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The Angular Distribution of Mass by Weighted Bergman Functions

Distribución Angular de Masa mediante Funciones de Bergman con Peso

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Abstract

Let $\mathbb{D}$ be the open unit disk in the complex plane. For $\varepsilon > 0$ we consider the sector $\Sigma_\varepsilon = \{z \in \mathbb{C} : |\arg z| < \varepsilon\}$. We prove that for every $\alpha \geq 0$ and for each $\varepsilon > 0$ there is a constant $K > 0$ depending only on α and ε such that for any function f in the weighted Bergman space A^1_α univalent on $\mathbb{D}$, and $f(0) = 0$, then

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)| dA_\alpha(z) > K\|f\|_{1,\alpha}.$$

This result extends a theorem of Marshall and Smith in [MS] for functions belonging to the unweighted Bergman space. We also prove that a such extension for α negative fails.
Key words and phrases: Bergman space, univalent functions, harmonic measure, hyperbolic metric.

Resumen

Sea $\mathbb{D}$ el disco unitario en el plano complejo. Sea $\varepsilon > 0$ y consideremos el sector $\Sigma_\varepsilon = \{z \in \mathbb{C} : |\arg z| < \varepsilon\}$. Probaremos que para cada $\alpha \geq 0$ y para cada $\varepsilon > 0$ existe una constante $K > 0$, que depende sólo

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de α y ε tal que para cualquier función f en el espacio de Bergman con peso A^1_α, univalente sobre $\mathbb{D}$, y con $f(0) = 0$, se cumple

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)| dA_\alpha(z) > K\|f\|_{1,\alpha}.$$

Este resultado extiende un teorema de Marshall y Smith [MS] para funciones en el espacio de Bergman sin peso. También probaremos que tal extensión falla para $\alpha < 0$.
Palabras y frases clave: Espacio de Bergman, funciones univalentes, medida armónica, métrica hiperbólica.

1 Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane. For $\alpha > -1$, the weighted Bergman space $A^1_\alpha := A^1_\alpha(\mathbb{D})$ is the class of all analytic functions in $\mathbb{D}$ which are in $L^1(\mathbb{D}, dA_\alpha)$, where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z),$$

$dA(z)$ being the usual two-dimensional Lebesgue measure on $\mathbb{D}$. Equipped with the norm

$$\|f\|_{1,\alpha} = \int_{\mathbb{D}} |f(z)| dA_\alpha(z) < +\infty,$$

$L^1(\mathbb{D}, dA_\alpha)$ is a Banach space containing A^1_α as a closed subspace.

For each $\varepsilon > 0$, we define the sector

$$\Sigma_\varepsilon = \{w \in \mathbb{C} \ : \ |\arg w| < \varepsilon\}.$$

In [MS], D. Marshall and W. Smith proved the following result

Theorem 1.1. *For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $f \in A^1$ is univalent and f(0) = 0, then*

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)| dA(z) > \delta\|f\|_1. \tag{1.1}$$

They also showed that A^1 cannot be replaced by A^p, $p > 1$ in Theorem 1.1. Moreover, it is an open problem whether or not Theorem 1.1 holds without assuming the hypothesis of univalence for the functions in A^1. In this paper, we extend Theorem 1.1 for weighted Bergman spaces in the unit disk, $0 < \alpha < \infty$, and give a counterexample showing that this result fails as $\alpha \in (-1, 0)$. We state our main result.

Theorem 1.2. *For every $\varepsilon > 0$ and for any $\alpha \geq 0$ there exists a $\delta > 0$, $\delta = \delta(\varepsilon, \alpha)$ such that if $f \in A^1_\alpha$ is univalent and f(0) = 0, then*

$$\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)| dA_\alpha(z) > \delta \|f\|_{1,\alpha}. \tag{1.2}$$

As in the proof of Theorem 1.1, the hyperbolic metric and harmonic measure play a crucial role in the proof of Theorem 1.2 as well. The *hyperbolic metric* on $\mathbb{D}$ is defined by

$$\beta(z_1, z_2) = \inf \left\{ \int_\gamma \frac{2|dz|}{1 - |z|^2} \ : \gamma \text{ is an arc in } \mathbb{D} \text{ from } z_1 \text{ to } z_2 \right\}.$$

In particular, the shortest distance from 0 to any point z is along the radius, and

$$\beta(0, z) = \log \left(\frac{1 + |z|}{1 - |z|} \right). \tag{1.3}$$

The hyperbolic metric is invariant under conformal self-maps of $\mathbb{D}$ and hence, for any z_1, z_2 in the unit disk, we have

$$\beta(z_1, z_2) = \log \frac{|1 - z_1 \overline{z_2}| + |z_1 - z_2|}{|1 - z_1 \overline{z_2}| - |z_1 - z_2|}.$$

The hyperbolic geodesics are diameters of the disk together with circles orthogonal to the unit circle. On any simple connected proper domain subset Ω of $\mathbb{C}$, the hyperbolic distance is defined in a natural way. If $\varphi : \mathbb{D} \to \Omega$ is any conformal map, the hyperbolic metric β_Ω in Ω is defined as

$$\beta_\Omega(w_1, w_2) = \beta(z_1, z_2),$$

where $w_i = \varphi(z_i)$, $i = 1, 2$. The shortest arc in $\mathbb{D}$ from z_1 to z_2 is the arc of the unique circle orthogonal to $\partial\mathbb{D}$ passing through z_1 and z_2. Its image by φ is the shortest arc in Ω from w_1 to w_2. If $E \subset \Omega$, then the hyperbolic distance from w to E will be denoted by $\beta_\Omega(w, E)$.

In general, the hyperbolic distance is not explicitly computable in terms of the geometry of Ω. A useful substitute is the *quasi-hyperbolic distance* on Ω introduced by F. Gehring and B. Palka in [GP]. For w_1 and w_2 in Ω, the quasi-hyperbolic distance from w_1 to w_2 is defined to be

$$k_\Omega(w_1, w_2) := \inf_\gamma \int_\gamma \frac{|dw|}{\delta_\Omega(w)}, \tag{1.4}$$

where the infimum is taken over all arcs γ in Ω from w_1 to w_2, and $\delta_\Omega(w)$ being the Euclidean distance from w to the boundary $\partial\Omega$ of Ω. For any $w_1, w_2 \in \Omega$ the following property holds (see [Po, page 92])

$$\frac{1}{2}\beta_\Omega(w_1, w_2) \le k_\Omega(w_1, w_2) \le 2\beta_\Omega(w_1, w_2). \tag{1.5}$$

Assume that O is an open set and let E be a closed set of the Riemann sphere. We denote by Γ that part of the boundary of $O \setminus E$ which is contained in E. We can assume that the geometrical situation is so simple that for $z \in O \setminus E$ we can calculate $\omega(z, \Gamma, O \setminus E)$. In these conditions, we define the harmonic measure of E in $z \in O \setminus E$ to be

$$\omega(z, E, O) := \omega(z, \Gamma, O \setminus E).$$

It should be noted that ω is the unique bounded harmonic function in $O \setminus E$ that is identically 1 on E, and vanishing on $\partial(\Omega \setminus E) \setminus E$.

We use arguments by Marshall and Smith in [MS] but introducing the necessary modifications because of the presence of the weight. We have analyzed their methods and techniques in depth and we have put especial care in illustrating some geometric constructions. Moreover, it has been our task to give the constants explicitly enough so that, in each occurrence, the parameters on which they depend are rather clear. In order that the paper becomes self-contained, in Section 2 we reproduce in detail some tools used by Marshall and Smith. The proof of Theorem 1.2 is given in Section 3, while in Section 4 we supply a counterexample showing that our main result cannot be extended when α is negative.

2 Background

In this section we collect several results that we will need for our goals. Firstly, we explain a contruction due to Marshall and Smith providing a covering for the range of a univalent function in A^1. After, we include in detail some useful area estimates for subsets of the unit disk which will be applied to establish further properties for the pieces of the covering of the image domain.

2.1 A covering for the image domain.

Fix $\varepsilon < 1/10$, for any conformal map f keeping invariant the origin, we can cover the domain $\Omega = f(\mathbb{D})$ by a countable colection of subsets Ω_n according to a subtle geometric construction by Marshall and Smith introduced in [MS].

Since such construction plays a fundamental role in the rest of the paper, we will give it in detail.

Without loss of generality we can assume that $\delta_\Omega(0) = 1$. We put $A_0 = \mathbb{D}$, and for $n \geq 1$, let us consider the rings

$$A_n = \left\{ w \in \mathbb{C} : (1+\varepsilon)^{n-1} < |w| < (1+\varepsilon)^n \right\}, \qquad n = 1, 2, \ldots \tag{2.6}$$

For each $n \in \mathbb{N}$, we choose, if possible, an Euclidean square $Q_n \subset A_n \cap \Sigma_\varepsilon \cap \Omega$ satisfying

$$\begin{cases} (i) & \operatorname{diam}(Q_n) \geq \frac{\varepsilon}{4}(1+\varepsilon)^{n-1}, \\ (ii) & \dfrac{1}{2} \leq \dfrac{\operatorname{dist}(Q_n, \partial(A_n \cap \Sigma_\varepsilon \cap \Omega))}{\operatorname{diam}(Q_n)} \leq 2, \end{cases} \tag{2.7}$$

where $w_n = f(z_n)$ denotes the centre of the square Q_n.

It is not difficult to check that $\delta_\Omega(0) = 1$ implies that there is a square $Q_0 \subset \mathbb{D} \cap \Sigma_\varepsilon$ satisfying (i) and (ii) in (2.7). We remark that, according to the geometry of Ω, many annuli may not contain one of these squares Q_n (satisfying the properties in (2.7)).

For those $n \in \mathbb{N}$ for which we can construct a square Q_n with the properties in (2.7), we define the set $N(Q_n)$ as

$$N(Q_n) := \left\{ w \in \mathbb{C} : \beta_\Omega(w, w_n) < \frac{100}{\varepsilon} \right\}.$$

Note that $N(Q_n)$ is a hyperbolic neighbourhood of Q_n. Indeed, choosing any $w \in Q_n$, by (1.5), we can write

$$\beta_\Omega(w, w_n) \leq 2k_\Omega(w, w_n) \leq 2 \int_{[w, w_n]} \frac{|ds|}{\delta_{\Omega \cap A_n \cap \Sigma_\varepsilon}(s)},$$

where we have used that the segment $[w, w_n] \subset Q_n \subset \Omega \cap A_n \cap \Sigma_\varepsilon$. Now, it is easy to see that

$$\beta_\Omega(w, w_n) \leq \frac{16}{\varepsilon(1+\varepsilon)^{n-1}} \int_{[w, w_n]} |ds| \leq \frac{8}{\varepsilon(1+\varepsilon)^{n-1}} \operatorname{diam}(Q_n) \leq 8, \tag{2.8}$$

so that $Q_n \subset N(Q_n)$.

We are ready to define the covering $\{\Omega_n\}$ for the domain Ω. For those $n \in \mathbb{N}$ for which we can construct a square Q_n, we say that $w \in \Omega$ is an element of Ω_n if

(a) $\beta_\Omega(\gamma_w, w_n) < 100/\varepsilon$, and

(b) If γ_w^n is the component of $\gamma_w \backslash N(Q_n)$ containing w, then either $\beta(\gamma_w, w_m) \geq 100/\varepsilon$ for all $n \neq m$ or else γ_w^n is empty.

Here γ_w denotes the hyperbolic geodesic from 0 to w. If there is no Q_n in A_n, we set $\Omega_n = \emptyset$. In other words, a point $w \in \Omega_n$ iff $N(Q_n)$ is the first hyperbolic neighbourhood that γ_w finds when running from w to the origin. It is easy to see that $N(Q_n) \subset \Omega_n$, and since $0 \in N(Q_0)$, the family $\{\Omega_n\}$ is a covering of Ω. Note that if $w \in \Omega$ and $w \notin \Omega_n$, $n > 0$, γ_w just meets $N(Q_0)$ which means that $w \in \Omega_0$.

2.2 Some area estimates.

We start with some upper and lower estimates for areas of subsets of the unit disk involving the harmonic measure and the hyperbolic metric. Interesting by their own right, they will also be useful in the proof of Theorem 1.2 and lead to prove other estimates for the Q_n squares.

Lemma 2.1 (Upper area estimate). *Let E be a measurable subset of the unit disk $\mathbb{D}$. Then*

$$\operatorname{Area}(E) \leq 6\pi e^{-\beta(0,E)} \omega(0, E). \tag{2.9}$$

PROOF: We observe that if $0 \in E$, then $e^{-\beta(0,E)} = \omega(0, E) = 1$ and, since $\operatorname{Area}(E) \leq \operatorname{Area}(\mathbb{D}) = \pi$, (2.9) follows. Hence we can assume that $0 \notin E$. Let $E^* = \left\{ \frac{z}{|z|} : z \in E \right\}$ be the radial projection of E on $\partial\mathbb{D}$ and set $C := \left\{ z \in \mathbb{D} \,:\, \beta(0, z) \geq \beta(0, E) \text{ and } \frac{z}{|z|} \in E^* \right\}$. Since $\beta(0, z) \geq \beta(0, E)$ and $\frac{z}{|z|} \in E^*$ whenever for $z \in E$, we have that $E \subset C$. Moreover

$$\operatorname{Area}(E) \leq \operatorname{Area}(C) = \frac{|E^*|}{2}\left(1 - d^2\right),$$

where $|E^*|$ denotes the length of E^* and $d := \inf\{|z| : z \in E\}$ so that

$$e^{-\beta(0,E)} = \frac{1-d}{1+d}.$$

(cf. (1.3)). Therefore

$$\operatorname{Area}(E) \leq \operatorname{Area}(C) = \frac{|E^*|}{2}(1 + d)^2 e^{-\beta(0,E)} \leq 2e^{-\beta(0,E)} |E^*|. \tag{2.10}$$

Now, by Hall's Lemma (cf. [Du, p. 209]) we deduce

$$\frac{|E^*|}{2\pi} = \omega\left(0, E^*\right) \leq \frac{3}{2}\omega\left(0, E\right),$$

which, with (2.10), implies

$$\operatorname{Area}\left(E\right) \leq 6\pi e^{-\beta(0,E)}\omega\left(0, E\right).$$

Lemma 2.2 (Lower area estimate). *Let $E \subset \mathbb{D}$ be a hyperbolic ball with hyperbolic radius bigger than ρ_0. Then*

$$\operatorname{Area}\left(E\right) \geq \frac{\pi}{64}\left(1 - e^{-\rho_0}\right)^2 e^{-\beta(0,E)}\omega\left(0, E\right). \tag{2.11}$$

PROOF: Suppose that $0 \notin E$ and let $d = \inf\{|z| : z \in E\}$ so that

$$e^{-\beta(0,E)} = \frac{1-d}{1+d}. \tag{2.12}$$

We denote by Γ the circle orthogonal to the unit circle separating E from 0 with $\beta\left(0, \Gamma\right) = \beta\left(0, E\right)$, and let I be the subarc of $\partial\mathbb{D}$, subtended by Γ and separated from 0 by Γ. Since Γ is orthogonal to $\partial\mathbb{D}$, it is well known that

$$\omega\left(z, I\right) \equiv \frac{1}{2},$$

for any $z \in \Gamma$.

In this setting, the function u defined as

$$u\left(z\right) = \omega\left(z, \Gamma, V\right) - 2\omega\left(z, I, \mathbb{D}\right),$$

where V is the region bounded by Γ and $\partial\mathbb{D} \setminus I$ is harmonic on V and satisfies that $u\left(z\right) \equiv 0$ for $z \in \Gamma \cup (\partial\mathbb{D} \setminus I)$, and by the maximum principle, $u\left(z\right) \equiv 0$ for all z in V. In particular, $u\left(0\right) = 0$, and since E is inside the disk whose boundary is Γ, we can write

$$\omega\left(0, E\right) \leq \omega\left(0, \Gamma\right) = 2\omega\left(0, I\right) = \frac{|I|}{\pi}.$$

We claim that

$$\omega\left(0, E\right) \leq \frac{|I|}{\pi} \leq 4\left(1 - d\right). \tag{2.13}$$

Indeed, for $0 < d < \frac{1}{2}$, (2.13) is obvious since $|I| \leq 2\pi$., On the other hand, for $\frac{1}{2} \leq d < 1$, by construction, the radius of the circle Γ is $\dfrac{1-d^2}{2d}$; therefore, one-half of the area of the sector in the disc $\mathbb{D}$ (whose border is I) is less than the area of the triangle whith vertices the origin, the center of the circle Γ, and one point of intersection of $\partial\mathbb{D}$ and Γ. So that

$$\frac{|I|}{4} \leq \frac{1-d^2}{4d} \leq 1-d.$$

Hence (2.13) holds for all $d \in (0,1)$.

Now, since E is a hyperbolic ball with hyperbolic radius $r > \rho_0 > \rho_0/2$ and hyperbolic centre at a, in particular E is an Euclidean ball whose Euclidean radius and centre are

$$R = \frac{\left(1-|a|^2\right)}{1-s^2|a|^2}s, \qquad C = \frac{1-s^2}{1-s^2|a|^2}a,$$

respectively, where $s = \tanh\ r$. Moreover, by construction, $d = |C| - R$ and so

$$1-d = \frac{1-s^2|a|^2-\left(1-s^2\right)|a|+\left(1-|a|^2\right)s}{1-s^2|a|^2} \leq \frac{2\left(1-|a|^2\right)}{1-s^2|a|^2}.$$

Then

$$\begin{aligned}
\operatorname{diam}(E) &= 2R \geq s(1-d) \\
&\geq (1-d)\tanh\left(\frac{\rho_0}{2}\right) \geq \frac{1}{2}(1-d)\left(1-e^{-\rho_0}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\operatorname{Area}(E) &= \frac{\pi}{4}\operatorname{diam}(E)^2 \geq \frac{\pi}{16}(1-d)^2\left(1-e^{-\rho_0}\right)^2 \\
&= \frac{\pi}{16}\left(1-e^{-\rho_0}\right)^2(1+d)\,e^{-\beta(0,E)}(1-d) \qquad \text{(by 2.12)} \\
&\geq \frac{\pi}{64}\left(1-e^{-\rho_0}\right)^2 e^{-\beta(0,E)}\omega(0,E),
\end{aligned}$$

which gives (2.11). In the last inequality we have used (2.13).

If $0 \in E$, then $\beta(0,E) = \omega(0,E) = 1$, $d = 0$ and, in this case, we obtain

$$\operatorname{Area}(E) = \frac{\pi}{4}\operatorname{diam}(E)^2 \geq \frac{\pi}{64}\left(1-e^{-\rho_0}\right)^2.$$

We are done.

Remark 2.3. A similar estimate to (2.11) also holds for a closed square $Q \subset \mathbb{D}$ containing a hyperbolic ball with radius bigger than ρ_0. Indeed, an application of the same argument yields

$$\omega(0, Q) \leq \omega(0, \Gamma) \leq 4(1 - d), \tag{2.14}$$

where $d = \inf\{|z| : z \in Q\}$, and satisfying

$$e^{-\beta(0,Q)} = \frac{1 - d}{1 + d}.$$

Note that Q is contained in a hyperbolic (and therefore Euclidean) disk $E \subset \mathbb{D}$ so that $\operatorname{diam}(Q) = \operatorname{diam}(E)$. Then, if $d_E = \inf\{|z| : z \in E\}$ so that

$$e^{-\beta(0,E)} = \frac{1 - d_E}{1 + d_E},$$

then $d_E \leq d$, and by Lemma 2.2, it follows that

$$\operatorname{diam}(Q) = \operatorname{diam}(E) \geq \frac{1}{2}(1 - d_E)\left(1 - e^{-\rho_0}\right) \geq \frac{1}{2}(1 - d)\left(1 - e^{-\rho_0}\right).$$

Since $\operatorname{Area}(Q) = \sqrt{2}\operatorname{diam}^2(Q)$, by (2.14) and the definition of d we have that

$$\begin{aligned} \operatorname{Area}(Q) &\geq \frac{\sqrt{2}}{4}(1 - d)^2\left(1 - e^{-\rho_0}\right)^2 \\ &\geq \frac{\sqrt{2}}{16}\left(1 - e^{-\rho_0}\right)^2 e^{-\beta(0,Q)}\omega(0, Q). \end{aligned}$$

2.3 Further Properties for the Q_n's Squares.

Lemma 2.4. *Suppose that $n \in \mathbb{N}$ is such that we can build up its respective Q_n. Then, Q_n contains a hyperbolic ball with radius bigger than $\frac{\sqrt{2}}{32}\varepsilon$.*

PROOF: By (2.7) we know that Q_n contains an Euclidean disk Δ_n with radius at least $\frac{1}{8\sqrt{2}}\varepsilon(1 + \varepsilon)^{n-1}$ (note that $\operatorname{diam}(Q_n) \geq \frac{\varepsilon}{4}(1 + \varepsilon)^{n-1}$). Take a curve γ connecting the centre of Δ_n with its boundary and observe that for $s \in \gamma$ we have

$$\begin{aligned} \delta_\Omega(s) &= \operatorname{dist}(s, \partial\Omega) \leq \delta_\Omega(0) + |s| \\ &\leq 1 + (1 + \varepsilon)^n \\ &\leq 2(1 + \varepsilon)^n, \end{aligned}$$

where we have used that $s \in Q_n \subset A_n$. If $w \in \Delta_n$ then

$$\begin{aligned}\beta_{\mathbb{D}}(z, z_n) = \beta_\Omega(w, w_n) &\geq \frac{1}{2} k_\Omega(w, w_n) = \frac{1}{2} \inf_\gamma \int_\gamma \frac{|ds|}{\delta_\Omega(s)} \\ &\geq \frac{1}{2(1+\varepsilon)^n} |\gamma| \geq \frac{\sqrt{2}}{32}\varepsilon.\end{aligned}$$

Here we have used that $0 < \varepsilon < \frac{1}{10}$ and that the curve γ is longer than any radius in Δ_n. Consequently, it is clear that Q_n contains a hyperbolic ball with radius at least $\frac{\sqrt{2}}{32}\varepsilon$.

Lemma 2.5. *Suppose that Γ_1 is a circle orthogonal to $\partial\mathbb{D}$ separating 0 from a subset $E \subset \mathbb{D}$, and so that $e^{-\beta(0,\Gamma_1)} \leq 1/4$. If $R = \left[0, e^{i\alpha}\right]$ is the radius orthogonal to Γ_1, let Γ_0 be the circle orthogonal both to $\partial\mathbb{D}$ and R satisfying*

$$e^{-\beta(0,\Gamma_0)} = 2e^{-\beta(0,\Gamma_1)}.$$

Let $\xi_0 = \Gamma_0 \cap R$ so that $\beta(0, \xi_0) = \beta(0, \Gamma_0)$. Then

$$\sup_{\xi \in \Gamma_0} \omega(\xi, E) \leq C\omega(\xi_0, E).$$

PROOF: Due to the conformal invariance, we can assume that $e^{-\beta(0,\Gamma_1)} = 1/4$ and $\xi_0 > 0$. Note that it determines Γ_0 and Γ_1; in fact, $\xi_0 = 1/3$, $\xi_1 = \Gamma_1 \cap R = 3/5$ and, consequently, the Euclidean distance from $\Gamma_0 \cap \mathbb{D}$ to $\Gamma_1 \cap \mathbb{D}$ is $4/15$. Let U be the region in $\mathbb{D}$ bounded by $\partial\mathbb{D}$ and Γ_1, and containing Γ_0. Let $\varphi : U \to \mathbb{D}$ be a conformal map with $\varphi(\xi_0) = 0$ and put $I = \varphi(\Gamma_1 \cap \mathbb{D}) \subset \partial\mathbb{D}$.

If $\varphi^{-1}(z) \in \Gamma_0 \cap \mathbb{D}$, then $\omega\left(\varphi^{-1}(z), E\right)$ is a harmonic function in $\mathbb{D}$ vanishing on $\partial\mathbb{D}\backslash I$. This means that it is the solution of the Dirichlet problem, with boundary data 0, for $\xi \in \partial\mathbb{D}\backslash I$, and $u(\xi) \geq 0$, if $\xi \in I$, where u is some continuous function. Then

$$\omega\left(\varphi^{-1}(z), E\right) = \int_I \frac{1 - |z|^2}{|\xi - z|^2} u(\xi) \frac{|d\xi|}{2\pi} = \int_I \frac{1 - |z|^2}{|\xi - z|^2} d\mu(\xi),$$

where $d\mu(\xi) = u(\xi)\dfrac{|d\xi|}{2\pi}$ is a positive measure. Since $\xi = \varphi(s_1) \in I = \varphi(\Gamma_1 \cap \mathbb{D})$ and $z = \varphi(s_0) \in \varphi(\Gamma_0 \cap \mathbb{D})$ we can assert that the Euclidean distance from $\varphi(\Gamma_0 \cap \mathbb{D})$ to I is positive, so that there exists a constant $C > 0$ so that

$$\frac{1 - |z|^2}{|\xi - z|^2} \leq C,$$

for any $\xi \in I$ and $z \in \varphi(\Gamma_0 \cap \mathbb{D})$. Integrating this inequality over I against the measure $d\mu(\xi)$ we find that for all $\varphi^{-1}(z) \in \Gamma_0 \cap \mathbb{D}$,

$$\omega\left(\varphi^{-1}(z), E\right) \leq C \int_I d\mu(\xi) = C\omega(\xi_0, E)$$

since $\varphi^{-1}(0) = \xi_0$. We obtain that

$$\sup_{z \in \Gamma_0} \omega(z, E) \leq C\omega(\xi_0, E).$$

Lemma 2.6. *There exists a constant $K_0(\varepsilon) > 0$ so that for any $j, n \in \mathbb{N} \cup \{0\}$,*

$$\omega_\Omega(0, \Omega_n \cap A_j) \leq K_0(\varepsilon)\, \omega_\Omega(0, Q_n)\, \omega_\Omega(w_n, \Omega_n \cap A_j). \tag{2.15}$$

PROOF: Firstly, we will assume that $\beta_\Omega(0, N(Q_n)) < \log 4$. By Lemma 2.1 and the conformal invariance of the harmonic measure we can write

$$\omega_\Omega(0, Q_n) \geq \frac{1}{6\pi} e^{\beta_\Omega(0,Q_n)} \operatorname{Area}\left(f^{-1}(Q_n)\right);$$

according to the definition of $N(Q_n)$, and using the triangle inequality, we have that $\beta_\Omega(0, Q_n) \geq \beta_\Omega(0, z_n) - \frac{100}{\varepsilon}$, and by Definition 1.3 we obtain

$$e^{\beta_\Omega(0,Q_n)} \geq e^{-100/\varepsilon} \left(1 - |z_n|^2\right)^{-1}.$$

Now, by Lemma 2.4, $f^{-1}(Q_n)$ contains a hyperbolic ball Δ_n centered at z_n with hyperbolic radius at least $\frac{\sqrt{2}}{32}\varepsilon$. Then

$$\operatorname{Area}\left(f^{-1}(Q_n)\right) \geq \operatorname{Area}(\Delta_n) \geq \pi \tanh^2\left(\frac{\sqrt{2}}{32}\varepsilon\right)\left(1 - |z_n|^2\right)^2$$

since, we recall once again, any hyperbolic ball is also an Euclidean ball. We get that

$$\omega_\Omega(0, Q_n) \geq \frac{1}{6} e^{-100/\varepsilon} \tanh^2\left(\frac{\sqrt{2}}{32}\varepsilon\right)\left(1 - |z_n|^2\right). \tag{2.16}$$

Now, there is a $w \in N(Q_n)$ so that

$$\beta_\Omega(0, w) < \beta_\Omega(0, N(Q_n)) + \varepsilon < \log 4 + \varepsilon,$$

and using the triangle inequality we find that

$$\beta_\Omega(0, w_n) \leq \beta_\Omega(0, w) + \beta_\Omega(w, w_n) < \log(4) + \varepsilon + \frac{100}{\varepsilon}.$$

By Definition 1.3 and the conformal invariance of the hyperbolic metric, it results that

$$\frac{1-|z_n|}{1+|z_n|} > \frac{1}{4}e^{-\left(\varepsilon+\frac{100}{\varepsilon}\right)}.$$

From this inequality, using Harnack inequality and (2.16), it follows that

$$\begin{aligned}
&\omega_\Omega\left(0,Q_n\right)\omega_\Omega\left(w_n,\Omega_n\cap A_j\right)\\
&\qquad\geq \frac{1}{6}e^{-100/\varepsilon}\tanh^2\left(\frac{\sqrt{2}}{32}\varepsilon\right)\left(1-|z_n|\right)^2\omega_\Omega\left(0,\Omega_n\cap A_j\right)\\
&\qquad\geq \frac{1}{96}\tanh^2\left(\frac{\sqrt{2}}{32}\varepsilon\right)e^{-2\varepsilon-300/\varepsilon}\omega_\Omega\left(0,\Omega_n\cap A_j\right),
\end{aligned}$$

and the estimate holds for this case.

Now, let us go to assume that $\exp\left(-\beta_\Omega\left(0,N\left(Q_n\right)\right)\right)\leq\frac{1}{4}$. Let Γ_n be the hyperbolic geodesic in $\mathbb{D}$ separating 0 from $f^{-1}\left(N\left(Q_n\right)\right)$, orthogonal to the radius in $\mathbb{D}$ through z_n and satisfying $\exp\left(-\beta\left(0,\Gamma_n\right)\right)=2\exp\left(-\beta\left(0,N\left(Q_n\right)\right)\right)$. Put $B=\sup_{\xi\in\Gamma_n}\omega\left(\xi,f^{-1}\left(\Omega_n\cap A_j\right)\right)$, and let I be the subarc in $\partial\mathbb{D}$ subtended by Γ_n and consider the harmonic function in $\mathbb{D}$ given by

$$u\left(z\right)=\omega\left(z,f^{-1}\left(\Omega_n\cap A_j\right)\right)-B\omega\left(z,\Gamma_n\right).$$

Note that $u\left(z\right)\equiv 0$ for $z\in\partial\mathbb{D}\setminus I$, and for $z\in\Gamma_n$ we have that $u\left(z\right)=\omega\left(z,f^{-1}\left(\Omega_n\cap A_j\right)\right)-B\leq 0$, by the definition of B. By the maximum principle $u\left(z\right)\leq 0$ for all z in the region bounded by $\partial\mathbb{D}\setminus I$ and Γ_n; in particular, $u\left(0\right)\leq 0$ and so

$$\omega\left(0,f^{-1}\left(\Omega_n\cap A_j\right)\right)\leq\omega\left(0,\Gamma_n\right)\sup_{\xi\in\Gamma_n}\omega\left(\xi,f^{-1}\left(\Omega_n\cap A_j\right)\right). \tag{2.17}$$

To estimate the first factor in the right-hand side of (2.17), note that, by construction, $\beta\left(0,f^{-1}\left(Q_n\right)\right)\geq\beta\left(0,\Gamma_n\right)$, and, since $f^{-1}\left(Q_n\right)$ contains a hyperbolic ball of radius at least $\frac{\sqrt{2}}{32}\varepsilon$ we have

$$\begin{aligned}
\omega_\Omega\left(0,Q_n\right) &= \omega\left(0,f^{-1}\left(Q_n\right)\right)\geq\frac{1}{6\pi}e^{\beta\left(0,f^{-1}\left(Q_n\right)\right)}\operatorname{Area}\left(f^{-1}\left(Q_n\right)\right)\\
&\geq \frac{1}{6\pi}e^{\beta\left(0,\Gamma_n\right)}\operatorname{Area}\left(f^{-1}\left(Q_n\right)\right)\\
&\geq \frac{1}{96}\left(1-e^{-\frac{\sqrt{2}}{32}\varepsilon}\right)^2\frac{\operatorname{Area}\left(f^{-1}\left(Q_n\right)\right)}{\operatorname{Area}\left(\widetilde{\Gamma_n}\right)}\omega\left(0,\Gamma_n\right),
\end{aligned}$$

$\widetilde{\Gamma_n}$ being the disc whose boundary is Γ_n. Note that, by construction, the radius of $\widetilde{\Gamma_n}$ is $r = \left(1-d^2\right)/(2d)$; hence

$$\frac{1+d}{1-d} = e^{\beta(0,\Gamma_n)} \geq 2,$$

where we have used that $\exp\left(-\beta\left(0,\Gamma_n\right)\right) = 2\exp\left(-\beta_\Omega\left(0,N\left(Q_n\right)\right)\right)$; so $d \geq \frac{1}{3}$, $\operatorname{Area}\left(\widetilde{\Gamma_n}\right) \leq 9\pi\left(1-d\right)^2$ and

$$1-d = (1+d)\, e^{-\beta(0,\Gamma_n)} \leq 4e^{-\beta_\Omega(0,N(Q_n))}.$$

Take now $w \in N\left(Q_n\right)$ such that $\beta_\Omega\left(0,w\right) < \beta_\Omega\left(0,N\left(Q_n\right)\right) + \varepsilon$; by the definition of $N\left(Q_n\right)$ and the triangle inequality we have $\beta_\Omega\left(0,N\left(Q_n\right)\right) \geq \beta\left(0,z_n\right) - \left(\varepsilon + \frac{100}{\varepsilon}\right)$; therefore,

$$1-d \leq 4e^{\varepsilon+\frac{100}{\varepsilon}} e^{-\beta(0,z_n)} = 4e^{\varepsilon+\frac{100}{\varepsilon}} \frac{1-|z_n|}{1+|z_n|} \leq 4e^{\varepsilon+\frac{100}{\varepsilon}} \left(1-|z_n|^2\right),$$

and

$$\operatorname{Area}\left(\widetilde{\Gamma_n}\right) \leq 144\pi e^{2\varepsilon+200/\varepsilon}\left(1-|z_n|^2\right)^2.$$

Since

$$\operatorname{Area}\left(f^{-1}\left(Q_n\right)\right) \geq \pi\tanh^2\left(\frac{\sqrt{2}}{32}\varepsilon\right)\left(1-|z_n|^2\right)^2,$$

we get

$$\omega_\Omega\left(0,Q_n\right) \geq \frac{1}{13824}\left(1-e^{-\frac{\sqrt{2}}{32}\varepsilon}\right)^2 e^{-2\varepsilon-200/\varepsilon}\tanh^2\left(\frac{\sqrt{2}}{32}\varepsilon\right)\omega\left(0,\Gamma_n\right). \tag{2.18}$$

To estimate the second factor in the right-hand side of (2.17) we use Lemma 2.6 and Harnack's inequality to obtain

$$\begin{aligned} \sup_{\zeta\in\Gamma_n}\omega\left(\zeta, f^{-1}\left(\Omega_n\cap A_j\right)\right) &\leq C\omega\left(\zeta_n, f^{-1}\left(\Omega_n\cap A_j\right)\right) \\ &\leq \tfrac{2C}{|\zeta_n|}\omega\left(z_n, f^{-1}\left(\Omega_n\cap A_j\right)\right) \\ &\leq 6C\omega\left(z_n, f^{-1}\left(\Omega_n\cap A_j\right)\right) \\ &= 6C\omega_\Omega\left(w_n, \Omega_n\cap A_j\right) \end{aligned} \tag{2.19}$$

where $\xi_n \in \Gamma_n$ satisfies $\beta(0, \xi_n) = \beta(0, \Gamma_n)$ (note, then, that $|\zeta_n| \geq \frac{1}{3}$). Replacing (2.18) and (2.19) in (2.17) we have

$$\begin{aligned}\omega_\Omega(0, \Omega_n \cap A_j) &\leq \omega(0, \Gamma_n) \sup_{\xi \in \Gamma_n} \omega\left(\xi, f^{-1}(\Omega_n \cap A_j)\right) \\ &\leq C(\varepsilon)\, \omega_\Omega(0, Q_n)\, \omega_\Omega(w_n, \Omega_n \cap A_j),\end{aligned}$$

where

$$C(\varepsilon) = 256C\left(1 - e^{-\frac{\sqrt{2}}{32}\varepsilon}\right)^{-2} \tanh^{-2}\left(\frac{\sqrt{2}}{32}\varepsilon\right) e^{2\varepsilon + \frac{200}{\varepsilon}},$$

and the proof is complete.

3 Proof of Theorem 1.2.

The idea to prove (1.2) is to establish adequate integral inequalities for each pair (Q_n, Ω_n) with constants independent of n and f. This is done in Lemma 3.1 below.

Lemma 3.1 (Main Lemma). *For $\alpha > 0$ fixed and given $\varepsilon > 0$, there exists a constant $K(\varepsilon, \alpha) > 0$ such that for any $f \in A^1_\alpha$, univalent with $f(0) = 0$, then*

$$\int_{f^{-1}(Q_n)} |f(z)|\, dA_\alpha(z) \geq K(\varepsilon, \alpha) \int_{f^{-1}(\Omega_n)} |f(z)|\, dA_\alpha(z) \tag{3.20}$$

for all $n = 1, 2, \ldots$

Taking Lemma 3.1 for granted, Theorem 1.2 follows as a simple consequence of it. Indeed, since $\bigcup Q_n \subset \Sigma_\varepsilon$ and $\mathbb{D} = \bigcup f^{-1}(\Omega_n)$, we can write

$$\begin{aligned}\int_{f^{-1}(\Sigma_\varepsilon)} |f(z)|\, dA_\alpha(z) &\geq \int_{f^{-1}(\cup Q_n)} |f(z)|\, dA_\alpha(z) \\ &= \sum_{n \geq 0} \int_{f^{-1}(Q_n)} |f(z)|\, dA_\alpha(z) \\ &\geq K(\varepsilon, \alpha) \sum_{n \geq 0} \int_{f^{-1}(\Omega_n)} |f(z)|\, dA_\alpha(z) \\ &\geq K(\varepsilon, \alpha) \int_{\cup f^{-1}(\Omega_n)} |f(z)|\, dA_\alpha(z) \\ &= K(\varepsilon, \alpha) \int_{\mathbb{D}} |f(z)|\, dA_\alpha(z) \\ &= K(\varepsilon, \alpha) \|f\|_{1,\alpha}\end{aligned}$$

which is (1.2) and Theorem 1.2 would be proven.

To prove Lemma 3.1 we will require an estimate that implicitly appears in [MS, pp. 104-110] and that we bring out in Lemma 3.2 below.

Lemma 3.2 (Marshall-Smith). *There exists an absolute positive constant $K_1 \leq \frac{1}{3\pi 10^{10}}$ and a constant $K_2(\varepsilon) \geq \frac{16}{\pi}(1+\varepsilon)^{(1+K_1\varepsilon)}$ (depending only on ε) such that*

$$e^{-\beta_\Omega(w_n,\Omega_n\cap A_j)}\omega_\Omega(w_n,\Omega_n\cap A_j) \leq K_2(\varepsilon)(1+\varepsilon)^{-|j-n|(1+K_1\varepsilon)},$$

for all $j,n\in\mathbb{N}$. For $n=0$ we replace w_n by 0.

To start with the proof of Lemma 3.1, first we consider the case $n=0$. Let us see that there is a constant $K_3(\varepsilon,\alpha)>0$ such that

$$\int_{f^{-1}(Q_0)} |f(z)|\, dA_\alpha(z) \geq K_3(\varepsilon,\alpha). \tag{3.21}$$

In fact, by (2.7), for $w\in Q_0$ we have

$$\delta_\Omega(w) = \operatorname{dist}(w,\partial\Omega) \geq \operatorname{dist}(Q_0,\partial(A_0\cap\Sigma_\varepsilon)) \geq \frac{\varepsilon}{10};$$

therefore, $\delta_\Omega(s)\geq\delta_\Omega(w)\geq\frac{\varepsilon}{10}$, for any s in the radius connecting 0 and w. Since hyperbolic distance and quasi-hyperbolic distance are comparable (cf. (1.5)) we can put

$$\beta_\Omega(0,w) \leq 2k_\Omega(0,w) \leq 2\int_{[0,w_0]}\frac{|ds|}{\delta_\Omega(s)} \leq \frac{20}{\varepsilon}. \tag{3.22}$$

Now, using (1.3), the conformal invariance of β_Ω and (3.22) it results that

$$\log\left(\frac{1+|z|}{1-|z|}\right) = \beta(0,z) = \beta_\Omega(0,f(z)) \leq \frac{20}{\varepsilon}, \tag{3.23}$$

for any $z\in f^{-1}(Q_0)$ and, since $\alpha\geq 0$, we obtain

$$\left(1-|z|^2\right)^\alpha \geq e^{-\frac{20}{\varepsilon}\alpha}. \tag{3.24}$$

If $z\in f^{-1}(Q_0)$, and remaining that $0\in\partial(A_0\cap\Sigma_\varepsilon\cap\Omega)$ we get

$$|f(z)| \geq \operatorname{dist}(Q_0,\partial(A_0\cap\Sigma_\varepsilon\cap\Omega)) \geq \frac{\varepsilon}{8(1+\varepsilon)};$$

this inequality and (3.24) yield

$$\int_{f^{-1}(Q_0)} |f(z)|\, dA_\alpha(z) \geq \frac{\varepsilon}{8(1+\varepsilon)} e^{-\frac{20}{\varepsilon}\alpha} \int_{f^{-1}(Q_0)} dA(z). \tag{3.25}$$

On the other hand, by (3.23) and keeping in mind that $\delta_\Omega(0) = 1$, $f(0) = 0$, we can apply the Koebe Distortion Theorem to deduce that

$$|f'(z)| \leq 4e^{3\beta_\Omega(0,f(z))} \leq 4e^{60/\varepsilon}.$$

for any $z \in f^{-1}(Q_0)$. So, (3.25) can be written as

$$\begin{aligned} \int_{f^{-1}(Q_0)} |f(z)|\, dA_\alpha(z) &\geq \frac{\varepsilon e^{-120/\varepsilon}}{128(1+\varepsilon)} e^{-\frac{20}{\varepsilon}\alpha} \int_{f^{-1}(Q_0)} |f'(z)|^2\, dA(z) \\ &= \frac{\varepsilon}{128(1+\varepsilon)} e^{-\frac{20}{\varepsilon}(6+\alpha)} \operatorname{Area}(Q_0). \end{aligned}$$

Since $\operatorname{Area}(Q_0) = \frac{1}{2}\operatorname{diam}(Q_0)^2 \geq \frac{1}{32(1+\varepsilon)^2}\varepsilon^2$, we conclude

$$\int_{f^{-1}(Q_0)} |f(z)|\, dA_\alpha(z) \geq K_3(\varepsilon,\alpha),$$

where

$$K_3(\varepsilon,\alpha) = \frac{\varepsilon^3}{4096(1+\varepsilon)^3} e^{-\frac{20}{\varepsilon}(6+\alpha)}.$$

Next, we will prove that there is a constant $K_4(\varepsilon) > 0$ such that

$$\int_{f^{-1}(\Omega_0)} |f(z)|\, dA_\alpha(z) \leq K_4(\varepsilon). \tag{3.26}$$

Indeed, by the definition of A_j it is clear that

$$\int_{f^{-1}(\Omega_0)} |f(z)|\, dA_\alpha(z) = \sum_{j=0}^{+\infty} \int_{f^{-1}(\Omega_0 \cap A_j)} |f(z)|\, dA_\alpha(z),$$

and since $|f(z)| < (1+\varepsilon)^j$ whenever $z \in f^{-1}(\Omega_0 \cap A_j)$, and $\left(1 - |z|^2\right)^\alpha \leq 1$ for all $z \in \mathbb{D}$, it is clear that

$$\begin{aligned} \int_{f^{-1}(\Omega_0)} |f(z)|\, dA_\alpha(z) &\leq \sum_{j=0}^{+\infty} (1+\varepsilon)^j \int_{f^{-1}(\Omega_0 \cap A_j)} dA(z) \\ &= \sum_{j=0}^{+\infty} (1+\varepsilon)^j \operatorname{Area}\left(f^{-1}(\Omega_0 \cap A_j)\right). \end{aligned}$$

Now, by Lemma 2.1

$$\operatorname{Area}\left(f^{-1}\left(\Omega_0 \cap A_j\right)\right) \leq 6\pi e^{-\beta_\Omega\left(0,\Omega_0 \cap A_j\right)} \omega_\Omega\left(0, \Omega_0 \cap A_j\right),$$

which, with Lemma 3.2, yields

$$\int_{f^{-1}(\Omega_0)} |f(z)|\, dA_\alpha(z) \leq K_4(\varepsilon),$$

where

$$K_4(\varepsilon) = \frac{6\pi K_2(\varepsilon)}{1-(1+\varepsilon)^{-K_1\varepsilon}}$$

as was claimed. Comparing (3.21) and (3.26) we conclude that our assertion holds for $n=0$ with $\check{K}(\varepsilon,\alpha) = \dfrac{K_4(\varepsilon)}{K_3(\varepsilon,\alpha)}$.

Assume now that $n>0$. Note that if $z \in f^{-1}(Q_n)$ then $f(z) \in Q_n$, so that, by triangle inequality, we can write

$$\beta_\Omega(0, f(z)) \leq \beta_\Omega(0, w_n) + \beta_\Omega(w_n, f(z));$$

by definition (1.3), the conformal invariance and (2.8) we have

$$\log\left(\frac{1+|z|}{1-|z|}\right) \leq \log\left(\frac{1+|z_n|}{1-|z_n|}\right) + 8,$$

so that for all $z \in f^{-1}(Q_n)$,

$$\left(1-|z|^2\right)^\alpha \geq \left[\frac{1}{4e^8}\right]^\alpha \left(1-|z_n|^2\right)^\alpha. \tag{3.27}$$

Now, by Lemma 2.4, Q_n contains a hyperbolic ball with radius bigger than $\frac{\sqrt{2}}{32}\varepsilon$, and since $Q_n \subset A_n$, by Lemma 2.2 we can write

$$\begin{aligned}\int_{f^{-1}(Q_n)} |f(z)|\, dA_\alpha(z) &\geq (1+\varepsilon)^{n-1} \int_{f^{-1}(Q_n)} dA_\alpha(z) \\ &\geq \left[\frac{1}{4e^8}\right]^\alpha \left(1-|z_n|^2\right)^\alpha (1+\varepsilon)^{n-1} \operatorname{Area}\left(f^{-1}(Q_n)\right) \\ &\geq K_5(\varepsilon,\alpha)\left(1-|z_n|^2\right)^\alpha (1+\varepsilon)^n e^{-\beta_\Omega(0,Q_n)} \omega_\Omega(0,Q_n),\end{aligned} \tag{3.28}$$

where

$$K_5(\varepsilon,\alpha) = \frac{1}{1+\varepsilon}\left(1-e^{-\sqrt{2}\varepsilon/32}\right)^2 \left[\frac{1}{4e^8}\right]^\alpha.$$

We observe that for any $w = f(z) \in \Omega_n$, looking at the hyperbolic geodesic γ_w from w to 0, we can find a point $\tilde{w} \in \gamma_w$ such that

$$\beta_\Omega(w_n, \tilde{w}) \leq \frac{100}{\varepsilon}$$

and satisfying

$$\beta_\Omega(0, w) = \beta_\Omega(0, \tilde{w}) + \beta_\Omega(\tilde{w}, w_n).$$

Hence,

$$\beta_\Omega(0, w) \geq \beta_\Omega(0, \tilde{w}) \geq \beta_\Omega(0, w_n) - \beta_\Omega(w_n, \tilde{w}) \geq \beta_\Omega(0, w_n) - \frac{100}{\varepsilon}$$

and

$$\left(1 - |z|^2\right)^\alpha \leq \left[4e^{100/\varepsilon}\right]^\alpha \left(1 - |z_n|^2\right)^\alpha, \qquad z \in f^{-1}(\Omega_n). \tag{3.29}$$

Using (3.29) and arguing as in the case $n = 0$, we obtain

$$\begin{aligned} &\int_{f^{-1}(\Omega_n)} |f(z)| \, dA_\alpha(z) \\ &\quad \leq \left[4e^{100/\varepsilon}\right]^\alpha \left(1 - |z_n|^2\right)^\alpha \sum_{j=0}^{+\infty} (1+\varepsilon)^j \operatorname{Area}\left(f^{-1}(\Omega_n \cap A_j)\right). \end{aligned} \tag{3.30}$$

Note that, in this setting, Lemma 2.1 asserts that

$$\operatorname{Area}\left(f^{-1}(\Omega_n \cap A_j)\right) \quad \leq \quad 6\pi e^{-\beta_\Omega(0, \Omega_n \cap A_j)} \omega_\Omega(0, \Omega_n \cap A_j); \tag{3.31}$$

We also observe that there exists $w \in \Omega_n \cap A_j$ such that

$$\beta_\Omega(0, \Omega_n \cap A_j) \geq \beta_\Omega(0, w) - \varepsilon,$$

and, since for any $w \in \Omega_n$ there is a $s \in \gamma_w$ so that

$$\beta_\Omega(0, w) = \beta_\Omega(0, s) + \beta_\Omega(s, w)$$

and

$$\beta_\Omega(s, w_n) \leq \frac{100}{\varepsilon},$$

we may write

$$\begin{aligned}\beta_\Omega(0,Q_n)+\beta_\Omega(w_n,\Omega_n\cap A_j) &\le \beta_\Omega(0,w_n)+\beta_\Omega(w_n,w)\\ &\le \beta_\Omega(0,s)+\beta_\Omega(s,w)+2\beta_\Omega(w_n,s)\\ &\le \beta_\Omega(0,\Omega_n\cap A_j)+\frac{200}{\varepsilon}+\varepsilon,\end{aligned}$$

so that

$$e^{-\beta_\Omega(0,\Omega_n\cap A_j)}\le e^{\frac{200}{\varepsilon}+\varepsilon}e^{-\beta_\Omega(0,Q_n)}e^{-\beta_\Omega(w_n,\Omega_n\cap A_j)}. \tag{3.32}$$

Carrying (3.32), (2.15) and (3.31) in (3.30), and applying Lemma 3.2 we find

$$\begin{aligned}&\int_{f^{-1}(\Omega_n)}|f(z)|\,dA_\alpha(z)\\ &\le K_6(\varepsilon,\alpha)\left(1-|z_n|^2\right)^\alpha\sum_{j=0}^{+\infty}(1+\varepsilon)^{j-|j-n|(1+K_1\varepsilon)}\,e^{-\beta_\Omega(0,Q_n)}\omega_\Omega(0,Q_n),\end{aligned}$$

where

$$K_6(\varepsilon,\alpha)=6\pi\left[4e^{100/\varepsilon}\right]^\alpha e^{200/\varepsilon+\varepsilon}K_0(\varepsilon)\,K_2(\varepsilon);$$

substituting in (3.28) we obtain

$$\begin{aligned}&\int_{f^{-1}(\Omega_n)}|f(z)|\,dA_\alpha(z)\\ &\qquad\le K_7(\varepsilon,\alpha)\sum_{j=0}^{+\infty}(1+\varepsilon)^{j-n-|j-n|(1+K_1\varepsilon)}\int_{f^{-1}(Q_n)}|f(z)|\,dA_\alpha(z),\end{aligned}$$

where

$$K_7(\varepsilon,\alpha)=\frac{K_6(\varepsilon)}{K_5(\varepsilon,\alpha)};$$

finally,

$$\int_{f^{-1}(\Omega_n)}|f(z)|\,dA_\alpha(z)\le\hat{K}(\varepsilon,\alpha)\int_{f^{-1}(Q_n)}|f(z)|\,dA_\alpha(z),$$

with

$$\hat{K}(\varepsilon,\alpha)=\frac{2}{1-(1+\varepsilon)^{-K_1\varepsilon}}K_7(\varepsilon,\alpha).$$

So, our assertion also holds for $n>0$. Taking

$$K(\varepsilon,\alpha)=\max\left\{\check{K}(\varepsilon,\alpha),\hat{K}(\varepsilon,\alpha)\right\}$$

it follows that

$$\int_{f^{-1}(\Omega_n)} |f(z)|\, dA_\alpha(z) \leq K(\varepsilon,\alpha) \int_{f^{-1}(Q_n)} |f(z)|\, dA_\alpha(z),$$

for all $n = 0, 1, 2, \ldots$

4 Failure for α negative

In this section we provide a counterexample showing that Theorem 1.2 does not hold for $\alpha \in (-1, 0)$.

Suppose we have fixed $\alpha \in (-1, 0)$, and for each $n = 1, 2, \ldots$, and take

$$\varepsilon_n = \frac{-\alpha\pi}{2} + \frac{1}{n}.$$

For each n, let f_n be the Riemann mapping from $\mathbb{D}$ onto $\Omega_n := \mathbb{C} \setminus (1 + \Sigma_{\varepsilon_n})$ such that $f_n(0) = 0$ and $f_n'(0) > 0$. These functions are explicitly defined as

$$f_n(z) = 1 - \left(\frac{1-z}{1+z}\right)^{(2+\alpha-\frac{2}{n\pi})}.$$

To show that $f_n \in A^1_\alpha$, it is sufficient to check that $(1+z)^{-(2+\alpha-\frac{2}{n\pi})} \in A^1_\alpha$. To see this, we can look at the proof of Theorem 1.7 in [HKZ, page 7]), and taking there $\beta = \dfrac{-2}{n\pi}$ and $\lambda = \lambda_n = \dfrac{2+\alpha-\frac{2}{n\pi}}{2}$, we obtain that

$$\|(1+z)^{-(2+\alpha-\frac{2}{n\pi})}\|_{A^1_\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\lambda_n)^2} \sum_{k=0}^{+\infty} \frac{\Gamma(k+\lambda_n)^2}{k!\Gamma(k+\alpha+2)}.$$

By Stirling's formula

$$\frac{\Gamma(k+\lambda_n)^2}{k!\Gamma(k+\alpha+2)} \sim (k+1)^{-\frac{2}{n\pi}-1}, \qquad k \to +\infty,$$

we conclude that $\|f_n\|_{A^1_\alpha} < \infty$ for all $n = 1, 2, \ldots$ Note that

$$f_n(z) \to f(z) = 1 - \left(\frac{1-z}{1+z}\right)^{(2+\alpha)},$$

as $n \to \infty$, for any $z \in \mathbb{D}$, and $f \notin A^1_\alpha$ (this is due to the case $\beta = 0$ in the reference above). Applying Fatou Lemma we obtain that

$$\|f_n\|_{A^1_\alpha} \to \infty, \qquad \text{as } n \to \infty. \tag{4.33}$$

On the other hand, take $\varepsilon = -\dfrac{\alpha\pi}{4}$. Standard trigonometric calculations yield

$$\Sigma_\varepsilon \cup \Omega_n \subset \Sigma_\varepsilon \cap (\mathbb{C} \setminus (1 + \Sigma_{2\varepsilon})) \subset D(0, 2),$$

and we get that

$$\int_{f_n^{-1}(\Sigma_\varepsilon)} |f_n(z)| dA_\alpha(z) \leq 2 \int_{\mathbb{D}} dA_\alpha(z) = 2 \tag{4.34}$$

for all n. But, if δ is a constant such that

$$\|g\|_{A^1_\alpha} < \delta \int_{g^{-1}(\Sigma_\varepsilon)} |g(z)| dA_\alpha(z)$$

for *any* univalent function $g \in A^1_\alpha$ which fixes the origin, in particular we would get that

$$\|f_n\|_{A^1_\alpha} < \delta \int_{f_n^{-1}(\Sigma_\varepsilon)} |f_n(z)| dA_\alpha(z)$$

which gives a contradiction between (4.33) and (4.34).

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