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On an Integral Inequality of Feng Qi

Sobre una Desigualdad Integral de Feng Qi

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Abstract

In this note, we study a general version of a problem posed by Feng Qi in [10] in the context of a measured space endowed with a positive finite measure. For other studies and results, one can consult the papers [2], [3], [5], [8], [9], [12], [13] and [14]. Our basic tool is the classical Hölder inequality. By the convexity method (see [3]) we give an interpretation of the lower bound occuring in our main result (see Theorem 2.2 below).
Key words and phrases: Qi type integral inequalities, Hölder's inequality, Convexity method.

Resumen

En esta nota se estudia una versión general de un problema planteado por Feng Qi en [10], en el contexto de un espacio de medida provisto de una medida positiva finita. Para otros estudios y resultados se puede consultar los artículos [2], [3], [5], [8], [9], [12], [13] y [14]. Nuestra herramienta básica es la desigualdad clásica de Hölder. Por el método de convexidad (ver [3]) se da una interpretación de la cota inferior que aparece en nuestro resultado principal (ver Teorema 2.2).
Palabras y frases clave: desigualdades integrales de tipo Qi, desigualdad de Hölder, método de convexidad.

1 Introduction

In [10] Feng Qi obtained the following new integral inequality which is not found in [1], [4], [6] and [7]:

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Theorem A. *[10] Let $n \geq 1$ be an integer and suppose that f has a continuous derivative of the $n-th$ order on $[a, b]$, $f^{(i)}(a) \geq 0$ and $f^{(n)}(x) \geq n!$ where $0 \leq i \leq n-1$. Then*

$$\int_a^b [f(x)]^{n+2}\, dx \geq \left[\int_a^b f(x)\, dx \right]^{n+1}. \tag{1.1}$$

At the end of [10] the author proposed the following open problem

Problem 1. *Under what conditions does the inequality*

$$\int_a^b [f(x)]^t\, dx \geq \left[\int_a^b f(x)\, dx \right]^{t-1}. \tag{1.2}$$

hold for some $t > 1$?

Different answers and solutions can be found in [2], [3], [5], [8], [9], [12], [13] and [14].

N. Towghi in [13] has found sufficient conditions for (1.2) to hold. To recall the result of [13], we need some notations. Let $f^{(0)} = f$, $f^{(-1)} = \int_a^x f(s)\, ds$, and $[x]$ denote the greatest integer less than or equal to x. For $t \in (n, n+1]$, where n is a positive integer, let $\gamma(t) := t(t-1)(t-2)\dots(t-(n-1))$. For $t < 1$, let $\gamma(t) := 1$. With these notations, we have

Theorem B. *[13] Let $t > 1$, $x \in [a, b]$, and $f^{(i)}(a) \geq 0$ for $0 \leq i \leq [t-2]$. If $f^{([t-2])}(x) \geq \gamma(t-1)(x-a)^{t-[t]}$, then $(b-a)^{t-1} \leq \int_a^b f(x)\, dx$, and (1.2) holds.*

Let $n \geq 1$ be an integer and suppose that f satisfies the conditions of Theorem A. Then from $f^{(n)}(x) \geq n!$ and $f^{(i)}(a) \geq 0$ for $0 \leq i \leq n-1$, it follows that $f^{(i)}(t) \geq 0$ and are nondecreasing for $0 \leq i \leq n-1$. In particular, f is nonnegative. When $t \geq 2$, the assumptions of Theorem B also imply that f is nonnegative. The proof of (1.2) in this case is made by the use of the integral version of Jensen's inequality.
By using a lemma of convexity and Jensen's inequality, K.-W. Yu and F. Qi established in [14] the following result.

Theorem C. *[14] Let $t > 1$. Suppose that f is a continuous function on $[a, b]$ satisfying the following condition:*

$$\int_a^b f(x)\, dx \geq (b-a)^{t-1}. \tag{1.3}$$

Then we have

$$\int_a^b [f(x)]^t \, dx \geq \left[\int_a^b f(x) \, dx \right]^{t-1}. \tag{1.4}$$

In [9] T. Pogány found conditions sufficient for the more general inequality

$$\int_a^b [f(x)]^\alpha \, dx \geq \left[\int_a^b f(x) \, dx \right]^{\beta}. \tag{1.6}$$

to hold without assuming the differentiability on the function f and without using convexity criteria. In the paper [9], T. Pogány established some inequalities which are generalizations, reversed form, or weighted version of 1.2. In the paper [5], S Mazouzi and F. Qi established a functional inequality from which the inequality 1.2 and other integral or discrete inequalities can be deduced.

In this note we consider the following problem:

Problem 2. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω. Under what conditions does the inequality*

$$\int_\Omega [f(x)]^t \, d\mu(x) \geq \left[\int_\Omega f(x) \, \mu(x) \right]^{t-1}. \tag{1.6}$$

hold for $t > 1$?

This probleme is a particular case of the following

Problem 3. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω. Under what conditions does the inequality*

$$\int_\Omega [f(x)]^\alpha \, d\mu(x) \geq \left[\int_\Omega f(x) \, \mu(x) \right]^{\beta}. \tag{1.7}$$

hold for $\alpha, \beta \in (0, \infty)$?

The aim of this note is to study this problem. More precisely we shall provide sufficient conditions for inequality (1.8) to hold when $\alpha \geq \max\{1, \beta\}$ and nonnegative measurable function f.

This paper is organized as follows: In Section 2, we prove Theorem 2.2 in which we provide sufficient conditions for 1.8. We end this section by giving some consequences and corollaries. In Section 3, we use the convexity method to give an interpretation (see Theorem 3.1) of the bound $K_\Omega^{(\alpha,\beta)}(\mu)$ used in Thorem 2.2.

2 The result.

The basic tool we use here is Hölder's inequality (see for example [11], p. 60).

Theorem 2.1. *Let $(\Omega, \mathcal{F})$ be a measured space. Let ν be a positive measure on X. Let $p, q \in [1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f, g : \Omega \longrightarrow [0, \infty)$ be two measurable functions. Then we have the following inequality:*

$$\int_\Omega fg\, d\nu \leq \left[\int_\Omega f^p\, d\nu\right]^{\frac{1}{p}} \left[\int_\Omega g^q\, d\nu\right]^{\frac{1}{q}}. \tag{2.1}$$

Our solution to problem 3 is given by the following result.

Theorem 2.2. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω. Let α and β in $(0, \infty)$ be such that $\alpha > \max\{1, \beta\}$. Set*

$$K_\Omega^{(\alpha,\beta)}(\mu) := \big[\mu(\Omega)\big]^{\frac{\alpha-1}{\alpha-\beta}}.$$

Let $f : \Omega \to [0, \infty)$ be a measurable function satisfying the following condition:

$$\int_\Omega f\, d\mu \geq K_\Omega^{(\alpha,\beta)}(\mu). \tag{2.2}$$

Then we have

$$\int_\Omega f^\alpha\, d\mu \geq \left[\int_\Omega f\, d\mu\right]^\beta. \tag{2.3}$$

Proof. Let s be such that $\frac{1}{s} + \frac{1}{\alpha} = 1$. That is $s = \frac{\alpha}{\alpha-1}$. We apply Hölder's inequality in the space $(\Omega, \mathcal{F})$ for f and the constant function $g = 1$. Then we get

$$\int_\Omega f\, d\mu \leq [\mu(\Omega)]^{\frac{\alpha-1}{\alpha}} \left[\int_\Omega f^\alpha\, d\mu\right]^{\frac{1}{\alpha}}. \tag{2.4}$$

From (2.4) we get

$$\int_\Omega f^\alpha\, d\mu \geq < J_\Omega^{(\alpha,\beta)}(f) \left[\int_\Omega f\, d\mu\right]^\beta, \tag{2.5}$$

where

$$J_\Omega^{(\alpha,\beta)}(f) := [\mu(\Omega)]^{1-\alpha} \left[\int_\Omega f\, d\mu\right]^{\alpha-\beta}. \tag{2.6}$$

Now, from the assumption (2.2) it follows that $K_\Omega^{(\alpha,\beta)}(f) \geq 1$. Using this inequality in (2.5) we arrive at the desired result. □

By setting $\alpha = t > 1$ and $\beta = t - 1$, we obtain the following

Corollary 2.3. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω. Let $t > 1$. Suppose that f is a nonnegative measurable function on Ω satisfying the following condition:*

$$\int_\Omega f \, d\mu \geq (\mu(\Omega))^{t-1}$$

Then we have

$$\int_\Omega f^t \, d\mu \geq \left[\int_\Omega f \, d\mu \right]^{t-1}. \tag{2.7}$$

Corollary 2.4. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω. Let $g : \Omega \to [0, \infty)$ be a measurable function such that $\int_\Omega g \, d\mu < \infty$. Let α and β in $(0, \infty)$ be such that $\alpha > \max\{1, \beta\}$. Set*

$$K_\Omega^{(\alpha,\beta)}(g) := \left[\int_\Omega g \, d\mu \right]^{\frac{\alpha-1}{\alpha-\beta}}.$$

Let $f : \Omega \to [0, \infty)$ be a measurable function satisfying the following condition:

$$\int_\Omega fg \, d\mu \geq K_\Omega^{(\alpha,\beta)}(g) \tag{2.8}$$

Then we have

$$\int_\Omega g f^\alpha \, d\mu \geq \left[\int_\Omega g f \, d\mu \right]^\beta. \tag{2.9}$$

Proof. We use Theorem 2.2 for the finite measure $\nu := g\mu$ having g as density. □

The corollary 2.4 is a slight generalization of the result stated in corollary 3.1 of [5]. Below, we list other consequences of Theorem 2.2.

Corollary 2.5. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω. Let α and β in $(0, \infty)$ be such that $\alpha > \max\{1, \beta\}$. If $f \geq [\mu(\Omega)]^{\frac{\beta-1}{\alpha-\beta}}$, a.e in Ω. Then inequality (2.3) holds.*

Corollary 2.6. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω such that $\mu(\Omega) \leq 1$. Suppose that f is a nonnegative measurable function on Ω satisfying $\int_\Omega f \, d\mu \geq 1$. Then inequality (1.6) holds for all $t > 1$.*

Corollary 2.7. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω such that $\mu(\Omega) \leq 1$. Suppose that f is a measurable function on Ω satisfying $f \geq \frac{1}{\mu(\Omega)}$ $\mu-$a.e. on Ω. Then inequality (1.6) holds for all $t > 1$.*

Corollary 2.8. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω such that $\mu(\Omega) \leq 1$. Suppose that f is a nonnegative measurable function on Ω satisfying $\int_\Omega f\, d\mu \geq \mu(\Omega)$. Then inequality (1.6) holds for all $t \geq 2$.*

Corollary 2.9. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be a (nonzero) finite positive measure on Ω such that $\mu(\Omega) \geq 1$. Suppose that f is a nonnegative measurable function on Ω satisfying $\int_\Omega f\, d\mu \geq \mu(\Omega)$. Then inequality (1.6) holds for all $1 < t \leq 2$.*

Finally, let us apply Theorem 2.2 to derive the following generalization of the discrete inequality obtained in corollary 3.8 of [5].

Corollary 2.10. *Let α and β in $(0, \infty)$ be such that $\alpha > \max\{1, \beta\}$. Let $\{w_n\}$ and $\{z_n\}$ be two sequences of non negative real numbers such that*

$$\left[\sum_{n=1}^{\infty} w_n\right]^{\alpha-1} \leq \left[\sum_{n=1}^{\infty} w_n z_n\right]^{\alpha-\beta} < \infty, \tag{2.10}$$

then we have

$$\left(\sum_{n=1}^{\infty} w_n z_n\right)^{\beta} \leq \sum_{n=1}^{\infty} w_n z_n^{\alpha}. \tag{2.11}$$

Remark 1. In connection to the previous problems, the following problem should be considered:

Problem 4. *Let $(\Omega, \mathcal{F})$ be a measured space. Let μ be an infinite positive measure on Ω. (i.e. $\mu(\Omega) = \infty$). Under what conditions does the inequality $\int_\Omega [f(x)]^\alpha\, dx \geq \left[\int_\Omega f(x)\, dx\right]^\beta$ hold for $\alpha, \beta \in (0, \infty)$?*

3 A related problem treated by the convexity method.

In their paper [3], V. Csiszár and T. F. Móri proposed the use of the convexity method to treat moment-type inequalities and studied in particular (the direct or reversed) variants of inequality (1.7) of the problem posed by F. Qi. Although they stated their result (see Theorem 2.1 [3]) in the particular case of bounded intervals of the real line, it is easy to see that their result is valid in the context of measured space. Next, we shall discuss a problem related to Problem 3, by using the convexity method.

Let α and β in $(0,\infty)$ be such that $\alpha > \max\{1,\beta\}$. Let, as before, $(\Omega,\mathcal{F},\mu)$ be a measured space. Set $P := \frac{1}{\mu(\Omega)}\mu$, so that P is a probability on $(\Omega,\mathcal{F})$. Consider $f = X$ as a nonnegative random variable on Ω. Inequality (2.3) can be rewritten as

$$E(X^\alpha) \geq C(E(X))^\beta, \tag{3.1}$$

where $E(X) = \int_\Omega X\,dP$ is the expectation of X and $C := (\mu(\Omega))^{\beta-1}$.

Definition 3.1. Let $a(\alpha,\beta) \geq 0$. We say that $a(\alpha,\beta)$ is the abscissa of means of type I, if the following property holds true: for all nonnegative random variable X,

$$E(X) \geq a(\alpha,\beta) \Longrightarrow E(X^\alpha) \geq C(E(X))^\beta. \tag{$P(\alpha,\beta)$}$$

and $a(\alpha,\beta)$ is the smallest nonegative number satisfying this property.

We want to find the value of $a(\alpha,\beta)$. Our solution to this problem is given by the next result.

Theorem 3.1. *Let $(\Omega,\mathcal{F})$ be a measured space. Let μ be a finite positive measure on Ω. Let α and β in $(0,\infty)$ be such that $\alpha > \max\{1,\beta\}$. Then*

$$a(\alpha,\beta) = K_\Omega^{(\alpha,\beta)}(\mu) := \left[\mu(\Omega)\right]^{\frac{\alpha-1}{\alpha-\beta}}. \tag{3.2}$$

Proof. From Theorem 2.2 it follows that $K_\Omega^{(\alpha,\beta)}(\mu) \geq a(\alpha,\beta)$.
Let $\mathcal{P}_{(\alpha,\beta)}$ be the set of probability distributions of all random variables X on $(\Omega,\mathcal{F})$ such that $E(X) \geq a(\alpha,\beta)$. Then $\mathcal{P}_{(\alpha,\beta)}$ is a convex set. Consider the moment set

$$\mathcal{M}_{(\alpha,\beta)} := \{(E(X), E(X^\alpha)) : P_X \in \mathcal{P}_{(\alpha,\beta)}\},$$

where, P_X is the probability distribution of X. Since (3.1) holds then $\mathcal{M}_{(\alpha,\beta)}$ lies entirely above the graph of the function $x \geq a(\alpha,\beta)$, $x \mapsto Cx^\beta$. We denote $\mathcal{Q}_{(\alpha,\beta)}$ the set of the probability distributions:

$$\delta_s, \quad P_{x,y} := \frac{y-s}{y-s}\delta_x + \frac{s-x}{y-x}\delta_y,$$

where $s \geq a(\alpha, \beta)$, and $x < s < y$ (δ_z means the Dirac measure for all real z). Then $\mathcal{P}_{(\alpha,\beta)}$ coincides with the convex hull of $\mathcal{Q}_{(\alpha,\beta)}$. Therefore the moment set $\mathcal{M}_{(\alpha,\beta)}$ contains the convex hull of the set $\{(s, s^\alpha) : s \geq a(\alpha, \beta)\}$. Thus we have $s^\alpha \geq Cs^\beta$ for all $s \geq a(\alpha, \beta)$. In particular, we get $a(\alpha, \beta) \geq \mu\big(\Omega\big)^{\frac{\alpha-1}{\alpha-\beta}}$. Hence we have the desired equality (3.2). □

So Theorem 3.1 gives an explanation to the bound $K_\Omega^{(\alpha,\beta)}(\mu)$ used in Theorem 2.2.

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