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ON QUARTER-SYMMETRIC NON-METRIC CONNECTION ON AN ALMOST HERMITIAN MANIFOLD

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Abstract. The present paper deals with different geometrical properties of the Hermitian manifold equipped with the quarter-symmetric non-metric connection. In the end, we studied the properties of the contravariant almost analytic vector field with quarter-symmetric non-metric connection.

1. Introduction

The idea of quarter-symmetric linear connection in a differentiable manifold was introduced by S. Golab [4] (1975). Various properties of quarter-symmetric metric connections have studied by [8], [9], [10], [11], [12], [14], [15] and many others. In 1980, Mishra and Pandey [7] defined and studied the quarter-symmetric metric F-connections in Riemannian, Kahlerian and Sasakian manifolds. In 2003, Sengupta and Biswas [13] defined quarter-symmetric non-metric connection in a Sasakian manifold and studied their properties. In this series, the properties of quarter-symmetric non-metric connections have been studied by [1], [2], [3] and many others. In the present paper, we defined a quarter-symmetric non-metric connection in almost Hermitian manifold and have studied their properties. It has been also prooved that a contravariant almost analytic vector field V with respect to the Riemannian connection D is also contravariant almost analytic with respect to the quarter-symmetric non-metric connection ∇ in a Kähler manifold.

2. Preliminaries

If on an even dimensional differentiable manifold V_n, $n = 2m$, of differentiability class C^{r+1}, there exists a vector valued real linear function F of differentiability class C^r, satisfying

$$F^2X + X = 0, \tag{2.1}$$

for arbitrary vector field X, then V_n is said to be an almost complex manifold and $\{F\}$ is said to give an almost complex structure to V_n [6].

If g is a non singular Hermitian metric of type $(0, 2)$ satisfies

$$g(FX, FY) = g(X, Y) \tag{2.2}$$

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for arbitrary vector fields X and Y, then an almost complex manifold V_n endowed with Hermitian metric g is called an almost Hermitian manifold and the system $\{F, g\}$ is called an almost Hermitian structure [6].

An almost Hermitian manifold V_n is called

(a) a Kähler manifold if

$$(D_X{'F})(Y, Z) = 0, \tag{2.3}$$

(b) a Nearly Kähler manifold if

$$(D_X{'F})(Y, Z) = (D_Y{'F})(Z, X), \tag{2.4}$$

(c) an almost Kähler manifold if

$$(D_X{'F})(Y, Z) + (D_Y{'F})(Z, X) + (D_Z{'F})(X, Y) = 0, \tag{2.5}$$

(d) a Quasi-Kähler manifold if

$$(D_{FX}{'F})(FY, Z) + (D_X{'F})(Y, Z) = 0 \tag{2.6}$$

for arbitrary vector fields X, Y, Z.

If we define

$${'F}(X, Y) \overset{\text{def}}{=} g(FX, Y), \tag{2.7}$$

for arbitrary vector fields X and Y, then

$${'F}(FX, FY) = {'F}(X, Y). \tag{2.8}$$

3. Quarter-symmetric non-metric connection

A linear connection ∇ on (V_n, g) defined as

$$\nabla_X Y = D_X Y + u(Y)FX, \tag{3.1}$$

for arbitrary vector fields X and Y, is said to be a quarter-symmetric non-metric connection [13]. The torsion tensor S of the connection ∇ and the metric tensor g are given by

$$S(X, Y) = u(Y)FX - u(X)FY \tag{3.2}$$

and

$$(\nabla_X g)(Y, Z) = -u(Y)g(FX, Z) - u(Z)g(FX, Y) \tag{3.3}$$

for arbitrary vector fields X, Y, Z ; where u is 1-form on V_n with U as associated vector field, i.e. ,

$$u(X) \overset{\text{def}}{=} g(X, U) \tag{3.4}$$

and D being the Riemannian connection.

Let us put (3.1) as

$$\nabla_X Y = D_X Y + H(X, Y), \tag{3.5}$$

where

$$H(X, Y) = u(Y)FX. \tag{3.6}$$

If we define

$${'H}(X, Y, Z) \overset{\text{def}}{=} g(H(X, Y), Z), \tag{3.7}$$

then in view of (3.6), (3.7) becomes

$${'H}(X, Y, Z) = u(Y)g(FX, Z). \tag{3.8}$$

Theorem 3.1. *If an almost Hermitian manifold V_n admits a quarter-symmetric non-metric connection ∇, then the necessary and sufficient condition for an almost Hermitian manifold to be a Hermitian manifold is that $(\nabla_X F)(Y)$ is hybrid in both the slots, i.e.,*

$$(\nabla_{FX} F)(FY) = (\nabla_X F)(Y).$$

Proof. Covariant derivative of FY with respect to the connection ∇ gives

$$(\nabla_X F)(Y) + F(\nabla_X Y) = \nabla_X FY$$

In consequence of (2.1) and (3.1), last expression becomes

$$(\nabla_X F)(Y) = (D_X F)(Y) + u(Y)X + u(FY)FX \tag{3.9}$$

Replacing X by FX and Y by FY in (3.9) and then using (2.1), we obtain

$$(\nabla_{FX} F)(FY) = (D_{FX} F)(FY) + u(Y)X + u(FY)FX \tag{3.10}$$

Subtracting (3.9) from (3.10), we have

$$(\nabla_{FX} F)(FY) - (\nabla_X F)(Y) = (D_{FX} F)(FY) - (D_X F)(Y) \tag{3.11}$$

A necessary and sufficient condition for an almost Hermitian manifold to be a Hermitian manifold is [6]

$$(D_{FX} F)(FY) = (D_X F)(Y) \tag{3.12}$$

In view of (3.11) and (3.12), we obtain the statement of the theorem. □

Theorem 3.2. *An almost Hermitian manifold with a quarter-symmetric non-metric connection ∇ is an almost Kähler manifold if and only if $'F$ is closed with respect to the connection ∇.*

Proof. We have,

$$\begin{aligned} X('F(Y,Z)) &= (\nabla_X {}'F)(Y,Z) +' F(\nabla_X Y, Z) +' F(Y, \nabla_X Z) \\ &= (D_X {}'F)(Y,Z) +' F(D_X Y, Z) +' F(Y, D_X Z) \end{aligned}$$

Then

$$(\nabla_X {}'F)(Y,Z) = (D_X {}'F)(Y,Z) -' F(\nabla_X Y - D_X Y, Z) -' F(Y, \nabla_X Z - D_X Z)$$

In consequence of (2.1), (2.2) and (3.1), last expression becomes

$$(\nabla_X {}'F)(Y,Z) = (D_X {}'F)(Y,Z) + u(Y)g(X,Z) - u(Z)g(X,Y) \tag{3.13}$$

Taking cyclic sum of (3.13) in X, Y, Z, we have

$$\begin{aligned} (\nabla_X {}'F)(Y,Z) \quad &+ \quad (\nabla_Y {}'F)(Z,X) + (\nabla_Z {}'F)(X,Y) \\ &= \quad (D_X {}'F)(Y,Z) + (D_Y {}'F)(Z,X) + (D_Z {}'F)(X,Y) \end{aligned} \tag{3.14}$$

In consequence of (2.5) and (3.14), we see that $'F$ is closed with respect to the connection ∇. Converse part is obvious from (3.14). □

Theorem 3.3. *If an almost Hermitian manifold admits a quarter-symmetric non-metric connection ∇, then the Nijenhuis tensors of D and ∇ coincide.*

Proof. From (3.9), we have

$$(D_X F)(Y) = (\nabla_X F)(Y) - u(Y)X - u(FY)FX \tag{3.15}$$

Replacing X by FX in (3.15) and then using (2.1), we find

$$(D_{FX} F)(Y) = (\nabla_{FX} F)(Y) - u(Y)FX + u(FY)X \tag{3.16}$$

Interchanging X and Y in (3.16), we obtain

$$(D_{FY} F)(X) = (\nabla_{FY} F)(X) - u(X)FY + u(FX)Y \tag{3.17}$$

Operating F on whole equation of (3.15) and then using (2.1), we have

$$F((D_X F)(Y)) = F((\nabla_X F)(Y)) - u(Y)FX + u(FY)X \tag{3.18}$$

Interchanging X and Y in (3.18), we have

$$F((D_Y F)(X)) = F((\nabla_Y F)(X)) - u(X)FY + u(FX)Y \tag{3.19}$$

The Nijenhuis tensor in an almost Hermitian manifold is defined as [6]

$$N(X,Y) = (D_{FX} F)(Y) - (D_{FY} F)(X) - F((D_X F)(Y)) + F((D_Y F)(X)) \tag{3.20}$$

In view of (3.16), (3.17), (3.18) and (3.19), (3.20) becomes

$$N(X,Y) = (\nabla_{FX} F)(Y) - (\nabla_{FY} F)(X) - F((\nabla_X F)(Y)) + F((\nabla_Y F)(X))$$

$$\Rightarrow N(X,Y) = N^*(X,Y),$$

where

$$N^*(X,Y) = (\nabla_{FX} F)(Y) - (\nabla_{FY} F)(X) - F((\nabla_X F)(Y)) + F((\nabla_Y F)(X))$$

is the Nijenhuis tensor of the connection ∇. □

Corollary 3.4. *An almost Hermitian manifold V_n with a quarter-symmetric non-metric connection ∇ to be a Hermitian manifold if the Nijenhuis tensor of connection ∇ vanishes, i.e., $N^*(X,Y) = 0$.*

Since an almost Hermitian manifold with vanishing Nijenhuis tensor is a Hermitian manifold [6].

Corollary 3.5. *On a Kähler manifold, Nijenhuis tensor with respect to quarter-symmetric non-metric connection ∇ vanishes, i.e., $N^*(X,Y) = 0$.*

The Nijenhuis tensor of the Riemannian connection D vanishes on the Kähler manifold [6].

Theorem 3.6. *A Kähler manifold with a quarter-symmetric non-metric connection ∇ satisfies the relations*

(a) $$(\nabla_{FX} F)(FY) = (\nabla_X F)(Y), \tag{3.21}$$

i.e., $(\nabla_X F)(Y)$ is hybrid in both the slots.

(b) $$(\nabla_X F)(Y) = 0 \Leftrightarrow u(Y) = 0.$$

Proof. In view of (2.3), (3.9) becomes

$$(\nabla_X F)(Y) = u(Y)X + u(FY)FX \tag{3.22}$$

Substituting FX in place of X and FY in place of Y in (3.9) and then using (2.1), we can find

$$(\nabla_{FX} F)(FY) = u(FY)FX + u(Y)X \tag{3.23}$$

In consequence of (3.22) and (3.23), we can find (3.21).

Again, if $(\nabla_X F)(Y) = 0$, then (3.22) gives

$$u(Y)X + u(FY)FX = 0.$$

But X and FX are linearly independent. Hence $u(Y) = 0$, which proves the first part of the statement. Converse part is obvious. □

Theorem 3.7. *Let D be a Riemannian connection on an almost Hermitian manifold V_n and let ∇ be a quarter-symmetric non-metric connection satisfying (3.1) and $(\nabla_X {}'F) = 0$. Then V_n is*

(a) a Kähler manifold if and only if

$$'H(FX, Y, Z) =' H(FX, Z, Y), \tag{3.24}$$

(b) a Nearly Kähler manifold if and only if

$$2'H(FX, Z, Y) =' H(FX, Y, Z) +' H(FY, X, Z), \tag{3.25}$$

(c) a Quasi-Kähler manifold if and only if

$$2'H(X, Z, FY) =' H(X, FY, Z) -' H(FX, Y, Z). \tag{3.26}$$

Proof. In view of (3.8) and $(\nabla_X {}'F) = 0$, (3.13) becomes

$$(D_X {}'F)(Y, Z) =' H(FX, Y, Z) -' H(FX, Z, Y) \tag{3.27}$$

If V_n is a Kähler manifold, then in consequence of (2.3) and (3.27), we obtain (3.24). Conversely when (3.24) is satisfies, then V_n is a Kähler manifold.

From (3.27), we have

$$(D_Y {}'F)(Z, X) =' H(FY, Z, X) -' H(FY, X, Z) \tag{3.28}$$

In view of (3.27), (3.28) and

$$'H(FX, Y, Z) =' H(FZ, Y, X), \tag{3.29}$$

we find

$$\begin{aligned}(D_X {}'F)(Y, Z) \quad &- \quad (D_Y {}'F)(Z, X) =' H(FX, Y, Z) \\ &+ \quad 'H(FY, X, Z) - 2'H(FX, Z, Y)\end{aligned} \tag{3.30}$$

In consequence of (2.4), (3.30) gives (3.25). Converse part is obvious from (3.25) and (3.30).

Now, replacing X and Y by FX and FY in (3.27), we obtain

$$(D_{FX} {}'F)(FY, Z) = -'H(X, FY, Z) +' H(X, Z, FY) \tag{3.31}$$

Adding (3.27) and (3.31) and using $'H(X, Z, FY) +' H(FX, Z, Y) = 0$, we obtain

$$\begin{aligned}(D_{FX} {}'F)(FY, Z) \quad &+ \quad (D_X {}'F)(Y, Z) = -'H(X, FY, Z) \\ &+ \quad 2'H(X, Z, FY) +' H(FX, Y, Z)\end{aligned} \tag{3.32}$$

In consequence of (2.6) and (3.8), (3.32) gives (3.26). Converse part follows immediatly from (3.8) and (3.32). □

Theorem 3.8. *An almost Hermitian manifold V_n admitting a quarter-symmetric non-metric connection ∇ satisfying (3.1) and $(\nabla_X{}'F) = 0$ is an almost Kähler manifold.*

Proof. Cyclic sum of (3.27) in X,Y,Z, we have

$$\begin{aligned}(D_X{}'F)(Y,Z) \;\; + & \;\; (D_Y{}'F)(Z,X) + (D_Z{}'F)(X,Y) \\ = & \;\; {}'H(FX,Y,Z) +' H(FY,Z,X) -' H(FX,Z,Y) \\ - & \;\; {}'H(FY,X,Z) +' H(FZ,X,Y) -' H(FZ,Y,X) \quad (3.33)\end{aligned}$$

In view of (2.5) (3.29) and (3.33), we obtain the statement of the theorem. □

4. Contravariant almost analytic vector fields on a Kähler manifold

If the Lie-derivative of F with respect to a vector field V vanishes identically for all X, i.e.,

$$(L_V F)(X) = 0, \tag{4.1}$$

then V is said to be a contravariant almost analytic vector field [6].

The equation (4.1) is equivalent to

$$[V, FX] = F[V,X] \tag{4.2}$$

In a Kähler manifold, the equation (4.2) becomes

$$(D_{FX}V) - F(D_X V) = 0 \iff F(D_{FX}V) + D_X V = 0 \tag{4.3}$$

Thus, consequently we have the theorem

Theorem 4.1. *On a Kähler manifold, a contravariant almost analytic vector field V with respect to the Riemannian connection D is also contravariant almost analytic with respect to quarter-symmetric non-metric connection ∇.*

Proof. Replacing Y by V in equation (3.1), we have

$$\nabla_X V = D_X V + u(V)FX \tag{4.4}$$

Substituting FX in place of X in (4.4) and then using (2.1), we get

$$\nabla_{FX}V = D_{FX}V - u(V)X \tag{4.5}$$

Operating F on both sides of the equation (4.4) and using (2.1), we find

$$F(\nabla_X V) = F(D_X V) - u(V)X \tag{4.6}$$

Subtracting (4.6) from (4.5), we get

$$(\nabla_{FX}V) - F(\nabla_X V) = (D_{FX}V) - F(D_X V).$$

Since V is a contravariant almost analytic vector field with respect to the Riemannian connection D, therefore we have $D_{FX}V - F(D_X V) = 0$, and then $\nabla_{FX}V - F(\nabla_X V) = 0$. Thus, V is a contravariant almost analytic vector field with respect to the connection ∇. □

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