If  $T \in L(X)$  the following statements hold:*

- (i) *If  $T^*$  has SVEP at every  $\lambda \notin \sigma_{\text{wa}}(T)$  and  $T$  satisfies  $a$ -Weyl's theorem then property (w) holds for  $T$ .*
- (ii) *If  $T$  has SVEP at every  $\lambda \notin \sigma_{\text{ws}}(T)$  and  $T^*$  satisfies  $a$ -Weyl's theorem then property (w) holds for  $T^*$ .*

**Theorem 4.5.** *If  $T \in L(X)$  is generalized scalar then property (w) holds for both  $T$  and  $T^*$ . In particular, property (w) holds for every spectral operator of finite type.*

*Remark 4.6.* Property (w) is not intermediate between Weyl's theorem and  $a$ -Weyl's theorem. For instance, if  $T$  is a hyponormal operator  $T$  given by the direct sum of the 1-dimensional zero operator and the unilateral right shift  $R$  on  $\ell^2(\mathbb{N})$ , then  $T$  does not satisfy  $a$ -Weyl's theorem, while property (w) holds for  $T$ , see [12] for details. If  $R \in \ell^2(\mathbb{N})$  denote the unilateral right shift and

$$U(x_1, x_2, \dots) := (0, x_2, x_3, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}),$$



then  $T := R \oplus U$  does not satisfy property (w), while  $T$  satisfies  $a$ -Weyl's theorem ([12]).

However, Weyl's theorem,  $a$ -Weyl's theorem and property (w) coincide in some special cases:

**Theorem 4.7.** [12] *Let  $T \in L(X)$ . Then the following equivalences hold:*

(i) *If  $T^*$  has SVEP, the property (w) holds for  $T$  if and only if Weyl's theorem holds for  $T$ , and this is the case if and only if  $a$ -Weyl's theorem holds for  $T$ .*

(ii) *If  $T$  has SVEP, the property (w) holds for  $T^*$  if and only if Weyl's theorem holds for  $T^*$ , and this is the case if and only if  $a$ -Weyl's theorem holds for  $T^*$ .*

**Theorem 4.8.** [12] *Suppose that  $T \in L(H)$ ,  $H$  a Hilbert space. If  $T'$  has property  $H(p)$  then property (w) holds for  $f(T)$  for all  $f \in \mathcal{H}(\sigma(T))$ . In particular, if  $T'$  is generalized scalar then property (w) holds for  $f(T)$  for all  $f \in \mathcal{H}(\sigma(T))$ .*

From Theorem 4.8 it then follows that if  $T'$  belongs to each one of the classes of operators of examples (a)-(h) then property (w) holds for  $f(T)$ .

An operator  $T \in L(X)$  is said to be *polaroid* if every isolated point of  $\sigma(T)$  is a pole of the resolvent operator  $(\lambda I - T)^{-1}$ , or equivalently  $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ , see [30, Proposition 50.2]. An operator  $T \in L(X)$  is said to be  *$a$ -polaroid* if every isolated point of  $\sigma_a(T)$  is a pole of the resolvent operator  $(\lambda I - T)^{-1}$ , or equivalently  $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ , see [30, Proposition 50.2]. Clearly,

$$T \text{ } a\text{-polaroid} \Rightarrow T \text{ polaroid.}$$

and the opposite implication is not generally true.  $a$ -Weyl's theorem and property (w) are equivalent for  $a$ -polaroid operators. Note that an  $a$ -polaroid operator may fail SVEP, so Theorem 4.7 does not apply.

**Theorem 4.9.** [12] *Suppose that  $T$  is  $a$ -polaroid. Then  $a$ -Weyl's theorem holds for  $T$  if and only if  $T$  satisfies property (w).*

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