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ON GENERALIZED FRACTIONAL q-INTEGRAL OPERATORS INVOLVING THE q-GAUSS HYPERGEOMETRIC FUNCTION

(DEDICATED IN OCCASION OF THE 70-YEARS OF PROFESSOR HARI M. SRIVASTAVA)

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Abstract. In this paper, we introduce two generalized operators of fractional q-integration, which may be regarded as extensions of Riemann-Liouville, Weyl and Kober fractional q-integral operators. Certain interesting connection theorems involving these operators and q-Mellin transform are also discussed.

1. Introduction

The fractional integration operators involving various special functions, in particular the Gaussian hypergeometric functions, have found significant importance and applications in various sub-fields of applicable mathematical analysis. Since last three decades, a number of workers like Love [11], McBride [13], Kalla and Saxena [8, 9], Saigo [21-23], Saigo and Raina [24] etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Miller and Ross [14], Kiryakova [10] and Nishimoto [15-18] etc.

The fractional q-calculus is the q-extension of the ordinary fractional calculus. The theory of q-calculus operators in recent past have been applied in the areas like ordinary fractional calculus, optimal control problems, solutions of the q-difference (differential) and q-integral equations, q-transform analysis etc. Recently, Abu-Risha, Annaby, Ismail and Mansour [1] and Mansour [12] derived the fundamental set of solutions for the homogenous linear sequential fractional q-difference equations with constant coefficients. Fang [6] and Purohit [19] deduced several transformations and summations formulae for the basic hypergeometric functions as the applications of fractional q-differential operator. For more details one may refer the

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recent papers [4], [5] and [20] on the subject.

We propose to define and investigate the q-extensions of the hypergeometric operators of fractional integration due to Saigo [21].

In a series of papers [21-23], Saigo introduced the following pair of hypergeometric operators of fractional integration.

For $\alpha > 0$, real numbers β and η, we have:

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \; {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-t/x\right) f(t)dt, \tag{1.1}$$

$$J_{x,\infty}^{\alpha,\beta,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \; t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-x/t\right) f(t)dt, \tag{1.2}$$

where, the ${}_2F_1(.)$ function occurring in the right-hand side of the above equations, is the familiar Gaussian hypergeometric function defined as:

$${}_2F_1\left(a, b; c; x\right) \equiv \; {}_2F_1\left[\begin{matrix} a, \; b \\ \\ c \end{matrix} \; ; \; x\right] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \; \frac{x^n}{n!}. \tag{1.3}$$

The operator $I_{0,x}^{\alpha,\beta,\eta}(.)$ contains both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators, by means of the following relationships:

$$R_{0,x}^{\alpha} f(x) = I_{0,x}^{\alpha,-\alpha,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \tag{1.4}$$

and

$$E_{0,x}^{\alpha,\eta} f(x) = I_{0,x}^{\alpha,0,\eta} f(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \; t^{\eta} f(t)dt, \tag{1.5}$$

where as the operator (1.2) unifies the Weyl type and the Erdélyi-Kober fractional integral operators. Indeed we have

$$W_{x,\infty}^{\alpha} f(x) = J_{x,\infty}^{\alpha,-\alpha,\eta} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt, \tag{1.6}$$

and

$$K_{x,\infty}^{\alpha,\eta} f(x) = J_{x,\infty}^{\alpha,0,\eta} f(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \; t^{-\alpha-\eta} f(t)dt. \tag{1.7}$$

For real or complex a and $|q| < 1$, the q-shifted factorial is defined as:

$$(a;q)_0 = 1, \; (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i), \; n > 0, \; \textit{and} \; (a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i). \tag{1.8}$$

Equivalently

$$(a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \tag{1.9}$$

where the q-gamma function cf. Gasper and Rahman [7], is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty}{(q^a;q)_\infty (1-q)^{a-1}} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}}, \tag{1.10}$$

$$(a \neq 0, -1, -2, \cdots).$$

Also, the q-analogue of the power (binomial) function $(x+y)^n$ cf. Gasper and Rahman (see also Ernst [5]), is given by

$$(x+y)^{(n)} = \begin{cases} x^n(-\frac{y}{x};q)_n & , x \neq 0 \\ \\ q^{n(n-1)/2}\ y^n & , x = 0, \end{cases} \tag{1.11}$$

where the q-binomial coefficient is defined as:

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q;q)_n}{(q;q)_k\ (q;q)_{n-k}}\ . \tag{1.12}$$

For a bounded sequence $(A_n)_{n\in\mathbb{Z}}$ of real or complex numbers, let $f(x) = \sum\limits_{n=-\infty}^{+\infty} A_n x^n$ be a power series in x, then the q-translation operator is defined as:

$$\mathcal{T}_{q,y}(f(x)) = \sum_{n=-\infty}^{+\infty} A_n x^n (y/x;q)_n. \tag{1.13}$$

The generalized basic hypergeometric series cf. Gasper and Rahman [7] is given by

$${}_r\Phi_s \left[\begin{array}{ccc} a_1, \cdots, a_r & & \\ & ; & q, x \\ b_1, \cdots, b_s & & \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\ x^n\ \left\{(-1)^n q^{n(n-1)/2}\right\}^{(1+s-r)}\ , \tag{1.14}$$

where

$$(a_1,\cdots,a_r;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n,$$

and for convergence, we have $|q|<1$ and $|x|<1$ if $r=s+1$, and for any x if $r \leq s$.

A q-analogue of the familiar Riemann-Liouville fractional integral operator of a function $f(x)$ due to Agarwal [2] is defined as:

$$I_q^\alpha \{f(x)\} = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t)\ d_q t, \tag{1.15}$$

where $\Re(\alpha) > 0; |q| < 1$ and

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty},\ \ \alpha \in \mathbb{R}.$$

Also, the basic analogue of the Kober fractional integral operator cf. Agarwal [2] is defined by

$$I_q^{\eta,\alpha} \{f(x)\} = \frac{x^{-\eta-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} t^\eta f(t)\ d_q t, \tag{1.16}$$

where $\Re(\alpha) > 0; |q| < 1; \eta \in \mathbb{R}$.
A q-analogue of the Weyl fractional integral operator (1.6) due to Al-Salam [3], is defined as:

$$K_q^\alpha f(x) \ = \ \frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)} \int_x^\infty t^{\alpha-1}(x/t;q)_{\alpha-1} f(tq^{1-\alpha})\ d_q t, \tag{1.17}$$

where $\Re(\alpha) > 0; |q| < 1$.
In the same paper, Al-Salam [3] introduced the q-analogue of the operator (1.7) in

the following manner:

$$K_q^{\eta,\alpha}\left\{f(x)\right\} = \frac{q^{-\eta}x^{\eta}}{\Gamma_q(\alpha)}\int_x^{\infty}(x/t;q)_{\alpha-1}\ t^{-\eta-1}f(tq^{1-\alpha})\ d_qt, \tag{1.18}$$

where $\Re(\alpha) > 0; |q| < 1; \eta \in \mathbb{R}$.
Also the basic integrals (cf. Gasper and Rahman [7]), are defined as:

$$\int_0^x f(t)\ d_qt = x(1-q)\sum_{k=0}^{\infty} q^k f(zq^k), \tag{1.19}$$

$$\int_x^{\infty} f(t)\ d_qt = x(1-q)\sum_{k=1}^{\infty} q^{-k} f(xq^{-k}), \tag{1.20}$$

and

$$\int_0^{\infty} f(t)\ d_qt = (1-q)\sum_{k=-\infty}^{\infty} q^k f(q^k). \tag{1.21}$$

The q-binomial summation theorem is given by

$${}_1\Phi_0\left[a;-;q,z\right] = \frac{(az;q)_\infty}{(z;q)_\infty},\ \ |z| < 1. \tag{1.22}$$

Also the q-Chu-Vondermonde summation theorem cf. Gasper and Rahman [7]

$${}_2\Phi_1\left[\begin{array}{ccc} q^{-n},\ a & & \\ & ; & q,q \\ c & & \end{array}\right] = \frac{(c/a;q)_n}{(c;q)_n}(a)^n. \tag{1.23}$$

The object of this paper is to introduce two hypergeometric operators of fractional q-integration, which may be regarded as extensions of the fractional q-integral operators (1.15)-(1.18). Having defined a q-extensions of these operaotrs, we investigate their fundamental properties such as integration by parts and connection theorems with q-analogue of Mellin transform. Certain interesting special cases in the form of the known results have also been discussed.

2. The Fractional q-Integral Operators

In this section, we introduce the following fractional q-integral operators involving the Gaussian basic hypergeometric function, which may be regarded as q-extensions of the Saigo operators (1.1) and (1.2).

For α and real β, we define the fractional q-integral operators $I_q^{\alpha,\beta,\eta}(.)$ and $K_q^{\alpha,\beta,\eta}(.)$ in the following manner:

$$I_q^{\alpha,\beta,\eta}f(x) = \frac{x^{-\beta-1}q^{-\eta(\alpha+\beta)}}{\Gamma_q(\alpha)}$$

$$\times\int_0^x (tq/x;q)_{\alpha-1}\ \mathcal{T}_{q,\frac{q^{\alpha+1}t}{x}}\left({}_2\Phi_1\left[q^{\alpha+\beta},\ q^{-\eta};q^{\alpha};q,q\right]\right)f(t)\ d_qt,\ \ |t/x| < 1, \tag{2.1}$$

and

$$K_q^{\alpha,\beta,\eta}f(x) = \frac{q^{-\eta(\alpha+\beta)-\alpha(\alpha+1)/2-2\beta}}{\Gamma_q(\alpha)}$$

$$\times \int_{x}^{\infty} (x/t;q)_{\alpha-1}\, t^{-\beta-1}\, \mathcal{T}_{q,\frac{q^{\alpha+1}x}{t}} \left({}_2\Phi_1 \left[q^{\alpha+\beta},\ q^{-\eta}; q^{\alpha}; q, q \right] \right) f(tq^{1-\alpha})\, d_q t,\ |x/t| < 1, \tag{2.2}$$

where η is any non negative integer and the ${}_2\Phi_1(.)$ function occurring in the right-hand side of (2.1) and (2.2) is the Gaussian q-hypergeometric function defined as special case (for $r = 2$ and $s = 1$) of the power series (1.14). Using series definitions of the basic integrals given by (1.19)-(1.20) and q-translation operator (1.13), we define the series representation for the operators (2.1) and (2.2) as:

$$I_q^{\alpha,\beta,\eta} f(x) = x^{-\beta} q^{-\eta(\alpha+\beta)} (1-q)^{\alpha} \times \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n \sum_{k=0}^{\infty} \frac{q^k (q^{\alpha+n};q)_k}{(q;q)_k} f(xq^k), \tag{2.3}$$

and

$$K_q^{\alpha,\beta,\eta} f(x) = x^{-\beta} q^{-\eta(\alpha+\beta)-\alpha(\alpha+1)/2-\beta} (1-q)^{\alpha} \times \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n \sum_{k=0}^{\infty} \frac{q^{\beta k} (q^{\alpha+n};q)_k}{(q;q)_k} f(xq^{-\alpha-k}), \tag{2.4}$$

where $\alpha > 0$, β being real number, and η is any non negative integer.

3. Fractional q-Integral Images of $x^{\lambda-1}$

This section envisage the evaluation of the q-images of an elimentary function $x^{\lambda-1}$ under the generalized fractional q-integral operators introduced in the previous section.

Theorem 1. *If $|q| < 1$, $\lambda > 0$ and $(\lambda - \beta + \eta) > 0$, then*

$$I_q^{\alpha,\beta,\eta} \left\{ x^{\lambda-1} \right\} = \frac{\Gamma_q(\lambda)\Gamma_q(\lambda-\beta+\eta)}{\Gamma_q(\lambda-\beta)\Gamma_q(\lambda+\alpha+\eta)} x^{\lambda-\beta-1}. \tag{3.1}$$

Proof. To prove the theorem (3.1), we take $f(x) = x^{\lambda-1}$ in the series definition of fractional q-integral operator $I_q^{\alpha,\beta,\eta}(.)$, given by (2.3), the left-hand side yields to

$$I_q^{\alpha,\beta,\eta} \left\{ x^{\lambda-1} \right\} = x^{\lambda-\beta-1} q^{-\eta(\alpha+\beta)} (1-q)^{\alpha} \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n \times \sum_{k=0}^{\infty} \frac{q^{\lambda k} (q^{\alpha+n};q)_k}{(q;q)_k}. \tag{3.2}$$

On summing the inner ${}_1\Phi_0(.)$ series with the help of the equation (1.22), it reduces to

$$I_q^{\alpha,\beta,\eta} \left\{ x^{\lambda-1} \right\} = x^{\lambda-\beta-1} q^{-\eta(\alpha+\beta)} (1-q)^{\alpha} \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n (q^{\lambda};q)_{\alpha+n}} q^n, \tag{3.3}$$

on simplification and the usage of the q-Chu-Vondermonde summation theorem given by (1.23), the above equation leads to Theorem 1.

Theorem 2. *If* $|q| < 1$, $(\beta - \lambda + 1) > 0$ *and* $(\eta - \lambda + 1) > 0$, *then*

$$K_q^{\alpha,\beta,\eta}\left\{x^{\lambda-1}\right\} = \frac{\Gamma_q(\beta-\lambda+1)\Gamma_q(\eta-\lambda+1)}{\Gamma_q(1-\lambda)\Gamma_q(\beta+\alpha-\lambda+\eta+1)} x^{\lambda-\beta-1} q^{\alpha(1-\lambda)-\alpha(\alpha+1)/2-\beta}. \tag{3.4}$$

Proof. On employing the definition (2.4) with $f(x) = x^{\lambda-1}$, we obtain

$$K_q^{\alpha,\beta,\eta}\left\{x^{\lambda-1}\right\} = x^{\lambda-\beta-1} q^{\alpha(1-\lambda)-\eta(\alpha+\beta)-\alpha(\alpha+1)/2-\beta}(1-q)^\alpha$$
$$\times \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n \sum_{k=0}^{\infty} \frac{q^{(\beta-\lambda+1)k}(q^{\alpha+n};q)_k}{(q;q)_k}. \tag{3.5}$$

On summing the inner ${}_1\Phi_0(.)$ series with the help of the equation (1.22), it leads to

$$K_q^{\alpha,\beta,\eta}\left\{x^{\lambda-1}\right\} = x^{\lambda-\beta-1} q^{\alpha(1-\lambda)-\eta(\alpha+\beta)-\alpha(\alpha+1)/2-\beta}(1-q)^\alpha$$
$$\times \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n (q^{\beta-\lambda+1};q)_{\alpha+n}} q^n, \tag{3.6}$$

which, on using the q-Vondermonde summation theorem (1.23) and some simplifications, leads to the proof of the result (3.4).

Further, it is interesting to observe that the newly defined operators (2.1) and (2.2) can be regarded as extensions of Riemann-Liouville, Weyl and Kober fractional q-integral operators with the following functional relations:

$$I_q^{\alpha,0,\eta} f(x) = I_q^{\eta,\alpha} f(x), \tag{3.7}$$

$$I_q^{\alpha,-\alpha,\eta} f(x) = I_q^{\alpha} f(x), \tag{3.8}$$

$$K_q^{\alpha,0,\eta} f(x) = q^{-\alpha(\alpha+1)/2} K_q^{\eta,\alpha} f(x), \tag{3.9}$$

$$K_q^{\alpha,-\alpha,\eta} f(x) = K_q^{\alpha} f(x). \tag{3.10}$$

4. Fractional Integration by Parts

In this section, we shall prove a theorem involving an important relationship between the operators $I_q^{\alpha,\beta,\eta}(.)$ and $K_q^{\alpha,\beta,\eta}(.)$:

Theorem 3. *If* $\alpha > 0$, β *a real number, and* η *being a non negative integer, then*

$$\int_0^\infty f(x) K_q^{\alpha,\beta,\eta} g(x)\ d_q t = q^{-\alpha(\alpha+1)/2-\beta} \int_0^\infty g(xq^{-\alpha}) I_q^{\alpha,\beta,\eta} f(x)\ d_q t. \tag{4.1}$$

Provided that both of the q-integrals exist.

Proof. On using the series definition of q-Saigo operator $K_q^{\alpha,\beta,\eta}(.)$, given by (2.4), the left-hand side, say L of Equation (4.1) yields to

$$L = \int_0^\infty f(x)\ x^{-\beta} q^{-\eta(\alpha+\beta)-\alpha(\alpha+1)/2-\beta}(1-q)^\alpha \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n$$
$$\times \sum_{k=0}^{\infty} \frac{q^{\beta k}(q^{\alpha+n};q)_k}{(q;q)_k} g(xq^{-\alpha-k})\ d_q t. \tag{4.2}$$

On changing the order of integration and summations in the above expression, which is valid under conditions mentioned with (2.4) and using the integral (1.21), the above equation reduces to

$$L = q^{-\eta(\alpha+\beta)-\alpha(\alpha+1)/2-\beta}(1-q)^{\alpha+1} \sum_{r=-\infty}^{\infty} q^r f(q^r)\, q^{-r\beta+\beta} \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n$$
$$\times \sum_{k=0}^{\infty} \frac{q^{\beta k}(q^{\alpha+n};q)_k}{(q;q)_k} g(q^{r-\alpha-k})$$
$$= q^{-\eta(\alpha+\beta)-\alpha(\alpha+1)/2-\beta}(1-q)^{\alpha+1} \sum_{t=-\infty}^{\infty} q^t g(q^{t-\alpha})\, q^{-t\beta+\beta} \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n$$
$$\times \sum_{k=0}^{\infty} \frac{q^{k}(q^{\alpha+n};q)_k}{(q;q)_k} f(q^{t+k}), \tag{4.3}$$

on replacing the basic bilateral series in the above relation by the integral (1.21), we obtain

$$L = q^{-\eta(\alpha+\beta)-\alpha(\alpha+1)/2-\beta} \int_0^{\infty} g(xq^{-\alpha})\ x^{-\beta} q^{\beta} (1-q)^{\alpha} \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n$$
$$\times \sum_{k=0}^{\infty} \frac{q^{k}(q^{\alpha+n};q)_k}{(q;q)_k} f(xq^{k})\ d_q t. \tag{4.4}$$

On interpreting the above expression in light of the series definition (2.3) of the q-Saigo operator $I_q^{\alpha,\beta,\eta}(.)$, the above equation (4.4) finally reduces to the right-hand side of the Theorem 3.

Interestingly, on setting $\beta = 0$ and employing the relations (3.7) and (3.9), the Theorem 3 yields to the following Corollary:

Corollary 1. *For $\alpha > 0$ and η being a non negative integer, the following result holds:*

$$\int_0^{\infty} f(x) K_q^{\eta,\alpha} g(x)\ d_q t = \int_0^{\infty} g(xq^{-\alpha}) I_q^{\eta,\alpha} f(x)\ d_q t. \tag{4.5}$$

Provided both of the q-integrals exist.

Further, if we replace β by $-\alpha$ and make use of the relations (3.8) and (3.10), in the Theorem 3, we obtain yet another corollary providing interesting relationship between the operators $K_q^{\alpha}(.)$ and $I_q^{\alpha}(.)$ namely:

Corollary 2. *For $\alpha > 0$ and η being a non negative integer, the following result holds:*

$$q^{\alpha(\alpha-1)/2} \int_0^{\infty} f(x) K_q^{\alpha} g(x)\ d_q t = \int_0^{\infty} g(xq^{-\alpha}) I_q^{\alpha} f(x)\ d_q t. \tag{4.6}$$

Provided that both of the q-integrals exist.

Finally, it is worth mentioning that, if we remove the non negativity restriction on the parameter η, the corollaries (4.5) and (4.6) reduces to the known results due to Agarwal [2].

5. The q-Mellin Transform of the q-Saigo Operators

In this section, we shall prove two theorems, which exhibit the connection between the q-Mellin transform and the operators given by Equations (2.1) and (2.2).

Theorem 4. *If $\alpha > 0$ and $s < 1 + min\,\{0, \eta - \beta\}$, then*

$$M_q\left(x^{\beta}\ I_q^{\alpha,\beta,\eta} f(x)\right)(s) = \frac{\Gamma_q(1-s)\Gamma_q(\eta+1-s-\beta)}{\Gamma_q(1-s-\beta)\Gamma_q(\eta+1-s+\alpha)} M_q\left(f(x)\right)(s), \qquad (5.1)$$

where the q-Mellin transform of $f(x)$ is defined as:

$$M_q\left(f(x)\right)(s) = \frac{1}{(1-q)} \int_0^{\infty} x^{s-1} f(x)\ d_q t = \sum_{r=-\infty}^{\infty} q^{rs} f(q^r). \qquad (5.2)$$

Proof. On using the definition (5.2) and the series definition of fractional q-integral operator $I_q^{\alpha,\beta,\eta}(.)$ given by (2.3), the left-hand side (say L) becomes

$$L = \sum_{r=-\infty}^{\infty} q^{rs-\eta(\alpha+\beta)}(1-q)^{\alpha} \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n \sum_{k=0}^{\infty} \frac{q^k (q^{\alpha+n};q)_k}{(q;q)_k} f(q^{r+k})$$

$$= \sum_{r=-\infty}^{\infty} q^{rs-\eta(\alpha+\beta)}(1-q)^{\alpha} f(q^r) \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n} q^n \sum_{k=0}^{\infty} \frac{q^{k(1-s)} (q^{\alpha+n};q)_k}{(q;q)_k}. \qquad (5.3)$$

On summing the inner ${}_1\Phi_0(.)$ series with the help of the equation (1.22), it reduces to

$$L = \sum_{r=-\infty}^{\infty} q^{rs-\eta(\alpha+\beta)}(1-q)^{\alpha} f(q^r) \sum_{n=0}^{\eta} \frac{(q^{\alpha+\beta};q)_n (q^{-\eta};q)_n}{(q;q)_n (q^{1-s};q)_{\alpha+n}} q^n, \qquad (5.4)$$

which further simplifies to

$$L = \frac{\Gamma_q(1-s)\Gamma_q(\eta+1-s-\beta)}{\Gamma_q(1-s-\beta)\Gamma_q(\eta+1-s+\alpha)} \sum_{r=-\infty}^{\infty} q^{rs} f(q^r). \qquad (5.5)$$

On interpreting the basic bilateral series in light of the definition (5.2), the above equation yields to the right-hand side of the theorem (5.1).

Theorem 5. *If $\alpha > 0$ and $s > -min\,\{\beta, \eta\}$, then following relation holds:*

$$M_q\left(x^{\beta}\ K_q^{\alpha,\beta,\eta} f(x)\right)(s) = \frac{\Gamma_q(\beta+s)\Gamma_q(\eta+s)}{\Gamma_q(s)\Gamma_q(s+\alpha+\beta+\eta)} q^{-\alpha(\alpha+1)/2-\beta} M_q\left(f(xq^{-\alpha})\right)(s), \qquad (5.6)$$

where the q-Mellin transform of $f(x)$ is given by the relation (5.2).

The proof of the above theorem follows similarly.

If we set $\beta = 0$ and make use of relations (3.7) and (3.9), the results of Theorems 4 and 5 respectively give rise to the following corollaries involving relations between the q-Mellin transform and the Kober fractional q-integral operators:

Corollary 3. *If $\alpha > 0$ and $(1-s) > 0$, then*

$$M_q\left(I_q^{\eta,\alpha} f(x)\right)(s) = \frac{\Gamma_q(\eta+1-s)}{\Gamma_q(\eta+1-s+\alpha)} M_q\left(f(x)\right)(s), \tag{5.7}$$

and

Corollary 4. *If $\alpha > 0$ and $(\eta+s) > 0$, then following relation holds:*

$$M_q\left(K_q^{\eta,\alpha} f(x)\right)(s) = \frac{\Gamma_q(\eta+s)}{\Gamma_q(\eta+s+\alpha)} M_q\left(f(xq^{-\alpha})\right)(s). \tag{5.8}$$

Finaly, if we replace β by $-\alpha$ and make use of the relations (3.8) and (3.10), Theorems 4 and 5 yield the following corollaries:

Corollary 5. *For $\alpha > 0$ and $(1-s) > 0$, following result holds:*

$$M_q\left(x^{-\alpha}\ I_q^{\alpha} f(x)\right)(s) = \frac{\Gamma_q(1-s)}{\Gamma_q(1-s+\alpha)} M_q\left(f(x)\right)(s), \tag{5.9}$$

and

Corollary 6. *If $\alpha > 0$ and $(s-\alpha) > 0$, then:*

$$M_q\left(x^{-\alpha}\ K_q^{\alpha} f(x)\right)(s) = \frac{\Gamma_q(s-\alpha)}{\Gamma_q(s)}\ q^{-\alpha(\alpha-1)/2}\ M_q\left(f(xq^{-\alpha})\right)(s). \tag{5.10}$$

6. Concluding Observations

We briefly consider now some consequences of the results derived in the preceeding sections.

(i) If we let $q \to 1^-$, and make use of the limit formulae:

$$\lim_{q\to 1^-} \Gamma_q(a) \ = \ \Gamma(a) \ and \ \lim_{q\to 1^-} \frac{(q^a;q)_n}{(1-q)^n} \ = \ (a)_n \ , \tag{6.1}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1), \tag{6.2}$$

we observe that the operators (2.1) and (2.2) provides respectively, the q-extensions of the known hypergeometric operators (1.1) and (1.2) due to Saigo [21].

(ii) Further, it is interesting to observe that the results given by (5.1) and (5.6) are the q-extensions of the known results due to Saigo, Saxena and Ram [25, pp. 295-296, eqn. (4.1) and (4.3)].

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