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On Approximately Semiopen Maps in Topological Spaces

Sobre Aplicaciones Aproximadamente Semiabiertas en Espacios Topológicos

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Abstract

In this paper a new generalization of semi-open maps via the concept of sg-closed sets is considered. They are called *approximately semi-open maps* and are used to obtain a characterization of semi-$T_{1/2}$ spaces.
Key words and phrases: Topological space, sg-closed set, semi-open set, semi-open map, irresolute map, semi-$T_{1/2}$ space.

Resumen

En este trabajo se considera una nueva generalización de las aplicaciones semiabiertas via el concepto de conjuntos sg-cerrados. Se les llama *aplicaciones aproximadamente semiabiertas* y se usan para obtener una caracterizacin de los espacios semi-$T_{1/2}$.
Palabras y frases clave: espacio topológico, conjunto sg-cerrado, conjunto semiabierto, aplicación semiabierta, aplicación irresoluta, espacio semi-$T_{1/2}$.

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1 Introduction

The concept of semi-open set in topological spaces was introduced in 1963 by N. Levine [11]. After the works of Levine on semi-open sets, various mathematicians turned their attention to the generalizations of various concepts in topology by considering semi-open sets. When open sets are replaced by semi-open sets, new results are obtained in some occasions and in other occasions substantial generalizations are exhibited. In 1987 P. Bhattacharyya and P. K. Lahiri [2] generalized the concept of closed sets to semi-generalized closed sets (sg-closed sets) with the help of semi-openness. These sets were also considered by various authors (e.g. P. Sundaram, H. Maki and K. Balachandran [14], M. Caldas [5], J. Dontchev and H. Maki [9], others) .

In this direction we shall introduce the concept of semi-openness maps called ap-semi-open maps by using sg-closed sets and study some of their basic properties. This definition enables us to obtain conditions under which inverse maps preserve sg-open sets and also establish relationships between this map and other generalized forms of openness. Finally we characterize the class of semi-$T_{1/2}$ spaces in terms of ap-semi-open maps.

Throughout this paper $(X,\tau),(Y,\sigma)$ and (Z,γ) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. For a subset A of a space $(X,\tau), Cl(A)$ and $Int(A)$ denote the closure of A and the interior of A respectively. In order to make the contents of the paper as selfcontained as possible, we briefly describe certain definitions, notations and some properties. For those not described, we refer to [1]. A subset A of a space (X,τ) is said to be semi-open [11] if, there exists $O \in \tau$ such that $O \subseteq A \subseteq Cl(O)$. By $SO(X,\tau)$ we mean the collection of all semi-open sets in (X,τ).
The semi-interior [7] of A denoted by $sInt(A)$, is defined by the union of all semi-open sets of (X,τ) contained in A. A is semi-open [7] if and only if $sInt(A) = A$. Also, we have
$sInt(A) = A \cap Cl(Int(A))$ [10]. A subset B of (X,τ) is said to be semi-closed [3] if, its complement B^c is semi-open in (X,τ).
The semi-closure [3] of a set B of (X,τ) denoted by $sCl_X(B)$ (briefly: $sCl(B)$), is defined to be the intersection of all semi-closed sets of (X,τ) containing B. B is semi-closed [13] if and only if $sCl(B) = B$. Also, we have $sCl(B) = B \cup Int(Cl(B))$ [10].

A subset F of (X,τ) is said to be semi-generalized closed (written in short as sg-closed) in (X,τ) [2] if, $sCl(F) \subseteq O$ whenever $F \subseteq O$ and O is semi-open in (X,τ). A subset B is said to be semi-generalized open (written in short as sg-open) in (X,τ) [2] if, its complement B^c is sg-closed in (X,τ).

A map $f : (X, \tau) \to (Y, \sigma)$ is called irresolute [8] (resp. sg-irresolute [14]) if, $f^{-1}(O)$ is semi-open (resp. sg-closed) in (X, τ) for every $O \in SO(Y, \sigma)$ (resp. sg-closed in (Y, σ).

$f : (X, \tau) \to (Y, \sigma)$ is called pre-semi-closed (resp. pre-semi-open) [8] if, for every semi-closed (resp. semi-open) set B of $(X, \tau), f(B)$ is semi-closed (resp. semi-open) in (Y, σ)).

2 Ap-semi-open maps.

Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) into a topological space (Y, σ).

Definition 1. A map $f : (X, \tau) \to (Y, \sigma)$ is said to be approximately semi-open (or ap-semi-open) if, $sCl(B) \subseteq f(A)$ whenever B is a sg-closed subset of (Y, σ), A is a semi-open subset of (X, τ) and $B \subseteq f(A)$.

Recall that, a map $f : (X, \tau) \to (Y, \sigma)$ is said to be approximately semi-closed (or ap-semi-closed) [6] if, $f(B) \subseteq sInt(A)$ whenever A is a sg-open subset of (Y, σ), B is a semi-closed subset of (X, τ) and $f(B) \subseteq A$.

Example 2.1. Let $X = \{a, b, c\} = Y$ and $\tau = \{\emptyset, \{a, b\}, X\} = \sigma$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = f(b) = c$ and $f(c) = a$. Then, the sg-closed sets of (X, τ) i.e., $SGC(X, \tau) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. Obviously f is ap-semi-open but not ap-semi-closed.

question 2.2. Is there an example of an ap-semi-closed map but not ap-semi-open ?.

Remark 2.3. ap-semi-openness and ap-semi-closedness are equivalent if the map is bijective.

Clearly pre-semi-open maps are ap-semi-open, but not conversely. The proof follows from Definition 1 and Theorem 6 of [2].

The following example shows that the converse implications do not hold.

Example 2.4. Let $X = \{a, b\}$ be the Sierpinski space with the topology, $\tau = \{\emptyset, \{a\}, X\}$. Let $f : X \to X$ be defined by $f(a) = b$ and $f(b) = a$. Since the image of every semi-open set is semi-closed, then f is ap-semi-open. However $\{a\}$ is semi-open in (X, τ) but $f(\{a\})$ is not semi-open in (Y, σ). Therefore f is not pre-semi-open.

Theorem 2.5. *$f : (X, \tau) \to (Y, \sigma)$ is ap-semi-open if, $f(O) \in SC(Y, \sigma)$ for every semi-open subset O of (X, τ).*

Proof. Let $B \subseteq f(A)$,where A is a semi-open subset of (X, τ) and B is a sg-closed subset of (Y, σ). Therefore $sCl(B) \subseteq sCl(f(A)) = f(A)$. Thus f is ap-semi-open. □

This Theorem was used in Example 2.4.

Remark 2.6. Let (X, τ) the topological space defined in Example 2.4 (or in Example 2.1 with $f : (X, \tau) \to (X, \tau)$ such that $f(a) = f(c) = a$ and $f(b) = b$). Then the identity map on (X, τ) is ap-semi-open, it is clear that the converses of Theorem 2.5 do not hold.

In the following theorem, we get under certain conditions that the converse of Theorem 2.5 is true.

Theorem 2.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) in a topological space (Y, σ). If the semi-open and semi-closed sets of (Y, σ) coincide, then f is ap-semi-open if and only if, $f(A) \in SC(Y, \sigma)$ for every semi-open subset A of (X, τ).*

Proof. Assume f is ap-semi-open. Let A be an arbitrary subset of (Y, σ) such that $A \subseteq Q$ where $Q \in SO(Y, \sigma)$. Then by hypothesis $sCl(A) \subseteq sCl(Q) = Q$. Therefore all subset of (Y, σ) are sg-closed (and hence all are sg-open). So for any $O \in SO(X, \tau)$,$f(O)$ is sg-closed in (Y, σ). Since f is ap-semi-open $sCl(f(O)) \subseteq f(O)$. Therefore $sCl(f(O)) = f(O)$, i.e., $f(O)$ is semi-closed in (Y, σ). The converse is clear by Theorem 2.5. □

As immediate consequence of Theorem 2.7, we have the following.

Corollary 2.8. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) in a topological space (Y, σ). If the semi-closed and semi-open sets of (Y, σ) coincide, then f is ap-semi-open if and only if, f is pre-semi-open.*

Recall, that a map $f : (X, \tau) \to (Y, \sigma)$ is called contra pre-semi-open [4] if $f(O)$ is semi-closed in (Y, σ) for each set $O \in SO(X, \tau)$. Example 2.4 above shows that contra-pre-semi-open does not imply pre-semi-open. The identity map on the same topological space (X, τ) where $\tau = \{\emptyset, \{a\}, X\}$ (or also the space $X = \{a, b, c\} = Y$ with $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} = \sigma$ and $f : (X, \tau) \to (Y, \sigma)$ such that $f(a) = f(c) = a$ and $f(b) = b$) is an example of a pre-semi-open map which is not contra pre-semi-open. i.e., contra-semi-open maps and pre-semi-open are independent notions.

Remark 2.9. By Theorem 2.5 and Remark 2.6, we have that every contra pre-semi-open map is ap-semi-open, the converse implication do not hold

Clearly the following diagram holds :

$$
\begin{array}{ccccc}
 & & \text{contra pre semi-open map} & & \\
 & \not\nearrow\,\not\swarrow & & \not\nwarrow\,\searrow & \\
\text{semi-open map} & & \not\uparrow\,\not\downarrow & & \text{ap-semi-open map} \\
 & \not\searrow\,\nwarrow & & \not\swarrow\,\nearrow & \\
 & & \text{pre semi-open map} & &
\end{array}
$$

The next theorem establishes conditions under which inverse maps of every sg-open subset of the codomain is sg-open.

Theorem 2.10. *If a map $f : (X, \tau) \to (Y, \sigma)$ is surjective irresolute and ap-semi-open, then $f^{-1}(A)$ is sg-open whenever A is sg-open subset of (Y, σ).*

Proof. Let A be a sg-open subset of (Y, σ). Suppose that $F \subseteq f^{-1}(A)$ where $F \in SC(X, \tau)$. Taking complements we obtain $f^{-1}(A^c) \subseteq F^c$ or $A^c \subseteq f(F^c)$. Since f is an ap-semi-open and $sInt(A) = A \cap Cl(Int(A)$ and $sCl(A) = A \cup Int(Cl(A))$, then $(sInt(A))^c = sCl(A^c) \subseteq f(F^c)$. It follows that $(f^{-1}(sInt(A)))^c \subseteq F^c$ and hence $F \subseteq f^{-1}(sInt(A)$. Since f is irresolute $f^{-1}(sInt(A))$ is semi-open. Thus we have $F \subseteq f^{-1}(sInt(A)) = sInt(f^{-1}(sInt(A))) \subseteq sInt(f^{-1}(A))$. This implies that $f^{-1}(A)$ is sg-open in (X, τ). $\Box$

Theorem 2.11. *Let $f : (X, \tau) \to (Y, \sigma)$, $g : (Y, \sigma) \to (Z, \gamma)$ be two maps such that $g \circ f : (X, \tau) \to (Z, \gamma)$. Then*
(i) $g \circ f$ is ap-semi-open, if f is pre-semi-open and g is ap-semi-open.
(ii) $g \circ f$ is ap-semi-open, if f is ap-semi-open and g is bijective pre-semi-closed and sg-irresolute.

Proof. In order to prove the statement (i), suppose A is an arbitrary semi-open subset in (X, τ) and B a sg-closed subset of (Z, γ) for which $B \subseteq g \circ f(A)$. Then $f(A)$ is semi-open in (Y, σ) because f is pre-semi-open. Since g is ap-semi-open, $sCl(B) \subseteq g(f(A))$. This implies that $g \circ f$ is ap-semi-open.
In order to prove the statement (ii), suppose A is an arbitrary semi-open subset of (X, τ) and B a sg-closed subset of (Z, γ) for which $B \subseteq g \circ f(A))$. Hence $g^{-1}(B) \subseteq f(A)$. Then $sCl(g^{-1}(A)) \subseteq f(A)$ because $g^{-1}(B)$ is sg-closed and f is ap-semi-open. Hence we have ,
$sCl(B) \subseteq sCl(gg^{-1}(B)) \subseteq g(sCl(g^{-1}(B))) \subseteq g(f(A)) = (g \circ f)(A)$.
This implies that $g \circ f$ is ap-semi-open. $\Box$

Regarding the restriction f_A of a map $f : (X, \tau) \to (Y, \sigma)$ to a subset A of X, we have the following.

Theorem 2.12. *If $f : (X, \tau) \to (Y, \sigma)$ is ap-semi-open and A is a semi-open set of (X, τ), then its restriction $f_A : (A, \tau_A) \to (Y, \sigma)$ is ap-semi-open.*

Proof. Suppose O is an arbitrary semi-open subset of (A, τ_A) and B a sg-closed subset of (Y, σ) for which $B \subseteq f_A(O)$. By ([12]) O is semi-open of (X, τ) because A is semi-open of (X, τ). Then $B \subseteq f(O) = f_A(O)$. Using Definition 1, we have that $sCl(B) \subseteq f_A(O)$. Thus f_A is an ap-semi-open map. □

Observe, that restrictions of ap-semi-open maps can fail to be ap-semi-open. Really, let X be an indiscrete space. Then X and $\emptyset$ are the only semi-closed subsets of X. Hence the semi-open subsets of X are also X and $\emptyset$. Let A a nonempty proper subset of X. The identity map $f : X \to X$ is ap-semi-open, but $f_A : A \to X$ fails to be ap-semi-open.In fact: $f(A)$ is sg-closed (every subset of X is sg-closed) and A is open in A, therefore semi-open in (A, τ_A), but $sCl(f(A)) \not\subseteq f(A)$.

In recent years, the class of semi-$T_{1/2}$ spaces has been of some interest, (see by example, [2], [4], [9], [14]). In the following theorem we give a new characterization of semi-$T_{1/2}$ spaces by using the concepts of ap-semi-open maps. In order to achieve our purpose, we recall, that a topological space (X, τ) is said to be semi-$T_{1/2}$ space [2], if every sg-closed set is semi-closed.

Theorem 2.13. *Let (Y, σ) be a topological space. Then the following statements are equivalent.*
(i) (Y, σ) is a semi-$T_{1/2}$ space,
(ii) For every space (X, τ) and every map $f : (X, \tau) \to (Y, \sigma)$, f is ap-semi-open.

Proof. $(i) \to (ii)$: Let B be a sg-closed subset of (Y, σ) and suppose that $B \subseteq f(A)$ where $A \in SO(X, \tau)$. Since (Y, σ) is a semi-$T_{1/2}$ space, B is semi-closed (i.e., $B = sCl(B)$). Therefore $sCl(B) \subseteq f(A)$. Then f is ap-semi-open.
$(ii) \to (i)$: Let B be a sg-closed subset of (Y, σ) and let X be the set Y with the topology $\tau = \{\emptyset, B, X\}$. Finally let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. By assumption f is ap-semi-open. Since B is sg-closed in (X, τ) and semi-open in (X, τ) and $B \subseteq f(B)$, it follows that $sCl(B) \subseteq f(B) = B$. Hence B is semi-closed in (Y, σ). Therefore (Y, σ) is a semi-$T_{1/2}$ space. □

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